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ON THE VARIATIONAL STRUCTURE OF BREATHER SOLUTIONS I: SINE-GORDON EQUATION

MIGUEL A. ALEJO, CLAUDIO MUÑOZ, AND JOSÉ M. PALACIOS

Abstract. In this paper we describe stability properties of the Sine-gordon breather solution. These properties are first described by suitable variational elliptic equations, which also implies that the stability problem reduces in some sense to (i) the study of the spectrum of explicit linear systems, and (ii) the understanding of how bad directions (if any) can be controlled using low regularity conservation laws. Then we discuss spectral properties of a fourth-order linear matrix system. Using numerical methods, we confirm that all spectral assumptions leading to the $H^2 \times H^1$ stability of SG breathers are numerically satisfied, even in the ultra-relativistic, singular regime.

1. Introduction

1.1. **Setting of the problem.** In this paper we study stability properties of breathers, which are nontrivial solutions of integrable dispersive equations, different to solitons and multi-solitons. We will consider the Sine-Gordon (SG) equation

$$u_{tt} - u_{xx} + \sin u = 0, \quad (u, u_t) = (u, u_t)(t, x) \in \mathbb{R}^2, \ (t, x) \in \mathbb{R}^2. \tag{1.1}$$

The above equation is a well-known *completely integrable* model [16, 1, 26], with infinitely many conserved quantities, and a suitable Lax-pair formulation.

Solutions of (1.1) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $u(t - t_0, x - x_0)$ is also a solution. Furthermore, an additional feature of the SG equation is the invariance under Lorentz transformations: given any $v \in (-1, 1)$, then

$$u(\gamma(t - vx), \gamma(x - vt)), \quad \gamma := (1 - v^2)^{-1/2}, \tag{1.2}$$

is a new solution of (1.1).

On the other hand, standard conservation laws for the SG equation (1.1) at the $H^1 \times L^2$-level are the *energy*

$$E[u, u_t](t) := \frac{1}{2}\int_{\mathbb{R}} (u_x^2 + u_t^2)(t, x)dx + \int_{\mathbb{R}} (1 - \cos u(t, x))dx = E[u, u_t](0), \tag{1.3}$$

and the *momentum*

$$P[u, u_t](t) := \frac{1}{2}\int_{\mathbb{R}} u_t(t, x)u_x(t, x)dx = P[u, u_t](0). \tag{1.4}$$

It is well known that any suitable well-posedness theory must deal simultaneously with the pair (u, u_t) and not only u. Since the nonlinear term $\sin u$ is uniformly bounded independently of the size of u, one can find a satisfactory $H^1 \times L^2$ global well-posedness theory, see e.g. Bourgain [10].

On the other hand, the SG equation (1.1) has soliton solutions usually referred as *kinks*. Indeed, given $v \in (0, 1)$, $x_0 \in \mathbb{R}$ and γ defined as in (1.2), SG (1.1) has kinks of the form

$$u(t, x) = \varphi(\gamma(x - vt - x_0)), \quad \varphi(s) := 4 \arctan e^s. \tag{1.5}$$

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This work comes from the splitting of our paper at arxiv.org/pdf/1309.0625v2.pdf, see also [7] for the other part.

It is not difficult to see that these solutions can be associated to a well known functional defined in the $H^1 \times L^2$ topology.

Kink solutions of (1.1) are orbitally stable in the natural energy space $H^1 \times L^2$. Indeed, the energy (1.3) is a conserved quantity and kinks can be viewed as relative minimizers of a suitable energy functional. For the proofs of this result for the SG case and more general equations, we refer to the works by Henry-Perez-Wreszinski [21], Grillakis-Shatah-Strauss [18], Soffer-Weinstein [32], Cuccagna [12] and the recent results by Kowalczyk, Martel and the second author [24, 25].

1.2. **Breathers and their stability.**

In addition to the above mentioned special solutions (1.1), there exists another large family of explicit and oscillatory solutions, known in the physical and mathematical literature as the *breather* solution, which is a periodic in time, spatially localized real function. Although there is no universal definition for a breather, we will adopt the following convention, that will match the SG case.

Definition 1.1 (Aperiodic breather). *We say that $B = B(t,x)$ is a breather solution for a particular one-dimensional dispersive equation if there are $T > 0$ and $L = L(T) \in \mathbb{R}$ such that, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$, one has*

$$B(t+T,x) = B(t,x-L), \tag{1.6}$$

and moreover, the infimum among times $T > 0$ such that property (1.6) *is satisfied for such a time T is uniformly positive in space.*

Remark 1.1. Note that the last condition ensures that kinks are not breathers, since e.g. $\varphi(\gamma(x - v(t+T))) = \varphi(\gamma(x - vt - L))$ for $L := vT$ but T can be any real-valued time.[1]

For the Sine-Gordon (1.1) scalar field equation, the classical *standing* breather is the following

$$B(t,x;\beta) := 4\arctan\Big(\frac{\beta}{\alpha}\frac{\cos(\alpha t)}{\cosh(\beta x)}\Big), \quad \alpha^2+\beta^2 = 1. \tag{1.7}$$

(In Definition 2.1 it is also presented a general formula of the SG breather, involving all symmetries of the equation.) As far as we know, *there is no rigorous proof of stability for this solution.* Moreover, it is believed that SG breathers play an important role in the so-called asymptotic stability problem for the kink solution, see e.g. [32, 24, 25].

Solutions like (1.7) have become a canonical example of complexity in nonlinear integrable systems [26, 1]. Moreover, their surprising mixed behavior, combining oscillatory and soliton character, has focused the attention of many researchers since thirty years ago [31, 9, 13]. From the physical point of view, breather solutions seem to be relevant to localization-type phenomena in optics, condensed matter physics and biological processes [8]. They also play an important role in the modeling of freak and rogue waves events on surface gravity waves and also of internal waves in the stratified ocean, in Josephson junctions and even in nonlinear optics. See [14, 19, 20, 2] for a representative set of these examples.

From a mathematical point of view, breather solutions arise in different contexts. In a geometrical setting, modified KdV (mKdV) breathers appear in the evolution of closed planar curves playing the role of smooth localized deformations traveling along the closed curve [3]. Moreover, it is interesting to point out that mKdV breather solutions have also been considered by Kenig, Ponce and Vega in their proof of the non-uniform continuity of the mKdV flow in the Sobolev spaces H^s, $s < \frac{1}{4}$ [23]. On the other hand, they should play an important role in the *soliton-resolution* conjecture, according to the analysis developed by Schuur in [31].

[1] In the case of NLS equations and their solitons, Definition 1.1 must be repaired to exclude them because of the $U(1)$ invariance.

1.3. **Main results.** The purpose of this paper is to present compelling evidence that SG breathers are stable. We will show that, under two (numerically verified) spectral assumptions, SG breathers are stable. First of all, we will show the following

Theorem 1.2. *Any pair of SG breather $(B, B_t) \equiv (B, \partial_t B)$ satisfy the set of matrix-valued, nonlinear equations*

$$
\begin{aligned}
&B_{txx} + \frac{1}{8}B_t^3 + \frac{3}{8}B_x^2 B_t - \frac{1}{4}B_t \cos B - aB_t - \frac{b}{2}B_x = 0,\\
&B_{(4x)} + \frac{3}{8}B_x^2 B_{xx} + \frac{3}{4}B_t B_{tx} B_x + \frac{3}{8}B_t^2 B_{xx} + \frac{5}{8}B_x^2 \sin B - \frac{5}{4}B_{xx}\cos B\\
&\qquad + \frac{1}{4}\sin B \cos B - \frac{1}{8}B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2}B_{tx} = 0,
\end{aligned}
$$

for a set of constants a, b depending only on some particular parameters of the breather.

This last result is, as far as we know, not present in the literature, although some ideas are sketched in [27] for the case of mKdV solitons and multisolitons (but never rigorously verified for the case of breathers). See also [17, 4, 5, 6] for a particular stability statement in the "simpler" mKdV case, both in H^1 and H^2. For a complete and detailed statement of the previous result, see Theorem 2.3.

As the main consequence of this result, it is possible to construct the associated *linearized operator* around a breather, and try to compute its spectra. However, a rigorous description of this operator has escaped to us. In the following, we will need two spectral assumptions that are standard in the literature:

- **Assumption 1:** The kernel of a linearized operator around a breather is nondegenerate and it satisfies the gap condition; and
- **Assumption 2:** There is a unique simple negative eigenvalue associated to this linear operator.

(See p. 12 for a precise description.) With these assumptions on hand, in this paper we prove the following conditional result:

Theorem 1.3. *Under spectral Assumptions 1 and 2, SG breathers are orbitally stable for small $H^2 \times H^1$ perturbations.*

These two spectral conditions are numerically tested, showing agreement and compelling evidence for their validity, for any suitable region of parameters, including the cases of low and high velocity breathers, see Figs. 2, 3 and 4. In order to clarify the absence of a rigorous proof for the spectral properties, the main difficulty in proving these two assumptions rests in the fourth order, coupled, matrix-valued character of the linearized operator around a SG breather solution, which formally leads to consider the understanding of an eight-order linear scalar operator, instead of only dealing with a fourth order operator as in mKdV. A rigorous proof of these facts is an interesting open problem.

Although by using some IST techniques one could possibly obtain a better resolution in Theorem 1.3 (but under additional decay assumptions), working in a variational framework has some particular advantages. First of all, the methods and ideas are "stable" under variations of the equation: they also say something about nonintegrable models close to the integrable one, just by comparing their respective variational formulations. One example of this property is the method of proof in [9], based on the Bäcklund transformation, which is hard to extrapolate and make it rigorous for the case of nonintegrable equations.

We should also remark that the original seed of these ideas is certainty not new and it was introduced in a seminal paper by Lax [27], in the particular case of the two-soliton solution of the Korteweg-de Vries equation. This method has been generalized to several equations with soliton solutions [28, 21, 22, 29]. However, no previous result was available in the case of breathers, apparently because of their dynamics, which do not resemble any type of simple, decoupled 2-soliton solution. Compared with the previous results, proofs in [4] are more involved, since there is no mass splitting as $t \to +\infty$. Further results where this technique has been successfully applied to the understanding of stability of different soliton solutions are the works [30], see also references therein.

We also mention that, in addition to Theorem 1.3, the more involved problem of asymptotic stability for breathers could been also considered, as far as a good and rigorous understanding of the associated spectral problem is at hand. Usually, these spectral properties are usually harder to establish than the ones involved in the stability problem (because the convergence problem requires the use of weighted functions, which destroy most of breather's algebraic properties). In addition, breathers can have zero, positive or negative velocity, which means that they do not necessarily decouple from radiation. However, it is worth to mention that if the velocity of a periodic mKdV breather is positive, then there is local strong asymptotic stability in the energy space, see [4].

1.4. **Organization of this paper.** This paper is organized as follows. In Section 2 we prove that SG breathers satisfy an elliptic system of differential equations, revealing its variational character in the proper space $H^2 \times H^1$. Also, in Section 2, we show the validity of Theorem 1.3 (see Theorem 2.5 for more details). In Section 3 we study mathematical properties of the related SG spectral problem. Finally, in Section 4 we describe, implement and use standard numerical methods to understand and recover the SG spectral problem, the required assumptions will be showed to hold numerically.

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2. The Sine-Gordon breather

2.1. **Preliminaries.** One of the most important examples of breather solution is the one obtained from the SG equation (1.1). For this case, we have the following definition.

