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Finite topology self-translating surfaces for the mean curvature flow in $\mathbb{R}^3$

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Finite topology self-translating surfaces for the mean curvature flow constitute a key element in the analysis of Type II singularities from a compact surface because they arise as limits after suitable blow-up scalings around the singularity. We prove the existence of such a surface $M \subset \mathbb{R}^3$ that is orientable, embedded, complete, and with three ends asymptotically paraboloidal. The fact that M is self-translating means that the moving surface $S(t) = M + te_z$ evolves by mean curvature flow, or equivalently, that M satisfies the equation $H_M = \nu \cdot e_z$ where H_M denotes mean curvature, ν is a choice of unit normal to M, and e_z is a unit vector along the z-axis. This surface M is in correspondence with the classical three-end Costa–Hoffman–Meeks minimal surface with large genus, which has two asymptotically catenoidal ends and one planar end, and a long array of small tunnels in the intersection region resembling a periodic Scherk surface. This example is the first non-trivial one of its kind, and it suggests a strong connection between this problem and the theory of embedded complete minimal surfaces with finite total curvature.

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1. Introduction

We say that a family of orientable, embedded hypersurfaces $S(t)$ in $\mathbb{R}^{n+1}$ evolves by mean curvature if each point of $S(t)$ moves in the normal direction with a velocity proportional to its mean curvature at that point. More precisely, there is a smooth family of diffeomorphisms $Y(\cdot,t) : S(0) \to S(t) \subset \mathbb{R}^{n+1}$, $t > 0$, determined by the mean curvature flow (MCF) equation

$$\frac{\partial Y}{\partial t} = H_{S(t)}(Y)\nu(Y) \tag{1.1}$$

where $H_{S(t)}(Y)$ designates the mean curvature of the surface $S(t)$ at the point $Y(y,t)$, $y \in S(0)$, namely the trace of its second fundamental form, and ν is a choice of unit normal vector.

The mean curvature flow is one of the most important examples of parabolic geometric evolution of manifolds. Relatively simple in form, it generates a wealth of interesting phenomena, which are so far only partly understood. Extensive, deep studies on the properties of this equation have been performed in the last 25 years or so. We refer for instance the reader to the surveys [6] and [32].

A classical, global-in-time definition of a weak solution to mean curvature flow is due to Brakke [3]. These solutions typically develop finite time singularities. When they arise, the evolving manifold loses smoothness, and a change of topology of the surface may occur as the singular time is crossed.

The basic issue of the theory for the mean curvature flow is to understand the way singularities appear and to achieve an accurate description of the topology of the surface obtained after blowing-up the manifold around the singularity.

Singularities are usually classified as types I and II. If T is a time when a singularity appears, type I roughly means that the curvatures grow no faster than $(T-t)^{-\frac{1}{2}}$. In such a case, a blowing-up procedure, involving a time dependent scaling and translation leads in the limit to a "self shrinking" ancient solution, as established by Huisken in [16]. The appearance of these singularities turns out to be generic under suitable assumptions, see Colding and Minicozzi [7].

Instead, if the singularity is not of type I, it is called type II. In that case, a suitable normalization can lead in the limit to an *eternal solution* to the mean curvature flow. See Colding and Minicozzi [5], Huisken and Sinestrari [17,18]. An eternal solution to (1.1) is one that is defined at all times $t \in (-\infty, \infty)$.

The simplest type of eternal solutions are the *self-translating solutions*, which are surfaces that solve (1.1), do not change shape and travel at constant speed in some specific direction. A self-translating solution of the mean curvature flow (1.1), with speed $c > 0$ and direction $\mathsf{e} \in \mathbb{S}^{n+1}$ is a hypersurface of the form

$$S(t) = ct\mathsf{e} + S(0)\,,$$

that satisfies (1.1). Equivalently, such that

$$H_{S(0)} = c\mathsf{e} \cdot \nu. \tag{1.2}$$

When $c = 0$, this problem is just the minimal surface equation. A result by Hamilton [12] states that in the case of a compact convex surface, the limiting scaled singularity does indeed take place in the form of a self-translating solution. This fact makes apparent the importance of eternal self-translating solutions in the understanding of singularity formation. However, the result in [12] is not known without some convexity assumptions. An open, challenging issue is to understand whether or not a given "self-translator" (convex or non-convex) can arise as a limit of a type II singularity for (1.1).

A situation in which strong insight has been obtained is the mean convex scenario (namely, surfaces with non-negative mean curvature, a property that is preserved under the flow). In fact under quite general assumptions, mean convexity in the singular limit becomes full convexity for the blown-up surface, as has been established by B. White [36,37], and by Huisken and Sinestrari [17,18].

In spite of their importance in the theory for the mean curvature flow, relatively few examples of self-translating solutions are known, and a theory for their understanding, even in special classes is still far from achieved. In this direction, Ilmanen [19,20] proved that the genus of a surface is nonincreasing along the mean curvature flow. Therefore, self-translators originated from a singularity in the flow of a compact surface must have finite genus, or **finite topology**. Since for $c = 0$, equation (1.2) reduces to the minimal surface equation, it is natural to look for analogies with minimal surface theory in order to obtain new nontrivial examples.

The purpose of this paper is to construct new examples of self-translating surfaces to the mean curvature flow with finite topology in $\mathbb{R}^3$. More precisely, we are interested in tracing a parallel between the theory of embedded, complete minimal surfaces with finite total Gauss curvature (which are precisely those with finite topology) and self-translators with positive speed. Before stating our main result, we recall some classical examples of self-translators.

If $S(t) = S(0) + cte_{n+1}$ is a **traveling graph**, namely

$$S(0) = \{(x, x_{n+1}) \mid x_{n+1} = F(x),\ x \in \Omega \subset \mathbb{R}^n\}$$

then equation (1.2) reduces to the elliptic PDE for F,

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{c}{\sqrt{1 + |\nabla F|^2}} \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.3}$$

For instance for $n = 1$ and $c = 1$, Grayson [11] gives an explicit solution, the so-called *grim reaper* curve $\mathcal{G}$, given by the graph[1]

[1] It is believed that Matt Grayson coined the phrase *grim reaper*, but the solutions were already known in 1956 by Mullins [26].

$$x_2 = F(x_1) = -\log(\cos x_1), \quad x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{1.4}$$

In other words, $S(t) = \mathcal{G} + te_2$ solves (1.1).

For dimensions $n \geq 2$, there exist entire convex solutions to equation (1.3). Altschuler and Wu [1] found a radially symmetric convex solution $F(|x|)$ to (1.3) by blowing-up a type II singularity of the mean curvature flow. This solution can be obtained explicitly by solving the radial PDE (1.3) which becomes simply

$$\frac{F''}{1+(F')^2} + (n-1)\frac{F'}{r} = c. \tag{1.5}$$

See [2] and [4]. The resulting surface is asymptotically a paraboloid: at main order, when $c = 1$, it has the behavior

$$F(r) = \frac{r^2}{2(n-1)} - \log r + O(r^{-1}) \quad \text{as } r \to +\infty. \tag{1.6}$$

We shall denote by $\mathcal{P}$ the graph of this entire graphical self-translator (which is unique up to an additive constant) which we shall refer to as the **traveling paraboloid**. Of course, this means that $S(t) = \mathcal{P} + te_{n+1}$ solves (1.1).

Xu-Jia Wang [35] proved that for $n = 2$, solutions of (1.3) are necessarily radially symmetric about some point, and in particular, they are convex. Surprisingly, for dimensions $n \geq 3$, Wang was able to construct nonradial convex solutions of (1.3).

In dimension $n+1$, $n \geq 2$, Angenent and Velázquez [2] constructed an axially symmetric solution to (1.1) that develops a type II singularity with a tip that blows-up precisely into the paraboloid $\mathcal{P}$. Also, B. White proved that the convex surface in $\mathbb{R}^{n+1}$ given by the $\mathcal{G} \times \mathbb{R}^{n-1}$ where $\mathcal{G}$ is the grim reaper curve (1.4), cannot arise as a blow-up of a type II singularity for (1.1).

A non-graphical, two-end axially symmetric self translating solutions of (1.1) for $n \geq 2$ has been found by direct integration of the radial PDE (1.3) by Clutterbuck, Schnürer and Schulze [4]. It can be described as follows:

Given any number $R > 0$, there is a self-translating solution of (1.1)

$$S(t) = \mathcal{W} + te_{n+1},$$

where $\mathcal{W}$ is a two-end smooth surface of revolution of the form

$$\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^-, \quad \mathcal{W}^\pm = \{(x, x_{n+1}) \mid x_{n+1} = F^\pm(|x|)\}.$$

Here the functions $F^\pm(r)$ solve (1.5) for $c = 1$ and $r > R$, with $F^-(r) < F^+(r)$ and $F^+(R) = F^-(R)$. It is shown in [4] that the functions $F^\pm$ have the asymptotic behavior (1.6) of $\mathcal{P}$ up to an additive constant. See Fig. 2. We call the two-end translating surface $\mathcal{W}$ the **traveling catenoid**. The reason is natural: when $c = 0$ equation (1.5) is nothing but the minimal surface equation for an axially symmetric minimal surface around the

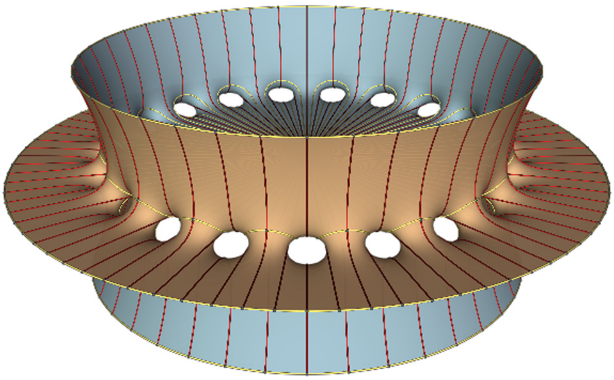

Fig. 1. Costa–Hoffman–Meeks surface (from www.indiana.edu/~minimal/).

x_{n+1}-axis. When $n = 2$ the equation leads (up to translations) to the plane $x_3 = 0$, or the standard catenoid $r = \cosh(x_3)$. The catenoid is exactly the parallel to $\mathcal{W}$. The plane is actually in correspondence with the paraboloid $\mathcal{P}$.

These simple but important examples are the only ones available with finite topology. Examples with infinite topology and periodic in one direction have been constructed by the third author [27–29].

Embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$ The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$ has seen a spectacular development in the last 30 years or so. For about two centuries, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa [8,9]. The Costa surface is a genus one minimal surface, complete and properly embedded, with exactly three components (or ends) outside a large ball. Two of these ends are asymptotically catenoids with the same axis and opposite directions; the third one is asymptotic to a plane perpendicular to that axis. Hoffman and Meeks [13–15] presented a class of three-end, embedded minimal surfaces, which look like the Costa surface far away, but they have an array of tunnels giving arbitrary genus k. These are known as the Costa–Hoffman–Meeks (CHM) surfaces, see Fig. 1. Many other examples of multiple-end embedded minimal surfaces have been found since.

All surfaces of this kind are constituted, away from a compact region, by the disjoint union of ends ordered along one coordinate axis, which are asymptotic to planes or to catenoids with parallel symmetry axes, as established by Osserman [30], Schoen [31] and Jorge and Meeks [21]. Such a surface is thus characterized by the genus of a compact region and the number of ends. Therefore, it has finite topology.

Main result: the traveling CHM surface of large genus In what follows, we restrict ourselves to the case $n + 1 = 3$.

Our purpose is to construct new complete and embedded surfaces in $\mathbb{R}^3$ which are self translating under mean curvature flow. After a rotation and dilation we can assume

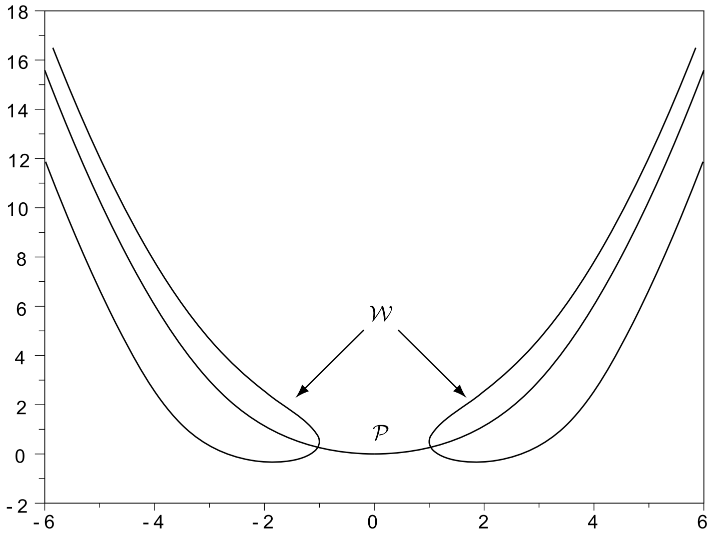

Fig. 2. Traveling paraboloid and catenoid.

that $c = 1$ and that the traveling direction is that of the positive x_3-axis. Thus we look for orientable, embedded complete surfaces M in $\mathbb{R}^3$ satisfying the equation

$$H_M = e_z \cdot \nu, \tag{1.7}$$

where $e_z = e_3$. In other words, the moving surface $S(t) = M + te_z$ satisfies equation (1.1). A major difficulty in extending the theory of finite total curvature minimal surfaces in $\mathbb{R}^3$ space to equation (1.7) is that much of the theory developed relies in the powerful tool given by the Weierstrass representation formula, which is not available in our setting. Unlike the static case, the traveling catenoid for instance is not asymptotically flat and does not have finite total Gauss curvature.

What we establish in our main result is the existence of a three-end surface M that solves (1.7), homeomorphic to a Costa–Hoffman–Meeks surface with large genus, whose ends behave like those of a traveling catenoid and a traveling paraboloid.

More precisely, let us consider the union of a traveling paraboloid $\mathcal{P}$ and a traveling catenoid $\mathcal{W}$, which intersect transversally on a circle C_ρ for some $\rho > 0$. See Fig. 2.

Our surface looks outside a compact set like $\mathcal{P} \cup \mathcal{W}$ in Fig. 2, while near the circle C_ρ the look is that of the static CHM surface in Fig. 1.

Theorem 1.1. *Let $\mathcal{P}$ and $\mathcal{W}$ be respectively a traveling paraboloid and traveling catenoid, which intersect transversally. Then for all $\varepsilon > 0$ small, there is a complete embedded 3-end surface M_ε satisfying equation (1.7), which lies within an ε-neighborhood of $\mathcal{P} \cup \mathcal{W}$. In addition, we have that*

$$\operatorname{genus}(M_\varepsilon) \sim \frac{1}{\varepsilon}.$$

The construction provides much finer properties of the surface M_ε. Let us point out that the CHM with large genus approaches a Scherk singly periodic minimal surface in the multiple-tunnel zone [15]. See Fig. 1.

Kapouleas [22–24], Traizet [33,34] and Hauswirth and Pacard [13] established a method for the reverse operation. Namely, starting with a union of intersecting catenoids and planes, they desingularize them using Scherk surfaces to produce smooth minimal surfaces (complete and embedded). A key element in those constructions is a fine knowledge of the Jacobi operator of the Scherk surface and along the asymptotically flat ends. This approach was used by the third author to construct translating surfaces in $\mathbb{R}^3$ built from a two dimensional picture of intersecting parallel grim reapers and vertical lines, trivially extended in an additional direction, and desingularized in that direction by infinite Scherk surfaces, see [27–29]. We shall use a similar scheme in our construction. The context here is considerably more delicate, since no periodicity is involved (the ultimate reason why the topology resulting is finite), and the fine interplay between the slowly vanishing curvatures and the Jacobi operators of the different pieces requires new ideas. Our method extends to the construction of more general surfaces built upon desingularization of intersection of multiple traveling catenoids and traveling paraboloids, but for simplicity in the exposition we shall restrict ourselves to the basic context of Theorem 1.1. Before proceeding into the detailed proof, we sketch below the core ingredients of it.

1.1. *Sketch of the proof of Theorem 1.1*

After a change of scale of $1/\varepsilon$, the problem is equivalent to finding a complete embedded surface $M \subset \mathbb{R}^3$ that satisfies

$$H_M = \varepsilon\nu \cdot e_z. \tag{1.8}$$

The first step is to construct a surface $\mathcal{M}$ that is close to being a solution to this equation. This is accomplished by desingularizing the union of $\mathcal{P}/\varepsilon$ and $\mathcal{W}/\varepsilon$ using singly periodic Scherk surfaces. At a large distance from C_ρ/ε, the approximation $\mathcal{M}$ is $\mathcal{P}/\varepsilon \cup \mathcal{W}_R/\varepsilon$ and in some neighborhood of C_ρ/ε, it is a slightly bent singly periodic Scherk surface. We call the core of $\mathcal{M}$ the region where the desingularization is made. The actual approximation $\mathcal{M}$ will depend on four real parameters: $\beta_1, \beta_4, \tau_1, \tau_4$, which are going to be small, of order ε.

Let ν denote a choice of unit normal of $\mathcal{M}$. We search for a solution of (1.8) in the form of the normal graph over $\mathcal{M}$ of a function $\phi : \mathcal{M} \to \mathbb{R}$, that is, of the form

$$\mathcal{M}_\phi = \{x + \phi\nu(x) : x \in \mathcal{M}\}.$$

Let H_ϕ and ν_ϕ denote the mean curvature and normal vector of $\mathcal{M}_\phi$, respectively, while H and ν denote those of $\mathcal{M}$. Then

$$H_\phi = H + \Delta\phi + |A|^2\phi + Q_1, \tag{1.9}$$
$$\nu_\phi = \nu - \nabla\phi + Q_2,$$

where Δ is the Laplace–Beltrami operator on $\mathcal{M}$, ∇ is the tangential component of the gradient, and Q_1, Q_2 are quadratic functions in $\phi, \nabla\phi, D^2\phi$. This allows us to write equation (1.8) as

$$\Delta\phi + |A|^2\phi + \varepsilon\nabla\phi\cdot e_z + H - \varepsilon\nu\cdot e_z + Q(x,\phi,\nabla\phi,D^2\phi) = 0 \quad \text{in } \mathcal{M}. \tag{1.10}$$

To solve (1.10), we linearize around $\phi = 0$, and the following linear operator becomes relevant:

$$\mathcal{L}_\varepsilon(\phi) = \Delta\phi + |A|^2\phi + \varepsilon\nabla\phi\cdot e_z.$$

We work with the following norms for functions ϕ, h defined on $\mathcal{M}$, where $0<\gamma<1$, $0<\alpha<1$ are fixed:

$$\|\phi\|_* = \sup_{s(x)\le\delta_s/\varepsilon} e^{\gamma s(x)}\|\phi\|_{C^{2,\alpha}(\overline{B}_1(x))} + \varepsilon^2 \sup_{s(x)>\delta_s/\varepsilon} e^{\gamma\delta_s/\varepsilon+\varepsilon\gamma s(x)}\|\phi\|_{C^{2,\alpha}(\overline{B}_1(x))} \tag{1.11}$$

and

$$\|h\|_{**} = \sup_{s(x)\le\delta_s/\varepsilon} e^{\gamma s(x)}\|h\|_{C^{\alpha}(\overline{B}_1(x))} + \sup_{s(x)>\delta_s/\varepsilon} e^{\gamma\delta_s/\varepsilon+\varepsilon\gamma s(x)}\|h\|_{C^{\alpha}(\overline{B}_1(x))}. \tag{1.12}$$

Here $\delta_s > 0$ is a small fixed parameter. The function $s : \mathcal{M} \to \mathbb{R}$ measures geodesic distance to the core of $\mathcal{M}$ and will be defined precisely later on, and $B_1(x)$ is the geodesic ball centered at x with radius 1.

The term in (1.10) that does not depend on ϕ is

$$E = H - \varepsilon\nu\cdot e_z.$$

We have the following approximation for it.

Proposition 1.2. *E can be decomposed as*

$$E = E_0 + E_d$$

with

$$\|E_0\|_{**} \le C\varepsilon$$

and

$$E_d = \tau_1 w_1 + \tau_4 w_2 + \beta_1 w_1' + \beta_4 w_2' + O\left(\sum_{i=1,4} \beta_i^2 + \tau_i^2\right).$$

The functions w_i, w_i' are defined later in (2.8), (2.9), but along with the ones appearing in $O(\sum_{i=1,4} \beta_i^2 + \tau_i^2)$, they are smooth with compact support. In particular, they all have finite $\| \|_{**}$ norm.

