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Smoothness of the metric projection onto nonconvex bodies in Hilbert spaces ☆

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ABSTRACT

Based on a fundamental work of R. B. Holmes from 1973, we study differentiability properties of the metric projection onto prox-regular sets. We show that if the set is a nonconvex body with a C^{p+1}-smooth boundary, then the projection is $\mathcal{C}^p$-smooth near suitable open truncated normal rays, which are determined only by the function of prox-regularity. A local version of the same result is established as well, namely, when the smoothness of the boundary and the prox-regularity of the set are assumed only near a fixed point. Finally, similar results are derived when the prox-regular set is itself a C^{p+1}-submanifold.

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1. Introduction

In his 1973 fundamental paper [13], R. B. Holmes showed that, whenever we have a closed convex set K in a Hilbert space X such that

(i) K has nonempty relative interior (namely, the interior of K as a subset of $Y = \overline{\text{aff}}(K)$ is nonempty), and
(ii) the boundary of K as a subset of Y, $\text{bd}\, K$, is a $\mathcal{C}^{p+1}$-submanifold at a point $x_0 \in \text{bd}\, K$, where p is a positive integer,

then the metric projection P_K is a mapping of class $\mathcal{C}^p$ in an open neighborhood W of the open normal ray

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$$\mathrm{Ray}_{x_0}(K) := \{x_0 + t\nu \ : \ t > 0\},$$

where ν denotes the unit exterior normal vector of K at x_0. The main steps of his approach to arrive to this theorem were:

1. It is enough to prove the theorem for convex bodies (namely, where K has nonempty interior), since under (i), restricting to the case $0 \in K$ (after suitable translation) we can write

$$P_K = \left(P_K\big|_Y\right) \circ \Pi_Y,$$

where Π_Y denotes the orthogonal projection to Y (which is a continuous linear mapping and therefore of class $\mathcal{C}^\infty$);
2. The smoothness of $\mathrm{bd}\, K$ at x_0 can be translated as the smoothness of the Minkowski functional ρ_K (independently of which translation is used to ensure that 0 is an interior point of K); furthermore, the equality $\nu = \|\nabla\rho_K(x_0)\|^{-1}\nabla\rho_K(x_0)$ holds true;
3. The distance function d_K is of class $\mathcal{C}^1$ in $X \setminus K$; and finally,
4. For any point $x \in \mathrm{Ray}_{x_0}(K)$ and a suitable choice of neighborhoods U and V of x and x_0 respectively, the mapping

$$\begin{aligned} F : U \times V &\to X \\ (u, v) &\mapsto u - v - d_K(u)\frac{\nabla\rho_K(v)}{\|\nabla\rho_K(v)\|} \end{aligned}$$

is well defined, of class $\mathcal{C}^1$, and for every $(u, v) \in U \times V$, one has

$$F(u, v) = 0 \iff v = P_K(u).$$

With all these features, Holmes concluded his theorem through an application of the well-known Implicit Function Theorem. Following the way opened by the strategy of Holmes, the aim of the present work is to establish, under the same hypotheses (i)–(ii), similar *local* results dropping the hypothesis of convexity and replacing it with *prox-regularity*, a notion first introduced as *positive reach* by H. Federer in [11] and widely studied in the literature.

The main motivation for this research came from the huge advances made in Proximal Analysis and from the 2000's paper by Poliquin, Rockafellar and Thibault [20], which allows us to replace the continuous differentiability of the distance function to convex bodies, with another suitable one related to prox-regular sets. Also we want to mention Mazade PhD Thesis [16], in which local prox-regularity was profoundly studied in a quantified sense.

The paper is organized as follows. In Section 2 we fix the notation, recall some fundamental results in Nonsmooth Analysis and formulate the problem of extension of Holmes' theorem (see Theorem 2.4) formally. Section 3 contains some variational results of submanifolds of Hilbert spaces that we will need. Some results of that section are probably known but explicit formulations of them are hard to find in the literature. In Section 4, we establish first a main result concerning the smoothness of the metric projection, and then we derive Theorem 2.4. In Section 5, we show how that theorem allows us, for any manifold, to get similar differentiability results on a suitable quantified neighborhood. We end this work with some final comments on possible future works and some comparisons with other related results in the literature.

2. Variational concepts and problem formulation

In the following, X will be a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$.

We will use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}_+ = [0, +\infty[$, and we will adopt the conventions $0^{-1} = +\infty$ and $(+\infty)^{-1} = 0$. Unless otherwise specified, p will be an *integer with* $p \geq 1$.

We will identify the dual space X^* with X, using the Riesz's representation theorem. For $x \in X$ and $\delta > 0$ we will write $B_X(x, \delta)$ and $B_X[x, \delta]$ to denote the open and closed δ-ball centered at x, respectively. Also, we will write $\mathbb{B}_X$ and $\mathbb{S}_X$ to denote the unit ball $B_X[0, 1]$ and the unit sphere $B_X[0, 1] \setminus B_X(0, 1)$. For $x \in X$, we will write $\mathcal{N}_X(x)$ to denote the family of neighborhoods of x. If there is no confusion we may omit the space X in the preceding notation. We use id_X to denote the identity map of X.

For a set $S \subseteq X$, we will put $S^c := X \setminus S$ and we will write $\operatorname{int} S$, $\operatorname{cl} S$ and $\operatorname{bd} S$ to denote its interior, its closure and its boundary, respectively. When S is included in a subset U of X, the latter topological concepts relative to the induced topology on U will be indexed with U, for example, $\operatorname{bd}_U S$ will stand for the boundary of S in U; it will be convenient to denote also by $\overline{S}$ the closure of S in the whole space X. The set S will be called a *closed body* (relative to X) near $x_0 \in \operatorname{bd} S$ provided there exists an open connected neighborhood U of x_0 such that $U \cap S = U \cap \left(\overline{\operatorname{int} S}\right)$ and $U \cap \operatorname{int} S$ is connected; note that in such a case $U \cap S$ is in turn connected. When $U = X$, that is, $S = \overline{\operatorname{int} S}$ and $\operatorname{int} S$ is connected, we will say that S is a *closed body* (relative to X).

By S^o we will mean the *(negative) polar set* of S, namely,

$$S^o = \{h \in X \ : \ \langle h, x\rangle \leq 1, \ \forall x \in S\}.$$

As usual (if $S \neq \emptyset$), we will denote by $d_S : X \to \mathbb{R}_+$ its distance function, namely,

$$\begin{aligned} d_S : X &\to \mathbb{R}_+ \\ x &\mapsto \inf_{y \in S}\{\|x - y\|\}. \end{aligned}$$

In some occasions, it will be useful to write $d(\cdot; S)$ instead of $d_S(\cdot)$. Also, for $x \in X$ we will denote by $\operatorname{Proj}_S(x)$ the set of all *nearest points* of x onto S; that means,

$$\operatorname{Proj}_S(x) = \{y \in S \ : \ \|x - y\| = d_S(x)\}.$$

Whenever $\operatorname{Proj}_S(x)$ is a singleton, we say that the point $\bar{y} \in \operatorname{Proj}_S(x)$ is the *(metric) projection of x onto S* and it is denoted by $P_S(x)$.

If Y is a closed subspace of X, we denote by Π_Y the orthogonal projection from X onto Y. It is known that Π_Y is a continuous linear operator (therefore, of class $\mathcal{C}^\infty$) and for each $x \in X$, $\Pi_Y(x)$ coincides with the metric projection of x onto Y.

Also, if Y and Z are two closed topologically complement subspaces of X, we will denote by π_Y and π_Z the parallel projections onto Y and onto Z, respectively, associated to the decomposition $X = Y \oplus Z$. Recall that $Z = Y^\perp$ if and only if π_Y and Π_Y coincide. In the case when $X = Y \times Z$, we will simply write π_Y and π_Z instead of $\pi_{Y \times \{0\}}$ and $\pi_{Z \times \{0\}}$, respectively.

If Y is another Hilbert space and $T : X \to Y$ is a continuous linear operator, we will denote by $T^* : Y \to X$ its adjoint operator, namely, the unique continuous linear operator that satisfies

$$\langle y, Tx\rangle = \langle T^* y, x\rangle, \ \forall (x, y) \in X \times Y.$$

It is known that the map $* : \mathcal{L}(X; Y) \to \mathcal{L}(Y; X)$ (where $\mathcal{L}(E; F)$ denotes the Banach space of continuous linear operators between the two Banach spaces E and F) which to any $T \in \mathcal{L}(X; Y)$ assigns the adjoint operator T^*, is linear and continuous and therefore, a mapping of class $\mathcal{C}^\infty$. In particular, for every mapping $A : V \subseteq E \to \mathcal{L}(X; Y)$ of class $\mathcal{C}^p$, where V is an open set of a Banach space E, one has that the mapping $v \mapsto A(v)^*$ is also of class $\mathcal{C}^p$.

Below we will adopt the notation in [8,9]. For a set $S \subseteq X$ and a point $x_0 \in S$, we will consider the *Clarke tangent cone* of S at x_0 and the *Bouligand tangent cone* (or *contingent cone*) of S at x_0 defined as

$$T^C(S;x_0) = \operatorname*{Liminf}_{S \ni u \to x_0;\ t \downarrow 0} \frac{1}{t}(S-u)$$
$$T^B(S;x_0) = \operatorname*{Limsup}_{t \downarrow 0} \frac{1}{t}(S-x_0),$$

where Liminf and Limsup denote the Peano–Painlevé–Kuratowski inferior and superior limit of sets. Particularly, the first equality means that $h \in T^C(S;x_0)$ if and only for any sequence $(x_n)_n$ in S converging to x_0 and any sequence $(t_n)_n$ of positive reals tending to 0, there is a sequence $(h_n)_n$ in X converging to h such that

$$x_n + t_n h_n \in S \quad \text{for all integers } n \geq 1. \tag{1}$$

We will say that S is *tangentially regular* at x_0 if $T^C(S;x_0) = T^B(S;x_0)$. Also, we will consider the *interior tangent cone* of S at x_0 given by

$$I(S;x_0) = \{h \in X \ : \ \exists \varepsilon > 0,\ U \in \mathcal{N}_X(x_0) \text{ and } V \in \mathcal{N}_X(h) \text{ such that } U \cap S + \,]0,\varepsilon[V \subset S\}.$$

It is known that if $I(S;x_0) \neq \emptyset$ then

$$T^C(S;x_0) = \overline{I(S;x_0)} \quad \text{and} \quad I(S;x_0) = \operatorname{int}[T^C(S;x_0)]. \tag{2}$$

Note that $I(S;x_0)$ could be empty even if $\operatorname{int}[T^C(S;x_0)]$ isn't. Nevertheless, when X is finite dimensional one always has $I(S;x_0) = \operatorname{int}[T^C(S;x_0)]$. For the proofs of the latter statements, we refer the reader to [22, Theorem 2 and Counterexample 1]. While the statement of [22, Theorem 2] is posed in the finite dimensional setting, the proof of sufficiency (which entails equation (2)) remains the same in the infinite dimensional case.

Given a subset U of X and a function $f : U \to \overline{\mathbb{R}}$, we recall the epigraph (resp. strict epigraph) and hypograph (resp. strict hypograph) are the sets

$$\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} : x \in U,\ f(x) \leq r\} \quad (\text{resp. } \operatorname{epi}_s f := \{(x,r) \in X \times \mathbb{R} : x \in U,\ f(x) < r\}),$$
$$\operatorname{hypo} f := \{(x,r) \in X \times \mathbb{R} : x \in U,\ f(x) \geq r\} \quad (\text{resp. } \operatorname{hypo}_s f := \{(x,r) \in X \times \mathbb{R} : x \in U,\ f(x) > r\}).$$

The graph of f will be denoted by $\operatorname{gph} f$, that is,

$$\operatorname{gph} f := \{(x,r) \in X \times \mathbb{R} : x \in U, f(x) = r\}.$$

A set S of X is said to be *epi-Lipschitz at $x_0 \in S$ in a nonzero direction $h \in X$* if there exists a neighborhood $U \in \mathcal{N}_X(x_0)$, a closed complement vector subspace Z of $\mathbb{R}h$, and a Lipschitz continuous function $f : Z \to \mathbb{R}$ such that, writing $X = Z \oplus \mathbb{R}h$, one has

$$U \cap S = \{z + rh \in U \ : \ (z,r) \in \operatorname{epi} f\}.$$

We simply say that S is *epi-Lipschitz at x_0* if there exists $h \in X \setminus \{0\}$ such that S is epi-Lipschitz at x_0 in the direction h.

