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Optimal fleet size, frequencies and vehicle capacities considering peak and off-peak periods in public transport

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A R T I C L E I N F O

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A B S T R A C T

The fleet for surface transit in urban areas - number of buses and vehicle size - is usually determined by the characteristics of demand during the peak period, adjusting frequencies for other periods where those characteristics are quite different. Here the problem is formulated analytically considering a representative single line where hourly passengers flow and average trip lengths differ between peak and off-peak periods. Analytical and numerical results show that minimizing social costs (operators and users) for the whole day yields a larger fleet of smaller buses than if only the peak period is considered. Contrary to previous results on this issue, optimal frequencies across periods differ by a large amount, as optimal peak and off-peak frequencies are larger and smaller respectively than what would be obtained modeling each period by itself. The optimal bus size lies in-between the capacities obtained when each period is independently optimized. Several other interesting counterintuitive results are shown.

1. Introduction

What is the adequate fleet of transit vehicles to satisfy users demand in a given city? A good design of public transport services in urban areas is very much dependent on the pattern of demand in space-time. Within the limits of a city this pattern, however, is far from homogeneous along a day, particularly when comparing the so-called peak and off-peak periods. The most notable structural differences are the hourly demand and both the distribution and average length of the trips; besides these, the traffic conditions under which buses circulate also varies importantly due to congestion. In principle, therefore, there is no reason to believe that a design that is optimal for one period is also optimal for the other regarding lines structure (buses network), fleet sizes and bus capacities. This poses an important challenge, as period-specific designs do not seem feasible in terms of some of its components such as the lines structure or bus size, but it does when it comes to determine frequencies; the main issue, then, is how to design the system making the best usage of a total fleet considering various periods. Even considering that routes must be fixed along the day out of convenience for the average user, there remains the question of optimal fleet size and vehicles capacity. In this paper, we deal with this problem by jointly optimizing a simplified system over two periods. We find novel results that contradict existing beliefs regarding the optimal levels of fleet size, frequencies and bus capacities.

In the remainder of this section, we summarize the treatment of the multiperiod design problem in the literature, showing that important simplifying assumptions are imposed to get results. In the next section we present a synthesis of the single line-one period model to determine optimal (social cost minimizing) fleet size, frequency and vehicles capacity. Then we expand the model to account for the two periods simultaneously in the third section, raising most of the simplifying assumptions and obtaining some analytical results regarding optimal fleet, vehicle size and period-specific frequencies and their relation with the most important

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exogenous parameters, notably the effect of demand flows in one period over design and operations in all periods. In the fourth section numerical results are shown for the cases representing peak and off-peak periods independently, which are then compared against the optimal design considering both periods; besides verifying some of the analytical findings, further results are obtained. In the final section, the main conclusions are presented.

The two-periods problem was formulated by Jansson (1980, 1984), highlighting that the incremental cost of running an already acquired bus during the off-peak is only operational. Jansson does not solve the problem for the general case with different optimal frequencies in peak and off-peak, but applies two approaches instead. In his first model operators' cost are assumed to be independent of bus size. Arguing that a single frequency is likely to be optimal in most cases, he obtains an extended "modified square root" expression for the optimal value of frequency. Peak and off-peak demands matter in the argument of the square root through two additive terms: one proportional to the average hourly demand and the other proportional to the product between this average and the peak-hour demand. In his second model - where bus cost does depend on size - he again solves the problem analytically assuming equality between frequencies, but pointing out that in this case where bus size is a variable of the problem, equal frequencies are less likely to be optimal. The common optimal frequency equation obtained follows the same general form, with a new term inside the square root associated to the cost of the marginal size of the bus and proportional to the square of the peak demand. The optimal bus size is obtained as the ratio between peak flow and the optimal frequency.

Chang and Schonfeld (1991) solved a multiperiod problem that optimizes the number of parallel routes in an area and, for each period, the frequency of the routes. Thanks to some simplifying assumptions (constant cycle and travel times, and bus cost not depending on size) they obtain analytical expressions for both optimization variables. These generalize an interesting rule obtained for the single-period case by Kocur and Hendrickson (1982): both the optimal number of routes and the optimal frequency depend on the cubic root of the demand. In the multiperiod case setting of Chang and Schonfeld (1991), the frequency of each period also depends on the demand of the other periods.

Design considering multiperiod problems have also been addressed in other transport contexts. Just to show some examples, Yang et al. (2005) and Gan et al. (2013) have studied how these different periods affect the taxi market, considering that both demand and congestion change during the day. Medina et al. (2013) face the problem of deciding where to locate bus stops, in a context of changing demand according to the period of the day. Ibarra-Rojas et al. (2015) tackle the timetabling problem also for a multiperiod context. Guo et al. (2017) analyze synchronization between metro lines, with an emphasis in the transition between peak and off-periods. Paulley et al. (2006) study the effect of different factors on public transport, and these effects are shown to change between peak and off-peak conditions.

