

Wandering intervals in affine extensions of self-similar interval exchange maps: the cubic Arnoux–Yoccoz map

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Abstract. In this article, we provide sufficient conditions on a self-similar interval exchange map, whose renormalization matrix has complex eigenvalues of modulus greater than one, for the existence of affine interval exchange maps with wandering intervals that are semi-conjugate with it. These conditions are based on the algebraic properties of the complex eigenvalues and the complex fractals built from the natural substitution emerging from self-similarity. We show that the cubic Arnoux–Yoccoz interval exchange map satisfies these conditions.

1. Introduction

The existence of wandering intervals in dynamical systems has been studied for a long time. Denjoy [Den32] proved that an orientation-preserving \mathcal{C}^1 -diffeomorphism of the circle with irrational rotation number is conjugate with an irrational rotation if and only if it has no wandering intervals, and he constructed examples of \mathcal{C}^r -diffeomorphisms of this type with wandering intervals for $r < 2$. The absence of wandering intervals is ensured for \mathcal{C}^2 -diffeomorphisms.

By suspending a rotation, one obtains a linear flow on a two-dimensional torus. A natural generalization of such a flow is a linear flow on surfaces of a higher genus, which, when restricted to a Poincaré section, induces an interval exchange map (i.e.m.). In this sense, an i.e.m. is a natural generalization of a rotation of the circle.

A bijective map $T : [0, 1) \rightarrow [0, 1)$ is said to be an *i.e.m.* if there exists a finite partition $(I_a; a \in \mathcal{A})$ of $[0, 1)$ made of intervals such that $T(t) = t + \delta_a$ for each $t \in I_a$ and

$a \in \mathcal{A}$. Clearly, T is a piecewise isometry of the unit interval exchanging the intervals $(I_a; a \in \mathcal{A})$. An i.e.m. T is said to be *self-similar* if there exists $0 < \alpha < 1$ such that the map $T^{(1)} : [0, \alpha) \rightarrow [0, \alpha)$ of first return by T to the interval $[0, \alpha)$ is, up to rescaling, equal to T . The natural symbolic extension of a self-similar i.e.m. is generated by a substitution. An *affine* interval exchange map (affine i.e.m.) $f : [0, 1) \rightarrow [0, 1)$ is a bijective piecewise affine map with positive slopes. The vector $(\ell_a; a \in \mathcal{A})$, where ℓ_a is the slope of f in the a th interval of continuity, is called *the slope vector* of f .

Levitt [Lev87] constructed an example of a non-uniquely ergodic affine i.e.m. with wandering intervals, showing that there exist Denjoy counterexamples of arbitrary smoothness in some surfaces of genus at least two. Given a slope vector and a self-similar i.e.m., Camelier and Gutiérrez [CG97] provided necessary and sufficient conditions for the existence of an affine i.e.m. with the same number of intervals and a slope vector that is semi-conjugate with the self-similar i.e.m. Namely, such map exists if and only if the logarithm of the given slope vector, $\log \ell = (\log \ell_a; a \in \mathcal{A})$, is orthogonal to the vector of interval lengths $\lambda = (|I_a|; a \in \mathcal{A})$. The self-similarity of the i.e.m. implies that $R\lambda = \alpha^{-1}\lambda$, where R is the renormalization matrix (recall that $[0, \alpha)$ is the interval of renormalization) and $\alpha^{-1} > 1$ is the Perron–Frobenius eigenvalue of R . Thus, basic linear algebra implies that $\log \ell$ is orthogonal to λ if and only if $\log \ell$ belongs to the invariant subspace corresponding to all the eigenvalues of $M = R^t$ that are different from α^{-1} . We remark that M is the matrix associated with the substitution associated with the self-similar i.e.m.

If $\log \ell$ belongs to the stable space of M , Camelier and Gutiérrez [CG97] proved that any semi-conjugate affine i.e.m. with this slope vector is, in fact, conjugate with an i.e.m. (that is, has no wandering intervals). The resulting conjugacy is of class $\mathcal{C}^{1+\varepsilon}$, with $\varepsilon > 0$ depending on the particular eigenspace [Bar99]. They also built an example of a uniquely ergodic affine i.e.m. with wandering intervals that is strictly semi-conjugate with a self-similar i.e.m. This last example and extensions of the results in [CG97] were considered more deeply by Cobo in [Cob02], where a generalization is also obtained by introducing the Rauzy–Veech–Zorich Oseledets decomposition (see [Vee82, Zor96]). In the case when $\log \ell$ belongs to the unstable space of M , Bressaud, Hubert and Maass proved, in [BHM10], that if it also lies in the eigenspace associated with a real eigenvalue of modulus strictly greater than one, which is different from the Perron–Frobenius eigenvalue α^{-1} but Galois-conjugate with it, then one can choose a semi-conjugate affine i.e.m. with such a slope vector and wandering intervals. If the given vector of logarithms lies in an eigenspace of M associated with the eigenvalue 1 or -1 , or if it lies in an invariant subspace corresponding to a conjugate pair of non-real eigenvalues of modulus one, then Bressaud, Bufetov and Hubert, in [BBH14], proved that any semi-conjugate affine i.e.m. with such a slope vector is indeed conjugate to the i.e.m. and thus has no wandering intervals. Finally, in [MMY10], Marmi, Moussa and Yoccoz proved that the existence of an affine i.e.m. with wandering intervals that is semi-conjugate with a given i.e.m. is generic. Thus many non-self-similar examples arise.

In this article, we study the remaining case of the program stated in [CG97]. That is, we consider a self-similar i.e.m. and a slope vector ℓ whose logarithm $\log \ell$ lies in an invariant subspace of M corresponding to a conjugate pair of non-real eigenvalues of

modulus strictly larger than one. We will heavily rely on the strategy of [BHM10] and the geometrical models for substitutions defined by Arnoux, Bernat and Bressaud in [ABB11]. These geometrical models are often of fractal nature. We will see that some properties of these fractals are sufficient conditions for the existence of an affine i.e.m. with wandering intervals that is semi-conjugate with a given self-similar i.e.m. having the aforementioned properties. Specifically, we will prove the following theorem.

THEOREM A. *Let T be a self-similar i.e.m. Assume that M has an eigenvalue β with $|\beta| > 1$ such that $\beta/|\beta|$ is not a root of unity, and that there exists an eigenvector Γ for β such that T has the unique representation property for β and Γ . Then, for almost every γ in the complex subspace generated by Γ , $\exp(-\operatorname{Re}(\gamma))$ can be realized as the slope vector of an affine i.e.m. that is semi-conjugate with T and has wandering intervals.*

The unique representation property, which will be stated in §6, is related to the different ways in which the extreme points of the dual Rauzy fractal, in the sense of [ABB11], can be written as certain sums $\sum_{m \geq 1} z_m \beta^{-m}$ with coefficients z_m belonging to a finite subset of the field $\mathbb{Q}[\beta]$.

Most of the properties that we will develop before the proof of the main theorem are of purely symbolic nature, in the sense that they can be proved for a primitive substitution and a non-real eigenvalue together with an eigenvector of the matrix associated to such substitution.

The unique representation property needed in this article is, in some way, analogous to the explicit algebraic condition required for the case treated in [BHM10]. We think that similar algebraic conditions imply the unique representation property, but we did not succeed in establishing a proof, even though interesting examples can be found. However, we found that an algebraic condition similar to the one needed in [BHM10] provides a simplification in the hypotheses of Theorem A. Indeed, assume that either β is Galois-conjugate with α^{-1} or that β is Galois-conjugate with α and the self-similarity comes from Rauzy–Veech renormalizations: that is, some iteration of the Rauzy–Veech renormalization returns to the original map. Then β is a simple eigenvalue. Moreover, if β is not real, then $\beta/|\beta|$ is not a root of unity (see Lemma 7.9). In these cases, since the corresponding eigenspace is one-dimensional, the unique representation property depends only on T and β . Since it is possible to construct an i.e.m. that is periodic for the Rauzy–Veech algorithm from most cycles of a Rauzy class, such an i.e.m. belongs to a natural family of self-similar maps.

Therefore, we have the following consequence of Theorem A.

THEOREM B. *Let T be an i.e.m. and let β with $|\beta| > 1$ be a non-real eigenvalue of M . Assume that either β is Galois-conjugate to α or that β is Galois-conjugate with α^{-1} and that T is periodic for the Rauzy–Veech renormalization algorithm on the interval $[0, \alpha)$. If T has the unique representation property for β , then, for almost every associated eigenvector γ , $\exp(-\operatorname{Re}(\gamma))$ can be realized as the slope vector of an affine i.e.m. that is semi-conjugate with T and has wandering intervals.*

Finally, we apply the techniques developed along this work to the cubic Arnoux–Yoccoz i.e.m. [AY81]. When defined on the circle, this map is self-similar and its renormalization

matrix has an eigenvalue β with $|\beta| > 1$ such that $\beta/|\beta|$ is not a root of unity. We will discuss this example in §8 and prove the following theorem.

THEOREM C. *Let β be a non-real eigenvalue of the renormalization matrix associated to the cubic Arnoux–Yoccoz i.e.m. satisfying that $|\beta| > 1$ and that $\beta/|\beta|$ is not a root of unity. For almost every eigenvector γ for β , there exists a semi-conjugate affine i.e.m. with slope vector $\exp(-\operatorname{Re}(\gamma))$ exhibiting wandering intervals.*

The fact that Theorem A is valid for almost every eigenvector for β when considering the cubic Arnoux–Yoccoz i.e.m. comes from the simplicity of the eigenvalue β in this case.

The article is organized as follows. In §2, we outline the general strategy introduced in [CG97] that will be used to prove the main theorem. In §3, we present some basic properties and definitions concerning a self-similar i.e.m. In §4, the notion of a minimal sequence is presented. In §5, the fractals associated with β and the concept of extreme points are defined. In §6, the unique representation property is introduced. In §7, we restate and prove the main theorems. In §8, we prove that the cubic Arnoux–Yoccoz i.e.m. satisfies the unique representation property and thus that the main theorem can be applied to it.

2. Strategy

Let $T : [0, 1) \rightarrow [0, 1)$ be a self-similar i.e.m. with continuity intervals $(I_a; a \in \mathcal{A})$. Our goal is to prove that, under the hypotheses of Theorem A, there exists an affine i.e.m. $f : [0, 1) \rightarrow [0, 1)$ with wandering intervals that is semi-conjugate with T or, equivalently, is a topological extension of T .

In order to achieve this, we will follow the strategy devised by Camelier and Gutiérrez in [CG97] that was used by Cobo in [Cob02] and by Bressaud, Hubert and Maass in [BHM10]: that is, we search for a Borel probability measure μ with atoms on $[0, 1)$ that assigns positive measure to every open interval with the following property. For each $a \in \mathcal{A}$, there exists a positive real ℓ_a such that

$$\mu(T(J)) = \ell_a \mu(J) \tag{1}$$

for every Borel set $J \subseteq I_a$. By following the proof of Lemma 3.6 of [CG97] and using such a measure, one constructs an affine i.e.m. that is semi-conjugate with T and has a wandering interval. Indeed, assume that μ satisfies the aforementioned properties. Let $g : [0, 1) \rightarrow [0, 1)$ be defined by $g(t) = \mu([0, t])$ and $g(0) = 0$. We know that g is strictly increasing, since μ is positive on open intervals. It is also right-continuous. Therefore, $G = [0, 1) \setminus g([0, 1))$ is a union of countably many intervals of the form $[t_0, t_1)$. Define the map $h : [0, 1) \rightarrow [0, 1)$ by $h = g^{-1}$ on $[0, 1) \setminus G$ and $h(t) = g^{-1}(t_1)$ if $t \in [t_0, t_1)$, where $[t_0, t_1)$ is a maximal interval in G . We know that h is right-continuous, non-decreasing and surjective. Then we define $f : [0, 1) \rightarrow [0, 1)$ in the following way. If $t \in [0, 1) \setminus G$, we put $f(t) = h^{-1} \circ T \circ h(t)$. If $[t_0, t_1)$ is a maximal interval in G , we define f to be linear and increasing between $[t_0, t_1)$ and $h^{-1}(T \circ h([t_0, t_1)))$. A straightforward computation shows that $h \circ f = T \circ h$ and that f is an affine i.e.m. with the desired slope vector.

One way to construct such a measure is finding a sequence $\omega \in \Omega_T$, where Ω_T is the natural symbolic extension of T constructed from codings of itineraries of points by T with respect to the continuity intervals $(I_a; a \in \mathcal{A})$, a complex vector $\gamma \in \mathbb{C}^{\mathcal{A}}$ and a number $\rho > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_n(\omega))}{n^\rho} > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_{-n}(\omega))}{n^\rho} > 0, \tag{2}$$

where $\gamma_n(\omega) = \gamma_{\omega_0} + \dots + \gamma_{\omega_{n-1}}$ and $\gamma_{-n}(\omega) = -(\gamma_{\omega_{-n}} + \dots + \gamma_{\omega_{-1}})$ for $n \geq 1$. This implies that $K = \sum_{n \in \mathbb{Z}} \exp(-\operatorname{Re}(\gamma_n(\omega))) < \infty$. Thus, if $\omega \in \Omega_T$ is the coding by T of $t \in [0, 1)$, then the measure $\mu = (1/K) \sum_{n \in \mathbb{Z}} \exp(-\operatorname{Re}(\gamma_n(\omega))) \mathbf{1}_{T^n(t)}$ satisfies the desired properties for the slope vector $\ell = (\exp(-\operatorname{Re}(\gamma_a))); a \in \mathcal{A}$. This can be proved in an analogous way to [BHM10, Lemma 22].

Otherwise, if ω is not the coding by T of any point in $[0, 1)$, then it is the coding by $T' : [0, 1) \rightarrow [0, 1)$ of some point in $[0, 1)$, where T' is equal to T up to some discontinuity points. The strategy can be still applied to find $f' : [0, 1) \rightarrow [0, 1)$ and $h : [0, 1) \rightarrow [0, 1)$ such that $h \circ f' = T' \circ h$, where h is continuous, surjective, non-decreasing and non-injective. Let f be the right-continuous function that is equal to f' up to a finite number of points. Then f is an affine i.e.m. By right-continuity, $h \circ f = T \circ h$. Moreover, f has wandering intervals since $h(J)$ is a point for some interval $J \subseteq [0, 1)$. Therefore, it is enough to prove (2) for a sequence ω in Ω_T . This last fact can also be proved using the classical construction of Keane in [Kea75].

3. Background and preliminaries

3.1. *The i.e.m. and the affine i.e.m.* A bijective map $T : [0, 1) \rightarrow [0, 1)$ is said to be an i.e.m. if there exists a finite partition $(I_a; a \in \mathcal{A})$ of $[0, 1)$ made of intervals such that $T(t) = t + \delta_a$ for each $t \in I_a$ and $a \in \mathcal{A}$. Clearly, T is a piecewise isometry of the unit interval exchanging the intervals $(I_a; a \in \mathcal{A})$. The vector $\delta = (\delta_a; a \in \mathcal{A})$ is called the *translation vector* of T . An i.e.m. T is determined by the following combinatorial data: that is, a *length vector* $\lambda = (\lambda_a; a \in \mathcal{A})$ of positive entries corresponding to the length of each interval I_a and a pair of bijections $\pi_0, \pi_1 : \{1, \dots, |\mathcal{A}|\} \rightarrow \mathcal{A}$ encoding the order of the intervals $(I_a; a \in \mathcal{A})$ before and after the transformation.

An i.e.m. T is said to be *self-similar* if there exists $0 < \alpha < 1$ such that the map $T^{(1)} : [0, \alpha) \rightarrow [0, \alpha)$ of first return by T to the interval $[0, \alpha)$ is, up to rescaling, equal to T . These maps are called *renormalizable* in [CG97]. In [LPV07], they are called *scale-invariant* and the term self-similar is used for maps such that the induced map on some interval is, up to rescaling and rotation, the same map T . Here, for convenience, we keep the notation used in [CG97] and used implicitly in [BHM10].

For each $a \in \mathcal{A}$, we define the interval $I_a^{(1)} = \alpha I_a$ and denote by R the *renormalization matrix* given by $R_{a,b} = |\{0 \leq k \leq r_b - 1; T^k(I_b^{(1)}) \subseteq I_a\}|$, where r_b is the first-return time of $I_b^{(1)}$ to $[0, \alpha)$.

Given an i.e.m., the *Rauzy–Veech renormalization algorithm* produces a new i.e.m. by considering the first-return map on a specific interval and rescaling the domain to $[0, 1)$. A *Rauzy class* consists of the possible combinatorics for the starting and ending position of $(I_a)_{a \in \mathcal{A}}$ that can be reached from a fixed permutation by applying the Rauzy–Veech algorithm (for details, see [Via06]). As explained later, cycles on Rauzy classes provide

a natural way to construct a self-similar i.e.m. Nevertheless, we will not assume that the self-similar i.e.m. comes from this construction, unless otherwise stated.

A self-similar i.e.m. is always uniquely ergodic and therefore minimal (see [Vee78]). Recall that T is minimal if any point in $[0, 1)$ has a dense orbit. For more details on a minimal i.e.m., see [Kea75, Via06].

An *affine i.e.m.* $f : [0, 1) \rightarrow [0, 1)$ is a bijective, piecewise affine map with positive slopes. If $(J_a; a \in \mathcal{A})$ are the continuity intervals of f , we say that $\ell = (\ell_a; a \in \mathcal{A})$ is its *slope vector*, where $\ell_a > 0$ is the slope of f restricted to J_a : $f(t) = \ell_a t + d_a$ for every $t \in J_a$ and for some translation vector $d = (d_a; a \in \mathcal{A})$. Clearly, an i.e.m. is an affine i.e.m. with slope vector $\ell = (1, \dots, 1)$.

We are interested in affine i.e.m. extensions of an i.e.m. T with wandering intervals: that is, an affine i.e.m. f with wandering intervals such that there exists a continuous, surjective, non-decreasing and non-injective map $h : [0, 1) \rightarrow [0, 1)$ satisfying $h \circ f = T \circ h$. As mentioned in the introduction, the existence of such extensions has already been studied in [CG97, Cob02, BHM10, BBH14, MMY10].

