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On the Central Paths in Symmetric Cone Programming

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Abstract This paper is devoted to the study of optimal solutions of symmetric cone programs by means of the asymptotic behavior of central paths with respect to a broad class of barrier functions. This class is, for instance, larger than that typically found in the literature for semidefinite positive programming. In this general framework, we prove the existence and the convergence of primal, dual and primal–dual central paths. We are then able to establish concrete characterizations of the limit points of these central paths for specific subclasses. Indeed, for the class of barrier functions defined at the origin, we prove that the limit point of a primal central path minimizes the corresponding barrier function over the solution set of the studied symmetric cone program. In addition, we show that the limit points of the primal and dual central paths lie in the relative interior of the primal and dual solution sets for the case of the logarithm and modified logarithm barriers.

Keywords Symmetric cone programming · Central paths · Barrier functions · Recession functions · Euclidean Jordan algebra

Mathematics Subject Classification 90C25 · 90C51 · 17C27

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1 Introduction

A usual (linear) symmetric cone program (SCP) consists of minimizing a linear function subject to the intersection of an affine subspace with a symmetric cone. Prominent optimization frameworks, such as linear programming, second-order cone programming and semidefinite programming, can be cast as SCPs. The machinery of Euclidean Jordan algebras (EJAs) has become essential in the study of symmetric cone programming because of the well-known property that states that every symmetric cone in a Euclidean vector space can be cast as the cone of square elements of a suitably chosen EJA (see, for instance, [1, Theorem III.3.1]).

Interior-point methods (IPMs) are efficient tools for solving an SCP. Typically, these methods use a barrier scheme defined via the logarithm barrier function. The solution of this barrier scheme is called the central path, and the good properties exhibited by the asymptotic behavior of this central path are fundamental to achieving convergence and good performance of IPMs. This topic has been very productive in the last two decades. We refer the reader to the pioneering works of Nesterov and Todd [2,3], Faybusovich [4,5] and Schmieta and Alizadeh [6]. For recent developments in this subject, we refer the reader to the works of Zhang and Zhang [7] and Yang et al. [8].

As noted in [5], the choice of the logarithm barrier function for obtaining the central path is optimal in the sense of available complexity estimates of corresponding interior-point algorithms. However, other choices of barrier functions occasionally have theoretical advantages concerning the characterization of the limit points of their central paths. This is the case, for instance, in Yu et al. [9], in which the central path, with respect to the entropy function, is analyzed.

In this work, we consider a generalization of the SCP, allowing the objective function to be convex (but smooth). To construct the barrier schemes, we use spectral functions associated with a large class of real-valued barrier functions, which include the logarithm barrier and the entropy function, among others. We also study the dual problem of the SCP via a central path, obtained from the optimality conditions of the barrier schemes associated with the primal problem SCP. Hence, we work with three central paths: the primal central path, the dual central path and the primal–dual central path. Our aim is thus to study their asymptotic behavior. More precisely, we prove the well-definedness of these central paths, we present convergence results, and we provide characterizations of their limit points in specific situations.

To the best of our knowledge, no general and systematic treatment of central paths with respect to broad classes of barrier functions for solving symmetric cone programs exists in the literature. Our main motivation is to bring well-known conic programs into a common framework for which one can detect the advantages and limitations of the EJA approach and the considered barrier functions.

The outline of this manuscript is as follows. In Sect. 2, we provide preliminary materials concerning EJAs. We also present some properties of spectral functions and introduce the class of barrier functions we work with. In Sect. 3, we show that the primal central paths are well defined, and we introduce the dual and the primal–dual central paths. In Sect. 4, we present convergence results for the primal, dual and primal–dual central paths. Finally, Sect. 5 is devoted to the study of the characterization of the limit points of the central paths.

2 Preliminaries

2.1 Euclidean Jordan Algebras

We assume that the reader has some familiarity with Euclidean Jordan algebras. For an exposition of the definitions and main results of this algebra, we refer the reader to the book by Faraut and Korányi [1].

Throughout this paper, we assume that $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$, shortly denoted by $\mathbb{V}$, is a (real) Euclidean Jordan algebra (EJA) of rank r and unit element $e \in \mathbb{V}$. For $x, y \in \mathbb{V}$, the symbol $\langle x, y \rangle$ denotes the inner product, and $x \circ y$ denotes the Jordan product. We use the notation $x^2 := x \circ x$, and we denote the corresponding *symmetric cone* as $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$.

An element $c \in \mathbb{V}$ is an *idempotent* if $c^2 = c$. An idempotent c is *primitive* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A *Jordan frame* is a collection $\{c_1, \ldots, c_r\}$ of primitive idempotents satisfying $\sum_{i=1}^r c_i = e$ and $c_i \circ c_j = 0$ when $i \neq j$. Every element $x \in \mathbb{V}$ admits a *spectral decomposition* (see [1, Theorem III.1.2]); that is, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that $x = \sum_{i=1}^r \lambda_i c_i$. The λ_i's values are uniquely determined by x and are called the *eigenvalues* of x.

Let $x \in \mathbb{V}$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i c_i$. We define the *trace* of x by $\operatorname{tr}(x) := \sum_{i=1}^r \lambda_i$ and the *determinant* of x by $\det(x) := \Pi_{i=1}^r \lambda_i$. The element x is *invertible* if no eigenvalue of x is equal to zero, in which case one defines the inverse of x as $x^{-1} := \sum_{i=1}^r \lambda_i^{-1} c_i$.

Two elements $a, b \in \mathbb{V}$ *operator commute* if $a \circ (b \circ z) = b \circ (a \circ z)$ for all $z \in \mathbb{V}$. From [6, Theorem 27], the operator commutation property is equivalent to the existence of a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_r$ such that $a = \sum_{i=1}^r \lambda_i c_i$ and $b = \sum_{i=1}^r \mu_i c_i$.

An EJA is said to be *scalarizable* if there exists a positive constant θ such that $\langle x, y \rangle = \theta \operatorname{tr}(x \circ y)$ for all $x, y \in \mathbb{V}$. This constant is unique and called the *scaling factor* of the EJA. Of course, it is given by $\theta = \frac{\langle e, e \rangle}{\operatorname{tr}(e)}$. Without any loss of generality, we assume that a scalarizable EJA has scaling factor $\theta = 1$. Indeed, we can work with the inner product $\frac{1}{\theta}\langle \cdot, \cdot \rangle$ on $\mathbb{V}$, which is compatible with the given Jordan structure. Throughout this work, we assume that the EJA is scalarizable with $\theta = 1$.

2.2 Spectral Sets and Functions

For $x \in \mathbb{V}$, the notation $\lambda(x)$ refers to the (column) vector of the eigenvalues of x arranged in nondecreasing order (i.e., $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_r(x)$). A set $\Omega \subseteq \mathbb{V}$ is said to be a *spectral set* if there exists a *permutation invariant set* $Q \subseteq \mathbb{R}^r$ such that $\Omega = \{x \in \mathbb{V} : \lambda(x) \in Q\}$. A function $\Phi : \mathbb{V} \to \mathbb{R} \cup \{+\infty\}$ is said to be a *spectral function* if there exists a *permutation invariant function* $\varphi : \mathbb{R}^r \to \mathbb{R} \cup \{+\infty\}$ such that $\Phi(x) = \Phi_\varphi(x) := \varphi(\lambda(x))$. We note that the domain of Φ_φ is given by

$$\operatorname{dom}\Phi_\varphi = \left\{ \sum_{i=1}^r \xi_i c_i : (c_1, \ldots, c_r) \in \mathcal{O}_{\mathbb{V}},\ \xi \in \operatorname{dom}\varphi \right\}, \tag{1}$$

where $\mathcal{O}_{\mathbb{V}}$ denotes the set of vectors $(c_1, \ldots, c_r) \in \mathbb{V}^r$ such that $\{c_1, \ldots, c_r\}$ is a Jordan frame. Hence, $\mathrm{dom}\Phi_\varphi$ is a spectral set, whose closure is given by

$$\mathrm{cl}\left(\mathrm{dom}\Phi_\varphi\right) = \left\{ \sum_{i=1}^{r} \xi_i c_i : (c_1, \ldots, c_r) \in \mathcal{O}_{\mathbb{V}},\ \xi \in \mathrm{cl}\,(\mathrm{dom}\varphi) \right\}. \tag{2}$$

