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Self-contracted curves in Riemannian manifolds

Aris Daniilidis [a,*,1], Robert Deville [b,2], Estibalitz Durand-Cartagena [c,3], Ludovic Rifford [d,e,4]

[a] *DIM–CMM, UMI CNRS 2807, Beauchef 581, Torre Norte, piso 5, Universidad de Chile, Santiago 8370456, Chile*
[b] *Laboratoire Bordelais d'Analyse et Geométrie, Institut de Mathématiques de Bordeaux, Université de Bordeaux 1, 351 cours de la Libération, Talence Cedex 33405, France*
[c] *Departamento de Matemática Aplicada, ETSI Industriales, UNED, Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain*
[d] *CMM, UMI CNRS 2807, Blanco Encalada 2120, Universidad de Chile, Santiago, Chile* [5]
[e] *Laboratoire J.A. Dieudonné, UMR CNRS 7351, Université Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France*

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A B S T R A C T

It is established that every self-contracted curve in a Riemannian manifold has finite length, provided its image is contained in a compact set.

Contents

* Corresponding author.
E-mail addresses: arisd@dim.uchile.cl (A. Daniilidis), Robert.Deville@math.u-bordeaux1.fr (R. Deville), edurand@ind.uned.es (E. Durand-Cartagena), ludovic.rifford@math.cnrs.fr (L. Rifford).
URLs: http://www.dim.uchile.cl/~arisd (A. Daniilidis), http://www.uned.es/personal/edurand (E. Durand-Cartagena), http://math.unice.fr/~rifford/ (L. Rifford).

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[5] Visiting Researcher.

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1. Introduction

This work is devoted to the study of self-contracted curves on Riemannian manifolds $\mathcal{M}$.

Definition 1.1 *(Self-contracted curve)*. Let $\mathcal{M}$ be a Riemannian manifold and let d_g denote its geodesic distance. Given an interval $I = [0, T_\infty)$ with $T_\infty \in (0, \infty) \cup \{\infty\}$, a curve $\gamma : I \to \mathcal{M}$ is called *self-contracted*, if for every $t_1 \leq t_2 \leq t_3$ in I we have

$$d_g(\gamma(t_1), \gamma(t_3)) \geq d_g(\gamma(t_2), \gamma(t_3)). \tag{1.1}$$

In other words, for every $\tau \in [0, T_\infty)$ the function $t \mapsto d_g(\gamma(t), \gamma(\tau))$ is nonincreasing on $[0, \tau]$.

Self-contracted curves were introduced in [3, Definition 1.2.]. The motivation of this definition comes from the following example.

Example 1.2. If $f : \mathbb{R}^n \to \mathbb{R}^+$ is a $\mathcal{C}^1$-smooth convex function and if $\gamma : (0, +\infty) \to \mathbb{R}^n$ is smooth and satisfies $\gamma'(t) = -\nabla f\big(\gamma(t)\big)$ for all $t > 0$, then γ is a self-contracted curve.

Indeed, observe first that $\big(f(\gamma(t))\big)' = -\|\nabla f\big(\gamma(t)\big)\|^2 \leq 0$, thus the function $t \mapsto f\big(\gamma(t)\big)$ is nonincreasing. Therefore, since f is convex, if $\tau \geq t$, then

$$\frac{d}{dt}\Big(\frac{1}{2}\|\gamma(\tau) - \gamma(t)\|^2\Big) = \langle \gamma(\tau) - \gamma(t), \nabla f(\gamma(t))\rangle \leq f(\gamma(\tau)) - f(\gamma(t)) \leq 0.$$

This proves that the function $t \mapsto \|\gamma(t) - \gamma(\tau)\|$ is nonincreasing on $[0, \tau]$.

One of the main interests in studying self-contracted curves lies in its applications. Rectifiability of self-contracted curves has been applied in different areas, including continuous and discrete dynamical systems, optimization and convergence of algorithms. See for example [3] and [4].

The definition of self-contractedness is purely metric: if φ is a nondecreasing function from an interval J onto I, then $\gamma \circ \varphi$ is also self-contracted, so this notion does not depend on the particular parametrization of the oriented graph $\{\gamma(t);\, t \in I\}$. Self-contractedness does not require prior smoothness or continuity assumption on the curve as shown by the following example.

Example 1.3. Let $\gamma : \mathbb{R} \to \mathbb{C}$ defined by $\gamma(t) = t$ if $t \leq -1$, $\gamma(t) = -t$ if $-1 < t \leq 0$ and $\gamma(t) = it$ if $t > 0$. The curve γ is self-contracted, is not smooth at $t = 0$, is discontinuous at $t = -1$, and moreover does not admit a continuous self-contracted extension, *i.e.* there exists no continuous self-contracted curve $\Gamma : \mathbb{R} \to \mathbb{C}$ such that $\{\Gamma(t) : t \in \mathbb{R}\} \supset \{\gamma(t) : t \in \mathbb{R}\}$.

In a Euclidean setting it has been established in [4, Section 3] (and independently in [8] for continuous curves) that bounded self-contracted curves have finite length. In both cases the proof was based on an old result of Manselli–Pucci [10] which allows to deduce that all self-contracted curves lying in a given ball have lengths which are uniformly bounded. Applications of this fact have been discussed in [4, Section 4], [2], [9].

The results of [9,10,3,4,8] are all heavily based on the Euclidean structure. In [7] the author establishes rectifiability for planar curves in the non-Euclidean case, while in [5] the authors consider (under a different terminology) absolutely continuous self-contracted curves in a bounded convex subset of a two-dimensional complete surface of constant Gaussian curvature, and provide an upper bound for the length, but in case of a surface of positive curvature (sphere), they made the additional assumption that the diameter of this subset was strictly less than $\pi/2$.

In this work we establish that any self-contracted curve in a compact set of a smooth Riemannian manifold has finite length. This result generalizes the results mentioned above. In particular, comparing to [5] it does not require any assumption on the curvature or on the dimension of the manifold. Moreover, our result holds in the case of discontinuous self-contracted curves.

2. Main result

2.1. Statement of the main result

Let $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold whose geodesic distance is denoted by d_g. Given an interval $I = [0, T_\infty)$ with $T_\infty \in [0, \infty) \cup \{\infty\}$, the *length* of a curve $\gamma : I \to \mathcal{M}$ is defined as

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{m-1} d_g(\gamma(t_i), \gamma(t_{i+1})) \right\}, \tag{2.1}$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \cdots < t_m$ that lie in the interval I. We say that a (possibly discontinuous) curve $\gamma : I \to \mathcal{M}$ has *finite length* if $\ell(\gamma)$ is finite. Any continuous curve $\gamma : I \to \mathcal{M}$ with finite length can be reparameterized into a Lipschitz curve on $[0, \ell(\gamma)]$ with speed of constant norm *a.e.* equal to 1. The following extends previous results by [5,4,8].

Theorem 2.1 *(Main result). Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, $\mathcal{K}$ be a compact subset of $\mathcal{M}$ and $\gamma : I \to \mathcal{K}$ be a self-contracted curve. Then γ has finite length.*

This result cannot further extend to an infinite dimensional setting.

Example 2.2. Let $\gamma : [1, +\infty) \to L^2(\mathbb{R})$ given by $\gamma(t)(s) = \frac{1}{\sqrt{s}}$ if $s \in (t, t+1)$ and 0 otherwise. It is easy to see that γ is a self-contracted curve, its closure is equal to $\{\gamma(t);\, t \in [1, +\infty)\} \cup \{0\}$ (therefore it is compact in $L^2(\mathbb{R})$) and its length is infinite (indeed, since $\|\gamma(n+1) - \gamma(n)\|_2 = \sqrt{\ln(1 + 2/n)} \geq 1/n$, the series $(\sum \|\gamma(n+1) - \gamma(n)\|_2)$ diverges).

The rest of the paper is devoted to the proof of Theorem 2.1.

2.2. Notation and sketch of the proof

The symbol $\mathcal{M}$ will always stand for a smooth manifold of dimension $n \geq 2$ whose tangent bundle is denoted by $T\mathcal{M}$. Elements of $T\mathcal{M}$ are denoted by $\xi = q_y = (y, q)$ with $q \in T_y\mathcal{M}$. Given a smooth Riemannian metric g, we denote the metric at $x \in \mathcal{M}$ by $\langle \cdot, \cdot \rangle_x$ and its norm by $|\cdot|_x$. We sometimes omit x if no ambiguity arises. The geodesic distance is denoted by d_g and the open geodesic ball centered at x of radius $r \geq 0$ is denoted by $B_g(x, r)$. For every $x \in \mathcal{M}$, we denote by $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ the exponential mapping at x. We denote by $\widehat{B}_x$ the balls in $T_x\mathcal{M}$ (with respect to the Euclidean metric in $T_x\mathcal{M}$). We denote the unit tangent bundle associated with g by $U\mathcal{M}$, that is,

$$U\mathcal{M} := \{u_x \in T\mathcal{M} : |u|_x = 1\}.$$

If $\mathcal{K}$ is a compact subset of $\mathcal{M}$, then

$$U\mathcal{K} := \{u_x \in T\mathcal{M} : x \in \mathcal{K} \text{ and } |u|_x = 1\}$$

is a compact subset of $U\mathcal{M}$. We consider a canonical Riemannian metric on the unit bundle, whose associated distance is denoted by D_g. We may assume that for every p_x, q_y in $U\mathcal{M}$ it holds

$$D_g(p_x, q_y) \geq d_g(x, y). \tag{2.2}$$

We refer to [1,6] for prerequisites on Riemannian manifolds.