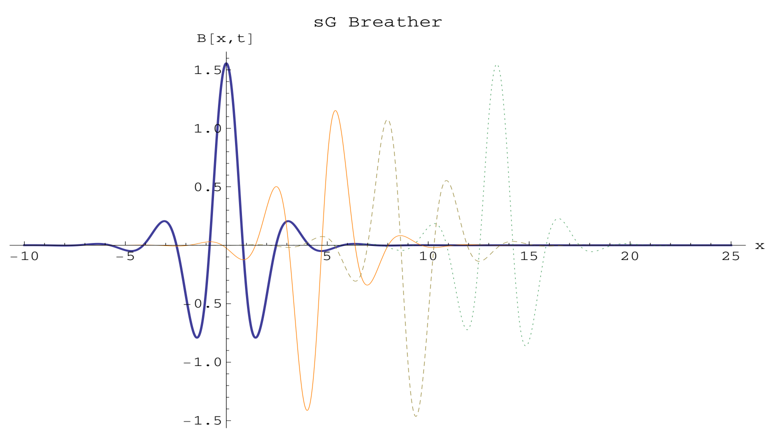

Figure 1. Evolution of a Sine-Gordon breather moving rightward in space as time evolves (blue color to green color).

Definition 2.1 ([26], see also Fig. 1)**.** *Let* $v \in (-1,1)$, $\gamma := (1-v^2)^{-1/2}$, $\beta \in (0,\gamma)$ *and* $x_1, x_2 \in \mathbb{R}$. *Any breather solution of Sine-Gordon* (1.1) *is given by the expression*

$$B := B_{\beta,v}(t,x;x_1,x_2) := 4\arctan\Big(\frac{\beta}{\alpha}\frac{\cos(\alpha y_1)}{\cosh(\beta y_2)}\Big), \tag{2.1}$$

with

$$y_1 := t - vx + x_1, \quad y_2 := x - vt + x_2, \quad \alpha := \sqrt{\gamma^2 - \beta^2}. \tag{2.2}$$

The parameters β and v correspond to the scaling and velocity of the breather, and the case $v = 0$ represents a standing SG breather.

Note that the SG breather satisfies Definition 1.1 with $T = \frac{2\pi}{\alpha(1-v^2)} > 0$ and $L = vT \in \mathbb{R}$. Also, the previous definition takes into account the velocity v of the breather via a Lorentz boost, therefore it is slightly different to the one written in [26]. However, after redefinition of the parameters, it is not difficult to check that they represent the same solution. Additionally, note that in the SG case the two parameters α and β are not independent, unlike the mKdV case. In order to make sense for a suitable Cauchy theory, our previous definition requires additionally a description of the time derivative of a breather solution. Since we are going to work with several time-dependent parameters, it is certainly necessary to give a precise definition of this second nonlinear mode.

Definition 2.2. *For any $x_1, x_2 \in \mathbb{R}$ fixed, we define the time derivative of B, denoted by $B_t = (B_{\beta,v})_t$, as follows*

$$B_t(t, x; x_1, x_2) := -4\alpha\beta \left[\frac{\alpha \sin(\alpha y_1) \cosh(\beta y_2) - \beta v \cos(\alpha y_1) \sinh(\beta y_2)}{\alpha^2 \cosh^2(\beta y_2) + \beta^2 \cos^2(\alpha y_1)} \right]. \tag{2.3}$$

We introduce now some useful notation. Recall that $v \in (-1, 1)$, $\gamma = (1 - v^2)^{-1/2} \geq 1$, and $\beta \in (0, \gamma)$. Define the parameters

$$a := -\frac{1}{4} + \beta^2 + v^2(2\gamma^4 - \gamma^2 + \beta^2), \qquad b := 4v(\gamma^4 - \beta^2). \tag{2.4}$$

Note that $a + \frac{1}{4} > 0$ and $b \in \mathbb{R}$. The reader may observe that whenever $v = 0$ (the static breather), we have the simplified expressions $a + \frac{1}{4} = \beta^2 \in (0, 1)$, and $b = 0$.

2.2. **Variational characterization.** As it was announced in the introduction of this work (see Theorem 1.2), in this paper we will prove the following generalization of [4, eqn.(3.6)] for the SG case.

Theorem 2.3. *Let (B, B_t) be any SG breather of parameters $v \in (-1, 1)$, $\beta \in (0, \gamma)$, and a, b as in (2.4). Then, for any fixed $t \in \mathbb{R}$, (B, B_t) satisfy the nonlinear equations*

$$B_{txx} + \frac{1}{8}B_t^3 + \frac{3}{8}B_x^2 B_t - \frac{1}{4}B_t \cos B - aB_t - \frac{b}{2}B_x = 0, \tag{2.5}$$

and

$$\begin{aligned} B_{(4x)} + \frac{3}{8}B_x^2 B_{xx} + \frac{3}{4}B_t B_{tx} B_x + \frac{3}{8}B_t^2 B_{xx} + \frac{5}{8}B_x^2 \sin B - \frac{5}{4}B_{xx} \cos B \\ + \frac{1}{4}\sin B \cos B - \frac{1}{8}B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2}B_{tx} = 0. \end{aligned} \tag{2.6}$$

In particular, (B, B_t) is a critical point of the functional

$$\mathcal{H}[u, u_t] = F[u, u_t] + aE[u, u_t] + bP[u, u_t], \tag{2.7}$$

where the energy $E[u, u_t]$ and momentum $P[u, u_t]$ are defined in (1.3) *and* (1.4) *respectively, and*

$$\begin{aligned} F[u, u_t](t) \; := \; & \frac{1}{2}\int_{\mathbb{R}} (u_{xx}^2 + u_{tx}^2)(t, x)dx - \frac{1}{32}\int_{\mathbb{R}} (u_t^4 + u_x^4)(t, x)dx - \frac{3}{16}\int_{\mathbb{R}} u_t^2(t, x)u_x^2(t, x)dx \\ & + \frac{5}{8}\int_{\mathbb{R}} u_x^2(t, x)\cos u(t, x)dx + \frac{1}{8}\int_{\mathbb{R}} (\sin^2 u(t, x) + u_t^2(t, x)\cos u(t, x))dx, \end{aligned} \tag{2.8}$$

is the third conserved quantity for the sine-Gordon equation.

Proof of (2.6)**, assuming** (2.5)**.** Taking time derivative in equation (2.5), and replacing B_{tt} using (1.1), we have

$$
\begin{aligned}
0 &= (B_{tt})_{xx} + \frac{3}{8}B_t^2 B_{tt} + \frac{3}{4}B_t B_x B_{tx} + \frac{3}{8}B_x^2 B_{tt} \quad - \frac{1}{4}B_{tt}\cos B + \frac{1}{4}B_t^2 \sin B - aB_{tt} - \frac{b}{2}B_{tx} \\
&= (B_{xx} - \sin B)_{xx} + \frac{3}{8}B_t^2(B_{xx} - \sin B) + \frac{3}{4}B_t B_x B_{tx} + \frac{3}{8}B_x^2(B_{xx} - \sin B) \\
&\quad - \frac{1}{4}(B_{xx} - \sin B)\cos B + \frac{1}{4}B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2}B_{tx} \\
&= B_{(4x)} - (B_x \cos B)_x + \frac{3}{8}B_t^2 B_{xx} - \frac{3}{8}B_t^2 \sin B + \frac{3}{4}B_t B_x B_{tx} + \frac{3}{8}B_x^2 B_{xx} - \frac{3}{8}B_x^2 \sin B \\
&\quad - \frac{1}{4}B_{xx}\cos B + \frac{1}{4}\sin B \cos B + \frac{1}{4}B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2}B_{tx} \\
&= B_{(4x)} + \frac{3}{8}B_x^2 B_{xx} + \frac{3}{4}B_t B_{tx} B_x + \frac{3}{8}B_t^2 B_{xx} + \frac{5}{8}B_x^2 \sin B - \frac{5}{4}B_{xx}\cos B \\
&\quad + \frac{1}{4}\sin B \cos B - \frac{1}{8}B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2}B_{tx} = \text{ l.h.s. of (2.6)}.
\end{aligned}
$$

For the proof of (2.5), see the Appendix A. Finally, the proof that (B, B_t) is a critical point of the functional $\mathcal{H}$ in (2.7) is given in Appendix B. Finally, the proof that the functional $F[u, u_t](t)$ (2.8) is a conserved quantity is a tedious but direct computation. □

Remark 2.1. The structure of equations (2.5)-(2.6) is inherent to solutions of (1.1). For example, the static kink solution $(\varphi(x), 0)$ defined in (1.5) is also a solution of the same set of equations, no matter what are a and b. In some sense, the proof of (2.5) is independent of the breather itself, but the proof of (2.6) explicitly involves the structure of the breather.

Although the parameters x_1, x_2 are chosen independent of time, a simple argument ensures that the previous lemma still hold under time dependent, translation parameters $x_1(t)$ and $x_2(t)$.

Corollary 2.4. *Let (B^0, B_t^0) be any SG breather as in (2.1), and $x_1(t), x_2(t) \in \mathbb{R}$ two continuous functions, defined for all t in a given interval I. Consider the modified breather*

$$(B, B_t)(t, x) := (B^0, B_t^0)(t, x; x_1(t), x_2(t)), \qquad (\textit{cf. } (2.1)).$$

Then (B, B_t) satisfy (2.5) and (2.6), for all t in the considered interval I.

Proof. A direct consequence of the invariance of equations (2.5) and (2.6) under translations in the variables x_1 and x_2. □

2.3. **A rigorous statement for Theorem 1.3.** The main consequence of previous results is the possibility of proving stability, independent of how complicated breather dynamics can be. After a suitable understanding of the linearized problem associated to (2.5)-(2.6), we are able to prove the following conditional result:

Theorem 2.5. *Under Assumptions 1 and 2 in page 12, SG breathers are stable under small $H^2 \times H^1$ perturbations. More precisely, there are $\eta_0 > 0$ and $K_0 > 0$, only depending on β, such that if $0 < \eta < \eta_0$ and*

$$\|(u_0, u_1) - (B_{\beta,v}, (B_{\beta,v})_t)(t = 0, \cdot; 0, 0)\|_{H^2 \times H^1} < \eta, \tag{2.9}$$

then there are real-valued parameters $x_1(t)$ and $x_2(t)$ for which the global $H^2 \times H^1$-solution $(u, u_t)(t)$ of (1.1) with initial data (u_0, u_1) satisfies

$$\sup_{t \in \mathbb{R}} \|(u, u_t)(t) - (B_{\beta,v}, (B_{\beta,v})_t)(t, \cdot; x_1(t), x_2(t))\|_{H^2 \times H^1} < K_0 \eta,$$

with similar estimates for the derivatives of the shift parameters x_1, x_2.

A mathematical proof of Assumptions 1 and 2 has escaped to us, even in the case $v = 0$,[2] mainly due to the matrix, highly coupled, fourth-order character of the linearized system associated to SG breathers. However, as explained in the Introduction, in this paper we present compelling numerical evidence revealing that Assumptions 1 and 2 do hold for any breather solution, regardless its size, initial velocity or position, see Figs. 2, 4 and 1.

The proof of Theorem 2.5 follows the ideas of our proof in [4], with some interesting changes due to the matrix-valued character of the solution. The proof is decomposed in several steps, and it is finally done in Section 3.4.

Remark 2.2. We find that Theorem 2.5 is in fact a surprise, since from Grillakis-Shatah-Strauss [18], it was expected that real-valued, non topological solutions to scalar field equations were certainly unstable. However, the fact that breathers have no sign, do not satisfy a simple second order ODE, and the more important fact that the equation is completely integrable reveal deep obstructions to a more general behavior of solutions. We emphasize that our results above are independent of the velocity v: standing and highly relativistic breathers should be both stable.

Remark 2.3. Looking for previous contributions to the stability problem we have found the work by Ercolani, Forest and McLaughlin [15], where a sketch of the stability proof is presented. To be more precise, their argument states that, by using the inverse scattering theory applied to a small perturbation of the breather, the corresponding solution to (1.1) must remain uniformly close in time to a modulated breather solution. However, it is important to stress that any rigorous argument involving the inverse scattering method requires a *nontrivial amount of decay* on the initial data, an assumption that is not needed in our case.

The purpose of the next Section is to give the main ingredients for the Proof of Theorem 2.5. In a first subsection, we prove some energy and momentum identities, allowing us to show that, at least formally, there is only one negative direction (see Definition 3.6 for a rigorous explanation of this term), and moreover, it is possible to control such a bad direction by using only the breather (B, B_t) as replacement.