2. The one period model

The stylized representation of the one period design problem follows the developments by Mohring (1972), who introduced the "square root formula" in a simple model that optimized the frequency of a line considering that operators' cost increases with frequency, whereas waiting time decreases with it; the resulting optimal frequency is proportional to the square root of the demand. Many refinements have been made over this basic model. Probably the most important one was the inclusion of the impact of passengers boarding and alighting on both bus cycle and passengers' in-vehicle times in such a way that a closed analytical solution is obtained (Jansson, 1980, 1984). The relation between optimal frequency and demand becomes linear for large patronage volumes (modified square root formula). Jara-Díaz and Gschwender (2009) analyzed the impact of a financial constraint on the optimal frequency and vehicle size, expanding Jansson's model by making bus cost dependent on its size. Let us recall this general formulation for the one-period model over a single line, defining at the same time some notation that will be useful in the rest of the paper. Let us start with the Value of the Resources Consumed, VRC:

$$VRC = B(C_B + C_K K) + \frac{p_w}{2}\frac{Y}{f} + \frac{p_v l t_c Y}{L} \tag{1}$$

The first term corresponds to the operators' costs, while the second and third are those corresponding to the users (their time). Operators' costs is given by the fleet size B times the cost of acquiring and using each bus, which has been shown to have a linear dependency on bus capacity K (Jansson, 1980, 1984); C_B and C_K are the corresponding exogenous unit costs. Users' cost depends on total waiting and in-vehicle times, which are expressed as a function of the (parametrically known) number of users that enters the system per hour Y, the frequency of the line f, the cycle time t_c, and the ratio between average trip length l and the total route length L (which converts Y into the passengers' flow at every point of the cycle). p_w and p_v are waiting time and in-vehicle time values, respectively. Both users and buses are assumed to arrive at the bus stops at regular intervals, which makes average waiting time equal to $\frac{1}{2f}$.[1] Note that behind C_B and C_K there are two types of operators' expenses, operating and capital costs, representing usage and acquisition respectively. As this distinction will become relevant in the analysis of the two-periods case, it is convenient to recall that each of these can be expressed as linear in capacity functions as well (Jansson, 1980, 1984), i.e. capital cost is given by $c_{BC} + c_{KC}K$ and operating costs correspond to $c_{BO} + c_{KO}K$.

Eq. (1) may be solved by expressing all the design terms as a function of the frequency f. Following Jansson (1980, 1984), vehicle cycle time is given by the addition of time in motion T and time at stops, i.e. $t_c = T + tY/f$, where t is boarding-alighting time per passenger. As frequency is total fleet size divided by cycle time, we obtain that $B = fT + tY$. Regarding vehicles capacity, costs

[1] Regularity and the absence of randomness in demand simplifies the analysis and dismisses some possible effects such as overcrowding and denied boarding.

minimization induces a value of K not larger than the minimum necessary to carry all passengers, i.e. $K = \frac{lY}{Lf}$, such that buses are used at capacity, and there is neither overcrowding nor denied boarding. Replacing B and K in (1) yields expression (2) where only f is a variable

$$VRC = C_B Tf + \frac{tY^2\frac{l}{L}(C_K + p_v) + p_w Y/2}{f} + TC_K\frac{lY}{L} + tYC_B + \frac{p_v lYTP}{L} \tag{2}$$

The optimal values for f and K are

$$f^* = \sqrt{\frac{Y\left(\frac{1}{2}p_w + p_v tY\frac{l}{L}\right) + \frac{c_K tY^2 L}{L}}{c_B T}} \quad \text{and} \quad K^* = \frac{lY}{Lf^*} \tag{3}$$

According to this result, both optimal frequency and bus size increase with patronage at a decreasing rate but, as shown by Jara-Díaz and Gschwender (2003, 2009), frequency grows faster than capacity. For large patronage, capacity is asymptotic to a maximum value while frequency increases linearly.

This procedure can be applied to obtain optimal frequencies and capacities for both the peak period and the off-peak period separately. As evident, optimal single period capacities will be different, requiring two fleets with potentially different vehicle sizes to serve the whole day. This is the kernel of the problem we want to solve: to find a single total fleet of some capacity that can offer period-specific frequencies.

Before moving into the two periods formulation, note that defining $A = TC_B$ and $G = \frac{Y^2lt}{L}(C_K + p_v) + \frac{p_w Y}{2}$, the VRC can be re-written as $VRC = Af + \frac{G}{f} + W$, where W collects all terms that are independent of f and play no role in the optimization problem. With this notation, the result of the optimizations process yields $f^* = \sqrt{\frac{G}{A}}$, an expression that will prove useful in the analysis that follows.

3. Two periods' model

3.1. Formulation and first order conditions

Here we extend the previous model to the case of optimizing simultaneously peak and off-peak periods. While keeping the essence of Jara-Díaz and Gschwender (2009), we improve over the two periods' formulations by Jansson (1984, first model) and Chang and Schonfeld (1991) in four aspects: bus costs depend on vehicle size, frequencies can be different across periods, and neither cycle nor travel time are assumed constant. To be precise, the general scheme is preserved, with a single fleet composed by buses of the same size K, but offering frequencies f_P and f_N in the peak and off-peak periods, respectively.[2] The passengers per hour, the length of the trips, and the time needed to tour the whole circuit (related to the velocity of the buses) are also dependant on the period, so the following parameters are needed: Y_P,Y_N,l_P,l_N,T_P and T_N. The durations of each period are denoted by E_P and E_N.

