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ORIGINAL PAPER

Lyapunov pairs for perturbed sweeping processes

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Abstract We give a full characterization of nonsmooth Lyapunov pairs for perturbed sweeping processes under very general hypotheses. As a consequence, we provide an existence result and a criterion for weak invariance for perturbed sweeping processes. Moreover, we characterize Lyapunov pairs for gradient complementarity dynamical systems.

Keywords Sweeping process · Lyapunov pair · Differential inclusions · Invariance · Normal cone · Complementarity dynamical systems

1 Introduction

The aim of this paper is to study Lyapunov pairs for the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1}$$

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where $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N(S; x)$ denotes the Clarke normal cone to S at x and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

The differential inclusion (1) is known as perturbed sweeping process. This problem is a first-order differential inclusion involving the normal cone to a moving set depending on time. Roughly speaking, a point is swept by a moving closed set. The sweeping process was introduced and deeply studied, for convex sets, by Moreau (see [24–27]) to model an elasto-plastic mechanical system. Since then, many other applications of the sweeping process have been given, namely in, electrical circuits [1], crowd motion [23], hysteresis in elasto-plastic models [21], etc.

The seminal work of Moreau was the starting point of many other developments related to existence, stability and optimal control of perturbed sweeping process.

In this paper we are interested in the study of stability of perturbed sweeping processes with nonconvex moving sets.

An important approach to deal with stability of dynamical system is the so called "Lyapunov method". This indirect approach allows to address several stability properties of dynamical systems as finite or asymptotic stability, existence of equilibria, stabilization, etc. (see, for example, [13–15]). The idea behind the Lyapunov method is to construct a pair of functions, called Lyapunov pair, which constitute a kind of energy of the system which decrease along the solutions of the system. Moreover, since in complex real-world applications it is virtually impossible to find explicit solutions, it is very important to have explicit characterizations of Lyapunov pairs for dynamical systems.

While the initial Lyapunov method was developed for smooth functions, it became clear the necessity of working with nonsmooth Lyapunov pairs. This is mainly due to the flexibility of working with nonsmooth functions and to the unfortunate fact that some dynamical systems do not admit smooth Lyapunov functions (see [14]). To pass from smooth to nonsmooth Lyapunov functions several notions of directional derivatives where used in the past (see [5, Chapter 6]). Since directional derivatives are naturally associated with subdifferentials, other authors started to use subgradients and subdifferentials. In this context, among all these subdifferentials, the use of the proximal subdifferential became a benchmark because it allows to characterize nonsmooth Lyapunov pairs for differential inclusions (see [15]). In fact, it is recognized that the proximal subdifferential is the smallest reasonable subdifferential that allows a characterization of nonsmooth Lyapunov pairs. In this work, we follow this path and give an explicit characterization, involving the proximal subdifferential, of weak Lyapunov pairs for the perturbed sweeping process. It is worth pointing out that our result, in contrast with [2,3,17], does not involve the singular (horizon) subdifferential or interior conditions of the moving sets.

Characterizations of smooth and nonsmooth Lyapunov pairs have been considered for different dynamical systems by several authors (see [5,13–15] and the references given there). In the present case, the situation is more involved because the perturbed sweeping process is a constrained differential inclusion with unbounded right-hand side. Thus, the classical results for differential inclusions are not applicable. Nevertheless, we can mention the works of Adly, Hantoute and Théra [2,3] which give explicit criteria for Lyapunov pairs for maximal monotone evolution equations, which includes the perturbed sweeping process driven by a fixed convex set and the work of Hantoute and Mazade [17], where they give explicit criteria for Lyapunov functions for perturbed sweeping process governed by a fixed uniformly prox-regular set. In our main result, we extend all these results by considering the class of uniformly subsmooth sets. This class, introduced in [6], is a generalization of uniform prox-regularity which corresponds to the uniform submonotonicity of the truncated normal cones and can be seen as a variational behavior or order one. Finally, we will see how this class of sets is well adapted with perturbed sweeping processes and appears naturally in the study of gradient complementarity dynamical systems.

The paper is organized as follows. After some preliminaries, in Sect. 4 we give a characterization of weak Lyapunov pairs for the sweeping process. As a consequence, we give an existence of solutions and a criterion for weak invariance for perturbed sweeping processes. Finally, in Sect. 5, we apply our main result to characterize Lyapunov pairs for gradient complementarity dynamical systems.

2 Preliminaries

From now on H stands for a separable Hilbert space whose norm is denoted by $\|\cdot\|$. The closed ball centered at x with radius r is denoted by $\bar{B}(x;r)$ and the closed unit ball is denoted by $\mathbb{B}$. The notation H_w stands for H equipped with the weak topology and $x_n \rightharpoonup x$ denotes the weak convergence of a sequence $(x_n)_n$ to x.

