International Journal of Non-Linear Mechanics 105 (2018) 113–129

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Modelling residual stresses in elastic bodies described by implicit constitutive relations

R. Bustamante [a,*], K.R. Rajagopal [b]

[a] Departamento de Ingeniería Mecánica, Computational and Applied Mechanics Laboratory and Center for Modern Computational Engineering, Universidad de Chile, Beaucheff 851, Santiago Centro, Santiago, Chile

[b] Department of Mechanical Engineering, University of Texas A&M, College Station, TX, USA

ARTICLE INFO

Keywords:
Nonlinear elasticity
Initial stresses
Residual stresses
Small strains

ABSTRACT

In this paper we study the response of bodies that are residually stressed within the context of a new class of constitutive relations, wherein the strains are assumed to be functions of the stresses. Such bodies are said to have residual stresses if there are stresses within the bodies even though the bodies are unstrained in the configuration of interest in the absence of external traction. Problems within the context of the norm of the gradient of the displacement field being small are considered, with regard to the determination of the residual stresses in an anisotropic cylindrical annulus with two preferred directions, and the nature of residual stresses within an anisotropic slab. The residual stresses in a body that is subject to incremental stresses are also studied.

1. Introduction

This paper concerns the development of response relations for elastic bodies when one has to take into consideration 'residual stresses', within the context of implicit constitutive theories recently introduced by Rajagopal (see Rajagopal [1,2]; see also Rajagopal and Srinivasa [3]). By 'residual stress' one means a body not being free of stress in the interior of the body though it is free of traction on the boundary of the body. Before one can embark on such a venture, one needs to first come to grips with what one means by 'residual stresses'. According to the Oxford English Dictionary [4], the primary meaning ascribed to the word 'residue' is: *The remainder, rest; that which is left*. Thus, the terminology 'residual stresses' implies that a body, in some configuration was subject to deformations, and at the end of the process the stresses 'which are left', that is the stresses remaining within the body are the 'residual stresses' in the body. The terminology 'pre-stress' on the other hand refers to stresses that were present in the body before it is subject to a particular process. Of course, it is most likely, the 'pre-stress' in a body might be the 'residual stress' due to some prior process the body was subject to. We may never be able to decide on whether the state of stress in a body is a 'pre-stress' or 'residual stress'. However, since we are interested in describing the response of a body that we have in hand, when it is subject to deformations, it would be most appropriate to refer to the state of stress prior to our deforming the body as 'pre-stress' rather than 'residual stress'. We shall however refer to it as 'residual stress' in keeping with the current usage.

The stress free configuration that corresponds to the current 'residually stressed' configuration is determined experimentally by carrying out 'cuts' (see Fung [5]) that supposedly relieve the 'residual stresses' within the body. Unfortunately, such 'cuts' cannot be described within the context of classical continuum mechanics as they are not diffeomorphisms, they lack the smoothness which is usually required of the motion. Nonetheless, a stress-free configuration so obtained by making cuts is used to determine the stress in the 'residually stressed' configuration.

In the classical theory of Cauchy elasticity, the reference configuration is usually considered to be free of stress (see Truesdell and Noll [6]) but there are several applications, especially in biomechanics, geomechanics, manufacturing, etc., wherein one has to develop models for the response of bodies which are in a state that is not free of stress (see, for example, in biomechanics [5,7,8], in welded structures [9,10], and in manufacturing [11–13][1]).

Several methods have been used to study 'residually stressed' bodies: The first method models the whole process that creates such stresses in the body, such as the modelling of growth, adaptation, modification, remodelling, development, maintenance, and healing of soft tissue (see, for example, [8], §4.3 of [15] and [16–20]), and elastic–plastic deformations (including phase transition) that have been considered for welding [9]. In biomechanics most of the approaches merely appeal to geometric ideas though there has been some effort to describe

[1] See also the introduction in the recent paper [14].

* Corresponding author.
E-mail address: rogbusta@ing.uchile.cl (R. Bustamante).

https://doi.org/10.1016/j.ijnonlinmec.2018.06.007
Received 1 March 2018; Received in revised form 21 May 2018; Accepted 16 June 2018
Available online 21 June 2018

the complicated mixture that comprises tissues (see Humphrey and Rajagopal [21]).

One approach to study the response of bodies that are in a configuration in which they are not stress free, is to hypothetically associate a stress free configuration with the current stressed configuration, which is at the heart of the notion of a 'natural configuration' associated with the current configuration of a body (see [22] for a discussion of the notion of 'natural configurations'). While in these studies the 'natural configuration' and its evolution have thermodynamic origins, the idea is similar to the use of an intermediate configuration in studying the response of inelastic bodies (see [23]). As observed earlier, the stress free configuration of a body can be determined experimentally by carrying out 'cuts' to relieve the stresses. For a Cauchy elastic body that is defined through $\mathbf{T} = \mathfrak{G}(\mathbf{F})$ (where $\mathbf{F}$ is the deformation gradient tensor) we have $\mathbf{T}_{\mathrm{R}} = \mathfrak{G}(\mathbf{F}_{\mathrm{R}})$, where $\mathbf{F}_{\mathrm{R}}$ is calculated from the stress free configuration to the reference configuration which is residually stressed [24]. If the body is isotropic, then the stress is given by $\mathbf{T} = \mathfrak{G}(\mathbf{B})$ and $\mathbf{T}_{\mathrm{R}} = \mathfrak{G}(\mathbf{B}_{\mathrm{R}})$, where $\mathbf{B}$ is the left Cauchy–Green tensor and $\mathbf{B}_{\mathrm{R}} = \mathbf{F}_{\mathrm{R}}\mathbf{F}_{\mathrm{R}}^{\mathrm{T}}$. A problem with this method is that it is not always possible to determine the stress free configuration from which $\mathbf{B}_{\mathrm{R}}$ can be determined so that $\mathbf{0} = \operatorname{div}\mathbf{T}_{\mathrm{R}} + \rho\mathbf{b}$ for the body and $\mathbf{T}_{\mathrm{R}}\mathbf{n} = \mathbf{0}$ on the boundary of the body.[2]

Yet another method to study problems involving residual stresses is by using the residual stress as an additional variable, say $\mathbf{T}_{\mathrm{R}}$ and express the Cauchy stress as $\mathbf{T} = \mathfrak{G}(\mathbf{B}, \mathbf{T}_{\mathrm{R}})$, with the restriction that we must have $\mathbf{T}_{\mathrm{R}} = \mathfrak{G}(\mathbf{B}, \mathbf{T}_{\mathrm{R}})$ (see, for example, [27–30]). This approach could be useful especially if we do not have information about the stress-free configuration for the body.

In the present paper we depart from the above approaches by considering subclasses of some constitutive equations, which belong to the implicit constitutive theory for elastic bodies proposed by Rajagopal and co-workers [1–3,31–33]. One such implicit relation corresponds to $\mathfrak{F}(\mathbf{T}, \mathbf{B}) = \mathbf{0}$, the classical Cauchy elastic body $\mathbf{T} = \mathfrak{G}(\mathbf{B})$ along with the constitutive equation $\mathbf{B} = \mathfrak{H}(\mathbf{T})$ and its subclass $\boldsymbol{\epsilon} = \mathfrak{h}(\mathbf{T})$, where $\boldsymbol{\epsilon}$ is the linearized strain tensor [34,35] being special cases. For example, in the case of the class of models defined by $\mathbf{B} = \mathfrak{H}(\mathbf{T})$ the reference configuration is residually stressed if $\mathbf{I} = \mathfrak{H}(\mathbf{T}_{\mathrm{R}})$ and $\mathbf{0} = \operatorname{div}\mathbf{T}_{\mathrm{R}} + \rho\mathbf{b}$ for the body with $\mathbf{T}_{\mathrm{R}}\mathbf{n} = \mathbf{0}$ on the boundary of the body.

In Section 2 after some preliminary discussion of the kinematics, we turn to a discussion of some basic concepts concerning what we mean by a body being elastic and we introduce implicit constitutive theories and their subclasses that describe elastic bodies. In Sections 3.1 and 3.2 we present the basic aspects of the modelling of residually stressed bodies within the context of some of the subclasses of constitutive relations presented in the previous section. In Section 3.3 we speak briefly about the use of the stress potential for problems concerning residual stresses. In Section 4.1 we present explicit expressions for $\mathfrak{h}(\mathbf{T})$ when $\mathfrak{h}$ leads to a response that is isotropic, transversely isotropic or a function that depends on the stress as the response has two preferred directions. In Section 4.2 we study the problem of the opening of a residually stressed annulus, when the response is isotropic and transversely isotropic, respectively, while in Section 4.3 we analyse the case of an annulus whose response depends on two preferred directions. The results presented in those two sections are used in order to choose an expression for $\mathfrak{h}$, from which we could obtain more interesting results. In Section 5 we discuss the problem of residual stresses in a cylindrical annulus, whose response exhibits dependence on two preferred directions, and obtain some numerical results for the stresses and strains. In Section 6 the state of residual stresses within a slab is analysed, and finally, in Section 7 we present an incremental formulation that can be used to study the effect of residual stresses on elastic bodies, by analysing the behaviour of such bodies when a small additional external traction is applied to them. We conclude in Section 8 with some remarks concerning the results presented in this work.

[2] See [25,26] for different concepts on residual stresses, especially in the context of the classical theory of nonlinear elasticity.

2. Basic equations

2.1. Kinematics and the equations of motion

Let a particle $X \in \mathscr{B}$ in an abstract body $\mathscr{B}$ occupy the position $\mathbf{X} \in \kappa_{\mathrm{R}}(\mathscr{B})$ in the reference configuration $\kappa_{\mathrm{R}}(\mathscr{B})$, and $\mathbf{x} \in \kappa_t(\mathscr{B})$ in the configuration at time t, $\kappa_t(\mathscr{B})$. It is assumed that there exists a one-to-one mapping χ such that $\mathbf{x} = \chi(\mathbf{X}, t)$. The deformation gradient $\mathbf{F}$, the left Cauchy–Green tensor $\mathbf{B}$, the displacement vector $\mathbf{u}$, and the linearized strain tensor $\boldsymbol{\epsilon}$ are defined through:

$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \boldsymbol{\epsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}^{\mathrm{T}}\right). \tag{1}$$

The Cauchy stress tensor is denoted by $\mathbf{T}$ and satisfies the equations of motion

$$\rho\ddot{\mathbf{x}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \tag{2}$$

where ρ is the density of the body and $\mathbf{b}$ represents the specific body forces acting on the current configuration, and where we have used the notation $(\dot{\ })$ for the material time derivative. More details concerning kinematics and the basic balance laws can be found, for example, in [36].

2.2. Constitutive relations

In [1–3,31] Rajagopal and co-workers have proposed implicit constitutive relations to describe the behaviour of elastic bodies. One such relation for isotropic bodies is of the form[3]

$$\mathfrak{F}(\mathbf{T}, \mathbf{B}) = \mathbf{0}, \tag{3}$$

which includes as a sub-class the classical Cauchy elastic body [6] $\mathbf{T} = \mathfrak{G}(\mathbf{B})$ and the class of models (see, for example, [37])

$$\mathbf{B} = \mathfrak{H}(\mathbf{T}). \tag{4}$$

An important restriction that $\mathfrak{H}$ has to meet is that for any stress the eigenvalues of $\mathfrak{H}(\mathbf{T})$ are positive.

In the case of the approximation that $|\nabla\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ we have the approximation $\mathbf{B} \approx 2\boldsymbol{\epsilon} + \mathbf{I}$ and from (4) we have the subclass[4]

$$\boldsymbol{\epsilon} = \mathfrak{h}(\mathbf{T}), \tag{5}$$

which is an important new class of constitutive relations in its own right, and has been proposed for the modelling of the problem of fracture in elastic bodies, and in describing the behaviour of some metallic alloys and rock, see, for example, [35,39–42]. It is important to recognize that the linearization based on the displacement gradient being small leads to models wherein one can have a nonlinear relationship between the linearized strain and the stress, an impossibility within the context of Cauchy elasticity.

In the present work we can also consider inhomogeneous bodies namely the situation when the functions $\mathfrak{H}$ and $\mathfrak{h}$ can also depend on the position $\mathbf{X}$. In the rest of the work we assume that $\mathbf{T}$ has been divided by a characteristic stress σ_o, and for the sake of simplicity we continue to use $\mathbf{T}$ to denote the dimensionless stress.

2.3. Boundary value problems

In order to solve a boundary value problem concerning such materials, one needs to solve simultaneously the constitutive relations and the balance equations of mass and linear momentum, for example, in the

[3] The meaning of what is meant by anisotropy and its classification for bodies defined by implicit constitutive relations is provided in [51].

[4] See the papers by Rajagopal [35,38] concerning the status of such approximations.