We now present the strategy of proof of the main theorem. Every self-contracted curve has left limit and right limit at each point. We show that if such a curve is contained in a compact subset of $\mathcal{M}$, then the set of points of large discontinuities (*i.e.* the set of points where the oscillation of γ is greater than some fixed threshold $\eta > 0$) is finite and its cardinal depends only on $\mathcal{K}$. Then we do a detailed study of the local behaviour of the curve around points of continuity and/or points of small discontinuity. The main tool here is a uniform cosine law for small triangles having one vertex in the compact $\mathcal{K}$. This study allows us to construct an element $p^a \in U\mathcal{M}$, called almost secant, such that the curve γ grows in this direction around the point $x = \gamma(\tau)$. Finally, we consider an η-net $\mathcal{F}$ of $U\mathcal{K}$. For $\xi := q_y \in \mathcal{F}$ and for $z \in \mathcal{M}$ such that $d_g(z, y)$ is sufficiently small, we define the local width of γ at $x = \gamma(\tau)$ with respect to q_y by $W_\xi(\tau) := \operatorname{diam}\left\{\langle q, \exp_y^{-1}(\gamma(t))\rangle_y : t \geq \tau,\, \gamma(t) \in B_g(y, 2\eta)\right\}$. Notice that $\tau \mapsto W_\xi(\tau)$ is a nonincreasing (Lyapunov) function. We deduce from the study of the local behaviour of γ, that if τ is not a point of large discontinuity and if s, t are in a neighborhood of τ and $s \leq \tau \leq t$, then at least one of the Lyapunov functions W_ξ ($\xi \in \mathcal{F}$) satisfies $W_\xi(s) - W_\xi(t) \geq \alpha\, d_g(\gamma(s), \gamma(t))$, where

$$\alpha = \frac{1}{32(n+1)^2} \qquad (n = \dim \mathcal{M}). \tag{2.3}$$

Since the curve γ is contained in the compact set $\mathcal{K}$, all functions W_ξ are bounded. This together with the above inequality, implies the rectifiability of γ.

2.3. Exponential map – cosine law – external functions

We introduce here a few tools from Riemannian manifolds. We first notice that for every $x \in \mathcal{M}$, there exists $r > 0$, such that the exponential function $\exp_x$ is a smooth diffeomorphism between the open ball $\widehat{B}_x(0, r)$ of $T_x\mathcal{M}$ onto the open geodesic ball $B_g(x, r)$ in $\mathcal{M}$. The following lemma is an easy consequence of the compactness of $\mathcal{K}$ and the smoothness of the geodesic flow.

Lemma 2.3. *There exists $\rho > 0$ such that for every $x \in \mathcal{K}$, $\exp_x$ is a smooth diffeomorphism from the ball $\widehat{B}_x(0, 2\rho)$ to its image $B_g(x, 2\rho)$.*

Thus, we can define, for any $x \in \mathcal{K}$ and $z \in B_g(x, \rho)$,

$$u_x(z) := \frac{\exp_x^{-1}(z)}{\left|\exp_x^{-1}(z)\right|_x} \in U_x\mathcal{M} \quad (\text{provided } z \neq x). \tag{2.4}$$

By construction, $\exp_x^{-1}(z)$ is the initial velocity of the geodesic $\theta : [0, 1] \to \mathcal{M}$ joining x to z, so we have $\left|\exp_x^{-1}(z)\right|_x = d_g(x, z)$.

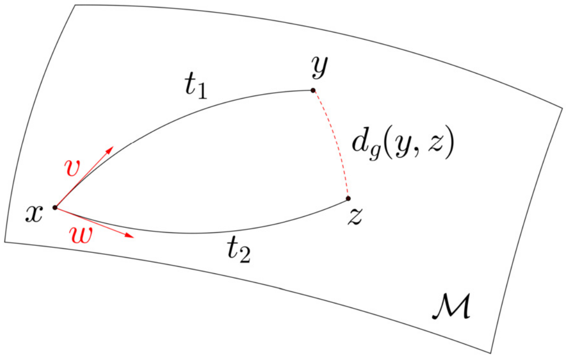

Fig. 1. Cosine law in Riemannian manifolds.

Recall that if x, y, z lie in a Euclidean space, the law of cosines asserts that

$$\|y-z\|^2 = \|y-x\|^2 + \|z-x\|^2 - 2\langle y-x, z-x\rangle \,.$$

The following result asserts that small geodesic triangles in a Riemannian manifold almost satisfy the law of cosines, see Fig. 1 for an illustration.

Lemma 2.4 *(Cosine law in manifolds). There exists $K > 0$ such that for every $x \in \mathcal{K}$ and every $y, z \in B(x,\rho)$,*

$$\left|d_g^2(y,z) - d_g^2(x,y) - d_g^2(x,z) + 2\langle \exp_x^{-1}(y), \exp_x^{-1}(z)\rangle_x\right| \le K d_g(x,y)^2 d_g(x,z)^2 \,. \tag{2.5}$$

Proof. By Lemma 2.3, there exist $t_1, t_2 \in (-\rho,\rho)$, $v, w \in U_x\mathcal{M}$ such that $y = \exp_x(t_1 v)$ and $z = \exp_x(t_2 w)$, precisely $t_1 = d_g(x,y) = |\exp_x^{-1}(y)|_x$ and $t_2 = d_g(x,z) = |\exp_x^{-1}(z)|_x$. For fixed $x \in \mathcal{K}$ and $v, w \in U_x\mathcal{M}$, we consider the function $\Phi : (-\rho,\rho)^2 \to \mathbb{R}$ defined by

$$\Phi(t_1,t_2) = d_g(\exp_x(t_1 v), \exp_x(t_2 w))^2 \,.$$

We check easily that for every $t_1, t_2 \in (-\rho,\rho)$,

$$\Phi(t_1,0) = t_1^2 \,, \quad \Phi(0,t_2) = t_2^2 \,,$$

and

$$\frac{\partial\Phi}{\partial t_1}(0,t_2) = -2\, t_2\, \langle v,w\rangle_x \,, \quad \frac{\partial\Phi}{\partial t_2}(t_1,0) = -2\, t_1\, \langle v,w\rangle_x \,.$$

Then we infer that

$$\frac{\partial^2\Phi}{\partial t_1 \partial t_2}(0,0) = -2\, \langle v,w\rangle_x$$

and for every integer $k \in \{2,3\}$,

$$\frac{\partial^{k+1}\Phi}{\partial t_1^k \partial t_2}(0,0) = \frac{\partial^{k+1}\Phi}{\partial t_1 \partial t_2^k}(0,0) = 0.$$

The Taylor expansion formula of order 4 for the function Φ shows that there exists $K > 0$ (depending on the fourth derivative of the exponential mapping at x) such that

$$\left|\Phi(t_1,t_2) - t_1^2 - t_2^2 + 2\,t_1\,t_2\,\langle v,w\rangle_x\right| \le K\,t_1^2\,t_2^2\,, \qquad \forall\, t_1,t_2 \in (-\rho,\rho).$$

By the definition of the exponential map, we have $\Phi(t_1,t_2) = d_g(y,z)^2$, $t_1 = d_g(x,y)$, $t_2 = d_g(x,z)$, $t_1 v = \exp_x^{-1}(y)$ and $t_2 w = \exp_x^{-1}(z)$ so the above formula implies equation (2.5). The compactness of $U\mathcal{K}$ shows that the constant K can be chosen independently of x, v, w. $\square$

Remark 2.5 *(Adapting the constant ρ).* Let $K > 0$ be given by Lemma 2.4 and $\alpha > 0$ be given by (2.3). We may always shrink $\rho > 0$ of Lemma 2.3 to ensure

$$4K\,\rho^2 \le \alpha\,. \tag{2.6}$$

In the following result, we introduce a parameter η that will be fixed throughout the paper, and we give a control of the difference of the growths of the mappings $z \mapsto \langle p, \exp_x^{-1}(z)\rangle_x$ and $z \mapsto \langle q, \exp_y^{-1}(z)\rangle_y$ around x whenever $D_g(p_x, q_y) < \eta$.

Lemma 2.6. *Let $\alpha > 0$ be given by (2.3). Then there exists $\eta \in (0, \rho/4)$ such that for every $x \in \mathcal{K}$, $y \in B_g(x,\rho)$, $z \in B_g(y, 2\eta)$ and for every $p_x \in U\mathcal{K}$, $q_y \in B_{D_g}(p_x, \eta)$ (Riemannian ball in the unit bundle $U\mathcal{M}$) we have*

$$\left|\langle q, \exp_y^{-1}(z)\rangle_y - \langle q, \exp_y^{-1}(x)\rangle_y - \langle p, \exp_x^{-1}(z)\rangle_x\right| \le \alpha d_g(x,z)\,.$$

Proof. Let us denote $b_{p_x}(z) := \langle p, \exp_x^{-1}(z)\rangle_x$ and $b_{q_y}(z) := \langle q, \exp_y^{-1}(z)\rangle_y$. We first claim that for every p_x in $U\mathcal{K}$,

$$\nabla b_{p_x}(x) = p \in T_x\mathcal{M}\,.$$

Since the differential $D\exp_x^{-1}(x)$ is the identity mapping on $T_x\mathcal{M}$ it follows by the chain rule that $Db_{p_x}(x) = \langle p, \cdot\rangle_x$. This proves the claim. Since the mapping

$$(q_y, x) \mapsto Db_{q_y}(x) = \langle q, D\exp_y^{-1}(x)(\cdot)\rangle_y$$

is continuous, we deduce easily from the compactness of $\mathcal{K}$ and $U\mathcal{K}$ and the claim that there exists $\eta > 0$ such that for all p_x, q_y in $U\mathcal{K}$ satisfying $D_g(p_x, q_y) < \eta$ we have

$$\left|\nabla b_{q_y}(x) - \nabla b_{p_x}(x)\right|_x = \left|\nabla b_{q_y}(x) - p\right|_x < \frac{\alpha}{2}. \tag{2.7}$$

We claim now that there exists $L > 0$ such that for every $q_y \in U\mathcal{K}$ and $x, z \in B_g(y,\rho)$ it holds

$$\left|b_{q_y}(z) - b_{q_y}(x) - \langle \nabla b_{q_y}(x), \exp_x^{-1}(z)\rangle_x\right| \le L|\exp_x^{-1}(z)|_x^2\,.$$