Operators' cost must be divided into two types: capital costs (buying the buses and terminals) that are the same for all buses, and operational costs (like maintenance or fuel) that depend on how many hours the bus is used. Each of these components is well described by a function that is linear in vehicle size, as in Eq. (1) (Jansson, 1980, 1984). The capital cost will be caused by the largest number of buses B_i needed considering both periods (usually the peak period); in the other period not necessarily all buses will be used. Operating costs will depend on the number of buses used during each period of length E_i. Therefore, the Value of the Resources Consumed in a day is now described by:

$$VRC_2 = max(B_P,B_N)(c_{BC} + c_{KC}K) + (B_P E_P + B_N E_N)(c_{BO} + c_{KO}K) + \frac{p_w}{2}\left(\frac{Y_P E_P}{f_P} + \frac{Y_N E_N}{f_N}\right) + \frac{p_v}{L}(l_P t_{cP} Y_P E_P + l_N t_{cN} Y_N E_N) \tag{4}$$

Optimality imposes that buses are fully loaded during one of the periods; we will assume that this happens during the peak, as is commonly observed, making $K = \frac{Y_P l_p}{f_P L}$, something that should be verified numerically (as in the following section). Also, the fleet size is given by the maximum number of buses, which is associated with the largest demand flow that defines the peak period, making $max(B_P,B_N) = B_P$. Replacing these and the values of B_i and t_{ci} ($i = P,N$) with the equations for B and t_c described in Section 2, we obtain Eq. (5) that depends only on peak and off-peak frequencies. After frequencies are found, the fleet size and vehicles capacity are obtained as explained above.

$$\begin{aligned} VRC_2 = {} & (f_p T_p + tY_P)\left(c_{BC} + c_{KC}\frac{Y_P l_P}{f_P L}\right) + [(f_P T_P + tY_P)E_P + (f_N T_N + tY_N)E_N]\left[c_{BO} + c_{KO}\frac{Y_P l_P}{f_P L}\right] + \frac{p_w}{2}\left(\frac{Y_P E_P}{f_P} + \frac{Y_N E_N}{f_N}\right) \\ & + \frac{p_v}{L}\left(l_P Y_P E_P\left[t\frac{Y_p}{f_P} + T_P\right] + l_N Y_N E_N\left[t\frac{Y_N}{f_N} + T_N\right]\right) \end{aligned} \tag{5}$$

Expanding the terms introduced in the single period case we define A_i and G_i ($i = P,N$) and δ as

[2] Note that extensions to a network of transit lines admit the possibility of different bus sizes, as in Fielbaum et al. (2016).

$$A_P = T_P c_{BC} + T_P E_P c_{BO} \tag{6}$$

$$G_P = \frac{tY_P^2 l_P (c_{KC} + c_{KO} E_P)}{L} + tY_N E_N c_{KO} \frac{Y_P l_P}{L} + \frac{p_w}{2} Y_P E_P + \frac{p_v}{L} l_p E_P Y_P^2 t \tag{7}$$

$$A_N = T_N E_N c_{B0} \tag{8}$$

$$G_N = \frac{p_w}{2} Y_N E_N + \frac{p_v}{L} l_N E_N Y_N^2 t \tag{9}$$

$$\delta = \frac{T_N E_N c_{KO} Y_P l_P}{L} \tag{10}$$

Using Eqs. (6)–(10) the value of the resources consumed can now be written as

$$VRC_2 = A_P f_P + \frac{G_P}{f_P} + A_N f_N + \frac{G_N}{f_N} + \delta \frac{f_N}{f_P} \tag{11}$$

The derivatives with respect to frequencies in compact form are

$$\frac{\partial VRC_2}{\partial f_N} = \left(A_N + \delta \frac{1}{f_P}\right) - \frac{G_N}{f_N^2} \tag{12}$$

$$\frac{\partial VRC_2}{\partial f_P} = A_P - \frac{G_P + \delta f_N}{f_P^2} \tag{13}$$

The minimum VRC is obtained by making these expressions equal to zero, which yields a system of equations where frequencies are to the power of 5, such that no analytical solution can be found as established by the Abel-Ruffini theorem (see, for example, Alekseev, 2004). Nevertheless, in what follows we show that some interesting properties can be deduced by inspection of the first order conditions. Optimal frequencies will be found and analyzed numerically in Section 4.

3.2. Comparing optimal and single period frequencies

Let us begin analyzing the differences between the optimal frequencies when jointly optimizing both periods and the sub-optimal solutions considering each period by itself. We will start with an inspection of frequency in the peak period. Making Eq. (13) equal 0 the optimal frequency f_P^* fulfills:

$$f_P^* = \sqrt{\frac{G_P + \delta f_N^*}{A_P}} \tag{14}$$

This expression can be compared with the solution for the one-period problem

$$f_P^1 = \sqrt{\frac{G}{A}} \tag{15}$$

Looking at the denominators, A and A_P are equal, as the latter is just the sum of the different components within the former. Regarding the numerators, $G_P > G$ due to the presence of the term that multiplies Y_N (involving c_{KO}); moreover, there is an additional positive term in (14) that also involves c_{KO}. This yields

$$f_P^* > f_P^1 \tag{16}$$

i.e., the optimal frequency in the peak period is higher if the buses are also going to be considered for the off-peak period.