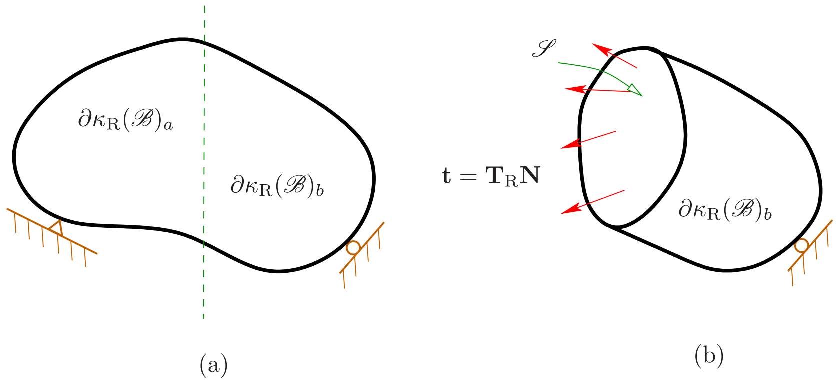

Fig. 1. (a) Residually stressed body in the reference configuration $\kappa_R(\mathscr{B})$. (b) After an imaginary slice.

case of (4) we need to look for ρ, $\mathbf{T}$ and χ such that they satisfy (2), $(1)_2$ and (4)

$$\dot{\rho} + \rho \operatorname{div} \dot{\chi} = 0, \quad \rho\ddot{\chi} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \quad \frac{\partial\chi}{\partial\mathbf{X}}\frac{\partial\chi}{\partial\mathbf{X}}^{\mathrm{T}} = \mathfrak{H}(\mathbf{T}). \tag{6}$$

While considering (5) we need to look for ρ, $\mathbf{T}$ and $\mathbf{u}$ such that (2), $(1)_4$ and (5) are satisfied, i.e.

$$\dot{\rho} + \rho \operatorname{div} \dot{\mathbf{u}} = 0 \quad \rho\ddot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \rho\mathbf{b}, \quad \frac{1}{2}\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}} + \frac{\partial\mathbf{u}}{\partial\mathbf{x}}^{\mathrm{T}}\right) = \mathfrak{h}(\mathbf{T}). \tag{7}$$

We shall only consider the balance of linear momentum and the constitutive equations to determine the stress and the displacement. Once the displacement is determined, the balance of mass can be used to determine the density. In the rest of this paper we will ignore the balance of mass and the determination of the density using the same.

3. Residually stressed bodies

3.1. The case of large elastic deformations

By a body in static equilibrium with residual stresses we mean a body which in a reference configuration $\kappa_R(\mathscr{B})$ is not free of stresses, i.e., there is a distribution of stresses, which we denote by $\mathbf{T}_R$, such that we have

$$\operatorname{Div}\mathbf{T}_R = \mathbf{0}, \quad \mathbf{I} = \mathfrak{H}(\mathbf{T}_R) \quad \text{in} \quad \mathbf{X} \in \kappa_R(\mathscr{B}),$$
$$\mathbf{T}_R\mathbf{N} = \mathbf{0} \quad \text{on} \quad \mathbf{X} \in \partial\kappa_R(\mathscr{B}), \tag{8}$$

where $\mathbf{N}$ is the outward normal unit vector to the surface $\partial\kappa_R(\mathscr{B})$. The above equation (8) must be understood not only as a restriction on $\mathbf{T}_R$, but also on $\mathfrak{H}$ as explained in detail in Section 5 for the case of $\mathfrak{h}$ (see (5)).

Let us assume that a body with residual stresses is cut by an imaginary surface $\mathscr{S}$ as shown in Fig. 1. Eq. $(8)_3$ implies that there is no external load (see Fig. 1(a)), however, when we cut the body along the imaginary surface, on that new surface (see Fig. 1(b)) we do have a distribution of traction $\mathbf{t} = \mathbf{T}_R\mathbf{N}$. If $\partial\kappa_R(\mathscr{B}) = \partial\kappa_R(\mathscr{B})_a \cup \partial\kappa_R(\mathscr{B})_b$, for the part of the body presented in Fig. 1(b) we have $\int_{\mathscr{S}\cup\partial\kappa_R(\mathscr{B})_b} \mathbf{t}\, \mathrm{d}A = \mathbf{0}$, which is equivalent to $\int_{\mathscr{S}} \mathbf{t}\, \mathrm{d}A = \mathbf{0}$ and thus we obtain

$$\int_{\mathscr{S}} \mathbf{T}_R\mathbf{N}\, \mathrm{d}A = \mathbf{0}, \tag{9}$$

which must be satisfied on every surface that results from an imaginary cutting of the body. It is necessary to observe that (9) in general is not an extra requirement placed on $\mathbf{T}_R$, as it is satisfied automatically if $\operatorname{Div}\mathbf{T}_R = \mathbf{0}$. In Section 6 we consider (9) for a particular boundary value problem for which such condition is not satisfied trivially.

Assume now it is possible to reach a stress-free configuration $\kappa_C(\mathscr{B})$ by performing a number of 'cuts'[5] (see, Fig. 2). Let $\mathbf{x}_C = \chi_C(\mathbf{X})$ be a map to $\kappa_C(\mathscr{B})$, so that

$$\mathbf{F}_C = \frac{\partial\chi_C}{\partial\mathbf{X}}, \quad \mathbf{B}_C = \mathbf{F}_C\mathbf{F}_C^{\mathrm{T}}. \tag{10}$$

If in that configuration we assume there is no residual stress, then from (4) we have

$$\mathbf{B}_C = \mathfrak{H}(\mathbf{0}), \tag{11}$$

3.2. The case of infinitesimal deformations

In this section we repeat briefly the theory presented above for the constitutive expression (5). A body in static equilibrium is said to be residually stressed if there exists a stress field $\mathbf{T}_R$ such that

$$\operatorname{div}\mathbf{T}_R = \mathbf{0}, \quad \mathbf{0} = \mathfrak{h}(\mathbf{T}_R) \quad \text{in} \quad \mathbf{x} \in \kappa_R(\mathscr{B}),$$
$$\mathbf{T}_R\mathbf{n} = \mathbf{0} \quad \text{on} \quad \mathbf{x} \in \partial\kappa_R(\mathscr{B}). \tag{12}$$

As in Section 3.1 the above equation could be interpreted as restrictions on $\mathbf{T}_R$ and $\mathfrak{h}$.

Let us assume again that the body is cut (see Fig. 2) and that all residual stresses are released producing as a result a displacement field $\mathbf{u}_C$. We can define the linearized strain tensor, associated with such deformation as

$$\varepsilon_C = \frac{1}{2}(\nabla\mathbf{u}_C + \nabla\mathbf{u}_C^{\mathrm{T}}). \tag{13}$$

and since there are no residual stresses we have (compare with (11))

$$\varepsilon_C = \mathfrak{h}(\mathbf{0}). \tag{14}$$

The above situation is possible if and only if $\mathfrak{h}(\mathbf{0})$ satisfies the compatibility equations (for the linearized case, see, for example, §4.10 in [43]).

3.3. On the stress potential

In the above two sections we have defined $\mathbf{T}_R$ as the residual stress that satisfies the equation of equilibrium, such that when replaced in the constitutive equation (10), (4) and (5) there is no associated deformation (see $(8)_2$, $(12)_2$). In the case $|\nabla\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, when we have

[5] Here we are speaking about a real cut, unlike in the discussion about Fig. 1. From now on when we speak about a body being cut we mean a real cut, and if we refer to an imaginary cut, we will state that explicitly.

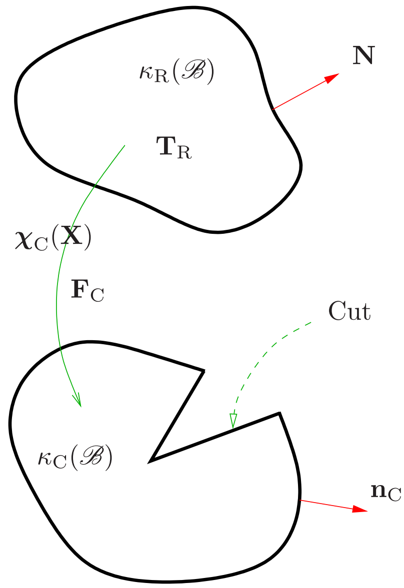

Fig. 2. Residually stressed body in the reference configuration $\kappa_{\mathrm{R}}(\mathscr{B})$, and in the cut configuration $\kappa_{\mathrm{C}}(\mathscr{B})$, where it is assumed that the body is stress-free.

infinitesimal deformations, (12) in index notation (assuming the use of, for example, Cartesian coordinates) becomes

$$\frac{\partial T_{\mathrm{R}_{ij}}}{\partial x_j} = 0, \quad 0 = \mathfrak{h}_{ij}(T_{\mathrm{R}_{kl}}) \quad \text{in} \quad x_i \in \kappa_r(\mathscr{B}),$$
$$T_{\mathrm{R}_{ij}} n_j = 0 \quad \text{on} \quad x \in \partial\kappa_r(\mathscr{B}). \tag{15}$$

It was stated that the six independent components of $\mathbf{T}_{\mathrm{R}}$ should satisfy the three equilibrium equations $(15)_1$ plus the six constitutive equations[6] $(15)_2$. Since in general that may not be possible, we have indicated that such equations should be seen also as restrictions on $\mathfrak{h}$ (we assume that the body is inhomogeneous, i.e., $\mathfrak{h} = \mathfrak{h}(\mathbf{T}, \mathbf{x})$, but we do not explicitly express the dependence on x).

On the other hand the stresses can be expressed in terms of a stress potential $\boldsymbol{\Phi}$, and in that case the number of equations to be solved is reduced. From §227 of [36] we have the representation

$$T_{\mathrm{R}_{ij}} = \epsilon_{ikl}\epsilon_{jmn}\frac{\partial^2 \Phi_{km}}{\partial x_l \partial x_n}, \tag{16}$$

where $\Phi_{km} = \Phi_{mk}$. Using (16) in $(15)_1$ that equation is satisfied, and the six independent components of $\boldsymbol{\Phi}$ should satisfy the six nonlinear partial differential equations $(15)_2$

$$\mathfrak{h}_{ij}\left(\epsilon_{pkl}\epsilon_{qmn}\frac{\partial^2 \Phi_{km}}{\partial x_l \partial x_n}\right) = 0, \quad x_i \in \kappa_r(\mathscr{B}), \tag{17}$$

where it should be understood that the functions $\mathfrak{h}_{ij}$ are evaluated at $\epsilon_{pkl}\epsilon_{qmn}\frac{\partial^2 \Phi_{km}}{\partial x_l \partial x_n}$. For the above equations we have the boundary conditions $(15)_3$

$$\epsilon_{ikl}\epsilon_{jmn}\frac{\partial^2 \Phi_{km}}{\partial x_l \partial x_n} n_j = 0, \quad x_i \in \partial\kappa_r(\mathscr{B}). \tag{18}$$

It can be seen that the nonlinear partial differential equations (17) must be solved in conjunction with the boundary conditions (18), where the order of the derivatives is the same as in the original equations (17). This is not a feature that is standard, and in general for (17) and (18) to be satisfied simultaneously some restrictions on $\mathfrak{h}_{ij}$ would come into play. In the following sections we choose not to work with the stress potential.

[6] We note that we have the 6 components of $\mathbf{T}_{\mathrm{R}}$ that would need to satisfy the 3 equilibrium equations plus the 6 components of the constitutive equation.

4. The opening of an annulus

In this section we study the problem of a cylindrical annulus with residual stresses, which after a radial cut along the axis displays an angular opening resulting in the release of all the stresses. The objective is use the information obtained from the opening of the annulus, in order to see which expression for $\mathfrak{h}(\mathbf{T})$ would be more interesting for the analysis to be carried out in the following sections. First, in Section 4.1 we present expressions for $\mathfrak{h}(\mathbf{T})$ in the case of bodies exhibiting isotropic, transversely isotropic and two directional anisotropy, then in Section 4.2 we study (14) in the case of an isotropic and transversely isotropic annulus, while in Section 4.3 we analyse (14) for the case of an annulus with two directional anisotropy.

4.1. Isotropic bodies, transversely isotropic bodies, and two directional anisotropy

We study a class of models for which there a exist scalar function $\Pi(\mathbf{T})$ such that (see [44])

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}) = \frac{\partial \Pi}{\partial \mathbf{T}}. \tag{19}$$

In the problems analysed in the following sections we discuss the solvability of (12), for which we will consider three cases, namely when Π is an isotropic function, a transversely isotropic function and a function with two preferred directions. If Π is an isotropic function, then $\Pi = \Pi(I_1, I_2, I_3)$, where

$$I_1 = \mathrm{tr}\,(\mathbf{T}), \quad I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{T}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{T}^3), \tag{20}$$

and from (19) we obtain the representation

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2, \tag{21}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3$.