Let us denote by $\Gamma_0(\mathbb{X})$ the set of convex, lower semicontinuous and proper functions defined on a Euclidean space $\mathbb{X}$ with values in $\mathbb{R} \cup \{+\infty\}$. The spectral function Φ_φ inherits many properties of φ. For instance, if $\varphi \in \Gamma_0(\mathbb{R}^r)$, then $\Phi_\varphi \in \Gamma_0(\mathbb{V})$ (see [10, Theorem 41]). Moreover, from [10, Theorem 30], the conjugate $\left(\Phi_\varphi\right)^*$ is also a spectral function with φ^* as the associated permutation invariant function satisfying the following relationship:

$$\left(\Phi_\varphi\right)^* = \Phi_{\varphi^*}. \tag{3}$$

Let $f \in \Gamma_0(\mathbb{V})$. A fundamental concept used throughout this work is the *recession function* of f (see [11,12]), which can be defined as

$$f_\infty(d) := \lim_{t \to +\infty} \frac{f(x + td) - f(x)}{t},\ \ \forall d \in \mathbb{V}, \tag{4}$$

where x is an arbitrary point in $\mathrm{dom}\, f := \{x : f(x) < +\infty\}$ (the domain of f).

The next proposition provides a formula to compute the recession function $\left(\Phi_\varphi\right)_\infty$. This result is a generalization of [13, Theorem 8.1] and [14, Proposition 3.3(2)] that, to our knowledge, has not previously been established for EJAs. Its proof follows the corresponding one in [13, Theorem 8.1], but it is necessary to apply a commutation principle that has been recently obtained for EJAs in Ramírez, Seeger and Sossa [15].

Proposition 2.1 *Let* $\varphi \in \Gamma_0\left(\mathbb{R}^r\right)$ *be a permutation invariant function and* $\Phi_\varphi : \mathbb{V} \to \mathbb{R} \cup \{+\infty\}$ *be the induced spectral function. Then,* $\left(\Phi_\varphi\right)_\infty = \Phi_{\varphi_\infty}$.

Proof According to [12, Theorem 2.5.4], the recession function of $f \in \Gamma_0(\mathbb{V})$ is equal to the support function of $\mathrm{dom}\, f^*$; that is,

$$f_\infty(d) = \sup\{\langle u, d\rangle : u \in \mathrm{dom}\, f^*\}. \tag{5}$$

Let $d \in \mathbb{V}$; because $\Phi_\varphi \in \Gamma_0(\mathbb{V})$, we have from (5) and (3) the following equality:

$$\left(\Phi_\varphi\right)_\infty (d) = \sup\left\{\langle u, d\rangle : u \in \mathrm{dom}\Phi_{\varphi^*}\right\}. \tag{6}$$

Denote by $d := \sum_{i=1}^{r} \lambda_i(d) c_i(d)$ a spectral decomposition of d, and take the elements $u_\xi := \sum_{i=1}^{r} \xi_i c_i(d)$, with $\xi \in \mathrm{dom}\varphi^*$. In view of (1), we have that $u_\xi \in \mathrm{dom}\Phi_{\varphi^*}$ for all $\xi \in \mathrm{dom}\varphi^*$. Note that $\langle u_\xi, d\rangle = \xi^T \lambda(d)$. Hence, from (6), we obtain

$$\left(\Phi_\varphi\right)_\infty (d) \geq \sup\left\{\xi^T \lambda(d) : \xi \in \mathrm{dom}\varphi^*\right\} = \varphi_\infty(\lambda(d)) = \Phi_{\varphi_\infty}(d),$$

where the first equality is because of (5). Reciprocally, by taking the closure of $\mathrm{dom}\Phi_{\varphi^*}$, (6) can be written as

$$\left(\Phi_{\varphi}\right)_{\infty}(d) = \sup\left\{\langle u, d\rangle : u \in \mathrm{cl}\left(\mathrm{dom}\Phi_{\varphi^*}\right)\right\}. \tag{7}$$

We recall that the closure of a spectral set is spectral, so $\mathrm{cl}\left(\mathrm{dom}\Phi_{\varphi^*}\right)$ is a spectral set. Assume first that $\left(\Phi_{\varphi}\right)_{\infty}(d) < +\infty$, and let $\bar{u} \in \mathrm{cl}\left(\mathrm{dom}\Phi_{\varphi^*}\right)$ be an optimal solution of problem (7). Hence, we can use the commutation principle [15, Theorem 2] to deduce that $\bar{u}$ and d operator commute. Therefore, $\bar{u}$ and d admit a simultaneous spectral decomposition, i.e., there exist $\bar{\xi} \in \mathbb{R}^r$ and a Jordan frame $\{\bar{c}_1, \ldots, \bar{c}_r\}$ such that $\bar{u} = \sum_{i=1}^r \bar{\xi}_i \bar{c}_i$, $d = \sum_{i=1}^r \lambda_i(d)\bar{c}_i$. Observe that $\bar{\xi} \in \mathrm{cl}\,(\mathrm{dom}\varphi^*)$ because $\bar{u} \in \mathrm{cl}\left(\mathrm{dom}\Phi_{\varphi^*}\right)$ and because of relationship (2). From (7), we then have

$$\left(\Phi_{\varphi}\right)_{\infty}(d) = \bar{\xi}^T\lambda(d) \leq \sup\left\{\xi^T\lambda(d) : \xi \in \mathrm{cl}\left(\mathrm{dom}\varphi^*\right)\right\} = \varphi_{\infty}(\lambda(d)) = \Phi_{\varphi_{\infty}}(d).$$

When $\left(\Phi_{\varphi}\right)_{\infty}(d) = +\infty$, from (6), we have that there exists a sequence $\{u_n\} \subset \mathrm{dom}\Phi_{\varphi^*}$ such that $\langle u_n, d\rangle \to +\infty$. By using the Von Neumann's trace inequality $\langle a, b\rangle \leq \lambda(a)^T\lambda(b)$ for all $a, b \in \mathbb{V}$ [10, Theorem 23], we conclude that the sequence $\{\lambda(u_n)\} \subset \mathrm{dom}\varphi^*$ satisfies $\lambda(u_n)^T\lambda(d) \to +\infty$, which implies that $\Phi_{\varphi_{\infty}}(d) = +\infty$. The proof is complete. □

2.3 Barrier Functions for Symmetric Cones

Let us denote by $\mathcal{C}$ the set of functions $v \in \Gamma_0(\mathbb{R}) \cap C^2(]0, \infty[)$ satisfying the following:

$$]0, \infty[\subseteq \mathrm{dom}v \subseteq [0, \infty[, \quad \lim_{s \to 0^+} v'(s) = -\infty \quad \text{and} \quad v''(s) > 0,\ \forall s > 0.$$

We also define the following hypothesis over v:

(H.1) v is sublevel bounded.
(H.2) $\lim_{s\to\infty} v'(s) = 0$.

Example 2.1 We provide examples of functions that belong to the class $\mathcal{C}$.
Functions with $\mathrm{dom}v = [0, \infty[$ (with the convention that these functions take the value $+\infty$ on negative numbers): $v_1(s) = s\log s - s$ (with the convention $0\log 0 = 0$), $v_2(s) = -\frac{1}{r}s^r$, and $v_3(s) = s - \frac{1}{r}s^r$ (with $r \in]0, 1[$). Note that v_1 and v_3 satisfy (H.1), whereas v_2 satisfies (H.2).