3.2. *Substitution subshifts and prefix-suffix decomposition.* We refer to [Que87] and [Fog02] and the references therein for the general theory of substitutions.

Let \mathcal{A} be a finite set or *alphabet*. A *word* is a finite string of symbols in \mathcal{A} , namely, $w = w_0 \dots w_{m-1}$, where $|w| = m$ is called the *length* of w . The *empty word* ε is defined as the word of length zero. The set of all words in \mathcal{A} is denoted by \mathcal{A}^* and the set of words of positive length is denoted by $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$.

We will need to consider words indexed by integers. We will write such a word as $w = w_{-m} \dots w_{-1} \cdot w_0 \dots w_n$, where m, n are non-negative integers and the dot separates negative and non-negative coordinates. The set of one-sided infinite sequences $\omega = (\omega_m)_{m \geq 0}$ in \mathcal{A} is denoted by $\mathcal{A}^{\mathbb{N}}$. Analogously, $\mathcal{A}^{\mathbb{Z}}$ denotes the set of two-sided infinite sequences $\omega = (\omega_m)_{m \in \mathbb{Z}}$.

A *substitution* is a map $\sigma : \mathcal{A} \rightarrow \mathcal{A}^+$. It naturally extends to $\mathcal{A}^+, \mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ by concatenation. For $\omega = (\omega_m)_{m \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, the extension is given by

$$\sigma(\omega) = \dots \sigma(\omega_{-2})\sigma(\omega_{-1}) \cdot \sigma(\omega_0)\sigma(\omega_1) \dots,$$

where the central dot separates the negative and non-negative coordinates of $\sigma(\omega)$. A further natural convention is that $\sigma(\varepsilon) = \varepsilon$.

Let M^σ be the matrix with indices in \mathcal{A} such that $M_{a,b}^\sigma$ is the number of times the letter b appears in $\sigma(a)$ for any $a, b \in \mathcal{A}$. The substitution is said to be *primitive* if there exists an integer $n \geq 1$ such that, for any $a \in \mathcal{A}$, $\sigma^n(a)$ contains every letter of \mathcal{A} , where σ^n denotes n consecutive iterations of σ .

Let $\Omega_\sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ be the subshift defined from σ : that is, $\omega \in \Omega_\sigma$ if and only if any subword of ω is a subword of $\sigma^n(a)$ for some integer $n \geq 0$ and $a \in \mathcal{A}$. We call Ω_σ the *substitution subshift* associated with σ . This subshift is minimal whenever σ is primitive.

Assume that σ is primitive. By the recognizability property (see [Mos92]), given a point $\omega \in \Omega_\sigma$, there exists a unique sequence $(p_m, c_m, s_m)_{m \geq 0} \in (\mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*)^{\mathbb{N}}$ such that, for each integer $m \geq 0$, $\sigma(c_{m+1}) = p_m c_m s_m$ and

$$\dots \sigma^3(p_3)\sigma^2(p_2)\sigma^1(p_1)p_0 \cdot c_0 s_0 \sigma^1(s_1)\sigma^2(s_2)\sigma^3(s_3) \dots$$

is the central part of ω , where the dot separates negative and non-negative coordinates. We remark that the p_m (respectively, s_m) are in the finite subset of \mathcal{A}^* containing all prefixes (respectively, suffixes) of $\sigma(a)$ for all a in \mathcal{A} . This sequence is called the *prefix-suffix decomposition* of ω (for more details, see, for instance, [CS01]).

If only finitely many suffixes s_m are non-empty, then there exist $a \in \mathcal{A}$ and non-negative integers n and q such that

$$\omega_{[0,\infty)} = c_0 s_0 \sigma^1(s_1) \dots \sigma^n(s_n) \lim_{m \rightarrow \infty} \sigma^{mq}(a).$$

Analogously, if only finitely many p_m are non-empty, then there exist $a \in \mathcal{A}$ and non-negative integers n and q such that

$$\omega_{(-\infty,-1]} = \lim_{m \rightarrow \infty} \sigma^{mq}(a) \sigma^n(p_n) \dots \sigma^1(p_1) p_0.$$

The recognizability property also implies that $\Omega_\sigma = \bigcup_{m=0}^n S^{-m}(\sigma(\Omega_\sigma))$ for some positive integer n , where $S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is the left shift map.

3.3. *Symbolic coding of a self-similar i.e.m.* Let T be a self-similar i.e.m. and let $(I_a; a \in \mathcal{A})$ be its associated intervals. Recall that, under the self-similar condition, T is minimal. Given $t \in [0, 1)$, we construct a symbolic sequence $\omega = (\omega_m)_{m \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, where $\omega_m = a$ if and only if $T^m(t) \in I_a$. The sequence ω is called the *itinerary* of t . Let $\Omega_T \subseteq \mathcal{A}^{\mathbb{Z}}$ be the closure of the set of sequences constructed in this way for every $t \in [0, 1)$. Clearly, the sequence associated with $T(t)$ corresponds to $S(\omega)$, where $S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is the left shift map. Moreover, it is classical that there exists a continuous and surjective map $\pi_T : \Omega_T \rightarrow [0, 1)$ such that $T \circ \pi_T = \pi_T \circ S$. The map π_T is invertible up to a countable set of points corresponding to the orbits of discontinuities of T . Since T is self-similar, the restriction of S to Ω_T is minimal and Ω_T is a substitutive subshift associated with a substitution $\sigma : \mathcal{A} \rightarrow \mathcal{A}^+$: that is, $\Omega_T = \Omega_\sigma$ for some substitution σ . The substitution is constructed in the following way: $\sigma(a) = w_0 \dots w_{r_a-1}$ if and only if $T^m(I_a^{(1)}) \subseteq I_{w_m}$ for all $0 \leq m \leq r_a - 1$ and $a \in \mathcal{A}$. Then $M^\sigma = R^t$ is the transpose of the renormalization matrix associated with T (for details, see [CG97]).

4. *Minimal sequences associated with a self-similar i.e.m.*

We fix a self-similar i.e.m. T that is self-induced on the interval $[0, \alpha)$ with $0 < \alpha < 1$. Let $M = R^t$ be the matrix of its associated substitution σ and let $\beta \in \mathbb{C}$ be an eigenvalue of M with $|\beta| > 1$ and such that $\beta/|\beta|$ is not a root of unity. We fix an eigenvector γ for β for the rest of the section.

Definition 4.1. For $w = w_0 \dots w_{n-1} \in \mathcal{A}^*$, we set $\gamma(w) = \gamma_{w_0} + \dots + \gamma_{w_{n-1}}$.

It is easy to see that, for any integer $n \geq 0$,

$$\gamma(\sigma^n(w)) = \beta^n \gamma(w). \tag{3}$$

For a sequence $\omega = (\omega_m)_{m \in \mathbb{Z}} \in \Omega_T$, we define $\gamma_0(\omega) = 0$, $\gamma_n(\omega) = \gamma(\omega_0 \dots \omega_{n-1})$ for $n \geq 1$ and $\gamma_n(\omega) = -\gamma(\omega_n \dots \omega_{-1})$ for $n \leq -1$.

Definition 4.2. A sequence $\omega \in \Omega_T$ is a *minimal sequence* for the vector γ if

$$\operatorname{Re}(\gamma_n(\omega)) \geq 0 \quad \text{for all } n \in \mathbb{Z}.$$

Our purpose is to prove that minimal sequences for some eigenvector γ satisfy equation (2): that is, that there exists a number $\rho > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_n(\omega))}{n^\rho} > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_{-n}(\omega))}{n^\rho} > 0.$$

This does not necessarily hold for an arbitrary eigenvector γ .

The next lemma illustrates a property of minimal sequences for finite words.

LEMMA 4.3. *Let $\omega \in \Omega_T$ be a minimal sequence for γ . Then, for all integers $n \leq -1$ and $m \geq n$,*

$$\operatorname{Re}(\gamma(\omega_n \dots \omega_{-1})) \leq \operatorname{Re}(\gamma(\omega_n \dots \omega_m)).$$

Proof. If $m \leq -1$, then

$$\operatorname{Re}(\gamma(\omega_n \dots \omega_{-1})) - \operatorname{Re}(\gamma(\omega_n \dots \omega_m)) = -\operatorname{Re}(\gamma_{m+1}(\omega)) \leq 0$$

by minimality of ω . If $m \geq 0$, then

$$\operatorname{Re}(\gamma(\omega_n \dots \omega_m)) - \operatorname{Re}(\gamma(\omega_n \dots \omega_{-1})) = \operatorname{Re}(\gamma_m(\omega)) \geq 0$$

by minimality of ω . □

LEMMA 4.4. *There exist minimal sequences for γ .*

Proof. We know that $\operatorname{Re}(\gamma) \neq 0$. Indeed, if $\operatorname{Re}(\gamma) = 0$, then $\gamma = i\chi$, where $\chi \in \mathbb{R}^{\mathcal{A}}$ is an eigenvector associated to β : that is, $M\chi = \beta\chi$. Since M is an integer matrix, $M\chi \in \mathbb{R}^{\mathcal{A}}$. This contradicts the fact that β is not real.

Recall that $\operatorname{Re}(\gamma)$ is orthogonal to λ . Then there exist letters $a, b \in \mathcal{A}$ with $\operatorname{Re}(\gamma_a) < 0$ and $\operatorname{Re}(\gamma_b) > 0$. Also, by minimality of T , there exists an itinerary of the form awb with $w \in \mathcal{A}^*$.

Let $\beta_0 = \beta/|\beta|$ and consider $(n_k)_{k \geq 1}$ to be a sequence of integers such that $\operatorname{Re}(\beta_0^{n_k}) > C$ for every integer k and some constant $C > 0$. In this way, there is a decomposition

$$\sigma^{n_k}(awb) = \sigma^{n_k}(a)\sigma^{n_k}(w)\sigma^{n_k}(b) = \omega_{-m_k}\omega_{-m_k+1} \dots \omega_{-1} \cdot \omega_0\omega_1 \dots \omega_{m'_k},$$

with the dot marking the minimal sum with respect to γ : that is,

$$\operatorname{Re}(\gamma(\omega_{-m_k} \dots \omega_{-1})) \leq \operatorname{Re}(\gamma(\omega_{-m_k} \dots \omega_m))$$

for every $-m_k \leq m \leq m'_k$. We claim that both sequences $(m_k)_{k \geq 0}$ and $(m'_k)_{k \geq 0}$ go to infinity when $k \rightarrow \infty$. For this, we only need to prove that both sequences $\operatorname{Re}(\gamma(\omega_{-m_k} \dots \omega_{-1}))$ and $\operatorname{Re}(\gamma(\omega_0 \dots \omega_{m'_k}))$ are unbounded. First, recall that, from (3), $\gamma(\sigma^{n_k}(a)) = \beta^{n_k}\gamma_a = |\beta|^{n_k}\beta_0^{n_k}\gamma_a$ and $\gamma(\sigma^{n_k}(b)) = |\beta|^{n_k}\beta_0^{n_k}\gamma_b$. Since the sequence $(\operatorname{Re}(\beta_0^{n_k}))_{k \geq 1}$ is positive and bounded from below, the choice of a and b implies that

$$\operatorname{Re}(\gamma(\sigma^{n_k}(a))) \rightarrow -\infty \quad \text{and} \quad \operatorname{Re}(\gamma(\sigma^{n_k}(b))) \rightarrow \infty.$$

But, from the definition of the decomposition,

$$\operatorname{Re}(\gamma(\sigma^{n_k}(a))) \geq \operatorname{Re}(\gamma(\omega_{m_k} \dots \omega_{-1})) \quad \text{and} \quad \operatorname{Re}(\gamma(\sigma^{n_k}(b))) \leq \operatorname{Re}(\gamma(\omega_0 \dots \omega_{m'_k})).$$

Thus, the claim follows.

By taking a subsequence, one can assume that $\omega_{-m_k}\omega_{-m_k+1} \dots \omega_{-1} \cdot \omega_0\omega_1 \dots \omega_{m'_k}$ converges to a sequence $\omega \in \mathcal{A}^{\mathbb{Z}}$. Since awb is a subword of a point in Ω_T and $\sigma(\Omega_T) \subseteq \Omega_T$, we obtain that $\omega \in \Omega_T$. Moreover, by construction, $\operatorname{Re}(\gamma_n(\omega)) \geq 0$ for every $n \in \mathbb{Z}$, which proves that $\omega \in \Omega_T$ is a minimal sequence for γ . □

5. Fractals associated with a self-similar i.e.m.

We continue with the notation of the previous section. So $\beta \in \mathbb{C}$ is an eigenvalue of M with $|\beta| > 1$ such that $\beta_0 = \beta/|\beta|$ is not a root of unity. Along this section γ is a fixed eigenvector of M for β .

For $a \in \mathcal{A}$, set

$$\mathcal{S}_a = \{(p_m, c_m, s_m)_{m \geq 1}; \sigma(a) = p_1 c_1 s_1 \text{ and } \sigma(c_m) = p_{m+1} c_{m+1} s_{m+1} \text{ for all } m \geq 1\}.$$

For $x \in \mathcal{S}_a$, write $x = (p_m^x, c_m^x, s_m^x)_{m \geq 1}$ and define functions $z_a, z_a^{(n)} : \mathcal{S}_a \mapsto \mathbb{C}$ for each $a \in \mathcal{A}$ and integer $n \geq 1$ by

$$z_a(x) = \sum_{m \geq 1} \beta^{-m} \gamma(p_m^x) \quad \text{and} \quad z_a^{(n)}(x) = \sum_{m=1}^n \beta^{-m} \gamma(p_m^x).$$

Clearly, the previous functions and the notation below depend on γ . To simplify, we will not include γ in the notation unless it becomes necessary.

Definition 5.1. The fractal associated with $a \in \mathcal{A}$ is the set $\mathfrak{F}_a = \{z_a(x); x \in \mathcal{S}_a\}$. We also define $\mathfrak{F}_a^{(n)} = \{z_a^{(n)}(x); x \in \mathcal{S}_a\}$. We say that $x \in \mathcal{S}_a$ is a representation of $z \in \mathfrak{F}_a$ if $z = z_a(x)$.

Let $a \in \mathcal{A}$. From the definition of \mathcal{S}_a , it is easy to see that, for any $x \in \mathcal{S}_a$, $\sigma^n(a) = \sigma^{n-1}(p_1^x) \dots p_n^x c_n^x s_n^x \dots \sigma^{n-1}(s_1^x)$ for any integer $n \geq 1$. Furthermore, $\mathfrak{F}_a = \bigcup_{n \geq 1} \mathfrak{F}_a^{(n)}$. Indeed, let $x \in \mathcal{S}_a$ and $n \geq 1$. We can ‘truncate’ x in the following way. Let $x|_n$ be the sequence defined by $(p_m^{x|_n}, c_m^{x|_n}, s_m^{x|_n}) = (p_m^x, c_m^x, s_m^x)$ for every $1 \leq m \leq n$. For each $m \geq n + 1$, we proceed inductively by defining $(p_m^{x|_n}, c_m^{x|_n}, s_m^{x|_n}) = (\varepsilon, c_m^{x|_n}, s_m^{x|_n})$ in a way such that $\sigma(c_m^{x|_n}) = c_{m+1}^{x|_n} s_{m+1}^{x|_n}$. It is easy to see that $x|_n \in \mathcal{S}_a$ and that $z_a(x|_n) = z_a^{(n)}(x)$, so $\mathfrak{F}_a^{(n)} \subseteq \mathfrak{F}_a$ and the claim follows. In a similar way, we can construct a point $x \in \mathcal{S}_a$ such that $p_m^x = \varepsilon$ for every $m \geq 1$. Clearly, $z_a(x) = 0$, so $0 \in \mathfrak{F}_a$. Moreover, $\mathfrak{F}_a \neq \{0\}$. In fact, by primitivity of σ and the fact that $\gamma \neq 0$, it is easy to see that, for some integer $n \geq 1$, there exists a prefix w of $\sigma^n(a)$ that satisfies $\gamma(w) \neq 0$. We can then choose $x \in \mathcal{S}_a$ such that $w = \sigma^{n-1}(p_1^x) \dots p_n^x$. Therefore

$$\begin{aligned} z_a^{(n)}(x) &= \sum_{m=1}^n \beta^{-m} \gamma(p_m^x) = \frac{1}{\beta^n} \sum_{m=1}^n \beta^{n-m} \gamma(p_m^x) = \frac{1}{\beta^n} \sum_{m=1}^n \gamma(\sigma^{n-m}(p_m^x)) \\ &= \frac{1}{\beta^n} \gamma(\sigma^{n-1}(p_1^x) \dots p_n^x) = \frac{1}{\beta^n} \gamma(w) \neq 0. \end{aligned} \tag{4}$$

Finally, consider $x \in \mathcal{S}_a$ such that $s_m^x = \varepsilon$ for each integer $m \geq 1$, which can be defined inductively, as before. We obtain that $\sigma^n(a) = \sigma^{n-1}(p_1^x) \dots p_n^x c_n^x$ for each integer $n \geq 1$, so a similar computation to (4) shows that $z_a(x) = \gamma_a \in \mathfrak{F}_a$.

An elementary computation yields that \mathfrak{F}_a is a particular case of a fractal built by projecting stepped lines to an expanding plane, as defined in [ABB11], and it is not necessarily a Rauzy fractal. Thus, the fractal \mathfrak{F}_a is a compact and path-connected subset of the plane. In addition, in [ABB11], the authors study the fractals associated with the cubic Arnoux–Yoccoz i.e.m., proving that they have non-empty interior and that they are somehow related to the so-called tribonacci fractal. In §8, we will find a parametrization

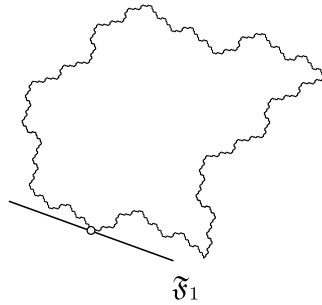


FIGURE 1. The dot indicates an extreme point of one Arnoux–Yoccoz fractal in the direction of the line.

of the boundary of the fractal associated with the cubic Arnoux–Yoccoz map. This map will serve as an example to illustrate our main result and the techniques developed along the article. Although this i.e.m. is not self-similar in the sense we are following here (see §3), Theorem A can be still applied by making some slight modifications.