Recall that a vector $h \in H$ belongs to the Clarke tangent cone $T(S;x)$ (see [12]) when for every sequence $(x_n)_n$ in S converging to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex, and its negative polar $N(S;x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N(S;x) := \{v \in H : \langle v, h\rangle \le 0 \quad \forall h \in T(S;x)\}.$$

As usual, $N(S;x) := \emptyset$ if $x \notin S$. Through that normal cone, the *Clarke subdifferential* of a function $f : H \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\partial f(x) := \{v \in H : (v,-1) \in N(\operatorname{epi} f, (x, f(x)))\},$$

where $\operatorname{epi} f := \{(y,r) \in H \times \mathbb{R} : f(y) \le r\}$ is the epigraph of f. When the function f is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [15]) in the following simple and amenable way

$$\partial f(x) = \left\{v \in H : \langle v, h\rangle \le f^{\circ}(x;h) \text{ for all } h \in H\right\},$$

where

$$f^{\circ}(x;h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}\left[f(y+th) - f(y)\right],$$

is the *generalized directional derivative* of the locally Lipschitzian function f at x in the direction $h \in H$. The function $f^{\circ}(x;\cdot)$ is in fact the support function of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of upper semicontinuity from H into H_w.

The *weak tangent cone* to a set S at $x \in S$ is defined by

$$T_S^w(x) := \{v \in H\colon \text{ there exists } t_n \searrow 0,\, v_n \rightharpoonup v \text{ such that } x + t_n v_n \in S\}.$$

The *support function* of $S \subseteq H$, is defined, for any $v \in H$, by

$$\sigma(v, S) := \sup_{s \in S} \langle v, s \rangle .$$

We say that a set-valued map $\Psi\colon H \rightrightarrows H$ with nonempty and closed is *scalarly upper semicontinuous* from H into H_w if its support function $x \mapsto \sigma(v, \Psi(x))$ is upper semicontinuous for all $v \in H$. If, in addition, the set-valued map Ψ has convex and bounded values, then scalarly upper semicontinuity of Ψ coincides with upper semicontinuity from H into H_w of Ψ (see [18, Proposition 2.32]).

$d_S(x) := \inf_{y \in S} \|x - y\|$ denotes the *distance function* to the set $S \subseteq H$ at $x \in H$ and $\text{Proj}_S(x)$ denotes the set (possibly empty)

$$\text{Proj}_S(x) := \{y \in S\colon d_S(x) = \|x - y\|\} .$$

The equality (see [15])

$$N(S; x) = \text{cl}^* (\mathbb{R}_+ \partial d_S(x)) \quad \text{ for } x \in S,$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x, S)$ in place of $\partial d(\cdot, S)(x)$.

Let $f\colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \text{dom}\, f$. An element ζ belongs to the *proximal subdifferential* $\partial^P f(x)$ of f at x (see [15]) if there exist two positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in B(x; \eta).$$

Moreover, the following property holds (see [15, Chapter 1]):

$$\zeta \in \partial^P f(x) \quad \Leftrightarrow \quad (\zeta, -1) \in N^P (\text{epi}\, f; (x, f(x))) . \tag{2}$$

An element ζ belongs to the *Fréchet subdifferential* $\partial^F f(x)$ of f at x (see [7]) if

$$\liminf_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle \zeta, h \rangle}{\|h\|} \geq 0.$$

Furthermore, the following formula holds:

$$\partial^F d_S(x) = N^F (S; x) \cap \mathbb{B} \quad \text{ for all } x \in S.$$

We say that a closed set S is *Fréchet normally regular* if $N^F(S;x) = N(S;x)$ for all $x \in S$. In particular, if S is closed and convex, then it is Fréchet normally regular.

The following result is a consequence of the chain rule [10, Proposition A.3].

Proposition 1 *Let $h\colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function, $K \subseteq \mathbb{R}^m$ a closed convex cone and $y \in \mathbb{R}^m$. Assume that there exists $k > 0$ such that*

$$\mathbb{B}_{\mathbb{R}^m} \subseteq Dh(x)k\mathbb{B}_{\mathbb{R}^n} - K \quad \textit{for all } x \in \mathbb{R}^n, \tag{3}$$

where Dh is the Jacobian of h. Then, the set $h^{-1}(K-y)$ is Fréchet normally regular and

$$N\left(h^{-1}(K-y);x\right) = [Dh(x)]^* N(K; h(x)+y)$$

Proof It follows from the chain rule [10, Proposition A.3] and the formula

$$I_{h^{-1}(K-y)}(x) = I_K(h(x)+y) \quad \text{for all } x \in \mathbb{R}^n,$$

where I_S is the indicator function of a set S. □

We recall the definition of the class of uniformly subsmooth sets. This notion includes strictly the notion of convex and uniformly prox-regular sets (see [6]).

