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Stress-Strength Reliability Analysis with Extreme Values based on *q*-Exponential Distribution

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When dealing with practical problems of stress–strength reliability, one can work with fatigue life data and make use of the well-known relation between stress and cycles until failure. For some materials, this kind of data can involve extremely large values. In this context, this paper discusses the problem of estimating the reliability index $R = P(Y < X)$ for stress–strength reliability, where stress *Y* and strength *X* are independent *q*-exponential random variables. This choice is based on the *q*-exponential distribution's capability to model data with extremely large values. We develop the maximum likelihood estimator for the index *R* and analyze its behavior by means of simulated experiments. Moreover, confidence intervals are developed based on parametric and nonparametric bootstrap. The proposed approach is applied to two case studies involving experimental data: The first one is related to the analysis of high-cycle fatigue of ductile cast iron, whereas the second one evaluates the specimen size effects on gigacycle fatigue properties of high-strength steel. The adequacy of the *q*-exponential distribution for both case studies and the point and interval estimates based on maximum likelihood estimator of the index *R* are provided. A comparison between the *q*-exponential and both Weibull and exponential distributions shows that the *q*-exponential distribution presents better results for fitting both stress and strength experimental data as well as for the estimated *R* index. Copyright © 2016 John Wiley & Sons, Ltd.

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1. Introduction

When assessing system reliability, a useful performance metric corresponds to the index $R = P(Y < X)$, where *X* is the strength of a component that is subjected to stress *Y*. Obviously, when *Y* is greater than *X*, the system failure occurs. Thus, the index *R* can be considered as a measure of system reliability. Several applications of physics and engineering such as strength failure and system collapse use stress–strength models.[1]

In most works where the index *R* is estimated, the authors assume that *X* and *Y* are independent random variables described by the same kind of probability distribution. For example, *X* and *Y* have been treated as normal,[2–5] Weibull,[6–8] and exponential[9,10] random variables. In addition, the generalized Pareto and generalized Rayleigh distributions are used to model *X* and *Y* in Rezaei *et al.*[11] and Fathipour *et al.*,[12] respectively. The Lomax distribution under general progressive censoring was treated by Al-Zahrani and Al-Harbi.[13] Panahi and Asadi[14] assume that *X* and *Y* follow Lomax distributions with different shape parameters and the same scale parameter. Some cases of generalized exponential distributions were considered by Kundu and Gupta[15] and Raqab *et al.*[16]

In recent years, a family of probability distributions based on non-extensive statistical mechanics, known as *q*-distributions, has experienced a surge in terms of applications to several fields of science and engineering. The basic properties of *q*-exponential, *q*-Gaussian, and *q*-Weibull were discussed by Picoli *et al.*[17] In a another work, Picoli *et al.*[18] also presented a comparison between *q*-exponential, *q*-Weibull, and Weibull distributions to model the frequency distributions of basketball baskets, cyclone victims, brand-name drugs by retail sales, and highway length. Moreover, complex systems have been satisfactorily described by *q*-distributions: cosmic rays,[19] cyclones,[20] financial markets,[21] gravitational systems,[22] earthquakes,[23] and the Internet.[24] Another important field of application of *q*-distributions is mechanical stress. For instance, it has been

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experimentally demonstrated that when a rock sample is subjected to mechanical stress, an electrical signal is emmited.[25,26] This electrical signal is related to the evolution of cracks' network within the stressed sample and is called pressure-stimulated current. In Vallianatos and Triantis,[27] pressure-stimulated current emissions in marble and amphibolite samples are considered to follow a q-exponential distribution.

In fact, q-exponential distribution is obtained by maximizing the non-extensive entropy under appropriate constraints.[28] This distribution has two parameters (q and η), differently from the exponential distribution that is one parametric. This feature gives more flexibility to q-exponential when it comes to decay of the probability density function (PDF) curve. Indeed, for a fixed parameter η, a slower or faster decay of the PDF is observed depending on the value of q. Besides, in addition to constant failure rate, the q-exponential distribution can model system improvement ($1 < q < 2$) and degradation ($q < 1$). In this way, q-exponential and Weibull distributions can be considered as alternative distributions given that both can model degradation or improvement behaviors. Nevertheless, these distributions have different origins. Indeed, Weibull distribution is a particular case of the generalized extreme value distribution (also known as Fisher–Tippett distribution). According to Goegebeur and Guillou,[29] extreme value theory studies the behavior of the largest observations in a sample and provides laws governing these values. The generalized extreme value distribution is a family of continuous probability distributions developed within the extreme value theory to combine the Gumbel, Fréchet, and Weibull families also known as type I, II and III extreme value distributions. The q-exponential distribution in turn derives from the non-extensive statistical mechanics, which is appropriate to systems where nonlinearity, long-range interactions, memory effects, and scaling are important.[30] Thus, it is natural to suggest q-exponential as a possible candidate to model systems that can present some kind of statistical dependency. Note that the q-exponential distribution has been considered by Sales Filho *et al.*[31] to obtain the index R; however, only the maximum likelihood estimator for the index R was developed.

For a given sample with extreme values, which can be thought of as the realizations of rare events, it is expected that both q-exponential and Weibull distributions can fit the data well: The entropic index of the q-exponential would lie within the interval (1,2), and the shape parameter of the Weibull distribution would be in (0,1). Indeed, $1 < q < 2$ characterizes a power law behavior for the q-exponential PDF, whereas a shape parameter between 0 and 1 indicates a stretched exponential behavior for the Weibull PDF.[18,32] As pointed out by Laherrère and Sornette,[32] a stretched exponential PDF has a tail that is heavier than the exponential PDF but lighter than a pure power law PDF. The stretched exponential provides a compromise between exponential and power law behaviors. Thus, we expect a superior performance of q-exponential over Weibull distribution in the characterization of data sets with extremely large values.

Several studies have investigated the presence of power laws in the behavior of data observed in fatigue analysis of materials. For instance, in Garcimartín *et al.*,[33] the acoustic emissions of microfractures before the breakup of the sample are evaluated, where the authors used samples made of composite inhomogeneous materials such as plaster, wood, or fiberglass. The experimental results were similar for all materials, and the authors conclude that statistics from acoustic energy measurements strongly suggest that the fracture can be viewed as a critical phenomenon and energy events are distributed in magnitude as a power law. Moreover, according to Basquin's law, the lifetime of a system increases as a power law with the reduction of the applied load amplitude.[34] Therefore, the alternating stress in terms of number of cycles to failure is expressed in a power law form[35] known as the Wöhler curve (SN curve). It has also been suggested that the underlying fracture dynamics in some systems might display self-organized criticality,[36] implying that long-range interactions between fracture events lead to a scale-free cascade of 'avalanches'.[37] For instance, in Zapperi *et al.*,[37] the authors present a scalar model of microfracturing that generates power law behavior in properties related to acoustic emission, and a scale-free hierarchy of avalanches characteristic of self-organized criticality.[38]

According to Shalizi[39] and Bercher and Vignat,[40] in order to obtain analytical expressions for the maximum likelihood estimation (MLE) of the q-exponential distribution parameters, it is necessary to reparameterize the q-exponential in order to transform this distribution in a generalized Pareto distribution. Unfortunately, this methodology allows obtaining analytical expressions for the MLE only when $1 < q < 2$, and as we shall show in the Section 2, the q-exponential distribution is also defined for $q < 1$. Thus, when we deal with degrading systems ($q < 1$), this reparameterization does not work, and an alternative approach should be investigated.

Thus, because it is generally very difficult to obtain analytical expressions for the MLE of q-exponential parameters because of the intricate derivatives of the log-likelihood function, in this paper, we will obtain the estimates of maximum likelihood numerically by the optimization algorithm proposed by Nelder and Mead.[41] Moreover, parametric and nonparametric bootstrap methods are coupled with the Nelder–Mead algorithm for the construction of confidence intervals for R. Note that we obtain expressions for R considering the general case, where the parameters can take any value, unlike other works[1,7,8,12,14,15,42] that assume that some of the parameters of a given model have the same values for both X and Y. In addition, we present two different equations for the MLE of R depending on the support of X: The first formula considers an unlimited support, whereas the second one involves a limited support for the strength variable.

Finally, simulation examples are here performed in order to assess the ability of the proposed method in providing point and interval estimates for the index R. Moreover, as application of the proposed approach, we develop two case studies in the context of stress–strength models by evaluating fracture data from applied tests in two different material types. The obtained data are related to the life cycle, which can be thought of as a measure of resistance to failure. Furthermore, the results from these case studies are also compared with the results from a classical Weibull distribution.

The remainder of this paper is structured as follows. The next section presents the q-exponential distribution along with its main characteristics and particularities. Then, we present the MLE for the index R considering the cases where X has limited and unlimited support. In the subsequent section, confidence intervals for the index R are developed by means of bootstrap approaches. Next, the computational simulations are presented to evaluate the performance of the MLE and bootstrap confidence intervals, followed by the section containing two case studies in the context of fatigue life distribution. Finally, some concluding remarks are presented.

2. The q-exponential distribution

The PDF of the q-exponential distribution is given by the following expression:

$$f_q(t) = \frac{(2-q)}{\eta}\exp_q\left[-\left(\frac{t}{\eta}\right)\right]; \quad q < 2 \quad \text{and} \quad \eta > 0,$$

where q is the parameter that determines the density shape and is known as the entropic index, η is the scale parameter, and $\exp_q(x)$ is the q-exponential function defined as

$$\exp_q(x) = \begin{cases} [1+(1-q)x]^{\frac{1}{1-q}}, & \text{if } [1+(1-q)x] \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where x and $q \in R$.

Note that the q-exponential PDF becomes an exponential PDF when $q \to 1$. Thus, the q-exponential distribution is a generalization of the exponential one. The parameters η and q determine how quickly the PDF decays. Note also that the q parameter dictates how the distribution deviates from exponentiality, and this deviation is also defined by the decay of the distribution. When compared with the decay of the exponential distribution with the same parameter η, the q-exponential presents a slower decay for $1 < q < 2$ (power law characteristic) and a faster decay for $q < 1$. By using the definition of the q-exponential function, it is possible to rewrite the density of q-exponential:

$$f_q(t) = \frac{(2-q)\left[1-\frac{(1-q)t}{\eta}\right]^{\frac{1}{1-q}}}{\eta}, \quad q < 2 \quad \text{and } \eta > 0.$$

Furthermore, depending on the value of the entropic index, we will have different results for the support t:

$$t \in \begin{cases} [0;\ \infty), & q \geq 1 \\ \left[0; \frac{1}{\frac{1}{\eta}(1-q)}\right), & q < 1 \end{cases} . \tag{1}$$

Figure 1(a–d) presents the q-exponential PDF for some possible values of q and η, illustrating the behavior that was previously commented.

The cumulative distribution function (CDF) of the q-exponential is defined by the following expression:

$$F_q(t) = \begin{cases} 1 - \exp_{q'}\left\{-\left[\frac{t.(2-q)}{\eta}\right]\right\}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $q' = \frac{1}{2-q}$. By inverting $F_q(t)$, we obtain a q-exponential random number generator:

$$T = \frac{\eta\left[1-(U)^{\frac{1-q}{2-q}}\right]}{1-q}, \tag{2}$$

where U is a uniform random variable defined in [0,1].