We are interested in the extreme points of the defined fractals along directions.

Definition 5.2. For $\tau \in \mathbb{S}^1$ and an integer $n \geq 1$, define

$$v_a(\tau) = \min_{z \in \mathfrak{F}_a} \operatorname{Re}(\tau z) \quad \text{and} \quad v_a^{(n)}(\tau) = \min_{z \in \mathfrak{F}_a^{(n)}} \operatorname{Re}(\tau z).$$

We call $E_a(\tau) = \{z \in \mathfrak{F}_a; \operatorname{Re}(\tau z) = v_a(\tau)\}$ the set of extreme points of \mathfrak{F}_a for the direction τ , as shown by Figure 1. Clearly, $v_a(\tau) \leq v_a^{(n)}(\tau) \leq 0$ since $\mathfrak{F}_a^{(n)} \subseteq \mathfrak{F}_a$ and $0 \in \mathfrak{F}_a^{(n)}$.

LEMMA 5.3. *The function $v_a : \mathbb{S}^1 \rightarrow \mathbb{R}$ is continuous for every $a \in \mathcal{A}$.*

Proof. Let $(\tau_n)_{n \geq 1}$ be a sequence in \mathbb{S}^1 that converges to τ . First, assume by contradiction that $u = \liminf_{n \rightarrow \infty} v_a(\tau_n) < v_a(\tau)$. Let $(n_k)_{k \geq 1}$ be a sequence such that $\lim_{k \rightarrow \infty} v_a(\tau_{n_k}) = u$. Let $z_{n_k} \in \mathfrak{F}_a$ such that $\operatorname{Re}(\tau_{n_k} z_{n_k}) = v_a(\tau_{n_k})$. Since \mathfrak{F}_a is compact, we can assume that $\lim_{k \rightarrow \infty} z_{n_k} = z \in \mathfrak{F}_a$. We know that $\lim_{k \rightarrow \infty} \operatorname{Re}(\tau_{n_k} z_{n_k}) = \operatorname{Re}(\tau z)$, so $\operatorname{Re}(\tau z) = u < v_a(\tau)$, which is a contradiction. Now, let $z \in \mathfrak{F}_a$ such that $\operatorname{Re}(\tau z) = v_a(\tau)$. Clearly, $v_a(\tau_n) \leq \operatorname{Re}(\tau_n z)$ and therefore $\limsup_{n \rightarrow \infty} v_a(\tau_n) \leq v_a(\tau)$. So

$$\limsup_{n \rightarrow \infty} v_a(\tau_n) \leq v_a(\tau) \leq \liminf_{n \rightarrow \infty} v_a(\tau_n),$$

which shows that $\lim_{n \rightarrow \infty} v_a(\tau_n) = v_a(\tau)$. □

5.1. *Basic properties of extreme points.* We will now present some important properties of extreme points that will be used to prove the main theorem. We start with a technical definition.

Definition 5.4. Let $w \in \mathcal{A}^*$ be an itinerary by T . A prefix w' of w is said to be a *minimal prefix* for w and γ if

$$\operatorname{Re}(\gamma(w')) = \min\{\operatorname{Re}(\gamma(w'')); w'' \text{ is a prefix of } w\}.$$

LEMMA 5.5. *Let $n \geq 1$ be an integer, let $a \in \mathcal{A}$ and let $\tau \in \mathbb{S}^1$. Let w be a minimal prefix for $\sigma^n(a)$ and the vector $\tau\gamma$. Let $x \in \mathcal{S}_a$ be any sequence that satisfies $w = \sigma^{n-1}(p_1^x) \dots \sigma(p_{n-1}^x)p_n^x$. Then $v_a^{(n)}(\beta_0^n \tau) = \operatorname{Re}(\beta_0^n \tau x_a^{(n)}(x))$.*

Proof. Take $y = (\bar{p}_m, \bar{c}_m, \bar{s}_m)_{m \geq 1} \in \mathcal{S}_a$ and let $w' = \sigma^{n-1}(\bar{p}_1) \dots \sigma(\bar{p}_{n-1})\bar{p}_n$. By hypothesis, $\text{Re}(\tau\gamma(w)) \leq \text{Re}(\tau\gamma(w'))$. By using (3),

$$\gamma(w) = \sum_{m=1}^n \beta^{n-m} \gamma(p_m^x) = \beta^n \sum_{m=1}^n \beta^{-m} \gamma(p_m^x) = \beta^n \varkappa_a^{(n)}(x).$$

Therefore, multiplying by τ , taking the real part and multiplying by $|\beta|^{-n}$,

$$\begin{aligned} \text{Re}(\beta_0^n \tau \varkappa_a^{(n)}(x)) &= |\beta|^{-n} \text{Re}(\tau\gamma(w)) \\ &\leq |\beta|^{-n} \text{Re}(\tau\gamma(w')) = \text{Re}(\beta_0^n \tau \varkappa_a^{(n)}(y)), \end{aligned}$$

where, in the last equality, we have used that the computation for w' is analogous to the one developed for w just above. Since y is arbitrary, the result follows. \square

LEMMA 5.6. (Continuation property) *Let $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$. If $x \in \mathcal{S}_a$ satisfies $\varkappa_a(x) \in E_a(\tau)$, then $\varkappa_{c_1^x}(S(x)) \in E_{c_1^x}(\beta_0^{-1}\tau)$, where S is the left shift map. Moreover,*

$$\text{Re}(\tau \varkappa_a(x)) = \text{Re}(\tau\beta^{-1}\gamma(p_1^x)) + |\beta|^{-1}v_{c_1^x}(\beta_0^{-1}\tau).$$

Similarly, if $x \in \mathcal{S}_a$ satisfies $\varkappa_a(x) \in E_a^{(n)}(\tau)$ for an integer $n \geq 1$, then $\varkappa_{c_1^x}(S(x)) \in E_{c_1^x}^{(n-1)}(\beta_0^{-1}\tau)$ and

$$\text{Re}(\tau \varkappa_a^{(n)}(x)) = \text{Re}(\tau\beta^{-1}\gamma(p_1^x)) + |\beta|^{-1}v_{c_1^x}^{(n-1)}(\beta_0^{-1}\tau).$$

Proof. Let $x \in \mathcal{S}_a$ as in the hypothesis and assume that $\varkappa_{c_1^x}(S(x)) \notin E_{c_1^x}(\beta_0^{-1}\tau)$. Then there exists $y \in \mathcal{S}_{c_1^x}$ such that $\text{Re}(\beta_0^{-1}\tau \varkappa_{c_1^x}(y)) < \text{Re}(\beta_0^{-1}\tau \varkappa_{c_1^x}(S(x)))$. Clearly, $(p_1^x, c_1^x, s_1^x)y \in \mathcal{S}_a$. Therefore, from the identities $\varkappa_a(x) = \beta^{-1}\gamma(p_1^x) + \beta^{-1}\varkappa_{c_1^x}(S(x))$ and $\varkappa_a((p_1^x, c_1^x, s_1^x)y) = \beta^{-1}\gamma(p_1^x) + \beta^{-1}\varkappa_{c_1^x}(y)$, we deduce that

$$\text{Re}(\tau \varkappa_a((p_1^x, c_1^x, s_1^x)y)) < \text{Re}(\tau \varkappa_a(x)),$$

which contradicts the fact that x is an extreme point in $E_a(\tau)$.

The equality

$$\text{Re}(\tau \varkappa_a(x)) = \text{Re}(\tau\beta^{-1}\gamma(p_1^x)) + |\beta|^{-1}v_{c_1^x}(\beta_0^{-1}\tau)$$

follows directly from the fact that $\varkappa_{c_1^x}(S(x)) \in E_{c_1^x}(\beta_0^{-1}\tau)$. The rest of the proof follows analogously. \square

LEMMA 5.7. (Exponential approximation) *Let $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$. For all $n \geq 1$, $|v_a^{(n)}(\tau) - v_a(\tau)| \leq C|\beta|^{-n}$, where $C = \max\{-v_b(\tau); b \in \mathcal{A}, \tau \in \mathbb{S}^1\} < \infty$.*

Proof. Let $x \in \mathcal{S}_a$ with $\varkappa_a(x) \in E_a(\tau)$. Then

$$v_a(\tau) = \text{Re}(\tau \varkappa_a(x)) \leq v_a^{(n)}(\tau) \leq \text{Re}(\tau \varkappa_a^{(n)}(x)).$$

Therefore

$$\begin{aligned} |v_a^{(n)}(\tau) - v_a(\tau)| &= v_a^{(n)}(\tau) - v_a(\tau) \leq -\text{Re}\left(\tau \sum_{m \geq n+1} \beta^{-m} \gamma(p_m^x)\right) \\ &= -|\beta|^{-n}v_{c_n^x}(\beta_0^{-n}\tau), \end{aligned}$$

where, in the last equality, we used the continuation property of Lemma 5.6. \square

LEMMA 5.8. For any $a \in \mathcal{A}$, eventually periodic elements of \mathcal{S}_a do not produce extreme points in \mathfrak{F}_a .

Proof. Let x be an eventually periodic element of \mathcal{S}_a . Then there exist non-negative integers n_0 and q such that, for every $k \geq 0$ and $1 \leq m \leq q$,

$$(p_{n_0+kq+m}^x, c_{n_0+kq+m}^x, s_{n_0+kq+m}^x) = (p_m, c_m, s_m).$$

Assume that $z_a(x)$ is an extreme point for the direction $\tau \in \mathbb{S}^1$. By definition and periodicity, for every $k \geq 1$,

$$\begin{aligned} \operatorname{Re}(\tau z_a(x)) &= \operatorname{Re}\left(\tau \sum_{m=1}^{n_0+kq} \beta^{-m} \gamma(p_m^x)\right) + |\beta|^{-n_0-kq} \operatorname{Re}\left(\beta_0^{-n_0-kq} \tau \sum_{j \geq 1} \beta^{-j} \gamma(p_{n_0+kq+j}^x)\right) \\ &= \operatorname{Re}\left(\tau \sum_{m=1}^{n_0+kq} \beta^{-m} \gamma(p_m^x)\right) + |\beta|^{-n_0-kq} \operatorname{Re}\left(\beta_0^{-n_0-kq} \tau \sum_{j \geq 0} \beta^{-jq} \sum_{m=1}^q \beta^{-m} \gamma(p_m)\right). \end{aligned}$$

Define $z = \sum_{j \geq 0} \beta^{-jq} \sum_{m=1}^q \beta^{-m} \gamma(p_m)$. Then

$$\operatorname{Re}(\tau z_a(x)) = \operatorname{Re}\left(\tau \sum_{m=1}^{n_0+kq} \beta^{-m} \gamma(p_m^x)\right) + |\beta|^{-n_0-kq} \operatorname{Re}(\beta_0^{-n_0-kq} \tau z).$$

By Lemma 5.6, $\operatorname{Re}(\beta_0^{-n_0-kq} \tau z) = v_{c_{n_0+kq}^x}(\beta_0^{-n_0-kq} \tau) = v_{c_q}(\beta_0^{-n_0-kq} \tau)$. Thus $\operatorname{Re}(\beta_0^{-n_0-kq} \tau z) \leq 0$ for any $k \geq 0$. Since $(\beta_0^{-kq})_{k \geq 0}$ is dense in \mathbb{S}^1 , we deduce that $z = 0$ and, consequently, $v_{c_q}(\beta_0^{-n_0-kq} \tau) = 0$ for every $k \geq 0$. Finally, by continuity of v_a (Lemma 5.3) and density, $v_{c_q}(\xi) = 0$ for every $\xi \in \mathbb{S}^1$. Since \mathfrak{F}_{c_q} is not reduced to $\{0\}$, we get a contradiction. \square

We remark that, from the previous lemma, one deduces that $v_a(\tau) < 0$ for any $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$. Indeed, we know that $v_a(\tau) \leq 0$, but if $v_a(\tau) = 0$, then we would have that $0 \in \mathfrak{F}_a$ is an extreme point, which contradicts the previous lemma.

5.2. The set Ψ and extreme points.

Definition 5.9. Let $a \in \mathcal{A}$ and $(p, c, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*$ such that $\sigma(a) = pcs$. We denote by $\mathcal{S}_{a,(p,c,s)}$ the set of sequences $x \in \mathcal{S}_a$ starting with $x_1 = (p, c, s)$. We define

$$\mathfrak{F}_{a,(p,c,s)} = \{z_a(x); x \in \mathcal{S}_{a,(p,c,s)}\} = \beta^{-1}(\gamma(p) + \mathfrak{F}_c) \subseteq \mathfrak{F}_a.$$

For a direction $\tau \in \mathbb{S}^1$, we define $v_{a,(p,c,s)}(\tau) = \min\{\operatorname{Re}(\tau z); z \in \mathfrak{F}_{a,(p,c,s)}\}$ and $E_{a,(p,c,s)}(\tau) = E_a(\tau) \cap \mathfrak{F}_{a,(p,c,s)}$.

It is easy to see that $\mathfrak{F}_a = \bigcup \mathfrak{F}_{a,(p,c,s)}$ and that $v_a(\tau) = \min\{v_{a,(p,c,s)}(\tau)\}$, where the union and the minimum are taken over the $(p, c, s) \in \mathcal{A} \times \mathcal{A}^* \times \mathcal{A}$ such that $\sigma(a) = pcs$.

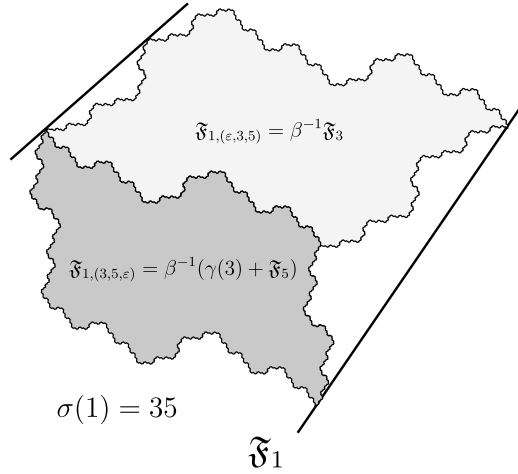


FIGURE 2. In the cubic Arnoux–Yoccoz map we show directions in Ψ_1 .

Definition 5.10. For each $a \in \mathcal{A}$, we define Ψ_a as the set of directions $\tau \in \mathbb{S}^1$ for which there exist distinct decompositions $\sigma(a) = pcs = \bar{p}\bar{c}\bar{s}$ in $\mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*$ with $E_{a,(p,c,s)}(\tau) \neq E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ and $v_a(\tau) = v_{a,(p,c,s)}(\tau) = v_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$: that is, Ψ_a is the set of directions for which there exist distinct extreme points in \mathfrak{F}_a belonging to distinct subfractals $\mathfrak{F}_{a,(p,c,s)}$ and $\mathfrak{F}_{a,(\bar{p},\bar{c},\bar{s})}$ (see Figure 2). Put $\Psi = \bigcup_{a \in \mathcal{A}} \Psi_a$.

LEMMA 5.11. *For each $a \in \mathcal{A}$, the set Ψ_a is at most countable.*

Proof. Let $a \in \mathcal{A}$. We will prove that the set $\Psi'_a = \{\tau \in \mathbb{S}^1; |E_a(\tau)| \geq 2\}$, which clearly contains Ψ_a , is at most countable.

Let $Z \subseteq \mathbb{C}$ and $\tau \in \mathbb{S}^1$. We define the set of extreme points of Z for the direction τ as $E_Z(\tau) = \{z \in Z; \operatorname{Re}(\tau z) = \min\{\operatorname{Re}(\tau z'); z' \in Z\}\}$. Similarly, we define the set of directions in \mathbb{S}^1 for which there exist at least two distinct extreme points as $\Psi'_Z(\tau) = \{\tau \in \mathbb{S}^1; |E_Z(\tau)| \geq 2\}$. We denote the convex hull of Z by $\operatorname{conv}(Z)$.

We know that $E_a(\tau) = E_{\mathfrak{F}_a}(\tau)$, $\Psi'_a(\tau) = \Psi'_{\mathfrak{F}_a}(\tau)$, $E_{\operatorname{conv}(\mathfrak{F}_a)}(\tau) = \operatorname{conv}(E_a(\tau))$ and that $\Psi'_{\operatorname{conv}(\mathfrak{F}_a)} = \Psi'_a$, since $|E_a(\tau)| \geq 2$ if and only if $|\operatorname{conv}(E_a(\tau))| \geq 2$.

Since $\operatorname{conv}(\mathfrak{F}_a)$ is convex and compact, it is either a point, a line segment or homeomorphic to a closed disk. Since, in the first two cases, $\Psi'_{\operatorname{conv}(\mathfrak{F}_a)}$ is finite, we can assume that the third case holds. Let $\Phi : \mathbb{S}^1 \rightarrow \partial\operatorname{conv}(\mathfrak{F}_a)$ be an homeomorphism between \mathbb{S}^1 and the boundary of $\operatorname{conv}(\mathfrak{F}_a)$. We know that $\tau \in \Psi'_a$ if and only if the map $\operatorname{Re}(\tau\Phi) : \mathbb{S}^1 \rightarrow \mathbb{R}$ is constant on an open interval of \mathbb{S}^1 . Since such open intervals must be disjoint for distinct τ and a family of disjoint open subsets of \mathbb{S}^1 is at most countable, we obtain that Ψ'_a is at most countable. \square

5.3. *Minimal sequences for γ gamma and limit extreme points.* Points in \mathcal{S}_a may be constructed by ‘reversing’ the prefix–suffix decomposition of elements in Ω_T . Moreover, minimal sequences in Ω_T for γ produce, by reversing the prefix–suffix decomposition, extreme points for any direction. We will make this statement precise in this section.