If Π is a transversely isotropic function, we have that $\Pi = \Pi(\mathbf{T}, \mathbf{a})$, where $\mathbf{a}$ is a vector field representing the direction with regard to which the body is transversely isotropic and $|\mathbf{a}| = 1$. In this case[7] $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5)$, where I_1, I_2 and I_3 are given in (20) and I_4, I_5 are defined as (see [47])

$$I_4 = \mathbf{a}\cdot(\mathbf{Ta}), \quad I_5 = \mathbf{a}\cdot(\mathbf{T}^2\mathbf{a}), \tag{22}$$

and from (19) we obtain

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2 + \Pi_4\mathbf{a}\otimes\mathbf{a} + \Pi_5[(\mathbf{Ta})\otimes\mathbf{a} + \mathbf{a}\otimes(\mathbf{Ta})], \tag{23}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3, 4, 5$.

Finally, let us consider the case $\Pi = \Pi(\mathbf{T}, \mathbf{a}, \mathbf{b})$, where there is another direction preference along the vector field $\mathbf{b}$ with $|\mathbf{b}| = 1$. In this case $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9)$, where I_j, $j = 1, 2, \dots, 9$ are given in (20), (22) and

$$I_6 = \mathbf{b}\cdot(\mathbf{Tb}), \quad I_7 = \mathbf{b}\cdot(\mathbf{T}^2\mathbf{b}), \quad I_8 = (\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\cdot(\mathbf{Tb}) + \mathbf{b}\cdot(\mathbf{Ta})], \tag{24}$$

$$I_9 = (\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\cdot(\mathbf{T}^2\mathbf{b}) + \mathbf{b}\cdot(\mathbf{T}^2\mathbf{a})], \tag{25}$$

and from (19) we have

$$\begin{aligned}\boldsymbol{\varepsilon} = {} & \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2 + \Pi_4\mathbf{a}\otimes\mathbf{a} + \Pi_5[(\mathbf{Ta})\otimes\mathbf{a} + \mathbf{a}\otimes(\mathbf{Ta})] + \Pi_6\mathbf{b}\otimes\mathbf{b} \\ & + \Pi_7[(\mathbf{Tb})\otimes\mathbf{b} + \mathbf{b}\otimes(\mathbf{Tb})] + \Pi_8(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}] \\ & + \Pi_9(\mathbf{a}\cdot\mathbf{b})[\mathbf{a}\otimes(\mathbf{Tb}) + (\mathbf{Tb})\otimes\mathbf{a} + \mathbf{b}\otimes(\mathbf{Ta}) + (\mathbf{Ta})\otimes\mathbf{b}],\end{aligned} \tag{26}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, \dots, 9$.

[7] In [45,46] a new set of invariants have been proposed for the case of transversely isotropic functions, and functions that depend on two vector fields. In these works it has been proved that some of the invariants presented in [47] are not independent.

4.2. The opening of an annulus, the case of an isotropic and a transversely isotropic annulus

Let us consider the cylindrical annulus in the reference configuration defined through

$$r_{\rm i}\leq r\leq r_{\rm o},\quad 0\leq\theta\leq 2\pi,\quad 0\leq z\leq L. \tag{27}$$

We assume that this tube has a residual stress distribution of the form

$$\mathbf{T}_{\rm R}=\mathbf{T}_{\rm R}(r). \tag{28}$$

For such a body the surfaces for which $(12)_3$ must be satisfied are the surfaces $r=r_{\rm i}$, $r=r_{\rm o}$ for which $\mathbf{n}=\mp\mathbf{e}_r$, respectively, and the surfaces $z=0$, $z=L$ for which we have $\mathbf{n}=\mp\mathbf{e}_z$, respectively. It follows from $(12)_3$ and (28) that

$$T_{{\rm R}_{rr}}(r_{\rm i})=0,\quad T_{{\rm R}_{rr}}(r_{\rm o})=0,\quad T_{{\rm R}_{r\theta}}(r_{\rm i})=0,\quad T_{{\rm R}_{r\theta}}(r_{\rm o})=0, \tag{29}$$

$$T_{{\rm R}_{rz}}(r)=0,\quad T_{{\rm R}_{\theta z}}(r)=0,\quad T_{{\rm R}_{zz}}(r)=0. \tag{30}$$

The non-zero components of the residual stress tensor must satisfy the equilibrium equations $(12)_1$, which in this case become

$$\frac{{\rm d}T_{{\rm R}_{rr}}}{{\rm d}r}+\frac{1}{r}(T_{{\rm R}_{rr}}-T_{{\rm R}_{\theta\theta}})=0,\quad \frac{{\rm d}T_{{\rm R}_{r\theta}}}{{\rm d}r}+\frac{2}{r}T_{{\rm R}_{r\theta}}=0, \tag{31}$$

and this last equation is satisfied if $T_{{\rm R}_{r\theta}}(r)=\frac{c_o}{r^2}$, which from $(29)_{3,4}$ is possible if and only if $c_o=0$.

We shall assume that Π has the form

$$\Pi=\Pi(\mathbf{T},r). \tag{32}$$

From $(12)_2$ and (21) and the above assumptions we obtain

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)+\Pi_2(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)T_{{\rm R}_{rr}}+\Pi_3(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)T^2_{{\rm R}_{rr}}=0, \tag{33}$$

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)+\Pi_2(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)T_{{\rm R}_{\theta\theta}}+\Pi_3(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)T^2_{{\rm R}_{\theta\theta}}=0, \tag{34}$$

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},r)=0, \tag{35}$$

where $I_{{\rm R}_1}=T_{{\rm R}_{rr}}+T_{{\rm R}_{\theta\theta}}$, $I_{{\rm R}_2}=\frac{1}{2}(T^2_{{\rm R}_{rr}}+T^2_{{\rm R}_{\theta\theta}})$ and $I_{{\rm R}_3}=\frac{1}{3}(T^3_{{\rm R}_{rr}}+T^3_{{\rm R}_{\theta\theta}})$. Eqs. $(31)_1$ and (33)–(35) must be satisfied by $T_{{\rm R}_{rr}}$, $T_{{\rm R}_{\theta\theta}}$ and Π. It is imperative to recognize that the residual stress has been treated as a part of the constitutive relation. If the body is cut and the residual stresses are relieved, the function Π will need to satisfy additional restrictions as shown below.

Let us assume now that the cylindrical annulus is cut in the radial direction along the axis, and that due to that cut all residual stresses are relieved, such that (14) is met. Furthermore, let us suppose that the displacement field $\mathbf{u}_{\rm C}$ is of the form

$$u_{{\rm C}_r}=U(r),\quad u_{{\rm C}_\theta}=(k-1)\theta r,\quad u_{{\rm C}_z}=(\lambda_z-1)z, \tag{36}$$

where $k=\frac{2\pi-\alpha}{2\pi}$, α being the opening angle and $\lambda_z>0$ is a constant, and leads to a diagonal ϵ as required by (14) and (21). It follows from (14) and (36) that

$$\epsilon_{{\rm C}_{rr}}=\frac{{\rm d}U}{{\rm d}r}=\Pi_1(\mathbf{0},r),\quad \epsilon_{{\rm C}_{\theta\theta}}=k-1+\frac{U}{r}=\Pi_1(\mathbf{0},r),$$
$$\epsilon_{{\rm C}_{zz}}=\lambda_z-1=\Pi_1(\mathbf{0},r). \tag{37}$$

From $(37)_3$ in $(37)_1$ we obtain $U=(\lambda_z-1)r+C_1$, where C_1 is a constant, whereas using $(37)_3$ in $(37)_2$ we obtain $U=(\lambda_z-k)r+C_2$, where C_2 is a constant. These two solutions are compatible if $k=1$, i.e., $\alpha=0$. Another way to solve (37) is to consider first $(37)_{(1,2)}$ from where we obtain $\frac{{\rm d}U}{{\rm d}r}=k-1+\frac{U}{r}$, whose solution is $U(r)=Cr+(k-1)r\ln r$, where C is a constant. But from this solution we have that $\Pi_1(\mathbf{0},r)=\frac{{\rm d}U}{{\rm d}r}=C+(k-1)(1+\ln r)$, which is compatible with the fact that λ_z is a constant (see $(37)_3$) if $k=1$.

There are three other possibilities wherein we can obtain more interesting results:

- In the case of anisotropic bodies.
- In the case the distributions of the residual stresses that would depend on the axial and angular position.
- In the case of a more general and complex form for the displacement $\mathbf{u}_{\rm C}$.

Let us explore briefly the case of a transversely isotropic body with a stress distribution as in (28) (in the following section we consider more general expressions for the residual stresses). Let us assume that $\mathbf{a}=\mathbf{e}_z$ and that $\Pi=\Pi(\mathbf{T},\mathbf{a},r)$ then from (23), $(12)_2$ we have

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)+\Pi_2(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)T_{{\rm R}_{rr}}$$
$$+\Pi_3(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)T^2_{{\rm R}_{rr}}=0, \tag{38}$$

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)+\Pi_2(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)T_{{\rm R}_{\theta\theta}}$$
$$+\Pi_3(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)T^2_{{\rm R}_{\theta\theta}}=0, \tag{39}$$

$$\Pi_1(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)+\Pi_4(I_{{\rm R}_1},I_{{\rm R}_2},I_{{\rm R}_3},I_{{\rm R}_4},I_{{\rm R}_5},r)=0. \tag{40}$$

Let us consider the same form for the displacement field, namely (36), if the tube is cut radially and axially, then from (14) and (23) we obtain

$$\frac{{\rm d}U}{{\rm d}r}=\Pi_1(\mathbf{0},r),\quad k-1+\frac{U}{r}=\Pi_1(\mathbf{0},r),\quad \lambda_z-1=\Pi_1(\mathbf{0},r)+\Pi_4(\mathbf{0},r). \tag{41}$$

From $(41)_{(1,2)}$ we have $\frac{{\rm d}U}{{\rm d}r}=k-1+\frac{U}{r}$ whose solution is $U(r)=Cr+(k-1)r\ln r$, where C is a constant. From $(41)_3$ such a solution is possible if $\Pi_4(\mathbf{0},r)=\lambda_z-1+C+(1-k)(\ln r+1)$. Other similar interesting cases can be obtained if we assume $\mathbf{a}=\mathbf{e}_r$ or $\mathbf{a}=\mathbf{e}_\theta$.

With the purpose of obtaining a more general expression for $U(r)$ in the next section we consider the case of an elastic annulus with two preferred directions.

4.3. The opening of an annulus comprised of an anisotropic body with two preferred directions

In this section the same problem presented in the previous section is studied for the special case

$$\mathbf{a}=c\mathbf{e}_\theta+s\mathbf{e}_z,\quad \mathbf{b}=c\mathbf{e}_\theta-s\mathbf{e}_z, \tag{42}$$

where $c=\cos\beta$ and $s=\sin\beta$.

In the event that $\mathbf{T}=\mathbf{T}(r)$ from (29), (30) we find that the nonzero components of the stress tensor are T_{rr} and $T_{\theta\theta}$. We will now assume that $\Pi=\Pi(\mathbf{T},\mathbf{a},\mathbf{b},r)$, and from (26) and $(12)_2$ we have

$$\Pi_1(I_{{\rm R}_j},r)+\Pi_2(I_{{\rm R}_j},r)T_{{\rm R}_{rr}}+\Pi_3(I_{{\rm R}_j},r)T^2_{{\rm R}_{rr}}=0, \tag{43}$$

$$\Pi_1(I_{{\rm R}_j},r)+\Pi_2(I_{{\rm R}_j},r)T_{{\rm R}_{\theta\theta}}+\Pi_3(I_{{\rm R}_j},r)T^2_{{\rm R}_{\theta\theta}}+\Pi_4(I_{{\rm R}_j},r)c^2$$
$$+2\Pi_5(I_{{\rm R}_j},r)c^2T_{{\rm R}_{\theta\theta}}+\Pi_6(I_{{\rm R}_j},r)c^2+2\Pi_7(I_{{\rm R}_j},r)c^2T_{{\rm R}_{\theta\theta}}$$
$$+2\Pi_8(I_{{\rm R}_j},r)(1-2s^2)c^2+4\Pi_9(I_{{\rm R}_j},r)(1-2s^2)c^2T_{{\rm R}_{\theta\theta}}=0, \tag{44}$$

$$\Pi_1(I_{{\rm R}_j},r)+\Pi_4(I_{{\rm R}_j},r)s^2+\Pi_6(I_{{\rm R}_j},r)s^2-2\Pi_8(I_{{\rm R}_j},r)(1-2s^2)s^2=0, \tag{45}$$

$$(\Pi_4(I_{{\rm R}_j},r)-\Pi_6(I_{{\rm R}_j},r))cs=0, \tag{46}$$

where $I_{{\rm R}_j}$, $j=1,2,\dots,9$ are defined through (20), (22) and (24). The four equations given above and $(31)_1$ must be satisfied by $\mathbf{T}_{\rm R}$ and the choice of Π. In particular, from (46) we find that the equation is satisfied for any r and β if

$$\Pi_4(I_{{\rm R}_j},r)=\Pi_6(I_{{\rm R}_j},r). \tag{47}$$

As mentioned in our discussion of the two cases considered earlier in Section 4, in general Eqs. (43)–(45) and $(31)_1$ must be considered as restrictions not only on $T_{{\rm R}_{rr}}$ and $T_{{\rm R}_{\theta\theta}}$ but also on $\Pi(\mathbf{T},r)$.