Definition 1 Let S be a closed subset of H. We say that S is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\left\langle x_1^* - x_2^*, x_1 - x_2 \right\rangle \geq -\varepsilon \|x_1 - x_2\|, \tag{4}$$

holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$. Moreover, if E is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is *equi-uniformly subsmooth*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (4) holds for each $t \in E$ and all $x_1, x_2 \in S(t)$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S(t); x_i) \cap \mathbb{B}$ for $i = 1, 2$.

The following result (see [31, Proposition 2.6]), known as horizontal approximation theorem of Rockafellar, will be useful in proof of Theorem 1.

Proposition 2 *Let $f\colon H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \operatorname{dom} f$, and let $x^* \in H$ with $(x^*, 0) \in N^P(\operatorname{epi} f; (x, f(x)))$. Then, for any $\varepsilon > 0$ there exist $x_\varepsilon \in \operatorname{dom} f$ and $(x_\varepsilon^*, -r_\varepsilon) \in N^P(\operatorname{epi} f; (x_\varepsilon, f(x_\varepsilon)))$ with $r_\varepsilon > 0$ along with*

$$\|x_\varepsilon - x\| + |f(x_\varepsilon) - f(x)| < \varepsilon \textit{ and } \|(x_\varepsilon^*, -r_\varepsilon) - (x^*, 0)\| < \varepsilon.$$

Now we give the notion of Lyapunov pairs for the perturbed sweeping process. Due to the fact that, in general, the perturbed sweeping process with subsmooth sets does not have a unique solution, we will use the notion of "weak" Lyapunov pairs.

Definition 2 Let $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ and $W\colon [T_0, T] \times H \to \mathbb{R}$ be two proper and lower semicontinuous functions. We say that (V, W) forms a *weak Lyapunov pair* for the perturbed sweeping process (1) if for every $x_0 \in C(T_0)$ there exists x solution of (1) such that

$$V(t, x(t)) + \int_{T_0}^{t} W(s, x(s))ds \leq V(T_0, x_0) \quad \text{for all } t \in [T_0, T].$$

Moreover, we say that V is a *weak Lyapunov function* for (1) if $(V, 0)$ is a Lyapunov pair for (1).

Remark 1 The function W is called a dissipation function in the theory of dissipative systems (see [9]).

3 Technical assumptions

For the sake of readability, in this section we collect the hypotheses used along the paper.

Hypotheses on the set-valued map $C\colon [T_0, T] \rightrightarrows H$: C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

($\mathcal{H}_1$) There exists $\kappa \geq 0$ such that for $s, t \in [T_0, T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq \kappa|t - s|.$$

($\mathcal{H}_2$) The family $\{C(t)\colon t \in [T_0, T]\}$ is equi-uniformly subsmooth.

($\mathcal{H}_3$) For all $t \in [T_0, T]$, the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Remark 2 If the sets $C(t)$ are convex for all $t \in [T_0, T]$, then ($\mathcal{H}_2$) holds.

Hypotheses on the functions V, W:

($\mathcal{H}^V$) $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous function.

($\mathcal{H}^W$) $W\colon [T_0, T] \times H \to \mathbb{R}$ is lower semicontinuous function such that for a.e. $t \in [T_0, T]$

$$|W(t, x)| \leq \beta(t)\,(1 + \|x\|) \quad \text{for all } x \in H,$$

for some $\beta \in C(T_0, T)$.

Hypotheses on the set-valued map $F\colon [T_0, T] \times H \rightrightarrows H$: F is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_1^F$) F is upper semicontinuous from $[T_0, T] \times H$ into H_w.

($\mathcal{H}_2^F$) There exist $\alpha \in C(T_0, T)$ and $\ell\colon H \to \mathbb{R}^+$ Lipschitz such that

$$\|F(t, x)\| := \sup\{\|w\|\colon w \in F(t, x)\} \leq \alpha(t)\ell(x),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

4 Lyapunov pairs and invariance

In this section we give an explicit criterion for weak Lyapunov pairs for the perturbed sweeping process (1).

The following result, which is a direct consequence of [20, Lemma 4.3], is a key element in the proof of Theorem 1.

