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Almost partitioning 2-edge-colourings of 3-uniform hypergraphs with two monochromatic tight cycles

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Abstract

We show that any 2-colouring of the 3-uniform complete hypergraph $K_n^{(3)}$ on n vertices contains two disjoint monochromatic tight cycles of distinct colours covering all but $o(n)$ vertices of $K_n^{(3)}$. The same result holds if we replace tight cycles with loose cycles.

Keywords: Monochromatic cycle partitioning, tight cycles.

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1 Introduction

Given a complete r-edge-colouring of graph or hypergraph $\mathcal{K}$, the problem of partitioning the vertices of $\mathcal{K}$ into the smallest number of monochromatic cycles has received much attention. Central to this area has been an old conjecture of Lehel [2] stating that two monochromatic disjoint cycles in different colours are sufficient to partition the vertex set of the complete graph $\mathcal{K}_n$ on n vertices, for all n. This was confirmed for large n in [10] and [1], and more recently, for all n, by Bessy and Thomassé [3].

For $r \geq 3$, there exist r-edge-colourings of $\mathcal{K}_n$ which do not allow for a partition of the vertex set into r monochromatic cycles [11]. On the other hand, the currently best bound (see [6]) shows that $100r \log r$ monochromatic cycles are sufficient to partition the vertex set of $\mathcal{K}_n$.

The problem transforms in the obvious way to hypergraphs, considering r-edge-colourings of the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ on n vertices and partitions into one of the many notions of cycles in hypergraphs. Here we deal with loose and tight cycles. Loose cycles are uniform hypergraphs with a cyclic ordering of its edges such that consecutive edges intersect in exactly one vertex and nonconsecutive edges have empty intersection. On the other hand, tight cycles are k-uniform hypergraphs with a cyclic ordering of its vertices such that the edges are all the sets of k consecutive vertices. For loose cycles, the best bound due to Sárközy in [12] shows that every r-edge-colouring of $\mathcal{K}_n^{(k)}$ admits a partition of its vertices into at most $50rk \log(rk)$ monochromatic loose cycles. Concerning tight cycles, to our best knowledge, nothing is known. We refer the reader to [5] for related results.

Our main result establishes an approximate version of the problem for the case of 3-uniform hypergraphs and two colours.

Theorem 1.1 *For every $\eta > 0$ there exists n_0 such that if $n \geq n_0$ then every 2-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ admits two vertex-disjoint monochromatic tight cycles, of distinct colours, which cover all but at most ηn vertices.*

We note that a 3-uniform tight cycle on n vertices contains a loose cycle if n is even. The proof of Theorem 1.1 guarantees that the two tight cy-

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cles obtained each have an even number of vertices. Hence, an analogue of Theorem 1.1 holds for loose cycles.

We believe that the error term ηn in the theorem can be improved and that every 2-colouring of the edges of $\mathcal{K}_n^{(3)}$ admits two disjoint monochromatic tight cycles which cover all but at most a constant number c of vertices (for some c independent of n). Furthermore, we believe that the previous statement holds for all k and not just $k = 3$. In a forthcoming article we confirm this for loose cycles, where the constant c depends only on k.

2 Outline of the proof

Due to lack of space we only give a sketch of the argument, referring to [4] for full details for the proof of Theorem 1.1.

The argument is inspired by the work of Haxell et al. [8] and relies on an application of Łuczak's method [9]. This reduces the problem at hand to that of finding, in any 2-colouring of the edges of an almost complete 3-uniform hypergraph, two disjoint *monochromatic connected matchings* which cover almost all vertices.

Here, as usual, a *matching* $\mathcal{M}$ in a hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges and $\mathcal{M} \subset \mathcal{H}$ is called *connected* if between every pair $e, f \in \mathcal{M}$ there is a *pseudo-path* in $\mathcal{H}$ connecting e and f, that is, there is a sequence $(e_1, \dots, e_p)$ of not necessarily distinct edges of $\mathcal{H}$ such that $e = e_1, f = e_p$ and $|e_i \cap e_{i+1}| = 2$ for each $i \in [p-1]$. Now, we call a matching $\mathcal{M}$ in a 2-coloured hypergraph a *monochromatic connected matching* if $\mathcal{M}$ is a subhypergraph of a monochromatic component induced by the considered relation of connectedness.