Let us assume that the tube is cut in the radial and axial directions so that all residual stresses are released, assuming that due to the cuts the tube deforms in a manner described by (36), then from (14) and (26) we obtain that

$$\frac{\mathrm{d}U}{\mathrm{d}r} = \Pi_1(\mathbf{0}, r), \tag{48}$$

$$k - 1 + \frac{U}{r} = \Pi_1(\mathbf{0}, r) + \Pi_4(\mathbf{0}, r)c^2 + \Pi_6(\mathbf{0}, r)c^2 + 2\Pi_8(\mathbf{0}, r)(1 - 2s^2)c^2, \tag{49}$$

$$\lambda_z - 1 = \Pi_1(\mathbf{0}, r) + \Pi_4(\mathbf{0}, r)s^2 + \Pi_6(\mathbf{0}, r)s^2 - 2\Pi_8(\mathbf{0}, r)(1 - 2s^2)s^2, \tag{50}$$

$$0 = [\Pi_4(\mathbf{0}, r) - \Pi_6(\mathbf{0}, r)]cs. \tag{51}$$

When $U(r)$, k and λ_z are known, these must be seen as restrictions on Π. In general we may have limited or partial knowledge concerning the function $U(r)$, i.e., from experiments we may be able to measure $U(r_\mathrm{i})$ and $U(r_\mathrm{o})$ but we might not have information with regard to U at other radial positions. On the other hand we should be able to measure k and λ_z. Then from (48) we would obtain

$$U(r) = \int_{r_\mathrm{i}}^{r} \Pi_1(\mathbf{0}, \xi)\,\mathrm{d}r + U(r_\mathrm{i}), \tag{52}$$

where $\Pi(\mathbf{T}, r)$ should be such that $\int_{r_\mathrm{i}}^{r_\mathrm{o}} \Pi_1(\mathbf{0}, \xi)\,\mathrm{d}r + U(r_\mathrm{i}) = U(r_\mathrm{o})$. In Section 5 we study this problem in more detail for a specific expression for Π. The special cases when the body is isotropic or transversely isotropic can be obtained as special subclasses of (26).

5. A specific model for Π and some numerical results for the problem of the opening of an annulus

In this section we study in more detail the problem presented in the previous section for the following special case for the function Π

$$\Pi(\mathbf{T}, \mathbf{a}, \mathbf{b}, r) = c_0 I_1 + c_1 I_1^2 + \sum_{i=2}^{9} c_i I_i, \tag{53}$$

wherein we assume $c_j = c_j(r)$, $j = 0, 1, 2 \ldots 9$.

In Section 5.1 we consider the case of an annulus with two preferred directions and residual stresses that only depend on the radial position. We do not use information from a radial cut. In Section 5.2 we study in more detail the problem presented in Section 4.3, in particular looking for the equations that the components of the residual stresses and the functions $c_j(r)$ have to satisfy, when assuming that some information from a radial cut of an annulus with two preferred directions is known. Finally, in Section 5.3 we study the behaviour of the cylindrical annulus which deforms under external loads, considering for simplicity the expressions for the functions $c_j(r)$ obtained from Section 5.1.

5.1. The case of a residually stressed cylindrical annulus when the residual stresses are known and depend on the radial position

When we assume that the residual stresses depend only on the radial position for a cylindrical annulus, from $(31)_1$ we find that $T_{\mathrm{R}_{\theta\theta}} = \frac{\mathrm{d}}{\mathrm{d}r}(r T_{\mathrm{R}_{rr}})$ and the boundary conditions $(29)_{1,2}$ are $T_{\mathrm{R}_{rr}}(r_\mathrm{i}) = 0$ and $T_{\mathrm{R}_{rr}}(r_\mathrm{o}) = 0$. In this section we do not consider any cuts that relieve the stresses in the body, thus we are only interested in solving $(31)_1$, (43)–(46). As a simplification we consider a distribution for $T_{\mathrm{R}_{rr}}$ that satisfies the conditions $(29)_{1,2}$. One such possibility is (see, for example Eq. (46) of [29])

$$T_{\mathrm{R}_{rr}}(r) = d_0(r - r_\mathrm{i})(r - r_\mathrm{o}), \tag{54}$$

where d_0 is a constant. The validity or usefulness of the above expression for $T_{\mathrm{R}_{rr}}$, and in particular for a possible value for d_0 can be assessed indirectly by studying the behaviour of the same cylindrical annulus subject to some known external loads, as investigated in Section 5.3, and also by studying the behaviour of the cylindrical annulus subject to incremental stresses as presented in Section 7.1.

From $T_{\mathrm{R}_{\theta\theta}} = \frac{\mathrm{d}}{\mathrm{d}r}(r T_{\mathrm{R}_{rr}})$ we obtain that

$$T_{\mathrm{R}_{\theta\theta}}(r) = d_0[(r - r_\mathrm{i})(r - r_\mathrm{o}) + r(r - r_\mathrm{o}) + r(r - r_\mathrm{i})]. \tag{55}$$

In the above system of equations $T_{\mathrm{R}_{zz}} = 0$ and $T_{\mathrm{R}_{ij}} = 0$ if $i \neq j$.

Considering the simplified expression for Π presented in (53), Eqs. (43)–(46) become

$$c_0 + 2c_1 I_{\mathrm{R}_1} + c_2 T_{\mathrm{R}_{rr}} + c_3 T_{\mathrm{R}_{rr}}^2 = 0, \tag{56}$$

$$c_0 + 2c_1 I_{\mathrm{R}_1} + c_2 T_{\mathrm{R}_{\theta\theta}} + c_3 T_{\mathrm{R}_{\theta\theta}}^2 + c_4 c^2 + 2c_5 c^2 T_{\mathrm{R}_{\theta\theta}} + c_6 c^2$$
$$+ 2c_7 c^2 T_{\mathrm{R}_{\theta\theta}} + 2c_8(1 - 2s^2)c^2 + 4c_9(1 - 2s^2)c^2 T_{\mathrm{R}_{\theta\theta}} = 0, \tag{57}$$

$$c_0 + 2c_1 I_{\mathrm{R}_1} + c_4 s^2 + c_6 s^2 - 2c_8(1 - 2s^2)s^2 = 0, \tag{58}$$

$$(c_4 - c_6)cs = 0, \tag{59}$$

where $I_{\mathrm{R}_1} = T_{\mathrm{R}_{rr}} + T_{\mathrm{R}_{\theta\theta}}$.

In this problem $T_{\mathrm{R}_{rr}}(r)$ and $T_{\mathrm{R}_{\theta\theta}}(r)$ are known (up to a constant d_0) and given by (54), (55), therefore (56)–(59) must be solved for some of the functions $c_i = c_i(r)$. Eq. (59) is solved easily if we assume that $c_4 = c_6$. Let us choose c_0, c_4 and c_8 as the functions to be found from (56)–(59) in terms of c_1, c_2, c_3, c_5 and c_9 (assuming as well that $c_7 = c_5$), we obtain that

$$c_0(r) = -2c_1 I_{\mathrm{R}_1} - T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}}), \tag{60}$$

$$c_4(r) = \frac{1}{4}\{T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}})\csc^2\beta - 4T_{\mathrm{R}_{\theta\theta}}[c_5 + c_9\cos(2\beta)]$$
$$+ (T_{\mathrm{R}_{rr}} - T_{\mathrm{R}_{\theta\theta}})[c_2 + c_3(T_{\mathrm{R}_{rr}} + T_{\mathrm{R}_{\theta\theta}})]\sec^2\beta\}, \tag{61}$$

$$c_8(r) = -\frac{1}{4}\{4T_{\mathrm{R}_{\theta\theta}}[c_5 + c_9\cos(2\beta)] + 4T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}})\cot(2\beta)\csc(2\beta)$$
$$+ T_{\mathrm{R}_{\theta\theta}}(c_2 + c_3 T_{\mathrm{R}_{\theta\theta}})\sec^2\beta\}\sec(2\beta). \tag{62}$$

5.2. A residual stress tensor that depends on the radial position and an opening angle that is constant due a radial cut of the annulus

In this section we explore the same problem as that considered in Section 5.1, but now assuming that the cylindrical annulus has been cut in order to relieve the residual stresses, which provide additional information about the body. We are in particular interested in obtaining the equations that some of the functions $c_j(r)$ and the components of the residual stress must satisfy for this problem.

In this problem the residual stress tensor $\mathbf{T}_\mathrm{R}$ has two components $T_{\mathrm{R}_{rr}}$, $T_{\mathrm{R}_{\theta\theta}}$, and from the equilibrium equations $(31)_1$ they have to satisfy $T_{\mathrm{R}_{\theta\theta}} = \frac{\mathrm{d}}{\mathrm{d}r}(r T_{\mathrm{R}_{rr}})$.

In this case, from the results presented in Section 4.3, it follows from (53) and (52) that

$$U(r) = \int_{r_\mathrm{i}}^{r} c_0(\xi)\,\mathrm{d}\xi + U(r_\mathrm{i}). \tag{63}$$

Eq. (51) (see also (46)) is satisfied if $\Pi_4 = \Pi_6$ which in the case of (53) is possible if

$$c_4(r) = c_6(r). \tag{64}$$

From (49) and (50), in virtue of the above results we obtain that

$$k - 1 + \frac{1}{r}\left[\int_{r_\mathrm{i}}^{r} c_0(\xi)\,\mathrm{d}\xi + U(r_\mathrm{i})\right] = c_0(r) + 2c_4(r)c^2 + 2c_8(r)(1 - 2s^2)c^2, \tag{65}$$

$$\lambda_z - 1 = c_0(r) + 2c_4(r)s^2 - 2c_8(r)(1 - 2s^2)s^2. \tag{66}$$

If $U(r_\mathrm{i})$ and $U(r_\mathrm{o})$ are known from experiments, then in view of (63), $c_0(r)$ should satisfy

$$\int_{r_\mathrm{i}}^{r_\mathrm{o}} c_0(\xi)\,\mathrm{d}\xi = U(r_\mathrm{o}) - U(r_\mathrm{i}). \tag{67}$$

The constants k and λ_z in (65), (66) could also be determined by corroboration against experimental data.

From (53) we have

$$\Pi_1(I_{\mathrm{R}_j}, r) = c_0(r) + 2c_1(r) I_{\mathrm{R}_1}, \tag{68}$$

where

$$I_{R_1}(r) = T_{R_{rr}}(r) + T_{R_{\theta\theta}}(r), \tag{69}$$

and thus from (45) we have

$$c_0(r) + 2c_1(r)I_{R_1}(r) + 2c_4(r)s^2 - 2c_8(r)(1-2s^2)s^2 = 0, \tag{70}$$

which in virtue of (66) becomes $2c_1(r)I_{R_1} + \lambda_z - 1 = 0$, which suggests that in general $c_1(r)$ should be found as a part of the solution to the problem rather than be given as a known function a priori. In this case we obtain

$$c_1(r) = \frac{1-\lambda_z}{2I_{R_1}(r)}. \tag{71}$$

It follows from (43) and (71) that

$$c_0(r) + 1 - \lambda_z + c_2(r)T_{R_{rr}}(r) + c_3(r)T^2_{R_{rr}}(r) = 0. \tag{72}$$

From (44), (71), $(31)_1$ and (64) we obtain that

$$c_0(r) + 1 - \lambda_z + c_2(r)\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}}) + c_3(r)\left[\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}})\right]^2$$
$$+2c_4(r)c^2 + 2c_5(r)c^2\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}}) + 2c_7(r)c^2\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}})$$
$$+2c_8(r)(1-2s^2)c^2 + 2c_9(1-2s^2)c^2\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}}) = 0. \tag{73}$$

In view of (65), (66), (72) and (73) we have four equations for $T_{R_{rr}}$, $c_0(r)$, $c_2(r)$, $c_3(r)$, $c_4(r)$, $c_5(r)$, $c_7(r)$, $c_8(r)$ and $c_9(r)$. Some of these functions $c_k(r)$ could be obtained by corroboration against experimental data. Let us assume that

$$c_5(r) = c_7(r), \tag{74}$$

and that $c_0(r)$, $c_3(r)$, $c_5(r)$ and $c_9(r)$ are obtained by correlations with experimental results. From (66) we have

$$2c_4(r) = \frac{\lambda_z - 1 - c_0(r)}{s^2} + 2c_8(r)(1-2s^2), \tag{75}$$

and using this in (65) we obtain

$$2c_8(r)(1-2s^2)c^2 = \frac{1}{2}\left\{k - 1 + \frac{1}{r}\left[\int_{r_i}^{r} c_0(\xi)\,\mathrm{d}\xi + U(r_i)\right]\right.$$
$$\left. + c_0(r)(\cot^2\beta - 1) + (1-\lambda_z)\cot^2\beta\right\}, \tag{76}$$

and replacing this expression for $2c_8(r)(1-2\sin^2\beta)\cos^2\beta$ in (75) we have

$$2c_4(r) = \frac{\lambda_z - 1 - c_0(r)}{s^2} + \frac{1}{2c^2}\left\{k - 1 + \frac{1}{r}\left[\int_{r_i}^{r} c_0(\xi)\,\mathrm{d}\xi + U(r_i)\right]\right.$$
$$\left. + c_0(r)(\cot^2\beta - 1) + (1-\lambda_z)\cot^2\beta\right\}. \tag{77}$$