Lemma 1 *Assume, in addition to* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$, *that* $\alpha \in C(T_0)$ *and* $\ell\colon H \to \mathbb{R}^+$ *is a Lipschitz function. Then, the set-valued map*

$$(t,x) \mapsto -(\kappa + \alpha(t)\ell(x))\partial d_{C(t)}(x) \quad (t,x) \in [T_0,T] \times H,$$

is upper semicontinuous from $[T_0,T] \times H$ *into* H_w.

The following theorem, which is the main result of this section, gives a full characterization of the weak Lyapunov pairs for the perturbed sweeping process (1).

Theorem 1 *Assume, in addition to* $(\mathcal{H}_1^F)$ *and* $(\mathcal{H}_2^F)$, *that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}^V)$ *and* $(\mathcal{H}^W)$ *hold. If* $\operatorname{dom} V(t,\cdot) \subseteq C(t)$ *for all* $t \in [T_0,T]$, *then the following conditions are equivalent:*

(i) For all $(t,x) \in \operatorname{dom} V$ *and* $(\theta,\zeta) \in \partial^P V(t,x)$

$$\theta + \inf\{\langle v,\zeta\rangle : v \in -(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t,x)\} \le -W(t,x).$$

(ii) (V,W) *forms a weak Lyapunov pair for the sweeping process* (1).

Proof Let $G\colon [T_0,T] \times H \times \mathbb{R} \rightrightarrows \mathbb{R} \times H \times \mathbb{R}$ defined by

$$\begin{aligned} G(t,x,y) = \{1\} \times (-(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t,x)) \\ \times [-\beta(t)(1+\|x\|), -W(t,x)] \end{aligned}$$

Then, due to Lemma 1 and $(\mathcal{H}^W)$, the set-valued map G is upper semicontinuous from $[T_0,T] \times H \times \mathbb{R}$ into $\mathbb{R} \times H_w \times \mathbb{R}$ with nonempty, closed and convex values. Moreover, for all $(t,x,y) \in [T_0,T] \times H \times \mathbb{R}$

$$\begin{aligned} \|G(t,x,y)\| &:= \sup\{\|v\| : v \in G(t,x,y)\} \\ &\le 1 + \kappa + \alpha(t)\ell(x) + \|F(t,x)\| + \beta(t)\,(1+\|x\|) \\ &\le 1 + \kappa + 2\alpha(t)\ell(x) + \beta(t)\,(1+\|x\|) \\ &\le (2\alpha(t)L_\ell + \beta(t))\|x\| + (2\alpha(t)L_\ell\ell(0) + \beta(t) + 1 + \kappa), \end{aligned} \tag{5}$$

where L_ℓ is the Lipschitz constant of ℓ.

Furthermore, due to $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $\operatorname{epi} V$ is ball compact. Indeed, on the one hand,

$$\begin{aligned} \operatorname{epi} V &= \{(t,x,\lambda) \in [T_0,T] \times H \times \mathbb{R} : V(t,x) \le \lambda\} \\ &\subseteq \{(t,x,\lambda) \in [T_0,T] \times H \times \mathbb{R} : x \in C(t)\} \\ &= \operatorname{graph} C \times \mathbb{R}, \end{aligned}$$

where we have used that $\operatorname{dom} V(t,\cdot) \subseteq C(t)$ for all $t \in [T_0, T]$. On the other hand, if $((t_n, x_n))_n \subseteq \operatorname{graph} C$ is a bounded sequence, then, after taking a subsequence, $t_n \to \bar{t}$ (without relabeling) for some $t \in [T_0, T]$ and

$$x_n \in C(t_n) \subseteq C(\bar{t}) + \kappa|\bar{t} - t_n|\mathbb{B},$$

that is, $x_n = y_n + \kappa|\bar{t} - t_n|b_n$ for some $y_n \in C(\bar{t})$ and $b_n \in \mathbb{B}$. Then, due to the boundedness of $(x_n)_n$ and $(\mathcal{H}_3)$, the sequence $(y_n)_n$ is relatively compact in H, which shows that $(x_n)_n$ is relatively compact in H. Therefore, graph C is ball compact.

Hence, due to [11, Theorem 1.1], the following conditions are equivalent.