Indeed, the mapping

$$(q_y, x) \mapsto b_{q_y}(x) := \langle q, \exp_y^{-1}(x)\rangle_y$$

is smooth (whenever it is well-defined, that is, $d_g(x,y) \le 2\rho$). The exact Taylor expansion of order 2 for the function $z \mapsto b_{q_y}(z)$ at the point x, together with the compactness of $\mathcal{K}$ and $U\mathcal{K}$ and a standard argument gives the above inequality. We now shrink $\eta > 0$ if necessary to ensure that $\eta \le \alpha/6L$. Pick any $z \in B_g(y, 2\eta) \subset B_g(y,\rho)$. It follows from (2.2) that if $D_g\big(p_x, q_y\big) < \eta$, then $d_g(x,y) < \eta$, and so $d_g(x,z) < 3\eta$. Since $|\exp_x^{-1}(z)|_x = d_g(x,z) < 3\eta \le \alpha/2L$ the above inequality becomes

$$\left|b_{q_y}(z) - b_{q_y}(x) - \langle \nabla b_{q_y}(x), \exp_x^{-1}(z)\rangle_x\right| \leq \frac{\alpha}{2} d_g(x,z)\,. \tag{2.8}$$

Equation (2.7) implies $\left|\langle \nabla b_{q_y}(x) - p, \exp_x^{-1}(z)\rangle_x\right| \leq (\alpha/2)\, d_g(x,z)$. This inequality combined with (2.8) yields the inequality of Lemma 2.6. $\square$

3. Geometrical description of self-contracted maps

3.1. Dealing with discontinuities

Let $\gamma : I \to \mathcal{M}$ be a self-contracted curve such that $\gamma(I) \subset \mathcal{K}$ with $\mathcal{K}$ compact. The results of this sub-section are valid assuming only that $\mathcal{M}$ is a metric space. For every $\tau \in I$, we denote by $\gamma(\tau^-)$ the left limit of γ at τ, that is,

$$\gamma(\tau^-) := \lim_{s<\tau, s\to\tau} \gamma(s).$$

Proposition 3.1. *The above limit always exists.*

Proof. Otherwise, by compactness, there would exist at least two accumulation points x_1 and x_2 with $l = d_g(x_1,x_2) > 0$. Let $t_1 < t_2 < t_3 < \tau$ be such that $d_g(\gamma(t_1),x_1) < l/4$, $d_g(\gamma(t_2),x_2) < l/4$ and $d_g(\gamma(t_3),x_1) < l/4$. Since γ is self-contracted, we have

$$l/2 > d_g(\gamma(t_1),\gamma(t_3)) \geq d_g(\gamma(t_2),\gamma(t_3)) > l/2$$

which is a contradiction. $\square$

We denote by $\mathcal{D}^- := \{\tau \in I : \gamma(\tau) \neq \gamma(\tau^-)\}$ the set of points where γ is not left-continuous. We fix $\eta > 0$ satisfying Lemma 2.6, and we set:

$$\text{(left-}\eta\text{-threshold)} \qquad \mathcal{D}^-(\eta) := \left\{\tau \in I : \quad d_g(\gamma(\tau),\gamma(\tau^-)) \geq \eta\right\}. \tag{3.1}$$

In the following lemma, the cardinality of a set S is denoted by $|S|$.

Proposition 3.2 *(Cardinality of $\mathcal{D}^-(\eta)$). Let $\gamma : I \to \mathcal{M}$ be a self-contracted map such that $\gamma(I) \subset \mathcal{K}$. If $N(\eta)$ is the minimal number of balls of radius $\eta/2$ that can cover $\mathcal{K}$, we have*

$$\left|\mathcal{D}^-(\eta)\right| = \left|\gamma(\mathcal{D}^-(\eta))\right| \leq N(\eta). \tag{3.2}$$

In particular, since $\mathcal{D}^- = \bigcup_{n\in\mathbb{N}} \mathcal{D}^-(1/n)$, $\mathcal{D}^-$ is at most countable.

Proof. We first claim that for any $x \in \mathcal{M}$ we have:

$$\left|\gamma(\mathcal{D}^-(\eta)) \cap B_g\left(x,\eta/2\right)\right| \leq 1.$$

Let $\tau_1,\tau_2 \in \mathcal{D}^-(\eta)$ with $\tau_1 < \tau_2$, be such that $\{\gamma(\tau_1),\gamma(\tau_2)\} \subset B_g(x,\frac{\eta}{2})$. Set $x_i = \gamma(\tau_i)$ and $x_i' = \gamma(\tau_i^-)$, $i \in \{1,2\}$. It follows that $\{x_1,x_2\} \subset B_g(x,\frac{\eta}{2})$ and $\{x_1',x_2'\} \subset \mathcal{M}\setminus B_g(x,\frac{\eta}{2})$. The fact that γ is self-contracted yields the following inequalities:

$$\eta \leq d_g(x_2',x_2) \leq d_g(x_1,x_2) \leq d_g(x_1,x) + d_g(x,x_2) < \eta\,.$$

This contradiction proves the claim.

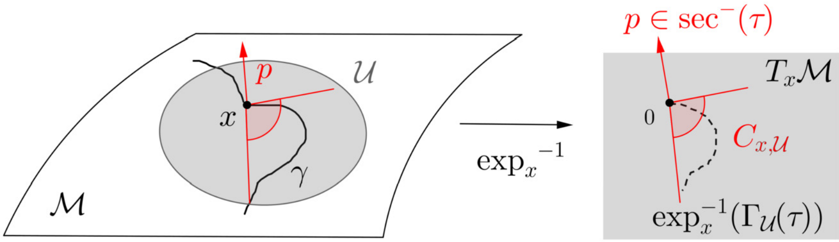

Fig. 2. $\sec^-(\tau) \subseteq N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x)$.

As a consequence of self-contractedness, the sets $\mathcal{D}^-(\eta)$ (subset of I) and $\gamma(\mathcal{D}^-(\eta))$ (subset of $\mathcal{K}$) have the same cardinality, for every $\eta > 0$. The claim yields that this cardinality is bounded by $N(\eta)$. Compactness of $\mathcal{K}$ guarantees that this latter is finite. $\square$

Remark 3.3 *(Cardinality of $\mathcal{D}^+(\eta)$).* Analogous results hold for right discontinuities. Let $\eta > 0$, and let us consider the set of points of I where γ has a large right discontinuity:

$$\text{(right-}\eta\text{-threshold)} \qquad \mathcal{D}^+(\eta) := \left\{\tau \in I : \quad d_g(\gamma(\tau), \gamma(\tau^+)) \geq \eta\right\}. \tag{3.3}$$

Then the cardinality of the set $\mathcal{D}^+(\eta)$ is bounded by $N(\eta)$.

3.2. Describing backward secants

Let us fix $\tau \in (0, T_\infty)$ and let us define the set of all possible limits of *backward secants* at $x = \gamma(\tau)$ as follows (recall notation (2.4)):

$$\sec^-(\tau) := \left\{p \in U_x\mathcal{M} : \quad p = \lim_{s_k \to \tau, s_k < \tau} u_x\big(\gamma(s_k)\big)\right\}.$$

Notice that $\sec^-(\tau) \neq \emptyset$ for every $\tau > 0$ (*cf.* compactness of the unit sphere).

For every $\tau \in I$, we define the set $\Gamma(\tau)$ (tail of γ at $x = \gamma(\tau)$) by

$$\Gamma(\tau) := \left\{\gamma(t) \, : \, t \geq \tau\right\},$$

and, given an open neighborhood $\mathcal{U}$ of $x = \gamma(\tau)$, we define the *$\mathcal{U}$-truncated tail* of γ at x by

$$\Gamma_{\mathcal{U}}(\tau) := \Gamma(\tau) \cap \mathcal{U}. \tag{3.4}$$

The cone in $T_x\mathcal{M}$ generated by $\{u_x(z) : \; z \in \Gamma_{\mathcal{U}}(\tau)\}$ will be denoted $C_{x,\mathcal{U}}$. The next result asserts that every backward secant at a point $x = \gamma(\tau)$ where the curve is left-continuous, is normal to $C_{x,\mathcal{U}}$.

Lemma 3.4 *(Backward secants). Let $\mathcal{U}$ be an open neighborhood of $x = \gamma(\tau)$ with* $\operatorname{diam}\mathcal{U} \leq \rho$. (I) *If γ is left-continuous at τ, then (Fig. 2)*

$$\sec^-(\tau) \subset N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x) \tag{3.5}$$

that is,

$$\langle p, u_x(z)\rangle_x \leq 0, \quad \textit{for all } p \in \sec^-(\tau) \textit{ and } z \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}.$$

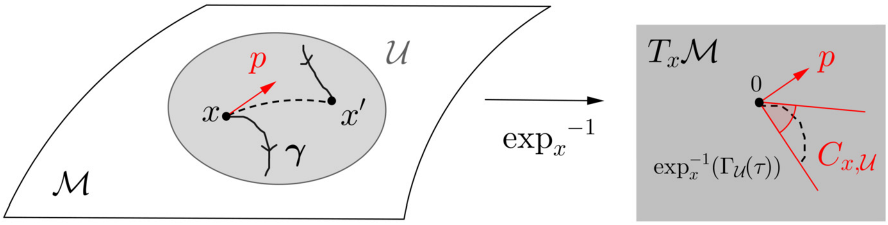

Fig. 3. $\sec^-(\tau) := \{u_x(\gamma(\tau^-))\} \nsubseteq N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x)$.