From (72) we can express, for example, $c_2(r)$ in terms of $T_{R_{rr}}(r)$, $c_3(r)$ and $c_0(r)$ as

$$c_2(r) = \frac{\lambda_z - 1 - c_0(r)}{T_{R_{rr}}} - c_3(r)T_{R_{rr}}. \tag{78}$$

From (73) we finally have

$$\left[c_3(r)\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}}) + \frac{\lambda_z - 1 - c_0(r)}{T_{R_{rr}}} - c_3(r)T_{R_{rr}} + 4c_5(r)c^2\right.$$
$$\left. +4c_9(1-2s^2)c^2\right]\frac{\mathrm{d}}{\mathrm{d}r}(rT_{R_{rr}}) = \lambda_z - k - \frac{1}{r}\left[\int_{r_i}^{r} c_0(\xi)\,\mathrm{d}\xi + U(r_i)\right]. \tag{79}$$

The above equation could be used to determine $T_{R_{rr}}(r)$; regarding the functions $c_0(r)$, $c_3(r)$, $c_5(r)$ and $c_9(r)$, we could assume they are given data. The functions $T_{R_{rr}}(r)$ and $c_0(r)$ would need to satisfy the conditions $T_{R_{rr}}(r_i) = 0$, $T_{R_{rr}}(r_o) = 0$ and $\int_{r_i}^{r_o} c_0(\xi)\,\mathrm{d}\xi = U(r_o) - U(r_i)$. The above equation is a first order nonlinear ordinary differential equation, and only one of the conditions $T_{R_{rr}}(r_i) = 0$, $T_{R_{rr}}(r_o) = 0$ could be applied directly, say $T_{R_{rr}}(r_i) = 0$, the other condition $T_{R_{rr}}(r_o) = 0$ should be satisfied indirectly, for example, by looking for appropriate expressions for $c_k(r)$, $k = 0, 3, 5, 9$ so that $T_{R_{rr}}(r_o) = 0$ would be satisfied.

There are other ways to solve the problem of determining the constants $c_j(r)$, where we assume we have information concerning the annulus when it is cut in the radial direction. For example, we could assume as in Section 5.1 that $T_{R_{rr}}$ is known and given by (54), and from (79) we could find, for example, $c_0(r)$. We choose not to study this problem further, under the understanding that the study of the opening of the annulus (see Section 4) has provided already useful information about the structure of the function.

5.3. The residually stressed annulus under inflation, torsion and axial extension when the residual stresses depend on the radial position

With knowledge of $\mathbf{T}_R$ and the expressions for $c_i = c_i(r)$ from the previous Sections 5.2, 5.1, we are now in a position to solve some boundary value problems. It is important to note that from the practical point of view, it is by carrying out such a study that we can actually assess the appropriateness of all the previous assumptions and simplifications regarding the specific expression for Π presented in (53). We should solve different boundary value problems and compare the results with the corresponding experimental results obtained, as a way to check if the many assumptions we have made are meaningful and physically faithful.

In this section for simplicity we study the case of the residually stressed annulus described in Section 5.1, using the expressions for c_j presented in (60)–(62), now being subjected to external loads on the surfaces $z = L$ and $r = r_i$, in particular we consider an axial load applied on $z = L$, a shear load along the circumferential direction on the same surface, and a radial load on the surface $r = r_i$. Under the influence of such external loads we assume that the stress tensor in the tube is of the form

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z$$
$$+T_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \tag{80}$$

This stress tensor must satisfy the equilibrium equation

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0. \tag{81}$$

Following the procedure described, for example, in [48], we assume now that the above stress field (80) produces a displacement field $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$ of the form

$$u_r = v(r), \quad u_\theta = \kappa z r, \quad u_z = (\lambda - 1)z, \tag{82}$$

where κ and λ are positive constants. It follows from (80), (82) and (26) that

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \mathfrak{h}_{11}(\mathbf{T}) = c_0(r) + 2c_1(r)I_1 + c_2(r)T_{rr} + c_3(r)T^2_{rr}, \tag{83}$$
$$\frac{v}{r} = \mathfrak{h}_{22}(\mathbf{T}) = c_0(r) + 2c_1(r)I_1 + c_2(r)T_{\theta\theta} + c_3(r)(T^2_{\theta\theta} + T^2_{\theta z})$$
$$+2c_4(r)c^2 + 4c_5(r)c^2T_{\theta\theta} + 2c_8(r)(1-2s^2)c^2$$
$$+4c_9(r)(1-2s^2)c^2T_{\theta\theta}, \tag{84}$$
$$\lambda - 1 = \mathfrak{h}_{33}(\mathbf{T}) = c_0(r) + 2c_1(r)I_1 + c_2(r)T_{zz} + c_3(r)(T^2_{\theta z} + T^2_{zz})$$
$$+2c_4(r)s^2 - 2c_8(r)(1-2s^2)s^2 - 4c_9(r)(1-2s^2)c^2T_{zz}, \tag{85}$$
$$\frac{\kappa r}{2} = \mathfrak{h}_{23}(\mathbf{T}) = c_2(r)T_{\theta z} + c_3(r)(T_{\theta\theta} + T_{zz})T_{\theta z} + 2c_5T_{\theta z}$$
$$+2c_9(r)(1-2s^2)^2T_{\theta z}, \tag{86}$$

where

$$I_1 = T_{rr} + T_{\theta\theta} + T_{zz}. \tag{87}$$

Eqs. (83)–(86) are solved using the finite element method in a manner similar to that carried out in [48]. From (81) we have

$$T_{\theta\theta} = \frac{\mathrm{d}}{\mathrm{d}r}(rT_{rr}), \tag{88}$$

Table 1
Constants for (53).

c_1	c_2	c_3	c_5	c_9
10^{-4}	10^{-4}	10^{-6}	10^{-4}	10^{-4}

therefore $\mathbf{T}$ is determined by knowing T_{rr}, $\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}$, T_{zz} and $T_{\theta z}$. From (84) we have $v = r\mathfrak{h}_{22}(\mathbf{T})$ so substituting in (83) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}r}[r\mathfrak{h}_{22}(\mathbf{T})] = \mathfrak{h}_{11}(\mathbf{T}). \tag{89}$$

In (85) we have $\lambda - 1 = \mathfrak{h}_{33}(\mathbf{T})$, taking the derivative of that equation with respect to r we obtain that

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}[\mathfrak{h}_{33}(\mathbf{T})]. \tag{90}$$

Finally from (86) we have $\frac{\kappa r}{2} = \mathfrak{h}_{23}(\mathbf{T})$ and taking the derivative of this equation with respect to r we have

$$\frac{\kappa}{2} = \frac{\mathrm{d}}{\mathrm{d}r}[\mathfrak{h}_{23}(\mathbf{T})]. \tag{91}$$

Let us introduce the auxiliary functions $\varsigma_z = \varsigma_z(r)$ and $\varsigma_{\theta z} = \varsigma_{\theta z}(r)$ through

$$T_{zz} = \frac{\mathrm{d}\varsigma_z}{\mathrm{d}r}, \quad T_{\theta z} = \frac{\mathrm{d}\varsigma_{\theta z}}{\mathrm{d}r}. \tag{92}$$

Eqs. (89)–(91) are solved using the finite element method (Comsol 3.4 [49]) for the functions $T_{rr}(r)$, $\varsigma_z(r)$ and $\varsigma_{\theta z}(r)$, subject to the boundary conditions

$$T_{rr}(r_{\mathrm{i}}) = -P_{\mathrm{i}}, \quad T_{rr}(r_{\mathrm{i}}) = 0, \quad \mathfrak{h}_{33}(\mathbf{T}(r_{\mathrm{i}})) = \lambda - 1, \quad \varsigma_z(r_{\mathrm{o}}) = 0, \tag{93}$$

$$\mathfrak{h}_{23}(\mathbf{T}(r_{\mathrm{i}})) = \frac{1}{2}\kappa r_{\mathrm{i}}, \quad \varsigma_{\theta z}(r_{\mathrm{o}}) = 0, \tag{94}$$

and using the notation $\bar{r} = r/r_{\mathrm{i}}$, where we have chosen the values $r_{\mathrm{i}} = 0.01$ and $r_{\mathrm{o}} = 0.011$. In Table 1 we have the set of values for some of the constants that appear in the model (53). Such values for the constants c_j, $j = 1,2,3,5,9$ have been chosen so that for the magnitude of the stresses obtained for the different problems studied in this paper, the magnitude of the strains from (26) are small.

In Fig. 3 we portray the results for the circumferential component of the stress $\mathbf{T}$ for different values of d_0, which from (54) indicates the 'intensity' of the residual stress, and for different cases, in the case of a traction applied on the inner surface of the tube (case (a)), torsion (case (b)), and axial extension (case (c)). In Fig. 4 we present results for all the non-zero components of the stress $\mathbf{T}$, for different values for d_0 when $\kappa = 0$, $\lambda = 1$ and $P_{\mathrm{i}} = 10$. In all the cases studied $T_{\theta z}(r) = 0$. In Fig. 5 results are shown for the different components of the stress tensor, for different values of d_0 for the case $\kappa = 2$, $\lambda = 1.01$ and $P_{\mathrm{i}} = 10$. Finally, in Fig. 6 results are presented for the components of the strain tensor for different values for d_0 (see (54)), for the case $\kappa = 2$, $\lambda = 1.01$ and $P_{\mathrm{i}} = 10$.

From Fig. 3 we see that as d_0 increases in value, the value of $T_{\theta\theta}$ at $r = r_{\mathrm{i}}$ decreases. For the problem of inflation of a cylindrical annulus in linearized elasticity, it is known that as P_{i} increases so does $T_{\theta\theta}(r_{\mathrm{i}})$, and that this stress can be very high and is usually responsible of the failure of the annulus. We see that with the residual stresses the value of $T_{\theta\theta}(r_{\mathrm{i}})$ would be lesser than in the case when there are no residual stresses, for a given value of P_{i}. The same effect is expected for the problem presented here where we have considered a nonlinear expression for $\mathfrak{h}(\mathbf{T})$. Interestingly the same effect is observed with respect to $T_{\theta\theta}(r_{\mathrm{i}})$ for the cases (b) and (c) of Fig. 3, i.e., for the nonlinear constitutive equation used here there is coupling between the torsion κ and the axial extension λ with the component $T_{\theta\theta}$ of the stress tensor. From Figs. 4 and 5 we can see again that the component of the stress T_{zz} is also affected by the residual stresses following a tendency similar to $T_{\theta\theta}$.