Our main result is the following, which might be of independent interest.

Theorem 2.1 *Let $\mathcal{H}$ be a 3-uniform hypergraph on t vertices and $(1-\gamma)\binom{t}{3}$ edges. Then any two-colouring of the edges of $\mathcal{H}$ admits two disjoint monochromatic connected matchings covering all but $o(t)$ vertices of $\mathcal{H}$.*

We first give an outline of the proof of Theorem 1.1 assuming that Theorem 2.1 holds, before dealing with Theorem 2.1 itself.

2.1 Proof of Theorem 1.1

For given $\eta > 0$ we apply the Strong Regularity Lemma (see [8] for details) for 3-uniform hypergraphs to $\mathcal{K}_n^{(3)}$ with suitable parameters to obtain a regular partition and the reduced hypergraph $\mathcal{K}$ on t vertices and $(1-\gamma)\binom{t}{3}$

edges, where γ depends on η. Consider the 2-edge-colouring of $\mathcal{K}$ given by the majority colouring over the triples of the regular partition.

Next, apply Theorem 2.1 to $\mathcal{K}$ to obtain the monochromatic connected matchings $\mathcal{M}_{\text{red}}$ and $\mathcal{M}_{\text{blue}}$ covering all but $o(t)$ vertices of $\mathcal{K}$.

By using $\mathcal{M}_{\text{red}}$ and $\mathcal{M}_{\text{blue}}$ as a frame and applying a suitable embedding strategy (see [4]) we find find two monochromatic disjoint tight cycles of even length covering at least $(1-\eta)n$ vertices of $\mathcal{K}_n^{(3)}$, as desired.

2.2 Proof of Theorem 2.1

We will need the following result concerning the existence of perfect matchings in 3-uniform hypergraphs with high minimum vertex degree.

Theorem 2.2 ([7]) *For all $\eta > 0$ there is a $n_0 = n_0(\eta)$ such that for all $n > n_0$, $n \in 3\mathbb{Z}$, the following holds. Suppose $\mathcal{H}$ is 3-uniform hypergraph on n vertices such that every vertex is contained in at least $\left(\frac{5}{9}+\eta\right)\binom{n}{2}$ edges. Then $\mathcal{H}$ contains a perfect matching.*

Denote by $\partial\mathcal{H}$ the *shadow* of $\mathcal{H}$, that is, the set of all pairs xy for which there exists z such that $xyz \in \mathcal{H}$. We call a pair of vertices xy *active* if there is an edge of $\mathcal{H}$ containing x and y. For convenience, we say that a set of vertices $U \subseteq V(\mathcal{H})$ is *negligible* in $\mathcal{H}$ if $|U| \leq 240\gamma^{1/6}|V(\mathcal{H})|$.

Lemma 2.3 ([8]) *Let $\gamma > 0$ and let $\mathcal{H}$ be a 3-uniform hypergraph on t vertices and at least $(1-\gamma)\binom{t}{3}$ edges. Then $\mathcal{H}$ contains a subhypergraph $\mathcal{K}$ such that the following holds. Every vertex x of $\mathcal{K}$ is in an active pair of $\mathcal{K}$, for all active pairs xy there are at least $(1-10\gamma^{1/6})t$ edges in $\mathcal{K}$ containing both x and y, and $V(\mathcal{H}) \setminus V(\mathcal{K})$ is negligible in $\mathcal{H}$.*

For our proof of Theorem 2.1, suppose we are given a 2-coloured 3-uniform hypergraph $\mathcal{H} = \mathcal{H}_{\text{red}} \cup \mathcal{H}_{\text{blue}}$ on $t_{\mathcal{H}}$ vertices and $(1-\delta)\binom{t_{\mathcal{H}}}{3}$ edges. Apply Lemma 2.3 to $\mathcal{H}$, with parameter γ depending on δ, to obtain $\mathcal{K}$ with the properties stated in the lemma. We want to find two monochromatic connected matchings covering all but a negligible set of vertices in $\mathcal{K}$. Let $\mathcal{K} = \mathcal{K}_{\text{red}} \cup \mathcal{K}_{\text{blue}}$ be the colouring of $\mathcal{K}$ inherited from $\mathcal{H}$.