(a) For all $(t, x, r) \in \operatorname{epi} V$

$$G(t, x, r) \cap T^w_{\operatorname{epi} V}(t, x, r) \neq \emptyset.$$

(b) For all $(t, x, r) \in \operatorname{epi} V$

$$G(t, x, r) \cap \overline{\operatorname{co}}\, T^w_{\operatorname{epi} V}(t, x, r) \neq \emptyset.$$

(c) For all $(\theta, \zeta, \mu) \in N^P(\operatorname{epi} V; (t, x, r))$

$$\theta + \inf\{\langle v, \zeta\rangle + s\mu \colon (1, v, s) \in G(t, x, r)\} \leq 0.$$

(d) $(\operatorname{epi} V, G)$ is weakly invariant, that is, for any $(T_0, x_0, r_0) \in \operatorname{epi} V$ there exists a solution (τ, x, r) over $[T_0, T]$ of the differential inclusion

$$(\dot{\tau}(t), \dot{x}(t), \dot{r}(t)) \in G(\tau(t), x(t), r(t))$$

with $(\tau(T_0), x(T_0), r(T_0)) = (T_0, x_0, r_0)$ such that $(\tau(t), x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$.

Therefore, to finish the proof, it suffices to show that (c) is equivalent to (i).
(c) $\Rightarrow$ (i): Let $(\theta, \zeta) \in \partial^P V(t, x)$. Then, by virtue of (2),

$$(\theta, \zeta, -1) \in N^P(\operatorname{epi} V; (t, x, V(t, x))).$$

Therefore, by using (c),

$$\begin{aligned}\theta + \inf\{\langle v, \zeta\rangle : v \in -(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t, x)\} + W(t, x)\\ \leq \theta + \inf\{\langle v, \zeta\rangle - s \colon (1, v, s) \in G(t, x, V(t, x))\}\\ \leq 0,\end{aligned}$$

which implies (i).
(i) $\Rightarrow$ (c): Let $(\theta, \zeta, \mu) \in N^P(\operatorname{epi} V; (t, x, r))$. Then, $\mu \leq 0$ and

$$(\theta, \zeta, \mu) \in N^P(\operatorname{epi} V; (t, x, V(t, x))).$$

First case: $\mu < 0$:
It is not difficult to prove that $r = V(t, x)$ (see [15, Exercise 2.1]). Then, since $\mu < 0$ and due to property (2),

$$\left(\frac{\theta}{|\mu|}, \frac{\zeta}{|\mu|}\right) \in \partial^P V(x, t).$$

Moreover, by using (i), we obtain

$$\begin{aligned}
&\theta + \inf\{\langle v, \zeta\rangle + s\mu : (1, v, s) \in G(t, x, V(t, x))\} \\
&\quad \le \frac{\theta}{|\mu|}|\mu| + \inf\left\{\left\langle v, \frac{\zeta}{|\mu|}\right\rangle : v \in -(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t, x)\right\}|\mu| \\
&\qquad - \mu W(t, x) \\
&\quad \le -W(t, x)|\mu| - \mu W(t, x) \\
&\quad = 0,
\end{aligned}$$

which proves (c).
Second case $\mu = 0$:
According to Proposition 2, for all $n \in \mathbb{N}$ there exist

$$(\theta_n, \zeta_n, \mu_n) \in N^P\,(\text{epi}\, V; (t_n, x_n, V(t_n, x_n)))$$

with $t_n \to t$, $x_n \to x$, $V(t_n, x_n) \to V(t, x)$, $\theta_n \to \theta$, $\zeta_n \to \zeta$, $\mu_n \to 0$ and $\mu_n < 0$. Thus, by the argument given in the first case, for all $n \in \mathbb{N}$

$$\theta_n + \inf\{\langle v, \zeta_n\rangle + s\mu_n : (1, v, s) \in G(t_n, x_n, V(t_n, x_n))\} \le 0. \tag{6}$$

Moreover, since $G(t_n, x_n, V(t_n, x_n))$ is closed, convex and bounded (see (5)), the infimum in (6) is attained at some point $(1, v_n, s_n)$ with $s_n \in [-\beta(t_n)(1 + \|x_n\|), -W(t_n, x_n)]$ and $v_n \in -(\kappa + \alpha(t_n)\ell(x_n))\,\partial d_{C(t_n)}(x_n) + F(t_n, x_n)$. This implies that

$$v_n \in (\kappa + 2\alpha(t_n)\ell(x_n))\,\mathbb{B}.$$

Hence, since $(t_n)_n$ and $(x_n)_n$ are bounded, $(v_n)_n$ is bounded and we can assume that $v_n \rightharpoonup \bar{v}$. The upper semicontinuity from $[T_0, T] \times H$ into H_w of F and $\partial d_{C(\cdot)}(:)$, shows that $\bar{v} \in -(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t, x)$. Therefore, by using (6), we get