Functions with $\mathrm{dom}v =]0, \infty[$ (with the convention that these functions take the value $+\infty$ on nonpositive numbers): $v_4(s) = s^{-1}$, $v_5(s) = -\log s$ (logarithm barrier), and $v_6(s) = s - \log s$ (modified logarithm barrier). Note that v_6 satisfies (H.1), whereas v_4 and v_5 satisfy (H.2).

The next proposition presents two technical results concerning the class $\mathcal{C}$.

Proposition 2.2 *Let $v \in \mathcal{C}$. The following properties are satisfied:*

(a) If $v(0) \in \mathbb{R}$, then for every $\bar{s} > 0$, there exists $L \in \mathbb{R}$ such that $-v'(s)s \leq L$ for all $s \in]0, \bar{s}]$.
(b) If v satisfies (H.2), then v_∞ is nonnegative.

Proof Part (a). The convexity of v and the fact that $v(0) \in \mathbb{R}$ imply that $-v'(s)s \leq v(0) - v(s)$ for all $s > 0$. Because v is lower semicontinuous on $[0, +\infty[$, the desired result follows from the fact that v is bounded below on $[0, \bar{s}]$ for every $\bar{s} > 0$.

Part (b). We use (4) to deduce that $v_\infty(0) = 0$ and $v_\infty(\delta) = +\infty$, $\forall \delta < 0$. Suppose that $\delta > 0$ and let $s > 0$. The convexity of v implies that

$$\frac{v(s)}{t} + v'(s)\delta - \frac{v'(s)s}{t} \leq \frac{v(t\delta)}{t}, \ \forall t > 0.$$

Letting t and s tend to infinity, we conclude that $0 \leq v_\infty(\delta)$. □

For $v \in \mathcal{C}$, we define the function $\Psi_v : \mathbb{V} \to \mathbb{R} \cup \{+\infty\}$ to be

$$\Psi_v(x) := \sum_{i=1}^{r} v(\lambda_i(x)). \tag{8}$$

Of course, Ψ_v is a spectral function with $\varphi_v(\xi) = \sum_{i=1}^r v(\xi_i)$, its associated permutation invariant function. Below, we list some properties that Ψ_v satisfies. A proof can be found in [10,16].

Proposition 2.3 *Let $v \in \mathcal{C}$. Ψ_v satisfies the following properties:*
(a) *Ψ_v is a spectral function that belongs to $\Gamma_0(\mathbb{V})$.*
(b) int $(\text{dom}\Psi_v) = \text{int}\mathcal{K}$, *where* int$C$ *denotes the* interior *of a set C.*
(c) *Ψ_v is strictly convex and twice continuously differentiable on* int$\mathcal{K}$.
(d) $\nabla\Psi_v(x) = \sum_{i=1}^r v'(\lambda_i(x))c_i(x)$ *for every* $x = \sum_{i=1}^r \lambda_i(x)c_i(x) \in \text{int}\mathcal{K}$.

Example 2.2 Consider v_5 and v_6 as in Example 2.1. We then have

$$\begin{aligned} \Psi_{v_5}(x) &= -\log\det(x), & \nabla\Psi_{v_5}(x) &= -x^{-1},\\ \Psi_{v_6}(x) &= \text{tr}(x) - \log\det(x), & \nabla\Psi_{v_6}(x) &= e - x^{-1}. \end{aligned}$$

3 Central Paths

The (convex) symmetric cone program is defined as follows:

$$\min_{x \in \mathbb{V}}\{f(x) : \mathcal{A}x = b, \ x \in \mathcal{K}\}, \tag{P}$$

where $f : \mathbb{V} \to \mathbb{R}$ is a convex, continuously differentiable function, $b \in \mathbb{R}^m$, and $\mathcal{A} : \mathbb{V} \to \mathbb{R}^m$ is a linear map. The classical Lagrangian dual problem associated with problem (P) is given by

$$\max_{(y,s) \in \mathbb{R}^m \times \mathcal{K}} b^T y + \inf_{x \in \mathbb{V}}\{f(x) - \langle x, \mathcal{A}^* y + s\rangle\}. \tag{D}$$

where $\mathcal{A}^*$ denotes the adjoint of the linear map $\mathcal{A}$. We denote by F_P (F_D) the primal (dual) feasible set and by S_P (S_D) the primal (dual) solution set.

Remark 3.1 Linear programming, semidefinite programming and second-order cone programming are particular cases of (P). In fact, this result is obtained by taking the algebra $\mathbb{R}^n$ of n-dimensional vectors, the algebra $\mathcal{S}^n$ of $n \times n$ symmetric matrices and the Jordan spin algebra $\mathcal{L}^n$, respectively, with f linear.

Throughout this paper, we suppose that the following standard assumptions are fulfilled:

(A.1) S_P is nonempty.
(A.2) $\mathcal{A}$ is surjective.
(A.3) $F_P^0 := F_P \cap \operatorname{int}\mathcal{K} \neq \emptyset$.

These assumptions ensure that S_D is a nonempty and compact set and that there is no duality gap between (P) and (D).

The KKT optimality conditions state that $(x, (y, s)) \in \mathcal{S}_P \times \mathcal{S}_D$ if and only if it solves the following system of equations

$$\mathcal{A}x = b, \quad \mathcal{A}^* y + s = \nabla f(x), \quad x, s \in \mathcal{K}, \quad \langle x, s\rangle = 0. \tag{9}$$

We study the solution of (P) by means of the following barrier schemes:

$$\min_{x\in\mathbb{V}}\{f(x) + \mu\Psi_v(x) : \mathcal{A}x = b,\ x \in \operatorname{dom}\Psi_v\}, \quad \mu > 0, \tag{P_μ}$$

where $v \in \mathcal{C}$ and Ψ_v was defined in (8). The *primal central path* associated with v is the set $\{x(\mu) : \mu > 0\}$, where $x(\mu)$ is a solution of problem P_μ.

Let $g : \mathbb{V} \to \mathbb{R} \cup \{+\infty\}$ be defined by $g := f + \delta_{\{x:\mathcal{A}x=b\}}$, where δ_C is the indicator function of $C \subset \mathbb{V}$. The problem P_μ can be written as

$$\min_{x\in\mathbb{V}} f_\mu(x) := g(x) + \mu\Psi_v(x).$$

Note that $F_p^0 \subseteq \operatorname{dom} g \cap \operatorname{dom}\Psi_v$. Hence, assumption (A.3) implies that f_μ is proper. Moreover, f_μ is strictly convex and lower semicontinuous.

The rest of the section is devoted to establishing that P_μ admits a unique solution for every $\mu > 0$. To do this, we need two technical lemmas. The first one is obtained from the continuity of the λ_i values and the compactness of the set of Jordan frames $\mathcal{O}_{\mathbb{V}}$.