It is known that S is epi-Lipschitz at x_0 if and only if $I(S;x_0) \neq \emptyset$. In such a case, for every nonzero $h \in I(S;x_0)$, one has that S is epi-Lipschitz at x_0 in the direction h (see, e.g., [22, Section 4], which stated this result in the finite dimensional setting, but its proof remains the same in the general case). When S is epi-Lipschitz at x_0, the Clarke tangent cone of $\operatorname{bd} S$ at $x_0 \in \operatorname{bd} S$ is related to that of S by the equality (see [10])

$$T^C(\operatorname{bd} S; x_0) = T^C(S; x_0) \cap -T^C(S; x_0). \tag{3}$$

We will also consider the *Clarke normal cone* and the *proximal normal cone* of S at x_0 as

$$N^C(S; x_0) = [T^C(S; x_0)]^o \quad \text{(the negative polar of the Clarke tangent cone)}$$
$$N^P(S; x_0) = \{\zeta \in X \ : \ \exists t > 0,\ x_0 \in \operatorname{Proj}_S(x_0 + t\zeta)\}.$$

For a bijective continuous linear mapping $A : X \to Y$ from X onto a Banach space Y and $S' \subset Y$ such that $S = A^{-1}(S')$, it is known (see, e.g., [19, Proposition 5.27]) that

$$N^C(S; x_0) = N^C(A^{-1}(S'); x_0) = A^*\big(N^C(S'; A(x_0))\big) := \{A^*(\xi) : \xi \in N^C(S'; A(x_0))\}. \tag{4}$$

More generally, given a C^1-diffeomorphism $\varphi : U \to \varphi(U) \subset Y$ from an open set U of X and a point $x_0 \in U \cap \varphi^{-1}(S')$, it is easily seen through the sequential characterization (1) that

$$T^C\big(\varphi^{-1}(S'); x_0\big) = D\varphi(x_0)^{-1}\big(T^C(S'; \varphi(x_0))\big). \tag{5}$$

From this it is clear, under the C^1-diffeomorphism property, that

$$N^C\big(\varphi^{-1}(S'); x_0\big) = D\varphi(x_0)^*\big(N^C(S'; \varphi(x_0))\big). \tag{6}$$

Besides the above geometrical definition of the proximal normal cone, it is worth mentioning the variational description saying that a vector $\zeta \in N^P(S; x_0)$ if and only if there is a real constant $\sigma \geq 0$ and a neighborhood U of x_0 such that

$$\langle \zeta, x - x_0 \rangle \leq \sigma \|x - x_0\|^2, \quad \text{for all } x \in U \cap S. \tag{7}$$

From this we also see that, for a bijective continuous linear mapping $A : X \to Y$ from X onto another Hilbert space Y and $S' \subset Y$ with $x_0 \in A^{-1}(S')$ one has

$$N^P(A^{-1}(S'); x_0) = A^*\big(N^P(S'; A(x_0))\big). \tag{8}$$

Recall that one always has

$$N^P(S; x_0) \subset N^C(S; x_0).$$

When the equality holds, that is, $N^P(S; x_0) = N^C(S; x_0)$, we will say that S is *normally regular* at x_0. Whenever the proximal normal cone of a set S at a point x_0 has the form

$$N^P(S; x_0) = \{t\nu \ : \ t \geq 0\},$$

for some unit vector $\nu \in \mathbb{S}_X$, we will define (for $\lambda > 0$) the sets:

$$\operatorname{Ray}_{x_0}(S) = \{x_0 + t\nu \ : \ t > 0\}$$
$$\operatorname{Ray}_{x_0,\lambda}(S) = \{x_0 + t\nu \ : \ t \in]0, \lambda[\ \},$$

which we call the *open normal ray* and the *λ-truncated open normal ray* of S at x_0, respectively.

With the two previous normal cones are associated two notions of subdifferential. For a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ and a point $x_0 \in X$ where f is finite, we define the *Clarke subdifferential* of f at x_0 and the *proximal subdifferential* of f at x_0 as

$$\partial_C f(x_0) = \{\zeta \in X \ : \ (\zeta, -1) \in N^C\big(\operatorname{epi} f, (x_0, f(x_0))\big)\}$$
$$\partial_P f(x_0) = \{\zeta \in X \ : \ (\zeta, -1) \in N^P\big(\operatorname{epi} f, (x_0, f(x_0))\big)\}.$$

We will then say that f is *tangentially regular* (resp. *normally regular*) at x_0 if $\operatorname{epi} f$ is tangentially regular (resp. normally regular) at $(x_0, f(x_0))$. When f is C^1 near x_0, one has (see, e.g., [8, Proposition 2.2.4])

$$\partial_C f(x_0) = \{\nabla f(x_0)\} \quad \text{and} \quad N^C\big(\operatorname{epi} f; (x_0, f(x_0))\big) = \{\lambda(\nabla f(x_0), -1) : \lambda \geq 0\},$$

and if in addition the gradient ∇f is Lipschitz near x_0 (that is, f is $C^{1,1}$ near x_0), then f is normally regular at x_0 and

$$\partial_C f(x_0) = \partial_P f(x_0) = \{\nabla f(x_0)\} \quad \text{and} \quad N^P\big(\operatorname{epi} f; (x_0, f(x_0))\big) = \{\lambda(\nabla f(x_0), -1) : \lambda \geq 0\}.$$

For further properties related to these objects, we refer to [8,9] and [17]. Let us now recall the definitions of some types of prox-regularity for sets.

Definition 2.1. Given an extended real $r \in]0, +\infty]$ and a real $\alpha > 0$, we say that a closed set S of X is *(r, α)-prox-regular at $x_0 \in S$* if for every $x \in S \cap B_X(x_0, \alpha)$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$ we have that

$$x \in \operatorname{Proj}_S(x + t\zeta), \quad \text{for every real } t \in [0, r]. \tag{9}$$

We say that S is *r-prox-regular at $x_0 \in S$* if it is (r, α)-prox-regular at x_0 for some $\alpha > 0$ and we simply say that S is *prox-regular at x_0* if there exists $r \in]0, +\infty]$ such that S is r-prox-regular at x_0.

Consequently, we say that S is *r-prox-regular* (resp. *prox-regular*) if it is r-prox-regular (resp. prox-regular) at every point $x \in S$.

It is clear that if S is (r, α)-prox-regular at x_0, then it is also (r', α')-prox-regular at x_0 for every $\alpha' \in]0, \alpha]$ and every $r' \in]0, r]$.

It is known (see, e.g., [9]) that S is prox-regular if and only if there exists a continuous function $\rho : S \to]0, +\infty]$ (that we will call *prox-regularity function*) such that for every $x \in S$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$ one has

$$x \in \operatorname{Proj}_S(x + t\zeta), \quad \text{for every real } t \in [0, \rho(x)].$$

It is also known (see, e.g., [9, Chapter 3, Propositions 4 and 11]) that whenever S is $\rho(\cdot)$-prox-regular, the enlargement of S

$$U_{\rho(\cdot)}(S) := \{u \in X \ : \ \exists y \in \operatorname{Proj}_S(u) \text{ with } d_S(u) < \rho(y)\}$$

is an open set, P_S is well-defined on $U_{\rho(\cdot)}(S)$ and $d_S^2(\cdot)$ is of class $\mathcal{C}^1$ on $U_{\rho(\cdot)}(S)$.

In the paper [20], Poliquin, Rockafellar and Thibault studied the local prox-regularity of a set S. We summarize their results (those that we will need) in the following theorem:

Theorem 2.2 *(PRT, 2000). Let S be a closed set of X and $x_0 \in S$. The following assertions are equivalent:*

(i) S is prox-regular at x_0;
(ii) There exists $O \in \mathcal{N}_X(x_0)$ such that P_S is well defined and locally Lipschitz continuous in O;
(iii) There exist two real constants $\sigma \geq 0$, $\delta > 0$ such that for every $x \in S \cap B_X(x_0, \delta)$ and every $\zeta \in N^P(S; x) \cap \mathbb{B}_X$, one has

$$\langle \zeta, y - x\rangle \leq \frac{\sigma}{2}\|y - x\|^2, \ \forall y \in S \cap B_X(x_0, \delta);$$

(iv) There exists $O \in \mathcal{N}_X(x_0)$ such that d_S is continuously differentiable in $O \setminus S$.

Moreover, if S is prox-regular at x_0, then S is tangentially and normally regular at x_0 and there exists a neighborhood $O \in \mathcal{N}_X(x_0)$ for which P_S is well defined in O, d_S is Fréchet differentiable in $O \setminus S$ and its gradient is given by

$$\nabla d_S(u) = \frac{u - P_S(u)}{d_S(u)}, \ \forall u \in O \setminus S. \tag{10}$$

In the PhD thesis [16] of M. Mazade, quantified versions are provided for the characterizations of local prox-regularity given in the PRT theorem. To do so, for $r \in]0, +\infty]$ and $\alpha > 0$ the following local enlargements of the set S at a point $x_0 \in S$ are introduced:

$$\mathcal{R}_S(x_0, r, \alpha) := \left\{x + tv \ : \ x \in S \cap B_X(x_0, \alpha), \ t \in [0, r[, \ v \in N^P(S; x) \cap \mathbb{B}_X\right\},$$
$$\mathcal{W}_S(x_0, r, \alpha) := \{u \in X \ : \ \mathrm{Proj}_S(u) \cap B_X(x_0, \alpha) \neq \emptyset, \ d_S(u) < r\}.$$

We summarize the results in [16] that we will use in the following theorem (see [16, Theorem 2.3.3 and Theorem 2.3.4]):

Theorem 2.3 *([16]). Let S be a closed set of X, $x_0 \in S$, $r \in]0, +\infty]$ and $\alpha > 0$. The following assertions are equivalent:*

(i) S is (r, α)-prox-regular at x_0;
(ii) The set $\mathcal{W}_S(x_0, r, \alpha)$ is open and P_S is well-defined and locally Lipschitz continuous on $\mathcal{W}_S(x_0, r, \alpha)$;
(iii) The set $\mathcal{W}_S(x_0, r, \alpha)$ is open and d_S is continuously differentiable on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$ with $\nabla d_S(u) = \frac{u - P_S(u)}{d_S(u)}$ for all $u \in \mathcal{W}_S(x_0, r, \alpha) \setminus S$;
(iv) For any $x \in S \cap B(x_0, \alpha)$ and $\zeta \in N^P(S; x)$ one has

$$\langle \zeta, x' - x\rangle \leq \frac{\|\zeta\|}{2r}\|x' - x\|^2 \quad \text{for all } x' \in S.$$

Moreover, if S is (r, α)-prox-regular at x_0, then $\mathcal{R}_S(x_0, r, \alpha)$ and $\mathcal{W}_S(x_0, r, \alpha)$ coincide.

Recall that we consider an integer $p \geq 1$. A set $M \subseteq X$ is said to be a $\mathcal{C}^p$*-submanifold of X at $m_0 \in M$* (see, e.g., [2, Ch. 9]) if there exists an open set $U \in \mathcal{N}_X(m_0)$, a closed subspace Z of X and a mapping $\varphi : U \to \varphi(U) \subset X$ such that

1. φ is a $\mathcal{C}^p$-diffeomorphism, that is, $\varphi(U)$ is an open set of X, $\varphi : U \to \varphi(U)$ is bijective and φ, φ^{-1} are both mappings of class $\mathcal{C}^p$;
2. $\varphi(m_0) = 0$ and $\varphi(U \cap M) = \varphi(U) \cap Z$.

In such a case, we call Z the model subspace. We say that M is a $\mathcal{C}^p$*-submanifold of* X if it is a $\mathcal{C}^p$-submanifold at every point $m_0 \in M$ with the same model space Z. We say that a set S has $\mathcal{C}^p$-smooth boundary (resp. $\mathcal{C}^p$-smooth boundary at $x_0 \in \operatorname{bd} S$) if $\operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold of X (resp. $\mathcal{C}^p$-submanifold of X at x_0).

For a $\mathcal{C}^p$-submanifold M of X at $m_0 \in M$, the *tangent (vector) space* of M at m_0 is defined as

$$T_{m_0}M := \left\{h \in X : \exists \gamma :]-1,1[\to M \ C^1\text{-curve with } \gamma(0) = m_0, \gamma'(0) = h\right\}.$$

For M and m_0 as above, it is known (see, e.g., [19, Propositions 2.88 and 5.26]) that

$$T^C(M; m_0) = T^B(M; m_0) = T_{m_0}M,$$

and that, if φ and Z are the $\mathcal{C}^p$-diffeomorphism and the model space of the definition of submanifold, one has

$$D\varphi^{-1}(0)Z = T_{m_0}M,$$

independently of the chosen diffeomorphism.

Now, in order to extend Holmes' theorem to the nonconvex setting we will study the sets S which are closed and such that

(i) S is a closed body relative to the subspace $Y = \overline{\operatorname{aff}}(S)$;
(ii) Considering S as a subset of Y, it has a $\mathcal{C}^{p+1}$-smooth boundary.

It is not hard to realize that for each $x \in X$ we have that

$$\operatorname{Proj}_S(x) = \operatorname{Proj}_S(\Pi_Y(x)),$$

since, for each $v \in S$, $\|x - v\|^2 = \|x - \Pi_Y(x)\|^2 + \|\Pi_Y(x) - v\|^2$. Thus, if we define the sets

$$\begin{aligned} O_X &= \{x \in X \ : \ P_S \text{ is well-defined and it is of class } \mathcal{C}^p \text{ near } x\} \\ O_Y &= \left\{y \in Y \ : \ P_S\big|_Y \text{ is well-defined and it is of class } \mathcal{C}^p \text{ near } y\right\} \end{aligned}$$

(which are open, relative to X and Y respectively), we get that $\Pi_Y^{-1}(O_Y) \subset O_X$. Indeed, for every $x \in \Pi_Y^{-1}(O_Y)$ we have

$$P_S(x) = P_S\big|_Y \circ \Pi_Y(x),$$

and therefore, the inclusion $\Pi_Y^{-1}(O_Y) \subset O_X$ is direct. On the other hand, let $x \in O_X$. Noting that for every $y \in Y$ and every $v \in S$ we have that $\Pi_Y(x) + y - v \in Y$, we can write

$$\|x + y - v\|^2 = \|x - \Pi_Y(x)\|^2 + \|\Pi_Y(x) + y - v\|^2$$

and so, we get that $P_S\big|_Y(\Pi_Y(x) + y) = P_S(x + y)$ for every $y \in Y$ such that $x + y \in O_X$, or equivalently

$$P_S\big|_Y(w) = P_S(x + w - \Pi_Y(x)) \quad \text{for all } w \in Y \text{ with } x + w - \Pi_Y(x) \in O_X.$$

By definition of O_X choose a real $\varepsilon > 0$ such that P_S is well defined on $x + B_X(0, \varepsilon)$ and of class C^p therein. Consider the mapping $\ell : \Pi_Y(x) + B_Y(0, \varepsilon) \to X$ defined by $\ell(w) := w + x - \Pi_Y(x)$ for all $w \in \Pi_Y(x) + B_Y(0, \varepsilon)$. Clearly, $\ell\big(\Pi_Y(x) + B_Y(0, \varepsilon)\big) \subset x + B_X(0, \varepsilon)$, hence $P_S \circ \ell$ is well defined on

$\Pi_Y(x) + B_Y(0,\varepsilon)$ and of class C^p therein. Since $P_S\big|_Y(w) = (P_S \circ \ell)(w)$ for every $w \in \Pi_Y(x) + B_Y(0,\varepsilon)$, the mapping $P_S\big|_Y$ is of class C^p near $\Pi_Y(x)$. This tells us that $\Pi_Y(x) \in O_Y$, or equivalently $x \in \Pi_Y^{-1}(O_Y)$. We derive that $O_X \subset \Pi_Y^{-1}(O_Y)$, which combined with the previous above inclusion gives the equality $O_X = \Pi_Y^{-1}(O_Y)$.