Note that differently from the exponential distribution, the q-exponential hazard rate is not constant. In fact, this is an important characteristic of the q-exponential distribution, especially in the reliability context. Next, we will show that for a q-exponential distribution, it is possible to model two additional behaviors for the hazard rate. Let us first define the hazard rate $h_q(t) = \frac{f_q(t)}{R_q(t)}$, where $R_q(t)$ is $1 - F_q(t)$.

Thus, we can write

$$h_q(t) = \frac{\frac{(2-q)\left[1-\frac{(1-q)t}{\eta}\right]^{\frac{1}{1-q}}}{\eta}}{\left[1-\frac{(1-q)t}{\eta}\right]^{\frac{2-q}{1-q}}} = \frac{(2-q)}{\eta}\left[1-\frac{(1-q)t}{\eta}\right]^{\frac{q-1}{1-q}}.$$

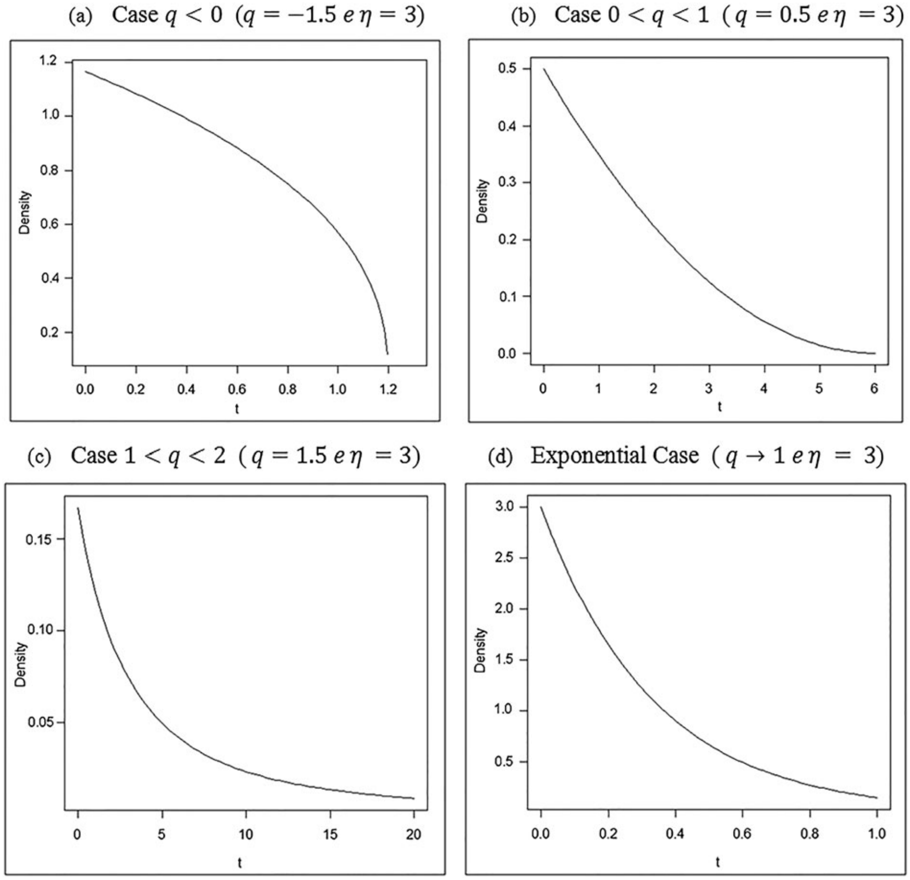

Figure 1. The q-exponential probability density function (PDF) for some possible values of q and η. (a) q-exponential PDF for $q = -1.5$ and $\eta = 3$, (b) q-exponential PDF for $q = 0.5$ and $\eta = 3$, (c) q-exponential PDF for $q = 1.5$ and $\eta = 3$, and (d) q-exponential PDF for $q \to 1$ and $\eta = 3$

Thus, the q-exponential distribution is able to represent two different types of hazard rate behaviors. That is, for $1 < q < 2$, $h_q(t)$ is a decreasing monotonic function (Figure 2(a)), while for $q < 1$, $h_q(t)$ is an increasing monotonic function (Figure 2(b)).

Nadarajah and Kotz[43] point out that many of the q-distributions that have emerged recently were known by other names, and they particularly discuss two families of distributions: Burr-type XII and Burr-type III, which have many q-distributions as special cases. However, it is worth noting that the q-exponential is a generalization of the Burr XII and not the opposite, as stated by Nadarajah and Kotz,[44] because the q-exponential is valid even for $q < 1$, which does not happen with the Burr XII.

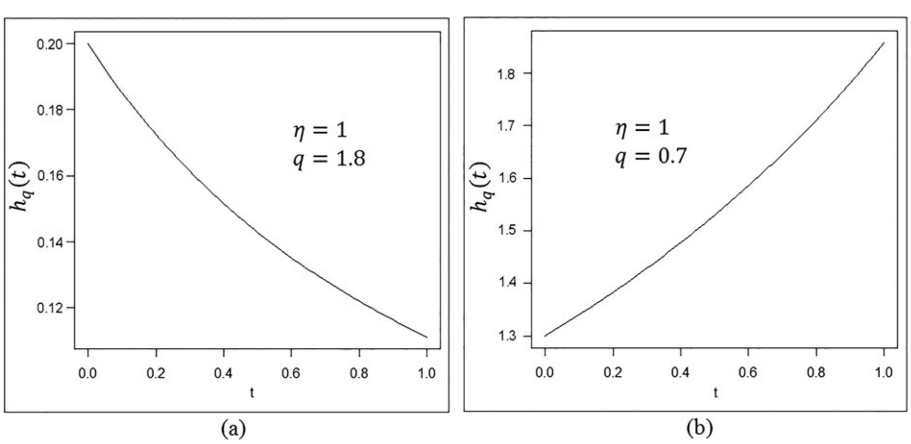

Figure 2. The q-exponential hazard rate. (a) q-exponential $h_q(t)$ with $\eta = 1$ and $q = 1.8$ and (b) q-exponential $h_q(t)$ with $\eta = 1$ and $q = 0.7$

3. Maximum likelihood estimators of index $R = P(Y < X)$

In this section, we estimate the index $R = P(Y < X)$ by using the maximum likelihood method. We assume that X and Y are independent random variables and follow q-exponential distributions with different parameters. We can write $Y \sim q\text{Exp}(q, \eta)$ and $X \sim q\text{Exp}(r, \beta)$, where q and r are the entropic indices (shape parameters) of stress and strength, respectively, and η and β are the scale parameters of stress and strength. As mentioned earlier, the support of the q-exponential can be limited ($q < 1$) or unlimited ($1 < q < 2$) (Equation (1)). Therefore, in order to calculate the index R, we will consider two cases:

Case 1: There is no limitation on the support of X (strength), that is, $1 < r < 2$:

$$R = P(Y < X) = \int_0^{\infty}\int_0^{x} \frac{\left\{(2-q)\left[\frac{(q-1)y}{\eta}+1\right]^{\frac{1}{1-q}}\right\}\left\{(2-r)\left[\frac{(r-1)x}{\beta}+1\right]^{\frac{1}{1-r}}\right\}}{\eta\beta} dydx =$$

$$= \frac{(r-2)\left\{\frac{\beta\left[(r-1)^3 ABC - \frac{DEF\left(\frac{r-1}{\beta}\right)^{\frac{r-2}{r-1}}\left(\frac{q-1}{\eta}\right)^{\frac{2-r}{r-1}}\left[1-\frac{\beta(q-1)}{\eta(r-1)}\right]^{\frac{1}{1-9}+\frac{1}{1-r}}[\eta(1-r)+\beta(q-1)]^2}{\eta^2}\right]}{A(r-1)^4} - \frac{\beta B}{r-1}\right\}}{\beta D}, \tag{3}$$

where $A = \Gamma\left(\frac{2-q}{q-1}\right)$, $B = \Gamma\left(\frac{2-r}{r-1}\right)$, $C = {}_2F_1\left(1, \frac{2-q}{q-1}; \frac{3-2r}{1-r}; \frac{(q-1)\beta}{(r-1)\eta}\right)$, $D = \Gamma\left(\frac{1}{r-1}\right)$, $E = \Gamma\left(\frac{1}{1-r}\right)$, $F = \Gamma\left(\frac{1}{r-1} - 2 + \frac{1}{q-1}\right)$ and ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt$ is the Gauss hypergeometric function.[45]

Case 2: The support of X (strength) is limited, that is, $r < 1$:

$$R = P(Y < X) = \int_0^{\frac{1}{\left(\frac{1}{\beta}\right)(1-r)}}\int_0^{x} \frac{\left\{(2-q)\left[\frac{(q-1)y}{\eta}+1\right]^{\frac{1}{1-q}}\right\}\left\{(2-r)\left[\frac{(r-1)x}{\beta}+1\right]^{\frac{1}{1-r}}\right\}}{\eta\beta} dydx = \tag{4}$$
$$= 1 - {}_2F_1\left(1, \frac{1}{q-1} - 1; 2 + \frac{1}{1-r}; \frac{(q-1)\beta}{(r-1)\eta}\right).$$

To compute the MLE of R, let $Y = \{y_1, y_2, \ldots, y_n\}$ be a random sample of size n and $X = \{x_1, x_2, \ldots, x_m\}$ be another random sample of size m. Because X and Y are independent variables, it is possible to write the likelihood function for the observed samples as

$$L(x, y, r, \beta, q, \eta) = \left\{(2-q)^n\left(\frac{1}{\eta}\right)^n\prod_{i=1}^{n}\left[1 - \frac{(1-q)y_i}{\eta}\right]^{\frac{1}{1-q}}\right\}\left\{(2-r)^m\left(\frac{1}{\beta}\right)^m\prod_{i=1}^{m}\left[1 - \frac{(1-r)x_i}{\beta}\right]^{\frac{1}{1-r}}\right\}$$
$$= (2-q)^n\left(\frac{1}{\eta}\right)^n(2-r)^m\left(\frac{1}{\beta}\right)^m\prod_{i=1}^{n}\left[1 - \frac{(1-q)y_i}{\eta}\right]^{\frac{1}{1-q}}\prod_{i=1}^{m}\left[1 - \frac{(1-r)x_i}{\beta}\right]^{\frac{1}{1-r}}.$$

Therefore, the log-likelihood function is written as follows:

$$l(x, y, r, \beta, q, \eta) = n\ln(2-q) + n\ln\left(\frac{1}{\eta}\right) + m\ln(2-r) + m\ln\left(\frac{1}{\beta}\right) + \frac{1}{1-q}\sum_{i=1}^{n}\ln\left[1 - \frac{(1-q)y_i}{\eta}\right]$$
$$+ \frac{1}{1-r}\sum_{i=1}^{m}\ln\left[1 - \frac{(1-r)x_i}{\beta}\right]. \tag{5}$$

Maximizing the log-likelihood function given in Equation (5) results in a convoluted system of equations, and thus, the derivation of analytical expressions for the MLE becomes impractical. So in this work, the maximization of the log-likelihood function in Equation (5) will be performed by the Nelder–Mead method[41] available in the software R (optim function).[46]