The interpretation of inequality (16) is quite interesting. Looking at Eqs. (14) and (15), the differences between the numerators in the squared roots are related with the optimal value of bus capacity (through c_{KO}), that has to be chosen taking into account that buses in this process are also going to run during the off-peak period. Note that the effect through G_P is related with the off-peak time at stops while passengers board and alight, while the effect through δ is due to the time in-motion. For short, bus size becomes more relevant as increasing K is now more costly than in a single peak-period analysis, making the optimal vehicle capacity smaller than what a single peak-period analysis would yield, with higher peak frequencies. This fits the intuition that bus sizes should be somewhere in between peak and off-peak optimal sizes when considered in isolation. This result was overlooked by Jansson (1980, 1984) because in his first two-periods model bus size is costless and argues that $f_P = f_N$ would be optimal in most cases; in his second model (where bus cost depends on size) he solves the problem analytically assuming equality between frequencies, which is something examined below.

The analysis for f_N^* is similar but the comparison with the single period case is inconclusive. To see this, note that making Eq. (12) equal 0 we get

$$f_N^* = \sqrt{\frac{G_N}{A_N + \delta / f_P^*}} \tag{17}$$

A_N does not include c_{BC} because fleet size (given by B_P) depends on peak frequency, implying that the daily fixed capital cost for a bus is not affected by off-peak operations. Therefore, A_N is smaller than A. However, there is a positive term involving c_{KO} that adds on A_N, provoked by the cost associated to bus sizes that are higher than the size calculated when optimizing this period by itself. Therefore, the denominator may be larger or smaller than A depending on the relative values of c_{BC} and c_{KO}. Regarding the numerator, $G_N < G$ because all the terms involving bus size related costs disappear. Hence, the relation between f_N^* and f_N^1 is unclear because there are elements that operate in opposing directions: on the one hand, an extra bus means no extra capital cost because total fleet is decided with respect to the peak-period (as in Jansson, 1980, 1984); this pushes f_N^* upwards. On the other hand, increasing f_N^* does not induce a reduction in bus size (given by the peak) and, therefore, there are no operators' savings because of this; in addition, the larger buses increase operating costs, which pushes f_N^* downwards. Which of these phenomena is going to dominate depends on whether daily fixed costs prevail over operating costs or vice versa. The answer is non-trivial, as some of the main components (namely drivers' wages) may be considered differently depending, for example, if it is possible to hire drivers just for some hours of the day. If daily fixed costs prevail, off-peak frequencies are going to be higher in the two-period scheme.

3.3. Crossed effects between optimal frequencies and period specific flows

We now study how f_N^* and f_P^* react to changes in the patronage of the other period. Let us first explain it intuitively. If Y_P increases, then both f_P^* and K will increase. From the point of view of off-peak operations, this means that each bus on the street is more expensive than before, so optimal frequency will diminish, that is, $\frac{\partial f_N^*}{\partial Y_p} < 0$. If the off-peak patronage increases, off-peak frequency will also increase. This means that the size of the buses become now more relevant, because more buses will be running the whole day. So capacities must go down, meaning the peak frequencies will increase, that is $\frac{\partial f_P^*}{\partial Y_N} > 0$.

To see the first phenomenon analytically, recall that when deriving VRC_2 with respect to f_N, we obtain Eq. (17), where the only element that depends on Y_P is the ratio Y_P/f_P (implicit in δ/f_P). f_P^* increases with Y_P at a decreasing rate (Jara-Díaz and Gschwender, 2003) because the system responds not only through frequency but also increasing bus size; therefore the ratio Y_P/f_P increases when Y_P increases, and so does K. So f_N^* will indeed decrease as Y_P increases.

In the case of f_P^* given by Eq. (14), both G_P and f_N^* increase with Y_N such that the optimal peak frequency increases. This happens because when off-peak flow increases, both off-peak frequency and cycle time increase and, therefore, the number of buses required (and operating costs) in off-peak increase; as vehicle size weighs more, this effect can be softened by reducing vehicle size. This, however, requires peak frequency to increase.

3.4. Comparison with previous approaches

Finally, the two-periods model presented and analyzed here can be used to investigate the conditions imposed by Jansson (1980, 1984), summarized in the first section. In his first model, Jansson assumes that each bus has the same cost, independently of its size. In our scheme, this is equivalent to put $c_{KC} = c_{KO} = 0$. From Eqs. (6)–(10) it is direct to observe that, under these conditions: $\delta = 0$; A_P,G_P and G_N become equal to the value they would have in the single period case; and A_N only considers operating costs. From the analysis in Section 3.2, the only element that still matters is that buses are already acquired when deciding off-peak frequencies; all the other effects depend on the role of bus size, which does not affect costs under Jansson's assumption. The result is quite direct, both analytically (from Eqs. (14) and (17)) and conceptually: for the peak period, all the conditions are exactly equal to the single period problem, such that optimal frequencies coincide; for the off-peak, as $A_N < A$, optimal frequencies are going to be higher than those obtained by solving this period in isolation. Therefore, even if size had no effect on buses costs, optimal frequencies are going to differ across periods, although off-peak frequency would be larger than in the single period analysis, getting closer to the peak period frequency (which seems to be what lies behind Jansson's intuition).