3. Mathematical Analysis of the SG breather

3.1. **Energy identities.** In the next lines, we summarize some well known identities for the energy and momentum of a fixed breather. For the sake of completeness, we give full proofs of these results. Recall the definition of breather (B, B_t) in (2.1)-(2.3).

Lemma 3.1. *Let (B, B_t) be any SG breather, with parameters for $\beta > 0$, $v \in (-1, 1)$, and $x_1, x_2 \in \mathbb{R}$. Let $P[B, B_t]$ be the momentum of a breather defined in (1.4). Then*

$$P[B, B_t](t) = -8\beta v. \tag{3.1}$$

Proof. From (2.3) and (A.1), we have

$$P[B, B_t] = -8\alpha^2\beta^2 \int_{\mathbb{R}} \frac{h_p(t,x)}{(\alpha^2\cosh^2(\beta y_2) + \beta^2\cos^2(\alpha y_1))^2},$$

where $h_p(t,x)$ is given by the expression

$$\begin{aligned} h_p(t,x) \;:=\;& \alpha^2 v\sin^2(\alpha y_1)\cosh^2(\beta y_2) - \alpha\beta(1+v^2)\sin(\alpha y_1)\cos(\alpha y_1)\sinh(\beta y_2)\cosh(\beta y_2) \\ & + \beta^2 v\cos^2(\alpha y_1)\sinh^2(\beta y_2). \end{aligned}$$

[2]Although we have a proof for the case where $v = x_1 = x_2 = 0$ and $\beta = \frac{1}{2}$, however this particular case is not sufficient to describe correctly a general dynamics, where nonzero shifts are always present.

Now the purpose is to use double angle formulae to avoid the squares.[3] We replace these identities in the previous expression above. We obtain

$$P[B,B_t] = -8\alpha^2\beta^2 \int_{\mathbb{R}} \frac{h_p(t,x)}{\Big(\frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2}{2}\cosh(2\beta y_2) + \frac{\beta^2}{2}\cos(2\alpha y_1)\Big)^2},$$

where now $h_p(t,x)$ is rewritten as

$$\begin{aligned} h_p(t,x) \quad &:= \quad \frac{v}{4}(\alpha^2-\beta^2) + \frac{v}{4}(\alpha^2+\beta^2)\cosh(2\beta y_2) - \frac{v}{4}(\alpha^2+\beta^2)\cos(2\alpha y_1) \\ &\quad -\frac{v}{4}(\alpha^2-\beta^2)\cos(2\alpha y_1)\cosh(2\beta y_2) - \frac{1}{4}\alpha\beta(1+v^2)\sin(2\alpha y_1)\sinh(2\beta y_2). \end{aligned}$$

Let $f_p(t,x) := \frac{1}{4\alpha\beta}[\beta\sin(2\alpha y_1) + v\alpha\sinh(2\beta y_2)]$. Then

$$(f_p)_x(t,x) = \frac{1}{2}v[\cosh(2\beta y_2) - \cos(2\alpha y_1)].$$

Recall the definition of g given in (A.6) but rewriten in terms of the double angle formulas (3.2),(3.3)

$$g(t,x) := \frac{1}{2}(\alpha^2+\beta^2) + \frac{\alpha^2}{2}\cosh(2\beta y_2) + \frac{\beta^2}{2}\cos(2\alpha y_1). \tag{3.4}$$

Then $(f_p)_x(t,x)g(t,x) - g_x(t,x)f_p(t,x) = h_p(t,x)$. From this identity we finally obtain

$$P[B,B_t] = -8\alpha^2\beta^2 \int_{\mathbb{R}} \Big(\frac{f_p}{g}\Big)_x (t,x)dx = -8\beta v.$$

□

We compute now the energy of a breather. See Lamb [26] for a similar result in the case where $v=0$.

Lemma 3.2. *Let (B,B_t) be any SG breather of parameters for $\beta>0$, $v\in(-1,1)$, and $x_1,x_2\in\mathbb{R}$. Then*

$$E[B,B_t](t) = 16\beta. \tag{3.5}$$

Moreover, this result does not change if we replace x_1 and x_2 by time dependent parameters.[4]

Proof. This is a classical result. From (2.3) and (A.1), we have

$$\begin{aligned} \frac{1}{2}(B_t^2+B_x^2) \quad &= \quad \frac{8\alpha^2\beta^2}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^2} \times \\ &\quad \times\Big[(1+v^2)\alpha^2\cosh^2(\beta y_2)\sin^2(\alpha y_1) + (1+v^2)\beta^2\sinh^2(\beta y_2)\cos^2(\alpha y_1) \\ &\quad -4\alpha\beta v\sin(\alpha y_1)\cos(\alpha y_1)\sinh(\beta y_2)\cosh(\beta y_2)\Big]. \end{aligned}$$

On the other hand, from (A.4),

$$1-\cos B = \frac{8\alpha^2\beta^2\cosh^2(\beta y_2)\cos^2(\alpha y_1)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^2}.$$

Therefore using double angle formulas, we have

$$\frac{1}{2}(B_t^2+B_x^2) + (1-\cos B) = \frac{8\alpha^2\beta^2\tilde{h}(t,x)}{(\frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2}{2}\cosh(2\beta y_2) + \frac{\beta^2}{2}\cos(2\alpha y_1))^2},$$

[3]More precisely, it is well known that

$$\cos^2(\alpha x) = \frac{1}{2}(1+\cos(2\alpha x)), \quad \sin^2(\alpha x) = \frac{1}{2}(1-\cos(2\alpha x)), \tag{3.2}$$

$$\cosh^2(\beta x) = \frac{1}{2}(1+\cosh(2\beta x)), \quad \sinh^2(\beta x) = \frac{1}{2}(\cosh(2\beta x)-1). \tag{3.3}$$

[4]But now (B,B_t) ceases being an exact solution of SG.

where

$$\begin{aligned}\tilde{h}(t,x) \quad &:= \quad \frac{1}{4}((1+v^2)(\alpha^2-\beta^2)+1)+\frac{1}{4}((1+v^2)(\alpha^2+\beta^2)+1)\cosh(2\beta y_2)\\ &\quad -\alpha\beta v\sin(2\alpha y_1)\sinh(2\beta y_2)-\frac{1}{4}((1+v^2)(\alpha^2+\beta^2)-1)\cos(2\alpha y_1)\\ &\quad +\frac{1}{4}((1+v^2)(\beta^2-\alpha^2)+1)\cosh(2\beta y_2)\cos(2\alpha y_1).\end{aligned}$$

Now with the relation among α, β (2.2) and g given by (3.4), a direct computation shows that

$$\tilde{h} = g(h_e)_x - h_e g_x,$$

where $h_e(t,x) := \frac{1}{2\alpha\beta}[\alpha\sinh(2\beta y_2) + \beta v\sin(2\alpha y_1)]$. From this identity we obtain

$$E[B,B_t] = 8\alpha^2\beta^2\int_{\mathbb{R}}\left(\frac{h_e}{g}\right)_x = 16\beta.$$

□

Remark 3.1. The reader may compare (3.5) with a similar result for the mass of the mKdV breather, see [4, eqn. (2.4)], where $M[B] = 4\beta$. In that sense, both results reveal that the mass (or energy) does not depend on the oscillatory parameter, but only on the main scaling β. This property seems inherent to aperiodic breathers.

3.2. **Linear operators and stability tests.** Some essential consequences of the last two identities are the following stability conditions, which will be useful when dealing with coercivity estimates in Subsection 3.3.

Corollary 3.3. *Let (B, B_t) be any SG breather of the form* (2.1)-(2.3). *For $t \in \mathbb{R}$ fixed, let*

$$\Lambda B := \partial_\beta B, \quad \Lambda B_t := \partial_\beta B_t. \tag{3.6}$$

Then $(\Lambda B, \Lambda B_t)$ are Schwartz in the space variable, and the following identities are satisfied

$$\partial_\beta E[B,B_t] = 16 > 0, \quad \partial_\beta P[B,B_t] = -8v, \tag{3.7}$$

independently of time.

Proof. By simple inspection one can see that for each time t one has that ΛB and ΛB_t are well-defined Schwartz functions. The proof of (3.7) is consequence of (3.5) and (3.1). □

Now, we introduce the following two directions associated to spatial translations. Let (B, B_t) as defined in (2.1)-(2.3), with main parameters $\beta > 0$ and $v \in (0,1)$. We define

$$\begin{cases} B_1(t;x_1,x_2) := \partial_{x_1}B(t;x_1,x_2),\\ B_2(t;x_1,x_2) := \partial_{x_2}B(t;x_1,x_2).\end{cases} \tag{3.8}$$

Similarly, we introduce the terms involving time derivatives

$$\begin{cases} (B_t)_1(t;x_1,x_2) := \partial_{x_1}B_t(t;x_1,x_2)\\ (B_t)_2(t;x_1,x_2) := \partial_{x_2}B_t(t;x_1,x_2).\end{cases} \tag{3.9}$$

It is clear that, for all $t \in \mathbb{R}$, $\beta > 0$ and $x_1, x_2 \in \mathbb{R}$, both B_1 and B_2 are real valued, Schwartz functions.[5] More explicitly,

$$B_1 = \frac{-4\alpha^2\beta\sin(\alpha y_1)\cosh(\beta y_2)}{\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1)}, \qquad B_2 = \frac{-4\alpha\beta^2\cos(\alpha y_1)\sinh(\beta y_2)}{\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1)}. \tag{3.10}$$

Now we introduce the main linear operator associated to the breather (B, B_t). As it will be clear below, this matrix operator will be of fourth order in one component ($\mathcal{L}_1$), and of second order in the second one ($\mathcal{L}_2$). Two nontrivial nondiagonal terms $\mathcal{B}_1$ and $\mathcal{B}_2$ couple both previous components in a nontrivial fashion, which is never zero in general, even in the case $v = 0$.

[5] Additionally, we can express B_t and B_x in terms of B_1 and B_2: we have

$$B_t = B_1 - vB_2, \qquad B_x = -vB_1 + B_2.$$

Definition 3.4. *Let $\mathcal{L}$ be matrix linear operator*

$$\mathcal{L}[z,w] := \begin{pmatrix} \mathcal{L}_1 & \mathcal{B}_1 \\ \mathcal{B}_2 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \tag{3.11}$$

where

$$\begin{aligned} \mathcal{L}_1[z] := z_{(4x)} - \Big[a - \frac{3}{8}(B_x^2 + B_t^2) + \frac{5}{4}\cos B\Big] z_{xx} + \Big[\frac{3}{4}B_xB_{xx} + \frac{3}{4}B_tB_{tx} + \frac{5}{4}\sin B B_x\Big] z_x \\ + \Big[a\cos B + \frac{5}{8}B_x^2\cos B + \frac{5}{4}B_{xx}\sin B + \frac{1}{4}(\cos^2 B - \sin^2 B) - \frac{1}{8}B_t^2\cos B\Big] z, \end{aligned} \tag{3.12}$$

$$\mathcal{L}_2[w] := -w_{xx} + \frac{1}{4}\Big[4a + \cos B - \frac{3}{2}(B_t^2 + B_x^2)\Big] w, \tag{3.13}$$

$$\mathcal{B}_1[w] := \frac{1}{4}\Big[3B_{tx}B_x + 3B_tB_{xx} - B_t\sin B\Big] w + \frac{1}{4}\Big[3B_tB_x - 2b\Big] w_x, \tag{3.14}$$

and finally

$$\mathcal{B}_2[z] := \frac{1}{2}(b - \frac{3}{2}B_tB_x)z_x - \frac{1}{4}B_t\sin B z. \tag{3.15}$$

In order to ensure a correct definition for $\mathcal{L}$, we will set $\mathcal{L}$ on the Hilbert space $L^2(\mathbb{R})^2$, with dense domain $H^4(\mathbb{R}) \times H^2(\mathbb{R})$.