The following claim illustrates the invertibility of the linear operator $\mathcal{L}_\varepsilon$, although it will not be used directly. Let us fix $\mathcal{M}$ by fixing the parameters $\beta_1, \beta_4, \tau_1, \tau_4$ sufficiently small and consider the problem

$$\Delta\phi + |A|^2\phi + \varepsilon\partial_z\phi = h + \sum_{i=1,4} \tilde{\beta}_i w_i' + \tilde{\tau}_i w_i \quad \text{in } \mathcal{M}. \tag{1.13}$$

Then, for $\varepsilon > 0$ small, there is a linear operator $h \mapsto \phi, \tilde{\beta}_i, \tilde{\tau}_i$ that produces for $\|h\|_{**} < \infty$ a solution of (1.13) with

$$\|\phi\|_* + |\tilde{\beta}_1| + |\tilde{\beta}_4| + |\tilde{\tau}_1| + |\tilde{\tau}_4| \le C\|h\|_{**},$$

where C is independent of ε.

Finally, the next result shows that the quadratic term Q in (1.10) is well adapted to the norms (1.11) and (1.12).

Proposition 1.3. *Assume $\phi_i \in C^{2,\alpha}(\mathcal{M})$ $(i = 1, 2)$ and $\|\phi_i\|_* \le 1$. Then, for $\varepsilon > 0$ small,*

$$\|Q(\cdot, \phi_1, \nabla\phi_1, D^2\phi_1) - Q(\cdot, \phi_2, \nabla\phi_2, D^2\phi_2)\|_{**} \le C(\|\phi_1\|_* + \|\phi_2\|_*)\|\phi_1 - \phi_2\|_*,$$

with C independent of ϕ_i and ε.

These results are used to prove Theorem 1.1 by the contraction mapping principle, which is done in Section 6. The preparatory steps are the construction of an initial approximate solution in Section 2 and some geometric computations in Section 3, which lead to the estimate of E in Proposition 1.2 and the estimate of Q in Proposition 1.3. In Section 4, we analyze the Jacobi equation for the Scherk surface and in Section 5.1, we study the Jacobi operator on the ends, which are the regions far from the desingularization.

2. Construction of an initial approximation

The purpose of this section is to construct a surface $\mathcal{M}$ that will serve as an initial approximation to (1.7).

Let F_0 be the unique radially symmetric solution of

$$\frac{F''}{1+(F')^2} + \frac{F'}{r} = 1, \quad F(0) = 0 \tag{2.1}$$

and let $\mathcal{P} \subset \mathbb{R}^3$ be the corresponding surface $z = F_0(r)$. Let $\mathcal{W}$ be a catenoidal self-translating solution of MCF, which can be written as $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^-$ where $\mathcal{W}^\pm$ is given by $z = F^\pm(r)$ and $F^\pm$ satisfies (2.1) for $r > R$, with $F^+(R) = F^-(R)$, $\lim_{r\to R^+}(F^+)'(r) = \infty$, $\lim_{r\to R^+}(F^-)'(r) = -\infty$.

We assume that $\mathcal{P}$ and $\mathcal{W}$ intersect transversally at a unique circle C_ρ of radius $\rho > 0$. To quantify the transversality, we fix a small constant $\delta_\alpha > 0$ so that all the intersection angles are greater than $4\delta_\alpha$. In this section, we are going to replace $\mathcal{P}\cup\mathcal{W}$ in a neighborhood of C_ρ with an appropriately bent Scherk surface. The number of periods used, and thus the number of handles, is of order ε^{-1}. Two of the three ends of the resulting approximate solution will differ slightly from the original ends.

2.1. *Self-translating rotationally symmetric surfaces*

We briefly recall some properties of self-translating rotationally symmetric surfaces. Let $\varepsilon > 0$ be a small constant, let $\gamma(s) = (\gamma_1(s), \gamma_3(s))$, $s \in [0,\infty)$ be a smooth planar curve parametrized by arc length and let $\mathcal{S}$ and $\mathcal{S}_\varepsilon$ be the surfaces of revolution parametrized by

$$\begin{cases} (s,\theta) \mapsto X(s,\theta) := (\gamma_1(ps)\cos(\theta), \gamma_1(ps)\sin(\theta), \gamma_3(ps)) \\ (s,\theta) \mapsto X_\varepsilon(s,\theta) := \varepsilon^{-1}(\gamma_1(\varepsilon ps)\cos(\varepsilon\theta), \gamma_1(\varepsilon ps)\sin(\varepsilon\theta), \gamma_3(\varepsilon ps)), \end{cases} \tag{2.2}$$

where $p = \gamma_1(0)$, and $s \in [0,\infty)$, $\theta \in [0, 2\pi]$. (The reason for introducing p in (2.2) is to make the parametrization conformal at $s = 0$.)

The surface $\mathcal{S}$ ($\mathcal{S}_\varepsilon$ respectively) is a self-translating surface under mean curvature flow with velocity e_z (εe_z respectively) if and only if γ, parametrized by arc length, satisfies the differential equation

$$-\gamma_1''\gamma_3' + \gamma_1'\gamma_3'' + \frac{\gamma_3'}{\gamma_1} - \gamma_1' = 0. \tag{2.3}$$

Another way to represent an axially symmetric self-translating solution is through the graph of a radial function, $z = F(r)$, where F satisfies (2.1) on some interval (R,∞). Then $\varphi = F'$ satisfies

$$\varphi' = (1+\varphi^2)\left(1 - \frac{\varphi}{r}\right). \tag{2.4}$$

Given $R > 0$ and an initial condition $\varphi(R) = \varphi_0 \in \mathbb{R}$, equation (2.4) has unique solution, which is defined for all $r \geq R$, see [4]. All solutions have the common asymptotic behavior

$$\varphi(r) = r - \frac{1}{r} - \frac{2}{r^3} + O\left(\frac{1}{r^5}\right), \quad \varphi'(r) = 1 + \frac{1}{r^2} + O\left(\frac{1}{r^4}\right), \tag{2.5}$$

as $r \to \infty$, see [2,4] (actually an expansion to arbitrary order is possible).

Using $\gamma_1(s) = r(s)$, $\gamma_3(s) = F(r)$, with $s(r) = \int_0^r \sqrt{1+\varphi(t)^2}\,dt$ and the asymptotic behavior (2.5), we can deduce the following estimates.

Lemma 2.1. *For a smooth planar curve $\gamma(s) = (\gamma_1(s), \gamma_3(s))$, $s \in [0,\infty)$ parametrized by arc length with γ_1 and γ_3 satisfying (2.3), we have*

$$\begin{aligned} \gamma_1(s) &= \sqrt{2s} + \frac{1}{2} + o(1) & \qquad \gamma_3(s) &= s + O(\sqrt{s}) \\ \gamma_1'(s) &= \frac{1}{\sqrt{2s}} + o(s^{-1/2}) & \qquad \gamma_3'(s) &= 1 + O(s^{-1/2}) \\ \gamma_1''(s) &= O(s^{-3/2}) & \qquad \gamma_3''(s) &= O(s^{-2}) \end{aligned}$$

as s tends to infinity.

2.2. The Scherk surfaces

Let x, y, z be Euclidean coordinates in $\mathbb{R}^3$ and consider the one parameter family of minimal surfaces $\{\Sigma(\alpha)\}_{\alpha\in(0,\pi/2)}$ given by the equation

$$\cos^2(\alpha)\cosh\left(\frac{x}{\cos\alpha}\right) - \sin^2(\alpha)\cosh\left(\frac{y}{\sin\alpha}\right) - \cos(z) = 0. \tag{2.6}$$

Outside of a large cylinder around the z-axis, $\Sigma(\alpha)$ has four connected components. We call these components the wings of $\Sigma(\alpha)$ and number them according to the quadrant where they lie. Each wing of $\Sigma(\alpha)$ is asymptotic to a half-plane forming an angle α with the xz-plane (note that the asymptotic half-planes do not contain the z-axis unless $\alpha = \pi/4$). Here, we will restrict the parameter α to $[\delta_\alpha, \pi/2 - \delta_\alpha]$ so that the geometry on all the $\Sigma(\alpha)$'s can be uniformly bounded as stated in the following lemma.

Let H^+ be the half-plane $\{(s,z) : s > 0\}$. Note that the parameter s here is on a different scale than the one used in the previous section. We construct approximate solutions satisfying (1.8) here, while the rotationally symmetric surfaces in Section 2.1 satisfy (1.7).

Lemma 2.2. *The surface $\Sigma(\alpha)$ is a singly periodic embedded complete minimal surface which depends smoothly on α. There is a constant $a = a(\delta_\alpha) > 0$ and smooth functions $f_\alpha : H^+ \to \mathbb{R}$ so that the wings of Σ_α can be expressed as the graph of f_α over half-planes. More precisely, the half-plane asymptotic to the first wing can be parametrized by $A^1_\alpha : H^+ \to \mathbb{R}^3$, with*

$$A^1_\alpha(s,z) := (a+s)((\cos\alpha)e_x + (\sin\alpha)e_y) + ze_z + b_\alpha\nu_\alpha,$$

where $b_\alpha = \sin(2\alpha)\log|\cot\alpha|$ and $\nu_\alpha = -(\sin\alpha)e_x + (\cos\alpha)e_y$. The wing itself is parametrized by $F^1_\alpha : H^+ \to \mathbb{R}^3$, which is defined by

$$F^1_\alpha(s,z) := A^1_\alpha(s,z) + f_\alpha(s,z)\nu_\alpha.$$

The functions f_α and F^1_α depend smoothly on α. Moreover, we have

$$\left\| e^s \frac{d^i f_\alpha}{d\alpha^i} \right\|_{C^k(\overline{H^+})} \leq C_{k,i} e^{-a}$$

for any $k, i \in \mathbb{N}$.

The function $f_\alpha(s,z)$ satisfies the minimal surface equation

$$\partial_s \left(\frac{\partial_s f}{\sqrt{1 + (\partial_s f)^2 + (\partial_z f)^2}} \right) + \partial_z \left(\frac{\partial_z f}{\sqrt{1 + (\partial_s f)^2 + (\partial_z f)^2}} \right) = 0, \tag{2.7}$$

for $s > 0$, $z \in \mathbb{R}$.

Definition 2.3. Let us denote by $\mathcal{R}_{yz}$ the reflection across the yz-plane and $\mathcal{R}_{xz}$ the reflection across the xz-plane. The parametrizations of the second, third, and fourth wings are given by

$$F^2_\alpha = \mathcal{R}_{yz} \circ F^1_\alpha, \quad F^3_\alpha = \mathcal{R}_{xz} \circ \mathcal{R}_{yz} \circ F^1_\alpha, \quad F^4_\alpha = \mathcal{R}_{xz} \circ F^1_\alpha.$$

The i**th wing** of $\Sigma(\alpha)$ is given by $F^i_\alpha(H^+)$ and is denoted by $W^i(\alpha)$. The parametrizations of the corresponding asymptotic half-planes are obtained by replacing F^1_α by A^1_α in the above formulas. We use A^i_α to denote the parametrization of the ith asymptotic half-plane as well as its image, $A^i_\alpha(H^+)$. The **inner core** of $\Sigma(\alpha)$ is the surface without its four wings.

Each half-plane $A^i_\alpha(H^+)$ starts close to the boundary of the corresponding wing W^i and intersects neither the xz-plane nor the yz-plane. Each wing and each asymptotic half-plane inherit the coordinates (s,z) from their descriptions in Lemma 2.2 and Definition 2.3.

2.3. Dislocation of the Scherk surfaces

We now perform dislocations on the first and fourth wings of $\Sigma(\alpha)$. These perturbations will help us deal with the kernel of the linear operator $\mathcal{L}_\varepsilon$ associated to normal perturbations of solutions to (1.8). Because translated solutions of (1.8) remain solutions, the functions $e_x \cdot \nu$, $e_y \cdot \nu$, and $e_z \cdot \nu$ are in the kernel of $\mathcal{L}_\varepsilon$. Here we have taken the normal component of the translations because we are considering normal perturbations. The last function, $e_z \cdot \nu$ does not satisfy our imposed symmetries so we can discard it from the kernel. The other two remain. In Section 4, we will show that the Dirichlet problem for the linear operator can be solved on a truncated piece of $\Sigma(\alpha)$, up to constants at the boundary. By adding a linear combination of the functions in the kernel, we can obtain a

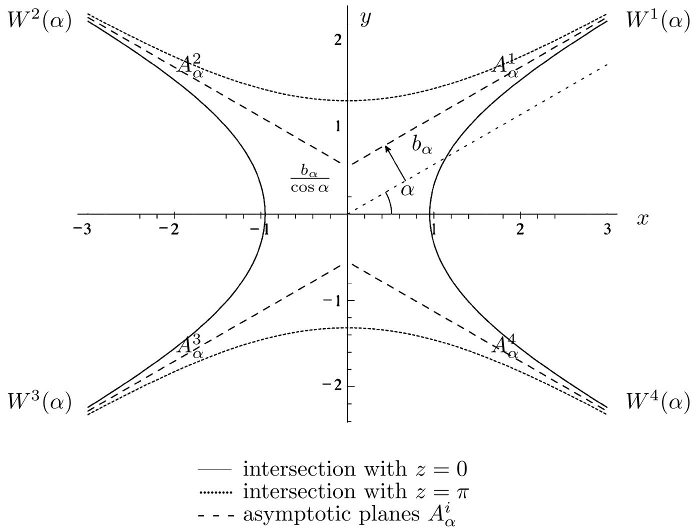

Fig. 3. Sections of the Scherk surface $\Sigma(\alpha)$.

solution that vanishes on the boundary of two adjacent wings, say the second and third wings. To obtain a solution that vanishes on all the connected pieces of the boundary, we will artificially translate the first and fourth wing by constants τ_1 and τ_4.

The linear operator $\mathcal{L}_\varepsilon$ is close to linear operator $L := \Delta + |A|^2$ associated to the equation $H = 0$, so we have small eigenvalues due to changes of the Scherk angle and rotation. Because there is a one parameter family of Scherk surfaces, we expect a function in the kernel of the Jacobi operator L, namely, the normal component of the motion associated to changing the angle α. One more dimension is generated by rotation of the Scherk surfaces around the z-axis. To summarize, besides the translations, we have two more dimensions in the kernel of L generated by linear functions along the wings. This is reasonable since L is close to the Laplace operator along the wings. By adding a linear combination of these two linear eigenfunctions, we can force exponential decay along the second and third wings again. As before, we will generate linear functions on the first and fourth wings through rotations by angles β_1 and β_4 respectively.

Definition 2.4. For $\beta \in \mathbb{R}$, we define the map $Z_\beta : \mathbb{R}^3 \to \mathbb{R}^3$ to be the rotation of angle β (counterclockwise in the xy-plane) around the z-axis:

$$Z_\beta(x, y, z) = (\cos(\beta)x - \sin(\beta)y, \sin(\beta)x + \cos(\beta)y, z).$$

In what follows, we will confine β to $(-\delta_p, \delta_p)$, where $\delta_p > 0$ is a small fixed number.

We consider two constants $R_{rot} > 10$, $R_{tr} > R_{rot} + 10$, and a family of smooth transition functions $\eta_b : \mathbb{R} \to \mathbb{R}$ such that $0 \le \eta_b(s) \le 1$, $\eta_b(s) = 0$ for $s < b$, and $\eta_b(s) = 1$ for $s > b + 1$. The numbers R_{rot}, R_{tr} will be fixed later to be large.

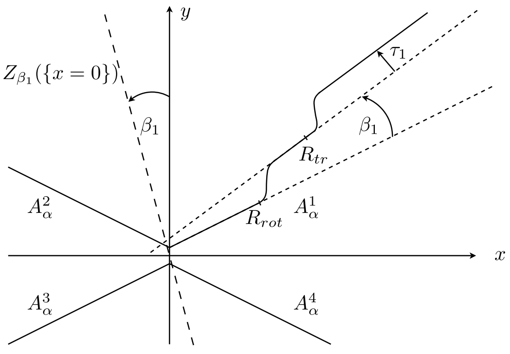

Fig. 4. Dislocations in wing 1.

Given $\alpha \in [\delta_\alpha, \pi/2 - \delta_\alpha]$, $\beta_1, \beta_4, \tau_1, \tau_4 \in (-\delta_p, \delta_p)$, we modify the first and fourth wings in the following way: the ith wing is shifted by τ_i at around $s = R_{tr}$, then it is rotated by an angle β_i at distance $s = R_{rot}$. The parametrization of the new ith wing, for $i = 1, 4$, is given by $F^i[\alpha, \beta_i, \tau_i] : H^+ \to \mathbb{R}^3$, where

$$F^1[\alpha,\beta_1,\tau_1](s,z) = (1-\eta_{R_{rot}}(s))F^1_\alpha(s,z) + \eta_{R_{rot}}(s) Z_{\beta_1}(F^1_\alpha(s,z) + \tau_1 \eta_{R_{tr}}(s)\nu_\alpha)$$

$$F^4[\alpha,\beta_4,\tau_4](s,z) = \mathcal{R}_{xz} \circ F^1[\alpha,\beta_4,\tau_4](s,z),$$

and $\mathcal{R}_{xz}$ is the reflection across the xz-plane. Note that the ith wing is moved away from the x-axis for positive constants β_i and τ_i. We denote the new wings by $W^i[\alpha,\beta_i,\tau_i] := F^i[\alpha,\beta_i,\tau_i](H^+)$, $i = 1, 4$ (see Fig. 4).

The wings have natural coordinates (s, z) given by the parametrizations F^1 and F^4. The surface $\Sigma'[\alpha,\beta_1,\beta_4,\tau_1,\tau_4]$ (or Σ' for short) is defined to be the union of the inner core of $\Sigma(\alpha)$ with the four wings $W^2(\alpha)$, $W^3(\alpha)$, $W^1[\alpha,\beta_1,\tau_1]$, and $W^4[\alpha,\beta_4,\tau_4]$. We will call the region of Σ' for which $s \in [0, R_{tr} + 10]$ the **outer core**.

Remark 2.5. The maps $F^i_\alpha \circ (F^i[\alpha,\beta_i,\tau_i])^{-1}$ and $F^i[\alpha,\beta_i,\tau_i] \circ (F^i_\alpha)^{-1}$, $i = 1, 4$ can be used to pullback tensors defined on $W^i(\alpha)$ to $W^i[\alpha,\beta_i,\tau_i]$ and vice versa: in the case of a function f defined on $W^i[\alpha,\beta_1,\tau_1]$, the composition $f \circ F^i[\alpha,\beta_i,\tau_i] \circ (F^i_\alpha)^{-1}$ is the corresponding pullback function on $W^i(\alpha)$. Taking each wing at a time, these maps transport functions and tensors between Σ' and $\Sigma(\alpha)$. This is very useful as it lets us work on a fixed surface, usually $\Sigma(\alpha)$. We will use the same notation for functions and tensors on Σ' or their pullback to $\Sigma(\alpha)$. For example, $H_{\Sigma'}$ could denote the mean curvature of Σ' as a function on Σ' or its pullback to $\Sigma(\alpha)$. The same notation convention applies to the unit normal vector ν, the metric g, and the second fundamental form A.

Let us define the following functions on $\Sigma(\alpha)$, which capture the contribution of the dislocations to the mean curvature:

$$w_1 := \frac{d}{d\tau_1}\Big|_{\beta_i=\tau_i=0} H_{\Sigma'}, \quad w_2 := \frac{d}{d\tau_4}\Big|_{\beta_i=\tau_i=0} H_{\Sigma'}, \tag{2.8}$$

$$w_1' := \frac{d}{d\beta_1}\Big|_{\beta_i=\tau_i=0} H_{\Sigma'}, \quad w_2' := \frac{d}{d\beta_4}\Big|_{\beta_i=\tau_i=0} H_{\Sigma'}. \tag{2.9}$$

These functions are compactly supported because rotations and translations do not change the mean curvature. They will later help us solve the Dirichlet problem associated to the Jacobi operator $\Delta + |A|^2$ on the Scherk surfaces in Section 4.

Because the parameters β_i are associated to rotations, the functions w_1' and w_2' can be written explicitly as the Jacobi operator on the normal component of rotation at a point $(x, y, z) \in \Sigma(\alpha)$:

$$\begin{cases} w_1'(x,y,z) = (\Delta_{\Sigma(\alpha)} + |A_{\Sigma(\alpha)}|^2)(\eta_{rot,1}\nu_{\Sigma(\alpha)} \cdot (-y, x, 0)), \\ w_2'(x,y,z) = (\Delta_{\Sigma(\alpha)} + |A_{\Sigma(\alpha)}|^2)(\eta_{rot,4}\nu_{\Sigma(\alpha)} \cdot (-y, x, 0)), \end{cases} \tag{2.10}$$

where $\eta_{rot,1}(s)$ is defined as $\eta_{R_{rot}}(s)$ on wing 1 and zero elsewhere and similarly for $\eta_{rot,4}$. We also have,

$$\begin{cases} w_1 = (\Delta_{\Sigma(\alpha)} + |A_{\Sigma(\alpha)}|^2)(\eta_{tr,1}), \\ w_2 = (\Delta_{\Sigma(\alpha)} + |A_{\Sigma(\alpha)}|^2)(\eta_{tr,4}), \end{cases} \tag{2.11}$$

on $\Sigma(\alpha)$, where $\eta_{tr,1}(s) = \eta_{R_{tr}}$ on wing 1 and zero elsewhere, and similarly for $\eta_{tr,4}$.