It is interesting to analyze the expressions that represent f_P^* and f_N^* under Jansson's strong assumption, i.e. when $c_{KC} = c_{KO} = 0$. These expressions are

$$f_P^* = \frac{Y_P}{T_P}\left[\frac{p_w/2 + Y_P p_v l_P t/L}{c_{BC}/E_P + C_{BO}}\right] \quad \text{and} \quad f_N^* = \frac{Y_N}{T_N}\left[\frac{p_w/2 + Y_N p_v l_N t/L}{C_{BO}}\right] \tag{18}$$

There are four period-specific elements that are worth looking at in this comparison: flow, trip length, time in motion and duration of the peak period. By definition, the peak flow is larger than the off-peak flow and trip length is usually longer during the peak as well. Both elements contribute to make the peak frequency larger than the off-peak one. However, time in motion could be larger in the peak due to congestion, which contributes to reduce the difference. A short peak period also works in that direction. This means that Jansson's claim that frequencies are equal under his strong assumption on costs, might hold under very specific circumstances, although observed relative values of these parameters make us expect a systematic positive difference between f_P^* and f_N^* even if this assumption holds, something that can be explored numerically as well.

3.5. Synthesis of the theoretical findings

In order to provide a concise view of the main elements discovered from an analytical perspective, we summarize the findings and intuitive explanations in Table 1.

Table 1
Theoretical findings from the two periods model.

Fact	Considerations	Effects
Peak bus sizes affect off-peak period	1. Bus size has to be chosen considering off-peak operating cost, which increases with K ($G_P + \delta f_N > G$) 2. Off-peak bus size must be sufficient to carry peak flows (Presence of δ/f_P in the denominator of f_N)	1. Optimal K is lower than in the single peak-period case and peak frequency is larger 2. Optimal K is larger than single off-peak period case; off-peak frequency is pushed downwards (buses might not be full)
Fleet is determined by the peak period, so enough buses for off-peak are available	There is no capital cost associated to off-peak ($A_N < A$)	Off-peak frequency is pushed upwards
The size of the buses is the same in both periods	Increasing off-peak frequency cannot be fully compensated by a reduction in vehicle size in order to reduce operators' costs ($G_N < G$)	Off-peak frequency is pushed downwards. Excess capacity appears
Increasing off-peak passengers increases off-peak frequency	Bus operating costs become more important; smaller buses are better	Peak frequency increases. $\frac{\partial f_P^*}{\partial Y_N} > 0$
Increasing peak passengers increases the size of the buses	Each bus running during the off-peak period becomes more costly.	Off-peak frequency decreases. $\frac{\partial f_N^*}{\partial Y_P} < 0$

4. Numerical analysis

As explained in Section 3, closed analytical solutions for the frequencies are impossible to obtain. In this section, we show the optimal values for frequencies, fleet and buses size obtained numerically using the parameters presented in Table 2, inspired on the case of Santiago, Chile. Patronage is treated parametrically, recalling that the model is a single-line representation and, therefore, devoid of spatial interrelations.

In Fig. 1 we show the behavior of total fleet (right axe, brown) and the size of buses (seats per bus, left axe, blue) as patronage (pax/hour) increases, keeping constant the ratio between peak and off-peak flows (10/3). Just as in the single period case, and as expected, both fleet and bus size increase with patronage. Let us analyze now period-specific frequencies and usage.

Table 2
Exogenous parameters for numerical simulation.
Source: SECTRA (2012), DTPM (2017), Espinoza (2017) and Fielbaum et al. (2016).

Parameter	E_P	E_N	T_P	T_N	t	l_P	l_N	L	c_{BC}
Value (units)	5 h	13 h	2 h	1.5 h	2.5 s	10 km	5 km	40 km	4.14 US$

Parameter	c_{KC}	c_{BO}	c_{KO}	p_v	p_w	Y_P (base)	Y_N (base)
Value (units)	0.45 US$/seat	1.32 US$/h	0.10 US$/h-seat	1.48 US$/h	4.44 US$/h	100,000 pax/h	30,000 pax/h