A simple consequence of Theorem 2.3 is the following partial kernel description.

Corollary 3.5. *Let a, b be defined as in (2.4), and B_1, B_2 given in* (3.10). *We have*

$$\mathcal{L}[B_1, (B_t)_1] = \mathcal{L}[B_2, (B_t)_2] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $(B_1, (B_t)_1)^T$ and $(B_2, (B_t)_2)^T$ are linearly independent in $\mathbb{R}$.

Now we consider the natural directions associated to the scaling parameters. First of all, we define the quadratic form associated to $\mathcal{L}$, namely

$$\mathcal{Q}[z,w] := \int_{\mathbb{R}} (z,w)\mathcal{L}[z,w]. \tag{3.16}$$

A more detailed version of $\mathcal{Q}$ is the following expression, obtained after integration by parts:

$$\begin{aligned} \mathcal{Q}[z,w] &= \int_{\mathbb{R}} \{z\mathcal{L}_1[z] + z\mathcal{B}_1[w] + w\mathcal{B}_2[z] + w\mathcal{L}_2[w]\} \\ &= \int_{\mathbb{R}} z_{xx}^2 + \int_{\mathbb{R}} w_x^2 + \int_{\mathbb{R}} \Big[a - \frac{3}{8}(B_x^2 + B_t^2) + \frac{5}{4}\cos B\Big] z_x^2 \\ &\quad + \int_{\mathbb{R}} \Big[\frac{5}{8}B_x^2\cos B + \frac{5}{4}B_{xx}\sin B + \frac{1}{4}(\cos^2 B - \sin^2 B) - \frac{1}{8}B_t^2\cos B + a\cos B\Big] z^2 \\ &\quad + \int_{\mathbb{R}} \Big[(a + \frac{1}{4})\cos B - \frac{3}{8}(B_t^2 + B_x^2) + \frac{3}{4}B_{tx}B_x + \frac{3}{4}B_tB_{xx} - \frac{1}{4}B_t\sin B\Big] w^2 \\ &\quad + \int_{\mathbb{R}} \Big[(b - \frac{3}{2}B_tB_x)z_xw - \frac{1}{2}B_t\sin B zw\Big]. \end{aligned} \tag{3.17}$$

The following concept is standard.

Definition 3.6. *Any nonzero pair $(z,w) \in H^4 \times H^2$ is said to be a positive (null, negative) direction for $\mathcal{Q}$ if we have $\mathcal{Q}[z,w] > 0$ $(= 0, < 0)$.*

Recall the definitions of ΛB and ΛB_t introduced in (3.6). For this direction, one has the result below.

Corollary 3.7. *The following are satisfied:*

(1) *Let (B, B_t) be any SG breather. Consider the* scaling *direction $(\Lambda B, \Lambda B_t)$ introduced in (3.6). Then $(\Lambda B, \Lambda B_t)$ is a negative direction for $\mathcal{Q}$. Moreover,*

$$\mathcal{Q}[\Lambda B, \Lambda B_t] = -32(1 + 3v^2)\beta < 0. \tag{3.18}$$

(2) *The quadratic functional* $\mathcal{Q}$ *is bounded below, namely*

$$\mathcal{Q}[z,w] \geq -c_{\beta,v}\|(z,w)\|^2_{H^2(\mathbb{R})\times H^1(\mathbb{R})}, \tag{3.19}$$

for some nonnegative constant $c_{\beta,v}$. *Moreover,* $\mathcal{L}$ *has* at least *one negative eigenvalue, and therefore a minimal one, which is simple.*

Proof. The proof of (3.19) is standard. On the other hand, from (2.6), we get after derivation with respect to β,

$$\mathcal{L}_1[\Lambda B] + \mathcal{B}_2[\Lambda B_t] = a'(\beta)(B_{xx} - \sin B) + \frac{1}{2}b'(\beta)B_{tx}, \tag{3.20}$$

and from (2.5),

$$\mathcal{L}_2[\Lambda B_t] + \mathcal{B}_1[\Lambda B] = -a'(\beta)B_t - \frac{1}{2}b'(\beta)B_x. \tag{3.21}$$

Integrating against ΛB and ΛB_t respectively, we get from (3.7),

$$\begin{aligned}\mathcal{Q}[\Lambda B, \Lambda B_t] &= -\frac{1}{2}a'(\beta)\partial_\beta \int_{\mathbb{R}} (B_x^2 + B_t^2 + 1 - \cos B) - \frac{1}{2}b'(\beta)\partial_\beta \int_{\mathbb{R}} B_t B_x\\ &= -a'(\beta)\partial_\beta E[B,B_t] - b'(\beta)\partial_\beta P[B,B_t]\\ &= -32(1+v^2)\beta - 64v^2\beta\\ &= -32(1+3v^2)\beta < 0,\end{aligned}$$

where we have used (2.4) to obtain $a'(\beta) = 2(1+v^2)\beta$ and $b'(\beta) = -8v\beta$. This last identity proves (3.18). □

From the previous result we conclude the following useful identity:

Corollary 3.8. *Let* $(B_0, \tilde{B}_0)$ *denote the following direction*

$$(B_0, \tilde{B}_0) := -\frac{1}{2\beta}(\Lambda B, \Lambda B_t).$$

Then

$$\mathcal{L}[B_0, \tilde{B}_0] = (1+v^2)\begin{pmatrix} \sin B - B_{xx} \\ B_t \end{pmatrix} + 2v \begin{pmatrix} B_{tx} \\ -B_x \end{pmatrix}, \tag{3.22}$$

and

$$-\int_{\mathbb{R}} (B_0, \tilde{B}_0)\cdot \mathcal{L}[B_0, \tilde{B}_0] = -\frac{1}{4\beta^2}\mathcal{Q}[\Lambda B, \Lambda B_t] = \frac{8}{\beta}(1+3v^2) > 0. \tag{3.23}$$

Proof. A direct consequence of (3.20) and (3.21). □

Remark 3.2. The previous result states that $(B_0, \tilde{B}_0)$ is a very good candidate to replace the first eigenfunction of $\mathcal{L}$ coming from Assumption 2 below. This idea, originally coming from Weinstein [33], has been recently used in several works (see [4] for example) where no knowledge of the ground state is at hand.

3.3. **Spectral analysis.** Let $z = z(x)$ and $w = w(x)$ be two functions, to be specified below, and let $(B, B_t)(t, x; x_1, x_2)$ be any breather solution, with parameters x_1, x_2 possibly depending on time. In this section we describe part of the spectrum of the operator $\mathcal{L}$ defined in (3.11)-(3.13). We start with the following result.

Lemma 3.9. $\mathcal{L}$ *is a linear, unbounded operator in* $L^2(\mathbb{R})^2$, *with dense domain* $H^4(\mathbb{R}) \times H^2(\mathbb{R})$. *Moreover,* $\mathcal{L}$ *is self-adjoint.*

The proof of this result is standard. A consequence of the above result is the fact that the spectrum of $\mathcal{L}$ is real valued. Furthermore, $\mathcal{L}$ is a compact perturbation of a constant coefficients operator. Now we look at the kernel of $\mathcal{L}$. Let us fix $t \in \mathbb{R}$, $v \in (-1, 1)$ and $\beta \in (0, \gamma)$. It is not difficult to check that any breather (B, B_t) as in (2.1)-(2.3) can be seen as a function independent of time, by considering

$$\begin{aligned}(B, B_t)(t, x; x_1, x_2) &= (B, B_t)(0, x; \tilde{x}_1, \tilde{x}_2), \quad \tilde{x}_1 := t + x_1,\ \tilde{x}_2 := -vt + x_2,\\ &=: (B, B_t)(x; \tilde{x}_1, \tilde{x}_2).\end{aligned} \tag{3.24}$$

Moreover, it turns out that if[6]

$$(z(x;\tilde{x}_1,\tilde{x}_2), w(x;\tilde{x}_1,\tilde{x}_2), \lambda(\tilde{x}_1,\tilde{x}_2)) \in H^4\times H^2\times \mathbb{R}$$

satisfy the eigenvalue-eigenfunction problem $\mathcal{L}[z,w]=\lambda(z,w)^T$ with (B,B_t) as in (3.24), then the new function

$$(\tilde{z},\tilde{w})(x;\tilde{x}_1,\tilde{x}_2) := (z,w)(x-\tilde{x}_2;\tilde{x}_1,\tilde{x}_2)$$

will satisfy the same spectral equation with (B,B_t) replaced by $(B,B_t)(x;\tilde{x}_1-\tilde{x}_2,0)$. After redefining $\tilde{x}_1$, we can assume without loss of generality that (B,B_t) is only depending on x, β, v and $\tilde{x}_1$, and therefore $(\tilde{z},\tilde{w})$ will also depend on the same variables. We will have $\lambda=\lambda(\beta,v,\tilde{x}_1)$ only. Finally, since the breather was periodic in the variable x_1, with period $2\pi/\alpha$, we can also assume

$$\tilde{x}_1\in\Big[0,\frac{2\pi}{\alpha}\Big], \quad \alpha=\sqrt{\gamma^2-\beta^2}.$$

In conclusion, with no loss of generality we set through this section $(B,B_t)=(B,B_t)(0,x;x_1,0)$, with x_1 lying in the interval $[0,2\pi/\alpha]$ and each breather-like potential periodic on that variable. The same property applies for the functions $(B_1,(B_t)_1)$ and $(B_2,(B_t)_2)$.

This being said, in this paper we will need the following two assumptions:

> **Assumption 1:** (*Nondegeneracy of the kernel*) For each $v\in(-1,1)$, $x_1\in\mathbb{R}$ and $\beta\in(0,\gamma)$, $\ker\mathcal{L}$ is spanned by the two elements $(B_1,(B_t)_1)^T$ and $(B_2,(B_t)_2)^T$; and there is a (uniform in x_1) gap between the kernel and the bottom of the positive spectrum.

> **Assumption 2:** (*Unique, simple negative eigenvalue*) For each $\beta\in(0,\gamma)$, $v\in(-1,1)$ and $x_1\in[0,2\pi\alpha^{-1}]$, the operator $\mathcal{L}$ has a unique simple, negative eigenvalue $\lambda_1=\lambda_1(\beta,v,x_1)<0$ associated to the unit $L^2\times L^2$-norm eigenfunction $(\tilde{B},\hat{B})^T$. Moreover, there is $\lambda_1^0<0$ depending on β and v only, such that $\lambda_1\le\lambda_1^0$ for all x_1.

In Section 4 we perform numerical computations that reveals that both conditions are naturally correct for every set of parameters that we numerically tested. The main consequence of the two preceding assumptions is the following direct coercivity property (recall the quadratic form $\mathcal{Q}$, associated to $\mathcal{L}$, and defined in (3.16)):

Lemma 3.10. *Let $(z,w)\in H^2(\mathbb{R})\times H^1(\mathbb{R})$, and (B,B_t) be any SG breather, and let $(\tilde{B},\hat{B})^T$ be the first eigenfunction from Assumption 2. If (z,w) are such that they satisfy the orthogonality conditions*

$$\int_{\mathbb{R}}(z,w)(B_1,(B_1)_t)^T=\int_{\mathbb{R}}(z,w)(B_2,(B_2)_t)^T=0, \tag{3.25}$$

then there is $\mu_0=\mu_0(\beta,v)>0$ such that

$$\mathcal{Q}[z,w]\ge\mu_0\|(z,w)\|^2_{H^2\times H^1}-\frac{1}{\mu_0}\left|\int_{\mathbb{R}}(z,w)(\tilde{B},\hat{B})^T\right|^2. \tag{3.26}$$

Following a similar strategy as in [4], we must use a different orthogonality condition in order to ensure a good control on the scaling parameter β, in such a form that we can run a stability argument without using modulations, which are very difficult to estimate for breather solutions[7] Indeed, consider the new direction

$$\begin{pmatrix} A\\ \tilde{A}\end{pmatrix} := (1+v^2)\begin{pmatrix}\sin B-B_{xx}\\ B_t\end{pmatrix}+2v\begin{pmatrix}B_{tx}\\ -B_x\end{pmatrix}\in H^2\times H^1. \tag{3.27}$$

Note that from (3.22), we have

$$\mathcal{L}[B_0,\tilde{B}_0]=\begin{pmatrix} A\\ \tilde{A}\end{pmatrix}.$$

[6] Of course, we also have a dependence on β and v, but it is not needed at this moment.