Because $\widehat{q}$, $\widehat{\eta}$, $\widehat{r}$, and $\widehat{\beta}$ are solutions that maximize the log-likelihood function of Equation (5) and using the property of invariance of the MLE, from Equations (3) and (4), we can obtain the MLE of R for the two previously mentioned cases:

Case 1: When the X (strength) has $1 < r < 2$,

$$\hat{R} = \frac{(\hat{r}-2)\left\{ \frac{\hat{\beta}\left[(\hat{r}-1)^3\hat{A}\hat{B}\hat{C} - \frac{\hat{D}\hat{E}\hat{F}\left(\frac{\hat{r}-1}{\hat{\beta}}\right)^{\frac{\hat{r}-2}{\hat{r}-1}}\left(\frac{\hat{q}-1}{\hat{\eta}}\right)^{\frac{2-\hat{r}}{\hat{r}-1}}\left[1-\frac{\hat{\beta}(\hat{q}-1)}{\hat{\eta}(\hat{r}-1)}\right]^{\frac{1}{1-\hat{q}}+\frac{1}{1-\hat{r}}}\left[\hat{\eta}(1-\hat{r})+\hat{\beta}(\hat{q}-1)\right]^2}{\eta^2}\right]}{\hat{A}(\hat{r}-1)^4} - \frac{\hat{\beta}\hat{B}}{\hat{r}-1} \right\}}{\hat{\beta}\hat{D}}, \tag{6}$$

where $\hat{A} = \Gamma\left(\frac{2-\hat{q}}{\hat{q}-1}\right)$, $\hat{B} = \Gamma\left(\frac{2-\hat{r}}{\hat{r}-1}\right)$, $\hat{C} = {}_2F_1\left(1, \frac{2-\hat{q}}{\hat{q}-1}; \frac{3-2\hat{r}}{1-\hat{r}}; \frac{(\hat{q}-1)\hat{\beta}}{(\hat{r}-1)\hat{\eta}}\right)$, $\hat{D} = \Gamma\left(\frac{1}{\hat{r}-1}\right)$, $\hat{E} = \Gamma\left(\frac{1}{1-\hat{r}}\right)$, and $\hat{F} = \Gamma\left(\frac{1}{\hat{r}-1} - 2 + \frac{1}{\hat{q}-1}\right)$.

Case 2: When the X (strength) has $r < 1$,

$$\hat{R} = 1 - {}_2F_1\left(1, \frac{1}{\hat{q}-1} - 1; 2 + \frac{1}{1-r}; \frac{(\hat{q}-1)\hat{\beta}}{(\hat{r}-1)\hat{\eta}}\right). \tag{7}$$

4. Bootstrap confidence intervals

In this section, we present the construction of confidence intervals for the index R by using bootstrap-p and nonparametric bootstrap methods.[47–49]

4.1. Bootstrap-p

The algorithm for constructing confidence intervals by using the bootstrap-p approach has the following steps:

- **Step 1:** From an initial sample for the variable $X = \{x_1, x_2, \ldots, x_m\}$ and another one for $Y = \{y_1, y_2, \ldots, y_n\}$, estimate the parameters (q, η, r, β) by maximizing Equation (5).
- **Step 2:** Use the estimates obtained in the previous step and Equation (2) to generate new samples for X and Y, that is, $\{x_1^*, x_2^*, \ldots, x_m^*\}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$. Based on these new samples, compute the bootstrap sample estimate of R, say R^*, using Equations (6) or (7) (depending on the r-value).
- **Step 3:** Repeat step 2, N times.
- **Step 4:** Using the N-values of R^* obtained in step 3 and by adopting a γ significance level, find the percentiles $R^*_{\gamma/2}$ and $R^*_{1-(\gamma/2)}$. Thus, it is possible to determine an approximate confidence interval, with confidence interval equal to $100^*(1-\gamma)\%$, for the index R, as

$$C.I. = \left[R^*_{\gamma/2},\ R^*_{(1-\gamma)/2}\right]. \tag{8}$$

4.2. Nonparametric bootstrap

The algorithm for constructing confidence intervals by using the nonparametric bootstrap approach is as follows:

- **Step 1:** From an initial sample for the variable $X = \{x_1, x_2, \ldots, x_m\}$ and another one for $Y = \{y_1, y_2, \ldots, y_n\}$, generate new samples for X and Y by sampling with replacement, that is, $\{x_1^*, x_2^*, \ldots, x_m^*\}$ and $\{y_1^*, y_2^*, \ldots, y_n^*\}$. Based on these new samples, compute the estimate of R, say R^*, using Equation (6) or (7) (depending on the r-value).
- **Step 2:** Repeat step 1, N times.
- **Step 3:** By using the N-values of R^* from step 2 and by adopting a γ significance level, the percentiles $R^*_{\gamma/2}$ and $R^*_{1-(\gamma/2)}$ are obtained; they determine an approximate confidence interval for the index R with confidence level equals to $100^*(1-\gamma)\%$ using Equation (8).

5. Numerical experiments

This section presents the performance evaluation of the MLE and bootstrap confidence intervals by means of simulation experiments. We consider different sample sizes and different parameter values. First, we analyze the MLE, and then we discuss the bootstrap confidence interval. Note that the simulations involved entropic indices ranging between 1 and 2 (cases 1–11), from 0 to 1 (cases 12–22), and with negative values (cases 23–33).

5.1. Analysis of the maximum likelihood estimation

Several combinations of sample sizes for the stress and strength are considered: $(n; m) = (100; 100)$, (250; 250), (500; 500), (1000; 1000), (5000; 5000), (100; 250), (100; 500), (100; 1000), (250; 100), (500; 100), and (1000; 100). Besides, we choose three sets of parameter values respecting three important situations for the entropic indices, that is, $1 < r,q < 2$; $0 < r,q < 1$, and $r,q < 0$. Thus, we have $(q, \eta, r, \beta) = (1.78; 0.15; 1.9; 0.1)$, (0.55; 22; 0.67; 30.5), and (−1.95; 0.1; −1.8; 0.18). Observe that 11 different combinations of sample sizes multiplied by three different parameter sets are equal to 33 initial samples. The samples for the simulations are generated by Equation (2). All results are based on 1000 replications. That is, we generate 1000 samples from each set of initial parameters for all the combinations of n and m. Thus, a total of 33,000 samples are generated. For each sample, we compute the MLE for q, η, r, β by maximizing Equation (5) via the Nelder–Mead method.

Thus, we obtain the MLE of index R by Equation (6) or (7) (depending on the r-value). This process is carried out for each of the 1000 replications. Subsequently, we obtain the average of the estimation results for parameters q, η, r, β, and also for the index R. Table IV presents the results for all simulation runs as well as the index R estimations. The mean squared error (MSE) and the average biases are calculated for $\hat{R}$ over the 1000 replications. Note that these quality indices are obtained for an estimator $\widehat{\theta}$ of θ as $bias\left(\hat{\theta}\right) = E\left(\hat{\theta}\right) - \theta$ and $MSE = \text{Var}\left(\hat{\theta}\right) + bias\left(\hat{\theta}\right)^2$.

From the simulation results (Table IV), the following findings are observed:

(i) When $(n; m)$ increase, the MSEs decrease. This suggests the consistency property of the MLE (Table I).
(ii) For a fixed n, MSEs decrease as m increases (Table II).
(iii) For a fixed m, MSEs decrease as n increases (Table III).

In order to illustrate behaviors (i)–(iii), excerpts of Table IV are reproduced in Tables I–III, respectively.

Table I. Examples of cases that present a decrease of the MSE when $(n; m)$ increase

Case	n	m	q	η	r	β	MSE
1	100	100	1.78	0.15	1.90	0.10	1.06E-03
12	100	100	0.55	22	0.67	30.5	1.28E-03
23	100	100	−1.95	0.1	−1.8	0.18	1.19E-03
5	5000	5000	1.78	0.15	1.90	0.10	2.77E-05
16	5000	5000	0.55	22	0.67	30.5	2.44E-05
27	5000	5000	−1.95	0.1	−1.8	0.18	4.87E-05

MSE, mean squared error.

Table II. Examples of cases that present a decrease of the MSE for a fixed n and an increase of m

Case	n	m	q	η	r	β	*MSE*
1	100	100	1.78	0.15	1.90	0.10	1.06E-03
6	100	250	1.78	0.15	1.90	0.10	8.36E-04
7	100	500	1.78	0.15	1.90	0.10	5.63E-04
8	100	1000	1.78	0.15	1.90	0.10	5.22E-04

MSE, mean squared error.

Table III. Examples of cases that present a decrease of the MSE for a fixed m and an increase of n

Case	n	m	q	η	r	β	*MSE*
12	100	100	0.55	22	0.67	30.5	1.28E-03
20	250	100	0.55	22	0.67	30.5	1.16E-03
21	500	100	0.55	22	0.67	30.5	9.4E-04
22	1000	100	0.55	22	0.67	30.5	8.8E-04

MSE, mean squared error.