2.4. Wrapping the dislocated Scherk surfaces around a circle

We first rotate our new surface Σ' so that its second and third wings match the directions of two chosen pieces of catenoid or paraboloid coming out of the intersection circle. The wrapping is performed by simply using a smooth map from a tubular neighborhood of the z-axis to a neighborhood of a large circle. The scaling factor is ε^{-1} so our target circle will have a radius of order ε^{-1}.

Definition 2.6. For $\varepsilon > 0$ and $\varrho > 0$, we define

$$B_{\varepsilon,\varrho}(x, y, z) = (\varepsilon^{-1}\varrho + x)(\cos(\varepsilon\varrho^{-1}z), \sin(\varepsilon\varrho^{-1}z), 0) + (0, 0, y).$$

This map takes a segment of length $2\pi\varepsilon^{-1}\varrho$ on the z-axis to the circle of radius $\varepsilon^{-1}\varrho$.

We can not wrap the whole surface Σ', so we cut its four wings at $s = R_{tr} + 10$ and denote the new surface by $\bar{\Sigma}'$, with a "bar" on top to indicate that it has a boundary. Our desingularizing surface is a dislocated rotated wrapped Scherk surface

$$\bar{\Sigma} := B_{\varepsilon,\rho_\varepsilon} \circ Z_\beta(\bar{\Sigma}'), \tag{2.12}$$

where the angle β has yet to be fixed and ρ_ε is the closest number in $\varepsilon\mathbb{Z}$ to ρ (the radius associated to the original intersection C_ρ).

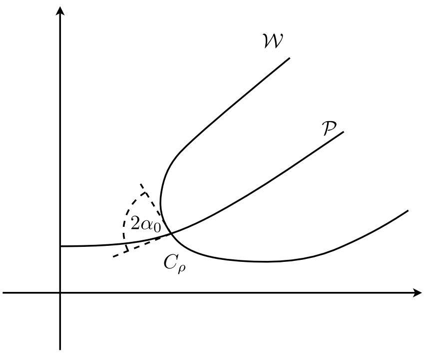

Fig. 5. Paraboloid $\mathcal{P}$ and catenoid $\mathcal{W}_R$ with transversal intersection at C_ρ.

We wish to prolong the wings of the desingularizing surface $\bar{\Sigma}$ with pieces of self-translating catenoids or paraboloids. At this point, it will be useful to record the boundary, not of the surface $\bar{\Sigma}$ itself, but of the asymptotic plane underneath at $s = R_{tr} + 10$. We will extend the asymptotic pieces first, then construct the approximate surface by adding the graph of f_α.

2.5. Fitting the Scherk surface

It is now time to examine the initial configuration in detail. We will work with cross-sections in the xz-plane. Let C_ρ denote the intersection of the paraboloid and catenoid and let $\alpha_0 \in [\delta_\alpha, \pi/2 - \delta_\alpha]$ be half of the angle of intersection between the top $\mathcal{W}_R$ and the inner part of $\mathcal{P}$ (see Fig. 5).

On the bounded part of the paraboloid, we take two points P and P' at distances $(a + b_{\alpha_0} \cot\alpha_0 + R_{tr} + 20)\varepsilon$ and $(a + b_{\alpha_0} \cot\alpha_0 + R_{tr} + 10)\varepsilon$ from C_ρ respectively and consider the half-line starting at P' tangent to the paraboloid and pointing to C_ρ. Recall that a was chosen in Lemma 2.2 and that the term $a + b_{\alpha_0} \cot\alpha_0$ is present because the distance from the intersection of the two (extended) asymptotic planes $A^2_\alpha(\mathbb{R}^2)$ and $A^3_\alpha(\mathbb{R}^2)$ to the line $A^2_\alpha(\{s = 0\})$ on is $a + b_\alpha \cot\alpha$ by Lemma 2.2. The new object $\tilde{\mathcal{P}}$ is formed by the paraboloid up to P, a smooth interpolating curve from P to P', and the tangent half-line after P' (see Fig. 6). We will also denote its corresponding surface of revolution by $\tilde{\mathcal{P}}$. We do a similar construction with the catenoid $\mathcal{W}$ and denote the new object $\tilde{\mathcal{W}}$ (see Fig. 6).

For $\varepsilon > 0$ small, the curves $\tilde{\mathcal{P}}$ an $\tilde{\mathcal{W}}$ intersect at a point $C_{\tilde{\rho}}$. We choose the angle α of the Scherk surface and the angle β of the rotation such that the lines A^2_α, A^3_α are parallel to the segments $C_{\tilde{\rho}}Q'$ and $C_{\tilde{\rho}}P'$ respectively. Note that $\alpha = \alpha_0(1+O(\varepsilon))$ and we did not dislocate the second and third wings of the Scherk surface, so α and β do not depend on β_1, β_2, τ_1, or τ_2.

Because we have approximated our original curves $\mathcal{W}$ and $\mathcal{P}$ up to first order, the new intersection point $C_{\tilde{\rho}}$ is at distance $O(\varepsilon^2)$ from C_ρ. By the same reasoning, the distance

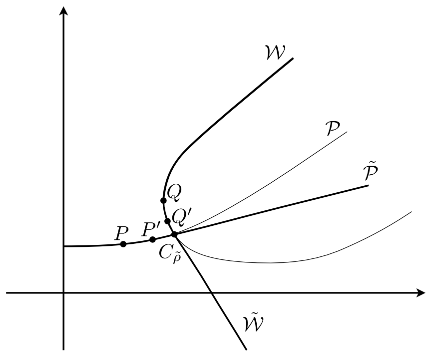

Fig. 6. Step in the construction.

from $C_{\tilde{\rho}}$ to either P' or Q' is $(a + b_{\alpha_0} \cot \alpha_0 + R_{tr} + 10)\varepsilon + O(\varepsilon^2)$. Combining with the estimate on α, we have

$$|C_{\tilde{\rho}}Q'|, |C_{\tilde{\rho}}P'| = (a + b_\alpha \cot \alpha + R_{tr} + 10)\varepsilon + O(\varepsilon^2). \tag{2.13}$$

We have to adjust the scale so that the dislocated bent Scherk surface $\bar{\Sigma}$ given in (2.12) fits around the circle of radius $\tilde{\rho}$ and so that its second and third asymptotic half-planes contain part of the line segments of $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{P}}$ respectively. Considering the image of the z-axis under $\varepsilon B_{\varepsilon,\rho_\varepsilon} \circ Z_\beta$ would be a mistake because in general, the second and third asymptotic planes do not meet there. Instead, we look at the image under $\varepsilon B_{\varepsilon,\rho_\varepsilon} \circ Z_\beta$ of the line $(b_\alpha / \sin \alpha, 0, z)$ (see Fig. 3) and obtain a circle of radius $\rho' = \rho_\varepsilon + \varepsilon \frac{b_\alpha \cos \beta}{\sin \alpha} = \rho_\varepsilon(1 + O(\varepsilon)) = \rho(1 + O(\varepsilon))$. This is the desired radius.

We had to wrap our Scherk surface around a circle of radius ρ_ε to get an embedded surface, so now we adjust the scale by defining $\lambda_\varepsilon = \frac{\tilde{\rho}}{\rho'} = 1 + O(\varepsilon^2)$. This function is not continuous in ε and the jumps occur when the number of periods of the desingularizing surface increases. We take $\lambda_\varepsilon \varepsilon \bar{\Sigma}$ and shift it vertically so that the asymptotic cone associated with the second wing matches the cone generated by the straight part of $\tilde{\mathcal{W}}$ on an open set. The cone associated to the third wing aligns automatically with the cone of $\tilde{\mathcal{P}}$ by our choice of α and β. We record the amount of vertical displacement with the constant d_ε and denote the shifted surface by $\lambda_\varepsilon \varepsilon \bar{\Sigma}^\uparrow$.

The scaled surface $\lambda_\varepsilon \varepsilon \bar{\Sigma}^\uparrow$ has a boundary at $s = R_{tr} + 10$. We wish to extend the underlying asymptotic cones with pieces of catenoidal ends that will match the cones in a C^1 manner. We take the curve γ_2 to be just a parametrization of $\mathcal{W}$ and γ_3 to be the curve generating the inner part of $\mathcal{P}$. Thanks to the estimate (2.13), the curves γ_2 and γ_3 match two of the underlying asymptotic cones to $\lambda_\varepsilon \varepsilon \bar{\Sigma}^\uparrow$ in a C^1 manner at some $s \in (R_{tr} + 9, R_{tr} + 11)$ if ε is small enough (the exact value of s is different for each wing). For $i = 1$ and 4, we consider the circle on the ith asymptotic cone of $\lambda_\varepsilon \varepsilon \bar{\Sigma}^\uparrow$ corresponding to $s = R_{tr} + 10$ and the tangent unit vector to the cone perpendicular to this circle, pointing away from the core. This gives us an initial position and velocity for the unique curve $\gamma_i = (\gamma_{i,1}, \gamma_{i,3}) : [0, \infty) \to \mathbb{R}^2$, $i = 1, 4$, generating a rotationally self-translating surface. The curves γ_1 and γ_4 are perturbations of sections of the original paraboloidal and lower catenoidal ends. We can assume without loss of generality that the γ_i's are parametrized by arclength.

The surface $\mathcal{M}$, which is an approximate solution to (1.8) is defined in the following way. In the region $s \geq R_{tr}+20$ the i-th wing of $\mathcal{M}$ is taken as a graph over the rotationally symmetric surface generated by γ_i. More precisely, let

$$X_i(t,\theta) := \varepsilon^{-1}(\gamma_{i,1}(\varepsilon pt)\cos(\varepsilon\theta), \gamma_{i,1}(\varepsilon pt)\sin(\varepsilon\theta), \gamma_{i,3}(\varepsilon pt)),$$

for $t \geq 0$, $\theta \in [0, 2\pi/\varepsilon]$, where $p = \gamma_1(0)$. The factor p is to make the parametrization conformal at $t = 0$, which we will take to be $s = R_{tr} + 20$. The unit normal vector is

$$\nu(t,\theta) = (-\gamma_{i,3}'(\varepsilon pt)\cos(\varepsilon\theta), -\gamma_{i,3}'(\varepsilon pt)\sin(\varepsilon\theta), \gamma_{i,1}'(\varepsilon pt)).$$

We parametrize the i-th wing of $\mathcal{M}$ in the region $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$ by

$$(s,\theta) \mapsto X_i(s - (R_{tr} + 20), \theta) + u(s,\theta)\nu(s - (R_{tr} + 20), \theta)$$

where the function u is given by

$$u(s,\theta) = pf_\alpha(s,\theta)\eta(\varepsilon s),$$

where f_α is as in Lemma 2.2 and η is a cut-off function satisfying $\eta(s) = 1$ for $s \leq 4\delta_s$ and $\eta(s) = 0$ for $s \geq 5\delta_s$. For $s \geq 5\delta_s/\varepsilon$, the surface $\mathcal{M}$ is the union of the four pieces of rotationally symmetric self-translating surfaces generated by the graphs of γ_i's.

In the region $R_{tr}+9 \leq s \leq R_{tr}+20$ we smoothly interpolate the previous parametrization for $s \geq R_{tr} + 20$ with the corresponding one for $s \leq R_{tr} + 9$, where the surface can be written as the graph of a function over a cone.

Lemma 2.7. *There exists a constant $\delta_p > 0$ depending only on δ_α so that the surface $\mathcal{M}[\varepsilon, \beta_1, \beta_4, \tau_1, \tau_4]$ is embedded for $\beta_1, \beta_4, \tau_1, \tau_4 \in (-\delta_p, \delta_p)$ and $\varepsilon \in (0, \delta_p)$. Moreover, $\mathcal{M}$ depends smoothly on β_1, β_4, τ_1, and τ_4. It also depends smoothly on ε, except on a countable set.*

Proof. The only point that needs an argument is that the unbounded ends do not intersect for all $\beta_1, \beta_4, \tau_1, \tau_4$ small. This is a consequence of the following observation: consider two solutions $\varphi_i = \varphi_i(r)$, $i = 1, 2$ of (2.4) defined for all $r \geq R$ with initial conditions

$$\varphi_i(R) = \varphi_{i,0}, \quad \varphi_{1,0} > \varphi_{2,0}.$$

By uniqueness of solutions to ODE,

$$\varphi_1(r) > \varphi_2(r) \quad \forall r \geq R.$$

Let $F_i(r)$ be such that $F_i' = \varphi_i$ and $F_1(0) = F_2(0)$. It follows that

$$F_1(r) > F_2(r) + m \quad \forall r \geq R,$$

with $m > 0$ and m remains positive if $\varphi_{1,0} - \varphi_{2,0}$ remains positive. $\square$

2.6. Summary of notation and terminology

We start by recalling the roles of the different parameters:

- ε controls the overall scale and the error in the construction.
- s is the distance to the inner core if we are on an underlying asymptotic surface. It is roughly the distance to the inner core if we are on $\mathcal{M}$.
- ρ_ε is the closest number in $\varepsilon\mathbb{Z}$ to ρ, which is the radius of the intersection of the paraboloid and catenoid in the original scale.
- α is the angle associated to the original Scherk surface (see Fig. 3).
- β_1 and β_4 are the angles of rotation of the first and fourth wings respectively.
- τ_1 and τ_4 are the amount by which the first and fourth wings are translated (along the normal of the asymptotic plane to the respective wing).

Then we have the different scaled and bent Scherk surfaces:

- $\Sigma(\alpha)$ is the original minimal Scherk surface given by (2.6). Its wings $W^i(\alpha)$ are asymptotic to the half-planes $A^i_\alpha(H^+)$.
- $\Sigma' = \Sigma'[\alpha, \beta_1, \beta_4, \tau_1, \tau_4]$ is $\Sigma(\alpha)$ with dislocations (see Fig. 4).
- $\bar{\Sigma} := B_{\varepsilon,\rho_\varepsilon} \circ Z_\beta(\bar{\Sigma}')$ is Σ' wrapped around a circle of radius $\rho_\varepsilon/\varepsilon$.
- $\lambda_\varepsilon\varepsilon\bar{\Sigma}^\uparrow$ is the previous Scherk surface scaled by a factor $\varepsilon\lambda_\varepsilon = \varepsilon(1+O(\varepsilon^2))$ and shifted by d_ε in the $\vec{e}_z$ direction so that it fits the configuration in Fig. 6.

Finally, let us recall the names of the different parts of the initial approximation $\mathcal{M} = \mathcal{M}[\varepsilon, \beta_1, \beta_4, \tau_1, \tau_4]$:

- The inner core is where the handles are.
- The middle core is where we perform all the dislocations. It corresponds to the region $0 \leq s \leq R_{tr} + 10$.
- The region $s \in (R_{tr} + 9, R_{tr} + 20)$ is a transition region called the outer core. Note that there is a change of scale in this region from $\lambda_\varepsilon\varepsilon$ to ε but this will not create too much error as the switch is contained in a bounded region with s small compared to ε^{-1}.
- The core is the union of the inner, middle, and outer core.
- In the region $\{s \in (R_{tr} + 20, 4\delta_s/\varepsilon)\}$, $\mathcal{M}$ is the graph of f_α over the rotationally symmetric surface generated by γ_i, $i = 1, \ldots, 4$.

- The region $\{s \in (4\delta_s/\varepsilon, 5\delta_s/\varepsilon)\}$ is a transition region where we cut-off the graph of f_α. We have to do it far enough so that the function f_α is small and the length of the interval has to be long enough so we don't create too much error.
- The wings are defined to be the region $\{s \in (0, 5\delta_s/\varepsilon)\}$.
- For $\{s \geq 5\delta_s/\varepsilon\}$, $\mathcal{M}$ is just the rotationally symmetric surface generated by the curves γ_i's.

3. Geometric computations

In this section, we perform computations related to the ansatz $\mathcal{M}$. In particular, we prove Proposition 1.2, which gives the error. We also use these computations to prove Proposition 1.3 for the quadratic terms appearing in the expansion of the mean curvature and normal vector.

3.1. Perturbation by normal graphs

Consider a surface M immersed in $\mathbb{R}^3$ with local parametrization of class C^2:

$$X : U \subset \mathbb{R}^2 \to M, \qquad X = X(x_1, x_2).$$

We use the notation

$$e_i = \partial_i X = \partial_{x_i} X$$

for tangent vectors and we take the normal unit vector to be

$$\nu = \frac{e_1 \times e_2}{|e_1 \times e_2|},$$

where $\times$ is the cross product in $\mathbb{R}^3$. The metric of M is denoted by

$$g_{ij} = \langle e_i, e_j \rangle,$$

and its inverse by g^{ij}, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^3$. We recall that

$$\partial_i \nu = -A_i^{\,j} e_j$$

where we used Einstein's convention of summation over repeated indices, and $A_i^{\,j}$ is the second fundamental form, which can be computed as

$$A_i^{\,j} = A_{ik} g^{kj}, \quad A_{ij} = \langle X_{ij}, \nu \rangle = -\langle e_i, \partial_j \nu \rangle.$$

The mean curvature H of M is given by

$$H = \text{trace of } A = A_1{}^1 + A_2{}^2 = g^{ij}\langle X_{ij}, \nu\rangle.$$

Consider a function $u \in C^2(M)$. We write

$$u_i = \partial_{x_i} u, \quad u_{ij} = \partial_{x_i x_j} u.$$

Let $\tilde{X} = X + \nu u$ be the graph of u over M. We then have

$$\tilde{e}_i = \partial_{x_i}\tilde{X} = e_i + u_i\nu - uA_i{}^k e_k$$

and

$$\tilde{g}_{ij} = \langle \tilde{e}_i, \tilde{e}_j\rangle = g_{ij} - u(A_j{}^l g_{il} + A_i{}^k g_{kj}) + u^2 A_i{}^k A_j{}^l g_{kl} + u_i u_j.$$

We compute the cross product

$$\begin{aligned}\tilde{e}_1 \times \tilde{e}_2 &= e_1 \times e_2(1 - uH - u^2 G) + u_1\nu \times e_2 + u_2 e_1 \times \nu \\ &\quad - uu_1 A_2{}^l \nu \times e_l - uu_2 A_1{}^k e_k \times \nu,\end{aligned}$$

where $G = A_1{}^1 A_2{}^2 - A_1{}^2 A_2{}^1$ is the Gauss curvature. We also compute the second derivatives of $\tilde{X}$:

$$\begin{aligned}\tilde{X}_{ij} &= \partial_j(e_i + \nu_i u + \nu u_i) \\ &= e_{ij} + u_{ij}\nu - u_i A_j{}^l e_l - u_j A_i{}^k e_k - u(A_i{}^k)_j e_k - uA_i{}^k e_{kj}.\end{aligned}$$

The mean curvature of M_u is given by

$$\tilde{H} = \tilde{g}^{ij}\left\langle \tilde{X}_{ij}, \frac{\tilde{e}_1 \times \tilde{e}_2}{|\tilde{e}_1 \times \tilde{e}_2|}\right\rangle.$$

Explicitly, the scalar product is

$$\begin{aligned}&\langle \tilde{e}_1 \times \tilde{e}_2, \tilde{X}_{ij}\rangle \\ &\quad = \langle e_1 \times e_2(1 - uH - u^2 G) + u_1\nu \times e_2 + u_2 e_1 \times \nu - uu_1 A_2{}^l \nu \times e_l - uu_2 A_1{}^k e_k \times \nu, \\ &e_{ij} + u_{ij}\nu - u_i A_j{}^l e_l - u_j A_i{}^k e_k - u(A_i{}^k)_j e_k - uA_i{}^k e_{kj}\rangle\end{aligned}$$

with $\det\tilde{g} = |\tilde{e}_1 \times \tilde{e}_2|^2$ and

$$\tilde{g}^{-1} = \frac{1}{\det\tilde{g}}\begin{bmatrix}\tilde{g}_{22} & -\tilde{g}_{12} \\ -\tilde{g}_{12} & \tilde{g}_{11}\end{bmatrix}.$$

3.2. Norms of tensors

We work with the following norms, which are independent of coordinates.

Definition 3.1. The pointwise norm of a tensor ${T_{a_1\ldots a_r}}^{b_1\ldots b_s}$ is given by

$$|T|^2 := {T_{a_1\ldots a_r}}^{b_1\ldots b_s}{T_{c_1\ldots c_r}}^{d_1\ldots d_s}g^{a_1c_1}\cdots g^{a_rc_r}g_{b_1d_1}\cdots g_{b_sd_s},$$

with summation over repeated indices.