Based on the latter equality and the above observations, the smoothness of P_S is characterized by the smoothness of $P_S\big|_Y$, and so our target problem can be reduced to prove the following extension of Holmes' theorem:

Theorem 2.4. *Let $S \subseteq X$ be a closed body near $x_0 \in \operatorname{bd} S$ and let an integer $p \geq 1$. Assume that there exist $r \in]0, +\infty]$ and $\alpha > 0$ such that $B_X(x_0, \alpha) \cap \operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold and that S is r-prox-regular at x_0. Then there exists a neighborhood V of $\operatorname{Ray}_{x_0,r}(S)$ such that*

- *d_S is of class $\mathcal{C}^{p+1}$ on V;*
- *P_S is of class $\mathcal{C}^p$ on V.*

Furthermore, if the set S is (r, α)-prox-regular at x_0, then

- *d_S is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$;*
- *P_S is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0, r, \alpha) \setminus S$.*

3. Variational and prox-regularity properties of submanifolds

We start this section with a property related to the codimension of the tangent space to the boundary when the latter is a smooth submanifold.

Proposition 3.1. *Let $S \subseteq X$ and $x_0 \in \operatorname{bd} S$ such that $x_0 \in \overline{\operatorname{int} S}$. If $\operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold of X at x_0, then $T_{x_0}(\operatorname{bd} S)$ is a closed subspace of X of codimension 1; that is, there exists a closed subspace Z of codimension 1, an open neighborhood U of x_0 in X and a C^p-diffeomorphism $\varphi : U \to \varphi(U) \subset X$ such that $\varphi(U \cap \operatorname{bd} S) = Z \cap \varphi(U)$.*

Proof. Let U be an open neighborhood of x_0 such that $M := U \cap \operatorname{bd} S$ is a $\mathcal{C}^p$-submanifold of X. Without loss of generality, we may assume that U is connected and also, that there exists a $\mathcal{C}^p$-diffeomorphism $\varphi : U \to \varphi(U) \subset X$ and a closed subspace Z of X such that $\varphi(x_0) = 0$ and

$$\varphi(U \cap M) = \varphi(U) \cap Z.$$

Denote $V = \varphi(U)$. As recalled above, we know that $T_{x_0}(\operatorname{bd} S) = T_{x_0}M = D\varphi^{-1}(0)Z$. It is enough to show that Z is a subspace of codimension 1. Assume the contrary and consider two distinct vectors $v_1, v_2 \in V \setminus Z$. Without loss of generality, we may assume that there exists $\delta > 0$ such that $V = B(0, \delta)$. Denoting by $\mathcal{H}_Z$ a Hamel basis of Z, we can distinguish two cases:

(I) The set $\mathcal{H}_Z \cup \{v_1, v_2\}$ is linearly independent: Then, for each $t \in [0, 1]$, putting

$$\gamma(t) = tv_1 + (1 - t)v_2 \in V \setminus Z,$$

the mapping $\gamma : [0, 1] \to V \setminus Z$ defines a continuous curve with $\gamma(0) = v_1$ and $\gamma(1) = v_2$.

(II) The set $\mathcal{H}_Z \cup \{v_1, v_2\}$ is not linearly independent: Since $\operatorname{codim}[Z] \geq 2$, there exists $v_3 \in V \setminus Z$ such that both sets $\mathcal{H}_Z \cup \{v_1, v_3\}$ and $\mathcal{H}_Z \cup \{v_2, v_3\}$ are linearly independent. Then, using the latter part, we can construct two continuous curves $\gamma_1 : [0, 1/2] \to V \setminus Z$ and $\gamma_2 : [1/2, 1] \to V \setminus Z$ such that $\gamma_1(0) = v_1$, $\gamma_1(1/2) = \gamma_2(1/2) = v_3$ and $\gamma_2(1) = v_2$. Then, considering the mapping $\gamma : [0, 1] \to V \setminus Z$ given by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [0, 1/2] \\ \gamma_2(t) & t \in (1/2, 1], \end{cases}$$

we arrive at the same conclusion as (I).

Since v_1 and v_2 are two arbitrary distinct points of $V \setminus Z$, the existence of such a continuous curve γ entails that $V \setminus Z$ is path-connected, and therefore is connected. Then, since $\varphi^{-1} : V \to U$ is continuous, we derive that $U \setminus M = \varphi^{-1}(V \setminus Z)$ is connected too. This is clearly a contradiction since the two open sets $(\operatorname{int} S) \cap U$ and $S^c \cap U$ are nonempty (according to the assumptions $x_0 \in \overline{\operatorname{int} S}$ and $x_0 \in \operatorname{bd} S$), and they satisfy the equality

$$U \setminus M = \big((\operatorname{int} S) \cap U\big) \cup (S^c \cap U).$$

The proof is therefore complete. □

The next proposition shows that a closed body whose boundary is a C^p-submanifold can be represented locally as the epigraph of a C^p-function.

Before proving the proposition we need some features for epi-Lipschitz sets whose boundaries are smooth. So, suppose that S is epi-Lipschitz at $x \in \operatorname{bd} S$ and that $\operatorname{bd} S$ is a C^p-submanifold at x. We first note that $T^C(S; x)$ is a half-space. Indeed, by the epi-Lipschitz property we know (see (3)) that $\operatorname{int}\big(T^C(S; x)\big) \neq \emptyset$ and

$$T^C(\operatorname{bd} S; x) = T^C(S; x) \cap -T^C(S; x).$$

Taking an orthogonal unit vector $\hat{n}_Z$ of $Z(x) := T_x(\operatorname{bd} S)$, Proposition 3.1 and the equality (6) tell us that

$$N^C(\operatorname{bd} S; x) = \mathbb{R}\hat{n}(x),$$

where $\hat{n}(x) := D\varphi(x)^*\hat{n}_Z/\|D\varphi(x)^*\hat{n}_Z\|$. It ensues that

$$T^C(S; x) \cap -T^C(S; x) = T^C(\operatorname{bd} S; x) = \{h \in X : \langle \hat{n}(x), h \rangle = 0\}.$$

Since the interior of the closed convex cone $T^C(S; x)$ is nonempty, it results that

$$\text{either } T^C(S; x) = \{h \in X : \langle \hat{n}(x), h \rangle \leq 0\} \quad \text{or} \quad T^C(S; x) = \{h \in X : \langle \hat{n}(x), h \rangle \geq 0\},$$

which confirms that $T^C(S; x)$ is a half-space. We may suppose that $\hat{n}_Z$ is chosen so that the second latter equality holds true. We then derive that

$$N^C(S; x) = \{-t\hat{n}(x) : t \geq 0\}. \tag{11}$$

The vector $\hat{n}(x)$ is called the *unit interior normal vector of* $\operatorname{bd} S$ *at* x, since it is orthogonal to $Z(x)$ and it "aims" to $\operatorname{int} S$. It is worth noting that $\hat{n}(x)$ doesn't depend on the diffeomorphism nor the model space chosen to describe $\operatorname{bd} S$ as submanifold, since it is fully determined by $Z(x)$ and $T^C(S; x)$. In what follows, we will preserve the notation $Z(x)$ and $\hat{n}(x)$ to denote the tangent space and the unit interior normal vector, respectively.

In view of the proof of the proposition we also state the following simple lemma.

Lemma 3.2. *Let S be a subset of X and U an open set of X.*
(a) The following equalities hold:

$$\mathrm{int}_U(U \cap S) = U \cap \mathrm{int}\, S, \quad \mathrm{cl}_U(U \cap S) = U \cap \mathrm{cl}\, S, \quad \mathrm{bd}_U(U \cap S) = U \cap \mathrm{bd}\, S.$$

(b) If $S = \overline{\mathrm{int}\, S}$, then

$$U \cap S = \mathrm{cl}_U\big(\mathrm{int}_U(U \cap S)\big) = \mathrm{cl}_U(U \cap \mathrm{int}\, S).$$

Proof. The first two equalities in (a) easily follow from the openness of U and the third is a consequence of the former equalities. Finally, if $S = \overline{\mathrm{int}\, S}$, then we see from (a) that

$$\mathrm{cl}_U\big(\mathrm{int}_U(U \cap S)\big) = \mathrm{cl}_U(U \cap \mathrm{int}\, S) = U \cap \mathrm{cl}(\mathrm{int}\, S) = U \cap S. \quad \square$$

Proposition 3.3. *Let $S \subseteq X$ be a closed body near $x_0 \in \mathrm{bd}\, S$. Assume that $\mathrm{bd}\, S$ is a $\mathcal{C}^p$-submanifold at x_0 with $p \geq 1$ and denote by $Z(x_0) := T_{x_0}(\mathrm{bd}\, S)$ the tangent space to the boundary of S at x_0. Then S is epi-Lipschitz at x_0, and there exist a neighborhood $U_0 \in \mathcal{N}_X(x_0)$ and a function $f : \pi_{Z(x_0)}(U_0) \subseteq Z(x_0) \to \mathbb{R}$ such that f is of class $\mathcal{C}^p$ on $\pi_{Z(x_0)}(U_0)$, $\nabla f(\pi_{Z(x_0)}(x_0)) = 0$ and*

$$U_0 \cap S = \{z + t\hat{n}(x_0) \in U_0 \ : \ z \in Z(x_0), f(z) \leq t\},$$

where $\hat{n}(x_0)$ denotes the unit interior normal vector of $\mathrm{bd}\, S$ at x_0. Furthermore, endowing $Z(x_0) \times \mathbb{R}$ with the inner product

$$\langle (z,t), (z',t') \rangle = \langle z, z' \rangle + tt',$$

if in addition S is r-prox-regular at x_0, then $\overline{\mathrm{epi}\, f}$ is also r-prox-regular at $(z_0, f(z_0))$.

Proof. By Proposition 3.1 choose an open neighborhood U of x_0, a C^p-diffeomorphism $\varphi : U \to \varphi(U) \subset X$, and a closed subspace Z of X of codimension 1 such that $\varphi(x_0) = 0$ and

$$\varphi(U \cap \mathrm{bd}\, S) = Z \cap \varphi(U).$$

By replacing φ by $D\varphi^{-1}(0) \circ \varphi$, we can choose $Z = Z(x_0)$ and $D\varphi(x_0) = \mathrm{id}_X$. Let ν be a unit vector of X orthogonal to Z. We have that $\mathbb{R}\nu$ is a topological vector subspace complement of Z in X, that is, $X = Z \oplus \mathbb{R}\nu$. Noticing that $Z = \{x \in X \ : \ \langle \nu, x \rangle = 0\}$, we see that, for $z + t\nu \in U$ with $z \in Z$ and $t \in \mathbb{R}$,

$$z + t\nu \in U \cap \mathrm{bd}\, S \Leftrightarrow \langle \varphi(z + t\nu), \nu \rangle = 0.$$

Consider the open set $W := \{(z,t) \in Z \times \mathbb{R} \ : \ z + t\nu \in U\}$ in $Z \times \mathbb{R}$, where Z is equipped with the induced norm, and consider also the $\mathcal{C}^p$ function $F : W \to \mathbb{R}$ defined by

$$F(z,t) := \langle \varphi(z + t\nu), \nu \rangle, \quad \text{for all } (z,t) \in W.$$

Write $x_0 = z_0 + t_0\nu$ with $z_0 \in Z$ and $t_0 \in \mathbb{R}$, and note that $F(z_0, t_0) = 0$ and that the derivative with respect to the second variable t at (z_0, t_0) satisfies

$$D_2F(z_0, t_0) = \langle D\varphi(z_0 + t_0\nu)\nu, \nu \rangle = \langle D\varphi(x_0)\nu, \nu \rangle = \|\nu\|^2 = 1.$$

We can apply the implicit function theorem to obtain a connected open neighborhood Q_0 of z_0 in Z, a real $\varepsilon > 0$ and a $\mathcal{C}^p$ function $f : Q_0 \to]t_0 - \varepsilon, t_0 + \varepsilon[$ such that

$$U_0 := \{z + t\nu : z \in Q_0,\ t \in]t_0 - \varepsilon, t_0 + \varepsilon[\ \} \subset U$$

and such that, for $z \in Z$ and $t \in \mathbb{R}$

$$\Big(z + t\nu \in U_0 \cap \operatorname{bd} S\Big) \Leftrightarrow \Big(z + t\nu \in U_0 \text{ and } F(z,t) = 0\Big) \Leftrightarrow \Big(z + t\nu \in U_0 \text{ and } t = f(z)\Big).$$

The set S being a closed body near x_0, shrinking Q_0 and ε if necessary we may and do suppose that $U_0 \cap \operatorname{int} S$ is connected and $U_0 \cap S = U_0 \cap \big(\overline{\operatorname{int} S}\big)$. Furthermore, for any $h \in Z$ we have

$$\langle \nabla f(z_0), h\rangle = -D_2F(z_0,t_0)^{-1} \circ D_1F(z_0,t_0)h = -D_1F(z_0,t_0)h = -\langle D\varphi(x_0)h, \nu\rangle = 0,$$

since $D_2F(z_0,t_0) = \mathrm{id}_{\mathbb{R}}$ and $D\varphi(x_0)\big|_Z = \mathrm{id}_Z$. Thus, $\nabla f(z_0) = 0$.

