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# Maximal monotonicity and cyclic monotonicity arising in nonsmooth Lur'e dynamical systems ☆

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ABSTRACT

We study a precomposition of a maximal monotone operator with linear mappings, which preserves the maximal monotonicity in the setting of reflexive Banach spaces. Instead of using the adjoint of such linear operators, as in the usual precomposition, we consider a more general situation involving operators which satisfy the so-called passivity condition. We also provide similar analysis for the preservation of the maximal cyclic monotonicity. These results are applied to derive existence results for nonsmooth Lur'e dynamical systems.



## 1. Introduction

In this paper, we investigate the preservation problem of maximal monotonicity and maximal cyclic monotonicity for the following precomposition, in the setting of reflexive Banach spaces,

$$H := -A + B(F^{-1} + D)^{-1}C,$$

where $F$ is a maximal (cyclically) monotone operator and $A$, $B$, $C$, $D$ are continuous linear mappings. We show that the resulting operator $H$ will remain maximal monotone provided that the classical Rockafellar qualification condition holds and the involved linear mappings are related through the so-called passivity condition; that is, for all $x$ and $y$,

$$\langle Ax, x\rangle + \langle (B - C^T)y, x\rangle - \langle Dy, y\rangle \leq 0.$$

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This condition may hold if $B$ is not necessarily the adjoint of $C$, as it is the case of the passive system $(-id, -id, id, id)$, with $id$ being the identity mapping in a Hilbert space. In this sense, our analysis provides a suitable and strict extension of the classical precomposition with linear operators corresponding to the particular case when $B = C^T$ and $A$, $D$ are the zero operators. Thus, the passivity condition allows one to go beyond the case $B = C^T$ whenever the difference $B - C^T$ remains under some control of mapping $A$ and $D$; in other words,

$$| \langle (B - C^T)y, x\rangle | \leq \langle Dy, y\rangle - \langle Ax, x\rangle.$$

The problem of the maximal cyclic monotonicity preservation follows the same way but with the use of a modified passivity condition, which will be referred to as the cyclic passivity condition.

The motivation in considering such an extension of the classical precomposition comes from the frequent use of passive systems in control theory and issues related to feedback systems; we refer to [10] for more details, namely in linear time-invariant systems connected to static (nonlinear) relations, also called Lur'e dynamical systems [1,7–9,12]. To explain this fact, let us consider the following set-valued Lur'e systems, given in the setting of Hilbert spaces,

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); \quad x(0) = x_0, \\ y(t) = Cx(t) + D\lambda(t), \quad \lambda(t) \in -F(y(t)),\ t \geq 0, \end{cases} \tag{1}$$

where $A : X \to X, B : Y \to X, C : X \to Y, D : Y \to Y$ are given linear bounded mappings, $F : Y \rightrightarrows Y$ is a given maximal monotone operator and $\lambda, y : \mathbb{R}_+ \to Y$ are two unknown connected mappings. It is easy to see that system (1) can be rewritten into the form of a first order differential inclusion involving our operator $H$:

$$\dot{x}(t) \in Ax(t) - B(F^{-1} + D)^{-1}(Cx(t)) = -Hx(t). \tag{2}$$

As a result of this translation, the persistence of the maximal monotonicity of $H$ will allow us to use the rich theory of maximal monotone operators to get existence of (weak and strong) solutions, and different related properties. As well, the preservation of the maximal cyclic monotonicity of $H$ will ensure the existence of strong solutions in the infinite dimensional Hilbert spaces rather than only weak ones.

To the best of our knowledge, set-valued Lur'e systems were firstly introduced and analyzed in a special case in [7], while the maximal monotonicity of $H$ under the passivity condition was firstly studied in [12] in finite dimensional spaces, where the existence and uniqueness of solutions are considered for these set-valued Lur'e systems.

The question of whether the maximal (and the maximal cyclic) monotonicity is preserved under a given algebraic operation has been a central problem in the theory of maximal monotone operators. The main issue is to find explicitly sufficient conditions for the maximality of operations like the sum of two maximal monotone operators or the precomposition of maximal monotone operators with linear continuous mappings. This relatively old topic of research is still active as long as the famous open question of whether the sum $F_1 + F_2$ of two maximal monotone operators $F_1, F_2 : X \rightrightarrows X^*$, defined on a general Banach space $X$ with values in its dual space $X^*$, and satisfying the Rockafellar condition,

$$\operatorname{int}\big(\operatorname{dom}(F_1)\big) \cap \operatorname{dom}(F_2) \neq \emptyset, \tag{3}$$

remains maximal monotone, is still unsolved. However, the answer to this problem is positive in the reflexive case (see, e.g., [18] and also [11]), as well as in some particular situations (see, e.g., [21–23]).

The same situation concerns the maximality problem of the precomposition $A^T FA$ of a maximal monotone operator $F : X \rightrightarrows X^*$ by a linear continuous mapping $A : Y \rightrightarrows X$ with adjoint $A^T : X^* \rightrightarrows Y^*$, where

$Y$ is another Banach space. Indeed, these two problems are equivalent in some sense and one can go forth and back from each problem to the other one: given $F_1$ and $F_2$ as above, we define the (continuous) linear mapping $A : X \to X \times X$ and the monotone operator $F : X \times X \rightrightarrows X^* \times X^*$ as

$$Ax := (x, x), \quad F(x_1, x_2) := F_1(x_1) \times F_2(x_2).$$

Then $F_1 + F_2 = A^T FA$. Conversely, given a maximal monotone operator $F : X \rightrightarrows X^*$ and a linear continuous mapping $A : Y \to X$, one defines the maximal monotone operators $F_1 := N_A$, the normal cone mapping to the graph of $A$, and $F_2 : Y \times X \rightrightarrows Y^* \times X^*$ as

$$F_2(y, x) := \{0\} \times Tx.$$

Then, one obtains $y^* \in A^T FAy \Leftrightarrow (y^*, 0) \in (F_1 + F_2)(y, Ay)$ (see, e.g., [20,14] for more details). Consequently, one easily deduces the maximality of the precomposition under Rockafellar's type condition in the reflexive setting, as well as under other weaker conditions like the Attouch–Brézis condition [2],

$$\mathbb{R}_+\Big(\mathrm{co}\big(\mathrm{dom}(F_1)\big) - \mathrm{co}\big(\mathrm{dom}(F_2)\big)\Big) = X, \tag{4}$$

where $\mathrm{co}(S)$ denotes the convex hull of $S$.