[7] In other words, we control the variations of the scaling parameter β at least at the second order in the error term (z,w).

We must have in mind that $(A,\tilde{A})^T$ is a sort of *generalized negative direction* (see (3.23)), that will replace $(\tilde{B},\hat{B})$ in (3.26). The advantage of taking this new set of orthogonality conditions comes from the fact that $(A,\tilde{A})^T$ are **naturally associated to the energy and momentum conservation laws**, which are only $H^1\times L^2$ based, and therefore, low-regularity conserved quantities compared with the $H^2\times H^1$ global dynamics. Indeed, will prove the following

Proposition 3.11. *Let $(z,w)\in H^2\times H^1$ satisfying the orthogonality conditions (3.25), and $(A,\tilde{A})$ the direction defined in* (3.27). *Then there is $\nu_0=\nu_0(\beta,v)>0$ such that*

$$\mathcal{Q}[z,w]\geq \nu_0\|(z,w)\|_{H^2\times H^1}^2-\frac{1}{\nu_0}\left|\int_{\mathbb{R}}(z,w)(A,\tilde{A})^T\right|^2.$$

Proof. We follow a similar strategy as stated in [4]. It is enough to prove that, under the orthogonality conditions (3.25), and the additional constraint

$$\int_{\mathbb{R}}\begin{pmatrix} z\\ w\end{pmatrix}\cdot\begin{pmatrix} A\\ \tilde{A}\end{pmatrix}=0,$$

we have

$$\mathcal{Q}[z,w]\geq \nu_0\|(z,w)\|_{H^2\times H^1}^2.$$

We write

$$(z,w)=(\tilde{z},\tilde{w})+\delta_0(\tilde{B},\hat{B}),\quad (B_0,\tilde{B}_0)=(b_0,\tilde{b}_0)+\gamma_0(\tilde{B},\hat{B}).$$

Note that

$$\int_{\mathbb{R}}(B_0,\tilde{B}_0)(B_1,(B_t)_1)^T=\int_{\mathbb{R}}(B_0,\tilde{B}_0)(B_2,(B_t)_2)^T=0,$$

and we can assume

$$\int_{\mathbb{R}}(\tilde{z},\tilde{w})(\tilde{B},\hat{B})^T=\int_{\mathbb{R}}(b_0,\tilde{b}_0)(\tilde{B},\hat{B})^T=0.$$

Therefore,

$$\begin{aligned}\mathcal{Q}[z,w]&=\mathcal{Q}[\tilde{z}+\delta_0\tilde{B},\tilde{w}+\delta_0\hat{B}]\\ &=\mathcal{Q}[\tilde{z},\tilde{w}]+\delta_0^2\mathcal{Q}[\tilde{B},\hat{B}]=\mathcal{Q}[\tilde{z},\tilde{w}]-\delta_0^2\lambda_0^2.\end{aligned}\tag{3.28}$$

Now we must replace δ_0 and λ_0 by suitable expressions. First of all,

$$0=\int_{\mathbb{R}}(z,w)(A,\tilde{A})^T=\int_{\mathbb{R}}(z,w)\mathcal{L}[B_0,\tilde{B}_0]=\int_{\mathbb{R}}(B_0,\tilde{B}_0)\mathcal{L}[z,w].$$

Since $\mathcal{L}[z,w]=\mathcal{L}[\tilde{z},\tilde{w}]-\delta_0\lambda_0^2(\tilde{B},\hat{B})^T$ and $(B_0,\tilde{B}_0)=(b_0,\tilde{b}_0)+\gamma_0(\tilde{B},\hat{B})$, we obtain

$$\int_{\mathbb{R}}(b_0,\tilde{b}_0)\mathcal{L}[\tilde{z},\tilde{w}]=\gamma_0\delta_0\lambda_0^2.\tag{3.29}$$

This last expression involves γ_0, δ_0 and λ_0. We want now an expression for γ_0. We have

$$\begin{aligned}\int_{\mathbb{R}}(A,\tilde{A})(B_0,\tilde{B}_0)^T&=\int_{\mathbb{R}}(B_0,\tilde{B}_0)\mathcal{L}[B_0,\tilde{B}_0]\\ &=\int_{\mathbb{R}}\{(b_0,\tilde{b}_0)+\gamma_0(\tilde{B},\hat{B})\}\{\mathcal{L}[b_0,\tilde{b}_0]-\gamma_0\lambda_0^2(\tilde{B},\hat{B})^T\}\\ &=\mathcal{Q}[b_0,\tilde{b}_0]-\gamma_0^2\lambda_0^2.\end{aligned}$$

Note in addition that from (3.22) and (3.23),

$$\int_{\mathbb{R}}(A,\tilde{A})(B_0,\tilde{B}_0)^T=\int_{\mathbb{R}}(B_0,\tilde{B}_0)\mathcal{L}[B_0,\tilde{B}_0]=-\frac{8}{\beta}(1+3v^2).$$

Therefore,

$$\gamma_0^2\lambda_0^2=\mathcal{Q}[b_0,\tilde{b}_0]+\frac{8}{\beta}(1+3v^2).$$

Replacing this last expression in (3.28), and using (3.29), we get

$$\begin{aligned}\mathcal{Q}[z,w] &= \mathcal{Q}[\tilde z,\tilde w] - \frac{(\gamma_0\delta_0\lambda_0^2)^2}{\gamma_0^2\lambda_0^2}\\ &= \mathcal{Q}[\tilde z,\tilde w] - \frac{\Big(\displaystyle\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[\tilde z,\tilde w]\Big)^2}{\mathcal{Q}[b_0,\tilde b_0]+\dfrac{8}{\beta}(1+3v^2)}.\end{aligned} \tag{3.30}$$

Note that the denominator above is **always positive**. If now $(\tilde z,\tilde w)=\lambda(b_0,\tilde b_0)$, for some $\lambda\neq 0$, we have

$$\Big(\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[\tilde z,\tilde w]\Big)^2 = \Big(\int_{\mathbb{R}}(\tilde z,\tilde w)\mathcal{L}[\tilde z,\tilde w]\Big)\Big(\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[b_0,\tilde b_0]\Big) = \mathcal{Q}[\tilde z,\tilde w]\mathcal{Q}[b_0,\tilde b_0],$$

and therefore,

$$\frac{\Big(\displaystyle\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[\tilde z,\tilde w]\Big)^2}{\mathcal{Q}[b_0,\tilde b_0]+\dfrac{8}{\beta}(1+3v^2)} = \frac{\mathcal{Q}[\tilde z,\tilde w]\mathcal{Q}[b_0,\tilde b_0]}{\mathcal{Q}[b_0,\tilde b_0]+\dfrac{8}{\beta}(1+3v^2)} = \rho\mathcal{Q}[\tilde z,\tilde w],$$

with $\rho\in(0,1)$ independent of (z,w). We have then,

$$\mathcal{Q}[z,w]=(1-\rho)\mathcal{Q}[\tilde z,\tilde w].$$

Now if $(\tilde z,\tilde w)$ lies in the orthogonal vector space spanned by $(b_0,\tilde b_0)$, $(\tilde B,\hat B)$, and the kernel $(B_1,(B_1)_t)$, $(B_2,(B_2)_t)$, we have that $\mathcal{Q}[\tilde z,\tilde w]$ defines an internal product, for which the Cauchy-Schwarz's inequality holds:

$$\Big(\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[\tilde z,\tilde w]\Big)^2 < \mathcal{Q}[\tilde z,\tilde w]\mathcal{Q}[b_0,\tilde b_0],$$

(note that there is no equality since $(b_0,\tilde b_0)$ and $(\tilde z,\tilde w)$ are not orthogonal). We conclude now that

$$\frac{\Big(\displaystyle\int_{\mathbb{R}}(b_0,\tilde b_0)\mathcal{L}[\tilde z,\tilde w]\Big)^2}{\mathcal{Q}[b_0,\tilde b_0]+\dfrac{8}{\beta}(1+3v^2)} < \frac{\mathcal{Q}[\tilde z,\tilde w]\mathcal{Q}[b_0,\tilde b_0]}{\mathcal{Q}[b_0,\tilde b_0]+\dfrac{8}{\beta}(1+3v^2)} = \rho\mathcal{Q}[\tilde z,\tilde w],$$

with $\rho\in(0,1)$. We conclude that

$$\mathcal{Q}[z,w] > (1-\rho)\mathcal{Q}[\tilde z,\tilde w]\geq 0.$$

Therefore, in (3.28) we get

$$\mathcal{Q}[\tilde z,\tilde w]\geq \delta_0^2\lambda_0^2.$$

Finally, we have

$$\begin{aligned}\mathcal{Q}[z,w] &> (1-\rho)\mathcal{Q}[\tilde z,\tilde w]\\ &= \frac12(1-\rho)\mathcal{Q}[\tilde z,\tilde w]+\frac12(1-\rho)\mathcal{Q}[\tilde z,\tilde w]\\ &\geq \frac12(1-\rho)\mu_0\|(\tilde z,\tilde w)\|_{H^1\times L^2}^2+\frac12(1-\rho)\delta_0^2\lambda_0^2\\ &\gtrsim \|(\tilde z,\tilde w)\|_{H^2\times H^1}^2+\|(\tilde B,\hat B)\|_{H^2\times H^1}^2\gtrsim \|(z,w)\|_{H^2\times H^1}^2.\end{aligned}$$

□

3.4. **Proof of the Main Theorem.** In this subsection we prove Theorem 2.5. This proof follows similar lines as in [4]. Assume (u_0, u_1) satisfy the hypothesis (2.9), for some $\eta < \eta_0$ small. Let (u, u_t) be the associated $H^2 \times H^1$ solution to (1.1) with initial data (u_0, u_1). Given $K^* > 1$ to be chosen later, we denote $T^* = T^*(K^*) > 0$ as the maximal time for which, for all time $T \in (0, T^*]$,

$$\sup_{t\in[0,T]} \|(u, u_t)(t) - (B, B_t)_{\beta,v}(0, \cdot; \tilde{x}_1(t), \tilde{x}_2(t))\|_{H^2\times H^1} < K^*\eta,$$

is satisfied for some choice of modulation parameters $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, not necessarily unique. For the sake of simplicity, and if no confusion arises, we denote

$$(B, B_t) := (B, B_t)_{\beta,v}.$$

If we assume that $T^*(K^*)$ is finite, we can choose (assuming η_0 smaller if necessary), using the Implicit Function Theorem, special parameters $x_1(t)$ and $x_2(t)$ such that

$$\int_{\mathbb{R}} (z, w)(B_1, (B_1)_t)^T = \int_{\mathbb{R}} (z, w)(B_1, (B_1)_t)^T = 0,$$

where

$$z(t, x) := u(t, x) - B(t, x; x_1(t), x_2(t)), \tag{3.31}$$

$$w(t, x) := u_t(t, x) - B_t(t, x; x_1(t), x_2(t)), \tag{3.32}$$

and B_1, B_2 are given in (3.8)-(3.9). These conditions are well-defined since the matrix with coefficients

$$\begin{pmatrix} \int_{\mathbb{R}} \left[B_1^2 + ((B_1)_t)^2\right] & \int_{\mathbb{R}} B_1B_2 + (B_1)_t(B_2)_t \\ \int_{\mathbb{R}} B_1B_2 + (B_1)_t(B_2)_t & \int_{\mathbb{R}} B_2^2 + ((B_2)_t)^2 \end{pmatrix}$$

has nonzero determinant everywhere (cf. Corollary 3.5). Moreover, thanks to (2.9), we have

$$\|(z, w)(0)\|_{H^2\times H^1} \lesssim \eta,$$

with constant independent of K^*.