Table IV. Simulation results and estimation for the index R

	Sample Size		Parameters					MLE Results					$\widehat{R}$	
Case	n	m	q	η	r	β	R	$\widehat{q}$	$\widehat{\eta}$	$\widehat{r}$	$\widehat{\beta}$	$\widehat{R}$	*bias*	*MSE*
1	100	100	1.78	0.15	1.90	0.10	0.6855	1.7765	0.1638	1.8989	0.1120	0.6850	−0.00044	0.00106
2	250	250	1.78	0.15	1.90	0.10	0.6855	1.7787	0.1548	1.8995	0.1059	0.6853	−0.00014	0.00046
3	500	500	1.78	0.15	1.90	0.10	0.6855	1.7793	0.1528	1.8997	0.1044	0.6860	0.00051	0.00026
4	1000	1000	1.78	0.15	1.90	0.10	0.6855	1.7798	0.1516	1.9000	0.1013	0.6854	−0.00009	0.00013
5	5000	5000	1.78	0.15	1.90	0.10	0.6855	1.7798	0.1506	1.8999	0.1006	0.6855	0.00007	0.00003
6	100	250	1.78	0.15	1.90	0.10	0.6855	1.7752	0.1667	1.8994	0.1082	0.6869	0.00145	0.00084
7	100	500	1.78	0.15	1.90	0.10	0.6855	1.7776	0.1633	1.8996	0.1039	0.6849	−0.00062	0.00056
8	100	1000	1.78	0.15	1.90	0.10	0.6855	1.7759	0.1621	1.8999	0.1014	0.6869	0.00147	0.00052
9	250	100	1.78	0.15	1.90	0.10	0.6855	1.7788	0.1551	1.8985	0.1139	0.6839	−0.00158	0.00097
10	500	100	1.78	0.15	1.90	0.10	0.6855	1.7794	0.1523	1.8984	0.1160	0.6853	−0.00016	0.00085
11	1000	100	1.78	0.15	1.90	0.10	0.6855	1.7797	0.1517	1.8991	0.1139	0.6855	−0.00001	0.00085
12	100	100	0.55	22	0.67	30.5	0.6259	0.5086	23.2133	0.4804	37.9860	0.6349	0.00894	0.00128
13	250	250	0.55	22	0.67	30.5	0.6259	0.5380	22.3628	0.5901	33.6764	0.6311	0.00522	0.00047
14	500	500	0.55	22	0.67	30.5	0.6259	0.5436	22.1900	0.6218	32.4722	0.6299	0.00400	0.00026
15	1000	1000	0.55	22	0.67	30.5	0.6259	0.5472	22.0755	0.6461	31.4549	0.6279	0.00194	0.00013
16	5000	5000	0.55	22	0.67	30.5	0.6259	0.5488	22.0306	0.6655	30.6923	0.6264	0.00047	0.00002
17	100	250	0.55	22	0.67	30.5	0.6259	0.5487	22.1981	0.5895	33.7513	0.6325	0.00658	0.00085
18	100	500	0.55	22	0.67	30.5	0.6259	0.5722	21.5316	0.6248	32.2672	0.6314	0.00547	0.00066
19	100	1000	0.55	22	0.67	30.5	0.6259	0.5820	21.3467	0.6460	31.4455	0.6298	0.00392	0.00060
20	250	100	0.55	22	0.67	30.5	0.6259	0.5330	22.4011	0.4775	38.5020	0.6395	0.01354	0.00116
21	500	100	0.55	22	0.67	30.5	0.6259	0.5347	22.4039	0.4926	37.6015	0.6354	0.00942	0.00094
22	1000	100	0.55	22	0.67	30.5	0.6259	0.5412	22.2352	0.4841	38.1364	0.6373	0.01139	0.00088
23	100	100	−1.95	0.1	−1.8	0.18	0.7656	−1.7945	0.0932	−1.7836	0.1757	0.7515	−0.01408	0.00119
24	250	250	−1.95	0.1	−1.8	0.18	0.7656	−2.0991	0.1041	−2.0037	0.1911	0.7675	0.00187	0.00061
25	500	500	−1.95	0.1	−1.8	0.18	0.7656	−2.0924	0.1042	−1.9563	0.1890	0.7689	0.00329	0.00040
26	1000	1000	−1.95	0.1	−1.8	0.18	0.7656	−2.0592	0.1033	−1.9143	0.1867	0.7687	0.00310	0.00026
27	5000	5000	−1.95	0.1	−1.8	0.18	0.7656	−1.9609	0.1003	−1.8256	0.1815	0.7659	0.00029	0.00005
28	100	250	−1.95	0.1	−1.8	0.18	0.7656	−1.8330	0.0945	−1.9804	0.1896	0.7622	−0.00341	0.00101
29	100	500	−1.95	0.1	−1.8	0.18	0.7656	−1.8214	0.0941	−1.9770	0.1902	0.7641	−0.00145	0.00099
30	100	1000	−1.95	0.1	−1.8	0.18	0.7656	−1.7832	0.0928	−1.9025	0.1860	0.7642	−0.00141	0.00086
31	250	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0941	0.1039	−1.7519	0.1737	0.7557	−0.00993	0.00084
32	500	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0575	0.1031	−1.8193	0.1782	0.7564	−0.00921	0.00077
33	1000	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0255	0.1023	−1.7842	0.1759	0.7543	−0.01123	0.00072

MLE, maximum likelihood estimation; MSE, mean squared error.

5.2. Bootstrap confidence interval

For the simulations of the confidence intervals based on bootstrap-*p* and on nonparametric bootstrap, we generated 33 initial samples using the same combinations of sample sizes and initial parameters presented in the Section 5.2. For the case of the bootstrap-*p*, we obtain, from the 33 initial samples, the MLE of the parameters by maximizing the log-likelihood function (Equation (5)). Given that we have the parameters' estimates (obtained from the initial samples) for each different combination of parameters and sample size, we can use these estimates to generate $N = 1000$ new samples from Equation (2). For the case of the nonparametric bootstrap, we use the 33 initial samples to generate $N = 1000$ samples by sampling with replacement (for each different combination of parameters and sample sizes).

From the samples generated by the bootstrap-*p* or by nonparametric bootstrap, we estimate the index *R* from Equation (6) or (7) (depending on the *r*-value). That is, for each method, we generate $N = 1000$ bootstrap estimates of *R*. We present the mean of $N = 1000$ bootstrap estimates of *R,* and based on the percentile method, the corresponding 90% and 95% confidence intervals are also provided.

Tables VIII and IX present, respectively, the results and estimation of bootstrap-*p* and nonparametric bootstrap confidence intervals for index *R*. From the simulations for the bootstrap-*p* and nonparametric bootstrap confidence intervals, we observe that

(i) When (*n*; *m*) increase, the amplitude of the interval (width) decreases. In order to illustrate this behavior, Table V presents an excerpt of Table VIII with the interval widths for the index *R* obtained by bootstrap-*p*, considering a 95% confidence level, for the three different combinations of parameters when $(n, m) = (100, 100)$ and $(n, m) = (5000, 5000)$.
(ii) For a fixed *n*, the widths decrease as *m* increases. For example, the results in Table VI are taken from Table IX (nonparametric bootstrap and 95% of confidence level) and demonstrate this behavior.
(iii) For a fixed *m*, the interval widths decrease as *n* increases. Table VII, which is an excerpt of Table VIII, exemplifies the decrease of interval widths for bootstrap-*p* and 90% of confidence level.

Yet note that the method of bootstrap-*p* showed greater efficiency in the simulations because the confidence intervals (90% or 95%) contain the parameter value in all the simulations. On the other hand, the nonparametric method resulted in some intervals that did not contain the real value of the parameter *R*. For example, this was observed for the case 3 (for $1 - \gamma = 0.90$), case 8 (for $1 - \gamma = 0.90$ and 0.95), case 20 (for $1 - \gamma = 0.90$ and 0.95), case 24 (for $1 - \gamma = 0.90$ and 0.95), case 26 (for $1 - \gamma = 0.90$), and case 33 (for $1 - \gamma = 0.90$), as shown in Table IX. The intervals that do not contain the parameters correspond to 14% of the simulated cases with the nonparametric method.

Table V. Examples of cases that present a decrease of interval widths when (*n*; *m*) increase

Case	n	m	q	η	r	β	$(1-\gamma) = 0.95$	Width
1	100	100	1.78	0.15	1.90	0.10	[0.6214; 0.7524]	1.310E-01
12	100	100	0.55	22	0.67	30.5	[0.5581; 0.7108]	1.527E-01
23	100	100	−1.95	0.1	−1.8	0.18	[0.6915; 0.8210]	1.296E-01
5	5000	5000	1.78	0.15	1.90	0.10	[0.6756; 0.6951]	1.945E-02
16	5000	5000	0.55	22	0.67	30.5	[0.6167; 0.6358]	1.905E-02
27	5000	5000	−1.95	0.1	−1.8	0.18	[0.7443; 0.7802]	3.590E-02

Table VI. Examples of cases that present a decrease of interval widths for a fixed *n* and an increase of *m*

Case	n	m	q	η	r	β	$(1-\gamma) = 0.95$	Width
1	100	100	1.78	0.15	1.90	0.10	[0.6111; 0.7430]	1.319E-01
6	100	250	1.78	0.15	1.90	0.10	[0.6701; 0.7670]	9.693E-02
7	100	500	1.78	0.15	1.90	0.10	[0.6456; 0.7407]	9.510E-02
8	100	1000	1.78	0.15	1.90	0.10	[0.6926; 0.7754]	8.279E-02

Table VII. Examples of cases that present a decrease of interval widths for a fixed *m* and an increase of *n*

Case	n	m	q	η	r	β	$(1-\gamma) = 0.90$	Width
12	100	100	0.55	22	0.67	30.5	[0.5720; 0.6973]	1.2533E-01
20	250	100	0.55	22	0.67	30.5	[0.5944; 0.6957]	1.0130E-01
21	500	100	0.55	22	0.67	30.5	[0.5926; 0.6929]	1.0023E-01
22	1000	100	0.55	22	0.67	30.5	[0.5948; 0.6931]	9.8340E-02

Table VIII. Simulation results and estimation of bootstrap-*p* confidence interval for the index *R*