With the linear isomorphism $L : Z \times \mathbb{R} \to X$ defined by $L(z,t) := z + t\nu$, clearly $(L^{-1}(U_0)) \cap \operatorname{epi}_s f$ and $(L^{-1}(U_0)) \cap \operatorname{hypo}_s f$ are the two connected components of $L^{-1}(U_0) \setminus \operatorname{gph} f$. It results that $U_0 \cap L(\operatorname{epi}_s f)$ and $U_0 \cap L(\operatorname{hypo}_s f)$ are the two connected components of $U_0 \setminus \operatorname{bd} S$. Since $U_0 \cap \operatorname{int} S$ is a connected component of $U_0 \setminus \operatorname{bd} S$ according to the above lemma, it ensures that either $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{epi}_s f)$ or $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{hypo}_s f)$. Noticing that

$$\begin{aligned} U_0 \cap L(\operatorname{hypo}_s f) &= \{z + t\nu : z \in Q_0,\ t \in]t_0 - \varepsilon, t_0 + \varepsilon[,\ t < f(z)\} \\ &= \{z + t(-\nu) : z \in Q_0,\ t \in]-t_0 - \varepsilon, -t_0 + \varepsilon[,\ (-f)(z) < t\}, \end{aligned}$$

and changing ν by $-\nu$ and t_0 by $-t_0$ if necessary, we may suppose that the equality $U_0 \cap \operatorname{int} S = U_0 \cap L(\operatorname{epi}_s f)$ holds true. By the above lemma again we derive that $U_0 \cap S = U_0 \cap L(\operatorname{epi} f)$, which also says that S is epi-Lipschitz at any point in $U_0 \cap S$.

Let us denote $A := L^{-1}$ and endow $Z \times \mathbb{R}$ with the canonical inner product, that is,

$$\langle (z,r), (z',r')\rangle_{Z\times\mathbb{R}} := \langle z, z'\rangle + rr'.$$

Writing any $x \in X$ as $x = \pi_Z(x) + \pi_{\mathbb{R}}(x)\nu$ with $\pi_Z(x) \in Z$ and $\pi_{\mathbb{R}}(x) \in \mathbb{R}$, the bijective linear mapping $A : X \to Z \times \mathbb{R}$ satisfies $A(x) := (\pi_Z(x), \pi_{\mathbb{R}}(x))$ and it is an isomorphism such that $A(U_0 \cap S) = A(U_0) \cap (\operatorname{epi} f)$. Since f is of class $\mathcal{C}^1$, at any $z \in \pi_Z(U_0)$ we have $\partial_C f(z) = \{\nabla f(z)\}$ and therefore

$$N^C\big(\operatorname{epi} f; (z, f(z))\big) = \{\lambda(\nabla f(z), -1) \ : \ \lambda \geq 0\}.$$

Further, taking the linear isomorphism A into account, we have for any $x \in S \cap U_0$ (see (4))

$$N^C(S;x) = N^C(U_0 \cap S; x) = A^*\big(N^C(A(U_0) \cap (\operatorname{epi} f); A(x))\big) = A^*\big(N^C(\operatorname{epi} f; A(x))\big),$$

where A^* denotes the adjoint of A. This yields by (11)

$$\{-\lambda\hat{n}(x_0) \ : \ \lambda \geq 0\} = N^C(S;x_0) = \{\lambda A^*(\nabla f(z_0), -1) \ : \ \lambda \geq 0\} = \{A^*(0, -\lambda) \ : \ \lambda \geq 0\}.$$

Observing that $A^* = L$, we get that $\hat{n}(x_0) = \nu$, which finishes the first part of the proof.

For the second part of the proof, assume also that S is r-prox-regular at x_0. Then, by Theorem 2.3 there exists $\delta > 0$ such that for all $x \in S \cap B_X(x_0, \delta)$ and every $\xi \in N^P(S;x)$, one has

$$\langle \xi, x' - x \rangle \le \frac{1}{2r}\|\xi\|\|x' - x\|^2, \ \forall x' \in S.$$

By shrinking U_0 if necessary, we may and do assume that $U_0 \subseteq B(x_0, \delta)$. Now, fix $(z,t) \in A(U_0) \cap (\operatorname{epi} f)$ and $\zeta \in N^P\big(\overline{\operatorname{epi} f}; (z,t)\big) \cap \mathbb{B}_X = N^P\big(\operatorname{epi} f; (z,t)\big) \cap \mathbb{B}_X$. For every $(z',t') \in A(U_0) \cap (\operatorname{epi} f)$, we have by (8) that

$$\begin{aligned}
\langle \zeta, (z',t') - (z,t) \rangle &= \langle (A^*)^{-1}A^*\zeta, (z',t') - (z,t) \rangle \\
&= \langle A^*\zeta, A^{-1}(z',t') - A^{-1}(z,t) \rangle \\
&\le \frac{1}{2r}\|A^*\zeta\|\|A^{-1}\big((z',t') - (z,t)\big)\|^2 \\
&\le \frac{1}{2r}\|(z',t') - (z,t)\|^2,
\end{aligned}$$

where the last inequality follows from the equalities $A^{-1} = A^*$ and $\|A^*\| = 1$. Now, consider $(z',t') \in (\operatorname{epi} f) \setminus A(U_0)$. Since $z' \in Q_0$ (keep in mind that f is defined only on Q_0) and since

$$A(U_0) = Q_0 \times]t_0 - \varepsilon, t_0 + \varepsilon[$$

we have necessarily that $t' \notin]t_0 - \varepsilon, t_0 + \varepsilon[$ and in fact, $t' \ge t_0 + \varepsilon > t$ because $t' \ge f(z') > t_0 - \varepsilon$. Since $\max\{t, f(z')\} < t_0 + \varepsilon \le t'$, we can define $t'' = \max\{t, f(z')\}$ and, noting that $\pi_{\mathbb{R}}(\zeta) \le 0$ and $(z', t'') \in A(U_0) \cap \operatorname{epi} f$, we can write by what precedes

$$\begin{aligned}
\langle \zeta, (z',t') - (z,t) \rangle &= \langle \zeta, (z',t'') - (z,t) \rangle + \langle \zeta, (0, t' - t'') \rangle \\
&\le \langle \zeta, (z',t'') - (z,t) \rangle \le \frac{1}{2r}\|(z',t'') - (z,t)\|^2 \\
&= \frac{1}{2r}\big(\langle z' - z, z' - z \rangle + (t'' - t)^2\big) \\
&\le \frac{1}{2r}\big(\langle z' - z, z' - z \rangle + (t' - t)^2\big) = \frac{1}{2r}\|(z',t') - (z,t)\|^2,
\end{aligned}$$

where the last inequality is due to the fact that $t \le t'' < t'$. We then obtain that, for all $(z',t') \in \operatorname{epi} f$

$$\langle \zeta, (z',t') - (z,t) \rangle \le \frac{1}{2r}\|(z',t') - (z,t)\|^2.$$

Taking limits, we see that the inequality still holds for all $(z',t') \in \overline{\operatorname{epi} f}$. This justifies the r-prox-regularity of $\overline{\operatorname{epi} f}$ at $(z_0, f(z_0))$ and finishes the proof. □

4. Smoothness of the metric projection onto nonconvex bodies

Theorem 4.1. *Let $O_0 \subseteq X$ be an open set and $f : O_0 \subseteq X \to \mathbb{R}$ be a function of class $\mathcal{C}^{p+1}$ (with $p \ge 1$) near $x_0 \in X$ such that $\nabla f(x_0) = 0$. Assume that $\overline{\operatorname{epi} f}$ is r-prox-regular at $(x_0, f(x_0))$. For the constant*

$$\lambda = \min\Big\{r, \ \big(-2\inf\big\{\langle u, D^2 f(x_0)u \rangle \ : \ u \in \mathbb{B}_X\big\}\big)^{-1}\Big\}$$

there exists an open neighborhood W of $\operatorname{Ray}_{(x_0, f(x_0)),\lambda}(\operatorname{epi} f)$ such that

(a) $d_{\operatorname{epi} f}$ is of class $\mathcal{C}^{p+1}$ on W;
(b) $P_{\operatorname{epi} f}$ is of class $\mathcal{C}^p$ on W.

Proof. Let us denote $S = \overline{\mathrm{epi}\, f}$, and $\pi_X : X \times \mathbb{R} \to X$, $\pi_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}$ the parallel projections associated to the product $X \times \mathbb{R}$. Also, for simplicity, we will write $u = (u_1, u_2)$ for each $u \in X \times \mathbb{R}$. According to the convention $0^{-1} = +\infty$ and noting that $\inf_{u \in \mathbb{B}_X} \langle u, D^2 f(x_0) u \rangle \leq 0$, one sees that $\lambda > 0$.

Since S is r-prox-regular at $v_0 := (x_0, f(x_0))$, by Theorem 2.3 there exists $\alpha > 0$ small enough for which, by denoting $O := \mathcal{W}_S(v_0, r, \alpha)$, we have that O is open, $\pi_X(O) \subseteq O_0$ (so $O \cap S = O \cap \mathrm{epi}\, f$), P_S is single-valued on O, f is of class $\mathcal{C}^{p+1}$ on $\pi_X(O)$, d_S is continuously differentiable in $O \setminus S$ and

$$\nabla d_S(v) = \frac{v - P_S(v)}{d_S(v)}, \ \forall v \in O \setminus S. \tag{12}$$

Also, since f is of class $\mathcal{C}^{p+1}$, for $x \in \pi_X(O)$ we have that $\partial_P f(x) = \{\nabla f(x)\}$ and so

$$N^P\big(S; (x, f(x))\big) = \{t(\nabla f(x), -1) \ : \ t \geq 0\}. \tag{13}$$

Since O coincides with $\mathcal{R}_S(v_0, r, \alpha)$, we have that for each $v \in S \cap O$

$$P_S\left[\big(v + N^P(S; v)\big) \cap O\right] = v, \tag{14}$$

and that $\mathrm{Ray}_{v_0,\lambda}(S) \subseteq O$. Let any $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$, and choose three convex neighborhoods $U \in \mathcal{N}_{X\times\mathbb{R}}(u_0)$ and $V, V' \in \mathcal{N}_{X\times\mathbb{R}}(v_0)$ such that

- $U \subseteq O \setminus S$, $V' \subseteq O$, $V \subset V'$;
- $(v_1, f(v_1)) \in V'$ for every $v_1 \in \pi_X(V)$;
- there exists $\delta > 0$ such that $U + (\{0\} \times \,]-\delta, \delta[\,) \subseteq O \setminus S$; and
- $\mathrm{diam}(\pi_{\mathbb{R}}(V')) < \delta$.

From those assumptions, we have that for each $v \in V$, $(v_1, f(v_1)) \in V'$ and $U - (0, v_2 - f(v_1)) \subseteq O \setminus S$. Let us define the mapping

$$\begin{aligned} F : U \times V &\to X \times \mathbb{R} \\ (u, v) &\mapsto u - v - d_S(u)\varphi(v), \end{aligned}$$

where

$$\varphi(v) = \frac{(\nabla f(v_1), -1)}{\|(\nabla f(v_1), -1)\|} \quad \text{for all } v \in V.$$

We claim that $F(u, v) = 0$ if and only if $v = P_S(u)$. For the sufficiency, let us suppose that $v = P_S(u)$. Then, $u - v \in N^P(S; v)$ and by (13) and the definition of φ, there exists $t \geq 0$ such that

$$u = v + t\varphi(v).$$

Thus, noting that $d_S(u) = \|u - P_S(u)\| = t\|\varphi(v)\| = t$, we conclude that $F(u, v) = 0$. On the other hand, to prove the necessity, let us suppose that $F(u, v) = 0$, so $\|u - v\| = d_S(u)$. Putting $v' = (v_1, f(v_1))$ and noting that $\varphi(v') = \varphi(v)$, we can write

$$u = v + d_S(u)\varphi(v) = v' + d_S(u)\varphi(v') + (0, v_2 - f(v_1)).$$

Therefore, with $u' := u - (0, v_2 - f(v_1))$, we have $u' - v' \in N^P(S; v')$ and so, since $v' \in O \cap S$ and $u' \in \big(v' + N^P(S; v')\big) \cap O$, by (14) we get $P_S(u') = v'$ and

$$d_S(u') = \|u' - v'\| = \|u - v\| = d_S(u).$$

Define the mapping

$$g :]-1, 1+\delta'[\to O \setminus S, \quad \text{given by} \quad g(t) = u - \big(0, t(v_2 - f(v_1))\big),$$

with some $\delta' > 0$ for which g is well-defined. Then,

$$(d_S \circ g)'(t) = -Dd_S(g(t))\big(0, v_2 - f(v_1)\big) = -\pi_{\mathbb{R}}\left(\frac{g(t) - P_S(g(t))}{d_S(g(t))}\right)(v_2 - f(v_1)).$$

Noting that $g(t) - P_S(g(t)) \in N^P\big(S; P_S(g(t))\big)$ and recalling that $g(t) \notin S$, by (13) we obtain that $\pi_{\mathbb{R}}\left(\frac{g(t)-P_S(g(t))}{d_S(g(t))}\right) < 0$. Thus, $\operatorname{sgn}((d_S \circ g)'(t)) = \operatorname{sgn}(v_2 - f(v_1))$ for all $t \in]-1, 1+\delta'[$ (where $\operatorname{sgn}(\cdot)$ denotes the sign function on $\mathbb{R} \setminus \{0\}$), and we get that if $v_2 \neq f(v_1)$, then

$$(d_S \circ g)(1) \neq (d_S \circ g)(0), \quad \text{that is, } d_S(u') \neq d_S(u),$$

since $d_S \circ g$ is strictly monotone. Since $d_S(u) = d_S(u')$, we conclude that $v_2 = f(v_1)$ and therefore $u = u'$ and $v = v'$. In particular, $P_S(u) = v$, which proves our claim.