We use the notation ∇_i to denote the covariant derivative with respect to $\frac{\partial}{\partial x_i}$. In the case where the metric is diagonal, the norms of the gradient and Hessian of a function u are

$$|\nabla u|^2 = |u_iu_jg^{ij}| = \frac{(u_1)^2}{g_{11}} + \frac{(u_2)^2}{g_{22}},$$
$$|\nabla^2 u|^2 = (g^{11}\nabla_{11}u)^2 + (g^{22}\nabla_{22}u)^2 + 2g^{11}g^{22}(\nabla_{12}u)^2.$$

For the second fundamental form, we have

$$|A|^2 = (g^{11}A_{11})^2 + (g^{22}A_{22})^2 + 2(g^{11}g^{22}A_{12}A_{12}),$$
$$|\nabla A|^2 = \sum_{i,j,k=1}^{2}(\nabla_i {A_j}^k)^2 g^{ii}g^{jj}g_{kk}.$$

3.3. Geometry of rotationally symmetric self-translating surfaces

We compute various geometric quantities attached to the parametrization X_ε given in (2.2). We use ∂_i or $(\ )_i$ to denote regular differentiation with respect to the variables s $(i = 1)$ or θ $(i = 2)$. Let $\{e_1, e_2\}$ be the tangent vectors to $\mathcal{S}_\varepsilon$ given by

$$e_1 = \partial_1 X_\varepsilon = p(\gamma_1'(\varepsilon ps)\cos(\varepsilon\theta), \gamma_1'(\varepsilon ps)\sin(\varepsilon\theta), \gamma_3'(\varepsilon ps)), \tag{3.1}$$

$$e_2 = \partial_2 X_\varepsilon = (-\gamma_1(\varepsilon ps)\sin(\varepsilon\theta), \gamma_1(\varepsilon ps)\cos(\varepsilon\theta), 0). \tag{3.2}$$

We recall that $(\gamma_1(s), \gamma_3(s))$ is parametrized by arc length and that $p := \gamma_1(0) > 0$. The associated metric is then

$$g_{11} = p^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = \gamma_1^2. \tag{3.3}$$

The only nonzero Christoffel symbols are

$$\Gamma_{22,1} = -\varepsilon p\gamma_1\gamma_1', \quad \Gamma_{12,2} = \Gamma_{21,2} = \varepsilon p\gamma_1\gamma_1',$$
$$\Gamma_{22}^1 = -\varepsilon\frac{\gamma_1}{p}\gamma_1', \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \varepsilon\frac{p}{\gamma_1}\gamma_1'.$$

Using $A_{ij} = \langle \partial_i e_j, \nu \rangle$, we obtain the coordinates of the second fundamental form

$$\begin{cases} A_{11} = -\varepsilon p^2(-\gamma_1''\gamma_3' + \gamma_1'\gamma_3''), A_{12} = 0, \qquad A_{22} = \varepsilon\gamma_1\gamma_3' \\ A_1{}^1 = -\varepsilon(-\gamma_1''\gamma_3' + \gamma_1'\gamma_3'') \quad A_1{}^2 = A_2{}^1 = 0\, A_2{}^2 = \varepsilon\gamma_1^{-1}\gamma_3' \end{cases} \tag{3.4}$$

where all the functions are taken at εps.

The following proposition is an immediate corollary of (3.4) and the growth of γ_1, γ_3 given in Lemma 2.1.

Proposition 3.2. *In the coordinates given by X_ε, the second fundamental form A and Christoffel symbols on $\mathcal{S}_\varepsilon$ satisfy*

$$\left| \frac{d^k}{ds^k} A_i{}^j(\varepsilon ps) \right| \le C\varepsilon^{k+1}, \tag{3.5}$$

$$|\Gamma_{ij}^k| \le C\varepsilon. \tag{3.6}$$

Proposition 3.3. *We have*

$$|\nabla^k A| \le C\varepsilon^{k+1}, \quad k = 0, 1, 2. \tag{3.7}$$

Proof. The fact that $|A|^2 \le C\varepsilon^2$ is straightforward from (3.4), (3.3) and Proposition 3.2. For the first covariant derivative of A, we recall $\nabla_k A_i{}^j = \partial_k A_1{}^j - \Gamma_{ki}^l A_l{}^j + \Gamma_{kl}^j A_i{}^l$ with implied summation over $l = 1, 2$. Upon inspection, $\nabla_1 A_1{}^2$ vanishes. If $(i, j, k) \ne (1, 2, 1)$, the quantity $\sqrt{g^{kk} g^{ii} g_{jj}}$ is bounded, therefore

$$\begin{aligned} \sqrt{g^{kk} g^{ii} g_{jj}} |\nabla_k A_i{}^k| &\le C(|\partial_k A_1{}^j| + |\Gamma_{ki}^l A_l{}^j| + |\Gamma_{kl}^j A_i{}^l|) \\ &\le C\varepsilon^2 \end{aligned}$$

and the estimate for $|\nabla A|$ is proved.

For the second covariant derivative, we argue similarly. Recall that $\nabla_{lk}^2 A_i{}^j = \partial_l(\nabla_k A_i{}^j) - \Gamma_{lk}^m \nabla_m A_i{}^j - \Gamma_{li}^m \nabla_k A_m^j + \Gamma_{lm}^j \nabla_k A_i{}^m$. Note that $\nabla_{11}^2 A_1{}^2 = 0$. As before, if $(i, j, k, m) \ne (1, 2, 1, 1)$, the product $\sqrt{g^{kk} g^{ii} g_{jj} g^{ll}}$ is bounded and we prove $|\nabla^2 A| \le C\varepsilon^3$ by combining (3.5), (3.6), and (3.7) for $k = 0, 1$. □

3.4. Computation of the error

Proof of Proposition 1.2. The initial approximation $\mathcal{M}$ consists of three parts. The core of $\mathcal{M}$ is a smooth perturbation of a compact piece of Scherk minimal surface, when we consider one period only. This introduces curvatures of order ε, together with some dislocations, so the statement of Proposition 1.2 follows directly for the error restricted to this part.

The region $s \geq 5\delta_s/\varepsilon$ of $\mathcal{M}$ is a rotationally symmetric self-translating surface, so E is zero there.

Where $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$, the surface $\mathcal{M}$ is a graph over a self-translating rotationally symmetric surface $\mathcal{S}_\varepsilon$. We parametrize $\mathcal{S}_\varepsilon$ with X_ε defined in (2.2), which we will write for convenience as X, so that

$$X(s,\theta) := \varepsilon^{-1}(\gamma_1(\varepsilon ps)\cos(\varepsilon\theta), \gamma_1(\varepsilon ps)\sin(\varepsilon\theta), \gamma_3(\varepsilon ps)),$$

where $p = \gamma_1(0)$, and $s \in [0, \infty)$, $\theta \in [0, 2\pi]$.

Then $\mathcal{M}$ in this region is parametrized by

$$(s,\theta) \mapsto \tilde{X}(s,\theta) = X(s,\theta) + u(s,\theta)\nu, \tag{3.8}$$

where the function u is given by

$$u(s,\theta) = pf_\alpha(s,\theta)\eta(\varepsilon s), \tag{3.9}$$

with f_α given in Lemma 2.2 and η is a cut-off function satisfying $\eta(s) = 1$ for $s \leq 4\delta_s$ and $\eta(s) = 0$ for $s \geq 5\delta_s$. We observe that

$$|e^s \partial^k u(s,\theta)| \leq Ce^{-a}, \quad k = 0, \ldots, 5,$$

and that $f_\alpha(s,\theta)$ satisfies the minimal surface equation (2.7).

In the rest of this proof, g, A, H, ν denote the metric, second fundamental form, mean curvature and Gauss map of the rotationally symmetric surface $\mathcal{S}_\varepsilon$ and $\tilde{g}, \tilde{A}, \tilde{H}, \tilde{\nu}$ the ones of $\mathcal{M}$ given by the parametrization above.

In the rest of this section, we shall work in the region $s \leq 5\delta_s/\varepsilon$. Following the calculations of Section 3.1 with the parametrization (3.8) and using (3.7) we have

$$\begin{aligned}
\tilde{e}_1 &= (1 - u{A_1}^1)e_1 - u{A_1}^2 e_2 + u_1\nu = e_1 + \nu u_1 + O(\varepsilon e^{-s}), \\
\tilde{e}_2 &= -u{A_2}^1 e_1 + (1 - u{A_2}^2)e_2 + u_2\nu = e_2 + \nu u_2 + O(\varepsilon e^{-s}),
\end{aligned}$$

where $O(\varepsilon e^{-s})$ is in the C^2 sense on the region $s \leq 5\delta_s/\varepsilon$ as $\varepsilon \to 0$, and e_1, e_2 are given in (3.1), (3.2). We compute the metric $\tilde{g}_{ij}$:

$$\tilde{g}_{ij} = g_{ij} - 2uA_{ij} + u^2{A_i}^l{A_j}^m g_{lm} + u_i u_j = g_{ij} + u_i u_j + O(\varepsilon e^{-s}). \tag{3.10}$$

Using (3.3), we have

$$\tilde{g}^{-1} = \frac{1}{\det(\tilde{g})}\begin{pmatrix} \gamma_1^2 + u_2^2 & -u_1u_2 \\ -u_1u_2 & p^2 + u_1^2 \end{pmatrix} + O(\varepsilon e^{-s}),$$

where γ_1 is evaluated at $p\varepsilon s$, and

$$\det(\tilde{g}) = p^2\gamma_1^2 + \gamma_1^2 u_1^2 + p^2 u_2^2 + O(\varepsilon e^{-s}). \tag{3.11}$$

For the normal direction, we recall that e_1 and e_2 are orthogonal and obtain

$$\tilde{e}_1 \times \tilde{e}_2 = \det(g)^{1/2}\nu - u_1 e_1 - u_2 e_2 + O(\varepsilon e^{-s}). \tag{3.12}$$

Next, we compute

$$\tilde{X}_{ij} = e_{ij} + u_{ij}\nu + O(\varepsilon e^{-s})$$

and

$$\tilde{A}_{ij} = \langle \tilde{X}_{ij}, \tilde{\nu} \rangle = \frac{\det(g)^{1/2}}{\det(\tilde{g})^{1/2}}(A_{ij} + u_{ij}) + O(\varepsilon e^{-s}).$$

Since $\frac{\det(g)}{\det(\tilde{g})} = 1 + O(e^{-s})$ and A_{ij} are $O(\varepsilon)$, we get

$$\tilde{A}_{ij} = A_{ij} + u_{ij} + O(\varepsilon e^{-s}). \tag{3.13}$$

With this, the second fundamental form can be expressed as

$$\begin{aligned}
\tilde{A}_1{}^1 &= \frac{1}{\det(\tilde{g})}\left[(A_{11} + u_{11})(\gamma_1^1 + u_2^2) - (A_{12} + u_{12})u_1 u_2\right] + O(\varepsilon e^{-s}) \\
\tilde{A}_2{}^2 &= \frac{1}{\det(\tilde{g})}\left[-(A_{21} + u_{21})u_1 u_2 + (A_{22} + u_{22})(p^2 + u_1^2)\right] + O(\varepsilon e^{-s}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{H} = {} & \frac{1}{\det(\tilde{g})}\left[A_{11}\gamma_1^2 + A_{22}p^2 + u_{11}(\gamma_1^2 + u_2^2) - 2u_{12}u_1u_2 + u_{22}(p^2 + u_1^2)\right] \\
& + O(\varepsilon e^{-s}).
\end{aligned}$$

Let $\bar{u} = \frac{1}{p}u = f_\alpha(s,\theta)\eta(\varepsilon s)$. We recall that $H = \frac{1}{\det(g)}\left[A_{11}\gamma_1^2 + A_{22}p^2\right]$ and expand $\gamma_1(p\varepsilon s) = p + O(\varepsilon s)$ in the region $s \leq 5\delta_s/\varepsilon$ to get

$$\tilde{H} = H + \frac{p^3}{\det(\tilde{g})}\left[\bar{u}_{11}(1 + \bar{u}_2^2) - 2\bar{u}_{12}\bar{u}_1\bar{u}_2 + \bar{u}_{22}(1 + \bar{u}_1^2)\right] + O(\varepsilon s e^{-s})$$

Because $\eta(\varepsilon s) = 1$ for $s \leq 4\delta_s/\varepsilon$ and f_α satisfies (2.7), we actually have

$$\bar{u}_{11}(1 + \bar{u}_2^2) - 2\bar{u}_{12}\bar{u}_1\bar{u}_2 + \bar{u}_{22}(1 + \bar{u}_1^2) = 0$$

in this region. Thus we obtain

$$\tilde{H} = H + O(\varepsilon s e^{-s}), \quad s \leq 4\delta_s/\varepsilon.$$

Also in this region, from (3.11), (3.12) and $|\tilde{e}_1 \times \tilde{e}_2| = \det(\tilde{g})^{1/2}$, we have $\varepsilon\tilde{\nu} \cdot e_z = \varepsilon\nu \cdot e_z + O(\varepsilon e^{-s})$ and since S_ε is a self translating surface, we get

$$\tilde{H} - \varepsilon\tilde{\nu} \cdot e_z = O(\varepsilon e^{-\gamma s}).$$

The same estimate holds for the derivatives of $\tilde{H} - \varepsilon\tilde{\nu} \cdot e_z$ (all the parametrizations here are smooth), which implies the corresponding estimate in the weighted C^α norm.

When $4\delta_s/\varepsilon \leq s \leq 5\delta_s/\varepsilon$, we have $|\partial^k u(s)| \leq Ce^{-s} \leq \varepsilon e^{-\gamma s}$ for ε small enough, so

$$\tilde{H} = H + O(\varepsilon e^{-\gamma s}), \quad \varepsilon\tilde{\nu} \cdot e_z = \varepsilon\nu \cdot e_z + O(\varepsilon e^{-\gamma s}),$$

and the desired estimate holds. □

3.5. Estimate of the Jacobi operator

Here we use the following notation: The metric, Christoffel symbols, and second fundamental forms on a rotationally symmetric piece of self-translating surface are denoted by g, Γ^k_{ij}, and A, the corresponding quantities for the ansatz are $g_{\mathcal{M}}$, $\Gamma^k_{ij,\mathcal{M}}$, and $A_{\mathcal{M}}$ while the ones on the corresponding original Scherk surface are g_Σ, $\Gamma^k_{ij,\Sigma}$, and A_Σ. For short, we write $\Delta_{g_{\mathcal{M}}} = \Delta_{\mathcal{M}}$, $\Delta_{g_\Sigma} = \Delta_\Sigma$. In the following proposition, the operators on $\mathcal{M}$ (the left-hand side) are pulled back to Σ using the transformations of Section 2.

Proposition 3.4. *For $s \leq 5\delta_s/\varepsilon$, we have*

$$\Delta_{\mathcal{M}} + |A_{\mathcal{M}}|^2 + \varepsilon\langle e_z, \nabla_{g_{\mathcal{M}}}\rangle = \Delta_\Sigma + |A_\Sigma|^2 + L'$$

where L' is a second order differential operator with coefficients with C^1 norm bounded by $O(\delta_s + \delta_p + \varepsilon)$.

Proof. We again divide into several regions. For $s \leq R_{tr} + 20$, $\mathcal{M}$ is obtained from the Scherk surface by a bending, which introduces terms of order ε and dislocations of order δ_p, so the estimate for L' here follows.

When $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$, $\mathcal{M}$ can be described by the parametrization (3.8). In this region, we express all geometric quantities of $\mathcal{M}$ and Σ as functions of the coordinates s and θ.

We note that $\Delta_{\mathcal{M}} - \Delta_\Sigma$ is a second order operator with coefficients whose C^1 norm can be estimated from the C^2 norm of $g_{\mathcal{M}} - g_\Sigma$. The ansatz is the graph of u from (3.9), so to be consistent, we take the Scherk surface parametrized by $(s, \theta) \mapsto (ps, p\theta, pf_\alpha(s, \theta))$. Then (3.10) gives

$$g_{ij,\Sigma} - g_{ij,\mathcal{M}} = p^2\delta_{ij} + p^2 f_{\alpha,i} f_{\alpha,j} - (g_{ij} + u_i u_j) + O(\varepsilon e^{-s}).$$

We use (3.3), $\|u\|_{C^3} \leq C$, and the expansion $\gamma_1(p\varepsilon s) = p + O(\varepsilon s)$ to deduce

$$\|g_{ij,\Sigma} - \bar{g}_{ij,\mathcal{M}}\|_{C^2} \leq C(\delta_s + \varepsilon),$$

where the norm is computed over $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$. Because the metrics are uniformly equivalent, in order to bound $|A_{\mathcal{M}}|^2 - |A_\Sigma|^2$, it suffices to control $|A_{ij,\mathcal{M}}(s,\theta) - A_{ij,\Sigma}(s,\theta)|$. By (3.13),

$$A_{ij,\Sigma} - A_{ij,\mathcal{M}} = pf_{ij} - (u_{ij} + A_{ij}) + O(\varepsilon e^{-s}),$$

where $O(\varepsilon e^{-s})$ is in C^1 norm. The functions u and f are equal if $s \leq 4\delta_s/\varepsilon$ and we have $e^{-s} \leq C\varepsilon$ otherwise. Moreover, $A_{ij} = O(\varepsilon)$ when $s \leq 5\delta_s/\varepsilon$ by (3.4). Therefore,

$$\|A_{ij,\Sigma} - A_{ij,\mathcal{M}}\|_{C^1} \leq C\varepsilon, \tag{3.14}$$

where the norm is over $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$.

Finally, the term $\varepsilon\langle e_z, \nabla_{g_{\mathcal{M}}}\rangle$ has coefficients of order ε in C^1 norm over $R_{tr} + 20 \leq s \leq 5\delta_s/\varepsilon$. $\square$

3.6. Estimates of the quadratic terms

Here, we prove Proposition 1.3 for functions defined on the surface $\mathcal{M}$. We recall from (3.14) and Proposition 3.3 that each $|\nabla^i A|$ remains uniformly bounded on $\mathcal{M}$, for $i = 0, 1, 2$.

Let Q_1 be defined by (1.9) and assume $|uA| < 1$. For this computation it is convenient to work with coordinates that are normal at a certain point $x_0 \in \mathcal{M}$, which means

$$g_{ij}(x_0) = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}(x_0) = 0.$$

This implies

$$\langle X_{ij}, e_k\rangle = 0,$$

at x_0. Moreover, by a further rotation $\langle X_{12}, \nu\rangle = 0$ at x_0 so that ${A_1}^2(x_0) = 0$. With these properties, following the computation in Section 3.1, we obtain at the point x_0:

$$\tilde{g}^{-1} = \frac{1}{\det(\tilde{g})}\begin{bmatrix} 1 - 2u{A_2}^2 + u^2(A_2^2)^2 + u_2^2 & -u_1u_2 \\ -u_1u_2 & 1 - 2u{A_1}^1 + u^2(A_1^1)^2 + u_1^2 \end{bmatrix}$$

with

$$\det(\tilde{g}) = |\tilde{e}_1 \times \tilde{e}_2|^2 = (1 - uH - u^2G)^2 + u_1^2(1 - uA_2^2)^2 + u_2^2(1 - uA_1^1)^2,$$

where G is the Gaussian curvature.

We will use Q to denote different functions of u, u_i, u_{ij}, x with the properties:

$$\begin{cases} Q \text{ is } C^\infty \text{ in } u, u_i, u_{ij}, \\ Q(0,0,0,x) = 0, \quad D_u Q(0,0,0,x) = 0, \\ D_{u_i} Q(0,0,0,x) = 0, \quad D_{u_{ij}} Q(0,0,0,x) = 0, \\ Q \text{ is linear in } u_{ij}, \\ \text{second derivatives with respect to } u, u_i, u_{ij} \text{ are bounded by} \\ \text{universal functions of } |A| \text{ and } |\nabla A| \text{ for } |uA| < 1/2. \end{cases} \tag{3.15}$$

Then we can write

$$\tilde{g}^{-1} = \frac{1}{\det(\tilde{g})} \begin{bmatrix} 1 - 2uA_2{}^2 & 0 \\ 0 & 1 - 2uA_1{}^1 \end{bmatrix} + Q$$

and

$$\det(\tilde{g}) = 1 - 2uH + Q.$$

Similarly,

$$\langle \tilde{X}_{ij}, \tilde{e}_1 \times \tilde{e}_2 \rangle = (1 - uH)A_i{}^j + u_{ij} - u(A_i{}^j)^2 + Q,$$

therefore

$$\tilde{H} = H + u_{11} + u_{22} + ((A_1{}^1)^2 + (A_2{}^2)^2)u + Q_1,$$

where Q_1 satisfies the properties (3.15). Let u, v be $C^{2,\alpha}$ functions on $\mathcal{M}$ with $|uA| < 1/2$, $|vA| < 1/2$. To simplify notation, let $U(x) = (u(x), \nabla u(x), \nabla^2 u(x))$. From the properties of Q_1 and Taylor's formula

$$|Q_1(U(x), x) - Q_1(V(x), x)| \le C(|U(x)| + |V(x)|)(|U(x) - V(x)|).$$

To estimate the Hölder norm of Q_1, we note that the expression for $\tilde{H} - \varepsilon\tilde{\nu} \cdot e_z$ is linear in the second derivative of u and we have C^1 bounds on all the other terms. We have C^α bounds on $\nabla_g u$ and C^1 bounds on everything else (u, $\nabla_g u$, A, and $\nabla_g A$).