$$\begin{aligned}
&\theta + \inf\{\langle v, \zeta\rangle : (1, v, s) \in G(t, x, r)\} \\
&\quad = \theta + \inf\{\langle v, \zeta\rangle : (1, v, s) \in G(t, x, V(t, x))\} \\
&\quad \le \theta + \langle \bar{v}, \zeta\rangle \\
&\quad = \lim_{n\to\infty} (\theta_n + \langle v_n, \zeta_n\rangle + s_n\mu_n) \\
&\quad = \lim_{n\to\infty} \inf\{\theta_n + \langle v, \zeta_n\rangle + s\mu_n : (1, v, s) \in G(t, x_n, V(t_n, x_n))\} \\
&\quad \le 0,
\end{aligned}$$

which proves (c). □

As an immediate consequence of Theorem 1, we obtain, by taking V as the indicator function of $C(\cdot)$ and W equal to 0, the existence of global solutions for the perturbed sweeping process (1). The following result was proved by Noel and Thibault [28] and Jourani and Vilches [20] by very different methods.

Theorem 2 *Assume that C satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ and that F satisfies $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$. Then, for any $x_0 \in C(T_0)$, there exists at least one Lipschitz solution x of the perturbed sweeping process (1). Moreover,*

$$\|\dot{x}(t)\| \leq \kappa + 2\alpha(t)\ell(x(t)) \quad a.e.\ t \in [T_0, T].$$

Proof Let $V\colon [T_0, T] \times H \to \mathbb{R} \cup \{+\infty\}$ be defined by $V(t, x) = I_{C(t)}(x)$ and $W \equiv 0$. Then $\operatorname{dom} V(t, \cdot) = C(t)$ for all $t \in [T_0, T]$. Let $(\theta, \zeta) \in \partial^P V(t, x)$. Then, there exist $\delta, \sigma > 0$ such that for all $(s, x) \in \bar{B}((t, x); \delta)$

$$I_{C(s)}(y) \geq I_{C(t)}(x) + \theta(s-t) + \langle \zeta, y - x\rangle - \sigma\left(|s-t|^2 + \|y-x\|^2\right). \tag{7}$$

Hence, if $s = t$, we obtain that $\zeta \in N^P\ (C(t); x)$. Moreover, due to $(\mathcal{H}_1)$,

$$x \in C(t) \subseteq C(s) + \kappa|t-s|\mathbb{B}.$$

Thus, there exists $b \in \mathbb{B}$ such that $y := x - \kappa|t-s|b \in C(s)$. Then, by virtue of (7), for all $|t-s| \leq \max\{\delta, \delta/\kappa\}$

$$\theta(s-t) \leq \langle \zeta, \kappa|t-s|b\rangle + \sigma|s-t|^2\left(1 + \kappa^2\|b\|^2\right).$$

Therefore, dividing by $|s-t|$ with $s \neq t$ and taking $s \to t$, we obtain that $|\theta| \leq \kappa\|\zeta\|$.

Then, for all $(\theta, \zeta) \in \partial^P V(t, x)$ with $\zeta \neq 0$,

$$\begin{aligned}
&\theta + \inf\{\langle v, \zeta\rangle : v \in -(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t, x)\} \\
&\leq \kappa\|\zeta\| - (\kappa + \alpha(t)\ell(x))\left\langle \frac{\zeta}{\|\zeta\|}, \zeta\right\rangle + \alpha(t)\ell(x)\|\zeta\| \\
&\leq 0.
\end{aligned}$$

Therefore, all the conditions of Theorem 1 hold. Thus, for all $x_0 \in C(T_0)$, there exists at least one solution x of the sweeping process (1). □

When H is a finite-dimensional Hilbert space and $C(t) \equiv H$ we obtain, as a consequence of Theorem 1, the well known criteria for Lyapunov pairs for differential inclusions with convex, upper semicontinuous right-hand side (see for example [15]).

Corollary 1 *Let H be a finite-dimensional Hilbert space. Assume, in addition to $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$, that $(\mathcal{H}^V)$ and $(\mathcal{H}^W)$ hold. Then the following conditions are equivalent:*

(i) For all $(t, x) \in \operatorname{dom} V$ *and* $(\theta, \zeta) \in \partial^P V(t, x)$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in F(t, x)\} \leq -W(t, x).$$

(ii) (V, W) *forms a weak Lyapunov pair for the differential inclusion*

$$\dot{x}(t) \in F(t, x(t)).$$

We end this section with an application of Theorem 1 to weak invariance of perturbed sweeping processes.

Definition 3 *(weak invariance)* We say that K is weakly invariant with respect to the perturbed sweeping process (1) if for all $(T_0, x_0) \in \operatorname{graph}(C) \cap \operatorname{graph}(K)$ there exists a solution of (1) with $x(T_0) = x_0$ and $x(t) \in K(t)$ for all $t \in [T_0, T]$.

The following result extends [16, Theorem 4.3].