Consider now the decomposition (3.31)-(3.32), with (B, B_t) depending on the modulation parameters $x_1(t)$ and $x_2(t)$. A simple argument reveals that $\mathcal{H}[B, B_t](t)$ is still independent of time (see [4] for a similar proof). Therefore we can apply Corollary 2.4 and Appendix A.9 at times $t = 0$ and $t > 0$ fixed to obtain

$$\mathcal{Q}[z, w](t) \leq \mathcal{Q}[z, w](0) + \sup_{t\in[0,T^*]} \mathcal{N}[z, w](t) \lesssim \eta^2 + (K^*)^3\eta^3,$$

Now, from Proposition 3.11 we have

$$\|(z, w)(t)\|^2_{H^2\times H^1} \lesssim \eta^2 + (K^*)^3\eta_0^3 + \left|\int_{\mathbb{R}} (z, w)(A, \tilde{A})^T(t)\right|^2, \tag{3.33}$$

Recall that $(A, \tilde{A})$ was defined in (3.27). Finally, using the energy and momentum conservation laws (1.3)-(1.4), evaluated at two different times, we obtain a good control on the term

$$\left|\int_{\mathbb{R}} (z, w)(A, \tilde{A})^T(t)\right|^2.$$

Indeed, we have

$$\frac{1}{2}\int_{\mathbb{R}} ((B_x + z_x)^2 + (B_t + w)^2)(t) + \int_{\mathbb{R}} (1 - \cos(B + z))(t) = \\ = \frac{1}{2}\int_{\mathbb{R}} ((B_x + z_x)^2 + (B_t + w)^2)(0) + \int_{\mathbb{R}} (1 - \cos(B + z))(0),$$

from which

$$\int_{\mathbb{R}} z(\sin B - B_{xx})(t) + \int_{\mathbb{R}} B_t w(t) = \int_{\mathbb{R}} z(\sin B - B_{xx})(0) + \int_{\mathbb{R}} B_t w(0) \\ + O\Big(\sup_{t\in[0,T^*]} \|(z, w)(t)\|^2_{H^1\times L^2}\Big).$$

Similarly,

$$\int_{\mathbb{R}}(B_x+z_x)(B_t+w)(t)=\int_{\mathbb{R}}(B_x+z_x)(B_t+w)(0),$$

so that

$$\int_{\mathbb{R}}(B_{tx}z-B_xw)(t)=\int_{\mathbb{R}}(B_{tx}z-B_xw)(0)+O\Big(\sup_{t\in[0,T^*]}\|(z,w)(t)\|^2_{H^1\times L^2}\Big).$$

Consequently, reconstructing $(A,\tilde{A})$ as it was defined in (3.27),

$$\begin{aligned}\left|\int_{\mathbb{R}}(z,w)(A,\tilde{A})^T(t)\right| &\lesssim \left|\int_{\mathbb{R}}(z,w)(A,\tilde{A})^T(0)\right| + \sup_{t\in[0,T^*]}\|(z,w)(t)\|^2_{H^2\times H^1}\\ &\lesssim \eta+(K^*)^2\eta^2.\end{aligned}$$

Finally, replacing in (3.33), we get

$$\|(z,w)(t)\|^2_{H^2\times H^1}\lesssim \eta^2+(K^*)^3\eta^3.$$

Taking K^* large enough, and then $\eta(K^*)>0$ small, we have

$$\|(z,w)(t)\|^2_{H^2\times H^1}\le \frac{1}{4}(K^*)^2\eta^2,$$

so we improve the original estimate on (z,w), which contradicts the finiteness of T^*. Therefore, for all K^* large enough, $T^*=+\infty$.

4. Numerical analysis

The purpose of this paragraph is to give enough evidence of the fact that Assumptions 1 and 2 in page 12 do hold. Since we consider only breather-like solutions, which are exponentially decreasing in space, non-exact numerical schemes give good approximations of the resulting exact values.

4.1. **Numerical scheme for SG.** In order to obtain a numerical approximation of the eigenvalues of $\mathcal{L}$ in (3.11), we follow the approach based in the approximate Galerkin method for the Hilbert space $L^2(\mathbb{R})^2$. First, we consider the subspace $\mathbb{V}_N$ generated by a finite dimensional subset of an orthonormal basis of $L^2(\mathbb{R})$:

$$\mathbb{V}_N:=\operatorname{span}\{f_0,f_1,f_2,\dots,f_N\}\subset L^2(\mathbb{R})^2.$$

For practical and computational purposes, the best candidate for an orthonormal basis is the one generated by the Hermite polynomials: for $i=1,2$, $f_n=(f_{n,1},0)$ or $f_n=(0,f_{n,2})$, where

$$f_{n,i}(x):=p_n(x)e^{-x^2/2},\quad x\in\mathbb{R},\quad n\ge 0,\qquad \int_{\mathbb{R}}f_n^2(x)dx=1.$$

Here $p_n(x)$ are the Hermite polynomials [34] suitably normalized such that the f_n have unit norm.

We will approximate the spectrum of $\mathcal{L}$ by the spectrum of the finite dimensional operator

$$\mathcal{L}_N:=\mathbb{P}_N\mathcal{L}\mathbb{P}_N,$$

where $\mathbb{P}_N$ is the projection operator into the subspace $\mathbb{V}_N$. It is expected that, as long as N is large enough, $\mathcal{L}_N$ will approach the discrete spectrum, in particular, the eigenvalues of $\mathcal{L}$, any spectral gap present in the exact operator $\mathcal{L}$, and even the multiplicity of the eigenvalues (seen as two very close approximate eigenvalues).[8]

The above scheme reduces matters to the understanding the eigenvalues of the matrix

$$(\mathcal{M}_N)_{i,j}:=\int_{\mathbb{R}}f_i\mathcal{L}f_j,\quad i,j=0,\dots,N,\quad f_i,f_j\in\mathbb{V}_N.$$

Note that this $(N+1)\times(N+1)$ matrix is symmetric, so its eigenvalues are real-valued. Using some symbolic computations done by Mathematica for some simple cases (periodic mKdV, [7]), we compute $\mathcal{M}_N$ and its eigenvalues. We emphasize that, similarly to the work in [11], each set

[8]Additionally, as far as N is large, the continuous spectrum is also discretely approximated on compact intervals.

of eigenvalues computation took in principle 20-30 minutes, but after some fine code reworking and improvement of the algorithm, we were able to reduce the time of computations to a more realistic value.

For the operator (3.11), it was important to understand the first four eigenvalues, ordered by size. We recovered the two-dimensional kernel, along with the unique negative eigenvalue. Additionally, at each computation performed, we recovered the spectral gap between the kernel and the bottom of the positive spectrum.[9] As it was revealed in the numerical computations, for α large the code becomes unstable, and a larger N is needed to recover the expected results.

First test. Recall that we can always assume $x_2 = 0$ and $x_1 \in [0, 2\pi/\alpha]$. We have run a primary test with $v = 0.7$, different values of β, and x_1 varying between -0.4 and 0.3. It seems that the SG case is numerically simpler than the mKdV one, because of the fact that the SG breather requires one derivative less for its definition. Therefore, with $N = 25$ (that is to say, 50 test functions), we have obtained the results summarized in Fig. 2, in agreement with Assumptions 1 and 2 up to two significative digits. We also check the expected oscillatory behavior of the minimal eigenvalue of the linearized operator $\mathcal{L}$ around a SG breather for variations of the shift x_1. We show this result in Fig. 3 where in that example, we consider $\beta = 0.5$ and $v = 0.7$ and shift parameter x_1 varying from -3.0 to 3.0.

Second test. When β is fixed (e.g. $\beta = 0.5$), and we move the velocity v of the relativistic SG breather from 0.0 to 0.7, we obtain the results written in Fig. 1 (table of eigenvalues). The corresponding graphics are shown in Fig. 4.

v	1st eig.	2nd	3rd	4th		x_1	1st eig.	2nd	3rd	4th
0.0	-0.3932	0.0020	0.0124	0.2783		-0.4	-5.555	0.0004	0.0011	1.658
0.1	-0.4246	0.0023	0.0107	0.2883		-0.3	-5.404	0.0003	0.0012	1.612
0.2	-0.5206	0.0040	0.0069	0.3191		-0.2	-5.290	0.0002	0.0013	1.576
0.3	-0.6871	0.0037	0.0085	0.3698		-0.1	-5.218	0.0001	0.0014	1.553
0.4	-0.9367	0.0023	0.0162	0.4292		0.0	-5.194	0.0001	0.0014	1.545
0.5	-1.2892	0.0026	0.0180	0.5143		0.1	-5.218	0.0001	0.0014	1.553
0.6	-1.7724	0.0067	0.0094	0.6994		0.2	-5.290	0.0002	0.0013	1.576
0.7	-2.4203	0.0036	0.0254	0.9489		0.3	-5.404	0.0003	0.0012	1.612

TABLE 1. The first four eigenvalues of $\mathcal{L}$: (left), for $\beta = 0.5$, $x_1 = 0.1$, $x_2 = 0$, and the velocity v of the relativistic SG breather varying from 0.0 to 0.7, as corresponding to Fig. 4. (right), for $\beta = 0.8$, $v = 0.7$, $x_2 = 0$, and x_1 varying from -0.4 to 0.3, as corresponding to bottom right Fig. 2. All computations were made with $N = 50$. The third and fourth columns (left) and eigth and ninth columns (right) represent the approximate kernel of $\mathcal{L}$ respectively.

Additionally, from the numerical tests we have found the following

(1) For v fixed, as long as β approaches zero, we see that the negative eigenvalue converges to zero, a phenomenon that it is in concordance with the convergence to zero in L^∞ norm of the breather as $\beta \to 0$.

(2) As v increases to the speed of light 1, the quality and accuracy of the resulting numerical results strongly decreases, implying that one needs to take N even larger to recover the desired spectral stability.

In conclusion, our numerical computations reveal that, in any of parameters regime considered, **there is only one negative eigenvalue, as well as a well-defined two dimensional kernel**, and a spectral gap between the kernel and the continuum spectrum, supporting Assumptions 1 and 2 in page 12.

[9]We compared our results with some standard linear operators that have continuous spectrum $[0, +\infty)$ and resonances at the origin, recovering the absence of a spectral gap by having plenty of numerical eigenvalues near zero as N was larger and larger.