6. Deformation of a residually stressed slab

In this section we study residual stresses in a slab defined through

$$-L_x \leq x \leq L_x, \quad 0 \leq y \leq H, \quad -L_z \leq z \leq L_z, \tag{95}$$

where we have used the notation x, y, z for x_1, x_2 and x_3, respectively. Let us assume that $L_z \gg L_x$ and $L_z \gg H$ and that the distribution of residual stresses only depend on the coordinates x, y, i.e.

$$T_{\mathrm{R}_{ij}} = T_{\mathrm{R}_{ij}}(x,y), \quad i,j = 1,2,3. \tag{96}$$

In this case the equations of equilibrium, under the assumption that there is no body force, reduce to

$$\frac{\partial T_{\mathrm{R}_{11}}}{\partial x} + \frac{\partial T_{\mathrm{R}_{12}}}{\partial y} = 0, \quad \frac{\partial T_{\mathrm{R}_{12}}}{\partial x} + \frac{\partial T_{\mathrm{R}_{22}}}{\partial y} = 0, \quad \frac{\partial T_{\mathrm{R}_{13}}}{\partial x} + \frac{\partial T_{\mathrm{R}_{23}}}{\partial y} = 0. \tag{97}$$

Regarding the boundary condition $(12)_3$ we demand that such conditions be satisfied on the surfaces $y = 0$, $y = H$ and $x = \pm L_x$. Recalling that in the z-direction the slab is assumed to be very long, we enforce

$$T_{\mathrm{R}_{12}}(x,0) = 0, \quad T_{\mathrm{R}_{12}}(x,H) = 0, \quad T_{\mathrm{R}_{22}}(x,0) = 0, \quad T_{\mathrm{R}_{22}}(x,H) = 0, \tag{98}$$

$$T_{\mathrm{R}_{23}}(x,0) = 0, \quad T_{\mathrm{R}_{23}}(x,H) = 0, \tag{99}$$

$$T_{\mathrm{R}_{11}}(L_x,y) = 0, \quad T_{\mathrm{R}_{11}}(-L_x,y) = 0, \quad T_{\mathrm{R}_{12}}(L_x,y) = 0, \quad T_{\mathrm{R}_{12}}(-L_x,y) = 0, \tag{100}$$

$$T_{\mathrm{R}_{13}}(L_x,y) = 0, \quad T_{\mathrm{R}_{13}}(-L_x,y) = 0. \tag{101}$$

The assumptions $L_z \gg L_x$ and $L_z \gg H$ are used in order to simplify the distribution of stresses so that we can assume the simplified form given in (96). Under such assumptions the condition $\mathbf{T}_{\mathrm{R}}\mathbf{n} = \mathbf{0}$ is not required on the surface $z = \pm L_z$ since they are considered to be essentially located at infinity; however, because the condition (9) is not satisfied trivially, we need to impose such a restriction explicitly on the surface (obtained from an imaginary cut) $z = constant$, $-L_x \leq x \leq L_x$, $0 \leq y \leq H$. In this case Eq. (9) becomes $\int_0^H \int_{L_x}^{L_x} \mathbf{T}_{\mathrm{R}}\mathbf{n}\,\mathrm{d}x\mathrm{d}y = \mathbf{0}$, which implies that

$$\int_0^H \int_{L_x}^{L_x} T_{\mathrm{R}_{33}}\,\mathrm{d}x\mathrm{d}y = 0, \quad \int_0^H \int_{L_x}^{L_x} T_{\mathrm{R}_{13}}\,\mathrm{d}x\mathrm{d}y = 0, \quad \int_0^H \int_{L_x}^{L_x} T_{\mathrm{R}_{23}}\,\mathrm{d}x\mathrm{d}y = 0. \tag{102}$$

Regarding the specific expressions for $\mathbf{a}$ and $\mathbf{b}$, we assume that $\mathbf{a} = \sum_{i=1}^3 a_i(x,y)\mathbf{e}_i$ and $\mathbf{b} = \sum_{i=1}^3 b_i(x,y)\mathbf{e}_i$.

It follows from $(12)_2$, (96) and (26) that

$$\begin{aligned}
&\Pi_1 + \Pi_2 T_{\mathrm{R}_{11}} + \Pi_3(T_{\mathrm{R}_{11}}^2 + T_{\mathrm{R}_{12}}^2 + T_{\mathrm{R}_{13}}^2) + \Pi_4 a_1^2 + 2\Pi_5(T_{\mathrm{R}_{11}}a_1 \\
&+ T_{\mathrm{R}_{12}}a_2 + T_{\mathrm{R}_{13}}a_3)a_1 + \Pi_6 b_1^2 + 2\Pi_7(T_{\mathrm{R}_{11}}b_1 + T_{\mathrm{R}_{12}}b_2 + T_{\mathrm{R}_{13}}b_3)b_1 \\
&+ 2\Pi_8(\mathbf{a}\cdot\mathbf{b})a_1 b_1 + 2\Pi_9(\mathbf{a}\cdot\mathbf{b})[a_1(b_1 T_{\mathrm{R}_{11}} + b_2 T_{\mathrm{R}_{12}} + b_3 T_{\mathrm{R}_{13}}) \\
&+ b_1(a_1 T_{\mathrm{R}_{11}} + a_2 T_{\mathrm{R}_{12}} + a_3 T_{\mathrm{R}_{33}})] = 0,
\end{aligned} \tag{103}$$

$$\begin{aligned}
&\Pi_1 + \Pi_2 T_{\mathrm{R}_{22}} + \Pi_3(T_{\mathrm{R}_{12}}^2 + T_{\mathrm{R}_{22}}^2 + T_{\mathrm{R}_{23}}^2) + \Pi_4 a_2^2 + 2\Pi_5(T_{\mathrm{R}_{12}}a_1 \\
&+ T_{\mathrm{R}_{22}}a_2 + T_{\mathrm{R}_{23}}a_3)a_2 + \Pi_6 b_2^2 + 2\Pi_7(T_{\mathrm{R}_{12}}b_1 + T_{\mathrm{R}_{22}}b_2 \\
&+ T_{\mathrm{R}_{23}}b_3)b_2 + 2\Pi_8(\mathbf{a}\cdot\mathbf{b})a_2 b_2 + 2\Pi_9(\mathbf{a}\cdot\mathbf{b})[a_2(b_1 T_{\mathrm{R}_{12}} + b_2 T_{\mathrm{R}_{22}} + b_3 T_{\mathrm{R}_{23}}) \\
&+ b_2(a_1 T_{\mathrm{R}_{12}} + a_2 T_{\mathrm{R}_{22}} + a_3 T_{\mathrm{R}_{23}})] = 0,
\end{aligned} \tag{104}$$

$$\begin{aligned}
&\Pi_1 + \Pi_2 T_{\mathrm{R}_{33}} + \Pi_3(T_{\mathrm{R}_{13}}^2 + T_{\mathrm{R}_{23}}^2 + T_{\mathrm{R}_{33}}^2) + \Pi_4 a_3^2 + 2\Pi_5(T_{\mathrm{R}_{13}}a_1 \\
&+ T_{\mathrm{R}_{23}}a_2 + T_{\mathrm{R}_{33}}a_3)a_3 + \Pi_6 b_3^2 + 2\Pi_7(T_{\mathrm{R}_{13}}b_1 + T_{\mathrm{R}_{23}}b_2 \\
&+ T_{\mathrm{R}_{33}}b_3)b_3 + 2\Pi_8(\mathbf{a}\cdot\mathbf{b})a_3 b_3 + 2\Pi_9(\mathbf{a}\cdot\mathbf{b})[a_3(b_1 T_{\mathrm{R}_{13}} + b_2 T_{\mathrm{R}_{23}} + b_3 T_{\mathrm{R}_{33}}) \\
&+ b_3(a_1 T_{\mathrm{R}_{13}} + a_2 T_{\mathrm{R}_{23}} + a_3 T_{\mathrm{R}_{33}})] = 0,
\end{aligned} \tag{105}$$

$$\begin{aligned}
&\Pi_2 T_{\mathrm{R}_{12}} + \Pi_3(T_{\mathrm{R}_{11}}T_{\mathrm{R}_{12}} + T_{\mathrm{R}_{12}}T_{\mathrm{R}_{22}} + T_{\mathrm{R}_{13}}T_{\mathrm{R}_{23}}) + \Pi_4 a_1 a_2 \\
&+ \Pi_5[(T_{\mathrm{R}_{11}}a_1 + T_{\mathrm{R}_{12}}a_2 + T_{\mathrm{R}_{13}}a_3)a_2 + (T_{\mathrm{R}_{12}}a_1 + T_{\mathrm{R}_{22}}a_2 + T_{\mathrm{R}_{23}}a_3)a_1]
\end{aligned}$$

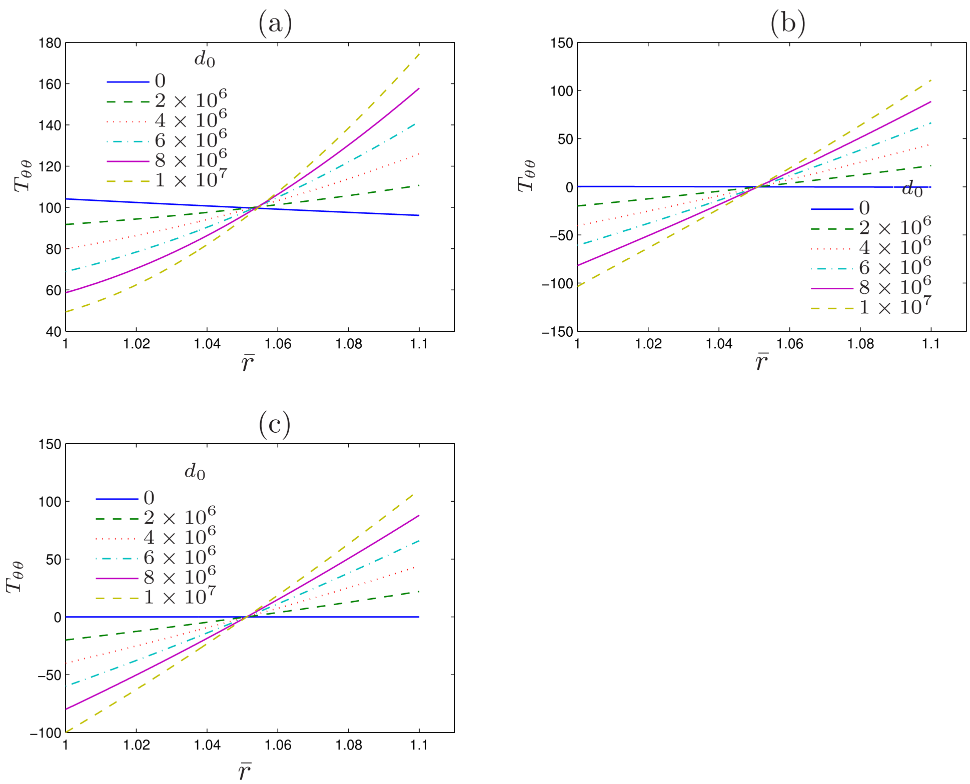

Fig. 3. Results for the circumferential component of the stress $T_{\theta\theta}$ for different values of d_0 (see (54)). Case (a) $\kappa = 0$, $\lambda = 1$ and $P_i = 10$. Case (b) $\kappa = 2$, $\lambda = 1$ and $P_i = 0$. Case (c) $\kappa = 0$, $\lambda = 1.01$ and $P_i = 0$.

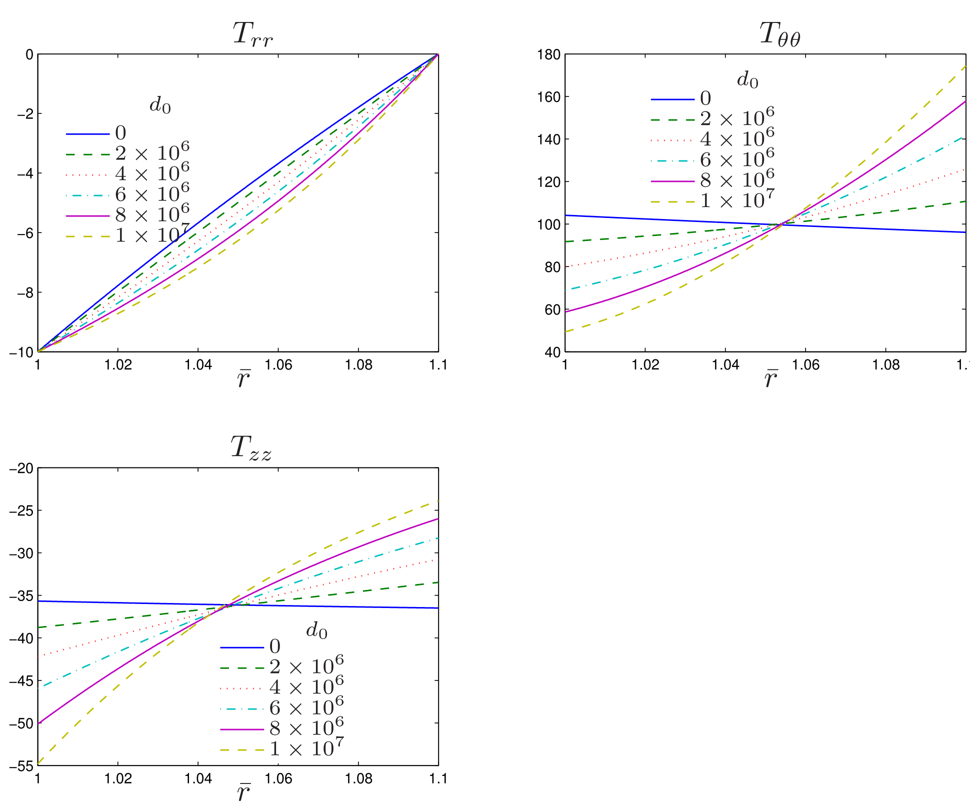

Fig. 4. Results for the different components of the stress tensor **T** for different values for d_0 (see (54)), when $\kappa = 0$, $\lambda = 1$ and $P_i = 10$.