Lemma 3.1 *Let $\{x_k\}$ be a sequence of $\mathbb{V}$ such that it converges to $x \in \mathbb{V}$. Denote by $x_k := \sum_{i=1}^r \lambda_i(x_k)c_i(x_k)$ a spectral decomposition of x_k, for every k. There then exists a spectral decomposition of x, given by $x = \sum_{i=1}^r \lambda_i c_i$, and a subsequence $\{x_{k_\ell}\}$ such that for every $i \in \{1, \ldots, r\}$,*

$$\lim_{\ell\to\infty} \lambda_i(x_{k_\ell}) \to \lambda_i, \quad \lim_{\ell\to\infty} c_i(x_{k_\ell}) = c_i.$$

Lemma 3.2 *Let $v \in \mathcal{C}$ such that $v(0) \in \mathbb{R}$. For every sequence $\{x_k\} \subset \mathrm{int}\mathcal{K}$ converging to a point on $\partial\mathcal{K} := \mathcal{K} \setminus \mathrm{int}\mathcal{K}$, there then exists a subsequence $\{x_{k_\ell}\}$ such that $\lim_{\ell\to\infty}\langle \nabla\Psi_v(x_{k_\ell}), \tilde{x} - x_{k_\ell}\rangle = -\infty$ for every $\tilde{x} \in \mathrm{int}\mathcal{K}$.*

Proof Let $\{x_k\} \subset \mathrm{int}\mathcal{K}$ be a sequence such that it converges to $x \in \partial\mathcal{K}$, and let $\tilde{x} \in \mathrm{int}\mathcal{K}$. Denote by $x_k := \sum_{i=1}^r \lambda_i(x_k)c_i(x_k)$ a spectral decomposition of x_k. We then have

$$\langle \nabla\Psi_v(x_k), \tilde{x} - x_k\rangle = \sum_{i=1}^r v'(\lambda_i(x_k))\langle c_i(x_k), \tilde{x}\rangle - \sum_{i=1}^r v'(\lambda_i(x_k))\lambda_i(x_k).$$

Note that for every i, the sequence $\{\lambda_i(x_k)\}$ is bounded because $\{x_k\}$ is bounded. Hence, there exists $\bar{s} > 0$ such that for every i, we have that $\{\lambda_i(x_k)\} \subset]0, \bar{s}]$. Proposition 2.2(a) implies that there exists $M \in \mathbb{R}$ such that

$$\langle \nabla\Psi_v(x_k), \tilde{x} - x_k\rangle \leq \sum_{i=1}^r v'(\lambda_i(x_k))\langle c_i(x_k), \tilde{x}\rangle + M, \ \forall k.$$

Observe that $0 < \langle c_i(x_k), \tilde{x}\rangle \leq \|\tilde{x}\|$ for every k because $c_i(x_k)$ is a nonzero element of $\mathcal{K}$ and $\tilde{x} \in \mathrm{int}\mathcal{K}$. Moreover, from Lemma 3.1, we have that there exists a spectral decomposition of x, $x = \sum_{i=1}^r \lambda_i c_i$, and a subsequence $\{x_{k_\ell}\}$ such that $\lim_{\ell\to\infty} \lambda_i(x_{k_\ell}) = \lambda_i$ for every i. Let $I_0 := \{i : \lambda_i = 0\}$; note that $I_0 \neq \emptyset$ because $x \in \partial\mathcal{K}$. Hence, because $v \in C^1(]0, \infty[)$ and $\{\langle c_i(x_k), \tilde{x}\rangle\}_k$ is bounded, the sum $\sum_{i\notin I_0} v'(\lambda_i(x_{k_\ell}))\langle c_i(x_{k_\ell}), \tilde{x}\rangle$ remains bounded when ℓ goes to ∞, whereas the sum $\sum_{i\in I_0} v'(\lambda_i(x_{k_\ell}))\langle c_i(x_{k_\ell}), \tilde{x}\rangle$ goes to $-\infty$ when ℓ goes to ∞ because $\lim_{s\to 0^+} v'(s) = -\infty$. Hence,

$$\lim_{\ell\to\infty}\langle \nabla\Psi_v(x_{k_\ell}), \tilde{x} - x_{k_\ell}\rangle \leq \lim_{\ell\to\infty}\sum_{i=1}^r v'(\lambda_i(x_{k_\ell}))\langle c_i(x_{k_\ell}), \tilde{x}\rangle + M = -\infty.$$

□

To prove that the primal central path is well defined (i.e., P_μ admits a unique solution for all $\mu > 0$), we recall the following proposition [12, Proposition 3.1.3] taken from recession analysis.

Proposition 3.1 *Let $f \in \Gamma_0(\mathbb{V})$. The following statements are equivalents:*

(a) *f is sublevel bounded (i.e., $\{x : f(x) \leq \gamma\}$ is bounded for all $\gamma > \inf f$).*
(b) *f is coercive (i.e., $f_\infty(d) > 0$ for all $d \neq 0$).*
(c) *The optimal set $\{x \in \mathbb{V} : f(x) = \inf f\}$ is nonempty and compact.*

In the next proposition, we establish the main result of the section, which states that the primal central path is well defined.

Proposition 3.2 *Let $v \in \mathcal{C}$. Suppose one of the following conditions holds:*

(a) *v satisfies (H.1).*

(b) v *satisfies (H.2) and* S_P *is bounded.*

The primal central path $\{x(\mu) : \mu > 0\}$ *with respect to* v *is then well defined and is contained in the interior of* $\mathcal{K}$.

Proof We divide the proof into two parts; in Part 1, we prove the existence and uniqueness of $x(\mu)$, and in Part 2, we prove that this path is in int$\mathcal{K}$. *Part 1* By using [12, Theorem 2.3.4], we can show that $(\varphi+\alpha\psi)_\infty = \varphi_\infty+\alpha\psi_\infty$, for all $\varphi, \psi \in \Gamma_0(\mathbb{V})$, and $\alpha > 0$. This fact and Proposition 2.1 imply that $(f_\mu)_\infty(d) = g_\infty(d)+\mu\Psi_{v_\infty}(d),\ \forall d \in \mathbb{V}$. According to Proposition 3.1, to show the existence of $x(\mu)$, it is sufficient to prove that f_μ is coercive, that is,

$$g_\infty(d) + \mu\Psi_{v_\infty}(d) > 0,\ \forall d \neq 0. \tag{10}$$

Suppose that v satisfies (H.1). This implies that Ψ_v is sublevel bounded and thus is coercive, i.e., $\Psi_{v_\infty}(d) > 0,\ \forall d \neq 0$. Moreover, because $S_P \neq \emptyset$, we have that g is bounded below, so g_∞ is nonnegative. These facts imply (10).
Suppose that v satisfies (H.2) and S_P is bounded. The boundedness of the primal solution set implies that g is coercive (Proposition 3.1), which means that $(g_\infty)(d) > 0,\ \forall d \neq 0$. From Proposition 2.2(b), we obtain that Ψ_{v_∞} is nonnegative. These facts imply the desired result (10). The strict convexity of f_μ ensures the uniqueness of $x(\mu)$.
Part 2 If $v(0) = +\infty$, then the result is clear because dom $f_\mu = F_P \cap \text{int}\mathcal{K}$. Suppose that $v(0) \in \mathbb{R}$. The proof that we present is similar to the proof of [17, Theorem 3.1]. Assume by contradiction that $x(\mu) \in \partial\mathcal{K}$, and define the sequence $x_k := \epsilon_k x_0 + (1-\epsilon_k)x(\mu)$, with $x_0 \in F_P \cap \text{int}\mathcal{K}$ and $\{\epsilon_k\} \subset]0, 1[$ a sequence satisfying $\lim_{k\to\infty}\epsilon_k = 0$. Clearly, $x_k \in F_P \cap \text{int}\mathcal{K}$ for all k. From the optimality of $x(\mu)$ and the convexity of f and Ψ_v, we obtain

$$\begin{aligned} 0 &\leq f(x_k) + \mu\Psi_v(x_k) - f(x(\mu)) - \mu\Psi_v(x(\mu)) \\ &\leq \langle\nabla f(x_k), x_k - x(\mu)\rangle + \mu\langle\nabla\Psi_v(x_k), x_k - x(\mu)\rangle. \end{aligned}$$

This inequality and the definition of x_k imply that

$$[(1-\epsilon_k)/\mu]\langle\nabla f(x_k), x(\mu) - x_0\rangle \leq \langle\nabla\Psi_v(x_k), x_0 - x_k\rangle. \tag{11}$$

Note that $\{x_k\}$ converges to $x(\mu)$. The left-hand side of (11) then remains bounded when $k \to \infty$ because $f \in C^1(\mathbb{V})$. However, from Lemma 3.2 we have that the right-hand side of (11) goes to $-\infty$ for a subsequence $\{x_{k_\ell}\}$, which is a contradiction. □