4. The Jacobi equation on Scherk surfaces

Let $\Sigma = \Sigma(\alpha)$ be the singly periodic Scherk surface defined by (2.6). In this section, we want to solve the problem involving the Jacobi operator on Σ,

$$\Delta\phi + |A|^2\phi = h \quad \text{in } \Sigma,$$

where Δ is the Laplace–Beltrami operator and A is the second fundamental form of Σ.

We let s and z denote the coordinates on the wings $W^i(\alpha)$, $i = 1, \dots, 4$, described in Lemma 2.2. In the rest of the section, we will work with right-hand sides h defined on Σ and such that $\|e^{\gamma s} h\|_{L^\infty(\Sigma)} < \infty$ for a fixed $\gamma \in (0, 1)$.

We will work with functions that are 2π periodic in z and even with respect to z, that is, ϕ and h satisfy

$$\phi(x, y, z) = \phi(x, y, z + 2\pi), \quad \phi(x, y, z) = \phi(x, y, -z), \quad \forall (x, y, z) \in \Sigma, \tag{4.1}$$

which is equivalent to symmetry with respect to the planes $z = k\pi$, $k \in \mathbb{Z}$.

We choose the unit normal vector to Σ such that

$$\nu \cdot e_y > 0 \text{ on wings 1 and 2, and } \nu \cdot e_y < 0 \text{ on wings 3 and 4.} \tag{4.2}$$

Because translating the surface Σ leaves its mean curvature unchanged, the functions $\nu \cdot e$ are in the kernel of $\Delta + |A|^2$ for any fixed $e \in \mathbb{R}^3$. Hence $\nu \cdot e_x$, $\nu \cdot e_y$, and $\nu \cdot e_z$ are in the kernel of the Jacobi operator. Of these functions, $\nu \cdot e_x$ and $\nu \cdot e_y$ satisfy the symmetries (4.1) and $\nu \cdot e_z$ does not because it is antisymmetric with respect to $z = 0$. We will write

$$z_1 = \nu \cdot e_x, \quad z_2 = \nu \cdot e_y. \tag{4.3}$$

The main results in this section are the following. First, we consider the problem of finding a bounded solution ϕ of

$$\begin{cases} \Delta\phi + |A|^2\phi = \displaystyle\sum_{i=1}^{2} c_i \eta_0 z_i + h & \text{in } \Sigma, \\ \phi \text{ satisfies the symmetries (4.1)} \end{cases} \tag{4.4}$$

for which

$$\int_\Sigma \phi \eta_0 z_i = 0, \quad i = 1, 2, \tag{4.5}$$

where $\eta_0 \in C^\infty(\Sigma)$ is a smooth function depending only on $x^2 + y^2$ such that

$$0 \le \eta_0 \le 1,\ \eta_0 = 1 \text{ on } x^2 + y^2 \le R_0^2, \text{ and } \eta_0 = 0 \text{ on } x^2 + y^2 \ge (R_0 + 1)^2, \tag{4.6}$$

where $R_0 > 1$ is fixed.

Proposition 4.1. *Let $0 < \gamma < 1$. Let h be defined in Σ, satisfy the symmetries (4.1), and $\|e^{\gamma s} h\|_{L^\infty(\Sigma)} < \infty$. Then there are unique $c_1, c_2 \in \mathbb{R}$ and $\phi \in L^\infty(\Sigma)$ satisfying (4.4) and (4.5). Moreover*

$$|c_1| + |c_2| + \|\phi\|_{L^\infty(\Sigma)} \le C \|e^{\gamma s} h\|_{L^\infty(\Sigma)} \tag{4.7}$$

and

$$|\nabla\phi| \le C(\|e^{\gamma s}h\|_{L^\infty(\Sigma)} + \|\phi\|_{L^\infty(\Sigma)})e^{-\gamma s}. \tag{4.8}$$

If ϕ is a bounded solution of (4.4), by (4.8) ϕ has a limit on each wing, that is, $L_i = \lim_{s\to\infty}\phi(s,z)$ on all wings $i = 1,\dots,4$. These limits define linear functionals of h and we have the estimate $|L_i| \le C\|e^{\gamma s}h\|_{L^\infty}$. For later consideration, it is desirable to find a solution to (4.4) with limit equal to zero on all wings. To achieve this, the right-hand side has to satisfy four restrictions, or equivalently, has to be projected onto a space of codimension 4. We do this by considering the main terms introduced by the dislocations. So we consider now the problem

$$\begin{cases} \Delta\phi + |A|^2\phi = h + \displaystyle\sum_{i=1}^{2}(\beta_i w_i' + \tau_i w_i) \quad \text{in } \Sigma, \\ \phi \text{ satisfies (4.1)}, \end{cases} \tag{4.9}$$

where the functions w_i, w_i' are defined in (2.8), (2.9).

Proposition 4.2. *Let $0 < \gamma < 1$ and h be a function satisfying $\|e^{\gamma s}h\|_{L^\infty(\Sigma)} < \infty$ and the symmetries (4.1). Then if R_{rot} and R_{tr}, which are the parameters in the construction associated to the dislocations, are fixed large enough, there exist β_i, τ_i, $i = 1, 2$ and ϕ a bounded solution of (4.9) such that ϕ has a limit equal to zero on all wings. Moreover ϕ, β_i, τ_i depend linearly on h and*

$$\|e^{\gamma s}\phi\|_{L^\infty} + \sum_{i=1}^{2}(|\beta_i| + |\tau_i|) \le C\|e^{\gamma s}h\|_{L^\infty}. \tag{4.10}$$

Using this proposition, we fix the parameters R_{tr}, R_{rot} of the construction of the initial approximation $\mathcal{M}$. The following non-degeneracy property of the Jacobi operator is crucial in the proof of the above results and was proved by Montiel and Ros [25].

Proposition 4.3. *Any bounded solution ϕ of*

$$\Delta\phi + |A|^2\phi = 0 \quad \text{in } \Sigma$$

is a linear combination of $\nu\cdot e_x$, $\nu\cdot e_y$ and $\nu\cdot e_z$.

The rest of the section is devoted to prove Propositions 4.1 and 4.2. We start by considering the problem

$$\begin{cases} \Delta\phi + |A|^2\phi = h \quad \text{in } \Sigma_R \\ \phi = 0 \quad \text{on } \partial\Sigma_R, \quad \phi \text{ satisfies (4.1)} \end{cases} \tag{4.11}$$

where $R > 0$ is large and

$$\Sigma_R \text{ is the union of the core of } \Sigma \text{ and } \cup_{i=1}^{4} W^i(\alpha) \cap \{s \leq R\}. \tag{4.12}$$

In the sequel, we work with $R >> R_0 + 1$.

Lemma 4.4. *Let $0 < \gamma < 1$. Let h be defined in Σ_R, satisfy the symmetries (4.1), and $\|e^{\gamma s} h\|_{L^\infty(\Sigma)} < \infty$. Then there are R_1, C such that for all $R \geq R_1$ and any solution ϕ of (4.11) such that*

$$\int_{\Sigma_R} \phi \eta_0 z_i = 0 \quad i = 1, 2, \tag{4.13}$$

we have

$$\|\phi\|_{L^\infty(\Sigma_R)} \leq C \|e^{\gamma s} h\|_{L^\infty(\Sigma_R)}.$$

Proof. We proceed by contradiction and assume that for any positive integer n, there are R_n, ϕ_n, h_n such that $R_n \to \infty$ as $n \to \infty$, (4.11), (4.13) hold and

$$\|\phi_n\|_{L^\infty(\Sigma_{R_n})} = 1, \quad \|e^{\gamma s} h_n\|_{L^\infty(\Sigma_{R_n})} \to 0 \quad \text{as } n \to \infty. \tag{4.14}$$

Let us show first that $\phi_n \to 0$ uniformly on compact sets of Σ_{R_n}. Otherwise, up to a subsequence and using standard local estimates for elliptic equations, $\phi_n \to \phi$ uniformly on compact sets of Σ, where ϕ is bounded, not identically zero, and satisfies

$$\Delta \phi + |A|^2 \phi = 0 \quad \text{in } \Sigma.$$

By Proposition 4.3, ϕ is a linear combination of $\nu \cdot e_x$, $\nu \cdot e_y$ and $\nu \cdot e_z$. But $\nu \cdot e_z$ is not present in this linear combination by the imposed symmetry (4.1), so

$$\phi = c_1 \nu \cdot e_x + c_2 \nu \cdot e_y$$

for some constants c_1 and c_2. But passing to the limit in (4.13), we deduce that ϕ satisfies (4.13). This implies that $c_1 = c_2 = 0$.

We note that $\psi = 1 - e^{-\gamma s}$ satisfies

$$\Delta \psi + |A|^2 \psi \leq -C e^{-\gamma s}$$

for some $C > 0$ and $s \geq s_0$ where s_0 is large enough. Using ψ as a barrier on each wing, we obtain that $\|\phi_n\|_{L^\infty(\Sigma_{R_n})} \to 0$ as $n \to \infty$, which contradicts (4.14). $\square$

With almost the same argument, we can prove the next result.

Lemma 4.5. *Let $0 < \gamma < 1$. Let h be defined in Σ, satisfy the symmetries (4.1), and $\|e^{\gamma s}h\|_{L^\infty(\Sigma)} < \infty$. Then there is a constant C such that for any bounded solution ϕ of*

$$\begin{cases} \Delta\phi + |A|^2\phi = h & \text{in } \Sigma \\ \phi \text{ satisfies (4.1)} \end{cases} \tag{4.15}$$

which also satisfies (4.5), we have

$$\|\phi\|_{L^\infty(\Sigma)} \le C\|e^{\gamma s}h\|_{L^\infty(\Sigma)}.$$

Proof. The proof changes only in the last step, when we use the maximum principle to show that $\phi \le C\psi$. We change slightly the barrier by considering

$$\psi + \delta Z,$$

where Z is an element in the kernel of the Jacobi operator that grows linearly, and then take $\delta \to 0$. $\square$

Lemma 4.6. *Let $0 < \gamma < 1$. Let h be defined in Σ, satisfy the symmetries (4.1), and $\|e^{\gamma s}h\|_{L^\infty(\Sigma)} < \infty$. Suppose ϕ is a bounded solution of (4.15). Then*

$$|\nabla\phi| \le C(\|e^{\gamma s}h\|_{L^\infty(\Sigma)} + \|\phi\|_{L^\infty(\Sigma)})e^{-\gamma s}.$$

Proof. Changing variables, we rewrite the equation on a fixed wing as

$$\Delta\phi = a_0(s,z)\phi + a_1(s,z)\nabla\phi + a_2(s,z)D^2\phi + h \quad \text{in } S \tag{4.16}$$

with boundary conditions

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on } (s_0,\infty)\times\{0,\pi\},$$

where S is the strip $(s_0,\infty)\times(0,\pi)$. We write the variables in S as (s,z), $s > s_0$, $z \in (0,\pi)$ and in (4.16), $\Delta = \partial_{ss} + \partial_{zz}$. The functions a_0, a_1, a_2 are smooth and have the decay

$$|a_i(s,z)| \le Ce^{-s}.$$

For $T > s_0$, we have

$$\|\phi\|_{L^2((T,T+5)\times(0,\pi))} \le C\|\phi\|_{L^\infty(S)}.$$

Since the coefficients of a_i are small as $s \to \infty$, we have from standard estimates

$$\|\phi\|_{H^2((T,T+1)\times(0,\pi))} \le C(\|\phi\|_{L^\infty(S)} + \|e^{\gamma s}h\|_{L^\infty(S)}).$$

Hence,

$$\|a_0\phi + a_1\nabla\phi + a_2 D^2\phi + h\|_{L^2((T,\infty)\times(0,\pi))} \le C(\|\phi\|_{L^\infty(S)} + \|e^{\gamma s}h\|_{L^\infty(S)})e^{-\gamma T}. \tag{4.17}$$

Let $g = a_0\phi + a_1\nabla\phi + a_2 D^2\phi + h$ and write

$$\phi(s,z) = \sum_{n=0}^{\infty} \phi_n(s)\cos(nz), \quad g(s,z) = \sum_{n=0}^{\infty} g_n(s)\cos(nz),$$

where, for $n \ge 0$, $\phi_n(s) = \frac{2}{\pi}\int_0^\pi \phi(s,z)\cos(nz)\,dz$ and $g_n(s) = \frac{2}{\pi}\int_0^\pi g(s,z)\cos(nz)\,dz$. We can write

$$\phi_0(s) = b_0 + \int_s^\infty (t-s)g_0(t)\,dt, \tag{4.18}$$

where $b_0 = \lim_{s\to\infty}\phi_0(s)$. The claim is that

$$\|\phi - b_0\|_{L^2((T,\infty)\times(0,\pi))} \le C\left(\|\phi\|_{L^\infty(S)} + \|e^{\gamma s}h\|_{L^\infty(S)}\right)e^{-\gamma T}. \tag{4.19}$$

To prove this, let $\tilde\phi(s,z) = \sum_{n=1}^\infty \phi_n(s)\cos(nz)$. We claim that, for T large

$$\|\tilde\phi\|_{L^2((T,\infty)\times(0,\pi))} \le C\left(\|\phi\|_{L^\infty(S)} + \|e^{\gamma s}h\|_{L^\infty}\right)e^{-\gamma T}. \tag{4.20}$$

Indeed, we have

$$\phi_n'' - n^2\phi_n = g_n \quad \text{for } s > s_0.$$

Note that $\phi_n(s)$ is bounded as $s \to \infty$, so that $\phi_n(s)$ must have the form (for $n \ge 1$)

$$\phi_n(s) = d_n e^{-n(s-s_0)} + \phi_{0,n}(s),$$

where

$$d_n = \phi_n(s_0), \quad \phi_{0,n}(s) = -e^{-nd}\int_{s_0}^s e^{2nt}\int_t^\infty g_n(\tau)e^{-n\tau}\,d\tau\,dt.$$

Let $\gamma < a < 1$ be fixed. By the Cauchy–Schwarz inequality, for $n \ge 1$, we have

$$|\phi_{0,n}(s)| \le \frac{1}{(4n(n-a))^{1/2}}e^{-as}\left(\int_{s_0}^s e^{2at}\int_t^\infty |g_n(\tau)|^2\,d\tau\,dt\right)^{1/2}. \tag{4.21}$$

Therefore

$$\|\tilde{\phi}\|^2_{L^2((T,\infty)\times(0,\pi))} \le Ce^{-2T}\sum_{n=1}^{\infty} d_n^2 + C\sum_{n=1}^{\infty}\int_T^{\infty} |\phi_{0,n}(s)|^2\, ds,$$

and using (4.17), (4.21), we deduce (4.20). The estimate above and a similar one for the integral in (4.18) give (4.19).

Note that $\bar{\phi} = \phi - b_0$ satisfies

$$\Delta\bar{\phi} = a_0(s,z)\bar{\phi} + a_1(s,z)\nabla\bar{\phi} + a_2(s,z)D^2\bar{\phi} + h + a_0(s,z)b_0 \quad \text{in } S.$$

From standard elliptic estimates, we get

$$\begin{aligned}\|\bar{\phi} + |\nabla\bar{\phi}|\|_{L^\infty((T+1,T+2)\times(0,\pi))} &\le C\left(\|\bar{\phi}\|_{L^2((T,T+3)\times(0,\pi))}\right.\\ &\qquad \left. + \|h + a_0 b_0\|_{L^\infty((T,T+3)\times(0,\pi))}\right)\\ &\le C\left(\|\phi\|_{L^\infty(S)} + \|h\|_{L^\infty(S)}\right)e^{-\gamma T}. \qquad \square\end{aligned}$$

To prove existence, let Σ_R be the truncated surface defined by (4.12) and consider the problem of finding ϕ and c_1, c_2 such that

$$\begin{cases} \Delta\phi + |A|^2\phi = \displaystyle\sum_{i=1}^{2} c_i\eta_0 z_i + h \quad \text{in } \Sigma_R, \\ \phi = 0 \quad \text{on } \partial\Sigma_R, \\ \phi \text{ satisfies (4.1).} \end{cases} \tag{4.22}$$

Lemma 4.7. *Let $0 < \gamma < 1$. Let h be defined in Σ_R, satisfy the symmetries (4.1), and $\|e^{\gamma s}h\|_{L^\infty(\Sigma_R)} < \infty$. Then there are unique $c_1, c_2 \in \mathbb{R}$ and $\phi \in L^\infty(\Sigma_R)$ satisfying (4.22) and (4.5). Moreover,*

$$|c_1| + |c_2| + \|\phi\|_{L^\infty(\Sigma_R)} \le C\|e^{\gamma s}h\|_{L^\infty(\Sigma_R)}, \tag{4.23}$$

with C independent of R.

Proof. We prove first (4.23). Indeed, by Lemma 4.4,

$$\|\phi\|_{L^\infty} \le C\left(|c_1| + |c_2| + \|e^{\gamma s}h\|_{L^\infty}\right).$$

Multiplying by z_j and integrating in Σ_R, we find

$$\int_{\partial\Sigma_R} \frac{\partial\phi}{\partial\nu} z_j = c_j \int_{\Sigma_R} \eta_0 z_j^2. \tag{4.24}$$

We claim that

$$\int_{\partial \Sigma_R} \frac{\partial \phi}{\partial \nu} z_j = o(1)(\|\phi\|_{L^\infty} + \|e^{\gamma s} h\|_{L^\infty}) \tag{4.25}$$

where $o(1) \to 0$ as $R \to \infty$. Combining (4.25) and (4.24), we get (4.23). To prove (4.25), we rewrite the equation on the ith wing as

$$L\phi = \sum_{i=1}^{2} c_i \eta_0 z_i + h \quad \text{in } S_R,$$

where $L\phi = \Delta\phi + a_0\phi + a_1\nabla\phi + a_2 D^2\phi$ and $S_R = \{(s,z) : 0 < s < R, 0 < z < \pi\}$. Here $\Delta = \partial_{ss} + \partial_{zz}$ and $a_i(s,z)$ are smooth with $|a_i(s,z)| \leq Ce^{-s}$. Let $R_1 >> R_0$ and $R >> R_1$. The function $\bar{\phi} = \left(\frac{R-s}{R-R_1}\right)^{\mu}$, where $0 < \mu < 1$ is fixed, satisfies

$$L\bar{\phi} \leq -c\left(\frac{R-s}{R-R_1}\right)^{\mu-2}$$

in $(R_1, R) \times (0,\pi)$ for some $c > 0$, if we take R_1 large. By the maximum principle, $|\phi| \leq C(\|\phi\|_{L^\infty} + \|e^{\gamma s}h\|_{L^\infty})\bar{\phi}$ in $(R_1, R) \times (0,\pi)$. It follows that

$$\left|\frac{\partial \phi}{\partial s}(R,z)\right| \leq \frac{C}{R-R_1}(\|\phi\|_{L^\infty} + \|e^{\gamma s}h\|_{L^\infty}),$$

and this proves (4.25).

For the existence of a solution of (4.23), let us define the Hilbert space

$$H = \left\{ \phi \in H^1(\Sigma_R \cap \{z \in (0,\pi)\}) : \phi\big|_{s=R} = 0, \int_{\Sigma_R \cap \{z \in (0,\pi)\}} \phi\eta_0 z_i = 0,\ i = 1,2 \right\}$$

with the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\Sigma_R \cap \{z \in (0,\pi)\}} \nabla\varphi_1 \nabla\varphi_2$$

where η_0 is the cut-off function with properties (4.6). We consider the following weak form of the equation to find $\phi \in H$:

$$\int_{\Sigma_R \cap \{z \in (0,\pi)\}} \nabla\phi \cdot \nabla\varphi - |A|^2\phi\varphi = -\int_{\Sigma_R \cap \{z \in (0,\pi)\}} h\varphi, \quad \forall \varphi \in H. \tag{4.26}$$

Let $T : H \to H$, $T(\phi) = \psi$ be the linear operator defined by the Riesz theorem from the relation

$$\langle \psi, \varphi \rangle = \int_{\Sigma_R \cap \{z \in (0,\pi)\}} |A|^2 \phi \varphi, \quad \forall \varphi \in H.$$

Then T is compact and we can formulate (4.26) as finding $\phi \in H$ such that

$$\phi = T(\phi) + L_h,$$

where $L_h(\varphi) = \int_{\Sigma_R \cap \{z \in (0,\pi)\}} h\varphi$. By the Fredholm theorem, this problem is uniquely solvable for any h provided the only solution of $\phi = T(\phi)$ in H is $\phi = 0$. This holds by (4.23). □

We now turn our attention to the problem on the whole Scherk surface Σ.