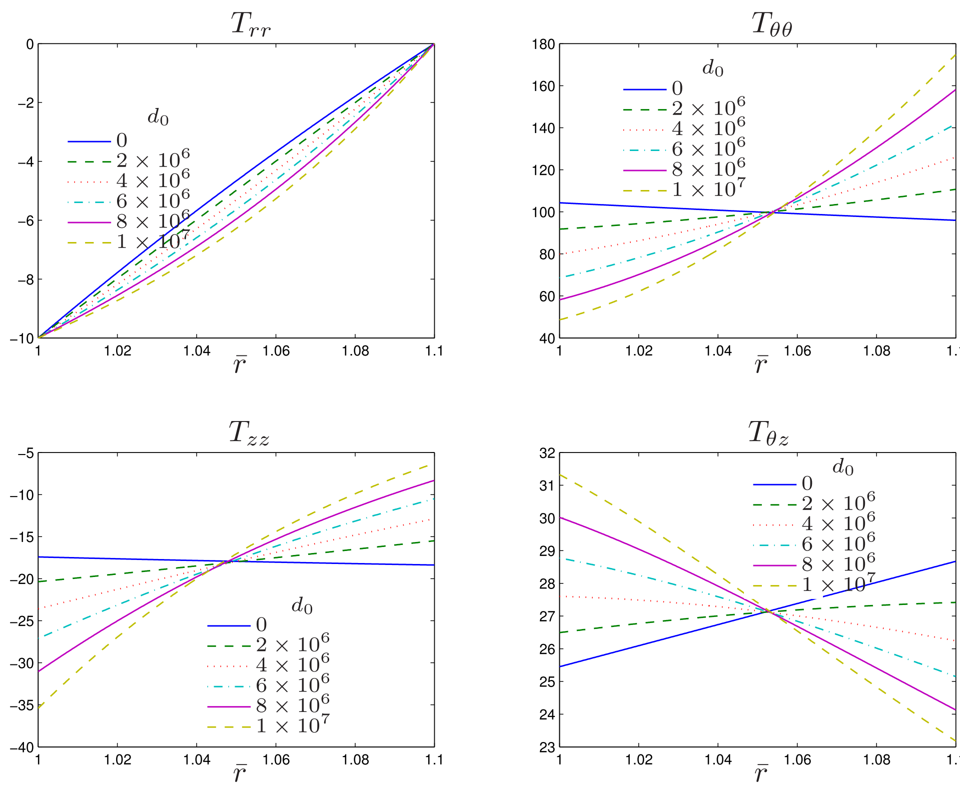

Fig. 5. Results for the different components of the stress tensor $\mathbf{T}$ for different values for d_0 (see (54)), for the case $\kappa = 2$, $\lambda = 1.01$ and $P_i = 10$.

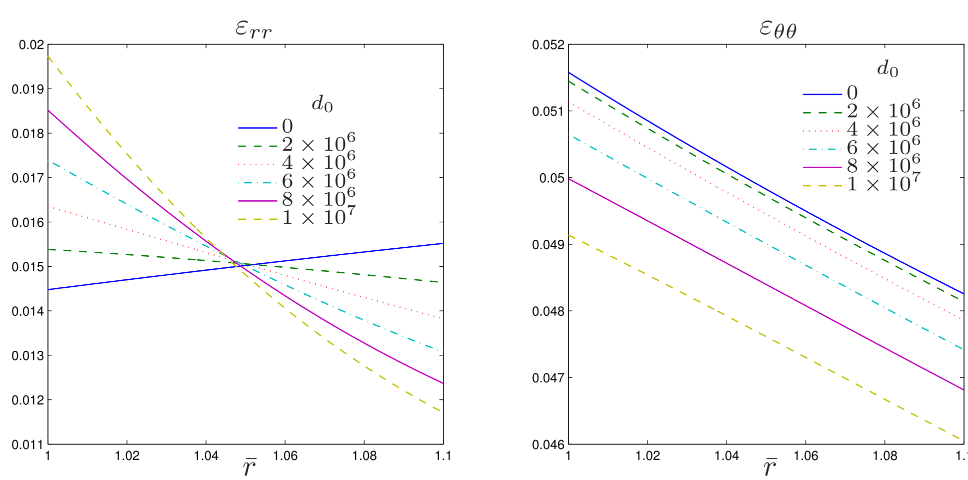

Fig. 6. Results for the components ε_{rr} and $\varepsilon_{\theta\theta}$ of the strain tensor, for different values for d_0 (see (54)), for the case $\kappa = 2$, $\lambda = 1.01$ and $P_i = 10$.

$$\begin{aligned}
&+\Pi_6 b_1 b_2 + 2\Pi_7[(T_{R_{11}} b_1 + T_{R_{12}} b_2 + T_{R_{13}} b_3) b_2 \\
&+(T_{R_{12}} b_1 + T_{R_{22}} b_2 + T_{R_{23}} b_3) b_1] + \Pi_8 (\mathbf{a}\cdot\mathbf{b})(a_1 b_2 + b_1 a_2) \\
&+\Pi_9 (\mathbf{a}\cdot\mathbf{b})[a_1(b_1 T_{R_{12}} + b_2 T_{R_{22}} + b_3 T_{R_{23}}) \\
&+a_2(b_1 T_{R_{11}} + b_2 T_{R_{12}} + b_3 T_{R_{13}}) + b_1(a_1 T_{R_{12}} + a_2 T_{R_{22}} + a_3 T_{R_{23}}) \\
&+b_2(a_1 T_{R_{11}} + a_2 T_{R_{12}} + a_3 T_{R_{13}})] = 0, \qquad (106)
\end{aligned}$$

$$\begin{aligned}
&\Pi_2 T_{R_{13}} + \Pi_3 (T_{R_{11}} T_{R_{13}} + T_{R_{12}} T_{R_{23}} + T_{R_{13}} T_{R_{33}}) \\
&+\Pi_4 a_1 a_3 + \Pi_5 [(T_{R_{11}} a_1 + T_{R_{12}} a_2 + T_{R_{13}} a_3) a_3 \\
&+(T_{R_{13}} a_1 + T_{R_{23}} a_2 + T_{R_{33}} a_3) a_1] + \Pi_6 b_1 b_3 \\
&+2\Pi_7 [(T_{R_{11}} b_1 + T_{R_{12}} b_2 + T_{R_{13}} b_3) b_3 + (T_{R_{13}} b_1 + T_{R_{23}} b_2 + T_{R_{33}} b_3) b_1] \\
&+\Pi_8 (\mathbf{a}\cdot\mathbf{b})(a_1 b_3 + b_1 a_3) + \Pi_9 (\mathbf{a}\cdot\mathbf{b})[a_1(b_1 T_{R_{13}} + b_2 T_{R_{23}} \\
&+b_3 T_{R_{33}}) + a_3(b_1 T_{R_{11}} + b_2 T_{R_{12}} + b_3 T_{R_{13}}) + b_1(a_1 T_{R_{13}} + a_2 T_{R_{23}} + a_3 T_{R_{33}}) \\
&+b_3(a_1 T_{R_{11}} + a_2 T_{R_{12}} + a_3 T_{R_{13}})] = 0, \qquad (107)
\end{aligned}$$

$$\begin{aligned}
&\Pi_2 T_{R_{23}} + \Pi_3 (T_{R_{12}} T_{R_{13}} + T_{R_{22}} T_{R_{23}} + T_{R_{23}} T_{R_{33}}) + \Pi_4 a_2 a_3 \\
&+\Pi_5 [(T_{R_{12}} a_1 + T_{R_{22}} a_2 + T_{R_{23}} a_3) a_3 + (T_{R_{13}} a_1 + T_{R_{23}} a_2 + T_{R_{33}} a_3) a_2] \\
&+\Pi_6 b_2 b_3 + 2\Pi_7 [(T_{R_{12}} b_1 + T_{R_{22}} b_2 + T_{R_{23}} b_3) b_3 + (T_{R_{13}} b_1 \\
&+T_{R_{23}} b_2 + T_{R_{33}} b_3) b_2] + \Pi_8 (\mathbf{a}\cdot\mathbf{b})(a_2 b_3 + b_2 a_3) \\
&+\Pi_9 (\mathbf{a}\cdot\mathbf{b})[a_2(b_1 T_{R_{13}} + b_2 T_{R_{23}} + b_3 T_{R_{33}}) \\
&+a_3(b_1 T_{R_{12}} + b_2 T_{R_{22}} + b_3 T_{R_{23}}) + b_2(a_1 T_{R_{13}} + a_2 T_{R_{23}} + a_3 T_{R_{33}}) \\
&+b_3(a_1 T_{R_{12}} + a_2 T_{R_{22}} + a_3 T_{R_{23}})] = 0. \qquad (108)
\end{aligned}$$

To find the distribution of the residual stresses in the slab, we need to solve (97) and (103)–(108). We have a total 9 equations for the 6 components of the residual stress tensor $\mathbf{T}_R$, therefore, in general apart from considering the components of the residual stress tensor as unknowns, we assume that some of the parameters of the constitutive equation Π are also unknowns.

In order to make some progress with this boundary value problem, we consider the same special expression for Π as in (53): $\Pi(\mathbf{T},\mathbf{a},\mathbf{b},r) = c_0 I_1 + c_1 I_1^2 + \sum_{i=2}^{9} c_i I_i$, from which we obtain

$$\Pi_1 = c_0 + 2c_1 I_1, \quad \Pi_k = c_k, \quad k = 2,3,\dots,9, \tag{109}$$

where in the present case we suppose that $c_k = c_k(x,y)$, $k = 0,1,2,\dots,9$ and where

$$I_1 = T_{11} + T_{22} + T_{33}. \tag{110}$$

Let us now explore a method for solving the problem, similar in spirit to one of the procedures used to determine the residual stresses in a cylindrical annulus (see Section 5.1). We only consider (97), (103)–(108) and we assume that we do not have information as a consequence of the slab being cut. Additionally, for simplicity we consider expressions for some of the components of $\mathbf{T}_\mathrm{R}$ as given a priori and from (103)–(108) we obtain some information associated with the functions $c_i(x,y)$ that characterize the model (109).

Let us now solve $(97)_{1,2}$, under the assumption that $T_{\mathrm{R}_{11}}$ and $T_{\mathrm{R}_{22}}$ are expressed in terms of $T_{\mathrm{R}_{12}}$. We have

$$T_{\mathrm{R}_{11}}(x,y) = -\int_{-L_x}^{x} \frac{\partial T_{\mathrm{R}_{12}}}{\partial y}(\xi,y)\,\mathrm{d}\xi + F(y), \qquad T_{\mathrm{R}_{22}}(x,y) = -\int_{0}^{y} \frac{\partial T_{\mathrm{R}_{12}}}{\partial x}(x,\eta)\,\mathrm{d}\eta + G(x), \tag{111}$$

which in virtue of the boundary conditions $(100)_{1,2}$, $(98)_{3,4}$ leads to $F(y) = 0$ and $G(x) = 0$. In view of the boundary conditions $(98)_{1,2}$, $(100)_{3,4}$ we impose the restrictions

$$\int_{-L_x}^{L_x} \frac{\partial T_{\mathrm{R}_{12}}}{\partial y}(\xi,y)\,\mathrm{d}\xi = 0, \qquad \int_{0}^{H} \frac{\partial T_{\mathrm{R}_{12}}}{\partial x}(x,\eta)\,\mathrm{d}\eta = 0. \tag{112}$$

Following a similar procedure we obtain $T_{\mathrm{R}_{13}}$ from (111), (112) and $(97)_3$ in terms of $T_{\mathrm{R}_{23}}$ as

$$T_{\mathrm{R}_{13}}(x,y) = -\int_{-L_x}^{x} \frac{\partial T_{\mathrm{R}_{23}}}{\partial y}(\xi,y)\,\mathrm{d}\xi, \quad \text{where} \quad \int_{-L_x}^{L_x} \frac{\partial T_{\mathrm{R}_{23}}}{\partial y}(\xi,y)\,\mathrm{d}\xi = 0. \tag{113}$$

We notice that the boundary conditions (101) are satisfied.

Let us assume an approximate expression for $T_{\mathrm{R}_{12}}$ of the form

$$T_{\mathrm{R}_{12}}(x,y) \approx \sum_{m=1}^{M}\sum_{n=1}^{N}\left\{\mathcal{P}_{mn}\cos\left[\frac{\pi(2m-1)x}{2L_x}\right] + \mathcal{Q}_{mn}\sin\left(\frac{\pi m x}{L_x}\right)\right\}\sin\left(\frac{\pi n y}{H}\right), \tag{114}$$

where $\mathcal{P}_{mn}$ and $\mathcal{Q}_{mn}$ are constants. It is easy to show that the boundary conditions $(100)_{3,4}$ $T_{\mathrm{R}_{12}}(\pm L_x, y) = 0$ are satisfied for all y such that $0 \le y \le H$. Similarly, for the boundary conditions $(98)_{1,2}$ $T_{\mathrm{R}_{12}}(x,0) = 0$, $T_{\mathrm{R}_{12}}(x,H) = 0$. From $(111)_1$ and (114) we have

$$T_{\mathrm{R}_{11}}(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{L_x n}{H}\left\{-\frac{2}{(2m-1)}\mathcal{P}_{mn}\left(\sin\left[\frac{\pi(2m-1)x}{2L_x}\right] + (-1)^{m+1}\right) + \frac{\mathcal{Q}_{mn}}{m}\left(\cos\left(\frac{\pi m x}{L_x}\right) + (-1)^{m+1}\right)\right\}\cos\left(\frac{\pi n y}{H}\right). \tag{115}$$

The condition $(100)_1$ (see $(112)_1$) that $T_{\mathrm{R}_{11}}(L_x,y) = 0$ is satisfied if, for example

$$\mathcal{P}_{1n} = \sum_{m=2}^{M}\frac{\mathcal{P}_{mn}(-1)^m}{(2m-1)}. \tag{116}$$

Regarding $T_{\mathrm{R}_{22}}$, from $(111)_2$ and (114) we have

$$T_{\mathrm{R}_{22}}(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{H}{L_x n}\left\{-\frac{(2m-1)}{2}\mathcal{P}_{mn}\sin\left[\frac{\pi(2m-1)x}{2L_x}\right] + m\mathcal{Q}_{mn}\cos\left(\frac{\pi m x}{L_x}\right)\right\}\left[\cos\left(\frac{\pi n y}{H}\right) - 1\right]. \tag{117}$$

The boundary conditions $(98)_{3,4}$ are satisfied automatically.