Consider a minimal sequence $\omega \in \Omega_T$ for γ and let $(p_m, c_m, s_m)_{m \geq 1}$ be its prefix–suffix decomposition. Fix some $\tau \in \mathbb{S}^1$ and recall that $\beta_0 = \beta/|\beta|$ is not a root of unity. Let $(n_k)_{k \geq 1}$ be an increasing sequence of integers such that $\beta_0^{n_k} \rightarrow \tau \in \mathbb{S}^1$ and $c_{n_k} = a \in \mathcal{A}$ for every $k \geq 1$. For each $k \geq 1$, consider a point $x_{n_k} \in \mathcal{S}_a$ starting with

$$(p_{n_k-1}, c_{n_k-1}, s_{n_k-1})(p_{n_k-2c_{n_k}-2}, s_{n_k-2}) \dots (p_0, c_0, s_0)$$

and such that $p_m^{x_{n_k}} = \varepsilon$ for any $m \geq n_k + 1$. This is a sequence obtained by reversing the indices of finitely many elements of the prefix–suffix decomposition of ω . Assume that $(x_{n_k})_{k \geq 1}$ converges to $x_\infty \in \mathcal{S}_a$. We will show that x_∞ is the representation of an extreme point in \mathfrak{F}_a for the direction τ . In fact, we can prove a little more. For this, we need to introduce the following concept.

Definition 5.12. A limit extreme point is an extreme point that has a representation $x \in \mathcal{S}_a$ with the following property. For each $j \geq 1$, there exists $y_j \in \mathcal{S}_{a_j}$ for some $a_j \in \mathcal{A}$ such that:

- (i) y_j is a representation of an extreme point for the direction $\beta_0^j \tau$;
- (ii) $c_j^{y_j} = a$; and
- (iii) $x = S^j(y_j)$, where S is the left shift map on $\bigcup_{a \in \mathcal{A}} \mathcal{S}_a$.

The set of limit extreme points will be denoted by $E_a^*(\tau)$.

LEMMA 5.13. Any limit $x_\infty \in \mathcal{S}_a$, constructed as above, is a representation of a limit extreme point in $E_a^*(\tau)$.

Proof. Assume that the sequence constructed as before $(x_{n_k})_{k \geq 1}$ converges to $x_\infty \in \mathcal{S}_a$. For each integer $j \geq 0$, let $y_{n_k}^j$ be the sequence in $\mathcal{S}_{c_{n_k+j}}$ starting with

$$(p_{n_k+j-1}, c_{n_k+j-1}, s_{n_k+j-1})(p_{n_k+j-2c_{n_k+j}-2}, s_{n_k+j-2}) \dots (p_0, c_0, s_0),$$

and such that $p_m^{y_{n_k}^j} = \varepsilon$ for every $m \geq n_k + j + 1$: that is, $y_{n_k}^j$ is the sequence obtained by reversing the first $n_k + j$ elements of the prefix–suffix decomposition $(p_m, c_m, s_m)_{m \geq 0}$ of ω . Clearly, $S^j(y_{n_k}^j) = x_{n_k}$. By taking a subsequence, we may assume that $c_{n_k+j} = a_j$ for every $k \geq 1$ and that $y_{n_k}^j$ converges to y_j when $k \rightarrow \infty$. Clearly, $c_j^{y_j} = a$ and $S^j(y_j) = x_\infty$.

We claim that both x_∞ and y_j are representations of extreme points for the letters a and a_j and the directions τ and $\beta_0^j \tau$, respectively. Indeed, since ω is a minimal sequence for γ , by Lemma 4.3 x_{n_k} satisfies the hypotheses of Lemma 5.5 (taking $\tau = 1$ in the lemma), so $v_a^{(n_k)}(\beta_0^{n_k}) = \text{Re}(\beta_0^{n_k} z_a(x_{n_k}))$. Thus, for every $x \in \mathcal{S}_a$, $\text{Re}(\beta_0^{n_k} z_a(x_{n_k})) \leq \text{Re}(\beta_0^{n_k} z_a^{(n_k)}(x))$. Finally, taking the limit when $k \rightarrow \infty$, $\text{Re}(\tau z_a(x_\infty)) \leq \text{Re}(\tau z_a(x))$ for any $x \in \mathcal{S}_a$, so x_∞ is an extreme point in $E_a(\tau)$. The proof for y_j is similar. \square

The previous lemma gives us the following important corollary.

COROLLARY 5.14. For every $\tau \in \mathbb{S}^1$, there exists $a \in \mathcal{A}$ such that $E_a^*(\tau)$ is non-empty. Moreover, if $E_a^*(\tau)$ is non-empty for some $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$, then $E_a^*(\tau) = E_a(\tau)$.

Proof. Take a minimal sequence $\omega \in \Omega_T$ and $\tau \in \mathbb{S}^1$. By reversing the prefix–suffix decomposition of ω , as above, on an appropriate sequence $(n_k)_{k \geq 1}$ such that $\beta_0^{n_k} \rightarrow \tau$, we

obtain by previous lemma that $E_a^*(\tau) \neq \emptyset$ for some $a \in \mathcal{A}$. Now, if $E_a^*(\tau)$ is non-empty for some $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$, let $x \in \mathcal{S}_a$ be a representation of a limit extreme point in $E_a^*(\tau)$, for every $j \geq 1$ let $y_j \in \mathcal{S}_{a_j}$ be the sequence from the definition of limit extreme point and let $x' \in \mathcal{S}_a$ be the representation of some extreme point in $E_a(\tau)$. For every $j \geq 1$, we define $y'_j \in \mathcal{S}_a$ to coincide with y_j in its first j coordinates and with x' in the rest of its coordinates. A direct computation shows that $y'_j \in \mathcal{S}_{a_j}$ and that it is the representation of an extreme point in $E_{a_j}(\beta_0^j \tau)$ such that $S^j(y'_j) = x'$ and $c_j^{y'_j} = a$. \square

Even though, for every $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$, $E_a(\tau)$ is non-empty, in general, $E_a^*(\tau)$ may be empty. Indeed, the Arnoux–Yoccoz fractals contain extreme points that are not limit extreme points.

Another interesting consequence of Lemma 5.13 is the following lemma.

LEMMA 5.15. *If $\omega \in \Omega_T$ has an eventually periodic prefix–suffix decomposition, then it cannot be minimal for any eigenvector γ associated with β .*

Proof. If it were, then, using the reversing procedure described above Lemma 5.13, we could construct a limit extreme point with periodic representation. This contradicts Lemma 5.8. \square

6. Unique representation property

We continue with the notation of the previous sections. So T is a self-similar i.e.m., which is self-induced on the interval $[0, \alpha)$, and β is an eigenvalue of M with $|\beta| > 1$ such that $\beta/|\beta|$ is not a root of unity. Consider an eigenvector γ of M associated to β . Recall that $x \in \mathcal{S}_a$ is a representation of $z \in \mathfrak{F}_a$ if $z = \varkappa_a(x)$.

Definition 6.1. We say that T satisfies the *unique representation property* (u.r.p.) for β and the eigenvector γ if every extreme point of the associated fractals has a unique representation. We say that T satisfies the *weak unique representation property* (weak u.r.p.) for β and the eigenvector γ if every limit extreme point of the associated fractals has a unique representation.

By Corollary 5.14, if T satisfies the weak u.r.p. for β and the eigenvector γ , then, for all $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$ such that $E_a^*(\tau)$ is non-empty, every extreme point of \mathfrak{F}_a for the direction τ has a unique representation. Indeed, in this case, $E_a(\tau) = E_a^*(\tau)$. In particular, if $\sigma(a) = pcs = \bar{p}\bar{c}\bar{s}$ for distinct decompositions in $\mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*$, then $E_{a,(p,c,s)}(\tau) \cap E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ is empty.

Before continuing, we need to comment on the dependence of the previous concepts on the eigenvector γ . Until now, we have fixed an eigenvector γ associated with β and defined the fractal set \mathfrak{F}_a , its extreme and limit extreme points for a given direction $E_a(\tau)$ and $E_a^*(\tau)$, and the directions with extreme points in distinct subfractals Ψ_a . These objects clearly depend on the choice of γ . We will temporarily make this dependence explicit by writing $\mathfrak{F}_a(\gamma)$, $E_a(\gamma, \tau)$, $E_a^*(\gamma, \tau)$ and $\Psi_a(\gamma)$, respectively. Our main concern is to see how these concepts vary in the one-dimensional space generated by γ . The following relations follow easily from definitions.

LEMMA 6.2. Let $\gamma, z\gamma \in \mathbb{C}^d$ be eigenvectors associated with β , with $z \in \mathbb{C} \setminus \{0\}$. Let $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$. Then:

- (i) $\mathfrak{F}_a(z\gamma) = z\mathfrak{F}_a(\gamma)$;
- (ii) $E_a(z\gamma, \tau) = zE_a(\gamma, z_0\tau)$, $E_a^*(z\gamma, \tau) = zE_a^*(\gamma, z_0\tau)$; and
- (iii) $\Psi_a(z\gamma) = z_0^{-1}\Psi_a(\gamma)$,

where $z_0 = z/|z|$.

From this lemma, we deduce that if the (weak) u.r.p. holds for an eigenvector γ associated to β , then it holds for $z\gamma$ for all $z \in \mathbb{C} \setminus \{0\}$. If the eigenvalue β is simple, then this condition is independent of the choice of the eigenvector, so, in this case, we can speak of the (weak) u.r.p. for β . We believe that there are some natural algebraic conditions that imply this fact.

The weak u.r.p. is, in some sense, analogous to the algebraic condition considered in the case when β is real. In fact, restating our definitions for the real case, [BHM10, Lemma 19] is equivalent to the weak u.r.p.

We will prove, in §8, that the cubic Arnoux–Yoccoz map satisfies the u.r.p. for some simple non-real eigenvalue β .

7. Proof of the main theorem

We restate our main theorem for completeness. We continue with the notation of the three previous sections.

THEOREM A. Let T be a self-similar i.e.m. Assume that M has an eigenvalue β with $|\beta| > 1$ such that $\beta/|\beta|$ is not a root of unity, and that there exists an eigenvector Γ for β such that T has the u.r.p. for β and Γ . Then, for almost every γ in the complex subspace generated by Γ , $\exp(-\text{Re}(\gamma))$ can be realized as the slope vector of an affine i.e.m. that is semi-conjugate with T and has wandering intervals.

Of course, an affine i.e.m. with wandering intervals cannot be conjugate with T , so the theorem asserts the existence of an affine i.e.m. that is strictly semi-conjugate with T . We remark that each affine i.e.m. is uniquely ergodic, since T is.

The weak u.r.p. is sufficient to prove Theorem A. In fact, the proof of this theorem relies completely on Lemma 7.8, which is also true under the weak u.r.p. as we point out at the end of the proof.

The (weak) u.r.p. may seem technical and difficult to check for a specific map. In the next section, we prove that this property is satisfied by the cubic Arnoux–Yoccoz i.e.m. (A–Y i.e.m.).

As discussed in §2, using the general strategy of [CG97], the proof of Theorem A is a consequence of the following more technical statement that we prove in the next section.

THEOREM 7.1. Under the same assumptions of Theorem A, there exists a number $\rho > 0$ such that, for almost all γ in the complex subspace generated by Γ , every minimal sequence $\omega \in \Omega_T$ for γ satisfies

$$\liminf_{n \rightarrow \infty} \frac{\text{Re}(\gamma_n(\omega))}{n^\rho} > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\text{Re}(\gamma_{-n}(\omega))}{n^\rho} > 0. \tag{5}$$

7.1. Proof of Theorem 7.1.

7.1.1. Differentiability of v_a . We start by describing some differentiability properties of the map $v_a : \mathbb{S}^1 \rightarrow \mathbb{R}$.

LEMMA 7.2. Let $a \in \mathcal{A}$ and $\tau \in \mathbb{S}^1$.

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{v_a(\tau \exp(it)) - v_a(\tau)}{t} &= -\text{Im}(\tau e_a^+(\tau)), \\ \lim_{t \rightarrow 0^-} \frac{v_a(\tau \exp(it)) - v_a(\tau)}{t} &= -\text{Im}(\tau e_a^-(\tau)), \end{aligned}$$

where $e_a^+(\tau), e_a^-(\tau)$ are the points of $E_a(\tau)$ such that $z \mapsto \text{Im}(\tau z)$ is maximal and minimal, respectively.

Proof. We will only prove the first equality, since the second one is analogous. Let $z = e_a^+(\tau), z_t \in E_a(\tau \exp(it))$ and write $\tau z = r \exp(i\theta)$ and $\tau z_t = r_t \exp(i\theta_t)$ for $r, r_t > 0, \theta, \theta_t \in [0, 2\pi)$. We will assume that $0 < t < \pi/2$.

Since $z = e_a^+(\tau)$ and $z_t \in E_a(\tau \exp(it))$, we know that $v_a(\tau) = \text{Re}(\tau z) = r \cos(\theta), v_a(\tau \exp(it)) = \text{Re}(\tau \exp(it)z_t) = r_t \cos(\theta_t + t)$ and $-\text{Im}(\tau e_a^+(\tau)) = -r \sin(\theta)$. Thus, we have to prove that

$$\lim_{t \rightarrow 0^+} \frac{r_t \cos(\theta_t + t) - r \cos(\theta)}{t} = -r \sin(\theta).$$

Since $z_t \in E_a(\tau \exp(it))$,

$$r \cos(\theta + t) = \text{Re}(\tau \exp(it)z) \geq \text{Re}(\tau \exp(it)z_t) = r_t \cos(\theta_t + t).$$

Hence

$$\begin{aligned} 0 &\geq r_t \cos(\theta_t + t) - r \cos(\theta + t) \\ &= (r_t \cos(\theta_t) - r \cos(\theta)) \cos(t) + (r \sin(\theta) - r_t \sin(\theta_t)) \sin(t). \end{aligned} \tag{6}$$

Moreover, since $z \in E_a(\tau), r \cos(\theta) = \text{Re}(\tau z) \leq \text{Re}(\tau z_t) = r_t \cos(\theta_t)$. We conclude that $(r_t \cos(\theta_t) - r \cos(\theta)) \cos(t) \geq 0$ for t small enough. Therefore, from $\sin(t) > 0$ and (6), we know that $r \sin(\theta) - r_t \sin(\theta_t)$ cannot be positive. This proves that $\text{Im}(\tau z) = r \sin(\theta) \leq r_t \sin(\theta_t) = \text{Im}(\tau z_t)$.

We claim that $\lim_{t \rightarrow 0^+} \text{Im}(\tau z_t) = \text{Im}(\tau z)$. Indeed, fix a real $\eta > 0$ and consider

$$u(t) = \min\{\text{Re}(\tau \exp(it)z'); z' \in \mathfrak{F}_a, \text{Im}(\tau z') \geq \text{Im}(\tau z) + \eta\}.$$

Since z is in $E_a(\tau)$ and is chosen with the maximal possible value for $\text{Im}(\tau z), \text{Re}(\tau z) < u(0)$. The map $u : [0, \pi/2) \rightarrow \mathbb{R}$ is continuous, so there exists $t_0 > 0$ such that $\text{Re}(\tau \exp(it)z) < \text{Re}(\tau \exp(it)z')$ for every $0 \leq t < t_0$ and $z' \in \mathfrak{F}_a$ such that $\text{Im}(\tau z') \geq \text{Im}(\tau z) + \eta$. Therefore, if $0 \leq t < t_0$ and $\text{Im}(\tau z_t) \geq \text{Im}(\tau z) + \eta$, then $\text{Re}(\tau \exp(it)z) < \text{Re}(\tau \exp(it)z_t)$, which contradicts the fact that $z_t \in E_a(\tau \exp(it))$. This shows that $\text{Im}(\tau z_t) < \text{Im}(\tau z) + \eta$ for $0 \leq t < t_0$, so we get the desired result by taking $\eta \rightarrow 0$.

The previous claim and (6) imply that

$$\lim_{t \rightarrow 0^+} \frac{r_t \cos(\theta_t + t) - r \cos(\theta + t)}{t} \geq \lim_{t \rightarrow 0^+} (\text{Im}(\tau z_t) - \text{Im}(\tau z)) \frac{\sin(t)}{t} = 0.$$

Finally, we write

$$\frac{v_a(\tau \exp(it)) - v_a(\tau)}{t} = \frac{r_t \cos(\theta_t + t) - r \cos(\theta + t)}{t} + \frac{r \cos(\theta + t) - r \cos(\theta)}{t},$$

and the result follows by taking $t \rightarrow 0^+$. □

7.1.2. *Good directions and good eigenvectors.* In order to prove the convergences in (5), we need to control the velocity at which β_0^n approaches a number τ in \mathbb{S}^1 . In what follows, $\llbracket \tau - \tau' \rrbracket$ is the natural distance between τ and τ' in \mathbb{S}^1 . We fix an eigenvector Γ of M for β . We will be interested in the complex subspace generated by Γ .

Definition 7.3. A direction $\xi \in \mathbb{S}^1$ is *good* for Γ if, for every constant $A > 1$ and every $\tau \in \Psi(\Gamma) = \bigcup_{a \in \mathcal{A}} \Psi_a(\Gamma)$, $\liminf_{n \rightarrow \infty} A^n \llbracket \tau - \beta_0^n \xi \rrbracket > 0$.

As shown by the next lemma, this property is generic.