	Sample size		Initial parameters					MLE results					Bootstrap estimative				
														Confidence interval			
Case	n	m	q	η	r	β	R	$\hat{q}$	$\hat{\eta}$	$\hat{r}$	$\hat{\beta}$	$\hat{R}$	R^*	$(1-\gamma)=0.90$	Width	$(1-\gamma)=0.95$	Width
1	100	100	1.78	0.15	1.90	0.10	0.6855	1.7765	0.1638	1.8989	0.1120	0.6850	0.6871	[0.6331; 0.7436]	0.11052	[0.6214; 0.7524]	0.13102
2	250	250	1.78	0.15	1.90	0.10	0.6855	1.7787	0.1548	1.8995	0.1059	0.6853	0.6867	[0.6479; 0.7232]	0.07534	[0.6389; 0.7302]	0.09127
3	500	500	1.78	0.15	1.90	0.10	0.6855	1.7793	0.1528	1.8997	0.1044	0.6860	0.6870	[0.6597; 0.7128]	0.05307	[0.6531; 0.7174]	0.06433
4	1000	1000	1.78	0.15	1.90	0.10	0.6855	1.7798	0.1516	1.9000	0.1013	0.6854	0.6859	[0.6682; 0.7045]	0.03629	[0.6652; 0.7082]	0.04299
5	5000	5000	1.78	0.15	1.90	0.10	0.6855	1.7798	0.1506	1.8999	0.1006	0.6855	0.6852	[0.6773; 0.6933]	0.01604	[0.6756; 0.6951]	0.01945
6	100	250	1.78	0.15	1.90	0.10	0.6855	1.7752	0.1667	1.8994	0.1082	0.6869	0.6876	[0.6443; 0.7309]	0.08658	[0.6375; 0.7394]	0.10182
7	100	500	1.78	0.15	1.90	0.10	0.6855	1.7776	0.1633	1.8996	0.1039	0.6849	0.6847	[0.6466; 0.7246]	0.07804	[0.6396; 0.7344]	0.09482
8	100	1000	1.78	0.15	1.90	0.10	0.6855	1.7759	0.1621	1.8999	0.1014	0.6869	0.6856	[0.6474; 0.7237]	0.07637	[0.6406; 0.7289]	0.08828
9	250	100	1.78	0.15	1.90	0.10	0.6855	1.7788	0.1551	1.8985	0.1139	0.6839	0.6875	[0.6400; 0.7360]	0.09597	[0.6333; 0.7437]	0.11047
10	500	100	1.78	0.15	1.90	0.10	0.6855	1.7794	0.1523	1.8984	0.1160	0.6853	0.6902	[0.6431; 0.7379]	0.09484	[0.6336; 0.7477]	0.11412
11	1000	100	1.78	0.15	1.90	0.10	0.6855	1.7797	0.1517	1.8991	0.1139	0.6855	0.6901	[0.6420; 0.7383]	0.09637	[0.6301; 0.7502]	0.12009
12	100	100	0.55	22	0.67	30.5	0.6259	0.5086	23.2133	0.4804	37.9860	0.6349	0.6358	[0.5720; 0.6973]	0.12533	[0.5581; 0.7108]	0.15269
13	250	250	0.55	22	0.67	30.5	0.6259	0.5380	22.3628	0.5901	33.6764	0.6311	0.6386	[0.5924; 0.6700]	0.07755	[0.5857; 0.6779]	0.09213
14	500	500	0.55	22	0.67	30.5	0.6259	0.5436	22.1900	0.6218	32.4722	0.6299	0.6307	[0.6037; 0.6557]	0.05200	[0.5966; 0.6606]	0.06401
15	1000	1000	0.55	22	0.67	30.5	0.6259	0.5472	22.0755	0.6461	31.4549	0.6279	0.6281	[0.6090; 0.6462]	0.03724	[0.6045; 0.6492]	0.04478
16	5000	5000	0.55	22	0.67	30.5	0.6259	0.5488	22.0306	0.6655	30.6923	0.6264	0.6264	[0.6185; 0.6341]	0.01564	[0.6167; 0.6358]	0.01905
17	100	250	0.55	22	0.67	30.5	0.6259	0.5487	22.1981	0.5895	33.7513	0.6325	0.6291	[0.5753; 0.6808]	0.10553	[0.5661; 0.6887]	0.12258
18	100	500	0.55	22	0.67	30.5	0.6259	0.5722	21.5316	0.6248	32.2672	0.6314	0.6241	[0.5781; 0.6732]	0.09510	[0.5691; 0.6817]	0.11263
19	100	1000	0.55	22	0.67	30.5	0.6259	0.5820	21.3467	0.6460	31.4455	0.6298	0.6236	[0.5803; 0.6658]	0.08553	[0.5699; 0.6773]	0.10738
20	250	100	0.55	22	0.67	30.5	0.6259	0.5330	22.4011	0.4775	38.5020	0.6395	0.6463	[0.5944; 0.6957]	0.10130	[0.5845; 0.7045]	0.11996
21	500	100	0.55	22	0.67	30.5	0.6259	0.5347	22.4039	0.4926	37.6015	0.6354	0.6413	[0.5926; 0.6929]	0.10023	[0.5820; 0.7001]	0.11810
22	1000	100	0.55	22	0.67	30.5	0.6259	0.5412	22.2352	0.4841	38.1364	0.6373	0.6252	[0.5948; 0.6931]	0.09834	[0.5808; 0.7023]	0.12145
23	100	100	−1.95	0.1	−1.8	0.18	0.7656	−1.7945	0.0932	−1.7836	0.1757	0.7515	0.7513	[0.7019; 0.8060]	0.10409	[0.6915; 0.8210]	0.12958
24	250	250	−1.95	0.1	−1.8	0.18	0.7656	−2.0991	0.1041	−2.0037	0.1911	0.7675	0.7730	[0.7318; 0.8093]	0.07747	[0.7233; 0.8148]	0.09146
25	500	500	−1.95	0.1	−1.8	0.18	0.7656	−2.0924	0.1042	−1.9563	0.1890	0.7689	0.7755	[0.7430; 0.8071]	0.06411	[0.7347; 0.8122]	0.07749
26	1000	1000	−1.95	0.1	−1.8	0.18	0.7656	−2.0592	0.1033	−1.9143	0.1867	0.7687	0.7743	[0.7486; 0.7986]	0.04998	[0.7442; 0.8033]	0.05913
27	5000	5000	−1.95	0.1	−1.8	0.18	0.7656	−1.9609	0.1003	−1.8256	0.1815	0.7659	0.7657	[0.7511; 0.7779]	0.02683	[0.7443; 0.7802]	0.03590
28	100	250	−1.95	0.1	−1.8	0.18	0.7656	−1.8330	0.0945	−1.9804	0.1896	0.7622	0.7671	[0.7109; 0.8219]	0.11092	[0.7027; 0.8330]	0.13022
29	100	500	−1.95	0.1	−1.8	0.18	0.7656	−1.8214	0.0941	−1.9770	0.1902	0.7641	0.7713	[0.7131; 0.8238]	0.11065	[0.7039; 0.8355]	0.13154
30	100	1000	−1.95	0.1	−1.8	0.18	0.7656	−1.7832	0.0928	−1.9025	0.1860	0.7642	0.7728	[0.7166; 0.8264]	0.10980	[0.7083; 0.8374]	0.12909
31	250	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0941	0.1039	−1.7519	0.1737	0.7557	0.7532	[0.7052; 0.7983]	0.09309	[0.6975; 0.8069]	0.10945
32	500	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0575	0.1031	−1.8193	0.1782	0.7564	0.7512	[0.7062; 0.7922]	0.08603	[0.6950; 0.8013]	0.10634
33	1000	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0255	0.1023	−1.7842	0.1759	0.7543	0.7468	[0.7024; 0.7852]	0.08278	[0.6911; 0.7907]	0.09956

Table IX. Simulation results and estimation of nonparametric bootstrap confidence interval for the index R

	Sample size		Initial parameters					MLE results					Bootstrap estimative				
														Confidence interval			
Case	n	m	q	η	r	β	R	$\hat{q}$	$\hat{\eta}$	$\hat{r}$	$\hat{\beta}$	$\hat{R}$	R^*	$(1-\gamma)=0.90$	Width	$(1-\gamma)=0.95$	Width
1	100	100	1.78	0.15	1.9	0.1	0.6855	1.7765	0.1638	1.8989	0.1120	0.6850	0.6780	[0.6215; 0.7350]	0.11356	[0.6111; 0.7430]	0.13186
2	250	250	1.78	0.15	1.9	0.1	0.6855	1.7787	0.1548	1.8995	0.1059	0.6853	0.7253	[0.6853; 0.7650]	0.07965	[0.6789; 0.7709]	0.09202
3	500	500	1.78	0.15	1.9	0.1	0.6855	1.7793	0.1528	1.8997	0.1044	0.6860	0.7135	[0.6859; 0.7402]	0.05427	[0.6819; 0.7440]	0.06215
4	1000	1000	1.78	0.15	1.9	0.1	0.6855	1.7798	0.1516	1.9000	0.1013	0.6854	0.6985	[0.6814; 0.7169]	0.03553	[0.6768; 0.7201]	0.04330
5	5000	5000	1.78	0.15	1.9	0.1	0.6855	1.7798	0.1506	1.8999	0.1006	0.6855	0.6902	[0.6824; 0.6983]	0.01584	[0.6807; 0.6996]	0.01894
6	100	250	1.78	0.15	1.9	0.1	0.6855	1.7752	0.1667	1.8994	0.1082	0.6869	0.7189	[0.6763; 0.7587]	0.08242	[0.6701; 0.7670]	0.09693
7	100	500	1.78	0.15	1.9	0.1	0.6855	1.7776	0.1633	1.8996	0.1039	0.6849	0.6944	[0.6548; 0.7348]	0.08004	[0.6456; 0.7407]	0.09510
8	100	1000	1.78	0.15	1.9	0.1	0.6855	1.7759	0.1621	1.8999	0.1014	0.6869	0.7349	[0.6995; 0.7694]	0.06985	[0.6926; 0.7754]	0.08279
9	250	100	1.78	0.15	1.9	0.1	0.6855	1.7788	0.1551	1.8985	0.1139	0.6839	0.7043	[0.6565; 0.7533]	0.09679	[0.6427; 0.7602]	0.11750
10	500	100	1.78	0.15	1.9	0.1	0.6855	1.7794	0.1523	1.8984	0.1160	0.6853	0.6802	[0.6283; 0.7302]	0.10189	[0.6183; 0.7419]	0.12362
11	1000	100	1.78	0.15	1.9	0.1	0.6855	1.7797	0.1517	1.8991	0.1139	0.6855	0.6819	[0.6333; 0.7263]	0.09296	[0.6237; 0.7376]	0.11382
12	100	100	0.55	22	0.67	30.5	0.6259	0.5086	23.2133	0.4804	37.9860	0.6349	0.6692	[0.6030; 0.7360]	0.13301	[0.5894; 0.7468]	0.15741
13	250	250	0.55	22	0.67	30.5	0.6259	0.5380	22.3628	0.5901	33.6764	0.6311	0.5923	[0.5538; 0.6314]	0.07763	[0.5465; 0.6388]	0.09222
14	500	500	0.55	22	0.67	30.5	0.6259	0.5436	22.1900	0.6218	32.4722	0.6299	0.6247	[0.5914; 0.6478]	0.05636	[0.5880; 0.6542]	0.06627
15	1000	1000	0.55	22	0.67	30.5	0.6259	0.5472	22.0755	0.6461	31.4549	0.6279	0.6373	[0.6199; 0.6540]	0.03406	[0.6164; 0.6581]	0.04170
16	5000	5000	0.55	22	0.67	30.5	0.6259	0.5488	22.0306	0.6655	30.6923	0.6264	0.6221	[0.6138; 0.6309]	0.01715	[0.6123; 0.6328]	0.02054
17	100	250	0.55	22	0.67	30.5	0.6259	0.5487	22.1981	0.5895	33.7513	0.6325	0.6696	[0.6102; 0.7198]	0.10960	[0.5994; 0.7302]	0.13081
18	100	500	0.55	22	0.67	30.5	0.6259	0.5722	21.5316	0.6248	32.2672	0.6314	0.6315	[0.5877; 0.6773]	0.08958	[0.5805; 0.6863]	0.10584
19	100	1000	0.55	22	0.67	30.5	0.6259	0.5820	21.3467	0.6460	31.4455	0.6298	0.6212	[0.5723; 0.6696]	0.09726	[0.5656; 0.6772]	0.11161
20	250	100	0.55	22	0.67	30.5	0.6259	0.5330	22.4011	0.4775	38.5020	0.6395	0.6861	[0.6382; 0.7300]	0.09179	[0.6280; 0.7381]	0.11009
21	500	100	0.55	22	0.67	30.5	0.6259	0.5347	22.4039	0.4926	37.6015	0.6354	0.6524	[0.6067; 0.6971]	0.09035	[0.5972; 0.7032]	0.10596
22	1000	100	0.55	22	0.67	30.5	0.6259	0.5412	22.2352	0.4841	38.1364	0.6373	0.6705	[0.6256; 0.7168]	0.09125	[0.6128; 0.7250]	0.11225
23	100	100	−1.95	0.1	−1.8	0.18	0.7656	−1.7945	0.0932	−1.7836	0.1757	0.7515	0.7282	[0.6889; 0.7961]	0.10721	[0.6852; 0.8068]	0.12163
24	250	250	−1.95	0.1	−1.8	0.18	0.7656	−2.0991	0.1041	−2.0037	0.1911	0.7675	0.8145	[0.7742; 0.8525]	0.07823	[0.7667; 0.8604]	0.09371
25	500	500	−1.95	0.1	−1.8	0.18	0.7656	−2.0924	0.1042	−1.9563	0.1890	0.7689	0.7490	[0.7164; 0.7826]	0.06622	[0.7123; 0.7916]	0.07933
26	1000	1000	−1.95	0.1	−1.8	0.18	0.7656	−2.0592	0.1033	−1.9143	0.1867	0.7687	0.7951	[0.7689; 0.8194]	0.05048	[0.7621; 0.8234]	0.06132
27	5000	5000	−1.95	0.1	−1.8	0.18	0.7656	−1.9609	0.1003	−1.8256	0.1815	0.7659	0.7649	[0.7533; 0.7763]	0.02299	[0.7502; 0.7780]	0.02777
28	100	250	−1.95	0.1	−1.8	0.18	0.7656	−1.8330	0.0945	−1.9804	0.1896	0.7622	0.7575	[0.6998; 0.8212]	0.12142	[0.6962; 0.8296]	0.13341
29	100	500	−1.95	0.1	−1.8	0.18	0.7656	−1.8214	0.0941	−1.9770	0.1902	0.7641	0.8028	[0.7300; 0.8627]	0.13267	[0.7147; 0.8727]	0.15803
30	100	1000	−1.95	0.1	−1.8	0.18	0.7656	−1.7832	0.0928	−1.9025	0.1860	0.7642	0.7423	[0.6964; 0.7946]	0.09814	[0.6939; 0.8040]	0.11013
31	250	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0941	0.1039	−1.7519	0.1737	0.7557	0.7775	[0.7338; 0.8214]	0.08764	[0.7269; 0.8314]	0.10448
32	500	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0575	0.1031	−1.8193	0.1782	0.7564	0.7662	[0.7160; 0.8059]	0.08994	[0.7027; 0.8143]	0.11162
33	1000	100	−1.95	0.1	−1.8	0.18	0.7656	−2.0255	0.1023	−1.7842	0.1759	0.7543	0.7313	[0.7026; 0.7605]	0.05786	[0.6960; 0.7669]	0.07093

MLE, maximum likelihood estimation.