(II) *If* $x \neq \gamma(\tau^-)$ *and* $\gamma(\tau^-) \in B_g(x, 2\rho)$ *then*

$$\sec^-(\tau) = \{u_x(\gamma(\tau^-))\}.$$

Proof. (I) Let $p \in \sec^-(\tau)$. Then for some $s_k \nearrow \tau$ we have

$$p := \lim_{k\to\infty} \frac{\exp_x^{-1}(\gamma(s_k))}{|\exp_x^{-1}(\gamma(s_k))|_x} \quad (\text{in } T_x\mathcal{M}).$$

Clearly $\mathcal{U} \subset B_g(x, 2\rho)$. We may also assume that $\Gamma_{\mathcal{U}}(\tau) \setminus \{x\} \neq \emptyset$ (else the conclusion follows trivially) and $\{\gamma(s_k)\}_k \subset \mathcal{U}$. Pick any $z \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}$. Applying the cosine law (2.5) we have

$$\left|d_g(\gamma(s_k), z)^2 - d_g(x, z)^2 - d_g(x, \gamma(s_k))^2 + 2\langle \exp_x^{-1}(\gamma(s_k)), \exp_x^{-1}(z)\rangle_x\right| \leq K d(x, \gamma(s_k))^2 d(x, z)^2.$$

On the other hand, since γ is self-contracted, we have

$$d_g\left(\gamma(s_k), z\right) \geq d_g\big(x, z\big),$$

thus

$$-d_g(x, \gamma(s_k))^2 + 2\, d_g(x, z)\, \langle \exp_x^{-1}(\gamma(s_k)), u_x(z)\rangle_x \leq K\, d_g(x, \gamma(s_k))^2\, d_g(x, z)^2.$$

Dividing by $|\exp_x^{-1}(\gamma(s_k))|_x = d_g(x, \gamma(s_k))$ and passing to the limit as $k \to \infty$ we conclude easily.

(II) It is straightforward since $x \neq \gamma(\tau^-)$ and $\gamma(\tau^-)$ is the limit of $\gamma(s)$ as $s \nearrow \tau$. $\square$

Remark 3.5. Notice that for $\tau \in \mathcal{D}^-$, the backward secant is unique (*cf.* Lemma 3.4 (II)), but (3.5) may fail. An illustration is given in Fig. 3.

3.3. Aperture of the truncated tail

Given any subset C of the unit sphere of $\mathbb{R}^n$, its *aperture* $A(C)$ is defined as follows:

$$A(C) := \inf \{ \langle u_1, u_2\rangle \ : \ u_1, u_2 \in C \} \ . \tag{3.6}$$

For every $y \in \mathcal{M}$ and $\Gamma \subset B_g(x, 2\rho)$, we define (the *aperture* of $\Gamma \subset \mathcal{M}$ at $y \in \mathcal{M}$):

$$A_y(\Gamma) := \inf\Big\{ \langle u_y(z_1), u_y(z_2)\rangle_y \ : \ z_1, z_2 \in \Gamma \setminus \{y\} \Big\}. \tag{3.7}$$

Roughly speaking, the aperture of a subset Γ of a manifold $\mathcal{M}$ (with respect to a point $y \in \mathcal{M}$) intends to measure the size of the cone generated by the unit tangents $u \in T_y\mathcal{M}$ at y corresponding to all points $z \in \Gamma \setminus \{y\}$ via the mapping $\exp_y^{-1}$.

The aperture will play a major role in the sequel. The set Γ will be taken to be the (truncated) tail $\Gamma_{\mathcal{U}}(\tau)$ of the self-contracted curve γ, see (3.4), and the point $y \in \mathcal{M}$ at which the aperture is taken will be either:

(i) the point $x = \gamma(\tau)$ if the curve γ is continuous at τ; or

(ii) a point $\bar{x}$ lying in the minimal geodesic joining $x = \gamma(\tau)$ to $x' = \gamma(\tau^-)$, if γ is left discontinuous at τ.

3.3.1. Left-continuous case

Proposition 3.6 *(Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at x). Let $\mathcal{U}$ be any nonempty open subset of $\mathcal{M}$ with* $\operatorname{diam}\mathcal{U} \le \rho$*. Then for every $\tau \in (0, T_\infty)$ with $x = \gamma(\tau) \in \mathcal{U}$ the following property holds:*

$$A_x(\Gamma_{\mathcal{U}}(\tau)) \ge -\alpha. \tag{3.8}$$

Proof. Set $x := \gamma(\tau)$ and for $i \in \{1, 2\}$ let $z_i = \gamma(t_i) \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}$ with $\tau < t_1 \le t_2$. Applying the law of cosines (2.5) we deduce

$$d_g(z_1, z_2)^2 - d_g(x, z_1)^2 - d_g(x, z_2)^2 + 2\langle \exp_x^{-1}(z_1), \exp_x^{-1}(z_2)\rangle_x \ge -K d(x, z_1)^2 d(x, z_2)^2.$$

Self-contractedness of γ yields that $d_g(x, z_2) \ge d_g(z_1, z_2)$, thus

$$2\langle \exp_x^{-1}(z_1), \exp_x^{-1}(z_2)\rangle_x \ge -K d(x, z_1)^2 d(x, z_2)^2.$$

Dividing by $|\exp_x^{-1}(z_1)|_x |\exp_x^{-1}(z_2)|_x = d_g(x, z_1) d_g(x, z_2)$, and then using (2.6) we obtain

$$\langle u_x(z_1), u_x(z_2)\rangle_x \ge -\frac{K\rho^2}{2} \ge -\alpha/8 \ge -\alpha. \qquad \square$$

Remark 3.7. The above result, in combination with forthcoming Lemma 3.14, will assert that the cone generated by the $\mathcal{U}$-truncated tail $\Gamma_{\mathcal{U}}(\tau)$ at $T_x\mathcal{M}$ has angle almost equal (a bit more than) $\pi/2$, for any open neighborhood $\mathcal{U}$ of x of sufficiently small diameter. This is the Riemannian analogue of [10, Section 3, Formula (2)] (see also [4, Fig. 1]).

3.3.2. Left-discontinuous case

Let $\tau \in \mathcal{D}^-$ (that is, γ is left-discontinuous at $x = \gamma(\tau)$). In this case, for reasons that will become transparent in Section 3.4 (see also Remark 3.5), we need to consider the aperture of the truncated tail $\Gamma_{\mathcal{U}}(\tau)$ with respect to a different point $\bar{x}$ (other than $x = \gamma(\tau)$). This point will be taken on the minimal geodesic joining x to x' and relatively close to $x' := \gamma(\tau^-)$. To define this geodesic, notice that $p := u_x(x')$ is the unique left secant of γ at τ (*cf.* Lemma 3.4 (II)), that is, the initial velocity of the unit speed geodesic $\theta : [0, d_g(x, x')] \to \mathcal{M}$ joining x to x'. We fix

$$\beta = \alpha/8 \tag{3.9}$$

and we denote

$$\bar{x} = \theta\left((1 - \beta)\, d_g(x, x')\right) \quad \text{and} \quad \bar{p} = \dot{\theta}\left((1 - \beta)\, d_g(x, x')\right) = u_{\bar{x}}(x'). \tag{3.10}$$

Notice that the value of β which determines the exact location of the point $\bar{x}$ is the same for all $\tau \in \mathcal{D}^- \backslash \mathcal{D}^-(\eta)$.

Proposition 3.8 *(Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at $\bar{x}$). Let $\tau \in \mathcal{D}^-$ and set $x = \gamma(\tau)$, $x' = \gamma(\tau^-)$ and $\bar{x}$ defined by (3.10). Then for every open subset $\mathcal{U}$ of $\mathcal{M}$ with* $\operatorname{diam}\mathcal{U} \le \rho$ *and $\{x, \bar{x}, x'\} \subset \mathcal{U}$ we have*

$$A_{\bar{x}}(\Gamma_{\mathcal{U}}(\tau)) \ge -\alpha.$$

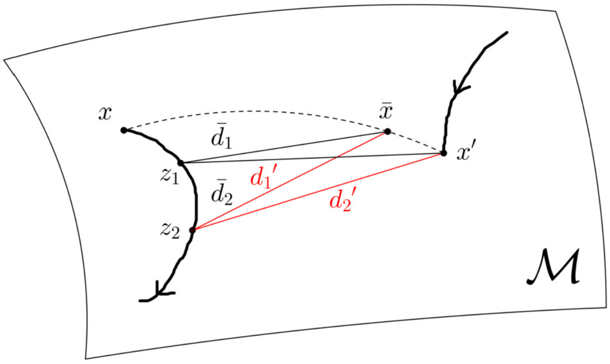

Fig. 4. Calculating the aperture $\Gamma_{\mathcal{U}}(\tau)$ at $\bar{x}$.

The proof of the above proposition will not be an easy task though. Indeed, since $\bar{x}$ is not a point of γ, the previous argument (*cf.* proof of Proposition 3.6), based on self-contractedness, is no longer valid. Our new task will require several technical estimations (see forthcoming Lemma 3.10 and Lemma 3.11), as well as estimating the aperture of $\Gamma_{\mathcal{U}}(\tau)$ at the point x' (Fig. 4) (which might not be a point of the curve, but belongs to its closure).