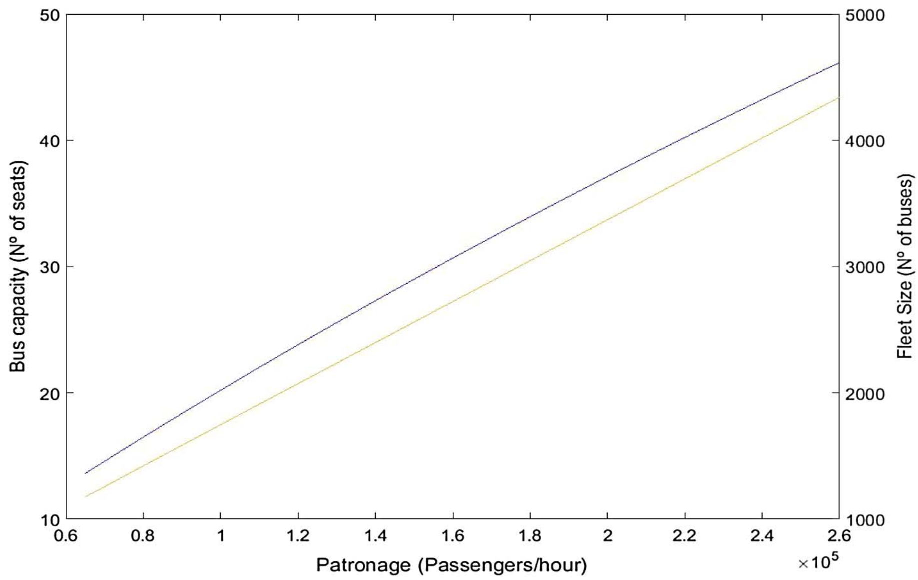

Fig. 1. Optimal fleet size and bus capacity as total patronage grows proportionally.

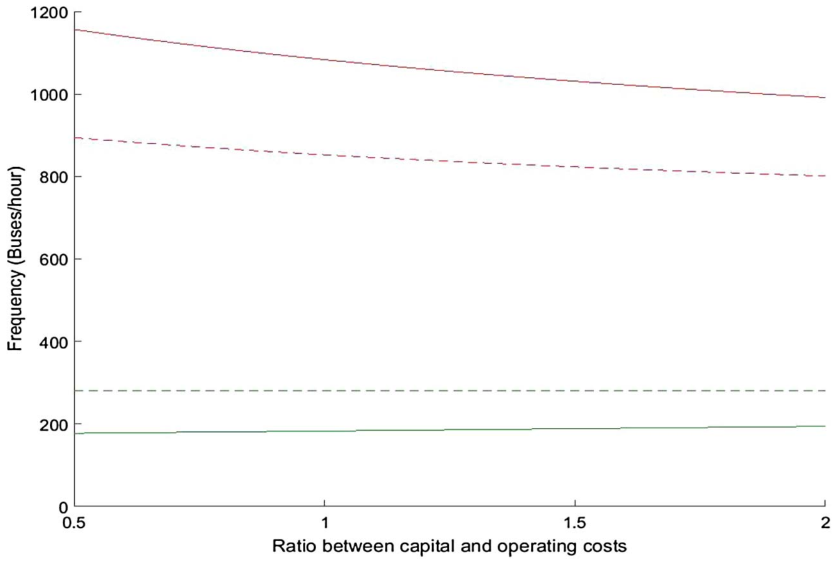

Fig. 2. Optimal frequencies for varying capital costs over operating costs ratio.

As explained, relations between peak and off-peak frequencies are strongly determined by the relationship between the different components of the operators' costs. So we studied the resulting frequencies when varying the ratio between total capital costs and total operating costs (Fig. 2), and the ratio between size-independent and size-dependent bus costs (Fig. 3), keeping flows at the base level shown in Table 2. In both cases the operators' cost of a standard bus (100 seats running 18 h on the streets) was kept constant. This is done in order to alter minimally the ratio between operators' and users' costs. In both figures, the scale varies from one half to double the original value of these ratios.

In these figures, we show the optimal frequencies (buses/hour) considering two periods (solid lines) and those that would be obtained considering each period in isolation (dotted lines). Peak-period frequencies are drawn in red and off-peak in green. As expected, peak-period frequencies are larger than off-peak ones considered in isolation, but the optimal peak frequency considering two periods are even larger (as summarized earlier in Table 1) because of the convenience of smaller buses due to the effect of the off-peak operating costs on vehicle size. The numerical novelty is that frequencies during the off-peak period are smaller than those obtained for a single period analysis. As a result, the difference between optimal frequencies considering two periods is larger than the difference between frequencies when periods are considered in isolation. As a consequence, the (single) optimal vehicle size is smaller than the single peak period case and larger than the single off-peak case.

Comparing our results with Jansson's (1980, 1984), he assumed that frequencies are equal, while we show that differences are even larger than when solving peak and off-peak separately, as predicted theoretically in the previous section. Following Fig. 2, the difference between the optimal frequencies (2 periods) diminishes as capital costs become more important, which fits intuition: once a bus is bought, it is better to use it.