Proof of Proposition 4.1. First, let us show that, for any bounded solution ϕ, c_1, c_2 of (4.4) satisfying (4.5), the estimate (4.7) is valid. Indeed, by Lemma 4.5,

$$\|\phi\|_{L^\infty} \le C(|c_1| + |c_2| + \|e^{\gamma s} h\|_{L^\infty}). \tag{4.27}$$

For $R >> 1$, let $\Sigma_R = \Sigma \cap \{(x, y, z) : x^2 + y^2 \le R^2\}$. Multiplying (4.4) by z_j and integrating in $\Sigma_R \cap \{z \in (0, \pi)\}$, we have

$$\int_{\partial\Sigma_R \cap \{z \in (0,\pi)\}} \frac{\partial \phi}{\partial \nu} z_j = c_j \int_{\Sigma_R \cap \{z \in (0,\pi)\}} \eta_0 z_j^2 + \int_{\Sigma_R \cap \{z \in (0,\pi)\}} h z_j \tag{4.28}$$

because $\int_{\Sigma_R \cap \{z \in (0,\pi)\}} \eta_0 z_1 z_2 = 0$ by symmetry. By Lemma 4.6,

$$\int_{\partial\Sigma_R \cap \{z \in (0,\pi)\}} \left| \frac{\partial \phi}{\partial \nu} z_i \right| \le C e^{-\gamma R} (|c_1| + |c_2| + \|e^{\gamma s} h\|_{L^\infty} + \|\phi\|_{L^\infty}). \tag{4.29}$$

Combining (4.27), (4.28), and (4.29), we deduce (4.7) and prove the uniqueness.

By Lemma 4.7, for R large, the problem (4.22) is uniquely solvable, yielding a solution to ϕ_R (and constants $c_{1,R}$ and $c_{2,R}$), which remains bounded as $R \to \infty$ by (4.23). By standard elliptic estimates, ϕ_R converges locally uniformly in Σ to a solution of (4.4). Estimate (4.8) follows from Lemma 4.6. □

Proof of Proposition 4.2. Given h as stated, let ϕ, c_1, c_2 be the solution to (4.4) provided by Proposition 4.1. Let $L_i = \lim_{s\to\infty} \phi(s, z)$ be the limit of ϕ on wing i. By adding appropriate multiples of z_1 and z_2, we can make two of these limits equal to zero. More precisely, by the choice of orientation (4.2) on Σ and the definition of z_1 and z_2 in (4.3), we have

$$\lim_{s\to\infty} z_1 = \begin{cases} -\sin\alpha & \text{on wing } 1, \\ \sin\alpha & \text{on wing } 2, \\ \sin\alpha & \text{on wing } 3, \\ -\sin\alpha & \text{on wing } 4, \end{cases} \qquad \lim_{s\to\infty} z_2 = \begin{cases} \cos\alpha & \text{on wing } 1, \\ \cos\alpha & \text{on wing } 2, \\ -\cos\alpha & \text{on wing } 3, \\ -\cos\alpha & \text{on wing } 4. \end{cases}$$

Let d_1, d_2 satisfy

$$d_1 \sin\alpha + d_2\cos\alpha = L_2, \quad d_1\sin\alpha - d_2\cos\alpha = L_3.$$

Then $\tilde\phi = \phi - d_1 z_1 - d_2 z_2$ satisfies (4.4) and has limit equal to zero on wings 2 and 3.

We remark that we could also achieve limit equal to zero on any two adjacent wings, but not on opposite ones in general. Also note that if the original ϕ satisfies the orthogonality conditions (4.5), the new $\tilde\phi$ does not in general.

Let $\tilde\eta_i$, $i = 1$ or $i = 4$, be smooth cut-off functions on Σ such that:

$$\begin{aligned} &\tilde\eta_i = 1 \text{ on wing } i \text{ and for } s \ge R_c + 1, \\ &\tilde\eta_i = 0 \text{ on wing } i \text{ for } 0 \le s \le R_c, \\ &\tilde\eta_i = 0 \text{ on the core and the rest of the wings.} \end{aligned}$$

Here $R_c > 0$ is a large constant to be fixed later. Define

$$\tilde z_1 = \nu \cdot \nu_\alpha, \quad \tilde z_4 = \nu\cdot(-\sin\alpha, -\cos\alpha, 0),$$

where $\nu_\alpha = (-\sin\alpha, \cos\alpha, 0)$ is the normal vector to the asymptotic plane of wing 1. Note that $\tilde z_1, \tilde z_4$ are in the kernel of $\Delta + |A|^2$, $\tilde z_1 \to 1$ as $s_1\to\infty$, and $\tilde z_4 \to 1$ as $s_4 \to \infty$ (they are convenient linear combinations of z_1, z_2). Define

$$\hat\phi = \tilde\phi - L_1\tilde\eta_1\tilde z_1 - L_4\tilde\eta_4\tilde z_1.$$

Then $\hat\phi$ satisfies

$$\Delta\hat\phi + |A|^2\hat\phi = h + \sum_{i=1}^2 c_i\eta_0 z_i - \sum_{i=1,4} L_i(\Delta + |A|^2)(\tilde\eta_i\tilde z_i) \quad \text{in } \Sigma$$

and $\hat\phi$ has a limit equal to zero on all the wings. Moreover, $\hat\phi$ satisfies the symmetries (4.1).

Suppose that β_i, τ_i, $i = 1, 2$, are given and let us consider the function $\hat\phi$ constructed previously with h replaced by $h + \sum_{i=1}^2 \beta_i w_i' + \tau_i w_i$. This $\hat\phi$ satisfies

$$\Delta\hat\phi + |A|^2\hat\phi = h + \sum_{i=1}^2 c_i\eta_0 z_i - \sum_{i=1,4} L_i(\Delta + |A|^2)(\tilde\eta_i\tilde z_i) + \sum_{i=1}^2(\beta_i w_i' + \tau_i w_i) \quad (4.30)$$

in Σ, has the symmetries (4.1) and the limits on all wings are equal to zero. In this construction, h, β_i, τ_i, $i = 1, 2$ are data and $\hat{\phi}$, c_i $(i = 1, 2)$, L_i $(i = 1, 4)$ are bounded linear functions of these data.

We claim that there is a unique choice of β_i, τ_i, $i = 1, 2$, such that $c_1 = c_2 = L_1 = L_4 = 0$. To prove this, we test (4.30) with functions that are linear combinations of z_1, z_2,

$$z_3 = \nu \cdot (-y, x, 0), \text{and } z_4 = \frac{\partial_\alpha S}{|\nabla_{x,y,z} S|},$$

where

$$S(x, y, z, \alpha) = \cos^2(\alpha) \cosh\left(\frac{x}{\cos\alpha}\right) - \sin^2(\alpha) \cosh\left(\frac{y}{\sin\alpha}\right) - \cos(z)$$

is the function (2.6) defining the Scherk surface, and $\nabla_{x,y,z} S = (\partial_x S, \partial_y S, \partial_z S)$. Note that z_3 arises from a rotation about the z-axis and z_4 is generated by the motion in α of the Scherk surfaces $\Sigma(\alpha)$, so these functions are in the kernel of the Jacobi operator $\Delta + |A|^2$ and have the symmetries (4.1). Also, z_3 and z_4 have linear growth.

We multiply (4.30) by z_i and integrate over $\Sigma \cap \{x \in (0, \pi)\}$. Since $\hat{\phi}$ has exponential decay,

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta\hat{\phi} + |A|^2\hat{\phi}) z_i = \int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta z_i + |A|^2 z_i)\hat{\phi} = 0,$$

for all $1 \le i \le 4$, while the right-hand side becomes an affine function of the numbers $c_1, c_2, L_1, L_4, \beta_1, \beta_2, \tau_1, \tau_2$. More precisely, we obtain

$$0 = \begin{bmatrix} \int hz_1 \\ \int hz_2 \\ \int hz_3 \\ \int hz_4 \end{bmatrix} + M_1 \begin{bmatrix} c_1 \\ c_2 \\ L_1 \\ L_4 \end{bmatrix} + M_2 \begin{bmatrix} \beta_1 \\ \beta_2 \\ \tau_1 \\ \tau_2 \end{bmatrix}.$$

The claim is that we can choose $\beta_1, \beta_2, \tau_1, \tau_2$ to achieve $c_1 = c_2 = L_1 = L_4 = 0$. For this, we will verify that M_1, M_2 are invertible (if the parameters R_c, R_{tr}, and R_{rot} are chosen adequately).

Note that by symmetry,

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} \eta_0 z_i z_j = c\delta_{ij}, \quad \text{for } i, j = 1, 2,$$

where $c > 0$ is some constant. Also thanks to symmetry, we have for $i = 1, 2$,

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} \eta_0 z_i z_3 = 0, \qquad \int_{\Sigma\cap\{z\in(0,\pi)\}} \eta_0 z_i z_4 = 0.$$

Let us compute for $i = 1$ or $i = 4$, and $j = 1, 2$:

$$\begin{aligned}\int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta + |A|^2)(\tilde{\eta}_i \tilde{z}_i) z_j &= \int_{\Sigma\cap\{z\in(0,\pi)\}} (2\nabla\tilde{\eta}_i \cdot \nabla\tilde{z}_i + \tilde{z}_i \Delta\tilde{\eta}_i) z_j \\ &= \int_{\Sigma\cap\{z\in(0,\pi)\}} \nabla\cdot(\tilde{z}_i^2 \nabla\tilde{\eta}_i)\frac{z_j}{\tilde{z}_i} \\ &= -\int_{\Sigma\cap\{z\in(0,\pi)\}} \tilde{z}_i^2 \nabla\tilde{\eta}_i \nabla\left(\frac{z_j}{\tilde{z}_i}\right) = O(e^{-R_c}),\end{aligned}$$

where the last equality holds because $\tilde{z}_i, z_j$ approach constants at an exponential rate.

For $i = 1, 4$, by the same computation, and using that z_3 has linear growth (for example $z_3 = \sin(\alpha)y + \cos(\alpha)x + O(se^{-s})$ on wing 1, where $s = \sqrt{x^2+y^2} + O(1)$), we obtain

$$\begin{aligned}\int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta + |A|^2)(\tilde{\eta}_i \tilde{z}_i) z_3 &= -\int_{\Sigma\cap\{z\in(0,\pi)\}} \tilde{z}_i^2 \nabla\tilde{\eta}_i \nabla\left(\frac{z_3}{\tilde{z}_i}\right) \\ &= -\bar{B}(-1)^{i-1} + o(1)\end{aligned}$$

where $\bar{B} > 0$ and $o(1) \to 0$ as $R_c \to \infty$. Similarly, for $i = 1, 4$,

$$\begin{aligned}\int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta + |A|^2)(\tilde{\eta}_i \tilde{z}_i) z_4 &= -\int_{\Sigma\cap\{z\in(0,\pi)\}} \tilde{z}_i^2 \nabla\tilde{\eta}_i \nabla\left(\frac{z_4}{\tilde{z}_i}\right) \\ &= B + o(1)\end{aligned}$$

where $B > 0$. This implies that the matrix M_1 is invertible if R_c is taken large (and fixed).

Let us now estimate the matrix M_2. This means, we need to estimate

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_i z_j \quad \text{and} \quad \int_{\Sigma\cap\{z\in(0,\pi)\}} w_i' z_j$$

for $i = 1, 2$, $j = 1, \dots, 4$. Using (2.11), for $j = 1, 2$, we have

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_1 z_j = \int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta + |A|^2)(\eta_{tr,1}) z_j = O(e^{-R_{tr}})$$

and similarly $\int w_2 z_j = O(e^{-R_{tr}})$. Next, we compute

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_1 z_3 = \int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta+|A|^2)(\eta_{tr,1})z_3$$
$$= \int_{\Sigma\cap\{z\in(0,\pi)\}} z_3\Delta\eta_{tr,1} + \int_{\Sigma\cap\{z\in(0,\pi)\}} |A|^2\eta_{tr,1}z_3.$$

The first integral is supported on $R_{tr} \le s \le R_{tr}+1$, so integrating by parts gives

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} z_3\Delta\eta_{tr,1} = -\int_{\{R_{tr}\le s\le R_{tr}+1\}} \nabla z_3\nabla\eta_{tr,1}$$
$$= -\nabla z_3\hat{n}\big|_{s=R_{tr}+1} + \int_{\{R_{tr}\le s\le R_{tr}+1\}} \eta_{tr,1}\Delta z_3,$$

where $\hat{n}$ is tangent to Σ and perpendicular to the curve $s = R_{tr}+1$. Therefore

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_1 z_3 = -\nabla z_3\hat{n}\big|_{s=R_{tr}+1} + \int_{\{s\ge R_{tr}+1\}} |A|^2 z_3$$
$$= -\pi + o(1),$$

where $o(1)\to 0$ as $R_{tr}\to\infty$, thanks to the behavior $z_3 = s + O(1)$ as $s\to\infty$ and a corresponding estimate for its derivative. Similarly,

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_2 z_3 = -\pi + o(1), \qquad \int_{\Sigma\cap\{z\in(0,\pi)\}} w_1 z_4 = -\pi + o(1),$$
$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_2 z_4 = \pi + o(1).$$

Let us now compute

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} w_1' z_1 = \int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta+|A|^2)(\eta_{rot,1}z_3)z_1,$$

where we have used (2.10). We have

$$\int_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta+|A|^2)(\eta_{rot,1}z_3)z_1 = \int_{\Sigma\cap\{z\in(0,\pi)\}} (z_3\Delta\eta_{rot,1} + 2\nabla\eta_{rot,1}\nabla z_3)z_1$$
$$= \int_{\Sigma\cap\{z\in(0,\pi)\}} z_1\nabla\eta_{rot,1}\nabla z_3 - z_3\nabla\eta_{rot,1}\nabla z_1$$
$$= -\pi\sin(\alpha) + o(1),$$

where $o(1) \to 0$ as $R_{rot} \to 0$. In a similar way,

$$\int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_1' z_2 = \pi\cos(\alpha) + o(1), \qquad \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_2' z_1 = \pi\sin(\alpha) + o(1),$$

$$\int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_2' z_2 = \pi\cos(\alpha) + o(1).$$

We also have

$$\int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_1' z_3 = 0 \text{ and } \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_2' z_3 = 0.$$

Indeed, consider

$$\begin{aligned}\int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_1' z_3 &= \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} (\Delta + |A|^2)(\eta_{rot,1} z_3) z_3 \\ &= \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} (z_3 \Delta\eta_{rot,1} + 2\nabla\eta_{rot,1}\nabla z_3) z_3 \\ &= \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} \nabla\cdot(z_3^2 \nabla\eta_{rot,1}) \\ &= 0.\end{aligned}$$

The integral $\int w_2' z_3 = 0$ is computed similarly.

Finally, we observe that

$$\int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_1' z_4 = O(1) \text{ and } \int\limits_{\Sigma\cap\{z\in(0,\pi)\}} w_2' z_4 = O(1),$$

as $R_{rot} \to \infty$. Then

$$\begin{aligned} M_2 &= \begin{bmatrix} \int w_1 z_1 & \int w_1 z_2 & \int w_1 z_3 & \int w_1 z_4 \\ \int w_2 z_1 & \int w_2 z_2 & \int w_2 z_3 & \int w_2 z_4 \\ \int w_1' z_1 & \int w_1' z_2 & \int w_1' z_3 & \int w_1' z_4 \\ \int w_2' z_1 & \int w_2' z_2 & \int w_2' z_3 & \int w_2' z_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\pi & -\pi \\ 0 & 0 & -\pi & \pi \\ -\pi\sin(\alpha) & \pi\cos(\alpha) & 0 & O(1) \\ \pi\sin(\alpha) & \pi\cos(\alpha) & 0 & O(1) \end{bmatrix} + o(1) \end{aligned}$$

This shows that M_2 is invertible if we fix both $R_{rot} + 10 < R_{tr}$ large, which finishes the proof. □

5. Linear theory

Let $\mathcal{E} = \mathcal{M} \cap \{s \geq \frac{\delta_s}{3\varepsilon}\}$. In this section we construct a right inverse of the operator $\Delta + |A|^2 + \varepsilon\partial_z$ on $\mathcal{E}$. More precisely, given h defined on $\mathcal{E}$ with some decay, we want a solution ϕ of

$$\Delta\phi + |A|^2\phi + \varepsilon\partial_z\phi = h \quad \text{on } \mathcal{E}. \tag{5.1}$$

Given $\alpha, \gamma \in (0, 1)$, let us define the following norms:

$$\begin{aligned} \|\phi_2\|_{*,\mathcal{E}} &= \varepsilon^2 \sup_{x\in\mathcal{E}} e^{\gamma\delta_s/\varepsilon+\gamma\varepsilon s(x)} \|\phi_2\|_{C^{2,\alpha}(\overline{B}_1(x))}, \\ \|h_2\|_{**,\mathcal{E}} &= \sup_{x\in\mathcal{E}} e^{\gamma\delta_s/\varepsilon+\gamma\varepsilon s(x)} \|h_2\|_{C^{2,\alpha}(\overline{B}_1(x))}, \end{aligned} \tag{5.2}$$

where $B_1(x)$ is the geodesic ball with center x and radius 1, and s is the function defined in the construction of $\mathcal{M}$, Section 2. The factor $e^{\gamma\delta_s/\varepsilon}$ in front of both norms is not immediately relevant; it will be useful later.

We have the following result.

Proposition 5.1. *Let $0 < \gamma < 1$. There is a linear operator that associates to a function h defined on $\mathcal{E}$ with $\|h\|_{**,\mathcal{E}} < \infty$ a solution ϕ to (5.1). Moreover,*

$$\|\phi\|_{*,\mathcal{E}} \leq C\|h\|_{**,\mathcal{E}}.$$

For the proof, we scale to size one, that is, we work on $\tilde{\mathcal{E}} = \varepsilon\mathcal{E}$. Then (5.1) becomes equivalent to

$$\Delta\phi + |A|^2\phi + \partial_z\phi = \tilde{h} \quad \text{on } \tilde{\mathcal{E}}, \tag{5.3}$$

with $\tilde{h}(x) = \varepsilon^{-2}h(x/\varepsilon)$.

We study the linear operator on the unbounded pieces in the following section, then we deal with the bounded piece in Section 5.2. We point out that in the radially symmetric case, a related linear theory for the Jacobi operator was developed on [10].

5.1. Linear theory on the ends

Let $\mathcal{E}_u$ be any of the unbounded components of $\mathcal{E}$ and $\tilde{\mathcal{E}}_u = \varepsilon\mathcal{E}_u$. We introduce coordinates on $\tilde{\mathcal{E}}_u$ as follows. Consider a curve

$$s \mapsto (\gamma_1(s), \gamma_3(s)),$$

parametrized by arc length, with $s \in [0, \infty)$ and $\gamma_1'(s) > 0$, that solves the ordinary differential equation (2.3) with initial conditions at $s = 0$ chosen to be compatible with the construction of the initial approximation $\mathcal{M}$ in Section 2.5. These initial conditions are functions of the parameters of the construction, in particular, the parameters $\beta_1, \beta_4, \tau_1, \tau_4$ used in the dislocations are all in $[-\delta_p, \delta_p]$. We note that the s here differs from the s in the construction of $\mathcal{M}$ in Section 2 by a shift and a scaling.

Then

$$X_0 = \nu_0(s, \theta) = (\gamma_1(s)\cos(\theta), \gamma_1(s)\sin(\theta), \gamma_3(s))$$

$s \in [0, \infty)$, $\theta \in [0, 2\pi]$, parametrizes part of the catenoid $\mathcal{W}$ or the paraboloid $\mathcal{P}$. Let

$$\nu_0(s, \theta) = (-\gamma_3'(s)\cos(\theta), -\gamma_3'(s)\sin(\theta), \gamma_1'(s))$$

be a unit normal vector. Then a parametrization of the unbounded end $\tilde{\mathcal{E}}_u$ is given by

$$X(s, \theta) = X_0(s, \theta) + \nu_0(s, \theta) f(s/\varepsilon, \theta/\varepsilon) \tag{5.4}$$

where f is essentially a cut-off function times f_α, the function that allows one to write the Scherk surface as a graph over a plane, see Lemma 2.2. The important properties of f are that $|\partial^k f(\tilde{s}, \tilde{\theta})| \le C_k e^{-\delta_s/(10\varepsilon)}$ for some C_k and that it vanishes for $\tilde{s} \ge 10\delta_s/\varepsilon$.