LEMMA 7.4. *Almost every direction $\xi \in \mathbb{S}^1$ is good for Γ .*

Proof. Let $A > 1$ and $\tau \in \mathbb{S}^1$. We will first prove that

$$K(A, \tau) = \left\{ \xi \in \mathbb{S}^1; \liminf_{n \rightarrow \infty} A^n \llbracket \tau - \beta_0^n \xi \rrbracket > 0 \right\}$$

has full Lebesgue measure. Consider the sets $B_n = \{\xi \in \mathbb{S}^1; \llbracket \tau - \beta_0^n \xi \rrbracket < A^{-n}\}$ for an integer $n \geq 0$. By the Borel–Cantelli lemma, the Lebesgue measure of $\limsup_{n \rightarrow \infty} B_n$ is zero: that is, the set of ξ which belong to infinitely many of the B_n has Lebesgue measure zero. This implies that, for a typical $\xi \in \mathbb{S}^1$, there exists some $N \geq 1$ such that $A^n \llbracket \tau - \beta_0^n \xi \rrbracket \geq 1$ if $n \geq N$. This proves the claim.

Now, since, by Lemma 5.11, $\Psi(\Gamma)$ is at most countable, the intersection $K(A) = \bigcap_{\tau \in \Psi(\Gamma)} K(A, \tau)$ also has full Lebesgue measure. Finally, the intersection $K = \bigcap_{n \geq 1} K(1 + 1/n)$ has full Lebesgue measure and it is easy to see that every element of K is a good direction. □

We can now define the eigenvectors for which our main result is valid.

Definition 7.5. An eigenvector γ of M associated with β is a *good eigenvector* if, for every $A > 1$ and every $\tau \in \bigcup_{a \in \mathcal{A}} \Psi_a(\gamma)$, $\liminf_{n \rightarrow \infty} A^n \llbracket \tau - \beta_0^n \rrbracket > 0$.

If $\xi \in \mathbb{S}^1$ is a good direction for Γ , then $\xi\Gamma$ is a good eigenvector by Lemma 6.2. Therefore, by Lemma 7.4, we conclude the following lemma.

LEMMA 7.6. *If Γ is an eigenvector of M for β , then almost every vector of the complex subspace generated by Γ is good.*

7.1.3. *Convergence in Theorem 7.1.* Assume the existence of an eigenvector Γ for β such that T has the (weak) u.r.p. for β and Γ . We show the convergence in (5) for the set of good eigenvectors in the subspace generated by Γ , which has full measure by the previous lemma. Fix a good eigenvector γ . As discussed at the end of §6, T also has the (weak) u.r.p. for β and γ .

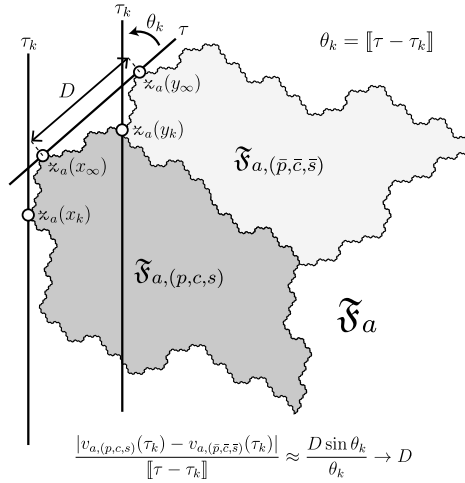


FIGURE 3. Illustration of the proof of Lemma 7.7.

In the rest of the section, the sets $\mathfrak{F}_a, E_a, \Psi_a$, etc., are computed with respect to γ .

The proof of the next lemma and proposition closes the proof of Theorem 7.1 and thus of Theorem A, as explained in §2.

LEMMA 7.7. Let $a \in \mathcal{A}$ and $\tau \in \Psi_a$. Assume that $(p, c, s), (\bar{p}, \bar{c}, \bar{s})$ are such that $E_{a,(p,c,s)}(\tau)$ and $E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ are non-empty but $E_{a,(p,c,s)}(\tau) \cap E_{a,(\bar{p},\bar{c},\bar{s})}(\tau) = \emptyset$. Put

$$D = \min\{|z - z'|; z \in E_{a,(p,c,s)}(\tau), z' \in E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)\}.$$

We know that $D > 0$ and, if $(\tau_k)_{k \geq 1}$ is a sequence in \mathbb{S}^1 such that $\tau_k \rightarrow \tau$ when $k \rightarrow \infty$, then

$$\liminf_{k \rightarrow \infty} \frac{|v_{a,(p,c,s)}(\tau_k) - v_{a,(\bar{p},\bar{c},\bar{s})}(\tau_k)|}{\|\tau - \tau_k\|} \geq D.$$

Proof. See Figure 3 for an insight into the proof, which is, in fact, a little technical.

The property $D > 0$ is a consequence of the fact that $E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ and $E_{a,(p,c,s)}(\tau)$ are non-empty, disjoint and compact.

Let $(x_k)_{k \geq 1}, (y_k)_{k \geq 1}$ be sequences in $\mathcal{S}_{a,(p,c,s)}$ and $\mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$, respectively, such that $x_k \rightarrow x_\infty$ and $y_k \rightarrow y_\infty$ when $k \rightarrow \infty$ for some sequences in $x_\infty \in \mathcal{S}_{a,(p,c,s)}$ and $y_\infty \in \mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$ and

$$v_{a,(p,c,s)}(\tau_k) = \text{Re}(\tau_k z_a(x_k)), \quad v_{a,(\bar{p},\bar{c},\bar{s})}(\tau_k) = \text{Re}(\tau_k z_a(y_k))$$

for all $k \geq 1$. We remark that $z_a(x_k)$ and $z_a(y_k)$ attain the minimum for the direction τ_k and the subfractals $\mathfrak{F}_{a,(p,c,s)}$ and $\mathfrak{F}_{a,(\bar{p},\bar{c},\bar{s})}$, respectively, but not necessarily for \mathfrak{F}_a , as Figure 3 illustrates.

By continuity and the fact that $E_{a,(p,c,s)}(\tau)$ and $E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ are non-empty by hypotheses, $z_a(x_\infty) \in E_{a,(p,c,s)}(\tau)$ and $z_a(y_\infty) \in E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$. Therefore

$$v_a(\tau) = \text{Re}(\tau z_a(x_\infty)) = \text{Re}(\tau z_a(y_\infty)). \tag{7}$$

From the definition of D ,

$$|\operatorname{Im}(\tau(z_a(x_\infty) - z_a(y_\infty)))| \geq D. \tag{8}$$

Since $x_k \rightarrow x_\infty$ and $y_k \rightarrow y_\infty$, there exists an increasing sequence $(n_k)_{k \geq 1}$ such that $(p_m^{x_k}, c_m^{x_k}, s_m^{x_k}) = (p_m^{x_\infty}, c_m^{x_\infty}, s_m^{x_\infty})$ and $(p_m^{y_k}, c_m^{y_k}, s_m^{y_k}) = (p_m^{y_\infty}, c_m^{y_\infty}, s_m^{y_\infty})$ for every $1 \leq m \leq n_k$. Without loss of generality, we may assume that $c_{n_k}^{x_k} = c_{n_k}^{x_\infty} = b$ and that $c_{n_k}^{y_k} = c_{n_k}^{y_\infty} = \bar{b}$ for every $k \geq 1$.

Let S be the left shift in $\bigcup_{a \in \mathcal{A}} \mathcal{S}_a$. Recall that, from the continuation property in Lemma 5.6, if $x \in \mathcal{S}_a$ represents an extreme point for the direction τ , then $S^m(x) \in \mathcal{S}_{c_m^x}$ represents an extreme point for the direction $\beta_0^{-m} \tau$, for every integer $m \geq 0$. Consequently, $v_b(\beta_0^{-n_k} \tau_k) = \operatorname{Re}(\beta_0^{-n_k} \tau_k z_b(S^{n_k}(x_k)))$ and $v_b(\beta_0^{-n_k} \tau) = \operatorname{Re}(\beta_0^{-n_k} \tau z_b(S^{n_k}(x_\infty)))$ for every $k \geq 1$.

Therefore, we can write

$$\begin{aligned} \operatorname{Re}(\tau z_a(x_\infty)) - \operatorname{Re}(\tau_k z_a(x_k)) &= \operatorname{Re}((\tau - \tau_k)(\beta^{-1} \gamma(p_1^{x_\infty}) + \dots + \beta^{-n_k} \gamma(p_{n_k}^{x_\infty})) \\ &\quad + |\beta|^{-n_k} (v_b(\beta_0^{-n_k} \tau) - v_b(\beta_0^{-n_k} \tau_k))). \end{aligned} \tag{9}$$

Analogously,

$$\begin{aligned} \operatorname{Re}(\tau z_a(y_\infty)) - \operatorname{Re}(\tau_k z_a(y_k)) &= \operatorname{Re}((\tau - \tau_k)(\beta^{-1} \gamma(p_1^{y_\infty}) + \dots + \beta^{-n_k} \gamma(p_{n_k}^{y_\infty})) \\ &\quad + |\beta|^{-n_k} (v_{\bar{b}}(\beta_0^{-n_k} \tau) - v_{\bar{b}}(\beta_0^{-n_k} \tau_k))). \end{aligned} \tag{10}$$

Thus, by taking (10)–(9), multiplying by $\llbracket \tau - \tau_k \rrbracket^{-1}$ and using (7),

$$\begin{aligned} \frac{v_{a,(p,c,s)}(\tau_k) - v_{a,(\bar{p},\bar{c},\bar{s})}(\tau_k)}{\llbracket \tau - \tau_k \rrbracket} &= \operatorname{Re} \left(\frac{\tau - \tau_k}{\llbracket \tau - \tau_k \rrbracket} (z_a^{(n_k)}(y_\infty) - z_a^{(n_k)}(x_\infty)) \right) \\ &\quad - |\beta|^{-n_k} \left(\frac{v_b(\beta_0^{-n_k} \tau) - v_b(\beta_0^{-n_k} \tau_k)}{\llbracket \tau - \tau_k \rrbracket} \right) \\ &\quad + |\beta|^{-n_k} \left(\frac{v_{\bar{b}}(\beta_0^{-n_k} \tau) - v_{\bar{b}}(\beta_0^{-n_k} \tau_k)}{\llbracket \tau - \tau_k \rrbracket} \right). \end{aligned} \tag{11}$$

Since $\llbracket \beta_0^{-n_k} \tau - \beta_0^{-n_k} \tau_k \rrbracket = \llbracket \tau - \tau_k \rrbracket$, Lemma 7.2 implies that the quotients

$$\frac{v_b(\beta_0^{-n_k} \tau) - v_b(\beta_0^{-n_k} \tau_k)}{\llbracket \tau - \tau_k \rrbracket} \quad \text{and} \quad \frac{v_{\bar{b}}(\beta_0^{-n_k} \tau) - v_{\bar{b}}(\beta_0^{-n_k} \tau_k)}{\llbracket \tau - \tau_k \rrbracket}$$

remain bounded for every $k \geq 1$, so the last two terms in the previous equality go to zero when $k \rightarrow \infty$. Now, if τ, τ' belong to \mathbb{S}^1 , then

$$\lim_{\tau' \rightarrow \tau} \frac{\tau - \tau'}{\llbracket \tau - \tau' \rrbracket} = i \tau$$

and, therefore,

$$\operatorname{Re} \left(\frac{\tau - \tau_k}{\llbracket \tau - \tau_k \rrbracket} (z_a^{(n_k)}(y_\infty) - z_a^{(n_k)}(x_\infty)) \right)$$

converges to $\operatorname{Re}(i \tau (z_a(y_\infty) - z_a(x_\infty)))$ when $k \rightarrow \infty$, which is the imaginary part of $\tau (z_a(y_\infty) - z_a(x_\infty))$. Finally, when $k \rightarrow \infty$,

$$\frac{|v_{a,(p,c,s)}(\tau_k) - v_{a,(\bar{p},\bar{c},\bar{s})}(\tau_k)|}{\llbracket \tau - \tau_k \rrbracket} \rightarrow |\operatorname{Im}(\tau(z_a(x_\infty) - z_a(y_\infty)))| \geq D,$$

where, in the last inequality, we have used (8). □

PROPOSITION 7.8. Assume the hypotheses of Theorem A. Consider $\omega \in \Omega_T$, a minimal sequence for γ . Let $\eta > 0$ with $|\beta| - \eta > 1$ and let $\rho = \log(|\beta| - \eta) / \log(\alpha^{-1} + \eta) > 0$. Then

$$\liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_n(\omega))}{n^\rho} > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\operatorname{Re}(\gamma_{-n}(\omega))}{n^\rho} > 0. \tag{12}$$

Proof. We will only prove the first inequality, since the other one is similar.

We denote the prefix–suffix decomposition of ω by $(p_m^\omega, c_m^\omega, s_m^\omega)_{m \geq 0}$. Let us assume that there exists an increasing sequence of positive integers $(n_k)_{k \geq 1}$ such that

$$\lim_{k \rightarrow \infty} \frac{\operatorname{Re}(\gamma_{n_k}(\omega))}{n_k^\rho} = 0. \tag{13}$$

Let $(p_m^{\omega_k}, c_m^{\omega_k}, s_m^{\omega_k})_{m \geq 0}$ be the prefix–suffix decomposition of $\omega_k = S^{n_k}(\omega)$, where, as usual, S is the left shift map on the corresponding subshift. We start by showing that ω must have infinitely many non-empty suffixes in its prefix–suffix decomposition. Indeed, assume by contradiction that $s_{n_0+m}^\omega = \varepsilon$ for some integer $n_0 \geq 0$ and every $m \geq 0$. We will show that $(p_m^\omega, c_m^\omega, s_m^\omega)_{m \geq 0}$ is eventually periodic, which contradicts Lemma 5.15. We know that $\sigma(c_{n_0+m+1}^\omega) = p_{n_0+m}^\omega c_{n_0+m}^\omega$ for every $m \geq 0$. Then, for every $m \geq 0$, the value of $c_{n_0+m+1}^\omega$ determines a unique possible value for $p_{n_0+m}^\omega$ and $c_{n_0+m}^\omega$. By induction, it is easy to see that $(p_m^\omega, c_m^\omega, s_m^\omega)_{m \geq n_0}$ is periodic.

Let N_k be the first integer such that $(p_m^{\omega_k}, c_m^{\omega_k}, s_m^{\omega_k})_{m \geq N_k} = (p_m^{\omega_k}, c_m^{\omega_k}, s_m^{\omega_k})_{m \geq N_k}$. By taking a subsequence, we can assume that $(N_k)_{k \geq 1}$ is an increasing sequence of integers. Moreover, we may assume that, for $k \geq 1$:

- (i) $c_{N_k}^\omega = c_{N_k}^{\omega_k} = a$;
- (ii) $(p_{N_k-1}^\omega, c_{N_k-1}^\omega, s_{N_k-1}^\omega) = (p, c, s)$;
- (iii) $(p_{N_k-1}^{\omega_k}, c_{N_k-1}^{\omega_k}, s_{N_k-1}^{\omega_k}) = (\bar{p}, \bar{c}, \bar{s})$;
- (iv) pc is a prefix of \bar{p} ; and
- (v) $\lim_{k \rightarrow \infty} \beta_0^{N_k} = \tau \in \mathbb{S}^1$.

Since pc is a prefix of \bar{p} ,

$$\sigma^{N_k-1}(p_{N_k-1}^{\omega_k}) \dots p_0^{\omega_k} = \sigma^{N_k-1}(p_{N_k-1}^\omega) \dots p_0^\omega \omega_0 \dots \omega_{n_k-1} \tag{14}$$

for every $k \geq 1$.

We will now proceed to reverse the indices of the prefix–suffix decompositions of ω and ω_k in order to obtain sequences in \mathcal{S}_a . Let $(x_{N_k})_{k \geq 1}$ and $(y_{N_k})_{k \geq 1}$ be the sequences in \mathcal{S}_a obtained by reversing the coordinates of $(p_m^\omega, c_m^\omega, s_m^\omega)_{m \geq 0}$ and $(p_m^{\omega_k}, c_m^{\omega_k}, s_m^{\omega_k})_{m \geq 0}$ up to the $(N_k - 1)$ th coordinate and such that $p_m^{x_{N_k}} = p_m^{y_{N_k}} = \varepsilon$ for each $m \geq N_k$, as detailed at the beginning of §5.3. By the assumptions above, $x_{N_k} \in \mathcal{S}_{a,(p,c,s)}$ and $y_{N_k} \in \mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$ for every $k \geq 1$.

Without loss of generality, we will assume that x_{N_k} converges to $x_\infty \in \mathcal{S}_{a,(p,c,s)}$, which is the representation of a limit extreme point in $E_a^*(\tau)$ by Lemma 5.13. We will show that any limit point of y_{N_k} in $\mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$ is the representation of an extreme point in $E_a(\tau)$ and therefore that τ belongs to Ψ_a .

Applying γ to (14), using the definitions of x_{N_k}, y_{N_k} and multiplying by $|\beta|^{-N_k}$, for every $k \geq 1$,

$$\beta_0^{N_k} \sum_{m=1}^{N_k} \beta^{-m} \gamma(p_m^{y_{N_k}}) = \beta_0^{N_k} \sum_{m=1}^{N_k} \beta^{-m} \gamma(p_m^{x_{N_k}}) + |\beta|^{-N_k} \gamma_{n_k}(\omega).$$

By taking real parts and rearranging the previous expression,

$$\operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) = |\beta|^{-N_k} \operatorname{Re}(\gamma_{n_k}(\omega)).$$

Furthermore, $\operatorname{Re}(\beta_0^{N_k} \varkappa_a(x_{N_k})) \leq \operatorname{Re}(\beta_0^{N_k} \varkappa_a(y_{N_k}))$ by Lemma 5.5. Then

$$0 \leq \operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) = |\beta|^{-N_k} \operatorname{Re}(\gamma_{n_k}(\omega)). \tag{15}$$

On the other hand, since $\omega_0\omega_1 \dots \omega_{n_k-1}$ is a subword of $\sigma^{N_k}(a)$ and $|\sigma^{N_k}(a)|$ grows as α^{-N_k} (recall that $\alpha^{-1} > 1$ is the Perron–Frobenius eigenvalue of M), $n_k \leq (\alpha^{-1} + \eta)^{N_k}$ for sufficiently large $k \geq 1$, where $\eta > 0$ was given in the statement of the proposition. Therefore, by definition of ρ ,

$$n_k^{-\rho} \geq (\alpha^{-1} + \eta)^{-N_k \rho} = (|\beta| - \eta)^{-N_k} \geq |\beta|^{-N_k}.$$

From the assumption (13),

$$\lim_{k \rightarrow \infty} (|\beta| - \eta)^{-N_k} \operatorname{Re}(\gamma_{n_k}(\omega)) = \lim_{k \rightarrow \infty} |\beta|^{-N_k} \operatorname{Re}(\gamma_{n_k}(\omega)) = 0.$$

In particular, from equation (15), we obtain that any limit point y_∞ of y_{N_k} in $\mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$ is such that $\varkappa_a(y_\infty)$ is an extreme point for the direction $\tau = \lim_{k \rightarrow \infty} \beta_0^{N_k}$: $v_a(\tau) = \operatorname{Re}(\tau \varkappa_a(y_\infty))$. Therefore τ belongs to Ψ_a .