To deal with our problem of the preservation of the maximal (and maximal cyclic) monotonicity of operator $H$, we shall follow a regularized approach based on the Moreau–Yoshida approximation of the initial operator $F$, and then the appeal to Minty's Theorem for both the approximated and the original systems will be the appropriate tool to get the desired result. On another hand, one could also make use of analysis based on Fitzpatrick functions to tackle this preservation problem. Such an approach would necessarily require a useful comparison between the Fitzpatrick functions associated to the operators $F$ and $H$, or at least a relation between the (Fenchel) subdifferential of these two functions (see, e.g., [4,5,16, 24] for the case of classical precomposition). However, we do not follow this root since it will involve a deep analysis, that we let for a future research.

The paper is organized as follows. In Section 2, we recall some definitions and useful results about maximal monotone operators in reflexive Banach spaces as well as the definition of linear passive systems. The main result about the maximal monotonicity (maximal cyclic monotonicity) of $H$ is stated and proved in Section 3 under the passivity (cyclic passivity, respectively) assumption and some interiority condition. Some conclusions and perspectives end the paper in Section 4.

## 2. Notations and mathematical background

In the sequel, $X$ is a real reflexive Banach space with dual $X^*$; we write $\langle x^*, x\rangle = x^*(x)$ for $x \in X$, $x^* \in X^*$ and denote both norms in $X$ and $X^*$ by $\|\cdot\|$ (if there is no confusion). We denote by $\mathbb{B}_X$ the unit ball in $X$ and by $\mathrm{int}(S)$ the interior of a set $S$. The strong and weak convergence in $X$ and $X^*$ are denoted by $\to$ and $\rightharpoonup$, respectively. The set of all proper, convex and lower semi-continuous functions from $X$ to $\mathbb{R} \cup \{+\infty\}$ is denoted by $\Gamma_0(X)$.

Let $J$ be the duality mapping of $X$ defined as

$$J(x) := \{x^* \in X^* : \|x^*\|^2 = \|x\|^2 = \langle x, x^*\rangle\};$$

equivalently, $J$ is the Fenchel subdifferential operator of the function $\frac{1}{2}\|\cdot\|^2$, i.e.,

$$\langle Jx, y - x\rangle \le \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2, \quad \text{for all } x, y \in X. \tag{5}$$

The mapping $J$ is a set-valued monotone operator. Using a renorming result [15], we may suppose that $X$ is equipped with a norm for which $X$ and $X^*$ are locally uniformly convex. Then $J$ is single-valued continuous, bijective, $J^{-1}$ is the duality mapping of $X^*$ and $X$ satisfies the Kadec–Klee property [13], i.e.,

$$\text{if} \quad x_n \rightharpoonup x \quad \text{and} \quad \|x_n\| \to \|x\|, \quad \text{then} \quad x_n \to x, \quad \text{as} \quad n \to +\infty.$$

The domain, the range and the graph of a set-valued mapping $F : X \rightrightarrows X^*$ are defined respectively by

$$\operatorname{dom}(F) = \{x \in X : \; F(x) \neq \emptyset\}, \quad \operatorname{rge}(F) = \bigcup_{x\in X} F(x),$$

and

$$\operatorname{gph}(F) = \{(x, y) : x \in X, y \in F(x)\}.$$

The inverse operator $F^{-1}$ is defined by

$$x \in F^{-1}(y) \Leftrightarrow y \in F(x) \quad \text{for all } x, y \in X;$$

hence, $\operatorname{dom}(F^{-1}) = \operatorname{rge}(F)$ and $\operatorname{rge}(F^{-1}) = \operatorname{dom}(F)$.

The operator $F$ is called $n$-cyclically monotone if for all $(x_i, x_i^*) \in \operatorname{gph}(F), i = 1, \dots, n$, one has that

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0, \quad \text{where} \quad x_{n+1} := x_1. \tag{6}$$

We say that $F$ is cyclically monotone if it is $n$-cyclically monotone for every positive integer $n$. If, in addition, there is no cyclically monotone $G$ such that $\operatorname{gph}(F)$ is strictly contained in $\operatorname{gph}(G)$, then $F$ is called maximal cyclically monotone. In particular, we say that $F$ is monotone if it is 2-cyclically monotone, and maximal monotone if in addition there is no monotone set-valued mapping $G$ such that $\operatorname{gph}(F)$ is contained strictly in $\operatorname{gph}(G)$. It is well-known [17] that a maximal cyclically monotone operator is necessarily the graph of the (Fenchel) subdifferential of a function $\varphi \in \Gamma_0(X)$. The converse of this statement is obviously true.

Let us remind some fundamental properties of maximal monotone operators (see, e.g., [3] for more details).

**Proposition 1.** *Let $F : X \rightrightarrows X^*$ be a maximal monotone operator. Then:*

(*i*) *$F^{-1} : X^* \rightrightarrows X$ is maximal monotone.*
(*ii*) *$F$ is weakly-strongly closed; i.e., if $y_n \in F(x_n)$, $x_n \rightharpoonup x$ and $y_n \to y$ as $n \to +\infty$, then $y \in F(x)$.*
(*iii*) *$F$ is locally bounded in* $\operatorname{int}(\operatorname{dom}(F))$.

Next, we recall in the setting of reflexive Banach spaces Minty's Theorem (see [11,18]) and Yosida approximation; see, also, [3].

**Proposition 2.** *Let $X$ be a reflexive Banach space, equipped with a norm for which $X$ and $X^*$ are locally uniformly convex. Let $F : X \rightrightarrows X^*$ be a monotone operator. Then $F$ is maximal monotone if and only if* $\operatorname{rge}(F + J) = X^*$.

Let $F : X \rightrightarrows X^*$ be a maximal monotone operator and let $\lambda > 0$. For each $x \in X$, let $x_\lambda$ be the unique solution of the inclusion

$$0 \in J(x_\lambda - x) + \lambda F(x_\lambda). \tag{7}$$

Define

$$J_\lambda(x) := x_\lambda \quad \text{and} \quad F_\lambda x := \lambda^{-1} J(x - x_\lambda).$$

Then the single-valued mappings $J_\lambda, F_\lambda : X \to X^*$ are called *resolvent* and *Yosida approximation* of $F$, respectively. It is known that $F_\lambda = (F^{-1} + \lambda J^{-1})^{-1}$ and

$$F_\lambda(x) \in F(J_\lambda x),\ J(x - J_\lambda x) = \lambda F_\lambda(x), \quad \text{for all } x \in X. \tag{8}$$

**Proposition 3.** *Let $F : X \rightrightarrows X^*$ be a maximal monotone operator and $\lambda > 0$. Then:*

*(i) $F_\lambda$ is single-valued, monotone, bounded, and demicontinuous (i.e., continuous from $X$ to $X^*_w$).*
*(ii) $\|F_\lambda(x)\| \le \|F^0(x)\|$ for all $x \in \mathrm{dom}(F)$, where $F^0(x)$ is the element of minimum norm in $F(x)$.*

Next, we introduce the concept of *passive systems* in infinite dimensional reflexive Banach spaces. This notion is well-used in control theory [8–10]. Given two reflexive Banach spaces $X$ and $Y$, let $A : X \to X^*$, $B : Y^* \to X^*$, $C : X \to Y$ and $D : Y^* \to Y$ be given linear bounded operators.