6. Case studies

In this section, we present two case studies in which stress (Y) and strength (X) follow q-exponential distributions. The first case study, which was originally described in Shirani and Härkegård,[50] deals with the experimental determination of high-cycle fatigue of ductile cast iron used for wind turbine components, and the second one, which was first reported in Furuya,[51] evaluates the gigacycle fatigue life of high-strength steel.

These two case studies are based on the well-known phenomenon[51,52] that the fatigue strength or endurance limit of large members is lower than that of small specimens made of the same material; in other words, a specimen size effect exists. That is, larger specimens fail at shorter fatigue lives than smaller specimens.[35,52] In fact, design of parts and structures against fatigue is based on laboratory-sized specimens that are usually smaller than the real ones. Therefore, it is of great importance to determine the reliability of larger specimens when data of smaller specimens are available. In our paper, we used the analogy that smaller specimens are stronger against fatigue and can be used as reference. Therefore, fatigue strength of smaller specimens was used as a reference to find the reliability of the larger specimens.

We use the stress–strength analysis in order to estimate the reliability of a specimen with large size by using a data set for small specimen. The reliability is evaluated based on the number of cycles to failure. Stress–cycle curves (SN curves) are often used to present fatigue resistance of materials at different stress levels. The SN curve simply represents the number of cycles to failure at a given stress level. Therefore, there is a one-to-one relationship between stress level and number of cycles to failure.[35,52] Therefore, using number of cycles instead of stress is a reasonable selection. That is, the number of cycles to failure can be understood as a measure of resistance to failure. In terms of stress–strength models, such measures are obtained in situations where the system has low resistance to fatigue failure (i.e., larger specimen) as well as in situations where the system has greater resistance to fatigue failure (i.e., smaller specimen).

Indeed, in the context of our work, Y refers to the number of cycles to failure in stress situation, that is, number of cycles to failure of larger specimen. Similarly, X refers to the number of cycles to failure in strength situation, that is, number of cycles to failure of smaller specimen. Therefore, the reliability index $R = P(Y < X)$ refers to the probability of the variable Y being less than variable X. In other words, the index R indicates a measure of the reliability of the larger specimen using the data set for the smaller specimen as reference.

As we have mentioned earlier, q-exponential distribution can be used to fit stress–strength data when they are represented by cycles to failure obtained from specimens made of the same material but with different sizes. Such an approach for stress and strength analysis has been previously used, for example, in Kundu and Gupta[7] and Shahsanaei and Daneshkhah.[53]

For each case study presented in this section, we estimate the parameters of two q-exponential distributions for data sets representing X and Y. The estimation method is the maximum likelihood method discussed in Section 3. Based on the estimates of the parameters, we analyze the goodness of fit of the q-exponential distributions by using both graphical analysis and hypothesis testing. Thus, we show the CDF for the two data sets along with the theoretical CDF of the q-exponential. Finally, we perform a bootstrapped version of the Kolmogorov–Smirnov (K–S boot) test in order to statistically check the fit of the q-exponential distribution to each data set.

6.1. Bootstrapped Kolmogorov–Smirnov test

The one-sample Kolmogorov–Smirnov test (K–S test) is not very useful in practice because it requires a simple null hypothesis. That is, the distribution must be completely specified with all parameters known.[54] A K–S boot test was proposed as an alternative to overcome this problem.[55] This method results in accurate asymptotic approximations of the p-values.[56] In this work, we will use this bootstrapped version to check the fit of the q-exponential distribution to each data set. This method follows the following steps:

- **Step 1:** From an initial sample for the variable $X = \{x_1, x_2, \ldots, x_n\}$, estimate the parameters $\Theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$ and construct the theoretical CDF: $F_n(X, \hat{\Theta})$.
- **Step 2:** Evaluate $D_0 = \max_{1 \le i \le n} \left|\left|\hat{F}_n(x_i) - F_n\left((x_i, \hat{\Theta}).\right|, \left|\hat{F}_n(x_{i-1}) - F_n\left((x_i, \hat{\Theta}).\right|\right|$, where $\hat{F}_n(X)$ is the empirical CDF.
- **Step 3:** Use the estimates obtained in the first step to generate new samples for X, that is, $\left\{x^*_{1,j}, x^*_{2,j}, \ldots, x^*_{n,j}\right\}$. Based on these new samples, compute the bootstrap sample estimate of Θ, say $\Theta^*_j = \left\{\theta^*_{1j}, \theta^*_{2j}, \ldots, \theta^*_{kj}\right\}$.
- **Step 4:** Repeat step 3, N times; $j = (1, 2, \ldots, N)$. The number of bootstrap samples N should be large to ensure a good approximation.
- **Step 5:** Evaluate $D^*_j = \max_{1 \le i \le n} \left|\left|\hat{F}^*_{n,j}\left(x^*_{i,j}\right) - F^*_{n,j}\left(x^*_{i,j}, \hat{\Theta}^*\right)\right|, \left|\hat{F}^*_{n,j}\left(x^*_{(i-1),j}\right) - F^*_{n,j}\left(x^*_{i,j}, \hat{\Theta}^*\right)\right|\right|$.

We reject the null hypothesis if $D_0 > D^*_{(N(1-\alpha)+1)}$ for a significance level α. An approximate p-value can be computed using

$$p = \frac{\#\left\{D^*_j \ge D_0\right\} + 1}{N + 1},$$

where $\#\left\{D^*_j \ge D_0\right\}$ indicates the quantity of D^*_j ($j = 1, \ldots, N$) that was larger than D_0.

6.2. Case study 1

From Shirani and Härkegård,[50] the size effect in ductile cast iron was studied using two sets of fatigue data for specimens with diameters 21 mm (Ø 21) and 50 mm (Ø 50). During the tests, the specimens were subjected to the same load condition. Figure 3 shows details of the drawings of the specimens.

For stress, we consider fatigue data of Ø 50 specimens, and for strength, the fatigue data of Ø 21 specimens are used. The data sets given in terms of number of cycles to failure are presented in Tables X and XI for diameters 21 and 50 mm, respectively.

Table XII presents the estimated parameters – entropic indices (shape parameters) and scale parameters, K–S distances between the empirical and fitted distribution functions, and the *p*-values of the K–S boot test; for this test, we use $N = 1000$. As mentioned in Section 2, the *q*-exponential distribution shows characteristics of a power law when the entropic index presents values between 1 and 2. In this case study, we can see that both *X* and *Y* present this behavior for the analyzed data sets.

Figure 4(a and b) presents the theoretical and empirical CDFs for *X* and *Y*, respectively. Note that the theoretical curve of the cumulative *q*-exponential distribution provides a very good fit to the data points of the empirical distribution for both the stress data and the strength data. In addition, by the K–S tests and the corresponding *p*-values (K–S boot) reported in Table XII, the *q*-exponential model adequately fits both the strength and stress data sets, as can also be seen in the graphs shown in Figure 4(a and b).

Given that $\widehat{r} = 1.1087$ $(1 < \widehat{r} < 2)$, there is no limitation on the support of *X*; thus, the index *R* is estimated by Equation (6) as 0.7579. The value obtained for *R* indicates that within the range of fatigue cycles here considered (i.e., high-cycle fatigue), there is a 0.7579 probability that fatigue life of specimens with Ø 21-mm diameter is longer than specimens with Ø 50-mm diameter. In terms of reliability, we can conclude that the value obtained for index *R* indicates the system performance. That is, based on the data presented for strength and stress, the reliability of the larger specimen is equal to 0.7579.

Moreover, the confidence intervals (bootstrap-*p* and nonparametric bootstrap approaches) are constructed by using the procedures presented in Section 4. In this case study, we obtained a large width for the confidence interval of *R* parameter because of the small size of the sample (Table XIII).

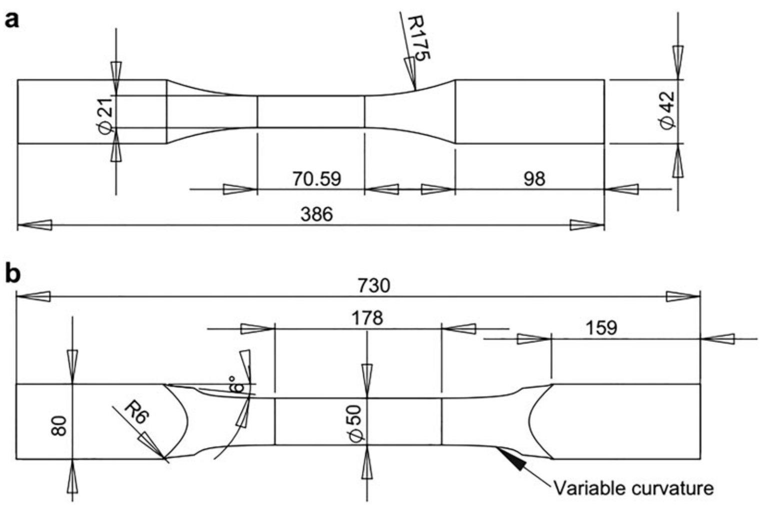

Figure 3. Detail drawings of (a) Ø 21 and (b) Ø 50 specimens (all dimensions are in millimeter)[50]

Table X. Ø 21 specimen fatigue test data (strength)

Specimen Number	Fatigue Life (number of cycles to failure)
1	3,000,000
2	716,400
3	1,674,100
4	679,400
5	801,000
6	1,076,600
7	4,181,701
8	619,200
9	469,500
10	83,200
11	92,500
12	107,700

Table XI. Ø 50 specimen fatigue test data (stress)

Specimen number	Fatigue life (number of cycles to failure)
1	295,000
2	869,000
3	869,900
4	1,573,335
5	151,400
6	152,000
7	183,700
8	218,000
9	30,200
10	45,100
11	46,900
12	47,300