Lemma 3.9 *(Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at x'). Let $\mathcal{U}$ be an open subset of $\mathcal{M}$ with $\operatorname{diam}\mathcal{U} \le \rho$ and let $\tau \in \mathcal{D}^-$ be such that both $x = \gamma(\tau)$ and $x' := \gamma(\tau^-)$ are in $\mathcal{U}$. Then*

$$A_{x'}(\Gamma_{\mathcal{U}}(\tau)) \ge -\alpha/8. \tag{3.11}$$

Proof. By Lemma 3.6 (and more precisely, using the estimate of the last line of its proof), the estimation $A_{\gamma(s)}(\Gamma_{\mathcal{U}}(\tau)) \ge -\alpha/8$ holds true for all $s \in (0,\tau)$ point of continuity of γ sufficiently close to τ so that $\mathcal{U} \subset B_g(\gamma(s),\rho)$. Since $x' := \lim_{s\nearrow\tau}\gamma(s)$ is a limit of points of continuity of γ, we conclude easily by a standard continuity argument. □

We now fix notations that will be used in Lemma 3.13, Lemma 3.11 and Proposition 3.12. Let $\tau \in \mathcal{D}^-$ and set $x = \gamma(\tau)$, $x' = \gamma(\tau^-)$ and $\bar{x} = \theta\left((1-\beta)\, d_g(x,x')\right)$ satisfying (3.9) and (3.10). We also fix an open $\mathcal{U}$ of $\mathcal{M}$ with $\operatorname{diam}\mathcal{U} < \rho$ and $\{x,\bar{x},x'\} \subset \mathcal{U}$. If $z \in \Gamma_{\mathcal{U}}(\tau)$, we denote:

$$\sigma := d_g(x,x'), \quad \bar{d} = d_g(\bar{x},z) \quad \text{and} \quad d' = d_g(x',z).$$

Lemma 3.10 *(Technical estimations – I). For every $z \in \Gamma_{\mathcal{U}}(\tau)$ one has:*

$$\frac{\sigma}{\bar{d}} \le \frac{2}{1-2\beta} \tag{3.12}$$

and

$$\frac{d'}{\bar{d}} \le \frac{1}{1-2\beta}. \tag{3.13}$$

Proof. Since γ is self-contracted, we have $d_g(x,z) \le d_g(x',z)$. Therefore

$$d_g(x,x') \le d_g(x,z) + d_g(x',z) \le 2d_g(x',z).$$

It follows by (3.10) that $d_g(\bar{x},x') = \beta d_g(x,x') = \beta\sigma$. Thus, we deduce

$$\frac{\sigma}{2} = \frac{1}{2} d_g(x, x') \le d_g(x', z) \le d_g(\bar{x}, z) + d_g(\bar{x}, x') = \bar{d} + \beta\sigma,$$

which yields (3.12). We now deduce from (3.12) that

$$d' = d_g(x', z) \le d_g(x', \bar{x}) + d_g(\bar{x}, z) \le \beta\sigma + \bar{d} \le \left(\frac{1}{1-2\beta}\right) \bar{d}.$$

This proves (3.13). □

Lemma 3.11 *(Technical estimations – II). For every $z \in \Gamma_{\mathcal{U}}(\tau)$ we have*

$$\bar{d}^2 - d'^{\,2} \ge -2(\beta\sigma)^2 - 2\beta\sigma\bar{d}, \tag{3.14}$$

and

$$\bar{d}^2 - d'^{\,2} \le 2(\beta\sigma)^2 + \beta\sigma d'\alpha/4. \tag{3.15}$$

Proof. Let $z \in \Gamma_{\mathcal{U}}(\tau)$. By the law of cosines (Lemma 2.4)

$$\left| d_g(x', z)^2 - d_g(\bar{x}, x')^2 - d_g(\bar{x}, z)^2 + 2\langle \exp_{\bar{x}}^{-1}(x'), \exp_{\bar{x}}^{-1}(z)\rangle_{\bar{x}} \right| \le K d_g(\bar{x}, x')^2 d_g(\bar{x}, z)^2.$$

Therefore, recalling that $d_g(\bar{x}, x') = \beta\sigma$,

$$\bar{d}^2 - d'^{\,2} \ge -(\beta\sigma)^2 \left[1 + K d_g(\bar{x}, z)^2\right] + 2\langle \exp_{\bar{x}}^{-1}(x'), \exp_{\bar{x}}^{-1}(z)\rangle_{\bar{x}}$$

Since $\bar{x}$, $z \in \mathcal{U}$, we have $d_g(\bar{x}, z) \le \rho$, so using (2.6), we have $K d_g(\bar{x}, z)^2 \le 1$. On the other hand, by the Cauchy–Schwarz inequality, we have also $\langle \exp_{\bar{x}}^{-1}(x'), \exp_{\bar{x}}^{-1}(z)\rangle_{\bar{x}} \ge -d_g(\bar{x}, x') d_g(\bar{x}, z)$. Thus (3.14) holds.

To establish (3.15), we use again the law of cosines:

$$\left| d_g(\bar{x}, z)^2 - d_g(\bar{x}, x')^2 - d_g(x', z)^2 + 2\langle \exp_{x'}^{-1}(\bar{x}), \exp_{x'}^{-1}(z)\rangle_{x'} \right| \le K d_g(\bar{x}, x')^2 d_g(x', z)^2. \tag{3.16}$$

Since $x \in \Gamma_{\mathcal{U}}(\tau)$ and $x \ne x'$ we deduce by Lemma 3.9 that

$$\langle u_{x'}(\bar{x}), u_{x'}(z)\rangle_{x'} = \langle u_{x'}(x), u_{x'}(z)\rangle_{x'} \ \ge -\alpha/8,$$

hence

$$\langle \exp_{x'}^{-1}(\bar{x}), \exp_{x'}^{-1}(z)\rangle_{x'} = d_g(\bar{x}, x') d_g(x', z) \langle u_{x'}(x), u_{x'}(z)\rangle_{x'} \ge \ -\beta\sigma d'\alpha/8\,.$$

Combining this inequality with (3.16) and recalling that $d_g(\bar{x}, x') = \beta\sigma$, we get

$$\bar{d}^2 - d'^{\,2} \le (1 + K d_g(x', z)^2)(\beta\sigma)^2 + \beta\sigma d'\alpha/4.$$

Since $d_g(x', z) \le \rho$ and $K\rho^2 \le 1$ (*cf.* (2.6)) we conclude easily. □

Proof of Proposition 3.8. Since $x' \notin \Gamma_{\mathcal{U}}(\tau)$ we deduce by Lemma 3.9 that for every $z_1, z_2 \in \Gamma_{\mathcal{U}}(\tau)$,

$$\langle u_{x'}(z_1), u_{x'}(z_2)\rangle_{x'} \ge \ -\alpha/8\,. \tag{3.17}$$

In order to simplify notation, let us set $\sigma := d_g(x, x')$ and

$$\begin{cases} d_i := d_g(x, z_i) \\ \bar{d}_i := d_g(\bar{x}, z_i) \\ d'_i := d_g(x', z_i) \end{cases} \quad \text{for } i \in \{1, 2\}.$$

Applying the law of cosines and setting

$$e := d_g(z_1, z_2)$$

we obtain

$$\left| e^2 - d_1'^2 - d_2'^2 + 2\, d_1'\, d_2' \, \langle u_{x'}(z_1), u_{x'}(z_2)\rangle_{x'} \right| \leq K d_1'^2\, d_2'^2, \tag{3.18}$$

and

$$\left| e^2 - \bar{d}_1^2 - \bar{d}_2^2 + 2\bar{d}_1 \bar{d}_2 \langle u_{\bar{x}}(z_1), u_{\bar{x}}(z_2)\rangle_{\bar{x}} \right| \leq K \bar{d}_1^2 \bar{d}_2^2. \tag{3.19}$$

Combining (3.17), (3.18) and (3.19) we deduce

$$2\bar{d}_1 \bar{d}_2 \langle u_{\bar{x}}(z_1), u_{\bar{x}}(z_2)\rangle_{\bar{x}} \geq -d_1' d_2' \alpha/4 - K\left(\bar{d}_1^2 \bar{d}_2^2 + d_1'^2 d_2'^2\right) + \bar{d}_1^2 - d_1'^2 + \bar{d}_2^2 - d_2'^2,$$

thus in particular

$$\langle u_{\bar{x}}(z_1), u_{\bar{x}}(z_2)\rangle_{\bar{x}} \geq -\left(\frac{d_1' d_2'}{\bar{d}_1 \bar{d}_2}\right)\frac{\alpha}{8} - \frac{K}{2}\bar{d}_1 \bar{d}_2 \left(1 + \left(\frac{d_1' d_2'}{\bar{d}_1 \bar{d}_2}\right)^2\right) + \frac{\bar{d}_1^2 - d_1'^2}{2\,\bar{d}_1\,\bar{d}_2} + \frac{\bar{d}_2^2 - d_2'^2}{2\,\bar{d}_1\,\bar{d}_2}. \tag{3.20}$$

To proceed, we need to bound the last two terms of (3.20). Applying Lemma 3.11 we obtain

$$\bar{d}_i^2 - d_i'^2 \geq -2\,(\beta\sigma)^2 - 2\,(\beta\sigma)\,\bar{d}_i\,, \quad \text{for } i \in \{1, 2\}\,,$$

thus, dividing by $2\,\bar{d}_1 \bar{d}_2$ we deduce in view of (3.12) and (2.6) that

$$\frac{\bar{d}_i^2 - d_i'^2}{2\,\bar{d}_1 \bar{d}_2} \geq -\frac{4\beta^2}{(1-2\beta)^2} - \frac{2\beta}{1-2\beta} = -\frac{2\beta}{(1-2\beta)^2}.$$

Using the above estimation, together with (3.12) and (3.13), we deduce from (3.20) that

$$\begin{aligned} \langle u_{\bar{x}}(z_1), u_{\bar{x}}(z_2)\rangle_{\bar{x}} &\geq -\frac{1}{(1-2\beta)^2}\frac{\alpha}{8} - \frac{K}{2}\rho^2\left(1 + \frac{1}{(1-2\beta)^4}\right) - \frac{4\beta}{(1-2\beta)^2} \\ &\geq -\frac{\alpha}{8}\left(1 + \frac{5}{(1-2\beta)^2} + \frac{1}{(1-2\beta)^4}\right). \end{aligned}$$

Since $2\beta = \dfrac{\alpha}{4} \leq \dfrac{1}{512}$, we obtain $\langle u_{\bar{x}}(z_1), u_{\bar{x}}(z_2)\rangle_{\bar{x}} \geq -\alpha$. $\square$

The following result is the analogue of Lemma 3.4 (I) for the left-discontinuous case. Roughly speaking, the result (almost) remedies the failure illustrated in Remark 3.5 by moving the point $x = \gamma(\tau)$ (where γ is left-discontinuous) to $\bar{x} := \theta\left((1-\beta)\, d_g(x, x')\right)$ (see (3.10)) and making a parallel transportation of the secant $p := u_x(x')$ at x to $\bar{p} = u_{\bar{x}}(x') \in T_{\bar{x}}\mathcal{M}$ along the geodesic θ joining x to $\bar{x}$.