**Definition 1.** The system $(A, B, C, D)$ is said to be *passive* if for all $x \in X$ and $y \in Y^*$, we have

$$\langle Ax, x\rangle + \langle (B - C^T)y, x\rangle - \langle Dy, y\rangle \le 0, \tag{9}$$

where $C^T : Y^* \to X^*$ is the adjoint of $C$.

**Remark 1.** The standard definition of passivity in systems theory is based on the trajectories of the system via the so-called dissipation inequality ([10,12], see, also, [1]); i.e., there exists a function $V : X \to \mathbb{R}_+$, called a storage function, such that

$$V(x(t_1)) + \int_{t_1}^{t_2} \langle \lambda(\tau), y(\tau)\rangle d\tau \le V(x(t_2)), \quad 0 \le t_1 \le t_2,$$

where $x(\cdot), y(\cdot)$ and $\lambda(\cdot)$ satisfy system (1), when stated in a Hilbert setting. Definition 1 gives an equivalent condition to that conceptual definition when the storage function is $x \mapsto \frac{1}{2}x^T x$.

We have the following lemma.

**Lemma 1.** *If $(A, B, C, D)$ is passive, then:*

*(i) $D$ is monotone.*
*(ii) For all $y \in Y^*$, one has $\|(B - C^T)y\|^2 \le 4\|A\|\langle Dy, y\rangle$.*
*(iii) For all $x \in X$, one has $\|(B^T - C)x\|^2 \le 4\|D\|\langle -Ax, x\rangle$.*

**Proof.** $(i)$ This assertion follows by taking $x = 0$ in (9).

$(ii)$ Fix $y \in Y^*$. If $(B - C^T)y = 0$, then the conclusion holds trivially. If $(B - C^T)y \ne 0$, then for $x := kJ^{-1}\frac{(B-C^T)y}{\|(B-C^T)y\|}$ with $k \in \mathbb{R}$, property (9) gives us

$$\|A\|k^2 - \|(B - C^T)y\|k + \langle Dy, y\rangle \ge 0, \quad \forall k \in \mathbb{R}. \tag{10}$$

Thus, we must have

$$\|(B - C^T)y\|^2 - 4\|A\|\langle Dy, y\rangle \leq 0,$$

and the conclusion follows.

($iii$) The proof of this assertion is similar to ($ii$). □

**Remark 2.** From Lemma 1, we obtain the following relation

$$\|B - C^T\| \leq 2\sqrt{\|A\|\|D\|}, \tag{11}$$

for every passive system $(A, B, C, D)$. Thus, operators $A$ and $D$ measure via the passivity condition (9) the gap between $B$ and the adjoint of $C$.

## 3. Main results

In this section, we consider full domain linear continuous mappings $A : X \to X^*$, $B : Y^* \to X^*$, $C : X \to Y$, and $D : Y^* \to Y$, together with a maximal monotone operator $F : Y \rightrightarrows Y^*$, where $X$ and $Y$ are reflexive Banach spaces with the renorming assumption. In what follows, we shall suppose for consistency purposes that

$$C^{-1}(\mathrm{rge}(F^{-1} + D)) \neq \emptyset. \tag{12}$$

We give the first main theorem.

**Theorem 1.** *Assume that $(A, B, C, D)$ is a passive system. If $\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \neq \emptyset$ or if $B$ is bijective, then the operator $H : X \rightrightarrows X^*$ defined by*

$$H(x) := -Ax + B(F^{-1} + D)^{-1}(Cx),$$

*is maximal monotone.*

**Proof.** Let us first observe that, thanks to (12), the domain of $H$ is not empty. To show that $H$ is monotone, we consider $x_i \in X$ and $y_i \in (F^{-1} + D)^{-1}(Cx_i)$ $(i = 1, 2)$ such that $Cx_i - Dy_i \in F^{-1}(y_i)$. Using the monotonicity of $F^{-1}$, we have

$$\langle C(x_1 - x_2) - D(y_1 - y_2), y_1 - y_2\rangle \geq 0.$$

Hence, invoking the passivity of $(A, B, C, D)$, we get

$$\langle -A(x_1 - x_2), x_1 - x_2\rangle + \langle B(y_1 - y_2), x_1 - x_2\rangle \geq \langle C(x_1 - x_2) - D(y_1 - y_2), y_1 - y_2\rangle \geq 0,$$

and the monotonicity of $H$ follows.

In order to establish that $H$ is maximal, according to Minty's Theorem we need to show that for every given $x^* \in X^*$, there exists $x \in X$ such that

$$x^* \in H(x) + J(x), \tag{13}$$

where $J$ is the duality mapping of $X$. Note that $D$ is maximal monotone because it is single-valued, monotone and continuous with full domain (see, e.g., [3] for more details). Hence, according to [18, Theorem 1], the sum operator $F^{-1}+D$ is maximal monotone, and so is its inverse $S := (F^{-1}+D)^{-1}$. For $\lambda > 0$, we consider the Yosida approximation of $S$,

$$S_\lambda = (S^{-1} + \lambda J^{-1})^{-1} = (F^{-1} + D + \lambda J^{-1})^{-1},$$

and define the operator $H^\lambda : X \to X^*$ as follows

$$H^\lambda(x) := -Ax + BS_\lambda(Cx),\ x \in X.$$

We can easily prove, thanks to the passivity condition, that $H^\lambda$ is a monotone operator. Moreover, like $S_\lambda$ the operator $H^\lambda$ is a single-valued demicontinuous operator with full domain. Thus, again by [3], it follows that $H^\lambda$ is maximal monotone. By Minty's Theorem, there exists a unique $x_\lambda \in X$ such that

$$x^* = H^\lambda(x_\lambda) + Jx_\lambda. \tag{14}$$

We fix $x_0 \in X$ such that (recall (12))

$$Cx_0 \in \operatorname{rge}(F^{-1} + D) = \operatorname{dom}(F^{-1} + D)^{-1} = \operatorname{dom}(S),$$

and set

$$x^*_\lambda := -Ax_0 + BS_\lambda(Cx_0) + Jx_0 = H^\lambda(x_0) + Jx_0. \tag{15}$$