Table XII. Estimated parameters, Kolmogorov–Smirnov distances, and *p*-values for the Kolmogorov–Smirnov test (K–S boot) – *q*-exponential distribution (case study 1)

	Entropic index	Scale parameter	K–S (D_0)	*p*-value
Data set 1 (strength)	$\hat{r} = 1.1087$	$\hat{\beta} = 884,013.7$	0.1477	0.7453
Data set 2 (stress)	$\hat{q} = 1.3005$	$\hat{\eta} = 161,904$	0.1554	0.6054

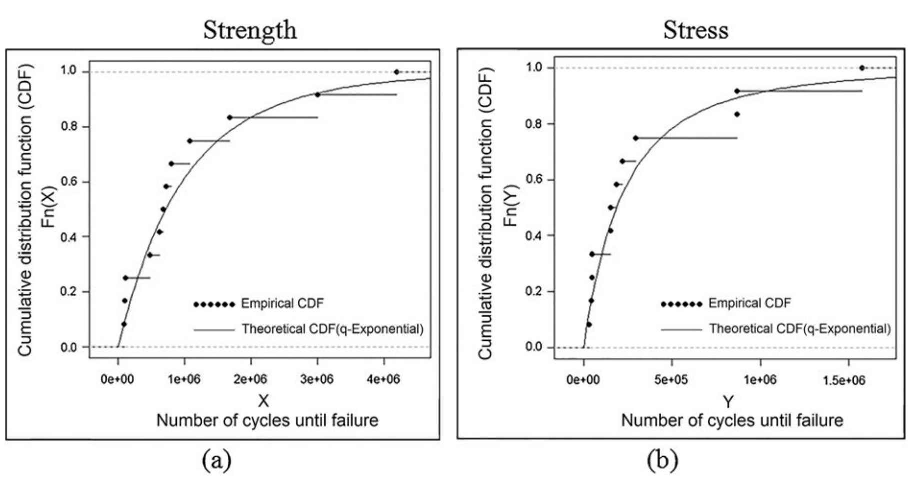

Figure 4. Theoretical (*q*-exponential) and empirical cumulative distribution function (CDF) for data sets of case study 1. (a) *X* – strength and (b) *Y* – stress

Table XIII. Point and interval estimates for $R = P(Y < X)$ – case study 1

Estimate of the parameter $R = P(Y < X)$	
$\hat{R} = 0.7579$	
Bootstrap-*p* confidence interval	
$n = 12$, $m = 12$	
C.I. (*R*,0.90) = [0.3258, 0.9078]	C.I. (*R*,0.95) = [0.1977, 0.9288]
Nonparametric bootstrap confidence interval	
$n = 12$, $m = 12$	
C.I. (*R*,0.90) = [0.4275, 0.9055]	C.I. (*R*,0.95) = [0.2495, 0.9260]

C.I., confidence interval.

6.3. Case study 2

In Furuya,[51] the size effect on gigacycle fatigue life of high-strength steel was evaluated using the following specimen geometries:

- Type A: Ø 8 mm × 10 mm specimen.
- Type B: Ø 3 mm hourglass-shaped specimens.

The specimens were subjected to the same load condition. Therefore, fatigue data for specimens type A and type B are selected as stress and strength, respectively. Figure 5 shows the detail drawings of the specimens. Data sets for strength and stress are presented in Tables XIV and XV, respectively.

In Table XVI, we present the estimated parameters (entropic indices and scale parameters), the K–S distances between the empirical and fitted distribution functions, and the corresponding *p*-values (K–S boot – for this test, we use $N = 1000$). Note that for stress and strength, the entropic indices present values that characterize a power law behavior, that is, $1 < q < 2$ for stress and $1 < r < 2$ for strength.

Figure 6(a and b) shows the theoretical and empirical CDF for X and Y, respectively. For the sake of visualization, we here use logarithmic scale to represent X and Y as the data sets include many extreme values. Analogously to the previous case study, one can conclude that, based on the plots presented in Figure 6, the curve of the cumulative distribution of the *q*-exponential shows close agreement with the empirical points for stress and strength data sets. We also report in Table XVI the bootstrapped K–S tests and the corresponding *p*-values. Based on those results, we notice that *q*-exponential model adequately fits both X and Y data sets, as can also be seen in Figure 6.

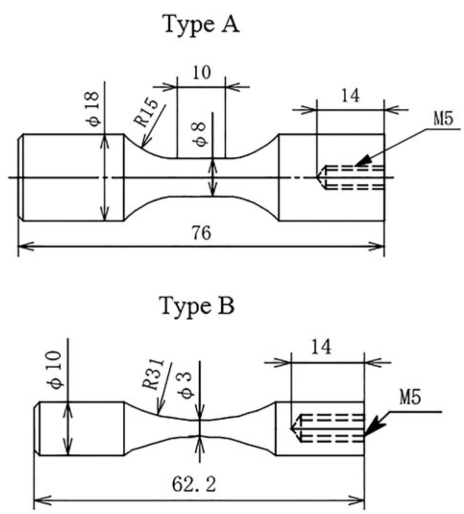

Figure 5. Detail drawings of (a) Ø 8 and (b) Ø 3 specimens (all dimensions are in millimeter). Adapted from Furuya[51]

Table XIV. Type B (Ø 3-mm hourglass-shaped specimen) fatigue test data (strength)

Specimen number	Fatigue life (number of cycles to failure)
1	1,017,286
2	2,989,152
3	4,059,346
4	4,256,299
5	8,376,572
6	9,560,400
7	13,007,977
8	25,303,118
9	33,621,704
10	55,951,560
11	101,155,984
12	144,322,192
13	376,711,232
14	731,957,760
15	9,444,513,800
16	9,912,163,300
17	9,918,688,300
18	9,921,105,900

Table XV. Type A (Ø 8 x 10 mm specimen) fatigue test data (stress)	
Specimen number	Fatigue life (number of cycles to failure)
1	289,867
2	1,291,756
3	6,404,257
4	7,848,468
5	9,374,890
6	31,500,474
7	211,678,768
8	5,575,744,500
9	5,926,607,400

Table XVI. Estimated parameters, Kolmogorov–Smirnov distances, and *p*-values for the Kolmogorov–Smirnov test (K–S boot) – *q*-exponential distribution (case study 2)

	Entropic index	Scale parameter	K–S (D_0)	*p*-value
Data set 1 (strength)	$\widehat{r} = 1.7519$	$\widehat{\beta} = 4,704,629$	0.1329	0.4955
Data set 2 (stress)	$\widehat{q} = 1.7643$	$\widehat{\eta} = 1,450,221$	0.1434	0.8501

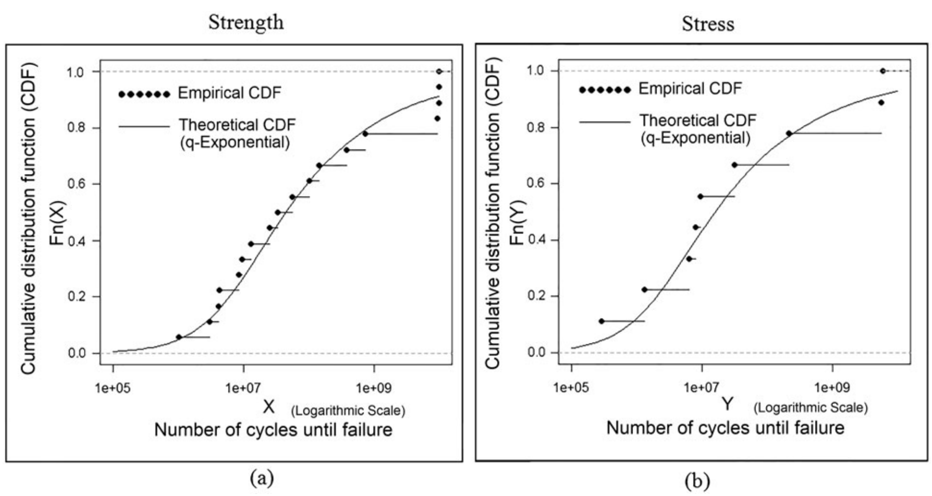

Figure 6. Theoretical (*q*-exponential) and empirical cumulative distribution function (CDF) for data sets of case study 2. (a) *X* – strength and (b) *Y* – stress

Given that $\widehat{r} = 1.7519$ $(1 < \widehat{r} < 2)$, there is no limitation on the support of *X*; thus, index *R* is estimated by Equation (6) as 0.5973. Thus, considering a component of strength and another of stress, obtained respectively when we measure the life cycle for specimens with 3-mm diameter and 8-mm diameter, there will be 59.73% chance that the larger specimen will not fail.

The confidence intervals are constructed using the bootstrap-*p* and nonparametric bootstrap approaches (presented in Section 4). Also in this case study, because of the small size of the samples, we obtain large width for the confidence intervals of the parameter *R* (Table XVII).

Table XVII. Point and interval estimates for $R = P(Y < X)$ – case study 2

Estimate of the parameter $R = P(Y < X)$		
	$\widehat{R} = 0.5973$	
	Bootstrap-*p* confidence interval	
	$n = 9$, $m = 18$	
C.I. $(R,0.90) = [0.3986, 0.7980]$		C.I. $(R,0.95) = [0.3681, 0.8242]$
	Nonparametric bootstrap confidence interval	
	$n = 9$, $m = 18$	
C.I. $(R,0.90) = [0.4090, 0.7665]$		C.I. $(R,0.95) = [0.3643, 0.7953]$

C.I., confidence interval.

6.4. *Comparing q-exponential with other distributions*

For the sake of comparison, both Weibull and exponential distributions were also considered to model the experimental strength and stress data sets presented in case studies 1 and 2. The results for the estimated parameters (scale and shape parameters for Weibull distribution parameter and for the exponential distribution parameter), K–S distances between empirical and fitted distribution functions, and the corresponding *p*-values (K–S boot – performed with $N = 1000$) obtained from the data sets are shown in Tables XVIII and XIX, which also include the K–S distance and *p*-values for the fit of the *q*-exponential distribution.

For case study 1, based on the K–S boot, the fit of the Weibull distribution resulted in *p*-values of 0.7322 and 0.5115 for the strength and stress data, respectively, clearly indicating that the Weibull is an appropriate distribution to describe the stress–strength data of this case study. In the case of the exponential distribution, the *p*-value for the K–S test was equal to 0.7212 for strength and 0.2757 for stress, resulting in a reasonable fit for the experimental data. However, for the strength data, we observed the most significant fit for *q*-exponential distribution (*p*-value = 0.7453), whereas Weibull distribution is the second (*p*-value = 0.7323), and exponential distribution also presents a good fit (*p*-value = 0.7212). For the stress, a similar behavior was observed. That is, *q*-exponential presents the most significant fit (*p*-value = 0.6054), while Weibull distribution is the second (*p*-value = 0.5115), and among the three distributions considered, the exponential presented the worst adjustment for the stress (*p*-value = 0.2757).