Proposition 3.12 *(Transported secant). Under the above notation and under the assumptions given before* Lemma 3.10,

$$\langle \bar{p}, u_{\bar{x}}(z)\rangle_{\bar{x}} := \langle u_{\bar{x}}(x'), u_{\bar{x}}(z)\rangle_{\bar{x}} < \alpha, \quad \textit{for all } z \in \Gamma_{\mathcal{U}}(\tau).$$

Proof. Let $z \in \Gamma_{\mathcal{U}}(\tau)$, and recall that $\sigma := d_g(x, x')$, $\bar{d} = d_g(\bar{x}, z)$ and $d' = d_g(x', z)$. We again apply the law of cosines to get

$$d'^2 - (\beta\sigma)^2 - \bar{d}^2 + 2\beta\sigma\,\bar{d}\,\langle u_{\bar{x}}(x'), u_{\bar{x}}(z)\rangle_{\bar{x}} \leq K\,(\beta\sigma)^2\,\bar{d}^2. \tag{3.21}$$

Notice that (3.15) yields

$$\frac{\bar{d}^2 - d'^2}{2\beta\sigma\,\bar{d}} \leq \frac{\beta\sigma}{\bar{d}} + \left(\frac{d'}{\bar{d}}\right)\frac{\alpha}{8}. \tag{3.22}$$

Combining (3.21) with (3.22) and using (3.12) and (3.13) we deduce (recall that $\beta\sigma \leq \rho$ and $\bar{d} \leq \rho$) we get

$$\langle u_{\bar{x}}(x'), u_{\bar{x}}(z)\rangle_{\bar{x}} \leq \frac{K}{2}\rho^2 + \frac{3\beta\sigma}{2\bar{d}} + \left(\frac{d'}{\bar{d}}\right)\frac{\alpha}{8} \leq \frac{\alpha}{8} + \frac{3\beta}{1-2\beta} + \left(\frac{1}{1-2\beta}\right)\frac{\alpha}{8}.$$

So $\langle u_{\bar{x}}(x'), u_{\bar{x}}(z)\rangle_{\bar{x}} < 4\,\beta + \dfrac{\alpha}{2} \leq \alpha$. $\square$

3.4. Estimations involving "almost secants"

We now show that at each point of left-continuity as well as at each point of left-discontinuity up to a certain discontinuity jump, a self-contracted curve grows backwards (with a uniform quantitative estimate) in some direction. We call this direction p^a an almost secant (because it is a modification of a secant p).

Theorem 3.13 *(Measuring growth using "almost secants"). Let $\gamma : [0, T_\infty) \to \mathcal{M}$ be a self-contracted curve and let us fix $x = \gamma(\tau)$ with $\tau \in (0, T_\infty)$.*

(i) *If γ is continuous at τ, for every $p \in \sec^-(\tau)$, there exists $p^a \in U_x\mathcal{M}$ such that for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $x \in \mathcal{U}$ and $\operatorname{diam}\mathcal{U} \leq \rho$, and every $z \in \Gamma_{\mathcal{U}}(\tau)$,*

$$\langle p^a, u_x(z)\rangle_x \leq -3\alpha \qquad \text{and} \qquad \langle p^a, p\rangle_x \geq 4\alpha\,.$$

(ii) *If $\tau \in \mathcal{D}^-\backslash\mathcal{D}^-(\eta)$, if we denote $\bar{p} = u_{\bar{x}}(x')$ the transported secant at $\bar{x}$, there exists $p^a \in U_{\bar{x}}\mathcal{M}$ such that for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $\{x, \bar{x}, x'\} \subset \mathcal{U}$ and $\operatorname{diam}\mathcal{U} \leq \rho$, and every $z \in \Gamma_{\mathcal{U}}(\tau)$,*

$$\langle p^a, u_{\bar{x}}(z)\rangle_{\bar{x}} \leq -3\alpha \qquad \text{and} \qquad \langle p^a, \bar{p}\rangle_{\bar{x}} \geq 4\alpha\,.$$

We need a separation lemma for subsets of the unit sphere of $\mathbb{R}^n$ with a controlled aperture.

Lemma 3.14 *(Strong separation lemma). Let C be a nonempty subset of the unit sphere of $\mathbb{R}^n$ satisfying*

$$A(C) \geq -\delta, \tag{3.23}$$

where

$$\delta = \frac{1}{2(n+1)}. \tag{3.24}$$

Then

$$\overline{\operatorname{conv}}\,(C) \bigcap B(0, \delta) = \emptyset\,. \tag{3.25}$$

Proof of Lemma 3.14. Let us assume, towards a contradiction, that for some $u \in \overline{\text{conv}}(C)$ we have $||u|| < \delta$. By Caratheodory's lemma there exist $\lambda_0, \dots, \lambda_n \in [0,1]$ with $\sum_{i=0}^n \lambda_i = 1$ and unit vectors $u_0, \dots, u_n \in C$ such that

$$\left\| \sum_{i=0}^{n} \lambda_i u_i \right\| < \delta.$$

Let $i_0 \in \{0, \dots, n\}$ be such that $\lambda_{i_0} \geq \lambda_i$ for any $i \in \{0, \dots, n\}$. Then $\lambda_{i_0} \geq 1/(n+1)$ and by the Cauchy–Schwarz inequality

$$\begin{aligned} \delta > \langle u_{i_0}, \sum_{i=0}^{n} \lambda_i u_i \rangle = \sum_{i=0}^{n} \lambda_i \langle u_{i_0}, u_i \rangle = \lambda_{i_0} + \sum_{i \neq i_0} \lambda_i \langle u_{i_0}, u_i \rangle \\ > \frac{1}{n+1} - \delta \left(\sum_{i \neq i_0} \lambda_i \right) > \frac{1}{n+1} - \delta = \delta, \end{aligned}$$

a clear contradiction. Thus the assertion holds true. □

Proof of Theorem 3.13. Both assertions follow by the same arguments and estimations. In order to present a common proof let us proceed to the following identification:

- If $x = \gamma(\tau) = \gamma(\tau^-)$, we identify the tangent space $T_x\mathcal{M}$ equipped with the scalar product $\langle \cdot, \cdot \rangle_x$ with the Euclidean space $\mathbb{R}^n$.
- If $x = \gamma(\tau) \neq \gamma(\tau^-)$, we identify the tangent space $T_{\bar{x}}\mathcal{M}$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{\bar{x}}$ with the Euclidean space $\mathbb{R}^n$.

In the sequel, we shall denote (in both cases) this scalar product by $\langle \cdot, \cdot \rangle$. We further set

$$C = \{u_x(z) : z \in \Gamma_{\mathcal{U}}(\tau)\} \qquad (\text{respectively} \quad C = \{u_{\bar{x}}(z) : z \in \Gamma_{\mathcal{U}}(\tau)\}).$$

Since $\alpha = \delta^2/8 \leq \delta$, Proposition 3.6 and Proposition 3.8 imply that $A(C) \geq -\delta$. Applying Lemma 3.14, we obtain that the projection of 0 to $\overline{\text{conv}}(C)$, denoted by $c \in T_x\mathcal{M}$, satisfies for every $u \in C$

$$||c|| \geq \delta \qquad \text{and} \qquad \langle -c, u - c \rangle \leq 0.$$

It follows

$$\langle -c, u \rangle \leq -||c||^2 \leq -\delta^2 = -8\alpha. \tag{3.26}$$

(i) Let $\tau \in (0, T_\infty) \setminus \mathcal{D}^-$ and fix any backward secant $p \in \sec^-(\tau) \in T_x\mathcal{M} \equiv \mathbb{R}^n$ and set

$$p^a := \frac{p - c}{||p - c||}.$$

By Lemma 3.4 (I) we get $\langle p, u \rangle \leq 0$, for all $u \in C$. Then for every $u \in C$ (unit vector) in view of (3.26) we deduce

$$\langle p^a, u \rangle = \frac{\langle p, u \rangle + \langle -c, u \rangle}{||p - c||} \leq \frac{0 - ||c||^2}{||p - c||} \leq \frac{-8\alpha}{||p - c||} \leq -3\alpha,$$

where the fact that $||p - c|| \leq 2$ is used. Finally, if $u \in U_x\mathcal{M}$ and $||p - u|| < \alpha$,

$$\langle p^a, p\rangle \geq \frac{||p||^2 + \langle -c, p\rangle}{||p-c||} \geq \frac{1+0}{2} \geq 4\alpha.$$

(ii) Let $\tau \in \mathcal{D}^-$ and consider the transported secant $\bar{p} = u_{\bar{x}}(x') \in T_x\mathcal{M} \equiv \mathbb{R}^n$ at $\bar{x}$. In an analogous manner to the above, we set

$$p^a := \frac{\bar{p}-c}{||\bar{p}-c||}.$$

By Proposition 3.12 we get

$$\langle \bar{p}, u\rangle \leq \alpha, \quad \text{for all } u \in C\,. \tag{3.27}$$

Since $c \in C$ we deduce

$$||\bar{p}-c||^2 = ||\bar{p}||^2 + ||c||^2 - \langle \bar{p}, c\rangle \geq 1 + \delta^2 - \alpha \geq 1.$$

In particular

$$1 \,\leq\, ||\bar{p}-c|| \,\leq\, 2\,.$$

For every $u \in C$ (unit vector) in view of (3.26) and (3.27) we deduce

$$\langle p^a, u\rangle \leq \frac{\langle \bar{p}, u\rangle + \langle -c, u\rangle}{||\bar{p}-c||} \leq \frac{\alpha - 8\alpha}{||\bar{p}-c||} \leq \frac{-7\alpha}{2} \leq -3\alpha.$$

On the other hand, if $u \in U_x\mathcal{M}$ and $||\bar{p}-u|| < \alpha$, using again (3.27), we get

$$\langle p^a, \bar{p}\rangle \geq \frac{||\bar{p}||^2 + \langle -c, \bar{p}\rangle}{||\bar{p}-c||} \geq \frac{1-\alpha}{||\bar{p}-c||} \geq \frac{1-\alpha}{2} \geq 4\alpha.$$

This concludes the proof of the assertion. $\square$

4. Proof of the main result

4.1. Width estimates via external functions

From now on, η is given by Lemma 2.6, $\mathcal{F}$ is a fixed finite η-net of $U\mathcal{K}$, and for each $\xi = q_y \in \mathcal{F}$, $\mathcal{U}_\xi := B_g(y, 2\eta)$. We recall that a finite subset $\mathcal{F}$ of $U\mathcal{K}$ is an η-net if $\mathcal{F}$ has a nonempty intersection with any ball (for the distance D_g) of radius η centered at a point of $U\mathcal{K}$. The existence of the finite η-net $\mathcal{F}$ follows from the compactness of $U\mathcal{K}$.