We would like now to apply the Implicit Function Theorem to F at (u_0, v_0), so we need to check that

$$D_2F(u_0, v_0) = -id_{X\times\mathbb{R}} - d_S(u_0) \cdot D\varphi(v_0)$$

is an isomorphism. Let us define the mappings $\varphi_1 : (X \times \mathbb{R}) \setminus \{0\} \to X \times \mathbb{R}$ and $\varphi_2 : X \to X \times \mathbb{R}$ given by

$$\varphi_1(y) = \frac{y}{\|y\|} \quad \text{and} \quad \varphi_2(x) = (x, -1).$$

We can write $\varphi = \varphi_1 \circ \varphi_2 \circ \nabla f \circ \pi_X$. Recalling that for all $h \in X \times \mathbb{R}$

$$D\varphi_1(y)h = \frac{\|y\|h - \langle \varphi_1(y), h\rangle y}{\|y\|^2}$$

we have that

$$\begin{aligned}
D\varphi(v_0)h &= D\varphi_1((\nabla f(x_0), -1)) \circ D\varphi_2(\nabla f(x_0)) \circ D^2 f(x_0) \circ \pi_X(h) \\
&= D\varphi_1((0,-1))(D^2 f(x_0)h_1, 0) \\
&= \|(0,-1)\|^{-2}\left(\|(0,-1)\|(D^2 f(x_0)h_1, 0) - \left\langle \frac{(0,-1)}{\|(0,-1)\|}, (D^2 f(x_0)h_1, 0)\right\rangle (0,-1)\right) \\
&= (D^2 f(x_0)h_1, 0).
\end{aligned}$$

Thus, $D\varphi(v_0) = (D^2 f(x_0) \circ \pi_X, 0)$. Let us then show that $\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0)$ is bijective. We may assume that $D^2 f(x_0) \neq 0$, since otherwise the bijectivity is trivial.

- **surjectivity:** Let us consider $h \in X \times \mathbb{R}$ with $h \neq 0$. Since

$$(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0))^* h = \mathrm{id}_{X\times\mathbb{R}}(h) + d_S(u_0)(D\varphi(v_0))^* h,$$

it follows that

$$\begin{aligned}
\|(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0))^* h\|^2 &= \|h\|^2 + 2d_S(u_0)\langle (D\varphi(v_0))^* h, h\rangle + d_S(u_0)^2 \|(D\varphi(v_0))^* h\|^2 \\
&= \|h\|^2 + 2d_S(u_0)\langle h_1, D^2 f(x_0) h_1\rangle + d_S(u_0)^2 \|(D\varphi(v_0))^* h\|^2 \\
&\geq \|h\|^2 + 2d_S(u_0)\left\langle \tfrac{h_1}{\|h\|}, D^2 f(x_0)\tfrac{h_1}{\|h\|}\right\rangle \|h\|^2 \\
&\geq \left(1 + 2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\} d_S(u_0)\right)\cdot \|h\|^2 \\
&\geq \left(1 - \frac{1}{\lambda} d_S(u_0)\right)\cdot \|h\|^2, \qquad (15)
\end{aligned}$$

where the last inequality is due to the definition of λ. Since $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S) \subset \mathcal{W}_S(v_0, \lambda, \alpha)$, we have that $c = 1 - \lambda^{-1} d_S(u_0) > 0$, and so, by for example [7, Theorem 2.20], the conclusion follows.

- **injectivity:** Let $h \in X \times \mathbb{R}$ such that $\big(\mathrm{id}_{X\times\mathbb{R}} + d_S(u_0)D\varphi(v_0)\big)h = 0$. Then necessarily $h_2 = 0$, provided $\pi_{\mathbb{R}}\,(D\varphi(v_0)h) = 0$, and so, recalling that $\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\} \leq 0$, we can write

$$\begin{aligned}
2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\}\|h\|^2 &\leq \langle h_1, D^2 f(x_0)h_1\rangle = \langle h, (D^2 f(x_0)h_1, 0)\rangle = \langle h, D\varphi(x_0)h\rangle \\
&= d_S(u_0)^{-1}\langle h, d_S(u_0)D\varphi(v_0)h\rangle = -d_S(u_0)^{-1}\|h\|^2,
\end{aligned}$$

where the last equality is due to the fact that we have supposed that $\big(\mathrm{id}_{X\times\mathbb{R}} + d_S(u)D\varphi(v_0)\big)h = 0$. But since $-d_S(u_0)^{-1} < -\lambda^{-1} \leq 2\inf_{x\in\mathbb{B}_X}\{\langle x, D^2 f(x_0)x\rangle\}$, we have that necessarily $h = 0$, which proves the injectivity.

Now, we can apply the Implicit Function Theorem in the following way. Since d_S is of class $\mathcal{C}^1$ in U, we have that F is of class $\mathcal{C}^1$ in $U \times V$. Therefore, there exist two neighborhoods $U_1 \in \mathcal{N}(u_0)$ and $V_1 \in \mathcal{N}(v_0)$ and a mapping $\phi : U_1 \to V_1$ such that

(i) ϕ is of class $\mathcal{C}^1$;
(ii) For each $u' \in U_1$, $F(u', \phi(u')) = 0$;
(iii) For each $(u', v') \in U_1 \times V_1$, $F(u', v') = 0 \Rightarrow v = \phi(u')$.

Then, by (*ii*) and (*iii*) we get that $P_S = \phi$ in U_1, and therefore, P_S is of class $\mathcal{C}^1$ on U_1, according to (*i*). Now, looking at the formula (12), we get that d_S is of class $\mathcal{C}^2$ on U_1 and so is F on $U_1 \times V_1$. We can apply recursively this argument as follows:

$$\begin{aligned}
d_S \text{ is of class } \mathcal{C}^2 \text{ in } U_1 &\Longrightarrow F \text{ is of class } \mathcal{C}^2 \text{ on } U_1 \times V_1 \\
&\underset{\text{IFT}}{\Longrightarrow} \exists U_2 \in \mathcal{N}(u_0),\ P_S \text{ is of class } \mathcal{C}^2 \text{ on } U_2 \\
&\ \ \vdots \\
&\Longrightarrow F \text{ is of class } \mathcal{C}^p \text{ on } U_{p-1} \times V_{p-1} \\
&\underset{\text{IFT}}{\Longrightarrow} \exists U_p \in \mathcal{N}(u_0),\ P_S \text{ is of class } \mathcal{C}^p \text{ on } U_p \\
&\Longrightarrow d_S \text{ is of class } \mathcal{C}^{p+1} \text{ on } U_p.
\end{aligned}$$

Since ∇f is of class $\mathcal{C}^p$, the argument ends at this iteration, since we can't ensure that F is of class $\mathcal{C}^{p+1}$. The proof is finished considering W as the union of the U_p obtained by this way for each $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$, and noting that P_S and $P_{\mathrm{epi}\, f}$ coincide on W since $W \subseteq O$. $\square$

Remark 4.2. Observe from the preceding proof that, for the point $u_0 \in \mathrm{Ray}_{v_0,\lambda}(S)$ we have

$$\begin{aligned} DP_S(u_0) &= -[D_2F(u_0,v_0)]^{-1} \circ D_1F(u_0,v_0) \\ &= -[D_2F(u_0,v_0)]^{-1} \circ \left(\mathrm{id}_X - \left\langle \frac{u_0 - P_S(u_0)}{d_s(u_0)}, \cdot \right\rangle \frac{u_0 - P_S(u_0)}{d_S(u_0)}\right) \\ &= -[D_2F(u_0,v_0)]^{-1} \circ \Pi_{X\times\{0\}}. \end{aligned}$$

Also, note that $-D_2F(u_0,v_0)$ maps $X\times\{0\}$ onto $X\times\{0\}$. In particular, we have that $DP_S(u_0)$ restricted to $X\times\{0\}$ is invertible as a mapping from $X\times\{0\}$ to $X\times\{0\}$.

The following lemma will be crucial in the development below.

Lemma 4.3. *Let U be an open set of X and $f : U \subseteq X \to \mathbb{R}$ be a function of class $\mathcal{C}^{p+1}$ near $x_0 \in X$ such that $\nabla f(x_0) = 0$. Assume that $\overline{\mathrm{epi}\, f}$ is r-prox-regular at $(x_0, f(x_0))$. Then, one has*

$$\inf\left\{\langle u, D^2 f(x_0)u\rangle \;:\; u \in \mathbb{B}_X\right\} \geq -\frac{1}{r}.$$

Proof. Let us denote $O := B_{X\times\mathbb{R}}((x_0, f(x_0)), \alpha)$ with $\alpha > 0$ small enough such that $\pi_X(O) \subseteq U$, f is of class $\mathcal{C}^{p+1}$ at $\pi_X(O)$ and $\overline{\mathrm{epi}\, f}$ is (r,α)-prox-regular at $(x_0, f(x_0))$. Then, for every $(x,s) \in O \cap \mathrm{epi}\, f$, and every $\xi \in N^P(\mathrm{epi}\, f; (x,s)) = N^P(\overline{\mathrm{epi}\, f}; (x,s))$ we have that

$$\langle \xi, (x',s') - (x,s)\rangle \leq \frac{1}{2r}\|\xi\|\|(x',s') - (x,s)\|^2,\ \forall (x',s') \in \mathrm{epi}\, f. \tag{16}$$

Fix $h \in X$. Since for every $x \in \pi_X(O)$, we have that

$$N^P\big(\mathrm{epi}\, f; (x, f(x))\big) = \{t(\nabla f(x), -1) \;:\; t \geq 0\},$$

so using the equality $\nabla f(x_0) = 0$ we can write

$$\begin{aligned} \langle h, D^2 f(x_0)h\rangle &= \lim_{t\searrow 0} \left\langle th, \frac{\nabla f(x_0+th) - \nabla f(x_0)}{t^2}\right\rangle \\ &= \lim_{t\searrow 0} \left\langle \big(th, f(x_0+th) - f(x_0)\big), \frac{\big(\nabla f(x_0+th), -1\big) - (0,-1)}{t^2}\right\rangle \\ &= \lim_{t\searrow 0} \left\langle \big(x_0+th, f(x_0+th)\big) - \big(x_0, f(x_0)\big), \frac{\big(\nabla f(x_0+th), -1\big)}{t^2}\right\rangle + \frac{f(x_0+th) - f(x_0)}{t^2}, \end{aligned}$$

thus, according to equation (16), we can write

$$\begin{aligned} \langle h, D^2 f(x_0)h\rangle &\geq \lim_{t\searrow 0} -\frac{1}{2rt^2}\|\big(\nabla f(x_0+th), -1\big)\| \left\|\big(th, f(x_0+th) - f(x_0)\big)\right\|^2 + \frac{f(x_0+th) - f(x_0)}{t^2} \\ &= \lim_{t\searrow 0} -\frac{1}{2r}\|\big(\nabla f(x_0+th), -1\big)\| \left\|\left(h, \frac{f(x_0+th) - f(x_0)}{t}\right)\right\|^2 + \frac{f(x_0+th) - f(x_0)}{t^2} \\ &= -\frac{1}{2r}\|\big(\nabla f(x_0), -1\big)\| \left\|\big(h, Df(x_0)h\big)\right\|^2 + \lim_{t\searrow 0} \frac{f(x_0+th) - f(x_0) - tDf(x_0)h}{t^2} \\ &= -\frac{1}{2r}\left\|\big(h, Df(x_0)h\big)\right\|^2 + \frac{1}{2}\langle h, D^2 f(x_0)h\rangle = -\frac{1}{2r}\|h\|^2 + \frac{1}{2}\langle h, D^2 f(x_0)h\rangle, \end{aligned}$$

where the last equality follows from the facts that $Df(x_0)h = 0$ and $\|(h,0)\|^2 = \|h\|^2$. The conclusion follows. $\square$

Another lemma is needed before proving Theorem 2.4.