We have the following expression for the operator $\Delta + |A|^2 + \partial_z$ in the coordinates s and θ:

$$\Delta\phi + |A|^2\phi + \partial_z\phi = \partial_{ss}\phi + \frac{1}{\gamma_1(s)^2}\partial_{\theta\theta}\phi + \left(\frac{\gamma_1'(s)}{\gamma_1(s)} + \gamma_3'(s)\right)\partial_s\phi + |A|^2\phi + \tilde{L}\phi \tag{5.5}$$

where $\tilde{L}$ is a second order differential operator in ϕ with coefficients that are $o(1)$ as $\varepsilon \to 0$ and supported in $s \in [0, 10\delta_s]$. Using (2.5) and the fact that the principal curvatures of a surface of revolution $z = F(r)$ are given by

$$\kappa_1 = \frac{F''(r)}{(1 + F'(r)^2)^{3/2}}, \quad \kappa_2 = \frac{F'(r)}{r(1 + F'(r)^2)^{1/2}},$$

we can write

$$\Delta\phi + |A|^2\phi + \partial_z\phi = \partial_{ss}\phi + a(s)\partial_{\theta\theta}\phi + b(s)\partial_s\phi + |A|^2\phi + \tilde{L}\phi,$$

where we have the following properties of the coefficients:

$$\begin{cases} a(s) = \dfrac{1}{2s}(1 + O(s^{-1/2})) \\ b(s) = 1 + O(s^{-1/2}) \\ |A|^2 = \dfrac{1}{2s}(1 + O(s^{-1/2})) \end{cases} \tag{5.6}$$

as $s \to \infty$, and $a(s) > 0$ for all $s \ge 0$.

Let

$$L_0\phi = \partial_{ss}\phi + a(s)\partial_{\theta\theta}\phi + b(s)\partial_s\phi + |A|^2\phi,$$

and consider the equation

$$L_0\phi = h, \quad s \in (0,\infty),\ \theta \in [0,2\pi]. \tag{5.7}$$

To prove Proposition 5.1, we will first construct an inverse operator for L_0.

Proposition 5.2. *Let $0 < \gamma < 1$. There is a linear operator $h \mapsto \phi$ that associates to a function $h = h(s,\theta)$ that is defined for $(s,\theta) \in (0,\infty) \times [0,2\pi]$, is 2π-periodic in θ, and satisfies $\|e^{\gamma s}h\|_{L^\infty} < \infty$, a solution ϕ of (5.7) that is 2π-periodic in θ and satisfies*

$$\|e^{\gamma s}\phi\|_{L^\infty} \le C\|e^{\gamma s}h\|_{L^\infty}.$$

For the proof of this result, we will write h and ϕ in Fourier series

$$\phi(s,\theta) = \sum_{k\in\mathbb{Z}} \phi_k(s)e^{ik\theta}, \quad h(s,\theta) = \sum_{k\in\mathbb{Z}} h_k(s)e^{ik\theta}. \tag{5.8}$$

Then, if ϕ is smooth with exponential decay, equation (5.7) is equivalent to

$$\phi_k'' + b(s)\phi_k' + (|A|^2 - a(s)k^2)\phi_k = h_k, \quad \text{for all } s > 0, k \in \mathbb{Z}. \tag{5.9}$$

We need a couple of lemmas before starting the proof of Proposition 5.2. They allow us to deal with the low modes, i.e. $|k| \le k_0$ for some fixed k_0, so we can focus our attention on the higher frequencies.

Lemma 5.3. *Let $0 < \gamma < 1$. If $|h_k(s)| \le Ce^{-\gamma s}$, then (5.9) has a unique solution ϕ_k with*

$$\|e^{\gamma s}\phi_k\|_{L^\infty} \le C_k\|e^{\gamma s}h_k\|_{L^\infty}.$$

Proof. As $s \to \infty$, equation (5.9) with $h_k = 0$ is asymptotic to

$$\phi_k'' + \phi_k' + \frac{b_k}{s}\phi_k = 0, \tag{5.10}$$

where $b_k = \frac{1-k^2}{2}$. Note that $b_0 = \frac{1}{2}$, $b_1 = 0$, and $b_k < 0$ for $k > 1$. We see that the homogeneous equation (5.10) has two independent solutions with the behaviors $s^{-b_k}(1 + O(s^{-1}))$ and $s^{b_k}e^{-s}(1 + O(s^{-1}))$ as $s \to \infty$. Using these solutions, it is possible to construct, for each $k \ge 0$, two elements $z_{1,k}$ and $z_{2,k}$ in the kernel of the operator $\phi'' + b(s)\phi' + (|A|^2 - a(s)k^2)\phi$ such that

$$z_{1,k}(s) = s^{-b_k}(1 + O(s^{-\sigma})), \quad z_{2,k}(s) = s^{b_k}e^{-s}(1 + O(s^{-\sigma})),$$

as $s \to \infty$, where $\sigma \in (0,1)$. Now we can construct the solution ϕ_k using the variation of parameters formula:

$$\phi_k = -z_{1,k} \int \frac{h_k z_{2,k}}{W_k} + z_{2,k} \int \frac{h_k z_{1,k}}{W_k},$$

where $W_k = z_{1,k} z'_{2,k} - z'_{1,k} z_{2,k} = e^{-s}(1+O(s^{-\sigma}))$ and the integrals are chosen to have the desired decay. For an alternative construction, one can use the super solution $\bar{\phi} = e^{-\gamma s}$ and the calculation as in Lemma 5.4. □

Let $k_0 \in \mathbb{N}$ and ϕ be a bounded measurable function on $[0,\infty) \times [0,2\pi]$. We will say that the Fourier coefficients of ϕ of order less than k_0 vanish if

$$\int_0^{2\pi} \phi(s,\theta) e^{-ik\theta}\, d\theta = 0, \quad \forall s > 0,\ \forall |k| < k_0.$$

Lemma 5.4. *There is a k_0 with the following property: Suppose that ϕ and h are two functions that are 2π-periodic in θ, satisfy (5.7) and*

$$\phi(0,\theta) = 0 \quad \forall \theta \in [0,2\pi],$$
$$\|e^{\gamma s}\phi\|_{L^\infty} < \infty, \quad \|e^{\gamma s} h\|_{L^\infty} < \infty.$$

In addition, if ϕ is continuous and the Fourier coefficients of order less than k_0 of ϕ and h are zero, then there is a constant C independent of ϕ and h such that

$$\|e^{\gamma s}\phi\|_{L^\infty} \le C \|e^{\gamma s} h\|_{L^\infty}.$$

Proof. We proceed by contradiction and assume that the statement fails. We then have two sequences ϕ_n, h_n such that ϕ_n, h_n are 2π-periodic in θ, with vanishing Fourier coefficients of order less than k_0 (k_0 will be fixed later), ϕ_n solves (5.7) with right hand side h_n, $\phi_n(0,\theta) = 0$ for $\theta \in [0,2\pi]$, and

$$\|e^{\gamma s}\phi_n\|_{L^\infty} = 1, \quad \|e^{\gamma s} h_n\|_{L^\infty} \to 0, \quad \text{as } n \to \infty. \tag{5.11}$$

Consider $\bar{\phi}(s) = e^{-\gamma s}$. By (5.6) we see that

$$L_0 \bar{\phi} = (\gamma^2 - \gamma) e^{-\gamma s}(1 + O(s)) \le \frac{\gamma^2 - \gamma}{2} e^{-\gamma s}$$

for $s \ge s_0$ (here we fix $s_0 > 0$ large). Using the maximum principle with $\bar{\phi} + \sigma e^{-\tilde{\gamma} s}$ for $0 < \tilde{\gamma} < \gamma$ and $\sigma > 0$, and then letting $\sigma \to 0$, we obtain

$$\|e^{\gamma s}\phi_n\|_{L^\infty((s_0,\infty)\times[0,2\pi]))} \le C\|e^{\gamma s} h_n\|_{L^\infty} + C\|e^{\gamma s}\phi_n\|_{L^\infty((0,s_0)\times[0,2\pi]))}.$$

From this and (5.11), we deduce that for a subsequence (also denoted ϕ_n)

$$\|e^{\gamma s}\phi_n\|_{L^\infty((0,s_0)\times[0,2\pi]))} \geq c_0$$

for some constant $c_0 > 0$. By standard elliptic estimates, up to a new subsequence, $\phi_n \to \phi$ uniformly on compact subsets of $[0,\infty) \times [0,2\pi]$ and by the previous remark, $\phi \not\equiv 0$. But because of (5.11), ϕ satisfies

$$L_0\phi = 0 \quad \text{in } (0,\infty) \times [0,2\pi],$$

with $\phi(0,\theta) = 0$ and $|\phi(s)| \leq e^{-\gamma s}$.

Let us now expand ϕ in Fourier series as in (5.8). Due to the hypotheses, $\phi_k = 0$ for $|k| < k_0$. Note that ϕ_k satisfies (5.9) with the right-hand side equal to 0 and $\phi_k(0) = 0$.

Once again, we use the function $\bar\phi(s) = e^{-\gamma s}$ as a barrier because

$$\bar\phi'' + b(s)\bar\phi' + (|A|^2 - a(s)k^2)\bar\phi = \Big(\gamma^2 - \gamma b(s) + |A|^2 - a(s)k^2\Big)e^{-\gamma s}.$$

From the fact that $a(s) > 0$ for all $s \geq 0$ and the estimates on the coefficients (5.6), we see that there is a k_0 such that

$$\bar\phi'' + b(s)\bar\phi' + (|A|^2 - a(s)k^2)\bar\phi < -c_k e^{-\gamma s}, \quad \forall |k| \geq k_0, \forall s \geq 0, \tag{5.12}$$

where $c_k > 0$. From the maximum principle, we deduce that $\phi_k \equiv 0$ for all $|k| \geq k_0$. This is a contradiction. $\square$

Proof of Proposition 5.2. Let k_0 be as in Lemma 5.4. Using Lemma 5.3, for each $|k| < k_0$, we find a solution ϕ_k of (5.9) in $(0,\infty)$. Then we need to prove the proposition only under the assumption that the Fourier coefficients of order less than k_0 of h vanish.

For the moment, let us assume in addition that h is C^2 and

$$|h_{\theta\theta}(s,\theta)| \leq Ce^{-\gamma s} \quad \text{for all } (s,\theta) \in [0,\infty) \times [0,2\pi]. \tag{5.13}$$

Write h in Fourier series as in (5.8) for $|k| \geq k_0$, we can find a solution ϕ_k of (5.9) in $(0,\infty)$ with right-hand side h_k, satisfying $\phi_k(0) = 0$. This can be done using the supersolution $\bar\phi(s) = e^{-\gamma s}$ and (5.12). Alternatively, one can use the variation of parameters formula and elements in the kernel as in Lemma 5.3. For $m > k_0$, let

$$\phi_m(s,\theta) = \sum_{k_0 \leq |k| \leq m} e^{ik\theta}\phi_k(s)$$

and similarly define h_m. By Lemma 5.4,

$$\|e^{\gamma s}\phi_m\|_{L^\infty} \leq C\|e^{\gamma s}h_m\|_{L^\infty},$$

with C independent of m. Note that

$$|h_k(s)| = \frac{1}{2\pi}\left|\int_0^{2\pi} h(s,\theta)e^{ik\theta}\, d\theta\right| = \frac{1}{2\pi k^2}\left|\int_0^{2\pi} h_{\theta\theta}(s,\theta)e^{ik\theta}\, d\theta\right| \le \frac{C}{k^2}e^{-\gamma s}.$$

Then

$$|h_m(s,\theta)| = \left|\sum_{k_0\le|k|\le m} h_k(s)e^{ik\theta}\right| \le \sum_{k_0\le|k|\le m} |h_k(s)| \le Ce^{-\gamma s}\sum_{k_0\le|k|\le m} k^{-2}.$$

Therefore

$$\|e^{\gamma s}\phi_m\|_{L^\infty} \le C,$$

with C independent of m. By standard elliptic estimates, for a subsequence $m \to \infty$ we find $\phi_m \to \phi$ uniformly on compact subsets of $[0,\infty)\times[0,2\pi]$ and ϕ is the desired solution.

Next, we lift the assumption (5.13). Indeed, assume only $\|e^{\gamma s}h\|_{L^\infty} < \infty$ and that the Fourier coefficients of h vanish for $|k| < k_0$. Let ρ_n be a sequence of mollifiers in $\mathbb{R}^2$ and

$$h_n = h * \rho_n$$

(extending h by 0 for $s \le 0$). We have $h_n \to h$ almost everywhere. The Fourier coefficients of order less than k_0 of h_n vanish and

$$|h_n(s,\theta)| \le Ce^{-\gamma s},$$
$$|\partial_{\theta\theta}h_n(s,\theta)| \le C_n e^{-\gamma s}.$$

Using the previous argument we find a solution ϕ_n with

$$\|e^{\gamma s}\phi_n\|_{L^\infty} \le C\|e^{\gamma s}h_n\|_{L^\infty} \le C,$$

with C independent of n and right-hand side h_n. Passing to a subsequence, we find the desired solution. □

5.2. Linear theory on the bounded piece

Let us consider the bounded component $\mathcal{E}_b$ of $\mathcal{E} = \mathcal{M} \setminus \{s \le \frac{\delta_s}{3\varepsilon}\}$ and let $\tilde{\mathcal{E}}_b = \varepsilon\mathcal{E}_b$.

Lemma 5.5. *For $h \in C^\alpha(\overline{\tilde{\mathcal{E}}_b})$, there is a unique solution ϕ of*

$$\begin{cases} \Delta\phi + |A|^2\phi + \partial_z\phi = h & \text{in } \tilde{\mathcal{E}}_b \\ \phi = 0 & \text{on } \partial\tilde{\mathcal{E}}_b. \end{cases}$$

with

$$\|\phi\|_{L^\infty} \le C\|h\|_{L^\infty}, \tag{5.14}$$

where the constant C is independent of h and ϕ.

Proof. Let $L\phi$ be $\Delta\phi + |A|^2\phi + \partial_z\phi$, where the geometric quantities and Laplacian are the ones for $\tilde{\mathcal{E}}_b$. We let L_0 denote the corresponding operator for the paraboloid $\mathcal{P}$.

We can parametrize $\mathcal{P}$ with polar coordinates

$$X_0(r,\theta) = (r\cos(\theta), r\sin(\theta), F(r)),$$

with $r \ge 0$, $\theta \in [0, 2\pi]$, where $F = F_0$ is the unique radially symmetric solution of (2.1) with $F(0) = 0$. Then, in the coordinates r and θ,

$$L_0\phi = B_1(r)\phi_{rr} + B_2(r)\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + A^2(r)\phi,$$

where

$$\begin{aligned} B_1(r) &= \frac{1}{1+F'(r)^2}, \\ B_2(r) &= \frac{1}{r(1+F'(r)^2)} - \frac{F'(r)F''(r)}{(1+F'(r)^2)^2}, \\ A^2(r) &= \frac{F''(r)^2}{(1+F'(r)^2)^3} + \frac{F'(r)^2}{r^2(1+F'(r)^2)}. \end{aligned}$$

As before, we denote by ν_0 the unit normal vector to $\mathcal{P}$ such that $\langle \nu_0, e_z\rangle > 0$. The surface $\tilde{\mathcal{E}}_b$ can then be parametrized by

$$X_0(r,\theta) + \nu_0(r,\theta) f(r/\varepsilon, \theta/\varepsilon)$$

for $r \in [0, R_1]$, $\theta \in [0, 2\pi]$, with some $R_1 > 0$, and where f has the property that $f(r/\varepsilon, \theta/\varepsilon)$ is supported where $r \in [R_1 - 10\delta_s, R_1]$ and f and its derivatives can be bounded by $e^{-\delta_s/(10\varepsilon)}$. This implies that

$$L\phi = L_0\phi + \tilde{L}\phi,$$

where $\tilde{L}$ is a second order differential operator in ϕ with coefficients that are $o(1)$ as $\varepsilon \to 0$ and supported in $r \in [R_1 - 10\delta_s, R_1]$.

Let $v(r) = \langle \nu_0, e_z\rangle$. Then $L_0 v = 0$ and $v(r) > 0$ for all $r \ge 0$. We define now

$$\bar{\phi}(r) = v(r) - \mu e^{-Kr},$$

where $\mu = \frac{1}{2}\inf_{r\in[0,R_1]} v(r) > 0$ and $K > 0$ is to be chosen. Then, we compute

$$L\bar{\phi} = \mu e^{-Ks}\left[-B_1(r)K^2 + KB_2(r) - A^2(r)\right] - \mu\tilde{L}(e^{-Ks}) + \tilde{L}v.$$

But $\tilde{L}v = O(\varepsilon)$, $\tilde{L}(e^{-Kr}) = o(1)K^2 e^{-Kr}$ where in the last expression $o(1)$ is uniform in K as $\varepsilon \to 0$. Since $B_1(r)$ is positive in $[0, R_1]$ we can choose K large so that

$$L\bar{\phi} \le -c, \quad \forall r \in [0, R_1],$$

for some $c > 0$ and all $\varepsilon > 0$ small. Then $\bar{\phi}$ is a super solution for the operator L and hence the bound (5.14) holds. The equation is solved then by super and subsolutions. $\square$

Proof of Proposition 5.1. Let $0 < \gamma < 1$.

Let us consider first one of the unbounded ends $\mathcal{E}_u$ and $\tilde{\mathcal{E}}_u = \varepsilon\mathcal{E}_u$. Recall that $\tilde{\mathcal{E}}_u$ is parametrized by (5.4) so $s \ge 0$, $\theta \in [0, 2\pi)$ are global coordinates on this surface. We write $s(x)$ for $x \in \tilde{\mathcal{E}}_u$. Using s, θ we may identify $\tilde{\mathcal{E}}_u$ with an unbounded piece of the paraboloid or catenoid. We write this piece as E_u.

Let $\alpha \in (0, 1)$ and define the norms for functions defined on E_u:

$$\|h\|_{k,\alpha} = \sup_{x \in E_u} e^{\gamma s(x)} \|h\|_{C^{k,\alpha}(\overline{B}_1(x))},$$

where $B_1(x)$ is the geodesic ball of center x and radius 1 in E_u.

If h is defined on E_u and $\|h\|_{0,\alpha} < \infty$, using Proposition 5.2, we obtain a solution $\phi = T_0(h)$ of (5.7), 2π-periodic in θ, and such that

$$\|e^{\gamma s} T_0(h)\|_{L^\infty} \le C\|e^{\gamma s} h\|_{L^\infty} \le C\|h\|_{0,\alpha}.$$

By considering ϕ and h as functions on E_u, equation (5.7) becomes

$$\Delta_{E_u}\phi + |A_{E_u}|^2\phi + \partial_z\phi = h \quad \text{on } E_u.$$

Standard elliptic estimates, applied on geodesic balls of radius 1 of E_u, give that

$$\|\phi\|_{2,\alpha} \le C(\|h\|_{0,\alpha} + \|\phi\|_{C^{2,\alpha}(\partial E_u)}).$$

Note that ∂E_u in the coordinates s, θ corresponds to $s = 0$. Consider the representations of ϕ and h as Fourier series as in (5.8). We recall that, by construction, if $\phi = T_0(h)$ then $\phi(0, \cdot)$ has Fourier modes of index $|k| \ge k_0$ equal to zero, where k_0 is a fixed integer. Hence

$$\|\phi\|_{C^{2,\alpha}(\partial E_u)} \le C \sum_{|k| \le k_0} |\phi_k(0)|.$$

The solutions ϕ_k are constructed in Lemma 5.3 and, in particular, we have $|\phi_k(0)| \le C_k\|e^{\gamma s} h_k\|_{L^\infty}$. From this, we deduce

$$\|\phi\|_{C^{2,\alpha}(\partial E_u)} \le C\|e^{\gamma s}h\|_{L^\infty},$$

and hence

$$\|T_0(h)\|_{2,\alpha} \le C\|h\|_{0,\alpha}. \tag{5.15}$$

To solve equation (5.3), we rewrite it as

$$\Delta_{E_u}\phi + |A_{E_u}|^2\phi + \partial_z\phi = h + \tilde{L}\phi \quad \text{on } E_u, \tag{5.16}$$

where $\tilde{L}$ a second order elliptic equation with coefficients that are $o(1)$ as $\varepsilon \to 0$ and with compact support. This translates to

$$\|\tilde{L}\phi\|_{0,\alpha} \le o(1)\|\phi\|_{2,\alpha}. \tag{5.17}$$

Using the operator T_0, we can find a solution of (5.16) by solving the fixed point problem

$$\phi = T_0(h + \tilde{L}\phi)$$

in the Banach space $\{\phi \in C^{2,\alpha}(\overline{E}_u) : \|\phi\|_{2,\alpha} < \infty\}$. By (5.15), (5.17) and the contraction mapping principle, this fixed point problem has a unique solution. This yields a solution of (5.3). By scaling we obtain therefore a solution of (5.1) in any of the unbounded ends.