Because the primal central path lies in the interior of the cone $\mathcal{K}$, the problem P_μ can be written as

$$\min_{x\in\mathbb{V}}\{f(x) + \mu\Psi_v(x) : \mathcal{A}x = b,\ x \in \text{int}\mathcal{K}\}, \quad \mu > 0.$$

Hence, the KKT conditions say that there exists $y(\mu) \in \mathbb{R}^m$ such that

$$\mathcal{A}x(\mu) = b, \quad f(x(\mu)) + \mu\nabla\Psi_v(x(\mu)) = \mathcal{A}^* y(\mu), \quad x(\mu) \in \operatorname{int}\mathcal{K}. \tag{12}$$

Note that, because $\mathcal{A}$ is surjective, we have the relationship

$$y(\mu) = \left(\mathcal{A}\mathcal{A}^*\right)^{-1} \mathcal{A}\left(\nabla f(x(\mu)) - s(\mu)\right),$$

where $s(\mu) := -\mu\nabla\Psi_v(x(\mu))$. Hence, the set $\{(y(\mu), s(\mu)) : \mu > 0\}$ is well defined and is called the *dual central path* with respect to v.

The *primal–dual central path* with respect to v is defined as the set $\{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$.

4 Limiting Behavior of Central Paths

In what follows, we are interested in the behavior of the three central paths defined in the previous sections when μ goes to 0. We thus focus our analysis only on small values of μ. Therefore, we henceforth assume that the different central paths are defined on an interval of the form $\bar{\mu} > \mu > 0$ for some given $\bar{\mu} > 0$.

Our first result concerns the boundedness of the primal central path (for small values of μ).

Proposition 4.1 *Suppose that S_P is bounded. Let $v \in \mathcal{C}$ such that it satisfies (H.1) or (H.2). The primal central path $\{x(\mu) : \bar{\mu} > \mu > 0\}$ with respect to v is then bounded, and all its limit points are optimal solutions of (P).*

Proof Let $x^0 \in F_P^0$. By the optimality of $x(\mu)$, we have that

$$g(x(\mu)) + \mu\Psi_v(x(\mu)) \leq g(x^0) + \mu\Psi_v(x^0). \tag{13}$$

We proceed by contradiction to prove that the primal central path is bounded. We assume that there exists a sequence $\{\mu_k\}$ of positive numbers such that

$$\mu_k \to 0, \quad \|x(\mu_k)\| \to \infty, \quad x(\mu_k)/\|x(\mu_k)\| \to d \neq 0.$$

Dividing (13) by $\|x(\mu_k)\|$ and passing to the limit, we obtain $g_\infty(d) \leq 0$, which contradicts the fact that S_P is compact (Proposition 3.1). Hence, the primal central path is bounded.

Let $\bar{x}$ be a limit point of $\{x(\mu) : \bar{\mu} > \mu > 0\}$; we claim that $\bar{x}$ lies in S_P. Indeed, let $\{\mu_k\}$ be a sequence of positive numbers such that $\lim_{k\to\infty} x(\mu_k) = \bar{x}$. Let $x^* \in S_P$ and $x^0 \in F_P^0$. For $\epsilon \in]0, 1]$, we define

$$z(\epsilon) := (1 - \epsilon)x^* + \epsilon x^0.$$

Note that $z(\epsilon) \in F_P^0$. Hence, by the optimality of $x(\mu_k)$, we have that

$$f(x(\mu_k)) + \mu_k\Psi_v(x(\mu_k)) \leq f(z(\epsilon)) + \mu_k\Psi_v(z(\epsilon)).$$

By using the convexity of Ψ_v, the above inequality implies that

$$\mu_k \langle \Psi_v(z(\epsilon)), x(\mu_k) - z(\epsilon) \rangle \leq f(z(\epsilon)) - f(x(\mu_k)).$$

By passing to the limit, we obtain that $f(\bar{x}) \leq f(z(\epsilon))$. Finally, letting $\epsilon \to 0$, we have $f(\bar{x}) \leq f(x^*)$. Thus, $\bar{x} \in S_P$. □

In the next proposition, we also present a boundedness result for the dual central path. The proof repeats word for word the arguments in [18, Proposition 3.3], substituting the EJA $\mathcal{S}^n$ with the general EJA $\mathbb{V}$.

Proposition 4.2 *Suppose that S_P is bounded. Let $v \in \mathcal{C}$ such that it satisfies (H.1) or (H.2). The dual central path with respect to v is then bounded (for small values of μ), and all of its limit points are optimal solutions of D.*

In the remainder of this section, we discuss the convergence of the primal–dual central path. The results that we present are obtained by assuming that the objective and barrier spectral functions are analytic. For instance, because the trace and determinant are polynomial functions on $\mathbb{V}$ (see [1, Proposition II.2.1]) and the logarithm function is analytic on the positive numbers, we deduce that the barrier spectral functions

$$\Psi_{v_5}(x) = -\log\det(x), \quad \Psi_{v_6}(x) = \operatorname{tr}(x) - \log\det(x),$$

are analytic on the interior of $\mathcal{K}$.

Recalling the KKT optimality conditions (12) for problem P_μ, we see that the primal–dual central path is well defined as the unique solution of

$$\mathcal{A}x = b, \quad x \in \operatorname{int}\mathcal{K}, \quad \mathcal{A}^* y + s = \nabla f(x), \quad s + \mu \nabla \Psi_v(x) = 0. \tag{14}$$

The strategy that we follow is to define a suitable semi-analytic set from the above conditions and apply a curve selection lemma [19]. This procedure was used, for instance, in [20,21]. The key factor for the construction of the semi-analytic set is to describe $\operatorname{int}\mathcal{K}$ by a set of elements that satisfy some polynomial inequalities. For instance, we know that a matrix in $\mathcal{S}^n$ is positive definite if and only if its leading principal minors are positives. We explain how this characterization is extended to the general context of EJA. Let $\{c_1, \ldots, c_r\}$ be any Jordan frame. We define $\mathbb{V}^{(k)} := \{x \in \mathbb{V} : (c_1 + \cdots + c_k) \circ x = x\}$, $k \in \{1, \ldots, r\}$, which are subalgebras such that $\mathbb{V}^{(1)} \subset \mathbb{V}^{(2)} \subset \cdots \subset \mathbb{V}^{(r)} = \mathbb{V}$. Let P_k be the orthogonal projection onto $\mathbb{V}^{(k)}$. The principal minor Δ_k is the polynomial function defined on $\mathbb{V}$ by $\Delta_k(x) := \det^{(k)}(P_k(x))$, where $\det^{(k)}$ denotes the determinant with respect to the subalgebra $\mathbb{V}^{(k)}$. The following equivalence [1, Section VI.3] provides us the desired characterization:

$$x \in \operatorname{int}\mathcal{K} \;\Leftrightarrow\; \Delta_k(x) > 0, \text{ for all } k = 1, \ldots, r.$$

Now, we are ready to enunciate the convergence result.