Lemma 4.4. *Let $S \subseteq X$ be a closed body near $x_0 \in \operatorname{bd} S$. Assume that there exist $r \in]0,+\infty]$ and $\alpha > 0$ such that $B_X(x_0,\alpha) \cap \operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold (with $p \geq 1$) and that S is r-prox-regular at x_0. Then, for $r' = r/2$, there exists a neighborhood V of $\operatorname{Ray}_{x_0,r'}(S)$ such that*

- *d_S is of class $\mathcal{C}^{p+1}$ on V;*
- *P_S is of class $\mathcal{C}^p$ on V.*

Furthermore, if the set S is (r,α)-prox-regular at x_0, then

- *d_S is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0,r',\alpha) \setminus S$;*
- *P_S is of class $\mathcal{C}^p$ on $\mathcal{W}_S(x_0,r',\alpha) \setminus S$.*

Proof. Shrinking α, we may suppose that S is r-prox-regular at each point in $B_X(x_0,\alpha) \cap \operatorname{bd} S$. Let $\bar{x} \in B_X(x_0,\alpha) \cap \operatorname{bd} S$. Recalling that $Z(\bar{x}) := T_{\bar{x}}(\operatorname{bd} S)$ and applying Proposition 3.3, there exist a neighborhood $U \in \mathcal{N}_X(\bar{x})$ and a function $f : \pi_{Z(\bar{x})}(U) \subseteq Z(\bar{x}) \to \mathbb{R}$ such that, denoting $\bar{z} := \pi_{Z(\bar{x})}(\bar{x})$, f is of class $\mathcal{C}^{p+1}$ in $\pi_{Z(\bar{x})}(U)$, $\nabla f(\bar{z}) = 0$,

$$U \cap S = \{z + t\hat{n}(\bar{x}) \in U \ : z \in Z(\bar{x}),\ f(z) \leq t\},$$

and also, $\overline{\operatorname{epi} f}$ is r-prox-regular at $(\bar{z}, f(\bar{z}))$; keep in mind that $\hat{n}(\bar{x})$ denotes the unit interior normal of $\operatorname{bd} S$ at $\bar{x}$. We may and do assume that $U \subseteq B_X(x_0,\alpha)$. By Theorem 4.1 and the inequality of Lemma 4.3, we have that $P_{\operatorname{epi} f}$ is of class $\mathcal{C}^p$ on a neighborhood W of $\operatorname{Ray}_{(\bar{z},f(\bar{z})),r'}(\operatorname{epi} f)$.

Choose $\delta \in]0,\alpha[$ small enough such that $B_X(\bar{x},\delta) \subseteq U$ and S is (r,δ)-prox-regular at $\bar{x}$. Let $L : Z(\bar{x}) \times \mathbb{R} \to X$ be the canonic isomorphism given by $L(z,t) = z + t\hat{n}(\bar{x})$. Noting by (11) that

$$\operatorname{Ray}_{\bar{x},r'}(S) = \{\bar{x} - t\hat{n}(\bar{x}) : t \in (0,r')\} = L\big(\{(\bar{z}, f(\bar{z})) + t(0,-1) : t \in (0,r')\}\big) = L\big(\operatorname{Ray}_{(\bar{z},f(\bar{z})),r'}(\operatorname{epi} f)\big),$$

we have that $W' := W \cap L^{-1}(\mathcal{W}_S(\bar{x},r',\delta))$ is also an open neighborhood of $\operatorname{Ray}_{(\bar{z},f(\bar{z})),r'}(\operatorname{epi} f)$. Since L is an isometry, we have that for each $w \in W'$, $P_S(L(w)) \in U \cap S$ and so

$$\begin{aligned}\|L(w) - P_S(L(w))\| &= \|w - L^{-1}\big(P_S(L(w))\big) \\ &\geq \|w - P_{\operatorname{epi} f}(w)\| = \|L(w) - L(P_{\operatorname{epi} f}(w))\| \geq \|L(w) - P_S(L(w))\|.\end{aligned}$$

Therefore, for each $v \in V_{\bar{x}} := L(W')$, we have that

$$P_S(v) = (L \circ P_{\operatorname{epi} f} \circ L^{-1})(v),$$

hence P_S is well-defined on $V_{\bar{x}}$ and it is of class $\mathcal{C}^p$ on $V_{\bar{x}}$. Further, since W can be assumed to be open, the set $V_{\bar{x}}$ is an open neighborhood of $\operatorname{Ray}_{\bar{x},r'}(S)$, proving the first part of the theorem.

The second part follows directly noting that $\mathcal{W}_S(x_0,r',\alpha) \setminus S \subseteq \bigcup\{V_x \ : \ x \in B_X(x_0,\alpha) \cap \operatorname{bd} S\}$, since $\mathcal{W}_S(x_0,r',\alpha)$ and $\mathcal{R}_S(x_0,r',\alpha)$ coincide and since we can write

$$\mathcal{R}_S(x_0,r',\alpha) \setminus S = \bigcup \big\{\operatorname{Ray}_{x,r'}(S) \ : \ x \in B_X(x_0,\alpha) \cap \operatorname{bd} S\big\}. \quad \square$$

From Remark 4.2, we see that, in the proof of the preceding lemma, for each $u_0 \in \mathrm{Ray}_{\bar{x},r'}(S)$, the operator $DP_{\mathrm{epi}\, f}(L^{-1}(u_0))$ restricted to $Z(\bar{x}) \times \{0\}$ is invertible as a mapping from $Z(\bar{x}) \times \{0\}$ onto $Z(\bar{x}) \times \{0\}$. From this observation, we can conclude that the operator

$$DP_S(u_0) = L \circ DP_{\mathrm{epi}\, f}(L^{-1}(u_0)) \circ L^{-1}$$

restricted to $Z(\bar{x})$ also is invertible as a mapping from $Z(\bar{x})$ onto $Z(\bar{x})$. This yields the following proposition, which will be useful in the study of the converse of Theorem 2.4. (See comments on Section 6.)

Proposition 4.5. *Under the assumptions and notation of Lemma 4.4, for each $u_0 \in \mathrm{Ray}_{x_0,r'}(S)$, the operator $DP_S(u_0)$ is invertible as a mapping from $Z(x_0)$ onto $Z(x_0)$.*

Furthermore, if S is (r,α)-prox-regular, then for each $u \in \mathcal{W}_S(x_0, r', \alpha) \setminus S$, the operator $DP_S(u)$ is invertible as a mapping from $Z(P_S(u))$ onto $Z(P_S(u))$.

Now, to prove Theorem 2.4, we will also need the following submanifold property of level sets. Let U_0 be an open set of X with $\bar{x} \in U_0$ and g be a mapping of class $\mathcal{C}^p$ (with $p \geq 1$) from U_0 into a Banach space Y such that $Dg(\bar{x})$ is surjective. With $\bar{y} := g(\bar{x})$, the level set $M := \{x \in U_0 \ : \ g(x) = \bar{y}\}$ is a $\mathcal{C}^p$-submanifold of X at $\bar{x}$.

Indeed, supposing (without loss of generality) $\bar{y} = 0$, we know by the Local Submersion theorem (see, e.g., [1, Theorem 2.5.13]) that, denoting by X_1 an orthogonal subspace of $X_2 := \mathrm{Ker}\, A$ with $A := Dg(\bar{x})$, there exist an open neighborhood $U \subset U_0$ of $\bar{x}$ in X, an open neighborhood V of $\big(g(\bar{x}), \pi_{X_2}(\bar{x})\big)$ in $Y \times X_2$, and a $\mathcal{C}^p$-diffeomorphism $\psi : V \to U$ from V onto U such that $g \circ \psi(v_1, v_2) = v_1$ for all $(v_1, v_2) \in V$. The continuous linear mapping $A_0 : X_1 \to Y$ with $A_0(x_1) := A(x_1)$ for all $x_1 \in X_1$ is bijective, hence an isomorphism from X_1 to Y by the closed graph theorem, so the mapping $j : X_1 \oplus X_2 \to Y \times X_2$ defined by $j(x_1 \oplus x_2) = \big(A_0(x_1), x_2\big)$ is also an isomorphism. Let $j_V : j^{-1}(V) \to V$ the bijective restriction from $j^{-1}(V)$ onto V and consider the $\mathcal{C}^p$-diffeomorphism $\varphi := j_V^{-1} \circ \psi^{-1}$ from U onto $j_V^{-1}(V)$. Then, with $\pi_i := \pi_{X_i}$ we have

$$\begin{aligned} x \in \varphi(U \cap M) &\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } g \circ \psi(j_V(x)) = 0 \Leftrightarrow \psi \circ j_V(x) \in U \text{ and } g \circ \psi\big(A_0 \circ \pi_1(x), \pi_2(x)\big) = 0 \\ &\Leftrightarrow \psi \circ j_V(x) \in U \text{ and } A_0 \circ \pi_1(x) = 0 \Leftrightarrow \psi \circ j_V(x) \in U \text{ and } \pi_1 x = 0 \Leftrightarrow x \in \varphi(U) \cap X_2, \end{aligned}$$

which means that $\varphi(U \cap M) = \varphi(U) \cap X_2$ and justifies that M is a $\mathcal{C}^p$-submanifold of X.

We can now proceed to the proof of Theorem 2.4:

Proof of Theorem 2.4. Let U be an open connected neighborhood of x_0 such that $U \cap \mathrm{int}\, S$ is connected and $U \cap S = U \cap \big(\overline{\mathrm{int}\, S}\big)$. Since S is r-prox-regular at x_0, there exist $\alpha' \in]0, \alpha[$ such that $B(x_0, \alpha') \subset U$ and S is (r, α')-prox-regular at x_0. We will show inductively that for every $n \in \mathbb{N}$, d_S is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0, r_n, \alpha') \setminus S$ with $r_n := \sum_{k=1}^{n} 2^{-k} r$. Noting that

$$\mathrm{Ray}_{x_0,r}(S) \subseteq \mathcal{W}(x_0, r, \alpha') \setminus S = \bigcup_{n=1}^{\infty} \mathcal{W}(x_0, r_n, \alpha') \setminus S,$$

and taking into account that $P_S(u) = (u - \nabla d_S(u))/d_S(u)$ for every $u \in \mathcal{W}(x_0, r, \alpha') \setminus S$, proving the latter assertion is enough to conclude the first part of the theorem.

The case $n = 1$ is contained in Lemma 4.4, so we only need to prove the inductive step. Consider then $n \geq 2$ and assume that d_S is already of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_S(x_0, r_{n-1}, \alpha') \setminus S$. It only rests to prove that d_S is of class $\mathcal{C}^{p+1}$ near each point of

$$\mathcal{W}_S(x_0, r_n, \alpha') \setminus \mathcal{W}_S(x_0, r_{n-1}, \alpha') = \{u \in \mathcal{W}_S(x_0, r_n, \alpha') \ : \ r_{n-1} \leq d_S(u) < r_n\}.$$

Fix $\bar{u} \in \mathcal{W}_S(x_0, r_n, \alpha') \setminus S$ with $r_{n-1} \le d_S(\bar{u}) < r_n = r_{n-1} + 2^{-n}r$. Let us denote $\bar{x} = P_S(\bar{u})$ and choose $\lambda \in]0, r_{n-1}[$ such that $d_S(\bar{u}) - \lambda < 2^{-n}r$. Note by definition of $\mathcal{W}_S(x_0, r_n, \alpha')$ that $\bar{x} \in B(x_0, \alpha')$.

Let us consider the set $S_\lambda := \{x \in X \ : \ d_S(x) \le \lambda\}$ and the point $\bar{y} := \bar{x} - \lambda \hat{n}(\bar{x})$ in $\operatorname{bd} S_\lambda$ (where we recall that $\hat{n}(\bar{x})$ denotes the unit interior normal vector of $\operatorname{bd} S$ at $\bar{x}$, which is well-defined since $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold at $\bar{x}$). Since $\lambda < r_{n-1}$ and $\bar{x} \in B(x_0, \alpha')$, we have that $\bar{y} \in \mathcal{W}_S(x_0, r_{n-1}, \alpha')$, and so, by hypothesis, d_S is of class $\mathcal{C}^{p+1}$ near $\bar{y}$. Choose $\delta \in]0, \alpha'[$ small enough such that $B_X(\bar{y}, \delta) \subset \mathcal{W}_S(x_0, r_{n-1}, \alpha') \setminus S$. We claim that

$$B_X(\bar{y}, \delta) \cap \{d_S < \lambda\} = B_X(\bar{y}, \delta) \cap \operatorname{int}(S_\lambda). \tag{17}$$

Denoting the second member by V it is clear that it contains the first member. Suppose there is some $u_0 \in V$ which is not in the first member. Then $d_S(u_0) = \lambda$, hence u_0 is a maximizer of d_S on the open set V, which yields $\nabla d_S(u_0) = 0$, contradicting the equality $\|\nabla d_S(u_0)\| = 1$. The claim is then justified. This says in particular that $B_X(\bar{y}, \delta) \cap \operatorname{bd}(S_\lambda) = B_X(\bar{y}, \delta) \cap \{d_S = \lambda\}$. Further, d_S is of class $\mathcal{C}^{p+1}$ on $B(\bar{y}, \delta)$ and $\nabla d_S(y)$ is surjective from X into $\mathbb{R}$ since $\|\nabla d_S(y)\| = 1$ for all $y \in B_X(\bar{y}, \delta)$. The set $\operatorname{bd}(S_\lambda)$ is then a C^{p+1}-submanifold of X as seen above for such a level set.