The proof for the bounded component $\mathcal{E}_b$ of $\mathcal{E} = \mathcal{M} \setminus \{s \le \frac{\delta_s}{3\varepsilon}\}$ is similar, if one uses Lemma 5.5, and the fact that the boundary condition is taken equal to 0. □

6. Proof of Theorem 1.1

To prove the theorem, it is sufficient to find a solution ϕ of (1.10), that is,

$$\Delta\phi + |A|^2\phi + \varepsilon\nabla\phi \cdot e_z + E + Q(x, \phi, \nabla\phi, D^2\phi, x) = 0 \quad \text{in } \mathcal{M}, \tag{6.1}$$

where $\mathcal{M}$ is the surface constructed in Section 2, which depends on $\beta_1, \beta_2, \tau_1, \tau_4 \in [-\delta_p, \delta_p]$, and $E = H - \varepsilon\nu \cdot e_z$. Later, we will verify that $\{x + \phi(x)\nu(x) : x \in \mathcal{M}\}$ is an embedded complete surface.

Thanks to Proposition 1.2, $E = E_0 + E_d$ with $\|E_0\|_{**} \le C\varepsilon$ and

$$E_d = \tau_1 w_1 + \tau_4 w_2 + \beta_1 w_1' + \beta_4 w_2' + O\left(\sum \beta_i^2 + \tau_i^2\right),$$

where $O\left(\sum \beta_i^2 + \tau_i^2\right)$ are smooth functions with compact support, with $\|\ \|_{**}$ bounded by $\sum_{i=1,4} \beta_i^2 + \tau_i^2$. Thus (6.1) takes the form

$$\begin{aligned} \Delta\phi + |A|^2\phi + \varepsilon\nabla\phi \cdot e_z + \tilde{E} + Q(\phi, \nabla\phi, D^2\phi) + \tau_1 w_1 \\ + \tau_4 w_2 + \beta_1 w_1' + \beta_4 w_2' = 0, \quad \text{in } \mathcal{M}, \end{aligned} \tag{6.2}$$

where $\tilde{E} = E_0 + O\left(\sum_{i=1,4} \beta_i^2 + \tau_i^2\right)$. Hence,

$$\|\tilde{E}\|_{**} \leq C\left(\varepsilon + \sum_{i=1,4}\left(\beta_i^2 + \tau_i^2\right)\right).$$

Note that $\mathcal{M} = \mathcal{M}[\beta_1, \beta_2, \tau_1, \tau_2]$ and the unknowns are ϕ and $\beta_1, \beta_2, \tau_1, \tau_2$.

We look for a solution ϕ of (6.2) of the form

$$\phi = \eta_1\phi_1 + \eta_2\phi_2,$$

where ϕ_1, ϕ_2 are new unknown functions, which solve an appropriate system, and η_1, η_2 are smooth cut-off functions such that

$$\eta_1(s) = \begin{cases} 1 & \text{if } s \leq \frac{2\delta_s}{\varepsilon} - 1 \\ 0 & \text{if } s \geq \frac{2\delta_s}{\varepsilon} \end{cases}$$

$$\eta_2(s) = \begin{cases} 0 & \text{if } s \leq \frac{\delta_s}{3\varepsilon} \\ 1 & \text{if } s \geq \frac{\delta_s}{2\varepsilon} \end{cases},$$

where $s = s(x)$ measures geodesic distance from the core of $\mathcal{M}$.

We introduce next the following system for ϕ_1, ϕ_2:

$$\begin{aligned} \Delta\phi_1 + |A|^2\phi_1 + \varepsilon\partial_z\phi_1 &= -\varepsilon\phi_2\partial_z\eta_2 - 2\nabla\phi_2\nabla\eta_2 - \phi_2\Delta\eta_2 \\ &\quad - \tilde{\eta}_1(\tilde{E} + Q(x, \phi, \nabla\phi, D^2\phi)) - \tau_1 w_1 + \tau_4 w_2 + \beta_1 w_1' + \beta_4 w_2' \quad \text{in } \mathcal{C} \end{aligned} \tag{6.3}$$

$$\begin{aligned} \Delta\phi_2 + |A|^2\phi_2 + \varepsilon\partial_z\phi_2 &= -\varepsilon\phi_1\partial_z\eta_1 - 2\nabla\phi_1\nabla\eta_1 - \phi_1\Delta\eta_1 \\ &\quad - \tilde{\eta}_2((\tilde{E} + Q(x, \phi, \nabla\phi, D^2\phi))) \quad \text{in } \mathcal{E}, \end{aligned} \tag{6.4}$$

where $\mathcal{E} = \mathcal{M} \cap \{s \geq \frac{\delta_s}{3\varepsilon}\}$ is the union of the ends, $\mathcal{C} = \mathcal{M} \cap \{s \leq \frac{2\delta_s}{\varepsilon}\}$ is close to a Scherk surface, and $\tilde{\eta}_1$, $\tilde{\eta}_2$ are smooth cut-off functions such that:

$$\tilde{\eta}_1(s) = \begin{cases} 1 & \text{if } s \leq \frac{\delta_s}{\varepsilon} - 1 \\ 0 & \text{if } s \geq \frac{\delta_s}{\varepsilon} \end{cases}$$

$$\tilde{\eta}_2(s) = 1 - \tilde{\eta}_1(s).$$

In the term $Q(\phi, \nabla\phi, D^2\phi)$ of (6.3), (6.4), ϕ means $\phi = \phi_1\eta_1 + \phi_2\eta_2$. If ϕ_1, ϕ_2 solve (6.3), (6.4), then multiplying (6.3) by η_1 and (6.4) by η_2 we see that $\phi = \phi_1\eta_1 + \phi_2\eta_2$ is a solution of (6.2).

Using the change of variables introduced in the construction of $\mathcal{M}$ in Section 2, we see that solving (6.3) is equivalent to finding a solution to

$$\Delta_\Sigma \phi_1 + |A_\Sigma|^2 \phi_1 = L' \phi_1 - \varepsilon \partial_z \phi_1 - \varepsilon \phi_1 \partial_z \eta_1 - \varepsilon \phi_2 \partial_z \eta_2 - 2\nabla \phi_2 \nabla \eta_2 - \phi_2 \Delta \eta_2$$
$$- \tilde\eta_1 (\tilde E + Q(x, \phi, \nabla\phi, D^2\phi)) - \tau_1 w_1 + \tau_4 w_2 + \beta_1 w_1' + \beta_4 w_2' \quad \text{in } \Sigma, \tag{6.5}$$

where now all the functions are considered on the Scherk surface Σ so that Δ_Σ and A_Σ refer to the Laplace operator and second fundamental form of Σ. By Proposition 3.4, L' is a second order operator with coefficients supported on $s \le 2\delta_s/\varepsilon$ and whose C^1 norms on the region $s \le 5\delta_s/\varepsilon$ are bounded by $\delta_s + \delta_p + \varepsilon$. In principle, by the change of variables, we need to solve (6.5) on a subset of Σ, but finding a solution in all Σ is sufficient. This solution is multiplied later by the cut-off η_1.

Similarly, we consider $\mathcal{E}_0 = \mathcal{M}_0 \cap \{s \ge \frac{\delta_s}{3\varepsilon}\}$, where $\mathcal{M}_0$ is the initial approximation corresponding to $\beta_1 = \beta_2 = \tau_1 = \tau_2 = 0$. For $|\beta_i| + |\tau_i| \le \delta_p$ and $\delta_p > 0$ fixed small, $\mathcal{E}$ is mapped onto $\mathcal{E}_0$ and this mapping allows us to write (6.4) as

$$\Delta_{\mathcal{E}_0} \phi_2 + |A_{\mathcal{E}_0}|^2 \phi_2 + \varepsilon \partial_z \phi_2 = L'' \phi_2 - 2\nabla\phi_1 \nabla \eta_1 - \phi_1 \Delta \eta_1$$
$$- \tilde\eta_2 ((\tilde E + Q(x, \phi, \nabla\phi, D^2\phi))) \quad \text{in } \mathcal{E}_0,$$

where now all functions are considered on the Scherk surface $\mathcal{E}_0$. In particular, $\Delta_{\mathcal{E}_0}$ and $A_{\mathcal{E}_0}$ refer to the Laplace operator and second fundamental form of $\mathcal{E}_0$. The operator L'' has coefficients whose C^1 norms are $o(\delta_p)$ as $\delta_p \to 0$. Indeed, one can see this using the form of the operator on the ends in (5.5) and the continuous dependence of ODE on initial conditions, because the parameters β_i, τ_i determine the initial condition for the differential equation (2.3).

We solve (6.5) on the Scherk surface using Proposition 4.2 with norms

$$\|\phi_1\|_{*,\Sigma} = \sup_{x\in\Sigma} e^{\gamma s(x)} \|\phi_1\|_{C^{2,\alpha}(\overline{B}_1(x))}, \tag{6.6}$$
$$\|h_1\|_{**,\Sigma} = \sup_{x\in\Sigma} e^{\gamma s(x)} \|h_1\|_{C^{\alpha}(\overline{B}_1(x))}.$$

We consider the system (6.4), (6.5) as a fixed point problem for $(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2)$ belonging to the subset $\mathcal{B}$ of $C^{2,\alpha}(\Sigma) \times C^{2,\alpha}(\mathcal{E}_0) \times \mathbb{R}^4$ defined by

$$\mathcal{B} = \{(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2) \mid \max_{i=1,2} (\|\phi_i\|_{*,\Sigma}, |\beta_i|, |\tau_i|) \le M\varepsilon\},$$

where $M > 0$ is a constant to be chosen later and the norms are defined in (5.2) and (6.6).

Consider $(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2) \in \mathcal{B}$. We define $F(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2)$ as follows. Using Proposition 4.2, we let $\bar\phi_1$, $\bar\beta_1$, $\bar\beta_2$, $\bar\tau_1$, $\bar\tau_2$ be the solution of

$$\Delta_\Sigma \bar\phi_1 + |A_\Sigma|^2 \bar\phi_1 = L' \phi_1 - \varepsilon \partial_z \phi_1 - \varepsilon \phi_1 \partial_z \eta_1 - \varepsilon \phi_2 \partial_z \eta_2 - 2\nabla \phi_2 \nabla \eta_2 - \phi_2 \Delta \eta_2$$
$$- \tilde\eta_1 (\tilde E + Q(x, \phi, \nabla\phi, D^2\phi)) - \bar\tau_1 w_1 + \bar\tau_4 w_2 + \bar\beta_1 w_1' + \bar\beta_4 w_2' \quad \text{in } \Sigma. \tag{6.7}$$

In the right-hand side of this equation, all terms are well defined in spite of the fact that ϕ_1, ϕ_2 (after changing variables) are defined only on some subsets, because of the cut-off functions. As we will verify later, the right-hand side has finite $\| \ \|_{**,\Sigma}$ norm, thus Proposition 4.2 applies. Next, using Proposition 5.1, we find a solution of

$$\begin{aligned} \Delta_{\mathcal{E}_0}\bar{\phi}_2 + |A_{\mathcal{E}_0}|^2\bar{\phi}_2 + \varepsilon\partial_z\phi_2 &= L''\phi_2 - 2\nabla\phi_1\nabla\eta_1 - \phi_1\Delta\eta_1 \\ &\quad - \tilde{\eta}_2(\tilde{E} + Q(x,\phi,\nabla\phi,D^2\phi)) \quad \text{in } \mathcal{E}_0, \end{aligned} \tag{6.8}$$

where the right-hand side will be proven to have finite $\| \ \|_{**,\mathcal{E}_0}$ norm. We define

$$(\bar{\phi}_1, \bar{\phi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{\tau}_1, \bar{\tau}_2) = F(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2).$$

Let us verify that F maps $\mathcal{B}$ into $\mathcal{B}$ and is a contraction. Consider

$$(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2) \in \mathcal{B}$$

and let $(\bar{\phi}_1, \bar{\phi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{\tau}_1, \bar{\tau}_2) = F(\phi_1, \phi_2, \beta_1, \beta_2, \tau_1, \tau_2)$. Using (4.10) and standard elliptic estimates, we deduce

$$\begin{aligned} &\|\bar{\phi}_1\|_{*,\Sigma} + |\bar{\beta}_i| + |\bar{\tau}_i| \\ &\quad \le C\|L'\phi_1 - \varepsilon\partial_z\phi_1 - \varepsilon\phi_1\partial_z\eta_1 - \varepsilon\phi_2\partial_z\eta_2 - 2\nabla\phi_2\nabla\eta_2 - \phi_2\Delta\eta_2\|_{**,\Sigma} \\ &\qquad + \|\tilde{\eta}_1(\tilde{E} + Q(x,\phi,\nabla\phi,D^2\phi))\|_{**,\Sigma}. \end{aligned} \tag{6.9}$$

We first remark that, by Proposition 1.2, we have

$$\|\tilde{\eta}_1\tilde{E}\|_{**,\Sigma} \le C_E\varepsilon.$$

If x lies in the support of η_2, that is, $\frac{\delta_s}{3\varepsilon} \le s(x) \le \frac{\delta_s}{2\varepsilon}$, then

$$\|\Delta\eta_2\|_{C^\alpha(\overline{B}_1(x))} \le C\varepsilon^2, \quad \|\phi_2\|_{C^\alpha(\overline{B}_1(x))} \le \varepsilon^{-2}e^{-\gamma\delta_s/\varepsilon - \varepsilon\gamma s(x)}\|\phi_2\|_{*,\mathcal{E}_0}.$$

Then

$$\|\phi_2\Delta\eta_2\|_{**,\Sigma} \le Ce^{-\frac{\delta_s}{2\varepsilon}}\|\phi_2\|_{*,\mathcal{E}_0}.$$

Similarly,

$$\|\nabla\phi_2\nabla\eta_2\|_{**,\Sigma} \le C\varepsilon^{-1}e^{-\frac{\delta_s}{2\varepsilon}}\|\phi_2\|_{*,\mathcal{E}_0} \quad \text{and} \quad \|\phi_2\partial_z\eta_2\|_{**,\Sigma} \le C\varepsilon^{-1}e^{-\frac{\delta_s}{2\varepsilon}}\|\phi_2\|_{*,\mathcal{E}_0}.$$

Because the C^1 norm of the coefficients of L' in $s \le 5\delta_s/\varepsilon$ are bounded by $C(\delta_p + \delta_s + \varepsilon)$, we have

$$\|L'\phi_1\|_{**,\Sigma} \le C(\delta_p + \delta_s + \varepsilon)\|\phi_1\|_{*,\Sigma}.$$

Also,

$$\|\varepsilon\partial_z\phi_1\|_{**,\Sigma} \le C\varepsilon\|\phi_1\|_{*,\Sigma}.$$

For the Q term, analogously to Proposition 1.3, we have

$$\|\tilde\eta_1 Q(x,\phi,\nabla\phi,D^2\phi)\|_{**,\Sigma} \le C\|\phi_1\|^2_{*,\Sigma} + C\varepsilon^{-4}e^{-\frac{\gamma\delta_s}{\varepsilon}}\|\phi_2\|^2_{*,\mathcal{E}_0}.$$

Therefore, using (6.9) and the previous inequalities, we obtain

$$\begin{aligned}\|\bar\phi_1\|_{*,\Sigma} &\le CC_E\varepsilon + C\varepsilon^{-1}e^{-\frac{\delta_s}{2\varepsilon}}\|\phi_2\|_{*,\mathcal{E}_0} + C(\delta_p+\delta_s+\varepsilon)\|\phi_1\|_{*,\Sigma}\\ &\quad + C\|\phi_1\|^2_{*,\Sigma} + C\varepsilon^{-4}e^{-\frac{\gamma\delta_s}{\varepsilon}}\|\phi_2\|^2_{*,\mathcal{E}_0},\end{aligned}$$

which gives

$$\begin{aligned}&\|\bar\phi_1\|_{*,\Sigma}\\ &\quad\le CC_E\varepsilon + C\varepsilon^{-1}e^{-\frac{\delta_s}{2\varepsilon}}M\varepsilon + C(\delta_p+\delta_s+\varepsilon)M\varepsilon + CM^2\varepsilon^2 + C\varepsilon^{-4}e^{-\frac{\gamma\delta_s}{\varepsilon}}M^2\varepsilon. \qquad (6.10)\end{aligned}$$

To estimate $\bar\phi_2$, we use Proposition 5.1 to obtain

$$\|\bar\phi_2\|_{*,\mathcal{E}_0} \le C\|L''\phi_2 - 2\nabla\phi_1\nabla\eta_1 - \phi_1\Delta\eta_1 - \tilde\eta_2(\tilde E + Q(\phi,\nabla\phi,D^2\phi))\|_{**,\mathcal{E}_0}. \qquad (6.11)$$

We first remark that, by Proposition 1.2, we have

$$\|\tilde\eta_2\tilde E\|_{**,\mathcal{E}_0} \le C_E\varepsilon.$$

We note that $\Delta\eta_1$ is supported on $\frac{2\delta_s}{\varepsilon} - 1 \le s \le \frac{2\delta_s}{\varepsilon}$. Using the smoothness of η_1, we get

$$\|\phi_1\Delta\eta_1\|_{**,\mathcal{E}_0} + \|\nabla\phi_1\nabla\eta_1\|_{**,\mathcal{E}_0} \le Ce^{-\frac{\gamma\delta_s}{\varepsilon}}\|\phi_1\|_{*,\Sigma}.$$

Again, as in Proposition 1.3, we have

$$\|\tilde\eta_2 Q(x,\phi,\nabla\phi,D^2\phi)\|_{**,\mathcal{E}_0} \le C\|\phi_1\|^2_{*,\Sigma} + C\varepsilon^{-4}e^{-\frac{\gamma\delta_s}{\varepsilon}}\|\phi_2\|^2_{*,\mathcal{E}_0}.$$

Combining (6.11) with the previous inequalities, we arrive at

$$\|\bar\phi_2\|_{*,\Sigma} \le CC_E\varepsilon + o(\delta_p)M\varepsilon + C\,e^{-\frac{\delta_s}{2\varepsilon}}M\varepsilon + CM^2\varepsilon^2 + C\varepsilon^{-4}e^{-\frac{\gamma\delta_s}{\varepsilon}}M^2\varepsilon. \qquad (6.12)$$

The right-hand sides of (6.10) and (6.12) are less than $M\varepsilon$ provided M is chosen large (for example $2CC_E$). Then we fix δ_p, δ_s small and work with small $\varepsilon > 0$. A similar estimate holds for $\bar\beta_i$, $\bar\tau_i$ and this shows that F maps $\mathcal{B}$ to itself.

Let us verify that F is a contraction in $\mathcal{B}$. For this, we first claim that

$$|F(\phi_1,\phi_2,\beta_1,\beta_2,\tau_1,\tau_2) - F(\psi_1,\psi_2,\beta_1,\beta_2,\tau_1,\tau_2)| \le o(1)(\|\phi_1-\psi_1\|_{*,\Sigma} + \|\phi_2-\psi_2\|_{*,\mathcal{E}_0})$$

for $(\phi_1,\phi_2,\beta_1,\beta_2,\tau_1,\tau_2), (\psi_1,\psi_2,\beta_1,\beta_2,\tau_1,\tau_2) \in \mathcal{B}$, where $o(1)$ is small if we choose the parameters $\delta_p, \delta_s > 0$ small and then let ε be small. The estimate relies on the same computations as before for the terms that are linear in ϕ_1, ϕ_2 in the right-hand side of equations (6.7) and (6.8). For the nonlinear terms, it is enough to have the following inequalities, whose proof is similar to Proposition 1.3: For $\phi = \eta_1\phi_1 + \eta_2\phi_2$, $\psi = \eta_1\psi_1 + \eta_2\psi_2$,

$$\begin{aligned}&\|\tilde\eta_1 Q(x,\phi,\nabla\phi,D^2\phi) - \tilde\eta_1 Q(x,\psi,\nabla\psi,D^2\psi)\|_{**,\Sigma}\\ &\qquad \le C(\|\phi_1\|_{*,\Sigma} + \|\phi_2\|_{*,\mathcal{E}_0} + \|\psi_1\|_{*,\Sigma} + \|\psi_2\|_{*,\mathcal{E}_0})(\|\phi_1-\psi_1\|_{*,\Sigma} + \|\phi_2-\psi_2\|_{*,\mathcal{E}_0})\end{aligned}$$

and there is a similar estimate for $\tilde\eta_2 Q$. The Lipschitz dependence of F on β_i, τ_i with small Lipschitz constant is proved using the fact that, in each term in the right-hand side of (6.7) and (6.8), either the dependence on the parameters is Lipschitz with small constant or is quadratic (this is the case of $\tilde E$).

By the contraction mapping principle, for $\varepsilon > 0$ small, F has a unique fixed point in $\mathcal{B}$. This gives the desired solution.

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