Amplifying equation (15) by A^{N_k} , where $A = |\beta|/(|\beta| - \eta) \in (1, |\beta|)$, we find that

$$0 \leq A^{N_k} \operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) = (|\beta| - \eta)^{-N_k} \operatorname{Re}(\gamma_{n_k}(\omega))$$

for all sufficiently large k . Hence

$$\lim_{k \rightarrow \infty} A^{N_k} \operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) = 0. \tag{16}$$

We know from Lemma 5.5 that

$$v_a^{(N_k)}(\beta_0^{N_k}) = v_{a,(p,c,s)}^{(N_k)}(\beta_0^{N_k}) = \operatorname{Re}(\beta_0^{N_k} \varkappa_a(x_{N_k})).$$

We also know that $\operatorname{Re}(\beta_0^{N_k} \varkappa_a(y_{N_k})) \geq v_{a,(\bar{p},\bar{c},\bar{s})}^{(N_k)}(\beta_0^{N_k})$ and, therefore, that

$$\operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) \geq v_{a,(\bar{p},\bar{c},\bar{s})}^{(N_k)}(\beta_0^{N_k}) - v_{a,(p,c,s)}^{(N_k)}(\beta_0^{N_k}) \geq 0. \tag{17}$$

On the other hand, since $x_\infty \in \mathcal{S}_{a,(p,c,s)}$ and $y_\infty \in \mathcal{S}_{a,(\bar{p},\bar{c},\bar{s})}$ are representations of extreme points for τ , the u.r.p. implies that $E_{a,(p,c,s)}(\tau)$ and $E_{a,(\bar{p},\bar{c},\bar{s})}(\tau)$ are disjoint, so, in particular, $\varkappa_a(x_\infty) \neq \varkappa_a(y_\infty)$. We notice at this point that the weak u.r.p. is sufficient. Indeed, the extreme points $\varkappa_a(x_\infty)$ and $\varkappa_a(y_\infty)$ are also limit extreme points in $E_a^*(\tau)$, so the weak u.r.p. implies that they are different. Also notice that, by Corollary 5.14, $E_a(\tau) = E_a^*(\tau)$.

Using Lemma 5.7, we conclude that, for each $k \geq 1$,

$$v_{a,(\bar{p},\bar{c},\bar{s})}^{(N_k)}(\beta_0^{N_k}) - v_{a,(p,c,s)}^{(N_k)}(\beta_0^{N_k}) \geq v_{a,(\bar{p},\bar{c},\bar{s})}(\beta_0^{N_k}) - v_{a,(p,c,s)}(\beta_0^{N_k}) - 2C|\beta|^{-N_k} \tag{18}$$

for a constant $C > 0$ which does not depend on k .

Finally, by (17), (18) and Lemma 7.7, we conclude that

$$\operatorname{Re}(\beta_0^{N_k} (\varkappa_a(y_{N_k}) - \varkappa_a(x_{N_k}))) \geq D\llbracket \tau - \beta_0^{N_k} \rrbracket - 2C|\beta|^{-N_k}$$

for infinitely many $k \geq 1$. Since γ is a good eigenvector for Γ and $\tau \in \Psi_a$, by definition, $\liminf_{k \rightarrow \infty} A^{N_k} \llbracket \tau - \beta_0^{N_k} \rrbracket = \infty$. This contradicts (16). \square

7.2. *Proof of Theorem B.* The theorem follows from Theorem A if we prove that β is a simple eigenvalue of M and that $\beta/|\beta|$ is not a root of unity. These two facts can be proved if either β is Galois-conjugate with α^{-1} or if β is Galois-conjugate with α and the self-similarity comes from Rauzy–Veech renormalizations: that is, if some iteration of the Rauzy–Veech algorithm on T and the interval $[0, \alpha)$ returns to T (see [Via06] for details about the Rauzy–Veech algorithm). This last condition is natural in the following sense. Let π be a vertex on a Rauzy class and consider a cycle starting at π in which every letter wins and loses at least once. Let R be the matrix of such a cycle. Then R is primitive and α^{-1} is its Perron–Frobenius eigenvalue. Also, $R\lambda = \alpha^{-1}\lambda$ for some positive eigenvector λ whose coordinates add up to one. The i.e.m. with combinatorial data (π, λ) is periodic for the Rauzy–Veech algorithm. Thus this algorithm provides a simple way to construct a self-similar i.e.m.

Let $\psi : \mathbb{Q}(\xi) \rightarrow \mathbb{Q}(\beta)$ be the natural field isomorphism coming from the Galois-conjugacy, where ξ is either α or α^{-1} . The following Lemma proves what we need.

LEMMA 7.9. *Let β be an eigenvalue of M that is either Galois-conjugate with α^{-1} or is Galois-conjugate with α and T is periodic for the Rauzy–Veech renormalization algorithm on $[0, \alpha)$. Then β has algebraic and geometric multiplicity one. Moreover, if β is not real, then $\beta/|\beta|$ is not a root of unity.*

Proof. If β is Galois-conjugate with α^{-1} , which is the Perron–Frobenius eigenvalue of M , then it has multiplicity one. We will now prove that if it is Galois-conjugate with α , then it also has multiplicity one when it is further assumed that T is periodic for the Rauzy–Veech algorithm. We will use the classical results of Veech in [Vee84], which use this fact.

Recall that $R = M^t$ is the renormalization matrix coming from the Rauzy–Veech induction. There exists a (possibly degenerate) antisymmetric integer matrix L^π such that

$$R^t L^\pi R = M L^\pi R = L^\pi. \tag{19}$$

Let $H(\pi) = L^\pi(\mathbb{C}^{\text{sd}})$ and $N(\pi) = \ker L^\pi$. It is easy to check from (19) that $H(\pi)$ is invariant for M and that $N(\pi)$ is invariant for R . The matrix L^π is non-degenerate when restricted to $H(\pi)$ and, therefore, $R|_{H(\pi)}$ is symplectic. We will show that the eigenspaces of α , α^{-1} and β are contained in $H(\pi)$.

By [Vee84, Lemma 5.6], R acts as a permutation on a basis of $N(\pi)$. Therefore every eigenvalue of $R|_{N(\pi)}$ has modulus one. Let V be the eigenspace of an eigenvalue z for R with $|z| \neq 1$. From (19), it is easy to see that $L^\pi(V) \subseteq H(\pi)$ is contained in the eigenspace of z^{-1} for M and, since $V \cap N(\pi) = \{0\}$, that $L^\pi(V)$ is the entire eigenspace of z^{-1} for M . Thus, the desired eigenspaces are contained in $H(\pi)$.

Let $p(t)$ be the characteristic polynomial of M restricted to $H(\pi)$. We know that $p(t) = t^{|\text{sd}|} p(1/t)$ by symplecticity. The Galois-conjugacy implies that β has the same algebraic multiplicity of α . Since α^{-1} is a simple root of $p(t)$, so is α by the equality $p(t) = t^{|\text{sd}|} p(1/t)$. The first part of the lemma is therefore proved.

Now, assume by contradiction that β is Galois-conjugate to ξ (with $\xi = \alpha$ or α^{-1}) and that β^n is real for some integer n . Since $\bar{\beta}$ is also an eigenvalue of M , which is different from β , $\beta^n = \bar{\beta}^n$ has algebraic multiplicity two for M^n . Moreover, ξ^n and β^n are also Galois-conjugate. Indeed, if $q(t)$ is the minimal polynomial of ξ^n , then

$q(\beta^n) = q(\psi(\xi^n)) = \psi(q(\xi^n)) = 0$. The matrix M^n is also primitive and corresponds to the induced map of T on $[0, \alpha^n)$. We can therefore replicate the proof of the first part of the lemma for M^n and conclude that β^n has algebraic multiplicity one, which is a contradiction. □

8. *The cubic Arnoux–Yoccoz map*

In this section, we illustrate Theorem A in the cubic A–Y i.e.m. This map is self-similar in the sense of [LPV08] but not in the sense of [CG97], which is the notion we are following in Theorem A. However, it satisfies all the other hypotheses of the theorem. Although the main theorem is written for a self-similar i.e.m. in the sense of [CG97], this precise notion plays a role in a very specific part of the proof of the theorem, which is ensuring that the symbolic system associated with the i.e.m. is substitutive. It is possible to prove this fact for the specific case of the A–Y i.e.m.

In addition, it is proved in [LPV08] that an induced system obtained from the A–Y i.e.m. with respect to a precise interval is a self-similar i.e.m. in the sense of [CG97], but the resulting substitution associated with the i.e.m. is unnecessarily complex to analyse. Nevertheless, some technical but not difficult modifications on the study that we will develop for the A–Y i.e.m. below allow us to prove that the induced system also satisfies the hypotheses of Theorem A.

8.1. *A–Y i.e.m.* Let α be the unique real number such that $\alpha + \alpha^2 + \alpha^3 = 1$ and let G_{t_0,t_1} be the map exchanging both halves of the interval $[t_0, t_1)$ while preserving orientation: that is,

$$G_{t_0,t_1}(t) = \begin{cases} t + (t_0 + t_1)/2 & t \in [t_0, (t_0 + t_1)/2), \\ t - (t_0 + t_1)/2 & t \in [(t_0 + t_1)/2, t_1), \\ t & t \notin [t_0, t_1). \end{cases}$$

Then the A–Y i.e.m. is given by $T = G_{0,1} \circ G_{0,\alpha} \circ G_{\alpha,\alpha+\alpha^2} \circ G_{\alpha+\alpha^2,1}$ (see Figures 4 and 5 for clarity). Properties of T were extensively discussed in [ABB11]. In particular, it is proved that the map T is equal, up to rescaling and rotation, to the map induced on the interval $[0, \alpha)$ and, by considering an appropriate refinement of continuity intervals of T into nine intervals, one may encode the relation of orbits by T for this partition and the orbits of the induced system for the induced partition by the substitution σ on the alphabet $\mathcal{A} = \{1, \dots, 9\}$ given by

$$\begin{aligned} \sigma(1) &= 35, & \sigma(4) &= 17, & \sigma(7) &= 29, \\ \sigma(2) &= 45, & \sigma(5) &= 18, & \sigma(8) &= 2, \\ \sigma(3) &= 46, & \sigma(6) &= 19, & \sigma(9) &= 3. \end{aligned}$$

Then $\Omega_T = \Omega_\sigma$. It is easy to check that σ is primitive. This solves the issue at the beginning of the section; the symbolic system is indeed substitutive.

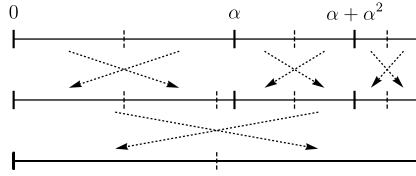


FIGURE 4. The compositions that produce the cubic Arnoux–Yoccoz interval exchange map T . The dashed lines show the midpoints of the respective intervals.

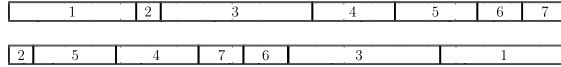


FIGURE 5. The cubic Arnoux–Yoccoz interval exchange map T .

Let M be the matrix associated with the substitution σ : that is,

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $(1 - t^3)(t^3 + t^2 + t - 1)(-t^3 + t^2 + t + 1)$, where the last two factors are irreducible. The roots of $t^3 + t^2 + t - 1$ are α , β and $\bar{\beta}$, whereas the roots of $-t^3 + t^2 + t + 1$ are α^{-1} , β^{-1} and $\bar{\beta}^{-1}$, where α^{-1} is the Perron–Frobenius eigenvalue. We assume that β is the eigenvalue with positive imaginary part. Numerically, $\beta \approx -0.771845 + 1.11514i$. It is proved in [Mes00] that $(\beta^{-1})^n$ is never real for any $n \in \mathbb{Z}$. Furthermore, the eigenvalue β is simple and the corresponding eigenspace is generated by

$$\gamma = (\beta^2 + \beta + 1, -\beta, -\beta, -\beta^2 - \beta - 1, \beta + 1, \beta + 1, -\beta^2 - \beta - 2, -1, -1).$$

Therefore, it is enough to prove the u.r.p. for β .

In what follows, β and γ are the corresponding eigenvalue and eigenvector of M used in the previous sections.

8.2. *Fractals associated with the A–Y i.e.m.* Theorem 6.4 of [ABB11] shows that the fractals defined in §5 exist and satisfy $\mathfrak{F}_2 = \mathfrak{F}_3$, $\mathfrak{F}_5 = \mathfrak{F}_6$ and $\mathfrak{F}_8 = \mathfrak{F}_9$. An illustration of each fractal is given in Figure 8. The boundaries of these fractals can be constructed by combining pieces of the boundary of the standard tribonacci fractal and can therefore be parametrized as we will see later.

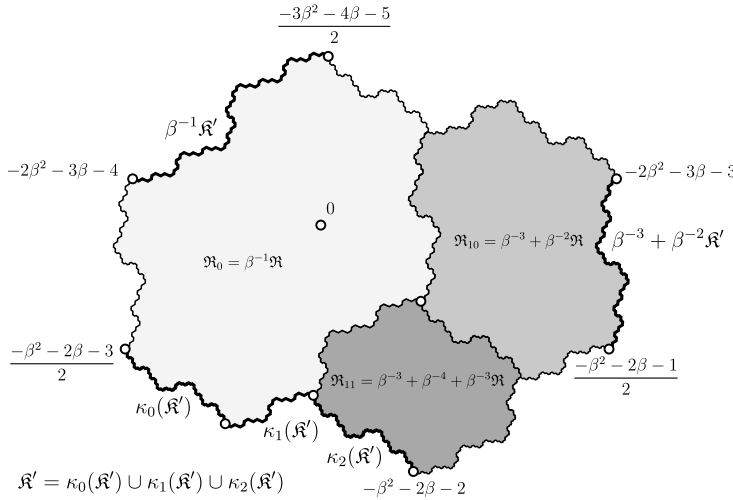


FIGURE 6. The tribonacci fractal with some important features.

8.2.1. *Parametrization of the boundary of tribonacci fractal.* First, we state some important properties of the tribonacci fractal. We will follow [Mes00] freely. Let \mathcal{N} be the set of sequences in $\{0, 1\}$ without three consecutive ones. The (standard) tribonacci fractal is defined by

$$\mathfrak{R} = \left\{ \sum_{m \geq 3} \beta^{-m} a_{m-2}; (a_m)_{m \geq 1} \in \mathcal{N} \right\}.$$

For $(a_m)_{m \geq 1} \in \mathcal{N}$, we define $r((a_m)_{m \geq 1}) = \sum_{m \geq 3} \beta^{-m} a_{m-2}$. For $z \in \mathfrak{R}$, we say that $(a_m)_{m \geq 1} \in \mathcal{N}$ is a *representation* of z if $z = r((a_m)_{m \geq 1})$. Clearly, any sequence in \mathcal{N} starts with either 0, 10 or 11.

$$\begin{aligned} \mathfrak{R}_0 &= \beta^{-1}\mathfrak{R} = \{z \in \mathfrak{R}; z \text{ has a representation starting with } 0\}, \\ \mathfrak{R}_{10} &= \beta^{-3} + \beta^{-2}\mathfrak{R} = \{z \in \mathfrak{R}; z \text{ has a representation starting with } 10\}, \\ \mathfrak{R}_{11} &= \beta^{-3} + \beta^{-4} + \beta^{-3}\mathfrak{R} = \{z \in \mathfrak{R}; z \text{ has a representation starting with } 11\}. \end{aligned}$$

These three subsets of \mathfrak{R} are scalings, rotations and translations of \mathfrak{R} and are disjoint except for a set of measure zero (see Figure 6).

Clearly, \mathcal{N} is a subshift. As before, S is the shift map in \mathcal{N} . For $z \in \mathfrak{R}$ and $(a_m)_{m \geq 1} \in \mathcal{N}$ being a representation of z , $r(S((a_m)_{m \geq 1})) \in \mathfrak{R}$ and

$$r(S((a_m)_{m \geq 1})) = \begin{cases} \beta z & \text{if } a_1 = 0, \\ \beta(z - \beta^{-3}) & \text{if } a_1 = 1. \end{cases}$$

It is easy to see that the points in \mathfrak{R}_{11} are mapped bijectively into \mathfrak{R}_{10} , that the points in \mathfrak{R}_{10} are mapped bijectively to \mathfrak{R}_0 and that the points in \mathfrak{R}_0 are mapped bijectively to \mathfrak{R} .

Now, the parametrization of the boundary of the tribonacci fractal is constructed as follows. Put $z_0 = \beta^{-4}/(1 - \beta^{-3})$ and for each $t \in [0, 1]$ let $(a_m)_{m \geq 1}$ be a sequence in $\{0, 1, 2\}$ such that $t = \sum_{m \geq 1} 3^{-m} a_m$. Put $\kappa(t) = \lim_{m \rightarrow \infty} \kappa_{a_1} \circ \kappa_{a_2} \circ \dots \circ \kappa_{a_m}(z_0)$,

where

$$\begin{aligned} \kappa_0(z) &= \beta^{-4} + \beta^{-3}z, \\ \kappa_1(z) &= \beta^{-4} + \beta^{-6} + \frac{\beta^{-10}}{1 - \beta^{-3}} - \beta^{-4}z, \\ \kappa_2(z) &= \beta^{-3} + \beta^{-4} + \beta^{-3}z. \end{aligned}$$

It is shown in [Mes00, §4] that κ is bijective and that

$$\mathfrak{K}' = \kappa([0, 1]) = \mathfrak{X} \cap (\mathfrak{X} + \beta^{-1}).$$

The set \mathfrak{K}' is also part of the boundary of the tribonacci fractal. The rest of the boundary is obtained by scaling, rotating and translating \mathfrak{K}' .