Then we have

$$\|x^*_\lambda\| \le \|Ax_0\| + \|B\|\|S_\lambda(Cx_0)\| + \|x_0\| \le \|Ax_0\| + \|B\|\|S^0(Cx_0)\| + \|x_0\|, \tag{16}$$

which shows that the net $(x^*_\lambda)_\lambda$ is bounded. Moreover, since $H^\lambda$ is monotone, from (14) and (15) we obtain

$$\langle x^* - x^*_\lambda, x_\lambda - x_0\rangle \ge \langle Jx_\lambda - Jx_0, x_\lambda - x_0\rangle \ge (\|x_\lambda\| - \|x_0\|)^2 \ge \|x_\lambda\|^2 - 2\|x_\lambda\|\|x_0\|.$$

Consequently

$$\|x_\lambda\|^2 \le (2\|x_0\| + \|x^*\| + \|x^*_\lambda\|)\|x_\lambda\| + \|x_0\|(\|x^*\| + \|x^*_\lambda\|). \tag{17}$$

Using the fact that $(x^*_\lambda)_{\lambda>0}$ is bounded, we deduce from (17) that $(x_\lambda)_{\lambda>0}$ is bounded. Moreover, since $x^* = -Ax_\lambda + BS_\lambda(Cx_\lambda) + J(x_\lambda)$, the net $\big(BS_\lambda(Cx_\lambda)\big)_{\lambda>0}$ is bounded. Consequently, if $B$ is bijective, then we conclude that $\big(S_\lambda(Cx_\lambda)\big)_{\lambda>0}$ is bounded. Let us show that this net is also bounded under the condition

$$\operatorname{rge}(C) \cap \operatorname{int}(\operatorname{rge}(F^{-1} + D)) \ne \emptyset.$$

Proceeding by contradiction, if this was not the case, then there would exist $\lambda_n > 0$ and $y_n = S_{\lambda_n}(Cx_{\lambda_n})$ such that $\|y_n\| \to +\infty$ as $n \to +\infty$. Take $u$ in $\operatorname{rge}(C) \cap \operatorname{int}(\operatorname{rge}(D + F^{-1}))$ and let $r > 0$ be such that

$$u + r\mathbb{B}_Y \subset \operatorname{rge}(F^{-1} + D) = \operatorname{dom}(F^{-1} + D)^{-1}, \tag{18}$$

and $(F^{-1} + D)^{-1}$ is bounded on $u + r\mathbb{B}_Y$ (recall Proposition 1). Set

$$w_n = u + rJ^{-1}\xi_n,$$

where $\xi_n := y_n/\|y_n\|$, so that $w_n \in u + r\mathbb{B}_Y$. By (18) we pick some $z_n$ in $(F^{-1}+D)^{-1}(w_n) \subset (F^{-1}+D)^{-1}(u + r\mathbb{B}_Y)$. Therefore the sequence $(z_n)$ is also bounded due to the choice of $r$ above. Since $w_n - Dz_n \in F^{-1}(z_n)$ and $y_n = S_{\lambda_n}(Cx_{\lambda_n})$; that is, $Cx_{\lambda_n} - Dy_n - \lambda_n J^{-1}y_n \in F^{-1}(y_n)$, from the monotonicity of $F^{-1}$, we get

$$\langle Cx_{\lambda_n} - w_n - D(y_n - z_n) - \lambda_n J^{-1}y_n, y_n - z_n\rangle \geq 0, \tag{19}$$

which, by the relation $J^{-1} = \partial(\frac{1}{2}\|\cdot\|^2)$, implies that

$$\langle Cx_{\lambda_n} - w_n - D(y_n - z_n), y_n - z_n\rangle \geq \langle \lambda_n J^{-1}y_n, y_n - z_n\rangle \geq \frac{\lambda_n}{2}(\|y_n\|^2 - \|z_n\|^2). \tag{20}$$

Dividing both sides of (20) by $\|y_n\|^2$, one has

$$\frac{\langle Cx_{\lambda_n} - w_n - D(y_n - z_n), y_n - z_n\rangle}{\|y_n\|^2} \geq \frac{\lambda_n}{2}(1 - \frac{\|z_n\|^2}{\|y_n\|^2}). \tag{21}$$

Thus for $n$ large enough, we obtain

$$\frac{\langle Cx_{\lambda_n} - w_n - D(y_n - z_n), y_n - z_n\rangle}{\|y_n\|^2} \geq 0. \tag{22}$$

Since $(x_{\lambda_n})$, $(w_n)$, and $(z_n)$ are bounded, by passing the limit as $n \to +\infty$ in (22), one gets (taking into account that $D$ is monotone by Lemma 1)

$$\lim_{n\to+\infty} \langle D\xi_n, \xi_n\rangle = 0.$$

Hence, thanks to Lemma 1, and the fact that $\lim\limits_{n\to+\infty} B\xi_n = 0$ due to the boundedness of the sequence $(By_n)_n = (BS_{\lambda_n}(Cx_{\lambda_n}))_n$, we conclude that

$$\lim_{n\to+\infty} C^T\xi_n = \lim_{n\to+\infty} (B - C^T)(\xi_n) = 0. \tag{23}$$

Therefore

$$\lim_{n\to+\infty} \langle Cx_{\lambda_n}, \xi_n\rangle = \lim_{n\to+\infty} \langle x_{\lambda_n}, C^T\xi_n\rangle = 0.$$

Using the monotonicity of $D$ and $J^{-1}$, we deduce from (19) that

$$\langle Cx_{\lambda_n} - w_n, y_n - z_n\rangle \geq \lambda_n\langle J^{-1}(y_n), y_n - z_n\rangle \geq \frac{\lambda_n}{2}(\|y_n\|^2 - \|z_n\|^2). \tag{24}$$

By dividing both sides of (24) by $\|y_n\|$ and for $n$ large enough, one has

$$\frac{\langle Cx_{\lambda_n} - w_n, y_n - z_n\rangle}{\|y_n\|} \geq 0. \tag{25}$$

Taking the lower limit as $n \to +\infty$ in (25), we get

$$\liminf_{n\to+\infty} \langle -w_n, \xi_n\rangle \geq 0.$$

Thus, recalling that $u \in \mathrm{rge}(C)$, (23) leads us to the contradiction

$$0 < r = \lim_{n\to+\infty}\langle rJ^{-1}\xi_n, \xi_n\rangle = \lim_{n\to+\infty}\langle u + rJ^{-1}\xi_n, \xi_n\rangle = \limsup_{n\to+\infty}\langle w_n, \xi_n\rangle \le 0.$$

Consequently, the net $\big(S_\lambda(Cx_\lambda)\big)_{\lambda>0}$ is bounded.