Proposition 4.3 *Suppose that S_P is bounded. Let $v \in \mathcal{C}$ such that it satisfies (H.1) or (H.2). Assume also that f and Ψ_v are analytic functions on the interior of their domains. The primal–dual central path with respect to v is then an analytic curve and it converges, when $\mu \downarrow 0$, to a point in $S_P \times S_D$.*

Proof The analytic curve property comes from [18, Proposition3.4]. We take any limit point (x^*, y^*, s^*) of the primal–dual central path and use the optimality conditions (9) and (14) to construct the set $\mathcal{V}$ as the set of elements $(\bar{x}, \bar{y}, \bar{s}, \mu)$ such that the following are satisfied:

$$\mathcal{A}\bar{x} = 0, \quad \mathcal{A}^*\bar{y} + \bar{s} + \nabla f(x^*) - \nabla f(\bar{x} + x^*) = 0, \quad \mu > 0,$$
$$\bar{s} + s^* + \mu\nabla\Psi_v(\bar{x} + x^*) = 0, \quad \Delta_k(\bar{x} + x^*) > 0 \text{ for } k = 1, \ldots, r.$$

For $\mu > 0$, the set of conditions that defines $\mathcal{V}$ provides a unique element $(\bar{x}, \bar{y}, \bar{s})$ given by $\bar{x} = x(\mu) - x^*$, $\bar{y} = y(\mu) - y^*$, and $\bar{s} = s(\mu) - s^*$. Observe that the zero point is in the closure of $\mathcal{V}$. Moreover, this set is semi-analytic because the partial derivatives of f and Ψ_v are analytic functions and, for $k = 1, \ldots, r$, $\Delta_k(\cdot)$ are polynomial functions. Hence, we can use the curve selection lemma for the semi-analytic set $\mathcal{V}$ [19, Proposition 2], and we can follow the same arguments as [20, Theorem 4.6.1] and [21, Theorem A.3] to deduce the convergence of the primal–dual central path. □

5 Characterization of Limits Points of Central Paths

The next theorem covers the case when v is defined at 0. The theorem shows that the primal central path is convergent and that its limit point minimizes the spectral barrier function over the primal solution set.

Theorem 5.1 *Let $v \in \mathcal{C}$ such that $v(0) \in \mathbb{R}$. Suppose that one of the following conditions holds:*

(a) v satisfies (H.1).
(b) v satisfies (H.2) and S_P is bounded.

The primal central path $\{x(\mu) : \mu > 0\}$ then converges, when $\mu \downarrow 0$, to the unique optimal solution of

$$\min\{\Psi_v(x) : x \in S_P\}. \qquad (AC)$$

Proof We start by noting that the problem *AC* admits a unique solution. Indeed, because $v(0) \in \mathbb{R}$, we get $\operatorname{dom}\Psi_v \cap S_P \neq \emptyset$. Therefore, $\Psi_v + \delta_{S_P} \in \Gamma_0(\mathbb{V})$. Hence, it is sufficient to show that $\Psi_v + \delta_{S_P}$ is coercive (Proposition 3.1). We note that $(\delta_C)_\infty = \delta_{C_\infty}$, where C is a nonempty, closed and convex set, and C_∞ denotes its recession set (see [11,12]). Therefore, we must prove the following:

$$\Psi_{v_\infty}(d) + \delta_{(S_P)_\infty}(d) > 0, \ \forall d \neq 0. \qquad (15)$$

If v satisfies (H.1), then Ψ_v is sublevel bounded; therefore, (15) is verified (Proposition 3.1). When v satisfies (H.2) and S_P is bounded, it follows from Proposition 2.2(b) that Ψ_{v_∞} is nonnegative, and the boundedness of S_P is equivalent to $(S_P)_\infty = \{0\}$ (see [12, Proposition 2.1.2]). Consequently, (15) is verified. The uniqueness of the optimal solution of *AC* follows from the strict convexity of Ψ_v.

Let x^* be the unique optimal solution *AC*. We now proceed to prove the convergence of the primal central path $\{x(\mu) : \mu > 0\}$ to x^* when μ goes to 0. By the optimality of $x(\mu)$, we have that

$$g(x(\mu)) + \mu\Psi_v(x(\mu)) \leq g(x^*) + \mu\Psi_v(x^*). \tag{16}$$

Because $x^* \in S_P$, (16) implies that

$$\Psi_v(x(\mu)) \leq \Psi_v(x^*). \tag{17}$$

Note that when v satisfies (H.2), it follows that S_P is bounded; therefore, the boundedness of the primal central path (for small values of μ) is ensured by Proposition 4.1. However, S_P is not necessarily bounded when v satisfies (H.1), but the boundedness of $\{x(\mu) : \bar{\mu} > \mu > 0\}$ is deduced from (17) and from the fact that Ψ_v is sublevel bounded. Hence, let $\bar{x}$ be an arbitrary limit point of $\{x(\mu) : \bar{\mu} > \mu > 0\}$ and denote by $\{\mu_k\}$ a sequence of positive values such that $x(\mu_k) \to \bar{x}$. Passing to the limit in (16) and (17), we obtain that $g(\bar{x}) \leq g(x^*)$ and $\Psi_v(\bar{x}) \leq \Psi_v(x^*)$. Thus, $\bar{x} \in S_P$, and by the uniqueness of x^*, we have that $\bar{x} = x^*$. We have proved that $\lim_{\mu\downarrow 0} x(\mu) = x^*$. □

Remark 5.1 An important portion of the results presented to this point were largely inspired by the works of Cruz Neto et al. [17] and López and Ramírez [18], which concern central paths in semidefinite programming. However, note that our results are valid for a larger class of barrier functions than those used in [17,18]. For instance, in [18], the authors work with the following two classes of barrier functions:

$$\mathcal{L}_1 := \{v \in \mathcal{C} : v(0) = +\infty,\ v \text{ is nonincreasing}, v \text{ satisfies } (H.2)\},$$
$$\mathcal{L}_2 := \{v \in \mathcal{C} : v(0) \in \mathbb{R},\ v \text{ satisfies } (H.1)\},$$

that do not include the functions v_2, v_3 and v_6. In addition, S_P is assumed to be bounded in articles [17,18]. However, this hypothesis is required neither in Proposition 3.2 nor in Theorem 5.1 when v satisfies (H.1).

We mention that Theorem 5.1 was proved in [9, Theorem 4.1] for the case Ψ_{v_1}, where $v_1(s) = s\log s - s$. For the general case of when v is not defined at 0, the characterization of the limit point of the primal central path remains an open question for symmetric cone programming. A complete characterization was shown in the case of linear programming by Auslender et al. [22]. For the case of semidefinite programming, Klerk et al. [23] showed that under the strict complementarity assumption, the central path converges to the analytic center of the optimal set. This characterization is no longer valid without assuming this hypothesis. However, Halická et al. [24] were able to provide another characterization for the limit point by means of an optimization problem. For the case of second-order cone programming, Terlaky and Wang [25] discussed the limiting behavior of the central path without assuming strict complementarity. Their analysis is based on an index set partition introduced by Bonnans and Ramírez [26].

Recall that the *Peirce decomposition* of $\mathbb{V}$ with respect to an idempotent $c \in \mathbb{V}$ is given by

$$\mathbb{V} = \mathbb{V}(c, 1) \oplus \mathbb{V}(c, 1/2) \oplus \mathbb{V}(c, 0),$$

where $\mathbb{V}(c, \ell) := \{x \in \mathbb{V} : c \circ x = \ell x\}$ with $\ell = 1, 1/2, 0$. In the remainder of this paper, we focus our attention on Peirce decompositions of $\mathbb{V}$ with respect to idempotents obtained from the limit points of the central paths. These decompositions will provide some information concerning the localization of these limit points. Special attention will be focused on barriers v_5 and v_6.