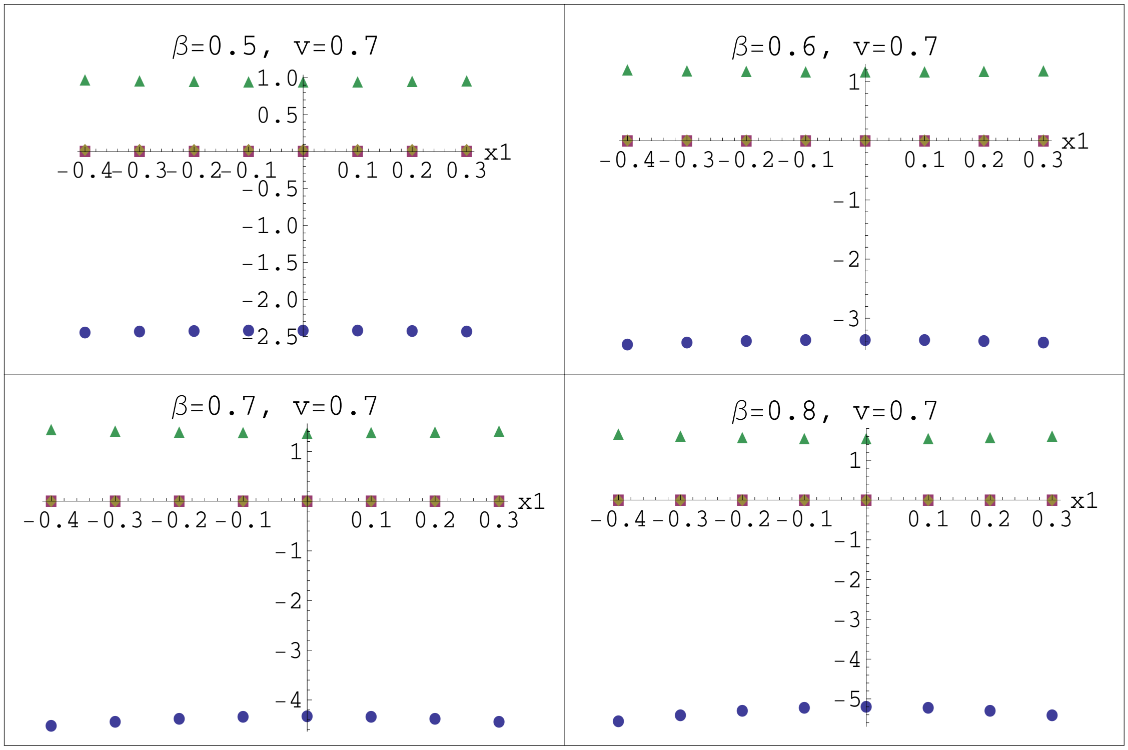

FIGURE 2. The graph of the four minimal eigenvalues (from the first negative eigenvalue(blue circle), to the double zero kernel (purple box and brown diamond) and the first positive eigenvalue (green triangle)) of the SG breather with $\beta = 0.5, 0.6, 0.7$ and 0.8, with x_1 varying between -0.4 and 0.3. The velocity of the relativistic breather is taken as $v = 0.7$. Note that two numerical eigenvalues are placed near zero, see also Fig. 1.

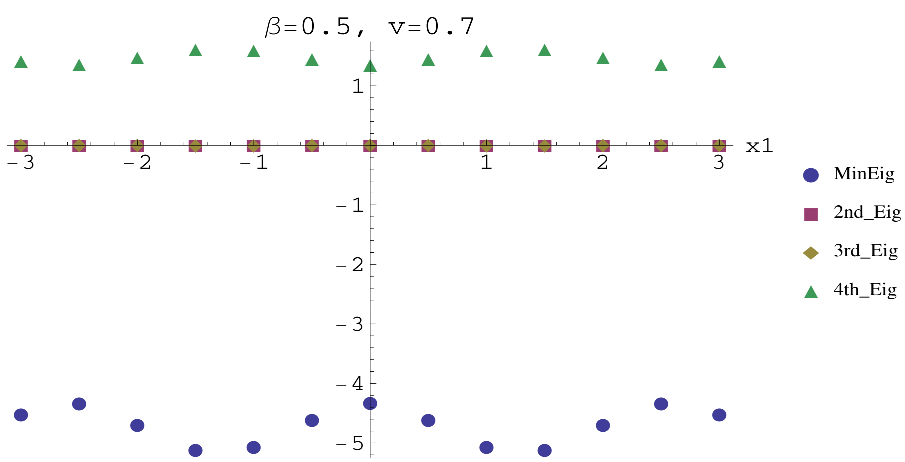

FIGURE 3. The graph of the four minimal eigenvalues of the SG breather case, with $\beta = 0.5,\ v = 0.7$ and with x_1 varying between -3.0 and 3.0.

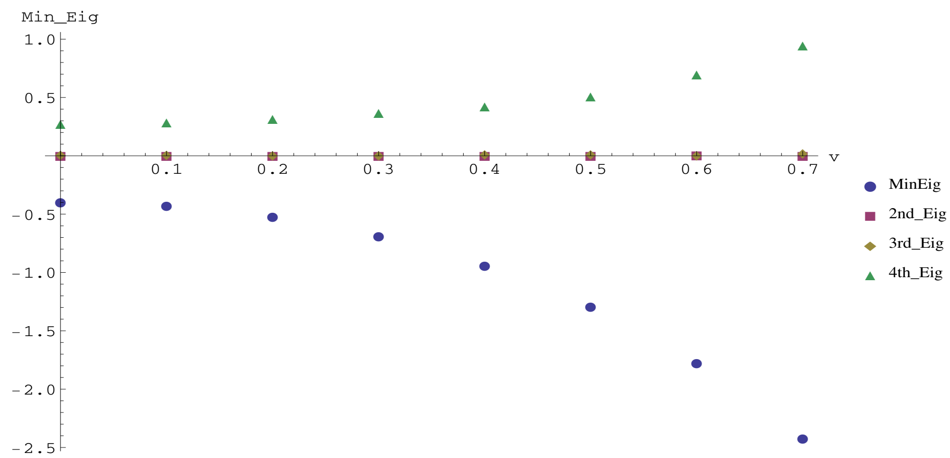

FIGURE 4. The graph of the four minimal eigenvalues of the SG breather with $\beta = 0.5$ and the velocity v of the relativistic breather varying between 0.0 to 0.7. The shifts $x_1 = 0.1,\ x_2 = 0$ and we take $N = 50$ eigenfunctions (see also Figs. 5 and 1, the table below left).

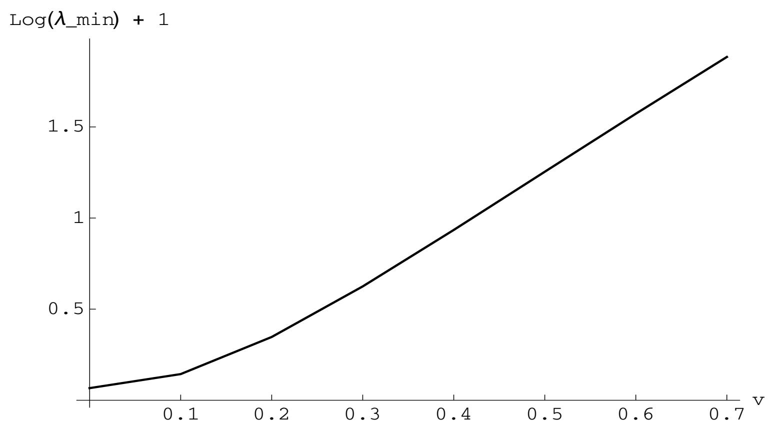

FIGURE 5. The graph of the logarithm of the *absolute* value of the minimal eigenvalue, plus one, for the SG breather with $\beta = 0.5$ and the velocity v of the relativistic breather varying between 0.0 to 0.7. The shifts are $x_1 = 0.1,\ x_2 = 0$. The continuous curve is the set of straight lines that join each pair of points, and the approximate slope among two adjacent points varies from 3.1 to 3.2.

APPENDIX A. PROOF OF (2.5)

We use the specific structure of the breather. From (2.3) and (2.1) we have

$$B_t = -4\alpha\beta\frac{\alpha\sin(\alpha y_1)\cosh(\beta y_2) - \beta v\cos(\alpha y_1)\sinh(\beta y_2)}{\alpha^2\cosh^2(\beta y_2) + \beta^2\cos^2(\alpha y_1)},$$

and

$$B_x = 4\alpha\beta\frac{v\alpha\sin(\alpha y_1)\cosh(\beta y_2) - \beta\cos(\alpha y_1)\sinh(\beta y_2)}{\alpha^2\cosh^2(\beta y_2) + \beta^2\cos^2(\alpha y_1)}. \tag{A.1}$$

From here we have

$$\begin{aligned}\frac{3}{8}B_x^2B_t &= \frac{-24\alpha^3\beta^3}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3}[v\alpha\cosh(\beta y_2)\sin(\alpha y_1)-\beta\sinh(\beta y_2)\cos(\alpha y_1)]^2\times\\ &\quad\times[\alpha\cosh(\beta y_2)\sin(\alpha y_1)-\beta v\sinh(\beta y_2)\cos(\alpha y_1)]\\ &= \frac{-24\alpha^3\beta^3\,h_1(t,x)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3},\end{aligned}$$

with

$$\begin{aligned}h_1(t,x) &:= \alpha^3v^2\cosh^3(\beta y_2)\sin^3(\alpha y_1)-\beta^3v\sinh^3(\beta y_2)\cos^3(\alpha y_1)\\ &\quad -v(2+v^2)\alpha^2\beta\sinh(\beta y_2)\cosh^2(\beta y_2)\sin^2(\alpha y_1)\cos(\alpha y_1)\\ &\quad +(1+2v^2)\alpha\beta^2\sinh^2(\beta y_2)\cosh(\beta y_2)\sin(\alpha y_1)\cos^2(\alpha y_1).\end{aligned}$$

Similarly,

$$\frac{1}{8}B_t^3 = \frac{-8\alpha^3\beta^3\,h_2(t,x)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3},$$

with

$$\begin{aligned}h_2(t,x) &:= \alpha^3\cosh^3(\beta y_2)\sin^3(\alpha y_1)-3\alpha^2\beta v\cosh^2(\beta y_2)\sin^2(\alpha y_1)\sinh(\beta y_2)\cos(\alpha y_1)\\ &\quad +3\alpha\beta^2v^2\cosh(\beta y_2)\sin(\alpha y_1)\sinh^2(\beta y_2)\cos^2(\alpha y_1)-\beta^3v^3\sinh^3(\beta y_2)\cos^3(\alpha y_1).\end{aligned}$$

We compute now the term B_{txx}. From (2.3) we have

$$B_{tx} = \frac{-4\alpha\beta\,h_3(t,x)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^2},$$

where

$$\begin{aligned}h_3(t,x) &:= [\alpha\beta(1-v^2)\sin(\alpha y_1)\sinh(\beta y_2)-v\gamma^2\cos(\alpha y_1)\cosh(\beta y_2)]\times\\ &\quad\times[\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1)]\\ &\quad -2\alpha\beta[\alpha\sin(\alpha y_1)\cosh(\beta y_2)-\beta v\cos(\alpha y_1)\sinh(\beta y_2)]\times\\ &\quad\times[\alpha\sinh(\beta y_2)\cosh(\beta y_2)+v\beta\sin(\alpha y_1)\cos(\alpha y_2)].\end{aligned}\tag{A.2}$$

Similarly, after a long computation,

$$B_{txx} = \frac{-4\alpha\beta\,h_4(t,x)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3},$$

where

$$\begin{aligned}h_4(t,x) &:= [\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1)](h_3)_x\\ &\quad -4\alpha\beta[\alpha\sinh(\beta y_2)\cosh(\beta y_2)+v\beta\sin(\alpha y_1)\cos(\alpha y_2)]h_3.\end{aligned}\tag{A.3}$$

On the other hand, using the well-known formulae

$$\cos(4\theta) = 1-8\sin^2\theta+8\sin^4\theta, \quad \sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta},$$

we have

$$\cos(4\arctan s) = \frac{(s^2-2s-1)(s^2+2s-1)}{(1+s^2)^2} = \frac{s^4-6s^2+1}{(1+s^2)^2}, \quad s\in\mathbb{R}.$$

Therefore, from (2.1) we obtain

$$\cos B = \frac{\beta^4\cos^4(\alpha y_1)-6\alpha^2\beta^2\cosh^2(\beta y_2)\cos^2(\alpha y_1)+\alpha^4\cosh^4(\beta y_2)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^2},\tag{A.4}$$

and thus

$$
\begin{aligned}
-\frac{1}{4}B_t\cos B &= \frac{\alpha\beta}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3}[\alpha\cosh(\beta y_2)\sin(\alpha y_1)-\beta v\sinh(\beta y_2)\cos(\alpha y_1)]\times\\
&\quad\times[\beta^4\cos^4(\alpha y_1)-6\alpha^2\beta^2\cosh^2(\beta y_2)\cos^2(\alpha y_1)+\alpha^4\cosh^4(\beta y_2)]\\
&= \frac{\alpha\beta\, h_5(t,x)}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3},
\end{aligned}
$$

where

$$
\begin{aligned}
h_5(t,x) \quad:=\quad & \alpha\beta^4\sin(\alpha y_1)\cos^4(\alpha y_1)\cosh(\beta y_2)-\alpha^4\beta v\cos(\alpha y_1)\sinh(\beta y_2)\cosh^4(\beta y_2)\\
&+\alpha^5\sin(\alpha y_1)\cosh^5(\beta y_2)-\beta^5 v\cos^5(\alpha y_1)\sinh(\beta y_2)\\
&+6\alpha^2\beta^3 v\cos^3(\alpha y_1)\sinh(\beta y_2)\cosh^2(\beta y_2)-6\alpha^3\beta^2\sin(\alpha y_1)\cos^2(\alpha y_1)\cosh^3(\beta y_2).
\end{aligned}
$$