We now show that $(x_\lambda)$ has a convergent subnet. Given $\lambda, \mu > 0$ we write

$$\begin{aligned} x^* = -Ax_\lambda + BS_\lambda(Cx_\lambda) + J(x_\lambda) = -Ax_\mu + BS_\mu(Cx_\mu) + J(x_\mu), \\ y_\lambda := S_\lambda(Cx_\lambda) = (F^{-1} + D + \lambda J^{-1})^{-1}(Cx_\lambda), \end{aligned} \tag{26}$$

and

$$y_\mu := S_\mu(Cx_\mu) = (F^{-1} + D + \mu J^{-1})^{-1}(Cx_\mu),$$

so that

$$J(x_\lambda) - J(x_\mu) = A(x_\lambda - x_\mu) - B(y_\lambda - y_\mu).$$

By multiplying both sides of this last equality by $x_\lambda - x_\mu$, the passivity of $(A, B, C, D)$ gives us

$$\begin{aligned} \langle J(x_\lambda) - J(x_\mu), x_\lambda - x_\mu\rangle &= \langle A(x_\lambda - x_\mu) - B(y_\lambda - y_\mu), x_\lambda - x_\mu\rangle \\ &\le \langle -C(x_\lambda - x_\mu) + D(y_\lambda - y_\mu), y_\lambda - y_\mu\rangle. \end{aligned} \tag{27}$$

Since $Cx_\lambda - Dy_\lambda - \lambda J^{-1}(y_\lambda) \in F^{-1}(y_\lambda)$ and $Cx_\mu - Dy_\mu - \mu J^{-1}(y_\mu) \in F^{-1}(y_\mu)$, the monotonicity of $F^{-1}$ and $J^{-1}$ gives us

$$\begin{aligned} &\langle -C(x_\lambda - x_\mu) + D(y_\lambda - y_\mu), y_\lambda - y_\mu\rangle \\ &\le -\langle \lambda J^{-1}(y_\lambda) - \mu J^{-1}(y_\mu), y_\lambda - y_\mu\rangle \\ &= -\lambda\langle J^{-1}(y_\lambda) - J^{-1}(y_\mu), y_\lambda - y_\mu\rangle - (\lambda - \mu)\langle J^{-1}(y_\mu), y_\lambda - y_\mu\rangle \\ &\le -(\lambda - \mu)\langle J^{-1}(y_\mu), y_\lambda - y_\mu\rangle \le |\lambda - \mu|\|y_\mu\|\|y_\lambda - y_\mu\|. \end{aligned} \tag{28}$$

Thus, as $y_\lambda$ and $y_\mu$ are bounded, we conclude from (27) that for some constant $c > 0$ it holds

$$\langle J(x_\lambda) - J(x_\mu), x_\lambda - x_\mu\rangle \le c|\lambda - \mu|, \ \forall\, \lambda, \mu > 0, \tag{29}$$

which implies, again by the monotonicity of $J$, that

$$\limsup_{\lambda,\mu\to 0}\langle J(x_\lambda) - J(x_\mu), x_\lambda - x_\mu\rangle = 0. \tag{30}$$

Furthermore, since $(x_\lambda)_{\lambda>0}$ and $(J(x_\lambda))_{\lambda>0}$ are bounded as we showed above, we may suppose that $(x_\lambda)_{\lambda>0}$ and $(J(x_\lambda))_{\lambda>0}$ weakly converge to some $x \in X$ and $y \in X^*$, respectively. Thus, according to [3, Lemma 2.3], (30) ensures that $y = J(x)$ and

$$\langle J(x_\lambda), x_\lambda\rangle \to \langle y, x\rangle = \langle J(x), x\rangle \quad \text{as } \lambda \to 0^+,$$

that is, $\lim_{\lambda\to 0^+}\|x_\lambda\| = \|x\|$. Then the Kadec–Klee property guarantees that $(x_\lambda)$ strongly converges to $x$.

To complete the proof, we need only to show that $x$ satisfies (13). Indeed, since $(y_\lambda) = (S_\lambda(Cx_\lambda))$ is bounded, we may suppose that it weakly converges to some $\eta \in Y^*$. Then by taking the limit as $\lambda \to 0^+$ in (26), we obtain

$$x^* = -Ax + B\eta + J(x).$$

Moreover, recalling (8), we have that $y_\lambda = S_\lambda(Cx_\lambda) \in S(J_\lambda(Cx_\lambda))$ and

$$\|J_\lambda(Cx_\lambda) - Cx\| \le \|J_\lambda(Cx_\lambda) - Cx_\lambda\| + \|Cx_\lambda - Cx\| \le \lambda\|S_\lambda(Cx_\lambda)\| + \|C\|\|x_\lambda - x\|,$$

which shows that $J_\lambda(Cx_\lambda)$ converges strongly to $Cx$. Consequently, by Proposition 1 it follows that $\eta \in S(Cx)$; that is,

$$x^* \in -Ax + B(F^{-1} + D)^{-1}(Cx) + J(x).$$

The proof is completed. □

**Remark 3.** (i) The conclusion of Theorem 1 is also true if the operators $A$ and $D$ used there are Lipchitz continuous with $A(0) = 0$ instead of just being linear and bounded. The proof of this fact is exactly the same as the one of Theorem 1.

(ii) Operators $B$ and $C$ play a symmetric role within the passivity condition. Indeed, it is easily seen that Definition 1 is equivalent to, for all $x \in X$ and $y \in Y^*$

$$\langle Ax, x\rangle + \langle (C^T - B)y, x\rangle - \langle Dy, y\rangle \le 0;$$

that is, the system $(A, C^T, B^T, D)$ is passive. Consequently, applying Theorem 1 to this new system, the operator

$$-A + C^T(F^{-1} + D)^{-1}B^T$$

is maximal monotone, provided that either $C$ is bijective or $\mathrm{rge}(B^T) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \ne \emptyset$.

(iii) The following corollary shows that in order to obtain the maximal monotonicity of $H = -A + B(F^{-1} + D)^{-1}C$, one can check the passivity of $(A, kB, C, D)$ for some $k \ge 1$ instead of the passivity of $(A, B, C, D)$.