If γ is a self-contracted map defined on $[0, +\infty)$ and if $\tau \in (0, +\infty)\backslash\mathcal{D}^-(\eta)$, we define an element $\xi^a \in U\mathcal{K}$ as follows. We denote p^a the almost secant given by Theorem 3.13, and we consider two cases:

- If τ is a point of left-continuity of γ, for every backward secant $p \in \sec^-(\tau)$ at $x = \gamma(\tau)$ we associate the almost secant $p^a \in U_x\mathcal{K}$ and we set

$$\xi^a := p^a_x = (x, p^a)\,. \tag{4.1}$$

  Notice that different secants at x might give rise to different $p^a \in U_x\mathcal{K}$ (therefore to different elements $\xi_a \in U\mathcal{K}$).

- If $\tau \in \mathcal{D}^- \setminus \mathcal{D}^-(\eta)$, then the backward secant $p := u_x(x')$ at $x = \gamma(\tau)$ is unique. Using the notation of (3.10), the almost secant associated to $\bar{p}$ is $p^a \in U_{\bar{x}}\mathcal{K}$. We set:

$$\xi^a := p^a_{\bar{x}} = (\bar{x}, p^a). \tag{4.2}$$

The following result is crucial for our purposes. Roughly speaking it will be used to associate to each ξ^a, constructed above, an element ξ from the finite set $\mathcal{F}$. In this way, instead of controlling the growth of γ by the infinite set of "almost secants", we shall control this growth by the finite set of external functions.

Lemma 4.1 *(Controlling the local growth of γ by external functions). Let $\mathcal{F}$ be a finite η-net of $U\mathcal{K}$, and, for each $\xi = q_y \in \mathcal{F}$, let us denote $\mathcal{U}_\xi := B_g(y, 2\eta)$. Then:*

(I) *Let $\tau \in (0, T_\infty) \setminus \mathcal{D}^-$ and $p \in \sec^-(\tau)$, let $\xi^a := p^a_x$ be defined as in (4.1), and let $\xi \in \mathcal{F}$ be such that $D_g(\xi^a, \xi) < \eta$. If $\gamma(s) \in \mathcal{U}_\xi$, $|p - u_x(\gamma(s))|_x \le \alpha$ and $z \in \Gamma_{\mathcal{U}_\xi}(\tau)$, then:*

$$\langle q, \exp_y^{-1}(\gamma(s))\rangle_y \ge \langle q, \exp_y^{-1}(z)\rangle_y + 2\alpha d_g(\gamma(s), z). \tag{4.3}$$

(II) *Let $\tau \in \mathcal{D}^- \setminus \mathcal{D}^-(\eta)$, let $\xi^a := p^a_{\bar{x}}$ be defined by (4.2), and let $\xi \in \mathcal{F}$ be such that $D_g(\xi^a, \xi) < \eta$. If $\gamma(s) \in \mathcal{U}_\xi$, $|\bar{p} - u_{\bar{x}}(\gamma(s))|_{\bar{x}} \le \alpha$ and $z \in \Gamma_{\mathcal{U}_\xi}(\tau)$, then:*

$$\langle q, \exp_y^{-1}(\gamma(s))\rangle_y \ge \langle q, \exp_y^{-1}(z)\rangle_y + 2\alpha d_g(\gamma(s), z). \tag{4.4}$$

Proof. Recall that ρ satisfies (2.6) and that $\eta \in (0, \rho/4)$ is given by Lemma 2.6, so $\operatorname{diam}(\mathcal{U}_\xi) < \rho$.

We shall first consider the case $\tau \in (0, T_\infty) \setminus \mathcal{D}^-$. We fix $p \in \sec^-(\tau)$ and set $\xi^a := p^a_x$. Let $\xi = q_y \in B_{D_g}(\xi^a, \eta)$. We know from Theorem 3.13 (i) that for all $z \in \Gamma_{\mathcal{U}_\xi}(\tau) \setminus \{x\}$, $\langle p^a, u_x(z)\rangle_x \le -3\alpha$ and $\langle p^a, p\rangle_x \ge 4\alpha$. If $\gamma(s) \in \mathcal{U}_\xi$ and $|p - u_x(\gamma(s))|_x \le \alpha$, we obtain from Lemma 2.6:

$$\begin{aligned}\langle q, \exp_y^{-1}(\gamma(s))\rangle_y - \langle q, \exp_y^{-1}(x)\rangle_y &\ge (\langle p^a, u_x(\gamma(s))\rangle_x - \alpha)\, d_g(x, \gamma(s)) \\ &\ge (\langle p^a, p\rangle_x - 2\alpha)\, d_g(x, \gamma(s)) \ge 2\alpha d_g(x, \gamma(s)).\end{aligned}$$

On the other hand, if $z \in \Gamma_{\mathcal{U}_\xi}(\tau)$, we deduce from Lemma 2.6 that

$$\langle q, \exp_y^{-1}(x)\rangle_y - \langle q, \exp_y^{-1}(z)\rangle_y \ge (-\langle p^a, u_x(z)\rangle_x - \alpha)\, d_g(x, z) \ge 2\alpha d_g(x, z).$$

Summing up these two inequalities, we obtain (4.3).

The case $\tau \in \mathcal{D}^- \setminus \mathcal{D}^-(\eta)$ is treated similarly. Theorem 3.13 (ii) gives that for all $z \in \Gamma_{\mathcal{U}_\xi}(\tau) \setminus \{x\}$, $\langle p^a, u_{\bar{x}}(z)\rangle_{\bar{x}} \le -3\alpha$ and $\langle p^a, \bar{p}\rangle_{\bar{x}} \ge 4\alpha$. If $\gamma(s) \in \mathcal{U}_\xi$ and $|\bar{p} - u_{\bar{x}}(\gamma(s))|_{\bar{x}} \le \alpha$, Lemma 2.6 gives:

$$\begin{aligned}\langle q, \exp_y^{-1}(\gamma(s))\rangle_y - \langle q, \exp_y^{-1}(x)\rangle_y &\ge (\langle p^a, u_{\bar{x}}(\gamma(s))\rangle_{\bar{x}} - \alpha)\, d_g(x, \gamma(s)) \\ &\ge (\langle p^a, \bar{p}\rangle_{\bar{x}} - 2\alpha)\, d_g(x, \gamma(s)) \ge 2\alpha d_g(x, \gamma(s)).\end{aligned}$$

On the other hand, if $z \in \Gamma_{\mathcal{U}_\xi}(\tau)$, Lemma 2.6 implies that

$$\langle q, \exp_y^{-1}(x)\rangle_y - \langle q, \exp_y^{-1}(z)\rangle_y \ge (-\langle p^a, u_{\bar{x}}(z)\rangle_{\bar{x}} - \alpha)\, d_g(x, z) \ge 2\alpha d_g(x, z).$$

Summing up these two inequalities, we obtain (4.4). $\square$

For each $\xi \in \mathcal{F}$, we define the local width of γ at $x = \gamma(\tau)$ with respect to ξ as follows:

$$W_\xi(\tau) := \operatorname{diam} \left\{ \langle q, \exp_y^{-1}(z) \rangle_y : z \in \Gamma_{\mathcal{U}_\xi}(\tau) \right\}, \tag{4.5}$$

using the convention that $\operatorname{diam} \emptyset = 0$. We are now ready to establish our fundamental result, which states that the growth of the length of a self-contracted curve is locally controlled by the decay of one of the functions W_ξ.

Theorem 4.2. *Let $\gamma : [0, T_\infty) \to \mathcal{M}$ be a self-contracted map such that its range is included in the compact $\mathcal{K}$, let $\mathcal{F}$ be a finite η-net of $U\mathcal{K}$, and, for each $\xi = q_y \in \mathcal{F}$, let us denote $\mathcal{U}_\xi := B_g(y, 2\eta)$. Let $\tau \in (0, T_\infty) \setminus \left(\mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta)\right)$. There exists $\delta > 0$ such that, for all s, t satisfying $\tau - \delta < s < \tau < t < \tau + \delta$, there exists $\xi \in \mathcal{F}$ such that:*

$$W_\xi(s) - W_\xi(t) \geq \alpha d_g(\gamma(s), \gamma(t)). \tag{4.6}$$

Proof. (i). Let $\tau \in (0, T_\infty) \setminus \mathcal{D}^-$ (point of left-continuity) and set $x = \gamma(\tau)$. Since $\sec^-(\tau)$ is the set of accumulation points of the subset $\{u_x(\gamma(s))\}$ of $U_x\mathcal{M}$ as $s \nearrow \tau$, and since $U_x\mathcal{M}$ is compact, there exists $\delta > 0$ such that for every $s \in (\tau - \delta, \tau)$, there exists $p^s \in \sec^-(\tau)$ such that $|p^s - u_x(\gamma(s))|_x < \alpha$. Applying Lemma 4.1 (for $x = \gamma(\tau)$ and $p^s \in \sec^-(\tau)$) we get that for all $s \in (\tau - \delta, \tau)$, there exists $\xi \in \mathcal{F}$ such that:

$$\forall z \in \Gamma_{\mathcal{U}_\xi}(\tau) \quad \langle q, \exp_y^{-1}(\gamma(s)) \rangle_y - \langle q, \exp_y^{-1}(z) \rangle_y \geq 2\alpha d_g(\gamma(s), z). \tag{4.7}$$