Collecting the above identities, we get

$$
B_{tx}+\frac{1}{8}B_t^3+\frac{3}{8}B_x^2B_t-\frac{1}{4}B_t\cos B=\alpha\beta\frac{[-4h_4-8\alpha^2\beta^2h_2-24\alpha^2\beta^2h_1+h_5]}{(\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1))^3}. \tag{A.5}
$$

In the following lines, we simplify the numerator in the previous expression. In order to carry out this computation, the key point will be the denominator, denoted by g:

$$
g(t,x):=\alpha^2\cosh^2(\beta y_2)+\beta^2\cos^2(\alpha y_1). \tag{A.6}
$$

First of all, note from (A.2) that h_3 obeys the unique decomposition

$$
h_3=:h_{31}g-h_{32}g_x,\quad g_x=2\alpha\beta[\alpha\sinh(\beta y_2)\cosh(\beta y_2)+v\beta\sin(\alpha y_1)\cos(\alpha y_1)].
$$

Therefore, from (A.3),

$$
\begin{aligned}
h_4 \quad&=\quad g[h_{31}g-h_{32}g_x]_x-2g_x[h_{31}g-h_{32}g_x]\\
&=\quad g^2(h_{31})_x+g[h_{31}g_x-(h_{32}g_x)_x-2h_{31}g_x]+2h_{32}g_x^2\\
&=\quad g^2(h_{31})_x-(h_{31}g_x+(h_{32}g_x)_x)g+2h_{32}g_x^2.
\end{aligned} \tag{A.7}
$$

On the other hand,

$$
\begin{aligned}
-8\alpha^2\beta^2(h_2+3h_1) &= -8\alpha^5\beta^2(1+3v^2)\sin^3(\alpha y_1)\cosh^3(\beta y_2)+8\alpha^2\beta^5v(3+v^2)\cos^3(\alpha y_1)\sinh^3(\beta y_2)\\
&\quad+24\alpha^4\beta^3v(3+v^2)\sin^2(\alpha y_1)\cos(\alpha y_1)\sinh(\beta y_2)\cosh^2(\beta y_2)\\
&\quad-24\alpha^3\beta^4(3v^2+1)\sin(\alpha y_1)\cos^2(\alpha y_1)\sinh^2(\beta y_2)\cosh(\beta y_2).
\end{aligned}
$$

Therefore, after some simplifications,

$$
\begin{aligned}
&h_5-8\alpha^2\beta^2(h_2+3h_1)-8h_{32}g_x^2=\\
&=g\Big[\alpha^3(1-32\beta^2)\sin(\alpha y_1)\cosh^3(\beta y_2)-\alpha^2\beta v(1-32\beta^2)\cos(\alpha y_1)\sinh(\beta y_2)\cosh^2(\beta y_2)\\
&\qquad-\beta^3v(1+32\alpha^2v^2)\cos^3(\alpha y_1)\sinh(\beta y_2)+\alpha\beta^2(1+32\alpha^2v^2)\sin(\alpha y_1)\cos^2(\alpha y_1)\cosh(\beta y_2)\\
&\qquad+24\alpha^3\beta^2(1-v^2)\sin(\alpha y_1)\cosh(\beta y_2)-24\alpha^2\beta^3v(1-v^2)\cos(\alpha y_1)\sinh(\beta y_2)\Big].
\end{aligned} \tag{A.8}
$$

On the other hand, we consider the second term in (A.7). We have

$$
\begin{aligned}
&4(h_{31}g_x+(h_{32}g_x)_x)g=\\
&=8\alpha\beta g\Big[2\alpha^2\beta(2-v^2)\sin(\alpha y_1)\cosh^3(\beta y_2)-3\alpha^2\beta(1-v^2)\sin(\alpha y_1)\cosh(\beta y_2)\\
&\qquad-2\alpha v(\beta^2+\gamma^2)\cos(\alpha y_1)\sinh(\beta y_2)\cosh^2(\beta y_2)+3\alpha\beta^2v(1-v^2)\cos(\alpha y_1)\sinh(\beta y_2)\\
&\qquad\quad+2\alpha\beta^2v(2v^2-1)\cos^3(\alpha y_1)\sinh(\beta y_2)-2\beta v^2(\alpha^2+\gamma^2)\sin(\alpha y_1)\cos^2(\alpha y_1)\cosh(\beta y_2)\Big].
\end{aligned} \tag{A.9}
$$

Collecting the last two estimates, we obtain

$$
(A.8)+(A.9)=g^2[\alpha(1-16\beta^2v^2)\sin(\alpha y_1)\cosh(\beta y_2)-\beta v(1+16\alpha^2)\cos(\alpha y_1)\sinh(\beta y_2)]. \tag{A.10}
$$

We finally add the first term in (A.7), such that

$$\begin{aligned}-4g^2(h_{31})_x+(A.10) \quad &= \quad \alpha\beta g^2[\alpha(1+4\alpha^2v^2-8\beta^2v^2-4\beta^2)\sin(\alpha y_1)\cosh(\beta y_2)\\ &\qquad -v\beta(1+8\alpha^2-4\beta^2+4\alpha^2v^2)\cos(\alpha y_1)\sinh(\beta y_2)].\end{aligned}$$

Comparing with (2.3) and (A.1), in order to obtain (2.5) we have

$$(A.5)=aB_t+\frac{b}{2}B_x,$$

provided a and b are chosen as in (2.4). The proof is complete.

Appendix B. Proof that (B,B_t) is a critical point of the functional $\mathcal{H}$.

Let's $\mathcal{H}[u,u_t]$ in (2.7), which is a real-valued, conserved quantity, well-defined for $H^2\times H^1$-solutions of (1.1). Moreover, if $(z,w)\in H^2(\mathbb{R})\times H^1(\mathbb{R})$ is any pair of functions with sufficiently small $H^2\times H^1$-norm, and (B,B_t) is any breather solution, with corresponding parameters $\beta>0$ and $v\in(-1,1)$, then for all $t\in\mathbb{R}$, one has

$$\mathcal{H}[B+z,B_t+w](t)-\mathcal{H}[B,B_t](t)=\frac{1}{2}\mathcal{Q}[z,w]+\mathcal{N}[z,w], \tag{B.1}$$

with $\mathcal{Q}$ being the quadratic form defined in (3.16), and $\mathcal{N}[z,w]$ a small, nonlinear term satisfying

$$|\mathcal{N}[z,w]|\lesssim p(\|z\|_{H^2(\mathbb{R})},\|w\|_{H^1(\mathbb{R})}),$$

where p is a positive, third order monomial on its corresponding variables. In order to prove that, we compute:

$$\begin{aligned}\mathcal{H}[B+z,B_t+w]=\ &\frac{1}{2}\int_{\mathbb{R}}[(B+z)_{xx}^2+(B_t+w)_x^2]-\frac{1}{32}\int_{\mathbb{R}}[(B_t+w)^4+(B+z)_x^4]\\ &-\frac{3}{16}\int_{\mathbb{R}}(B_t+w)^2(B+z)_x^2+\frac{5}{8}\int_{\mathbb{R}}(B+z)_x^2\cos(B+z)\\ &+\frac{1}{8}\int_{\mathbb{R}}[\sin^2(B+z)+(B_t+w)^2\cos(B+z)]\\ &+\frac{a}{2}\int_{\mathbb{R}}[(B+z)_x^2+(B_t+w)^2]+a\int_{\mathbb{R}}[1-\cos(B+z)]+\frac{b}{2}\int_{\mathbb{R}}(B_t+w)(B+z)_x.\end{aligned}$$

Expanding every term above, as is done in [4], we get

$$\begin{aligned}&\mathcal{H}[B+z,B_t+w]=\mathcal{H}[B,B_t]+\\ &\quad+\int_{\mathbb{R}}\Big[B_{4x}+\frac{3}{8}B_x^2B_{xx}+\frac{3}{8}B_t^2B_{xx}+\frac{5}{8}B_x^2\sin B+\frac{3}{4}B_tB_xB_{tx}-\frac{5}{4}B_{xx}\cos B+\frac{1}{4}\sin B\cos B\\ &\qquad\qquad-\frac{1}{8}B_t^2\sin B-a(B_{xx}-\sin B)-\frac{b}{2}B_{tx}\Big]z\\ &\quad-\int_{\mathbb{R}}\Big[B_{txx}+\frac{1}{8}B_t^3+\frac{3}{8}B_x^2B_t-\frac{1}{4}B_t\cos B-aB_t-\frac{b}{2}B_x\Big]w\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}\Big\{\Big[z_{4x}-[a-\frac{3}{8}(B_x^2+B_t^2)+\frac{5}{4}\cos B]z_{xx}+[\frac{3}{4}B_{xx}B_x+\frac{3}{4}B_tB_{tx}-\frac{5}{4}B_x\sin B]z_x\\ &\qquad+[a\cos B+\frac{1}{4}(\cos^2 B-\sin^2 B)-\frac{5}{8}B_x^2\cos B+\frac{5}{4}B_{xx}\sin B-\frac{1}{8}B_t^2\cos B]z\Big]z\\ &\qquad+[-w_{xx}+\frac{1}{4}(\cos B-\frac{3}{2}(B_x^2+B_t^2))]w-\frac{1}{2}B_t\sin Bwz+(b-\frac{3}{2}B_tB_x)z_xw\Big\}\\ &\quad-\frac{1}{8}\int_{\mathbb{R}}\Big[B_tw^3+B_xz_x^3+\frac{1}{4}(w^4+z_x^4)+3B_twz_x^2+3B_xz_xw^2+\frac{3}{2}w^2z_x^2+\frac{5}{2}\cos Bz^2z_x^2\\ &\qquad+5\sin Bzz_x^2-\frac{1}{4}\sin^2 Bz^4+\sin B\cos Bz^3+\frac{1}{2}\cos Bw^2z^2+B_twz^2+\sin Bzw^2\Big].\end{aligned}$$

Therefore, we have the decomposition

$$\mathcal{H}[B+z,B_t+w]=\mathcal{H}[B,B_t]+\int_{\mathbb{R}}(2.6)z-\int_{\mathbb{R}}(2.5)w+\frac{1}{2}\mathcal{Q}[z,w]+\mathcal{N}[z,w],$$

where $\mathcal{Q}$ is defined in (3.16). Taking into account the Lemma 2.3, the second and third term in the r.h.s of the above equation vanish. Finally, the term $N[z,w]$ is given by

$$\begin{aligned}\mathcal{N}[z,w] \quad := \quad & -\frac{1}{8}\int_{\mathbb{R}}\Big[B_tw^3+B_xz_x^3+\frac{1}{4}(w^4+z_x^4)+3B_twz_x^2+3B_xz_xw^2\\ &+\frac{3}{2}w^2z_x^2+\frac{5}{2}\cos Bz^2z_x^2+5\sin Bzz_x^2-\frac{1}{4}\sin^2Bz^4\\ &+\sin B\cos Bz^3+\frac{1}{2}\cos Bw^2z^2+B_twz^2+\sin Bzw^2\Big].\end{aligned}$$

Therefore, from direct estimates one has $\mathcal{N}[z,w]\lesssim p(\|z\|_{H^2(\mathbb{R})},\|w\|_{H^1(\mathbb{R})})$, where p is a positive, third order polynomial in its variables, and where the constant is independent of the size of (z,w) and the time, provided the former are chosen small.

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