**Corollary 1.** *Assume that the system $(A, kB, C, D)$ is passive for some $k \ge 1$. If $B$ is bijective or*

$$\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \ne \emptyset,$$

*for a maximal monotone operator $F : Y \rightrightarrows Y^*$, then $H = -A + B(F^{-1} + D)^{-1}C$ is maximal monotone.*

**Proof.** Theorem 1 implies the maximal monotonicity of the operator

$$H_k = -A + kB(F^{-1} + D)^{-1}C.$$

Then the operator

$$H = k^{-1}H_k - (1 - k^{-1})A$$

is also maximal monotone, since the mapping $-A$ is maximal monotone and $1 - k^{-1} \ge 0$. □

The statement of Corollary 1 is not true for $k < 1$, as we show in the following example.

**Example 1.** In $\mathbb{R}^2$ we take $F^{-1} \equiv 0$ and $-A = -kB = C = D = I_2$ for $0 < k < 1$. Then $(A, kB, C, D)$ is passive, but the operator

$$H = -A + B(F^{-1} + D)^{-1}C = (1 - k^{-1})I_2$$

is not monotone. For the case $k = 0$, the counterexample is $-A = -B = \frac{1}{2}C = D = I_2$, while for $k < 0$ it suffices to take $A = 0$ and $kB = C = D = I_2$.

Next, we analyze the preservation of the maximal cyclic monotonicity of the operator $H = -A+B(F^{-1}+D)^{-1}C$. More precisely if $F$ is maximal cyclically monotone, then, according to [17], there exists a proper lower semi-continuous convex function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$F = \partial\varphi.$$

Hence, if $A$ and $D$ are the zero operators, and $B = C^T$, so that the passivity condition obviously holds, the operator $H$ is written as

$$H = C^T\, \partial\varphi\, C.$$

Consequently, under the interiority assumption of Theorem 1, for instance, we obtain ([18])

$$H = \partial(\varphi \circ C),$$

which again by [18] shows that $H$ is a maximal cyclically monotone operator.

Our aim here is to enforce the passivity condition given in (9) in order to guarantee the preservation of the maximal cyclic monotonicity of the operator $H$. In some situations, the passivity condition alone could ensure this preservation, as it is the case when $B = kC^T$ for some constant $k > 0$ and the linear operators $-A$ and $D$ are symmetric and monotone. However, it is not true in general as the following example shows.

**Example 2.** Let us consider in $\mathbb{R}^2$ the system $(A, B, C, D)$ and the operator $F$, where $F^{-1} \equiv 0$ and

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we can easily verify that $(A, B, C, D)$ is passive and that $B$ is bijective, so that the hypothesis of Theorem 1 holds and the operator $H = -A + B(F^{-1} + D)^{-1}C$ is maximal monotone. However,

$$H = -A + BC = \begin{pmatrix} 1 & 0 \\ -1 & 5/4 \end{pmatrix}$$

is not maximal cyclically monotone.

**Definition 2.** The system $(A, B, C, D)$ is said to be *cyclically passive* if it is passive and for each positive integer $n \geq 3$, given $x_i \in X$, and $y_i \in Y^*$ $(i = 1, 2, \dots, n)$ satisfying that $\{(Cx_i - Dy_i, y_i) : \; i = 1, \dots, n\}$ is $n$-cyclically monotone, we have (with $x_{n+1} := x_1,\; y_{n+1} := y_1$)

$$\sum_{i=1}^{n} \langle -Ax_i, x_{i+1} - x_i\rangle + \sum_{i=1}^{n} \langle (B - C^T)y_i, x_{i+1} - x_i\rangle + \langle y_i, D(y_{i+1} - y_i)\rangle \leq 0, \tag{31}$$

or equivalently,

$$\sum_{i=1}^{n}\langle -Ax_i + By_i, x_{i+1} - x_i\rangle \leq \sum_{i=1}^{n}\langle y_i, C(x_{i+1} - x_i) - D(y_{i+1} - y_i)\rangle. \tag{32}$$

Definition 2 covers the classical setting when $B = C^T$ and $-A, D$ are monotone symmetric operators. It is easy to see that the cyclic passivity implies the passivity, but in general the two notions are not equivalent (see Example 2). We have the following theorem.

**Theorem 2.** *Let $F : Y \rightrightarrows Y^*$ be a maximal cyclically monotone operator. Assume that $(A, B, C, D)$ is a cyclically passive system such that either $\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \neq \emptyset$ or $B$ is bijective. Then the operator $H = -A + B(F^{-1} + D)^{-1}C$ is maximal cyclically monotone.*

**Proof.** Theorem 1 ensures that $H$ is maximal monotone, and so it remains to check that $H$ is cyclically monotone. Given $n \geq 3$, $x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_n \in Y^*$ such that $y_i \in (F^{-1} + D)^{-1}Cx_i$ $(i = 1, 2, \ldots, n)$, we have $y_i \in F(Cx_i - Dy_i)$. By the cyclic monotonicity of $F$, one obtains that $\{(Cx_i - Dy_i, y_i) :\ i = 1, \ldots, n\}$ is $n$-cyclically monotone, i.e.,

$$\sum_{i=1}^{n}\langle y_i, C(x_{i+1} - x_i) - D(y_{i+1} - y_i)\rangle \leq 0.$$

Then the cyclic passivity of the system $(A, B, C, D)$ ensures that

$$\sum_{i=1}^{n}\langle -Ax_i + By_i, x_{i+1} - x_i\rangle \leq \sum_{i=1}^{n}\langle y_i, C(x_{i+1} - x_i) - D(y_{i+1} - y_i)\rangle \leq 0.$$

The arbitrariness of $n$ allows us to conclude that $H$ is cyclically monotone. □

We give an example to illustrate Theorem 2.