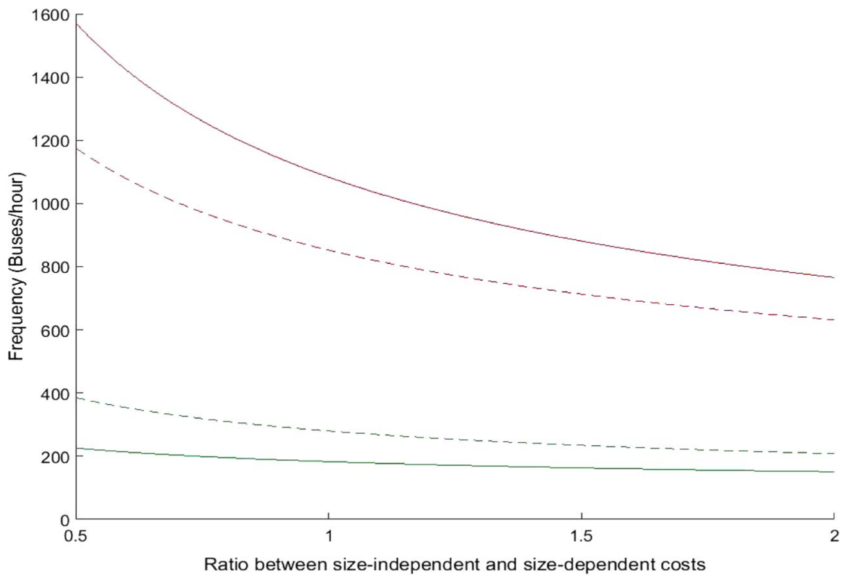

Fig. 3. Optimal and single period frequencies for varying size-independent over size-dependent bus costs ratio.

Fig. 3 confirms the same general conclusions. Besides, peak frequency drops significantly which means that bus size increases as costs related to size turn less relevant; this pushes optimal (two-periods) frequencies towards those from a single period analysis. Extending Fig. 3 to the right (where size dependent costs vanish, as in Jansson's model) the difference between peak and off-peak frequencies diminishes but remains significantly positive, as advanced at the end of Section 3.

Next, in Figs. 4 and 5 we analyze the impact of Y_P and Y_N on frequencies (right y-axis) and capacity of buses (left y-axis). The optimal capacity (number of seats of each bus) is presented in blue; the load of each bus in the off-peak period is also shown (diamonds line), in order to verify that it is always lower than the optimal (assumed to be commanded by the peak load).

Fig. 4 shows responses to changes in peak-period passengers flow. As expected, both peak-frequency and bus size increase when Y_P does, because buses run full in this period. As predicted analytically, off-peak frequency decreases (although very slowly) due to the rise in bus size. Accordingly, off-peak bus load increases but capacity is always enough.

Fig. 5 shows responses to changes in the off-peak period passengers flow. As predicted analytically, buses become smaller as Y_N increases, because each large bus becomes more costly as f_N grows in response to new passengers. These smaller sizes make peak frequencies increase as well. Note that off-peak load approaches capacity as patronage increases, so there will be a point where size begins to be determined by off-peak conditions, which we will not analyze here.

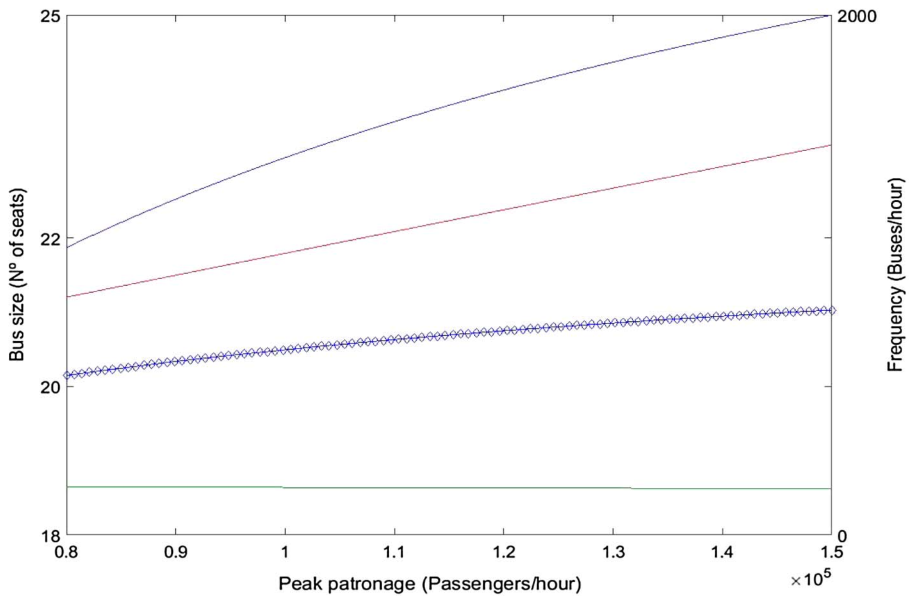

Fig. 4. Optimal frequencies and bus size for varying peak passengers flow.