Proposition 2.4 ([8]) *The hypergraph $\mathcal{K}$ admits a partition $\{X, V_{\text{red}}, V_{\text{blue}}\}$ such that the following holds. The set X is negligible in $\mathcal{K}$ and there is a red component $\mathcal{R}$ (a blue component $\mathcal{B}$) such that, for every $x \in V_{\text{red}}$ ($x \in V_{\text{blue}}$), there are at least $(1-\gamma)t$ vertices $y \in V(\mathcal{K})$ with $xy \in \partial R$ ($xy \in \partial B$).*

We start by choosing two disjoint monochromatic connected matchings,

$\mathcal{M}_{\text{red}} \subseteq \mathcal{R}$ and $\mathcal{M}_{\text{blue}} \subseteq \mathcal{B}$, where $\mathcal{R}$ and $\mathcal{B}$ are components from Proposition 2.4, which together cover as many vertices as possible. Let $V'_{\text{red}} = V_{\text{red}} \setminus (V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})$ and $V'_{\text{blue}} = V_{\text{blue}} \setminus (V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}))$. Notice that if both V'_{red} and V'_{blue} are negligible in $\mathcal{K}$ we are done. Also, observe that

there is no edge xy with $x \in V'_{\text{red}}$ and $y \in V'_{\text{blue}}$ such that $xy \in \partial\mathcal{R} \cap \partial\mathcal{B}$. (1)

Indeed, any such edge xy constitutes an active pair (by Lemma 2.3), and as $|V_{\text{red}}| > \delta t + 2$, there must be a vertex $z \in V'_{\text{red}}$ such that xyz. This yields a contradiction with the maximality of the matching $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$.

We show that if $|V'_{\text{red}}|$ and $|V'_{\text{blue}}|$ are both greater than $2\delta t$, then we can find a pair xy contradicting (1). So we can assume, by symmetry of the argument, that V'_{blue} is negligible in $\mathcal{K}$.

Next, because of the maximality of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$, each edge having all its vertices in V'_{red} is blue. By Lemma 2.3, V'_{red} is negligible in $\mathcal{K}$ (in which case we are done), or V'_{red} is sufficiently large to induce a dense monochromatic blue component $\mathcal{B}'$ such that $V'_{\text{red}} \setminus V(\mathcal{B}')$ is negligible in $\mathcal{K}$ and satisfying the hypothesis of Theorem 2.2. Therefore, the blue component $\mathcal{B}'$ contains a perfect matching.

At this point, we have three disjoint monochromatic connected matchings, one in red ($\mathcal{M}_{\text{red}} \subseteq \mathcal{R}$) and two in blue ($\mathcal{M}_{\text{blue}} \subseteq \mathcal{B}$ and $\mathcal{M}'_{\text{blue}} \subseteq \mathcal{B}'$). Together, these matchings cover all but a negligible set of vertices in $\mathcal{K}$. Notice that $\mathcal{B}$ and $\mathcal{B}'$ can not be the same component because of the maximality of $\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}$.

Our aim now is to dissolve the blue matching $\mathcal{M}_{\text{blue}}$ and cover all but a negligible set (in $\mathcal{K}$) of $V(\mathcal{M}_{\text{blue}})$ with edges in $\mathcal{R}$. To this end, we show that $V(\mathcal{M}_{\text{blue}})$ is negligible in $\mathcal{K}$ (in which case we are done) or, as a consequence of Lemma 2.3, $V(\mathcal{M}_{\text{blue}})$ is contained in V_{red}. Finally, by using the defect form of Hall's theorem, we cover the vertices of $\mathcal{M}_{\text{blue}}$ with a matching $\mathcal{M}'_{\text{red}}$ in ∂R. In other words, $\mathcal{M}_{\text{red}} \cup \mathcal{M}'_{\text{red}}$ and $\mathcal{M}'_{\text{blue}}$ are the two monochromatic connected matchings we had to find.

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