**Example 3.** We consider the case where $X = Y = L^2(0, 1)$. We fix an integer $m \geq 1$ and denote by $\{e_1, e_2, \ldots, e_m, \ldots\}$ the orthonormal basis of $L^2(0, 1)$. We define the operators $A, B, C, D : L^2(0, 1) \to L^2(0, 1)$ as follows

$$De_j = e_j \text{ for } 1 \leq j \leq m, \quad De_j = 0 \text{ for } j \geq m + 1,$$

and

$$A = 2B = -2C = -D.$$

Let $F : L^2(0, 1) \rightrightarrows L^2(0, 1)$ be a maximal cyclically monotone operator such that $\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \neq \emptyset$. According to Theorem 2, in order to conclude that the operator $H = -A + B(F^{-1} + D)^{-1}C$ is maximal cyclically monotone, we only need to show that the system $(A, B, C, D)$ is cyclically passive. Clearly $(A, B, C, D)$ is passive. Next, we consider any integer $n \geq 3$ and pick in $L^2(0, 1)$ the vectors $x_i = (x_{i1}, x_{i2}, \ldots, x_{im}, \ldots)$ and $y_i = (y_{i1}, y_{i2}, \ldots, y_{im}, \ldots)$, $i = 1, \ldots, n$ such that the set

$$\{(Cx_i - Dy_i, y_i) :\ i = 1, \ldots, n\} \text{ is } n\text{-cyclically monotone.}$$

It is sufficient to prove that

$$\sum_{i=1}^{n}\{\langle -Ax_i, x_{i+1}-x_i\rangle - 2\langle y_i, C(x_{i+1}-x_i)\rangle + \langle D(y_{i+1}-y_i), y_i\rangle\} \le 0.$$

Since $\{(Cx_i - Dy_i, y_i):\ i = 1,\dots,n\}$ is $n$-cyclically monotone, one has

$$\sum_{i=1}^{n}\langle y_{i+1}, C(x_{i+1}-x_i) - D(y_{i+1}-y_i)\rangle \ge 0,$$

which implies that

$$\sum_{i=1}^{n}\langle y_{i+1}, C(x_{i+1}-x_i)\rangle \ge \sum_{i=1}^{n}\langle y_{i+1}, D(y_{i+1}-y_i)\rangle \ge 0,$$

where the last inequality comes from the fact that $D$ is monotone and symmetric. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n}\{\langle -Ax_i, x_{i+1}-x_i\rangle - 2\langle y_i, C(x_{i+1}-x_i)\rangle + \langle D(y_{i+1}-y_i), y_i\rangle\}\\
&\le \sum_{i=1}^{n}\{\langle -Ax_i, x_{i+1}-x_i\rangle - 2\langle y_i - y_{i+1}, C(x_{i+1}-x_i)\rangle + \langle D(y_{i+1}-y_i), y_i\rangle\}\\
&= \sum_{j=1}^{m}\sum_{i=1}^{n}\Big\{x_{ij}(x_{(i+1)j}-x_{ij}) + (y_{(i+1)j}-y_{ij})(x_{(i+1)j}-x_{ij}) + y_{ij}(y_{(i+1)j}-y_{ij})\Big\}\\
&= -\frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\Big\{(x_{(i+1)j}-x_{ij})^2 - 2(y_{(i+1)j}-y_{ij})(x_{(i+1)j}-x_{ij}) + (y_{(i+1)j}-y_{ij})^2\Big\}\\
&\le 0.
\end{aligned}$$

If $F : Y \rightrightarrows Y^*$ is a maximal cyclically monotone operator, then $F = \partial\varphi$ for some $\varphi \in \Gamma_0(Y)$. Under the assumptions of Theorem 2, the operator $H$ is also maximal cyclically monotone operator, i.e., $H = \partial\Phi$ for some $\Phi \in \Gamma_0(X)$. According to [17], $\Phi$ is given by the following formula

$$\Phi(x) = \sup\{\langle x - x_n, x_n^*\rangle + \ldots + \langle x_1 - x_0, x_0^*\rangle\}, \tag{33}$$

where $x_i^* \in -Ax_i + B(\partial\varphi^{-1} + D)^{-1}Cx_i,\ i = 1,\dots,n$, and the supremum is taken over all possible finite sets of such pairs $(x_i, x_i^*)$.

In some particular cases, it is possible to find a function $\Phi$ explicitly in terms of the data $\varphi, A, B, C$ and $D$.

**Proposition 4.** *Assume that $X = Y = \mathbb{R}^n$ and $F = \partial\varphi$, where $\varphi(x) := \varphi_1(x_1) + \ldots + \varphi_n(x_n)$, $x = (x_1,\dots,x_n)^T$, $\varphi_i \in \Gamma_0(\mathbb{R})$, $i = 1,\dots,n$. Let $B = [b_1,\dots,b_n], C = [c_1,\dots,c_n], D = [d_1,\dots,d_n]$ be diagonal matrices $(b_i, c_i, d_i \in \mathbb{R}, c_i \ne 0, i = 1,\dots,n)$ and $A$ be symmetric such that $(A, B, C, D)$ is passive. Then $H = \partial\Phi$ with*

$$\Phi(x) = \frac{1}{2}\langle -Ax, x\rangle + \frac{b_1}{c_1}\varphi_{1,d_1}(c_ix_i) + \ldots + \frac{b_n}{c_n}\varphi_{n,d_n}(c_nx_n),$$

*where $\varphi_{i,d_i}(x_i) := \inf(\varphi_i(\cdot) + \frac{1}{2d_i}|x_i - \cdot|^2)$ for $d_i > 0$, and $\varphi_{i,0} := \varphi_i$.*

**Proof.** If we set

$$\omega_A(x) := \frac{1}{2}\langle -Ax, x\rangle, \quad \omega_D(x) := \frac{1}{2}\langle Dx, x\rangle, \quad x \in \mathbb{R}^n,$$

then we get, using [19, Theorem 23.9], together with the interior condition ($\mathrm{rge}(C) = \mathbb{R}^n$),

$$\begin{aligned} H &= \nabla\omega_A(x) + B(\partial\varphi^* + \nabla\omega_D)^{-1}Cx \\ &= \nabla\omega_A(x) + B\partial(\varphi^* + \omega_D)^*Cx \\ &= \nabla\omega_A(x) + BC^{-1}C\partial(\varphi \,\square\, \omega_D^*)Cx \\ &= \nabla\omega_A(x) + BC^{-1}\partial((\varphi \,\square\, \omega_D^*) \circ C)x, \end{aligned}$$

where $\square$ denotes the inf-convolution. Note that due to the passivity condition we have $d_i \geq 0$ and $b_i = c_i$ whenever $d_i = 0$. Thus simple computations show that

$$BC^{-1}\partial((\varphi \,\square\, \omega_D^*) \circ C)x = \partial(\frac{b_1}{c_1}\varphi_{1,d_1}(c_i x_i) + \ldots + \frac{b_n}{c_n}\varphi_{n,d_n}(c_n x_n)), \tag{34}$$

and the conclusion follows by using again [19, Theorem 23.9]. $\square$

In this last part, we give an application of Theorem 2 to the existence problem of solutions of Lur'e systems in infinite dimensional Hilbert space [1,8–10]. The originality of this work with respect to the previous ones, namely, the result of [1], resides in the fact that here the initial condition may be given by any point of the boundary of the domain of $H$, possibly not in the domain.