(ii). Let us now assume $\tau \in \mathcal{D}^- \setminus \mathcal{D}^-(\eta)$, set $x = \gamma(\tau)$, $x' = \gamma(\tau^-)$ and $\bar{x} := \theta\left((1 - \beta)\, d_g(x, x')\right)$. Since $\bar{p} := u_{\bar{x}}(x')$ and $x' = \lim_{s \nearrow \tau} \gamma(s)$, there exists $\delta > 0$ such that if $s \in (\tau - \delta, \tau)$, then $|\bar{p} - u_{\bar{x}}(\gamma(s))|_{\bar{x}} < \alpha$. In this case, Lemma 4.1 yields that, there exists $\xi \in \mathcal{F}$ such that for all $s \in (\tau - \delta, \tau)$,

$$\forall z \in \Gamma_{\mathcal{U}_\xi}(\tau) \quad \langle q, \exp_y^{-1}(\gamma(s)) \rangle_y - \langle q, \exp_y^{-1}(z) \rangle_y \geq 2\alpha d_g(\gamma(s), z). \tag{4.8}$$

Let us finally assume (in both cases (i) and (ii)) that $\tau \notin \mathcal{D}^+(\eta)$, that is, $d_g(\gamma(\tau), \gamma(\tau^+)) < \eta$. Shrinking if necessary δ, we can assume that for all $t \in (\tau, \tau + \delta)$, we have $d_g(\gamma(\tau), \gamma(t)) < \eta$. This implies $d_g(y, \gamma(t)) \leq d_g(y, \gamma(\tau)) + d_g(\gamma(\tau), \gamma(t)) < 2\eta$ and so $\gamma(t) \in \mathcal{U}_\xi$ and $\Gamma_{\mathcal{U}_\xi}(t) \neq \emptyset$. The first inequality below follows from (4.7) and (4.8) and the fact that $\Gamma_{\mathcal{U}_\xi}(t)$ is included in $\Gamma_{\mathcal{U}_\xi}(\tau)$ whenever $t > \tau$, while the second one comes from the triangle inequality and the fact that γ is self-contracted. For all $s \in (\tau - \delta, \tau)$, there exists $\xi \in \mathcal{F}$ such that, for all $t \in (\tau, \tau + \delta)$ and for all $z \in \Gamma_{\mathcal{U}_\xi}(t)$:

$$\langle q, \exp_y^{-1}(\gamma(s)) \rangle_y - \langle q, \exp_y^{-1}(z) \rangle_y \geq 2\alpha d_g(\gamma(s), z) \geq \alpha d_g(\gamma(s), \gamma(t)).$$

Hence $W_\xi(s) \geq W_\xi(t) + \alpha d_g(\gamma(s), \gamma(t))$. $\square$

4.2. Proof of finite length

Let $\mathcal{F} \subset U\mathcal{K}$ be the finite η-net defined in the previous section. Then for any $\xi = q_y \in \mathcal{F}$, the function $z \mapsto \langle q, \exp_y^{-1}(z) \rangle_y$ is well defined on

$$\mathcal{U}_\xi := B_g(y, 2\eta) \subset B_g(y, 2\rho).$$

We recall that $W_\xi(\tau) := \operatorname{diam} \left\{ \langle q, \exp_y^{-1}(z) \rangle_y : z \in \Gamma_{\mathcal{U}_\xi}(\tau) \right\}$. Notice that for $\tau_1 \leq \tau_2$ we have $\Gamma_{\mathcal{U}_\xi}(\tau_2) \subset \Gamma_{\mathcal{U}_\xi}(\tau_1)$, therefore $W_\xi(\tau_2) \leq W_\xi(\tau_1)$. In other words, the function $\tau \longmapsto W_\xi(\tau)$ is nonincreasing on $[0, T_\infty)$ for every $\xi \in \mathcal{F}$. Let us now consider the (decreasing) aggregate function

$$W_{\mathcal{F}}(\tau) := \sum_{\xi \in \mathcal{F}} W_\xi(\tau).$$

The following result holds.

Proposition 4.3. *Let $[a,b] \subset (0, T_\infty) \setminus (\mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta))$. Then for every partition*

$$a = t_0 < t_1 < \ldots < t_m = b$$

of $[a,b]$,

$$\sum_{j=1}^{m} d_g(\gamma(t_{i-1}), \gamma(t_i)) \leq \frac{1}{\alpha}\left(W_{\mathcal{F}}(a) - W_{\mathcal{F}}(b)\right). \tag{4.9}$$

Proof. If $\tau \notin \mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta)$ and $x = \gamma(\tau)$, Theorem 4.2 tells us that there exists $\delta_\tau > 0$ such that $(\tau - \delta_\tau, \tau + \delta_\tau) \cap (\mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta))$ is empty and for all $s, t \in (\tau - \delta_\tau, \tau + \delta_\tau)$ with $s \leq \tau \leq t$, there exists $\xi \in \mathcal{F}$ satisfying $W_\xi(s) \geq W_\xi(t) + \alpha\, d_g(\gamma(s), \gamma(t))$. We deduce easily from the definition of $W_{\mathcal{F}}$ that:

$$W_{\mathcal{F}}(s) - W_{\mathcal{F}}(t) \geq \alpha\, d_g(\gamma(s), \gamma(t)). \tag{4.10}$$

Using a standard compactness argument, for every fixed $i \in \{1, \ldots, m\}$, there exists a subdivision $\{s_{i,j}\}_{j=0}^{j_i}$ of $[t_{i-1}, t_i]$ such that (4.10) is true for $s = s_{i,j-1}$ and $t = t_{i,j}$. Summing up these inequalities for all j and using the triangular inequality, we obtain that (4.10) is true for $s = t_{i-1}$ and $t = t_i$. Summing up these inequalities for all i we obtain (4.9). $\square$

We are now ready to conclude the proof of Theorem 2.1 (Main result).

Proof of Theorem 2.1. Let $\gamma : [0, T_\infty) \to \mathcal{M}$ be a self-contracted curve. Set $\mathcal{N} := \mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta)$ and denote by $|\mathcal{N}|$ its cardinality. Fix $T < T_\infty$ and denote by γ_T the restriction of γ to the compact interval $[0,T]$. We shall prove that γ_T is rectifiable and its length is bounded by $W_{\mathcal{F}}(0) + |\mathcal{N}|\,\Sigma$, where Σ is a strict upper bound for the maximal left or right jump of γ, that is,

$$\Sigma > \max\left\{\max_{\sigma \in \widehat{D}} d_g(\gamma(\sigma), \gamma(\sigma^-)),\ \max_{\sigma \in \mathcal{D}^+(\eta)} d_g(\gamma(\sigma), \gamma(\sigma^+))\right\}.$$

By Proposition 3.2, $\mathcal{N}$ is finite (and the right and left limits exist at every point), so there exists $\delta' > 0$ such that for any $\sigma \in \mathcal{N}$ and any $s, t \in (\sigma - \delta', \sigma + \delta')$ with $s \leq \sigma \leq t$ it holds

$$d_g(\gamma(s), \gamma(t)) < \Sigma. \tag{4.11}$$

Notice that the compact set $[0,T] \setminus \bigcup_{\sigma \in \mathcal{N}} (\sigma - \delta', \sigma + \delta')$ is a finite union of intervals $[a_i, b_i]$, for each of which Proposition 4.3 applies. We deduce easily that

$$\ell(\gamma_T) \leq \frac{1}{\alpha} W_{\mathcal{F}}(0) + |\mathcal{N}|\,\Sigma.$$

Since the above bound is independent of T, passing to the limit as $T \to +\infty$ we obtain that the length of γ is bounded by the same constant. $\square$

Remark 4.4. Proposition 3.2 tells us that $|\mathcal{N}| \leq |\mathcal{D}^-(\eta)| + |\mathcal{D}^+(\eta)| \leq 2N(\eta)$, where $N(\eta)$ is the minimal number of balls of radius $\eta/2$ that can cover $\mathcal{K}$. On the other hand, for every $\xi \in \mathcal{F}$, $W_\xi(0) \leq$

$\sup\{|\exp_y^{-1}(z_1) - \exp_y^{-1}(z_2)|_y;\, z_1, z_2 \in \mathcal{U}_\xi\} \leq 4\eta$, so $W_{\mathcal{F}}(0) \leq 4\eta|\mathcal{F}|$. Finally, $\Sigma \leq 2\mathrm{diam}(\mathcal{K})$. Therefore, the above proof shows that the upper bound for the length of any self-contracted curve $\gamma : [0, T_\infty) \to \mathcal{K}$ only depends on the dimension of the manifold and the compact set $\mathcal{K}$.

References

[1] W. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd edition, Pure Appl. Math., vol. 120, Academic Press, 1986.
[2] A. Daniilidis, D. Drusvyatskiy, A.S. Lewis, Orbits of geometric descent, Canad. Math. Bull. 58 (2015) 44–50.
[3] A. Daniilidis, O. Ley, S. Sabourau, Asymptotic behaviour of self-contracted planar curves and gradient orbits of convex functions, J. Math. Pures Appl. 94 (2010) 183–199.
[4] G. David, A. Daniilidis, E. Durand-Cartagena, A. Lemenant, Rectifiability of self-contracted curves in the euclidean space and applications, J. Geom. Anal. 25 (2015) 1211–1239.
[5] C. Giannotti, A. Spiro, Steepest descent curves of convex functions on surfaces of constant curvature, Israel J. Math. 191 (2012) 279–306.
[6] J. Lee, Manifolds and Differential Geometry, Grad. Stud. Math., vol. 107, American Mathematical Society, Providence, RI, 2009.
[7] A. Lemenant, Rectifiability of non Euclidean planar self-contracted curves, Confluentes Math. 8 (2017) 23–38.
[8] M. Longinetti, P. Manselli, A. Venturi, On steepest descent curves for quasiconvex families in $\mathbb{R}^n$, Math. Nachr. 288 (2015) 420–442.
[9] P. Manselli, C. Pucci, Uniqueness results for evolutes and self-evolvents, Boll. Unione Mat. Ital. A 5 (1991) 373–379.
[10] P. Manselli, C. Pucci, Maximum length of steepest descent curves for quasi-convex functions, Geom. Dedicata 38 (1991) 211–227.