For case study 2, based on the K–S boot, the fit of Weibull distribution resulted in *p*-values of 0.2048 and 0.2297 for strength and stress data, respectively, indicating that despite the adjustment being significant, we cannot consider this as an excellent fit. In the case of exponential distribution, the *p*-value for the K–S test was equal to 9.99E-04 for strength and the same value for the stress, which yields a nonsignificant fit for exponential distribution. Note that *q*-exponential distribution presents the most significant fit for both strength (*p*-value = 0.4955) and stress (*p*-value = 0.8501), Weibull distribution provides the second most significant fit for strength (*p*-value = 0.2048) and stress (*p*-value = 0.2298), whereas exponential distribution was not significant for both data sets.

Note also that for both case studies, when we consider *q*-exponential distribution, a power law behavior is obtained for all analyzed cases, as the entropic indices for all data sets are greater than one. Moreover, when we consider the Weibull distribution, the shape parameter for all cases were between 0 and 1, which indicates a behavior of stretched exponential. As we mentioned in Section 1, *q*-exponential PDF with power law behavior presents a heavier tail than that of a Weibull PDF (with stretched exponential behavior). Thus, it is expected the *q*-exponential to have a superior performance over Weibull distribution when dealing with data sets that contain extremely large values. Thus, although the fit by *q*-exponential and Weibull distributions was comparable for the first case study, *q*-exponential is superior in the second one. This fact is due to the presence of extremely large values in the associated samples (magnitude in the order of 10^9).

The estimation of *R* when *X* and *Y* are Weibull-independent variables was presented by Kundu and Gupta.[7] In their work, the authors presented the expression $\widehat{R} = \frac{\widehat{\theta}_1}{\widehat{\theta}_1 + \widehat{\theta}_2}$, where $\widehat{\theta}_1$ is the estimate of scale parameter for *X* and $\widehat{\theta}_2$ is the estimate of scale parameter

Table XVIII. Comparing Weibull versus *q*-exponential – case studies 1 and 2

Case studies		Parameters (Weibull distribution): Shape parameter	Parameters (Weibull distribution): Scale parameter	(K–S boot) (Weibull distribution): K–S (D_0)	(K–S boot) (Weibull distribution): *p*-value	(K–S boot) (*q*-exponential distribution): K–S (D_0)	(K–S boot) (*q*-exponential distribution): *p*-value
Case study 1	Data set 1 (strength)	0.9331	1,088,102	0.1409	0.7322	0.1477	0.7453
	Data set 2 (stress)	0.8336	335,326.1	0.164	0.5115	0.1554	0.6054
Case study 2	Data set 1 (strength)	0.3366	417,229,706	0.1648	0.2048	0.1329	0.4955
	Data set 2 (stress)	0.3077	176,273,348	0.2222	0.2298	0.1434	0.8501

Table XIX. Comparing exponential versus *q*-exponential – case studies 1 and 2

Case studies		Parameter (exponential Distribution): Rate parameter	(K–S boot) (exponential distribution): K–S (D_0)	(K–S boot) (exponential distribution): *p*-value	(K–S boot) (*q*-exponential distribution): K–S (D_0)	(K–S boot) (*q*-exponential distribution): *p*-value
Case study 1	Data set 1 (strength)	8.89E-07	0.1587	0.7212	0.1477	0.7453
	Data set 2 (stress)	2.68E-06	0.2245	0.2757	0.1554	0.6054
Case study 2	Data set 1 (strength)	4.42E-10	0.6048	9.99E-4	0.1329	0.4955
	Data set 2 (stress)	7.65E-10	0.6429	9.99E-4	0.1434	0.8501

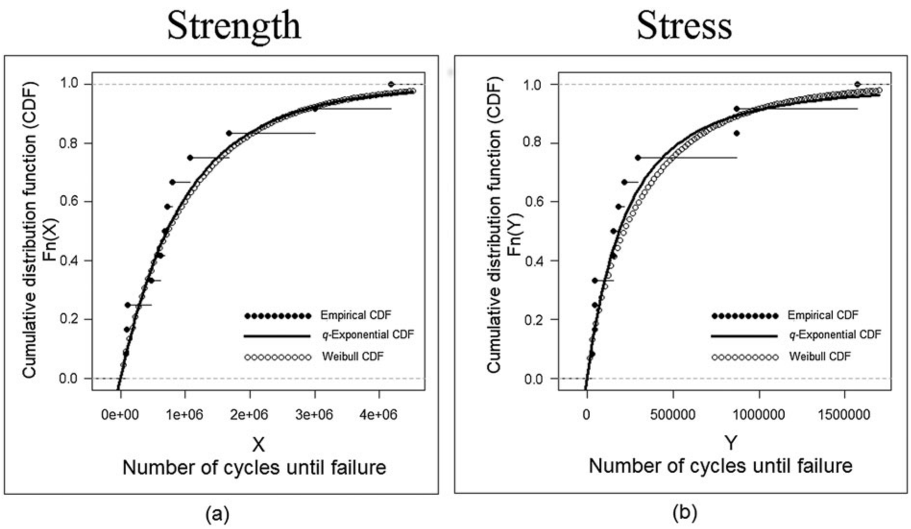

Figure 7. Empirical and theoretical (*q*-exponential and Weibull) cumulative distribution functions (CDFs) for case study 1 – (a) strength and (b) stress

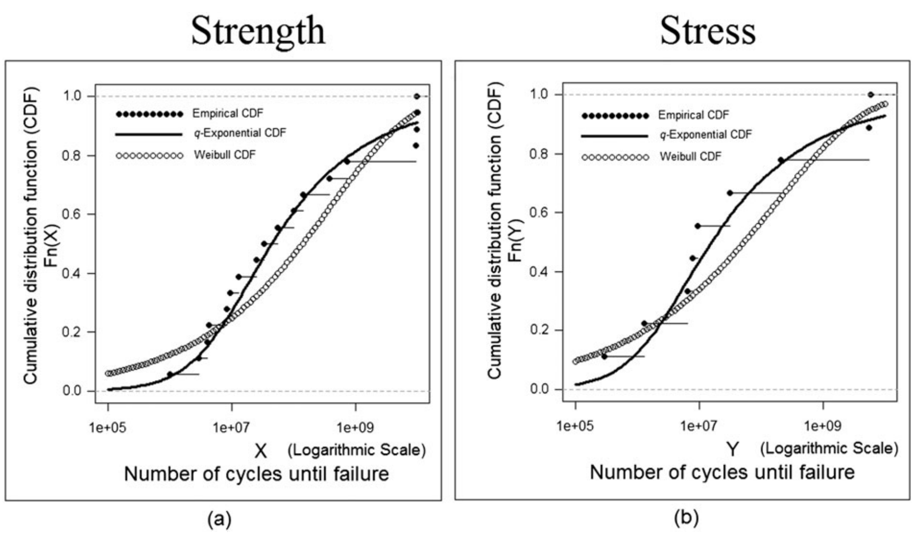

Figure 8. Empirical and theoretical (*q*-exponential and Weibull) cumulative distribution functions (CDFs) for case study 2 – (a) strength and (b) stress

for *Y*. Thus, once the data set for *X* and *Y* in case study 1 also presented a good fit for the Weibull distribution, we computed *R* considering a Weibull distribution as $\widehat{R} = 0.7649$. This result is very similar to the one when *X* and *Y* are modeled by two independent *q*-exponential distributions, that is, $\widehat{R} = 0.7579$. This indicates that both distributions can be used in order to estimate $R = P(Y < X)$ for the first case study. For case study 2, when *X* and *Y* are modeled by two independent Weibull distributions, the estimated *R* index is $\widehat{R} = 0.7029$, which is very different from the one obtained when we considered *q*-exponential ($\widehat{R} = 0.5973$). This difference is because Weibull distribution presented an inferior fit performance for both *X* and *Y* when compared with *q*-exponential. In fact, both *X* and *Y* present extremely large values, and as discussed previously, this kind of data is better modeled by a PDF that has the ability to model data with characteristic of power law as is the case of *q*-exponential when $1 < q < 2$.

From Figure 7, one can observe that, in the first case study, a good fit is obtained for both *q*-exponential and Weibull distributions. On the other hand, based on the plots shown in Figure 8 for case study 2, the empirical points for stress and strength are fitted very well by the curves of the theoretical CDF of the *q*-exponential distribution, whereas the same is not true for the curve of the Weibull cumulative distribution.

Therefore, the *q*-exponential distribution can be considered as an alternative to the Weibull distribution in stress–strength problems, especially when one is dealing with data with extreme large values. Thus, despite the more intricate set of equations underlying the estimation of the index *R* based on the *q*-exponential distribution, its use represents a viable way for providing more useful estimates for the index *R*.

7. Conclusions

In this paper, we have introduced the *q*-exponential distribution as a model for reliability data with extreme values in the relevant context of stress–strength reliability. More specifically, when we deal with fatigue life data that present a power law behavior and we need to estimate the performance index $R = P(Y < X)$, we have considered that the stress Y and strength X are *q*-exponential-independent random variables and have proposed a procedure for estimating index R by considering that the support of X is limited (i.e., entropic index or shape parameter of the strength is $r < 1$) and unlimited ($1 < r < 2$). Additionally, confidence intervals for the index R have been presented by means of parametric and nonparametric bootstrap approaches.

From the simulation experiments, the consistency of the MLE obtained for the Index R has been verified based on the *q*-exponential distribution, once the absolute bias and the MSE values related to the estimation of R via maximum likelihood decrease as the sample size increases. Furthermore, for different sample sizes for X and Y, the bootstrap-*p* confidence intervals showed to be very efficient in estimating the confidence interval of R given that for all simulations, the confidence intervals included the true parameter value. On the other hand, not all confidence intervals provided by the nonparametric bootstrap presented the true parameter value (about 14% of the simulated cases).

With respect to the first case study involving ductile cast iron specimens, *q*-exponential distribution properly fits both stress and strength data, as can be seen by the CDF and PDF plots and by the bootstrapped K–S test. In addition, for the sake of comparison, we have estimated the parameters for the situations where X and Y are both modeled by either Weibull or exponential distribution. The latter provided the worst fit, while *q*-exponential and Weibull models resulted in quite similar fits. Such a result was reinforced by the proximity of the estimates for the index R obtained from both models.

In relation to the second case study involving high-strength steel, *q*-exponential distribution presented an excellent fit for both strength and stress. The Weibull distribution, despite having a significant adjustment to the experimental data, presented smaller *p*-values for both strength and stress. The exponential distribution in turn was not significant for the data sets of this case study.

Thus, based on the results for case studies, the *q*-exponential distribution is more suitable for modeling data that include extreme large values when compared with the Weibull distribution. This result is evident in the second case study that deals with a very resistant material, and because of this characteristic, the number of cycles until failure presents a high order of magnitude. Consequently, the index R obtained with *q*-exponential model was more informative than the same index obtained by the Weibull distribution.

Therefore, based on the discussed results, it is natural to suggest the *q*-exponential as a candidate distribution to model stress–strength reliability problems, especially when dealing with extreme large values. We also note that the *q*-exponential distribution is able to model data that present a power law asymptotic behavior, which is an important characteristic of cycles until fatigue failure, as corroborated by the two case studies considered in this work, where the estimated entropic indices had values that characterize a power law behavior, that is, $1 < q < 2$ for stress and $1 < r < 2$ for strength.

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