Consider x^* and (y^*, s^*) arbitrary limit points of the primal and dual central paths, respectively, with respect to $v \in \mathcal{C}$. We have that the relationship $s(\mu) = -\mu\nabla\Psi_v(x(\mu))$ clearly implies that $x(\mu)$ and $s(\mu)$ admit a simultaneous spectral decomposition, namely

$$x(\mu) = \sum_{i=1}^{r} \lambda_i(\mu)c_i(\mu), \qquad s(\mu) = \sum_{i=1}^{r} \delta_i(\mu)c_i(\mu). \tag{18}$$

From Lemma 3.1, we deduce that there exists a simultaneous spectral decomposition of x^* and s^*, namely

$$x^* = \sum_{i=1}^{r} \lambda_i^* c_i^*, \qquad s^* = \sum_{i=1}^{r} \delta_i^* c_i^*, \tag{19}$$

and a sequence $\{\mu_k\}$ of positive numbers satisfying $\lim_{k\to\infty} \mu_k = 0$ such that for every $i \in \{1, \ldots, r\}$, one has

$$c_i(\mu_k) \to c_i^*, \qquad \lambda_i(\mu_k) \to \lambda_i^*, \qquad \delta_i(\mu_k) \to \delta_i^*, \qquad \text{when } k \to \infty. \tag{20}$$

For $x = \sum_{i=1}^{r} \lambda_i c_i$, we introduce the index notation

$$I := \{1, \ldots, r\},\ \ I_+(x) := \{i \in I : \lambda_i > 0\},\ \ I_0(x) := \{i \in I : \lambda_i = 0\}. \tag{21}$$

The Jordan frame in (19) can be partitioned as

$$\{c_i^* : i \in I_+(x^*)\} \cup \{c_i^* : i \in I_+(s^*)\} \cup \{c_i^* : i \in I_0(x^*) \cap I_0(s^*)\}.$$

We define

$$c_B^* := \sum_{i\in I_+(x^*)} c_i^*, \quad c_N^* := \sum_{i\in I_0(s^*)} c_i^*. \tag{22}$$

The Peirce decompositions of $\mathbb{V}$ with respect to the idempotents c_B^* and c_N^* are given, respectively, by

$$\mathbb{V} = \mathbb{V}(c_B^*, 1) \oplus \mathbb{V}(c_B^*, 1/2) \oplus \mathbb{V}(c_B^*, 0), \tag{23}$$
$$\mathbb{V} = \mathbb{V}(c_N^*, 1) \oplus \mathbb{V}(c_N^*, 1/2) \oplus \mathbb{V}(c_N^*, 0). \tag{24}$$

We use the words *primal* and *dual* Peirce decomposition of $\mathbb{V}$ to refer the decompositions (23) and (24), respectively.

In the next proposition, we show that the primal solution set S_P is contained in the subalgebra $\mathbb{V}(c_B^*, 1)$ under the assumption that v verifies the following:

(**H.3**) There exists $\bar{s} > 0$ and $L \in \mathbb{R}$ such that $-v'(s)s \leq L,\ \forall s \in]0, \bar{s}]$.

Note that, if $v \in \mathcal{C}$ such that $v(0) \in \mathbb{R}$, then hypothesis (H.3) is satisfied automatically (see Proposition 2.2(a)). Otherwise, when v is not defined at 0, hypothesis (H.3) is not necessarily fulfilled (see, e.g., v_4 in Example 2.1). However, particular choices of functions v can verify (H.3), even when they are not defined at 0 (see, for example, v_5 and v_6 in Example 2.1).

Proposition 5.1 *Suppose that S_P is bounded. Let $v \in \mathcal{C}$ such that it satisfies (H.1) or (H.2). Assume also that v satisfies (H.3). Let x^* be an arbitrary limit point of the primal central path associated with v. We then have that*

$$S_P \subset \mathbb{V}(c_B^*, 1) \cap F_P,$$

with $c_B^* = \sum_{i \in I_+(x^*)} c_i^*$, where $\{c_1^*, \ldots, c_r^*\}$ is some Jordan frame associated with x^*.

Proof Let $(x, (y, s)) \in S_P \times S_D$ and $(x(\mu), y(\mu), s(\mu))$ be a point of the primal–dual central path. From the optimality conditions (9) and (12), we obtain

$$\begin{aligned}\langle x - x(\mu), s - s(\mu)\rangle &= \langle x - x(\mu), \nabla f(x) - \mathcal{A}^* y - \nabla f(x(\mu)) + \mathcal{A}^* y(\mu)\rangle \\ &= \langle x - x(\mu), \nabla f(x) - \nabla f(x(\mu))\rangle \geq 0, \end{aligned}\tag{25}$$

where the last inequality is due to the monotonicity of ∇f. Because $\langle x, s\rangle = 0$ and $\langle x(\mu), s\rangle \geq 0$, the above relationship implies that $\langle x, s(\mu)\rangle \leq \langle x(\mu), s(\mu)\rangle$. Moreover, from the relationship $s(\mu) = -\mu \nabla \Psi_v(x(\mu))$, we obtain

$$\langle x, -\Psi_v(x(\mu))\rangle \leq \langle x(\mu), -\Psi_v(x(\mu))\rangle. \tag{26}$$

Let us take spectral decompositions $x(\mu) = \sum_{i=1}^r \lambda_i(\mu) c_i(\mu)$, $x^* = \sum_{i=1}^r \lambda_i^* c_i^*$, and the sequence $\{\mu_k\}$ such that they satisfy (20). Hence, (26) becomes

$$T(\mu) := -\sum_{i=1}^r v'(\lambda_i(\mu))\langle x, c_i(\mu)\rangle \leq -\sum_{i=1}^r v'(\lambda_i(\mu))\lambda_i(\mu) =: R(\mu).$$

By using the index notation (21), we have that $T(\mu) = T_1(\mu) + T_2(\mu)$, where

$$T_1(\mu) := -\sum_{i \in I_0(x^*)} v'(\lambda_i(\mu))\langle x, c_i(\mu)\rangle, \quad T_2(\mu) := -\sum_{i \in I_+(x^*)} v'(\lambda_i(\mu))\langle x, c_i(\mu)\rangle.$$

Let $\bar{s} > 0$ as in hypothesis (H.3); we introduce the index notation

$$I_{\leq \bar{s}}(x^*) := \{i \in I : \lambda_i^* \leq \bar{s}\}, \quad I_{>\bar{s}}(x^*) := \{i \in I : \lambda_i^* > \bar{s}\}.$$

$R(\mu)$ can be partitioned as $R(\mu) = R_1(\mu) + R_2(\mu)$, where

$$R_1(\mu) := -\sum_{i \in I_{\leq \bar{s}}(x^*)} v'(\lambda_i(\mu))\lambda_i(\mu), \quad R_2(\mu) := -\sum_{i \in I_{>\bar{s}}(x^*)} v'(\lambda_i(\mu))\lambda_i(\mu).$$

Taking the sequence $\{\mu_k\}$, for sufficiently large k, we have the following results: $T_1(\mu_k) \geq 0$ because $\lim_{s \to 0^+} v'(s) = -\infty$, $T_2(\mu_k)$ and $R_2(\mu_k)$ are bounded because of the continuity of v' on $]0, \infty[$, and $R_1(\mu_k)$ is also bounded by hypothesis (H.3). Hence, there exists $M \geq 0$ such that

$$0 \leq T_1(\mu_k) \leq M, \text{ for sufficiently large } k.$$

Because $\lim_{k \to \infty} v'(\lambda_i(\mu_k)) = -\infty$ for all $i \in I_0(x^*)$, we conclude that

$$\langle x, c_i^* \rangle = 0, \ \forall i \in I_0(x^*).$$

Consequently, because $c_B^* + \sum_{i \in I_0(x^*)} c_i^* = e$, one obtains

$$x \circ c_B^* = x \circ \left(e - \sum_{i \in I_0(x^*)} c_i^*\right) = x.$$

That is, $x \in \mathbb{V}(c_B^*, 1)$. We have thus proved that $S_P \subset \mathbb{V}(c_B^*, 1)$. □

The next theorem improves Proposition 5.1 for the cases of the logarithm barrier function $v_5(s) = -\log s$ and the modified logarithm barrier function $v_6(s) = s - \log s$. The theorem also allows us to conclude that the accumulation points of the primal and dual central paths lie in the relative interior of the primal and dual optimal solution sets, respectively.