We suppose that $X$ and $Y$ are two Hilbert spaces, and consider the following set-valued Lur'e dynamical system,

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); \\ y(t) = Cx(t) + D\lambda(t), \\ \lambda(t) \in -F(y(t)), \ t \geq 0; \\ x(0) = x_0, \end{cases} \tag{35}$$

where $A : X \to X, B : Y \to X, C : X \to Y, D : Y \to Y$ are as above, and $\lambda, y : \mathbb{R}_+ \to Y$ are two connected (unknown) mappings. It is not difficult to see that (35) can be recasted to the first-order differential inclusion

$$\dot{x}(t) \in Ax(t) - B(F^{-1} + D)^{-1}(Cx(t)) = -Hx(t). \tag{36}$$

**Theorem 3.** *Let $X$ and $Y$ be two Hilbert spaces, and let $F : Y \rightrightarrows Y$ be a maximal cyclically monotone operator. Suppose that $(A, B, C, D)$ is a cyclically passive system such that $B$ is bijective or*

$$\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(F^{-1} + D)) \neq \emptyset.$$

*Then, for each $x_0 \in \mathrm{cl}(C^{-1}(\mathrm{rge}(F^{-1} + D)))$, Lur'e system (35) has a unique strong solution.*

**Proof.** By Theorem 2 the operator $H$ is maximal cyclically monotone. Hence, $H = \partial\Phi$ for some $\Phi \in \Gamma_0(X)$, and, consequently, differential inclusion (36) has a unique absolutely continuous solution $x(\cdot)$, defined on $[0, +\infty)$, such that $x(0) = x_0$ (see, e.g., [6]). $\square$

**Example 4.** Consider first the diode bridge circuit in [12, Example 1] with resistors $R_1 > 0, R_2 > 0$, one capacitor $\mathcal{C} > 0$, one inductor $L > 0$, one voltage source $u$ and four ideal diodes $D_i$, $i = 1, 2, 3, 4$. Let $x = (x_1\ x_2)^T$ where $x_1$ is the current through the inductor and $x_2$ is the voltage across the capacitor. Let $(v_{D_i}\ i_{D_i})$ be the voltage–current of diode $D_i$. We obtain the governing circuit equations

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + (\frac{1}{L} \quad 0)^T u, \\ y(t) = Cx(t) + D\lambda(t), \\ \lambda(t) \in -F(y(t)),\ t \geq 0; \\ x(0) = x_0, \end{cases} \tag{37}$$

where $y(\cdot), \lambda(\cdot), A, B, C, D$ are defined in [12, Example 1]. Here $F = \partial\varphi$, where $\varphi : \mathbb{R}^4 \to \overline{\mathbb{R}}$ is defined as follows

$$\varphi(x_1, x_2, x_3, x_4) = \delta_{\mathbb{R}_+}(x_1) + \delta_{\mathbb{R}_+}(x_2) + \delta_{\mathbb{R}_+}(x_3) + \delta_{\mathbb{R}_+}(x_4), \tag{38}$$

and $\delta_S(\cdot)$ denotes the indicator function of $S$. By using change of variables if necessary (see, e.g., [1,12]), one observes that $(A, B, C, D)$ is passive (also cyclically passive) and $\mathrm{rge}(C) \cap \mathrm{int}(\mathrm{rge}(\partial\varphi^{-1} + D)) \neq \emptyset$.

Now let us consider the non-constant voltage source $u(\cdot)$ which depends on the temperature $s \in [0, 50]$ such that $u \in L^2(0, 50)$. For each $s$, one has

$$\begin{cases} \dot{x}_t(s,t) = Ax(s,t) + B\lambda(s,t) + (\frac{1}{L} \quad 0)^T u(s), \\ y(s,t) = Cx(s,t) + D\lambda(s,t), \\ \lambda(s,t) \in -\partial\varphi(y(s,t)),\ t \geq 0; \\ x(s,0) = x_0(s). \end{cases} \tag{39}$$

Let $X = L^2(0, 50)$ and $\tilde{x}(t) = x(\cdot, t), \tilde{y}(t) = y(\cdot, t), \tilde{\lambda}(t) = \lambda(\cdot, t), \tilde{u} = u(\cdot)$. The linear bounded operator $\tilde{A} : X \to X$ is defined as $\tilde{A}x(s) := Ax(s)$ and similar definitions for $\tilde{B}, \tilde{C}, \tilde{D}, \tilde{\varphi}$. Then we obtain

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{\lambda}(t) + (\frac{1}{L} \quad 0)^T \tilde{u}, \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{\lambda}(t), \\ \tilde{\lambda}(t) \in -\partial\tilde{\varphi}(y(t)),\ t \geq 0; \\ \tilde{x}(0) = \tilde{x}_0. \end{cases} \tag{40}$$

Clearly $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is cyclically passive and $\mathrm{rge}(\tilde{C}) \cap \mathrm{int}(\mathrm{rge}(\partial\tilde{\varphi}^{-1} + \tilde{D})) \neq \emptyset$. By using Theorem 3, one obtains that for each $\tilde{x}_0 \in \mathrm{cl}(\tilde{C}^{-1}(\mathrm{rge}(\partial\tilde{\varphi}^{-1} + \tilde{D}))) \subset L^2(0, 50)$, problem (40) has a unique solution $\tilde{x}(\cdot)$ in $L^2(0, 50)$. It means that the current through the inductor and the voltage across the capacitor are consistent in some sense with respect to the initial condition and the voltage source.

**Remark 4.** We may also consider the set-valued Lur'e dynamical systems in reflexive Banach spaces. In this case, the systems still have the form (35) but with linear continuous mappings $A : X \to X$, $B : Y \to X$, $C : X \to Y$, $D : Y \to Y$ and a *m-accretive* operator $F : Y \rightrightarrows Y$ (see, e.g., [3] for the definition). Let us note that the notions of $m$-accretion and maximal monotonicity coincide in Hilbert spaces.

## 4. Conclusion

In this paper, the maximal monotonicity and maximal cyclic monotonicity for the precomposition with a linear passive system are analyzed in the setting of reflexive Banach spaces. It can be considered as a generalization of the classical precomposition with a linear operator. This result is used to provide an existence theorem for set-valued Lur'e dynamical systems. It is also interesting to use the Fitzpatrick function to study the maximal monotonicity of this precomposition, which is out of scope of the current paper.

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