Theorem 3 *Assume, in addition to* $(\mathcal{H}_1^F)$ *and* $(\mathcal{H}_2^F)$ *, that* $(\mathcal{H}_1)$*,* $(\mathcal{H}_2)$*,* $(\mathcal{H}_3)$ *and* $K(t) \subseteq C(t)$ *for all* $t \in [T_0, T]$*. Then the following conditions are equivalent:*

(i) For all $(\theta, \zeta) \in N^P\,(\operatorname{graph} K; (t, x))$

$$\theta + \inf\{\langle v, \zeta\rangle : v \in -\,(\kappa + \alpha(t)\ell(x))\,\partial d_{C(t)}(x) + F(t, x)\} \leq 0.$$

(ii) For all $x_0 \in K$ *there exists a solution of the sweeping process (1) with* $x(T_0) = x_0$ *and* $x(t) \in K(t)$ *for all* $t \in [T_0, T]$*.*

Proof It follows directly from Theorem 1 by taking $V(t, x) := I_{\operatorname{graph} K}(t, x)$ and $W(t, x) \equiv 0$. □

5 An application to gradient complementarity dynamical systems

In this section, we illustrate our result with an application to Gradient Complementarity Dynamical Systems (GCDS). A GCDS consists of an ordinary differential equation coupled with complementarity conditions. More explicitly, given functions $F\colon [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $h\colon \mathbb{R}^n \to \mathbb{R}^m$ and $d\colon [T_0, T] \to \mathbb{R}^m$, the defining equations for the GCDS corresponding to these functions are

$$\begin{cases} \dot{x}(t) = F(t, x(t)) + [Dh(x(t))]^* \, u(t), \\ y(t) = h(x(t)) + d(t), \\ K \ni y(t) \perp u(t) \in K^*, \end{cases} \tag{8}$$

where $K \subseteq \mathbb{R}^m$ is a closed convex cone and

$$K^* = \{y \in \mathbb{R}^m \colon \langle v, y\rangle \geq 0 \text{ for all } v \in K\},$$

denotes the dual cone of K. A typical example of GCDS are the Linear Complementarity Dynamical Systems (LCDS) which correspond to the particular case

$$\begin{cases} \dot{x}(t) = Ax(t) + h^T u(t), \\ y(t) = hx(t) + d(t), \\ \mathbb{R}^m_+ \ni y(t) \perp u(t) \in \mathbb{R}^m_+. \end{cases}$$

GCDS, in particular LCDS, is an important class of dynamical systems with several applications, such as electrical circuits, dynamic traffic assignment problems, differential Nash games, etc. (see [10,29,30] and the references therein). GCDS has been studied by several authors. A usual approach to deal with GCDS is to transform the system into a perturbed sweeping process (see [10,19]). Indeed, the third line in (8) is a complementarity relation between $y(t)$ and $u(t)$ which are forced to remain always orthogonal one to each other. This fact can be expressed in an equivalent way as

$$K \ni y(t) \perp u(t) \in K^* \quad \Leftrightarrow \quad -u(t) \in N(K; y(t)).$$

Therefore, by using this equivalence and some chain rule for nonsmooth functions (see Proposition 1), the gradient complementarity dynamical system is formally equivalent (see [10,19] for more details) to the following perturbed sweeping process:

$$\dot{x}(t) \in -N(C(t); x(t)) + F(t, x(t)) \quad \text{a.e. } t \in [T_0, T],$$

where $C(t) := h^{-1}(K - d(t))$ for all $t \in [T_0, T]$. Thus, if the set-valued map C has a Lipschitz continuous variation and the perturbation term satisfies $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$, the existence of solutions for GCDS can be obtained from Theorem 2. To do that some smoothness hypotheses on h and constraint qualifications conditions must be imposed (see hypothesis (3)). This method was used in [4,10,30], where the authors gave sufficient conditions to assure the uniformly prox-regularity of the moving sets $C(t)$. In these papers, to assure the uniformly prox-regularity of the inverse images, the authors assume that the involved function is $C^{1,1}$ together some constraint qualification condition. The following result, proved in [19], give a sufficient condition so that the set-valued map C is Lipschitz continuous with uniformly subsmooth values.

Proposition 3 *Assume that $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function with uniformly continuous derivative, $d\colon [T_0, T] \to \mathbb{R}^m$ is κ_d-Lipschitz continuous and (3) holds. Then, the set valued map $C\colon [T_0, T] \rightrightarrows \mathbb{R}^n$ defined by $C(t) := h^{-1}(K - d(t))$ is $k \times \kappa_d$-Lipschitz continuous and satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.*

Therefore, due to Proposition 3 and Theorem 1, we get the following characterization of Lyapunov pairs for GCDS.