Furthermore, for every $y \in B_X(\bar{y}, \delta) \cap \operatorname{bd} S_\lambda$, the $C^{1,1}$-property of d_S near y gives $\partial_P d_S(y) = \{\nabla d_S(y)\}$ and by [6, Theorem 4.3] we know that

$$\partial_P d_S(y) = N^P(S_\lambda; y) \cap \mathbb{S}_X,$$

so it follows that

$$N^P(S_\lambda; y) = \{t \nabla d_S(y) \ : \ t \ge 0\} = \{-t \hat{n}(P_S(y)) \ : \ t \ge 0\}. \tag{18}$$

Fix $y \in B_X(\bar{y}, \delta) \cap \operatorname{bd} S_\lambda$ and denote $x := P_S(y)$. Noting that $\lambda + t < r$ for every $t \in \left]0, \frac{r}{2^{n-1}}\right]$ and recalling that S is r-prox-regular at x, we have

$$d_S(y - t\hat{n}(x)) = d_S(x - (\lambda + t)\hat{n}(x)) = \lambda + t.$$

Noting also that

$$d_{S_\lambda}(u) = d_S(u) - \lambda, \ \forall u \in X \setminus S_\lambda, \tag{19}$$

we can write $d_{S_\lambda}(y - t\hat{n}(x)) = t$, and so $y \in \operatorname{Proj}_{S_\lambda}(y - t\hat{n}(x))$ for every $t \in \left]0, \frac{r}{2^{n-1}}\right]$. In particular, by (18), for every $\zeta \in N^P(S_\lambda; y) \cap \mathbb{B}_X$,

$$y \in \operatorname{Proj}_{S_\lambda}(y + t\zeta), \ \forall t \in \left]0, \tfrac{r}{2^{n-1}}\right].$$

Since this last inclusion holds for every $y \in S_\lambda \cap B_X(\bar{y}, \delta)$ (the case of $y \in \operatorname{int}(S_\lambda)$ is trivial since $N^P(S_\lambda; y) = 0$), we conclude that S_λ is $\left(\frac{r}{2^{n-1}}, \delta\right)$-prox-regular at $\bar{y}$ according to Definition 2.1.

Since $B_X(\bar{y}, \delta) \subseteq \mathcal{W}_S(x_0, r_{n-1}, \alpha')$, we have that for all $y' \in B_X(\bar{y}, \delta)$, $P_S(y') \in U$. We derive that, for any $y' \in B_X(\bar{y}, \delta) \cap \{d_S < \lambda\}$ we have $y' \in P_S(y') + B_X(0, \lambda)$ with $P_S(y') \in S \cap U$, thus

$$B_X(\bar{y}, \delta) \cap \{d_S < \lambda\} = B_X(\bar{y}, \delta) \cap \bigcup_{u \in U \cap S} \big(u + B_X(0, \lambda)\big) = B_X(\bar{y}, \delta) \cap \bigcup_{u \in U \cap \operatorname{int} S} \big(u + B_X(0, \lambda)\big), \tag{20}$$

where the second equality is due to the fact $U \cap S = U \cap \overline{\operatorname{int} S}$. Taking any y_i in the latter set with $i = 1, 2$, there are $x_i \in U \cap \operatorname{int} S$ and $b_i \in B_X(0, \lambda)$ such that $y_i = x_i + b_i$. The set $U \cap \operatorname{int} S$ being arc-wise connected as an open connected set in the normed space X, there exists a continuous mapping $\gamma : [0, 1] \to U \cap \operatorname{int} S$ with

$\gamma(0) = x_1$ and $\gamma(1) = x_2$. The mapping $\gamma_0 : [0,1] \to B_X(\bar{y},\delta) \cap \{d_S < \lambda\}$ with $\gamma_0(t) = \gamma(t) + (1-t)b_1 + tb_2$ is well defined (by (20)) and continuous, and further $\gamma_1(0) = y_1$ and $\gamma_0(1) = y_2$. This tells us that the set $B_X(\bar{y},\delta) \cap \operatorname{int}(S_\lambda) = B_X(\bar{y},\delta) \cap \{d_S < \lambda\}$ is (arc-wise) connected.

To see that S_λ is a closed body near $\bar{y}$ it remains to show that $B_X(\bar{y},\delta) \cap S_\lambda = B_X(\bar{y},\delta) \cap \overline{\operatorname{int}(S_\lambda)}$. The second member is obviously included in the first. Take any $y' \in B_X(\bar{y},\delta)$ with $d_S(y') = \lambda$. Putting $v = \nabla d_S(y') \in \mathbb{S}_X$, for $t > 0$ small enough we have

$$d_S(y' - tv) = d_S(y') - t\big(\langle \nabla d_S(y'), v\rangle + \varepsilon(t)\big) = \lambda - t\big(1 + \varepsilon(t)\big),$$

where $\varepsilon(t) \to 0$ as $t \downarrow 0$, so for $t > 0$ small enough

$$y' - tv \in B_X(\bar{y},\delta) \cap \{d_S < \lambda\} = B_X(\bar{y},\delta) \cap \operatorname{int}(S_\lambda),$$

where the equality is due to (17). This entails that $y' \in B_X(\bar{y},\delta) \cap \overline{\operatorname{int}(S_\lambda)}$, hence the desired equality $B_X(\bar{y},\delta) \cap S_\lambda = B_X(\bar{y},\delta) \cap \overline{\operatorname{int}(S_\lambda)}$ is justified. The set S_λ is then a closed body near $\bar{y}$.

We can then apply the second part of Lemma 4.4 to S_λ at $\bar{y}$ to get that d_{S_λ} is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{S_\lambda}(\bar{y},\delta,2^{-n}r) \setminus S_\lambda$. Finally, by (19), we conclude that d_S itself is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{S_\lambda}(\bar{y},\delta,2^{-n}r) \setminus S_\lambda$ and in particular, it is so near $\bar{u}$. The induction (and therefore the proof of the first part of the theorem) is then completed.

The second part of the theorem follows directly from the first one, following the last observations of the proof of Lemma 4.4. □

The first corollary is concerned with $\rho(\cdot)$-prox-regular closed bodies.

Corollary 4.6. *Let $S \subseteq X$ be a closed body such that $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold with $p \geq 1$. If S is $\rho(\cdot)$-prox-regular, then*

- *d_S is of class $\mathcal{C}^{p+1}$ on $U_{\rho(\cdot)}(S) \setminus S$;*
- *P_S is of class $\mathcal{C}^p$ on $U_{\rho(\cdot)}(S) \setminus S$.*

Proof. Fix $u \in U := U_{\rho(\cdot)}(S) \setminus S$. Since S is $\rho(\cdot)$-prox-regular, we have that there exists $y \in \operatorname{Proj}_S(u)$ such that $d_S(u) < \rho(y)$. Let us fix a real r with $d_S(u) < r < \rho(y)$. Since ρ is continuous, there exists a neighborhood $V \in \mathcal{N}_X(y)$ on which $\rho(v) > r$ for each $v \in S \cap V$. Therefore, by properties related to $U_{\rho(\cdot)}(S)$ recalled in Section 2 and by Theorem 2.3 the set S is r-prox-regular at y. Then, by Theorem 2.4 there exists $\alpha > 0$ small enough such that P_S is well-defined on $\mathcal{W}_S(y,r,\alpha) \setminus S$ and it is of class $\mathcal{C}^p$ on this open set. Noting that

$$u \in (\mathcal{W}_S(y,r,\alpha) \setminus S) \cap U \subseteq U,$$

and that both sets $\mathcal{W}_S(y,r,\alpha) \setminus S$ and U are open, we conclude that P_S is well-defined near u and it is of class $\mathcal{C}^p$ near u. Since u is arbitrary, the conclusion follows. □

In the case that S is a convex body, recalling that all convex closed sets are $(+\infty)$-prox-regular, we can recuperate Holmes' theorem as corollary of Theorem 2.4:

Corollary 4.7 *(Holmes, 1973). Let $K \subseteq X$ be a convex body and suppose that $\operatorname{bd} K$ is a $\mathcal{C}^{p+1}$-submanifold at a point $x_0 \in \operatorname{bd} K$, with $p \geq 1$. Then there exists an open neighborhood W of $\operatorname{Ray}_{x_0}(K)$ such that*

- *d_S is of class $\mathcal{C}^{p+1}$ on W;*
- *P_S is of class $\mathcal{C}^p$ on W.*

5. Smoothness of the metric projection onto submanifolds

In their work of 1984, J-B. Poly and G. Raby proved that if a closed subset M of a finite dimensional Euclidean space is a $\mathcal{C}^{p+1}$-submanifold at $m_0 \in M$, then the function $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ near m_0 (see [21, Section 1]).

In their proof, they used the finite dimensional assumption only to ensure that, when M is a $\mathcal{C}^{p+1}$-submanifold at m_0, the set-valued mapping $\mathrm{Proj}_M(\cdot)$ is nonempty near m_0 (which is provided by the local compactness of M). Nevertheless, in every Hilbert space X, if a closed subset M of X is a $\mathcal{C}^{p+1}$-submanifold at $m_0 \in M$ with $p \geq 1$, then it is prox-regular at m_0.

Indeed, consider a neighborhood U of m_0, a $\mathcal{C}^{p+1}$-diffeomorphism $\varphi : U \to \varphi(U)$ and a closed subspace Z of X such that $\varphi(m_0) = 0$ and $\varphi(U \cap M) = \varphi(U) \cap Z$. Let us denote $S := \overline{\varphi(U) \cap Z}$. Clearly, for $z \in Z \cap \varphi(U)$ we have $N^C(S; z) = Z^\perp$, so $\langle \xi, z' - z\rangle = 0$ for all $\xi \in N^P(S; z) \subseteq N^C(S; z)$ and $z' \in S$. Theorem 2.2(ii) tells us that S is prox-regular at $\varphi(m_0) = 0$. Choose a real $\delta > 0$ such that $S' := S \cap B(0, \delta) \subseteq \varphi(U)$. We can apply equality (6) to get that for each $m \in M \cap \varphi^{-1}(B(0, \delta)) = \varphi^{-1}(S')$

$$N^P(\varphi^{-1}(S'); m) \subset N^C(\varphi^{-1}(S'); m) = D\varphi(m)^* \left(N^C(S'; \varphi(m))\right) = D\varphi(m)^* \left(Z^\perp\right).$$

Shrinking δ if necessary, we can suppose that φ is Lipschitz on $\varphi^{-1}(B(0, \delta))$ with a Lipschitz constant $\gamma \geq 0$ and that, for all $z, z' \in B(0, \delta)$

$$\|\varphi^{-1}(z') - \varphi^{-1}(z) - D\varphi^{-1}(z)(z' - z)\| \leq C\|z' - z\|^2$$

for some constant $C > 0$. Thus, for any $m, m' \in \varphi^{-1}(S')$ and $\zeta \in N^P(\varphi^{-1}(S'); m) \cap \mathbb{B}_X$ we conclude that

$$\begin{aligned}
\langle \zeta, m' - m\rangle &= \langle \zeta, \varphi^{-1}\big(\varphi(m')\big) - \varphi^{-1}\big(\varphi(m)\big)\rangle \\
&\leq \langle \zeta, D\varphi^{-1}(\varphi(m))(\varphi(m') - \varphi(m))\rangle + \gamma^2 C\|m' - m\|^2 \\
&= \langle (D\varphi(m)^*)^{-1}\zeta, \varphi(m') - \varphi(m)\rangle + \gamma^2 C\|m' - m\|^2 \\
&= \gamma^2 C\|m' - m\|^2
\end{aligned}$$

which, by Theorem 2.2(iii), proves the prox-regularity of M at m_0. Therefore, in order to follow the proof of Poly and Raby, the finite-dimensional assumption is not needed. So, we can reformulate the main result of [21] as follows:

Theorem 5.1 *(See Poly and Raby [21]). Let M be a closed subset of a Hilbert space X and let $m_0 \in M$. Given an integer $p \geq 1$, if the set M is a $\mathcal{C}^{p+1}$-submanifold at m_0, then there exists a neighborhood $U \in \mathcal{N}_X(m_0)$ such that*

(i) $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on U;
(ii) P_M is well-defined on U and it is of class $\mathcal{C}^p$ therein.

Based on this theorem, we will prove an analogous version of Theorem 2.4 when S is itself a $\mathcal{C}^{p+1}$-submanifold, instead of a nonconvex body with $\mathcal{C}^{p+1}$-smooth boundary.