We know that $\mathfrak{K}' \subseteq \mathfrak{X}_0 \cup \mathfrak{X}_{11}$: that is, every point in \mathfrak{K}' has a representation starting with either 0 or 11. Furthermore, $\mathfrak{K}' = \kappa_0(\mathfrak{K}') \cup \kappa_1(\mathfrak{K}') \cup \kappa_2(\mathfrak{K}')$, which is a consequence of [Mes00, Lemma 4], and

$$\mathfrak{K}' \cap \mathfrak{X}_0 = \kappa_0(\mathfrak{K}') \cup \kappa_1(\mathfrak{K}'), \quad \mathfrak{K}' \cap \mathfrak{X}_{11} = \kappa_2(\mathfrak{K}').$$

A simple computation shows that by applying the shift map to $\mathfrak{K}' \cap \mathfrak{X}_{11}$ one obtains $\beta^{-3} + \beta^{-2}\mathfrak{K}'$ and that by applying the shift map again one obtains $\beta^{-1}\mathfrak{K}'$, as Figure 6 shows.

Finally, we describe some basic additional properties of \mathfrak{X} and \mathfrak{K} that we will need.

LEMMA 8.1.

- (i) $\mathfrak{X} \cap (\mathfrak{X} + (\beta^2 - 1)/2) = \emptyset$;
- (ii) $\mathfrak{K}' \cap (\mathfrak{X} + (\beta^2 + 2\beta + 3)/2) = \emptyset$; and
- (iii) $\mathfrak{K}' \cap \beta^{-2}\mathfrak{X} = \emptyset$.

8.2.2. *Parametrization of the boundary of fractals associated with the cubic A–Y i.e.m.*

We will use $\mathfrak{K} = \mathfrak{K}' - z_0$ to parametrize the boundaries of each \mathfrak{F}_a for each $a \in \mathcal{A}$. It is not difficult to see (after a simple computation) that \mathfrak{K} is a curve with endpoints $\kappa(0) - z_0 = 0$ and $\kappa(1) - z_0 = (-\beta^2 - 2\beta - 1)/2$. To get our parametrization, we will need the following four lemmas that, at the end, show that the boundaries of the \mathfrak{F}_a are Jordan curves.

LEMMA 8.2. *The following equalities are satisfied: that is,*

$$\mathfrak{K} = \kappa(1) - \mathfrak{K} \quad \text{and} \quad \mathfrak{K} = \beta^{-3}\mathfrak{K} \cup (\beta^{-4}\mathfrak{K} + \beta^{-3}) \cup (\beta^{-3}\mathfrak{K} + \beta^{-3}).$$

Proof. The first equality comes from the fact that \mathfrak{K}' is symmetric, as shown in [Mes00, Lemma 4]. Furthermore, since $\mathfrak{K}' = \kappa_0(\mathfrak{K}') \cup \kappa_1(\mathfrak{K}') \cup \kappa_2(\mathfrak{K}')$, by subtracting z_0 from both sides, expanding and using the first equality, we get the desired result. □

The following two lemmas will serve to prove that the parts of the boundaries in the A–Y fractals coming from the tribonacci fractal intersect in a unique point (see Figure 8).

LEMMA 8.3. $\mathfrak{K} \cap (\beta\mathfrak{K} + (\beta^2 + 1)/2) = \mathfrak{K} \cap (\beta^2\mathfrak{K} + (-\beta^2 + 1)/2) = \{0\}$.

Proof. Recall that $z_0 = \beta^{-4}/(1 - \beta^{-3}) = (-\beta^2 - 2\beta - 3)/2$ and $\mathfrak{R}' = \mathfrak{R} \cap (\mathfrak{R} + \beta^{-1})$. Set $K_1 = \mathfrak{R} \cap (\beta\mathfrak{R} + (\beta^2 + 1)/2)$ and $K_2 = \mathfrak{R} \cap (\beta^2\mathfrak{R} + (-\beta^2 + 1)/2)$.

(i) Let $z \in K_1$. We will prove that $-\beta^2z \in K_2$. By definition, $z = \zeta - z_0 = \beta(\zeta' - z_0) + (\beta^2 + 1)/2$, where $\zeta, \zeta' \in \mathfrak{R}'$. Therefore

$$\zeta = \beta\zeta' - \beta z_0 + z_0 + \frac{\beta^2 + 1}{2} = \beta\zeta' + \frac{\beta^2 - 1}{2}.$$

From the discussion in §8.2.1, ζ' has a representation starting with either 0 or 11. Let a_1, a_2, a_3 be the first three letters of such representation and consider the point $\zeta'' = \beta(\zeta' - \beta^{-3}a_1) \in \mathfrak{R}$: that is, the point obtained by shifting the representation of ζ' . We get that $\zeta = \zeta'' + \beta^{-2}a_1 + (\beta^2 - 1)/2$, where $a_1 \in \{0, 1\}$.

By Lemma 8.1 item (i), $\mathfrak{R} \cap (\mathfrak{R} + (\beta^2 - 1)/2) = \emptyset$, so $a_1 \neq 0$ and thus $a_1a_2a_3 = 110$. Then, $\zeta' \in \mathfrak{R}' \cap \mathfrak{R}_{11}$, so $\zeta'' \in \beta^{-3} + \beta^{-2}\mathfrak{R}' = \beta^{-2}\mathfrak{R} + (-\beta^2 - 2\beta - 1)/2$ and, by replacing the value of a_1 in previous expressions, $\zeta = \zeta'' + (3\beta^2 + 4\beta + 3)/2$.

Finally, $z \in \mathfrak{R} \cap (\beta^{-2}\mathfrak{R} + (3\beta^2 + 4\beta + 5)/2)$, so

$$-\beta^2z \in \left(\frac{-\beta^2 - 2\beta - 1}{2} - \mathfrak{R} \right) \cap (-\beta^2\mathfrak{R}) = K_2,$$

where the last equality is obtained by using Lemma 8.2 and a simple computation.

(ii) Let $z \in K_2$. We will prove that $-\beta z \in K_1$. The proof is similar to (i) so we skip some details.

By definition, $z = \zeta - z_0 = \beta^2(\zeta' - z_0) + (-\beta^2 + 1)/2$, where $\zeta, \zeta' \in \mathfrak{R}'$. Therefore

$$\zeta = \beta^2\zeta' - \beta^2z_0 + z_0 + \frac{-\beta^2 + 1}{2} = \beta^2\zeta' + \frac{-\beta^2 - 2\beta - 1}{2}.$$

As in the proof of (i), ζ' has a representation that begins with either 0 or 11. Let a_1, a_2, a_3 be its first three letters and let $\zeta'' = \beta^2(\zeta' - \beta^{-3}a_1 - \beta^{-4}a_2) \in \mathfrak{R}$: that is, the point obtained by shifting the representation of ζ' twice. We know that $\zeta = \zeta'' + \beta^{-1}a_1 + \beta^{-2}a_2 + (-\beta^2 - 2\beta - 1)/2$, where $a_1, a_2 \in \{0, 1\}$.

We also know that $a_1a_2 \neq 10$. If $a_1a_2 = 01$, it would be that $\zeta = \zeta'' + (\beta^2 + 2\beta + 3)/2$. By Lemma 8.1 item (ii), $\mathfrak{R}' \cap (\mathfrak{R} + (\beta^2 + 2\beta + 3)/2) = \emptyset$, so this cannot happen. Furthermore, by item (iii) of the same lemma, $\mathfrak{R}' \cap \beta^{-2}\mathfrak{R} = \emptyset$, so $a_1a_2 \neq 00$. We obtain $a_1a_2a_3 = 110$, so $\zeta = \zeta'' + (3\beta^2 + 4\beta + 5)/2$ and we deduce that $\zeta'' \in \beta^{-1}\mathfrak{R}' = \beta^{-1}\mathfrak{R} + (-3\beta^2 - 4\beta - 5)/2$. Finally, $z \in \mathfrak{R} \cap (\beta^{-1}\mathfrak{R} + (\beta^2 + 2\beta + 3)/2)$, so

$$-\beta z \in \left(\frac{-\beta^2 - 2\beta - 1}{2} - \mathfrak{R} \right) \cap (-\beta\mathfrak{R}) = K_1,$$

where the last equality is obtained by using the previous lemma.

From (i) and (ii), we find that if z belongs to either K_1 or K_2 , then $\{\beta^n z, -\beta^n z\}$ meets $K_1 \cup K_2$ for infinitely many integers $n \geq 0$. Since $K_1 \cup K_2$ is bounded and $|\beta| > 1$, it must be that $z = 0$. □

LEMMA 8.4. $(\mathfrak{R} - 2\beta^2 - 3\beta - 4) \cap (\mathfrak{R} + (-3\beta^2 - 4\beta - 7)/2)$ is a unique point.

Proof. We illustrate the proof in Figure 7. Observe that $(\mathfrak{R} - 2\beta^2 - 3\beta - 4) \subseteq \mathfrak{R}$. By the previous lemma and a translation, a rotation and a scaling, $\mathfrak{R} - 2\beta^2 - 3\beta - 4$ is a subset of $\text{int } \mathfrak{R}$ up to one point. On the other hand, $(\mathfrak{R} + (-3\beta^2 - 4\beta - 7)/2) \subseteq \mathfrak{R} - 1 - \beta^{-1}$.

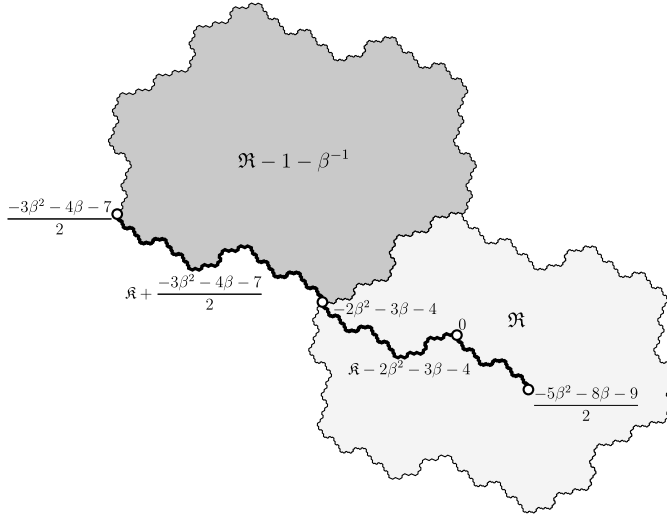


FIGURE 7. Illustration of the proof of Lemma 8.4.

We know from [Mes00] that $\mathfrak{R} - 1 - \beta^{-1} \subseteq \mathbb{C} \setminus \text{int } \mathfrak{R}$. Then $(\mathfrak{R} + (-3\beta^2 - 4\beta - 7)/2)$ is a subset of $\mathbb{C} \setminus \mathfrak{R}$ up to one point. \square

Now we give the parametrization of the boundaries of the Arnoux–Yoccoz fractals. Define

$$\begin{aligned} \mathcal{C}_1 &= (\beta^{-1}\mathfrak{R}) \cup (\beta^{-2}\mathfrak{R} + \beta^2 + \beta + 1) \cup \left(\mathfrak{R} + \frac{3\beta^2 + 4\beta + 3}{2}\right) \\ &\cup \left(\beta^{-2}\mathfrak{R} + \frac{3\beta^2 + 4\beta + 3}{2}\right) \cup \left(\beta^{-1}\mathfrak{R} + \frac{\beta^2 + 2\beta + 1}{2}\right) \\ &\cup \left(\mathfrak{R} + \frac{\beta^2 + 2\beta + 1}{2}\right), \\ \mathcal{C}_2 &= \mathfrak{R} \cup (\beta\mathfrak{R} - \beta) \cup \left(\mathfrak{R} + \frac{\beta^2 + 1}{2}\right) \cup \left(\beta\mathfrak{R} + \frac{\beta^2 + 1}{2}\right), \\ \mathcal{C}_4 &= \beta\mathfrak{R} \cup (\beta^{-1}\mathfrak{R} + \beta + 1) \cup (\beta^{-3}\mathfrak{R} + \beta + 1) \cup (\beta^{-2}\mathfrak{R} - \beta^2 - \beta - 1) \\ &\cup \left(\beta^{-2}\mathfrak{R} + \frac{\beta^2 + 2\beta + 3}{2}\right) \cup \left(\beta^{-1}\mathfrak{R} + \frac{\beta^2 + 2\beta + 3}{2}\right), \\ \mathcal{C}_5 &= \beta\mathfrak{R} \cup (\beta^{-1}\mathfrak{R} + \beta + 1) \cup \left(\beta\mathfrak{R} + \frac{\beta^2 + 2\beta + 3}{2}\right) \cup \left(\beta^{-1}\mathfrak{R} + \frac{\beta^2 + 2\beta + 3}{2}\right), \\ \mathcal{C}_7 &= \beta^{-1}\mathfrak{R} \cup (\mathfrak{R} - 1) \cup (\beta^{-2}\mathfrak{R} - 1) \cup (\beta^{-1}\mathfrak{R} - \beta^2 - \beta - 2) \\ &\cup \left(\beta^{-2}\mathfrak{R} + \frac{\beta^2 + 2\beta + 1}{2}\right) \cup \left(\mathfrak{R} + \frac{\beta^2 + 2\beta + 1}{2}\right), \\ \mathcal{C}_8 &= \beta^{-1}\mathfrak{R} \cup (\mathfrak{R} - 1) \cup \left(\beta^{-1} + \frac{\beta^2 + 2\beta + 1}{2}\right) \cup \left(\mathfrak{R} + \frac{\beta^2 + 2\beta + 1}{2}\right) \end{aligned}$$

and $\mathcal{C}_3 = \mathcal{C}_2$, $\mathcal{C}_6 = \mathcal{C}_5$, $\mathcal{C}_9 = \mathcal{C}_8$.

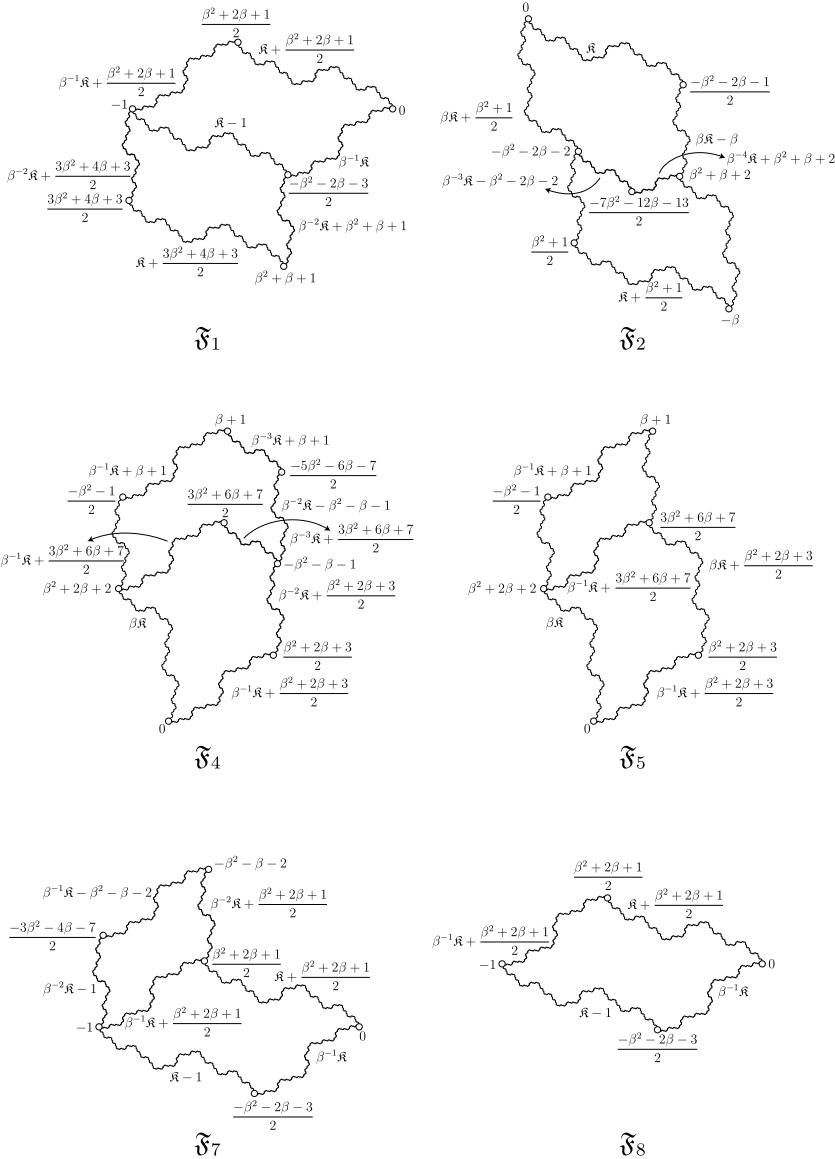


FIGURE 8. Description of Arnoux–Yoccoz fractals and their features.

LEMMA 8.5. For each $a \in \mathcal{A}$, \mathcal{C}_a is a Jordan curve.

Proof. In the definition of each \mathcal{C}_a , the terms between unions correspond to the segments in the boundary of \mathfrak{F}_a shown in Figure 8 in the clockwise order, starting at zero. So it is enough to see that the intersection of two consecutive segments is a single point. Indeed, using the results for the tribonacci fractal from [Mes00], the intersection of most of the pairs of contiguous segments have only one point. The conclusion of the lemma follows from the two previous lemmas.