Theorem 4 *Assume, in addition to the hypotheses of Proposition 3, that $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$, $(\mathcal{H}^V)$ and $(\mathcal{H}^W)$ hold. If* dom $V(t, \cdot) \subseteq C(t)$ *for all $t \in [T_0, T]$, then the following conditions are equivalent:*

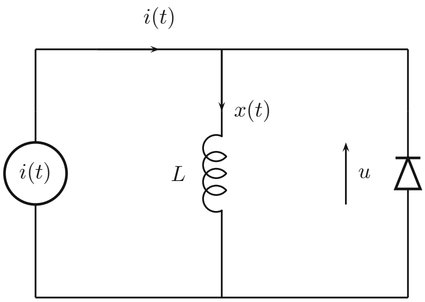

Fig. 1 A circuit with an ideal diode, an inductor and a current source

(i) For all $(t, x) \in \operatorname{dom} V$ *and* $(\theta, \zeta) \in \partial^P V(t, x)$

$$\theta + \inf\{\langle v + F(t, x), \zeta\rangle : v \in -\Gamma(t, x)\} \leq -W(t, x),$$

where κ_d *is the Lipschitz constant of* d, k *is given by (3) and*

$$\Gamma(t, x) := (\kappa_d \times k + \alpha(t)\ell(x)) \left([Dh(x)]^* N(K; h(x) + d(t))\right) \cap \mathbb{B}_{\mathbb{R}^n}.$$

(ii) (V, W) *forms a weak Lyapunov pair for (8).*

Proof According to Proposition 1, the sets $C(t)$ are Fréchet normally regular. Then, for every $x \in C(t)$

$$N(C(t); x) \cap \mathbb{B}_{\mathbb{R}^n} = N^F(C(t); x) \cap \mathbb{B}_{\mathbb{R}^n} = \partial^F d_{C(t)}(x) \subseteq \partial d_{C(t)}(x).$$

Thus,

$$N(C(t); x) \cap \mathbb{B}_{\mathbb{R}^n} = \partial d_{C(t)}(x) \quad \text{for all } x \in C(t). \tag{9}$$

Therefore, the result follows from Theorem 1, (9) and Proposition 1. □

Example 1 Let us consider a circuit with an ideal diode, an inductor and a current source (see Fig. 1), where x is the current through the inductance and a current κ-Lipschitz source $i(t)$. The dynamics is given by (see [8, Example 5.2] for more details)

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) - i(t) \\ \mathbb{R}_+ \ni y(t) \perp u(t) \in \mathbb{R}_+. \end{cases} \tag{10}$$

Hence, the system (10) is equivalent to

$$\dot{x}(t) \in -N(\mathbb{R}_+ + i(t); x(t)).$$

Let $V: [T_0, T] \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $\operatorname{dom} V(t, \cdot) \subseteq \mathbb{R}_+ + i(t)$ for all $t \in [T_0, T]$. Then, according to Theorem 1, V is a Lyapunov function for (10) if and only

$$\theta + \kappa\zeta \cdot \mathbb{1}_{\mathbb{R}_-}(\zeta)\,\mathbb{1}_{\{x=i(t)\}}(t,x) \leq 0 \text{ for all } (\theta,\zeta) \in \partial^P V(t,x).$$

where $\mathbb{1}_S$ is the characteristic function of a set S.

The following example concerns with an example in an infinite dimensional space.

Example 2 Let us consider the space $H = L^2(0,1)$, the usual space of square-integrable real-valued functions on $(0,1)$ with the norm defined by $\|\phi\| = \int_0^1 |\phi(x)|^2 dx$, and the set

$$C := \{\phi \in H_0^1(0,1) \colon |\phi'(x)| \leq \Phi(x) \text{ for a.e. } x \in (0,1)\},$$

with $H_0^1(0,1)$ being the Sobolev space of functions $\phi \in L^2(0,1)$ that possess a (distributional) derivative $\phi' \in L^2(0,1)$. The function $\Phi\colon (0,1) \to [0,\infty[$ in the definition of C is prescribed and bounded from above. Hence, since the embedding of $H_0^1(0,1)$ into $L^2(0,1)$ is relatively compact and the function Φ is bounded from above, the set C is relatively compact in $H = L^2(0,1)$, thus, the set C satisfies the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$. The study of perturbed sweeping process governed by this kind of set C comes from mechanical systems (see [22]). Therefore, by applying Theorem 1, we can characterize weak Lyapunov pairs for the perturbed sweeping process (1) governed by a fixed set C.

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