Lemma 5.2. *Let M be a closed set of X such that M is a $\mathcal{C}^1$-submanifold at $m_0 \in M$. If M is r-prox-regular at m_0, then for every $\lambda \in]0, r[$, the set*

$$M_\lambda := \{x \in X \ : \ d_M(x) \leq \lambda\}$$

is a closed body near each point $y_0 := m_0 + \lambda v$, where $v \in N^P(M;m_0) \cap \mathbb{S}_X$, and there exists $\delta > 0$ such that

$$B_X(y_0,\delta) \cap \{d_M < \lambda\} = B_X(y_0,\delta) \cap \operatorname{int}(M_\lambda) \quad \textit{and} \quad B_X(y_0,\delta) \cap M_\lambda = B_X(y_0,\delta) \cap \overline{\operatorname{int}(M_\lambda)}.$$

Proof. Note first that $y_0 \in M_\lambda$ and $d_M(y_0) = \lambda$ by the r-prox-regularity of M. Since M is a $\mathcal{C}^1$-submanifold at m_0, there exist a closed subspace Z of X, an open convex neighborhood $U \in \mathcal{N}(0)$ and a C^1-diffeomorphism $\varphi : U \to \varphi(U)$ such that $\varphi(0) = m_0$ and

$$\varphi(U \cap Z) = \varphi(U) \cap M.$$

Since $U \cap Z$ is arc-wise connected (as a convex set), we get that $\varphi(U) \cap M$ is also arc-wise connected. Now, choose $\alpha, \delta > 0$ small enough such that $B_X(m_0,\alpha) \subseteq \varphi(U)$, M is (r,α)-prox-regular at m_0 and $B_X(y_0,\delta) \subseteq \mathcal{W}_M(m_0,r,\alpha)$. As in the proof of Theorem 2.4, we have that

$$B_X(y_0,\delta) \cap \{d_M < \lambda\} = B_X(y_0,\delta) \cap \operatorname{int}(M_\lambda).$$

Now, since for each $y \in B_X(y_0,\delta)$ we have that $P_M(y) \in \varphi(U)$, we can write

$$B_X(y_0,\delta) \cap \{d_M < \lambda\} = B_X(y_0,\delta) \cap \bigcup_{u \in \varphi(U) \cap M} (u + B_X(0,\lambda)). \tag{21}$$

Taking any $y_1, y_2 \in B_X(y_0,\delta) \cap \{d_M < \lambda\}$, we can find $m_1, m_2 \in \varphi(U) \cap M$ and $b_1, b_2 \in B_X(0,\lambda)$ such that $y_i = m_i + b_i$ for $i = 1,2$. Since $\varphi(U) \cap M$ is arc-wise connected, there exists a continuous mapping $\gamma : [0,1] \to \varphi(U) \cap M$ with $\gamma(0) = m_1$ and $\gamma(1) = m_2$. Thus, the mapping $\gamma_0 : [0,1] \to B_X(y_0,\delta) \cap \{d_M < \lambda\}$ given by $\gamma_0(t) = \gamma(t) + (1-t)b_1 + tb_2$ is well-defined by (21), is continuous, $\gamma_0(0) = y_1$ and $\gamma_0(1) = y_2$. We get that the set $B_X(y_0,\delta) \cap \{d_M < \lambda\} = B_X(y_0,\delta) \cap \operatorname{int}(M_\lambda)$ is therefore (arc-wise) connected.

We can show that $B_X(y_0,\delta) \cap M_\lambda = B_X(y_0,\delta) \cap \overline{\operatorname{int}(M_\lambda)}$ following the same argument as in the end of the proof of Theorem 2.4. The proof is now complete. □

Theorem 5.3. *Let M be a closed set of X which is a $\mathcal{C}^{p+1}$-submanifold at $m_0 \in M$ with $p \geq 1$. If M is r-prox-regular at m_0, then there exists $\alpha > 0$ such that*

- *$d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_M(m_0,r,\alpha)$;*
- *P_M is of class $\mathcal{C}^p$ on $\mathcal{W}_M(m_0,r,\alpha)$.*

Proof. By Theorem 5.1, there exists $\varepsilon > 0$ small enough such that $d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $B_X(m_0,\varepsilon)$. Choose then $\alpha \in]0,\varepsilon[$ such that M is (r,α)-prox-regular at m_0 and choose also $\lambda \in]0,r[$ such that $\alpha + \lambda < \varepsilon$. In particular, we have that $\overline{\mathcal{W}_M(m_0,\lambda,\alpha)} \subseteq B_X(m_0,\varepsilon)$. Fix $u \in \mathcal{W}_M(m_0,r,\alpha) \setminus B_X(m_0,\varepsilon)$. We have that $\lambda < d_M(u) < r$.

By definition of $\mathcal{W}_M(m_0,r,\alpha)$ we can take $m \in B_X(m_0,\alpha) \cap M$ and $\nu \in N^P(M;m) \cap \mathbb{S}_X$ such that $u = m + d_M(u)\nu$. Put $y = m + \lambda\nu$. Defining M_λ as in Lemma 5.2 we have that $y \in \operatorname{bd} M_\lambda$ and M_λ is a closed body near y, and for some real $\delta > 0$ we have $B_X(y,\delta) \subset \mathcal{W}_M(m_0,r,\alpha)$ along with

$$B_X(y,\delta) \cap \operatorname{bd} M_\lambda = B_X(y,\delta) \cap \{d_M = \lambda\} \quad \text{and} \quad B_X(y,\delta) \cap M_\lambda = B_X(y,\delta) \cap \overline{\operatorname{int}(M_\lambda)}.$$

Fix any $y' \in B_X(y,\delta) \cap \operatorname{bd} M_\lambda$. By the remarks preceding the proof of Theorem 2.4, we also know that $\operatorname{bd} M_\lambda$ is a $\mathcal{C}^{p+1}$-submanifold at y', since d_M is of class $\mathcal{C}^{p+1}$ near y' with $\nabla d_M(y') \neq 0$. Further, by [6, Theorem 4.3], we have that

$$\{\nabla d_M(y')\} = \partial_P d_M(y') = N^P(M_\lambda; y') \cap \mathbb{S}_X,$$

so setting $\nu' := \nabla d_M(y')$, it follows that $N^P(M_\lambda; y') = \{t\nu' \ : \ t \geq 0\}$. Note that, setting $m' := P_M(y')$ so we can write $y' = m' + \lambda\nu'$, and hence

$$d_M(y' + t\nu') = d_M(m' + (t + \lambda)\nu') = t + \lambda,\ \forall t \in [0, r - \lambda[.$$

Also, noting that

$$d_{M_\lambda}(x) = d_M(x) - \lambda,\ \forall x \in X \setminus M_\lambda, \tag{22}$$

we conclude that $d_{M_\lambda}(y' + t\nu') = t$ for every $t \in [0, r - \lambda[$. In particular, fixing $r' \in]d_M(u) - \lambda, r - \lambda[$, we have that for all $y' \in B_X(y, \delta) \cap \operatorname{bd} M_\lambda$ and $\zeta \in N^P(M_\lambda; y') \cap \mathbb{B}_X$,

$$y' \in \operatorname{Proj}_{M_\lambda}(y' + t\zeta),\ \forall t \in [0, r'],$$

and so, M_λ is (r', δ)-prox-regular at y. Applying Theorem 2.4 it results that, for $\alpha' := \delta$, the function d_{M_λ} is of class $\mathcal{C}^{p+1}$ on $\mathcal{W}_{M_\lambda}(y, r', \alpha') \setminus M_\lambda$. By equation (22) and since $u \in \mathcal{W}_{M_\lambda}(y, r', \alpha') \setminus M_\lambda$, it ensues that $d_M(\cdot)$ (and therefore $d_M^2(\cdot)$) is of class $\mathcal{C}^{p+1}$ near u. Since the function d_M^2 is also of class $\mathcal{C}^{p+1}$ on $B_X(m_0, \varepsilon)$, we conclude that it is class $\mathcal{C}^{p+1}$ on the whole open set $\mathcal{W}_M(m_0, r, \alpha)$. $\square$

Observing the proof of Corollary 4.6, we can establish the following direct result from Theorem 5.3:

Corollary 5.4. *Let M be a closed set of X. Assume that M is a $\mathcal{C}^{p+1}$-submanifold. If M is $\rho(\cdot)$-prox-regular, then*

- *$d_M^2(\cdot)$ is of class $\mathcal{C}^{p+1}$ on $U_{\rho(\cdot)}(M)$;*
- *P_M is of class $\mathcal{C}^p$ on $U_{\rho(\cdot)}(M)$.*

6. Final comments

The study of differentiability properties of the metric projection onto convex sets was not limited to the work of Holmes in 1973. Before him, for a closed convex subset K of a Hilbert space X, a conical differentiabilty of P_K at $x_0 \in K$ was established by Zarantonello [24], and earlier J. B. Kruskal [15] provided examples of closed convex sets K for which the differentiability of P_K fails. After Holmes' paper, the study of the differentiability of P_K was continued by Fitzpatrick and Phelps [12] and Noll [18] for closed convex sets with smooth boundary. Even further, some approaches to the nonconvex case were made by Shapiro [23], whose paper is in fact one of the fundamental contributions to the modern theory of prox-regular sets. Under the light of our results, we would like to do some comments about those works.

First of all, when we work with general closed bodies, the hypothesis of $\mathcal{C}^2$-smooth boundary seems to be crucial in order to get differentiability of the metric projection in the usual sense: In [12], Fitzpatrick and Phelps gave a counterexample of a convex body such that the boundary is $\mathcal{C}^{1,1}$, but the metric projection is nowhere Fréchet differentiable. In that sense, our result (Theorem 2.4) cannot be improved.

Secondly, the Fréchet differentiability of the metric projection is not enough to guarantee the smoothness of the boundary of the set: In [12, Section 4], we can find a construction of polar convex sets which have nonsmooth boundary (not even $\mathcal{C}^1$) and such that their metric projections are of class $\mathcal{C}^1$ outside them. Nevertheless, in the same paper a converse of Holmes' theorem is established under an additional qualification hypothesis:

Theorem 6.1 *(Fitzpatrick and Phelps, 1982). Let K be a closed convex set of a Hilbert space X with* $\operatorname{int} K \neq \emptyset$, *and let $x \in X \setminus K$. Then,* $\operatorname{bd} K$ *is a $\mathcal{C}^{p+1}$-submanifold at $P_K(x)$ if and only if P_K is of class $\mathcal{C}^p$ near x and $DP(x)\big|_{H[x]} : H[x] \to H[x]$ is invertible, where $H[x]$ denotes the hyperplane*

$$H[x] := \{y \in X \ : \ \langle x - P_K(x), y\rangle = 0\}.$$

Unfortunately, we still lack a converse for Theorem 2.4. Clearly, it will not be enough to have the smoothness of the metric projection, since we already have counterexamples in the convex case, but Theorem 6.1 is rather suggestive of what to look for. In fact, when $\operatorname{bd} S$ is a $\mathcal{C}^{p+1}$-submanifold we have that $H[x] = T_{P_S(x)}(\operatorname{bd} S)$. This entails, by the remark after the proof of Lemma 4.4, that the invertibility of $DP_S(x)$ as a function from $H[x]$ onto $H[x]$ for x close enough to $P_S(x)$ is already a necessary condition for $\operatorname{bd} S$ to be a $\mathcal{C}^{p+1}$-submanifold at $P_S(x)$.

In the same line, we have that Poly and Raby also proved the converse of Theorem 5.1 in the finite dimensional setting (see [21, Section 3]), but this part of their proof cannot be directly extended to the infinite dimensional setting (at least to our knowledge). The converses of Theorem 2.4 and Theorem 5.1 require quite long developments and will be carried out in another paper.

In other direction, Noll studied in [18] a weaker notion of second-order smoothness of the metric projection onto convex sets. Namely, he was interested in knowing whether the function $f : X \to \mathbb{R}_+$ given by $f(x) = \frac{1}{2}\|x\|^2 + \frac{1}{2}d_K^2(x)$ has a second-order Taylor expansion. He focused his work on the *second-order Mosco differentiability* of f, which is sufficient for Taylor expansions, even when P_K is not Fréchet differentiable. Even though this notion of second-order smoothness is weaker, prox-regularity is not enough to guarantee it. Indeed, in his celebrated paper [15], Kruskal provided a counterexample of a closed convex set in $\mathbb{R}^3$ such that its metric projection fails to be one-side directionally differentiable at some points, and, following [12,18,23], a minimal condition to have second-order approaches is the existence of directional derivatives of the metric projection. We are then encouraged to follow this idea and search for other notions of smoothness in the nonconvex case. This could also allow us to study second-order conditions when the space X is assumed to be uniformly convex instead of being Hilbertian.

Implicitly, our work strongly uses the fact that the norm in a Hilbert space is infinitely differentiable off zero (which is necessary to ensure the smoothness of the function φ in the proof of Theorem 4.1). This is lost beyond the Hilbert setting. Nevertheless, the theory of prox-regular sets in uniformly convex spaces has been widely developed by Bernard, Thibault and Zlateva [3,4] and therefore, the smoothness of the metric projection deserves to be studied in this context as well. Extending the results of [18] to the nonconvex case and using them to further develop the theory in the uniformly convex setting will be one of our next objectives.

Finally, we believe that the study of smoothness of the metric projection can be developed as well in the Riemannian manifold setting. Recent works [5,14] have shown that proximal calculus and prox-regular theory can be developed when the Hilbert space X is replaced by a Riemannian submanifold, since the basic ingredients that are the differential calculus and the distance notion are both well-posed. We would like to obtain similar versions of Theorems 2.4 and 5.3 in this context. Also, we think that Theorem 5.3 can be improved. For example, if we consider the unit sphere $\mathbb{S}_2$ in $\mathbb{R}^2$ endowed with the Euclidean norm, Theorem 5.3 guarantees that the metric projection $P_{\mathbb{S}_2}$ is $\mathcal{C}^\infty$ on the set

$$U_1(\mathbb{S}_2) = \{x \in \mathbb{R}^2 \ : \ d_{\mathbb{S}_2}(x) < 1\} = \{x \in \mathbb{R}^2 \ : \ 0 < \|x\| < 2\}.$$

Further, using the convexity of the unit ball $\mathbb{B}_2$ and applying Holmes' theorem (Corollary 4.7), we also get that $P_{\mathbb{S}_2}$ is $\mathcal{C}^\infty$ on $\mathbb{R}^2 \setminus \mathbb{B}_2$. Therefore, the actual set of smoothness of $P_{\mathbb{S}_2}$ is $\mathbb{R}^2 \setminus \{0\}$; of course this can also be seen in a direct way. This asymmetry comes from the observation that the best prox-regularity function of $\mathbb{S}_2$ doesn't depend only on the point $x \in \mathbb{S}_2$ considered, but also on which direction of $N^P(\mathbb{S}_2; x)$ we are following. In a future work we will study Theorems 2.4 and 5.3 in Riemannian Manifolds and we will provide an improvement of Theorem 5.3.

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