Theorem 5.2 *Suppose that S_P is bounded. Let x^* and (y^*, s^*) be arbitrary limit points of the primal and dual central paths, respectively, associated with $v \in \{v_5, v_6\}$. There then exists a common Jordan frame of x^* and s^*, namely $\{c_1^*, \ldots, c_r^*\}$, such that*

$$S_P = \mathbb{V}(c_B^*, 1) \cap F_P,$$
$$S_D = \left(\mathbb{R}^m \times \mathbb{V}(c_N^*, 0)\right) \cap F_D,$$

where, $c_B^* = \sum_{i \in I_+(x^*)} c_i^*$ and $c_N^* = \sum_{i \in I_0(s^*)} c_i^*$. In particular, $x^* \in \mathrm{ri}(S_P)$ and $(y^*, s^*) \in \mathrm{ri}(S_D)$, where $\mathrm{ri}(C)$ denotes the relative interior of C.

Proof Let $v \in \{v_5, v_6\}$. Recall that the inclusion

$$S_P \subseteq \mathbb{V}(c_B^*, 1) \cap F_P, \tag{27}$$

was stated in Proposition 5.1. We now prove that

$$S_D \subseteq \left(\mathbb{R}^m \times \mathbb{V}(c_N^*, 0)\right) \cap F_D. \tag{28}$$

The barrier function v can be written as $v(s) = \alpha s - \log s$, with α equal to 0 or 1. Let $(x, (y, s)) \in S_P \times S_D$, and let $(x(\mu), y(\mu), s(\mu))$ be a point of the primal–dual central path associated with v. Similarly to (25), we obtain

$$\langle x, s(\mu)\rangle + \langle x(\mu), s\rangle \leq \langle x(\mu), s(\mu)\rangle. \tag{29}$$

From relationships $s(\mu) = -\mu \nabla \Psi_v(x(\mu))$ and $\nabla \Psi_v(x(\mu)) = \alpha e - x(\mu)^{-1}$, we have that

$$\langle x(\mu), s(\mu)\rangle = -\mu\alpha\langle x(\mu), e\rangle + \mu r, \tag{30}$$

$$x(\mu)^{-1} = \mu^{-1}s(\mu) + \alpha e, \tag{31}$$

$$(s(\mu) + \mu\alpha e)^{-1} = \mu^{-1}x(\mu). \tag{32}$$

By adding $\langle x, \alpha e\rangle$ to both sides of (29) and by using the relationship (30), we obtain

$$\langle x, \mu^{-1}s(\mu) + \alpha e\rangle + \langle \mu^{-1}x(\mu), s\rangle + \langle x(\mu), \alpha e\rangle \leq \langle x, \alpha e\rangle + r =: L. \tag{33}$$

By replacing (31) and (32) in (33), we obtain

$$\langle x, x(\mu)^{-1}\rangle + \langle (s(\mu) + \mu\alpha e)^{-1}, s\rangle + \langle x(\mu), \alpha e\rangle \leq L.$$

Because all terms in the foregoing inequality are nonnegative, we obtain

$$\langle (s(\mu) + \mu\alpha e)^{-1}, s\rangle \leq L. \tag{34}$$

We take a spectral decomposition of $s(\mu)$ and s^* as in (18) and (19), respectively, and a sequence $\{\mu_k\}$ as in (20). Observe that

$$(s(\mu_k) + \mu_k\alpha e)^{-1} = \sum_{i=1}^{r} (\delta_i(\mu_k) + \mu_k\alpha)^{-1} c_i(\mu_k).$$

Hence, inequality (34) becomes

$$\sum_{i=1}^{r} (\delta_i(\mu_k) + \mu_k\alpha)^{-1} \langle c_i(\mu_k), s\rangle \leq L.$$

Because $(\delta_i(\mu_k)+\mu_k\alpha)^{-1} \to +\infty$ when $k \to +\infty$ for all $i \in I_0(s^*)$, we conclude that $\langle c_i^*, s\rangle = 0,\ \forall i \in I_0(s^*)$, which implies $\langle c_N^*, s\rangle = \left\langle \sum_{i\in I_0(s^*)} c_i^*, s \right\rangle = 0$. Therefore, $s \in \mathbb{V}(c_N^*, 0)$. Hence, (28) holds.

Let $(x, (y, s)) \in \left[\mathbb{V}(c_B^*, 1) \cap F_P\right] \times \left[\left(\mathbb{R}^m \times \mathbb{V}(c_N^*, 0)\right) \cap F_D\right]$. To show that (27) and (28) are indeed equalities, it is sufficient to prove that $\langle x, s\rangle = 0$ because, in this case, the optimality conditions (9) ensure that $(x, (y, s))$ belongs to $S_P \times S_D$. We proceed to prove this orthogonality condition. Note that

$$\langle x, s\rangle = \langle x \circ c_B^*, s\rangle = \langle x, s \circ c_B^*\rangle, \tag{35}$$

where the first equality arises because $x \in \mathbb{V}(c_B^*, 1)$, and the second equality comes from the associative property of the inner product with respect to the Jordan product. Moreover, because $s \in \mathbb{V}(c_N^*, 0)$, we have

$$s \circ c_B^* + s \circ c_T^* = s \circ c_N^* = 0,$$

where, $c_T^* = \sum_{i\in I_0(x^*)\cap I_0(s^*)} c_i^*$. Hence, relationship (35) becomes

$$\langle x, s\rangle = \langle x, -s \circ c_T^*\rangle = -\langle x \circ c_T^*, s\rangle.$$

However, $x \circ c_T^* = 0$ because $x \in \mathbb{V}(c_B^*, 1)$, $c_T^* \in \mathbb{V}(c_B^*, 0)$ and these subalgebras are orthogonal, i.e., $\mathbb{V}(c_B^*, 1) \circ \mathbb{V}(c_B^*, 0) = \{0\}$ (see [1, Proposition IV.1.1]). We thus conclude that $\langle x, s\rangle = 0$. □

Remark 5.2 Because the accumulation points of the primal and dual central paths are localized in the relative interior of S_P and S_D, respectively, one can conjecture that, under some additional assumption (such as strict complementarity condition: there exists $(x, (y, s)) \in S_P \times S_D$ such that $x + s \in \operatorname{int}\mathcal{K}$), they are solutions of the following problems:

$$\min\{\Psi_v^P(x) : x \in \operatorname{ri}(S_P)\}, \tag{AC_P}$$

$$\min\{\Psi_{v^*}^D(s) : (y, s) \in \operatorname{ri}(S_D)\}. \tag{AC_D}$$

Here, $v \in \{v_5, v_6\}$ and Ψ_v^P and Ψ_v^D are functions defined similarly to (8), but with respect to the subalgebras $V(c_B^*, 1)$ and $V(c_N^*, 0)$, respectively.

Indeed, in the case of linear programming, it was shown in [22, Theorem 3.4] and [22, Theorem 3.1] that the primal and dual central paths, with respect to the foregoing v, converge to the unique solutions of AC_P and AC_D, respectively.

The same results were shown in [24, Theorem 3.2] for the (linear) semidefinite programming setting, provided we use the logarithm barrier function v_5 and that the strict complementarity assumption is fulfilled. Unfortunately, we are thus far unable to prove this conjecture.

6 Conclusions

Primal, dual and primal–dual central paths in symmetric cone programming associated with a large class of barrier functions have been considered in the paper. We have proved the existence, the boundedness and the convergence (under some analyticity assumptions) of these central paths. When the barrier function is defined at the zero point, we have also proved that the limit point of the central path coincides with the analytic center of the primal solution set. In a more general case, we have proved that the primal solution set is included in a subalgebra obtained from a Peirce decomposition of the EJA with respect to a suitable idempotent element, constructed from a limit point of the primal central path. For the logarithm and modified logarithm barrier functions, we have proved that the accumulation points of their primal and dual central paths belong to the relative interior of their primal and dual solution sets, respectively.

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