For example, consider the segments $\beta^{-1}\mathfrak{K}$ and $\beta^{-2}\mathfrak{K} + \beta^2 + \beta + 1$, which are part of \mathcal{C}_1 . By translating by $-\beta^{-1}z_0$ and replacing \mathfrak{K} by $\mathfrak{K}' - z_0$, we get that these segments intersect at a single point if and only if $\beta^{-1}\mathfrak{K}'$ and $\beta^{-2}\mathfrak{K}' + \beta^{-2} + \beta^{-1}$ do. These two segments are part of the boundary the tribonacci fractal, which is a Jordan curve. Further examples are the segments $\mathfrak{K} + (\beta^2 + 2\beta + 1)/2$ and $\beta^{-1}\mathfrak{K}$, which are also part of \mathcal{C}_1 . We can amplify by β to get that these segments intersect at a single point if and only if \mathfrak{K} and $\beta\mathfrak{K} + (\beta^2 + 1)/2$ do, which is implied by Lemma 8.3. \square

Now we are ready to prove that the \mathcal{C}_a are parametrizations of the boundaries of the Arnoux–Yoccoz fractals.

LEMMA 8.6. *For each $a \in \mathcal{A}$, \mathfrak{F}_a is the closure of the interior (in the Jordan sense) of \mathcal{C}_a . Furthermore, if $\sigma(a)$ can be written as $\sigma(a) = bc$ with $b, c \in \mathcal{A}$, then $\mathfrak{F}_b \cap \text{int}(\gamma(b) + \mathfrak{F}_c) = \emptyset$ and $\text{int } \mathfrak{F}_b \cap (\gamma(b) + \mathfrak{F}_c) = \emptyset$.*

Proof. We will make the identifications $3 \sim 2, 6 \sim 5, 9 \sim 8$. By the definition of \mathcal{C}_a ,

$$\begin{aligned} \beta^{-1}\mathcal{C}_1 &= (\beta^{-2}\mathfrak{K}) \cup (\beta^{-3}\mathfrak{K} + \beta^2 + 2\beta + 2) \cup \left(\beta^{-1}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2}\right) \\ &\quad \cup \left(\beta^{-3}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2}\right) \cup \left(\beta^{-2}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right) \\ &\quad \cup \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right), \\ \beta^{-1}\mathcal{C}_2 &= \beta^{-1}\mathfrak{K} \cup (\mathfrak{K} - 1) \cup \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 1}{2}\right) \cup \left(\mathfrak{K} + \frac{\beta^2 + 2\beta + 1}{2}\right), \\ \beta^{-1}\mathcal{C}_4 &= \mathfrak{K} \cup (\beta^{-2}\mathfrak{K} + \beta^2 + \beta + 2) \cup (\beta^{-4}\mathfrak{K} + \beta^2 + \beta + 2) \\ &\quad \cup (\beta^{-3}\mathfrak{K} - \beta^2 - 2\beta - 2) \cup \left(\beta^{-3}\mathfrak{K} + \frac{3\beta^2 + 4\beta + 5}{2}\right) \\ &\quad \cup \left(\beta^{-2}\mathfrak{K} + \frac{3\beta + 4\beta + 5}{2}\right), \\ \beta^{-1}\mathcal{C}_5 &= \mathfrak{K} \cup (\beta^{-2}\mathfrak{K} + \beta^2 + \beta + 2) \cup \left(\mathfrak{K} + \frac{3\beta^2 + 4\beta + 5}{2}\right) \\ &\quad \cup \left(\beta^{-2}\mathfrak{K} + \frac{3\beta^2 + 4\beta + 5}{2}\right), \\ \beta^{-1}\mathcal{C}_7 &= \beta^{-2}\mathfrak{K} \cup (\beta^{-1}\mathfrak{K} - \beta^2 - \beta - 1) \cup (\beta^{-3}\mathfrak{K} - \beta^2 - \beta - 1) \\ &\quad \cup (\beta^{-2}\mathfrak{K} - 2\beta^2 - 3\beta - 3) \cup \left(\beta^{-3}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right) \\ &\quad \cup \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right), \\ \beta^{-1}\mathcal{C}_8 &= \beta^{-2}\mathfrak{K} \cup (\beta^{-1}\mathfrak{K} - \beta^2 - \beta - 1) \cup \left(\beta^{-2}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right) \\ &\quad \cup \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 3}{2}\right). \end{aligned}$$

By using Lemma 8.2 when necessary,

$$\begin{aligned}
 \beta^{-1}\mathcal{C}_2 \cup \beta^{-1}(-\beta + \mathcal{C}_5) &= \mathcal{C}_1 \cup (\mathfrak{K} - 1), \\
 \beta^{-1}\mathcal{C}_4 \cup \beta^{-1}(-\beta^{-1} + \mathcal{C}_5) &= \mathcal{C}_2 \cup (\beta^{-4}\mathfrak{K} + \beta^2 + \beta + 2) \cup (\beta^{-3}\mathfrak{K} - \beta^2 - 2\beta - 2), \\
 \beta^{-1}\mathcal{C}_1 \cup \beta^{-1}(\beta^{-1} + \mathcal{C}_7) &= \mathcal{C}_4 \cup \left(\beta^{-1}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right) \\
 &\quad \cup \left(\beta^{-3}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right), \\
 \beta^{-1}\mathcal{C}_1 \cup \beta^{-1}(\beta^{-1} + \mathcal{C}_8) &= \mathcal{C}_5 \cup \left(\beta^{-1}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right), \\
 \beta^{-1}\mathcal{C}_2 \cup \beta^{-1}(-\beta + \mathcal{C}_8) &= \mathcal{C}_7 \cup \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 1}{2} \right), \\
 \beta^{-1}\mathcal{C}_2 &= \mathcal{C}_8.
 \end{aligned}
 \tag{20}$$

The following equations stated in [ABB11, (6-1)] produce a unique solution for the given γ .

$$\begin{aligned}
 \mathfrak{F}_1 &= \beta^{-1}\mathfrak{F}_2 \cup \beta^{-1}(-\beta + \mathfrak{F}_5), & \mathfrak{F}_2 &= \beta^{-1}\mathfrak{F}_4 \cup \beta^{-1}(-\beta^{-1} + \mathfrak{F}_5), \\
 \mathfrak{F}_4 &= \beta^{-1}\mathfrak{F}_1 \cup \beta^{-1}(\beta^{-1} + \mathfrak{F}_7), & \mathfrak{F}_5 &= \beta^{-1}\mathfrak{F}_1 \cup \beta^{-1}(\beta^{-1} + \mathfrak{F}_8), \\
 \mathfrak{F}_7 &= \beta^{-1}\mathfrak{F}_2 \cup \beta^{-1}(-\beta + \mathfrak{F}_8), & \mathfrak{F}_8 &= \beta^{-1}\mathfrak{F}_2.
 \end{aligned}
 \tag{21}$$

Finally, let \mathfrak{F}'_a be the closure of the Jordan interior of \mathcal{C}_a . We have to show that $(\mathfrak{F}'_a)_{a \in \mathcal{A}}$ satisfies (21). We know that (see Figure 8)

$$\begin{aligned}
 \mathfrak{K} - 1 &\subseteq \mathfrak{F}'_1, & (\beta^{-4}\mathfrak{K} + \beta^2 + \beta + 2) \cup (\beta^{-3}\mathfrak{K} - \beta^2 - 2\beta - 2) &\subseteq \mathfrak{F}'_2, \\
 \left(\beta^{-1}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right) \cup \left(\beta^{-3}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right) &\subseteq \mathfrak{F}'_4, \\
 \left(\beta^{-1}\mathfrak{K} + \frac{3\beta^2 + 6\beta + 7}{2} \right) &\subseteq \mathfrak{F}'_5, & \left(\beta^{-1}\mathfrak{K} + \frac{\beta^2 + 2\beta + 1}{2} \right) &\subseteq \mathfrak{F}'_7.
 \end{aligned}$$

So, by (20), we conclude that $(\mathfrak{F}'_a)_{a \in \mathcal{A}}$ satisfies (21), which proves that $\mathfrak{F}_a = \mathfrak{F}'_a$ for each $a \in \mathcal{A}$.

The second part of the proof follows directly from the parametrization (see Figure 8). \square

The following corollary is straightforward and follows from the previous lemma.

COROLLARY 8.7. $\partial\mathfrak{F}_a = \mathcal{C}_a$ for each $a \in \mathcal{A}$.

8.2.3. Unique representation property for the A–Y i.e.m. We finally prove that every extreme point of each fractal \mathfrak{F}_a has a unique representation.

LEMMA 8.8. For any $a \in \mathcal{A}$, extreme points in the boundary of \mathfrak{F}_a have a unique representation: that is, T has the unique representation property for β .

Proof. The proof is by contradiction. We prove that extreme points in \mathfrak{F}_a with more than one representation have an eventually periodic representation. This, together with Lemma 5.8, gives a contradiction.

Let $z \in \partial \mathfrak{F}_a$ be an extreme point with two representations. By shifting and using Lemma 5.6, we can assume that the first letters in these representations are different. This implies that $\sigma(a)$ cannot be a single letter, so we get that $\sigma(a) = bc$ for some $b, c \in \mathcal{A}$. Then $z \in \beta^{-1} \mathfrak{F}_b \cap \beta^{-1}(\gamma(b) + \mathfrak{F}_c)$ and, by Lemma 8.6,

$$z \in \partial \mathfrak{F}_a \cap (\beta^{-1} \mathfrak{F}_b \cap \beta^{-1}(\gamma(b) + \mathfrak{F}_c)) = \partial \mathfrak{F}_a \cap \partial \beta^{-1} \mathfrak{F}_b \cap \partial \beta^{-1}(\gamma(b) + \mathfrak{F}_c).$$

Now, to get the desired contradiction, we prove that these points have an eventually periodic representation. There are seven cases (see Figure 8 to understand the cases).

(i) If $a = 1$, then $z = -1$ or $z = (-\beta^2 - 2\beta - 3)/2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (3, 5, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = -1$, and the point $x_2 \in \mathcal{S}_a$ defined by

$$x_2 = (\varepsilon, 3, 5)(\varepsilon, 4, 6)(1, 7, \varepsilon)(\varepsilon, 2, 9)(\varepsilon, 4, 5)(1, 7, \varepsilon)(\varepsilon, 2, 9)(\varepsilon, 4, 5) \dots$$

is an eventually periodic representation of $z = (-\beta^2 - 2\beta - 3)/2$.

(ii) If $a = 2$, then $z = -\beta^2 - 2\beta - 2$ or $z = \beta^2 + \beta + 2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (4, 5, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = -\beta^2 - 2\beta - 2$, and the point $x_2 \in \mathcal{S}_a$ defined by $(p_1^{x_2}, c_1^{x_2}, s_1^{x_2}) = (4, 5, \varepsilon)$, $(p_2^{x_2}, c_2^{x_2}, s_2^{x_2}) = (1, 8, \varepsilon)$ and $p_m^{x_2} = \varepsilon$ for every $m \geq 3$ is an eventually periodic representation of $z = \beta^2 + \beta + 2$.

(iii) If $a = 3$, then $z = -\beta^2 - 2\beta - 2$ or $z = \beta^2 + \beta + 2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (4, 6, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = -\beta^2 - 2\beta - 2$, and the point $x_2 \in \mathcal{S}_a$ defined by $(p_1^{x_2}, c_1^{x_2}, s_1^{x_2}) = (4, 6, \varepsilon)$, $(p_2^{x_2}, c_2^{x_2}, s_2^{x_2}) = (1, 9, \varepsilon)$ and $p_m^{x_2} = \varepsilon$ for every $m \geq 3$ is an eventually periodic representation of $z = \beta^2 + \beta + 2$.

(iv) If $a = 4$, then $z = \beta^2 + 2\beta + 2$ or $z = -\beta^2 - \beta - 1$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (1, 7, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = \beta^2 + 2\beta + 2$, and the point $x_2 \in \mathcal{S}_a$ defined by $(p_1^{x_2}, c_1^{x_2}, s_1^{x_2}) = (\varepsilon, 1, 7)$, $(p_2^{x_2}, c_2^{x_2}, s_2^{x_2}) = (3, 5, \varepsilon)$ and $p_m^{x_2} = \varepsilon$ for every $m \geq 3$ is an eventually periodic representation of $z = -\beta^2 - \beta - 1$.

(v) If $a = 5$, then $z = \beta^2 + 2\beta + 2$ or $z = (3\beta^2 + 6\beta + 7)/2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (1, 8, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = \beta^2 + 2\beta + 2$, and the point $x_2 \in \mathcal{S}_a$ defined by

$$x_2 = (\varepsilon, 1, 8)(3, 5, \varepsilon)(\varepsilon, 1, 8)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9) \dots$$

is an eventually periodic representation of $z = (3\beta^2 + 6\beta + 7)/2$.

(vi) If $a = 6$, then $z = \beta^2 + 2\beta + 2$ or $z = (3\beta^2 + 6\beta + 7)/2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (1, 9, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = \beta^2 + 2\beta + 2$, and the point $x_2 \in \mathcal{S}_a$ defined by

$$x_2 = (\varepsilon, 1, 9)(3, 5, \varepsilon)(\varepsilon, 1, 8)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9) \dots$$

is an eventually periodic representation of $z = (3\beta^2 + 6\beta + 7)/2$.

(vii) If $a = 7$, then $z = -1$ or $z = (\beta^2 + 2\beta + 1)/2$. The point $x_1 \in \mathcal{S}_a$ defined by $(p_1^{x_1}, c_1^{x_1}, s_1^{x_1}) = (2, 9, \varepsilon)$ and $p_m^{x_1} = \varepsilon$ for every $m \geq 2$ is an eventually periodic representation of $z = -1$, and the point $x_2 \in \mathcal{S}_a$ defined by

$$x_2 = (\varepsilon, 2, 9)(4, 5, \varepsilon)(\varepsilon, 1, 8)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9)(\varepsilon, 3, 5)(4, 6, \varepsilon)(\varepsilon, 1, 9) \dots$$

is an eventually periodic representation of $z = (\beta^2 + 2\beta + 1)/2$. □

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REFERENCES

- [ABB11] P. Arnoux, J. Bernat and X. Bressaud. Geometrical models for substitutions. *Exp. Math.* **20**(1) (2011), 97–127.
- [AY81] P. Arnoux and J.-C. Yoccoz. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.* **292**(1) (1981), 75–78.
- [Bar99] R. Barreto. 1999 On C^r -class of conjugacy between affine interval exchange maps and its isometric model. *PhD Thesis*, Instituto Nacional de Matemática Pura e Aplicada.
- [BBH14] X. Bressaud, A. I. Bufetov and P. Hubert. Deviation of ergodic averages for substitution dynamical systems with eigenvalues of modulus 1. *Proc. Lond. Math. Soc. (3)* **109**(2) (2014), 483–522.
- [BHM10] X. Bressaud, P. Hubert and A. Maass. Persistence of wandering intervals in self-similar affine interval exchange transformations. *Ergod. Th. & Dynam. Sys.* **30**(3) (2010), 665–686.
- [CG97] R. Camelier and C. Gutiérrez. Affine interval exchange transformations with wandering intervals. *Ergod. Th. & Dynam. Sys.* **17**(6) (1997), 1315–1338.
- [Cob02] M. Cobo. Piece-wise affine maps conjugate to interval exchanges. *Ergod. Th. & Dynam. Sys.* **22**(2) (2002), 375–407.
- [CS01] V. Canterini and A. Siegel. Automate des préfixes-suffixes associé à une substitution primitive. *J. Théor. Nombres Bordeaux* **13**(2) (2001), 353–369.
- [Den32] A. Denjoy. Sur les courbes définies par les équations différentielles à la surface du tore. *J. Math. Pures Appl. (9)* **11**(9) (1932), 333–376.
- [Fog02] P. Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics (Lecture Notes in Mathematics, 1794)*. Springer, Berlin, 2002.
- [Kea75] M. Keane. Interval exchange transformations. *Math. Z.* **141**(1) (1975), 25–31.
- [Lev87] G. Levitt. La décomposition dynamique et la différentiabilité des feuilletages des surfaces. *Ann. Inst. Fourier (Grenoble)* **37**(3) (1987), 85–116.
- [LPV07] J. H. Lowenstein, G. Poggiaspalla and F. Vivaldi. Interval exchange transformations over algebraic number fields: the cubic Arnoux–Yoccoz model. *Dyn. Syst.* **22**(1) (2007), 73–106.
- [LPV08] J. H. Lowenstein, G. Poggiaspalla and F. Vivaldi. Geometric representation of interval exchange maps over algebraic number fields. *Nonlinearity* **21**(1) (2008), 149–177.
- [Mes00] A. Messaoudi. Frontière du fractal de Rauzy et système de numération complexe. *Acta Arith.* **95**(3) (2000), 195–224.
- [MMY10] S. Marmi, P. Moussa and J.-C. Yoccoz. Affine interval exchange maps with a wandering interval. *Proc. Lond. Math. Soc. (3)* **100**(3) (2010), 639–669.
- [Mos92] B. Mossé. Puissances de mots et reconnaissabilité des points fixes d’une substitution. *Theoret. Comput. Sci.* **99**(2) (1992), 327–334.
- [Que87] M. Queffélec. *Substitution Dynamical Systems—Spectral Analysis (Lecture Notes in Mathematics, 1294)*. Springer, Berlin, 1987.
- [Vee78] W. Veech. Interval exchange transformations. *J. Anal. Math.* **33**(1) (1978), 222–272.
- [Vee82] W. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)* **115**(1) (1982), 201–242.
- [Vee84] W. Veech. The metric theory of interval exchange transformations I. Generic spectral properties. *Amer. J. Math.* **106**(6) (1984), 1331–1359.
- [Via06] M. Viana. Ergodic theory of interval exchange maps. *Rev. Mat. Complut.* **19**(1) (2006), 7–100.
- [Zor96] A. Zorich. Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents. *Ann. Inst. Fourier (Grenoble)* **46**(2) (1996), 325–370.