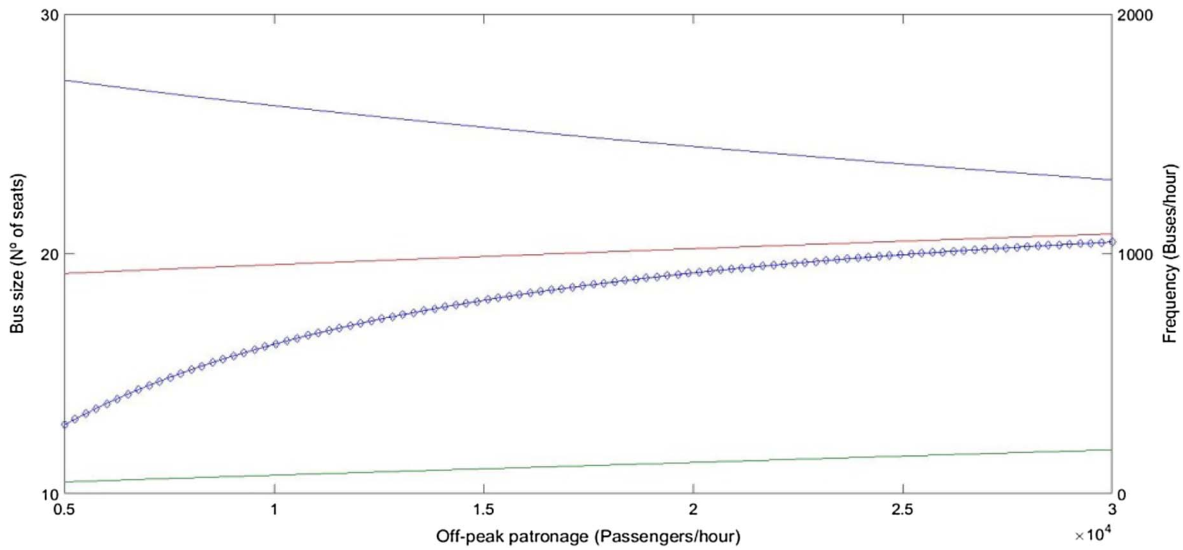

Fig. 5. Optimal frequencies and bus size for varying off-peak passengers flow.

5. Synthesis, conclusions and further research

In this paper, we have analyzed the problem of finding the optimal fleet for a single line to serve a pattern of transit users that changes during peak and off-peak periods. The main differences between periods are hourly patronage (which defines the peak), duration, average trip length and speed of the vehicles (due to traffic conditions). Previous analysis in the literature assume equality between frequencies across periods or impose very simplifying assumptions regarding vehicles cost and cycle times. Here we have shown analytically and numerically that optimal frequencies during peak and off-peak conditions are quite different in general, but capacity of the vehicles lies in between those that would prevail if each period were optimized independently. Interestingly, several counterintuitive results were obtained and could be explained from the model.

In a previous paper (Jara-Díaz and Gschwender, 2009) we have shown that imposing a self-financing policy on surface transit yields smaller frequencies and larger buses than optimal; this was shown using a single period model. This is indeed an issue in public transport design. One of the main results obtained here is that when peak and off-peak periods are considered, the optimal frequency in the peak is larger and bus size smaller than what is obtained when the peak is optimized in isolation. This is explained by a key element in the multiperiod model developed here: the same fleet is used longer, during both periods but under different conditions; as operating a vehicle has a cost that increases with its size, introducing the off-peak conditions induces a size reduction that increases the peak frequency.

Regarding off-peak frequency, we have shown analytically that it is affected by different factors that push it upwards and downwards with respect to the optimal frequency calculated when solving this period by itself. The fact that buses are already acquired pushes off-peak frequency upwards but, precisely because of this, increasing frequency cannot induce smaller buses, which pushes this frequency downwards. Also, buses are larger than what would be obtained in an isolated analysis; these higher operating costs pushes frequency downwards as well. The numerical analysis showed that these last factors prevail, and that off-peak optimal frequency is indeed lower than optimal frequency for the isolated period.

Crossed effects were analytically deduced and numerically verified. Off-peak frequency was shown to decrease when peak passengers increase, because buses are larger, hence more expensive to operate in that case. Peak frequency was shown to increase when off-peak passengers increase, because more buses operate in the off-peak period in that case, so it is better to decrease their size.

Although fleet is always calculated according to peak conditions (that present larger flows by definition), bus size could be determined by off-peak conditions if its flow is sufficiently large; this would require a re-examination of the solutions making size equal to the off-peak requirement.

The stylized model presented here to analyze the multiperiod design problem can be applied to explore other important issues. One emerges directly from our analysis: examination of the conditions under which bus capacity is determined by off-peak - instead of peak - characteristics. Also, a cost analysis could be helpful to examine the loss due to a design based upon a single period analysis (either peak or off-peak) with adjustments to operate the other period. Furthermore, as the cost function approach taken here deals with a parametrically given demand level, endogeneity could be analyzed by introducing sensitivity to quality of service of the public transport system (in this case frequency). Another aspect worth exploring is some randomness in the arrival of both vehicles and passengers to the bus stops, which would introduce new dimensions as overcrowding and denied boarding. In addition, when dealing with two periods there is a likely relation between peak and off-peak demand due to a joint modal decision that is taken on a daily basis by individuals; for example, a reduction in off-peak frequency could lead to a reduction in demand in both periods.

Finally, the complex decision of optimal lines structures (routes) should be studied as well; this requires moving into a spatial scenario, which we are studying using the parametric city previously developed and studied in Fielbaum et al. (2016, 2017). Usually the line structure is defined to serve the peak demand pattern, and the operation during the rest of the day considers a subset of the peak lines. An opposite approach has been proposed (Walker, 2012), i.e. to design the lines structure considering the off-peak pattern (as this period is much longer than the peak) and complement it in the peak with additional peak-specific lines, potentially considering the use of buses of a different capacity. What would the optimal lines structure look like, if it was built considering both demand patterns, in a similar way as the two-periods single line model proposed in this paper?

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