With regard to $T_{\mathrm{R}_{23}}$ let us assume an approximate expression of the form

$$T_{\mathrm{R}_{23}}(x,y) \approx \sum_{n=1}^{N}\mathcal{R}_{0n}\sin\left(\frac{\pi n y}{H}\right) + \sum_{m=1}^{M}\sum_{n=1}^{N}\left\{\mathcal{R}_{mn}\cos\left[\frac{\pi(2m-1)x}{2L_x}\right] + \mathcal{S}_{mn}\sin\left(\frac{\pi m x}{L_x}\right)\right\}\sin\left(\frac{\pi n y}{H}\right), \tag{118}$$

where $\mathcal{R}_{mn}$ and $\mathcal{S}_{mn}$ are constants. It possible to show that the conditions (99) are satisfied automatically. From $(113)_1$ and (118) we obtain that

$$T_{\mathrm{R}_{13}}(x,y) = -(x+L_x)\frac{\pi}{H}\sum_{n=1}^{N} n\mathcal{R}_{0n}\cos\left(\frac{\pi n y}{H}\right) + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{L_x n}{H}\left\{-\frac{2}{(2m-1)}\mathcal{R}_{mn}\left(\sin\left[\frac{\pi(2m-1)x}{2L_x}\right] + (-1)^{m+1}\right) + \frac{\mathcal{S}_{mn}}{m}\left(\cos\left(\frac{\pi m x}{L_x}\right) + (-1)^{m+1}\right)\right\}\cos\left(\frac{\pi n y}{H}\right). \tag{119}$$

The condition $(101)_1$ (see $(113)_2$) $T_{\mathrm{R}_{13}}(L_x,y) = 0$ is satisfied if, for example

$$\mathcal{R}_{0n} = \frac{1}{\pi}\sum_{m=1}^{M}\frac{\mathcal{R}_{mn}(-1)^m}{(2m-1)}. \tag{120}$$

Finally, regarding $T_{\mathrm{R}_{33}}$, as there is no boundary condition that the component of the stress has to satisfy (as the plate is infinitely long in the direction z), we assume the following approximate expression for that component of the residual stress

$$\begin{aligned} T_{\mathrm{R}_{33}}(x,y) \approx{}& \mathcal{U}_{00} + \sum_{m=1}^{M}\left\{\mathcal{U}_{m0}\cos\left(\frac{\pi m x}{L_x}\right) + \mathcal{V}_{m0}\sin\left(\frac{\pi m x}{L_x}\right)\right\} + \sum_{n=1}^{N}\left\{\mathcal{U}_{0n}\cos\left(\frac{\pi n y}{H}\right) + \mathcal{T}_{0n}\sin\left(\frac{\pi n y}{H}\right)\right\} \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{N}\left\{\mathcal{U}_{mn}\cos\left(\frac{\pi m x}{L_x}\right)\cos\left(\frac{\pi n y}{H}\right) + \mathcal{V}_{mn}\sin\left(\frac{\pi m x}{L_x}\right)\cos\left(\frac{\pi n y}{H}\right) + \mathcal{T}_{mn}\cos\left(\frac{\pi m x}{L_x}\right)\sin\left(\frac{\pi n y}{H}\right) + \mathcal{W}_{mn}\sin\left(\frac{\pi m x}{L_x}\right)\sin\left(\frac{\pi n y}{H}\right)\right\}, \end{aligned} \tag{121}$$

where $\mathcal{U}_{mn}$, $\mathcal{V}_{mn}$, $\mathcal{T}_{mn}$ and $\mathcal{W}_{mn}$ are constants.

In view of the above representations with regard to $T_{\mathrm{R}_{33}}$, the condition $(102)_1$ is satisfied if, for example

$$\mathcal{U}_{00} = \sum_{n=1}^{N}\frac{1}{\pi n}[(-1)^n - 1]\mathcal{T}_{0n}. \tag{122}$$

Regarding $T_{\mathrm{R}_{23}}$, from (118) we conclude that $(102)_3$ is satisfied if the following relation holds

$$\sum_{n=1}^{N}\mathcal{R}_{0n}\frac{1}{n}[(-1)^n - 1] = \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2\mathcal{R}_{mn}}{(2m-1)}\frac{(-1)^m}{\pi n}[(-1)^m - 1]. \tag{123}$$

In the case of $T_{\mathrm{R}_{13}}$ from (119) we note that $(102)_2$ is satisfied automatically.

Finally, (120) and (123) are satisfied if, for example

$$\mathcal{R}_{11} = \sum_{m=2}^{M}\left\{\frac{\mathcal{R}_{m1}}{(2m-1)}(-1)^{m+1} + \sum_{n=2}^{N}\frac{\mathcal{R}_{mn}}{2(2m-1)}\frac{(-1)^{m+1}}{n}[(-1)^n - 1]\right\}. \tag{124}$$

It follows from (114), (115), (117), (118), (119), (121) and (103)–(108) that we can obtain, c_0, c_4, c_5, c_6, c_7 and c_8. The system of equations

(103)–(108) can be rewritten as the following system of linear equations

$$[\mathbb{M}][\mathbb{C}] = [\mathbb{D}], \tag{125}$$

where the 6×6 matrix $[\mathbb{M}]$ and the 6×1 vectors $[\mathbb{C}]$ and $[\mathbb{D}]$ are defined as

$$[\mathbb{M}] = \begin{pmatrix} 1 & a_1^2 & 2l_1 & b_1^2 & 2w_1 & 2(\mathbf{a}\cdot\mathbf{b})a_1 b_1 \\ 1 & a_2^2 & 2l_2 & b_2^2 & 2w_2 & 2(\mathbf{a}\cdot\mathbf{b})a_2 b_2 \\ 1 & a_3^2 & 2l_3 & b_3^2 & 2w_3 & 2(\mathbf{a}\cdot\mathbf{b})a_3 b_3 \\ 0 & a_1 a_2 & l_4 & b_1 b_2 & w_4 & (\mathbf{a}\cdot\mathbf{b})(a_1 b_2 + b_1 a_2) \\ 0 & a_1 a_3 & l_5 & b_1 b_3 & w_5 & (\mathbf{a}\cdot\mathbf{b})(a_1 b_3 + b_1 a_3) \\ 0 & a_2 a_3 & l_6 & b_2 b_3 & w_6 & (\mathbf{a}\cdot\mathbf{b})(a_2 b_3 + b_2 a_3) \end{pmatrix}, \tag{126}$$

$$[\mathbb{C}] = \begin{pmatrix} c_0 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}, \quad [\mathbb{D}] = \begin{pmatrix} r_0 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{pmatrix},$$

where l_k, w_k and r_k, $k = 1,2,3,4,5,6$ are variables defined in terms of the components of $\mathbf{T}_{\mathrm{R}}$, $\mathbf{a}$, $\mathbf{b}$, and c_1, c_2, c_3 and c_9, and are given in Appendix A. The above system of linear equations can be solve uniquely if[8] $\det[\mathbb{M}] \neq 0$. Eq. (125) can be solved symbolically or numerically, in this last case we need to completely know an expression for $\mathbf{T}_{\mathrm{R}}$.

7. On the use of incremental equations to study the properties of residually stressed bodies

In the previous sections we assumed that information about the properties of residually stressed bodies can be obtained by assuming that such bodies on being cut attain a configuration that is free of stress. From the practical point of view it may not be possible or convenient to cut a body in order to obtain information about the material properties and the distributions of residual stresses and moreover a 'cut' is not a diffeomorphism. In the present section we explore a different method to obtain information concerning the residually stressed body, working with the incremental formulation presented in [50] (see also [30] for a similar analysis, but for a subclass of $\mathfrak{G}(\mathbf{B},\mathbf{T}_{\mathrm{R}})$). Recall the definitions presented in Section 3.2, the residual stresses $\mathbf{T}_{\mathrm{R}}$ must satisfy the equations (see (12))

$$\operatorname{div}\mathbf{T}_{\mathrm{R}} = \mathbf{0}, \quad \mathbf{0} = \mathfrak{h}(\mathbf{T}_{\mathrm{R}}) \quad \text{on } \kappa_r(\mathcal{B}), \quad \mathbf{T}_{\mathrm{R}}\mathbf{n} = \mathbf{0} \quad \text{on } \partial\kappa_r(\mathcal{B}).$$

Let us consider the application of a small external load $\Delta\hat{\mathbf{t}}$, which is applied on some parts of the boundary of the body, such that the stresses change according to $\mathbf{T} = \mathbf{T}_{\mathrm{R}} + \Delta\mathbf{T}$ and such that $|\Delta\mathbf{T}| \ll |\mathbf{T}_{\mathrm{R}}|$. Of course this last assumption in most cases can be only verified a posteriori, as in general the residual stresses $\mathbf{T}_{\mathrm{R}}$ are unknown. The total stress $\mathbf{T} = \mathbf{T}_{\mathrm{R}} + \Delta\mathbf{T}$ must satisfy the equation of motion[9] $\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \operatorname{div}\mathbf{T}$ and (12) and we obtain

$$\rho\frac{\partial^2\Delta\mathbf{u}}{\partial t^2} = \operatorname{div}\Delta\mathbf{T}, \tag{127}$$

where $\Delta\mathbf{u}$ is the displacement field that appears when the incremental external load $\Delta\hat{\mathbf{t}}$ is applied on the boundary of the body. Let $\Delta\boldsymbol{\epsilon}$ denote the strain tensor associated with such a displacement field, we have (in index notation and Cartesian co-ordinates)

$$\Delta\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial x_j} + \frac{\partial\Delta u_j}{\partial x_i}\right), \tag{128}$$

and from $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$ since $|\Delta\mathbf{T}| \ll |\mathbf{T}_{\mathrm{R}}|$ and since $(12)_1$ must be satisfied, we have the approximate expression

$$\Delta\epsilon_{ij} \approx \left.\frac{\partial\mathfrak{h}_{ij}}{\partial T_{kl}}\right|_{\mathbf{T}_{\mathrm{R}}} \Delta T_{kl}, \tag{129}$$

which in the case of (19) becomes

$$\Delta\epsilon_{ij} \approx \left.\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}}\right|_{\mathbf{T}_{\mathrm{R}}} \Delta T_{kl}. \tag{130}$$

From (127) and (130) we see that these linearized equations contain information concerning the residual stresses.

As a summary, for a given distribution of residual stresses $\mathbf{T}_{\mathrm{R}}$ we need to find $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$ solving in parallel the partial differential equations[10]

$$\rho\frac{\partial^2\Delta u_i}{\partial t^2} = \frac{\partial\Delta T_{ij}}{\partial x_j}, \quad \frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial x_j} + \frac{\partial\Delta u_j}{\partial x_i}\right) = \left.\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}}\right|_{\mathbf{T}_{\mathrm{R}}} \Delta T_{kl}, \tag{131}$$

subject to the boundary conditions

$$\Delta T_{ij}n_j = \Delta\hat{t}_i \quad \text{on } \partial\kappa_r(\mathcal{B})_t, \quad \Delta u_i = \Delta\hat{u}_i \quad \text{on } \partial\kappa_r(\mathcal{B})_u, \tag{132}$$

where $\Delta\hat{\mathbf{u}}$ is the prescribed boundary displacement on $\partial\kappa_r(\mathcal{B})_u$, and $\partial\kappa_r(\mathcal{B}) = \partial\kappa_r(\mathcal{B})_u \cup \partial\kappa_r(\mathcal{B})_t$ and $\partial\kappa_r(\mathcal{B})_u \cap \partial\kappa_r(\mathcal{B})_t = \emptyset$.

In principle, we could apply a variety of small external loads $\Delta\hat{\mathbf{t}}$ on the residually stressed body on the surface $\partial\kappa_r(\mathcal{B})_t$, and we should be able to measure the displacement field $\Delta\mathbf{u}$ that such loads generate on the same surface $\partial\kappa_r(\mathcal{B})_t$. This experimental information could be used to help us characterize the residual stresses $\mathbf{T}_{\mathrm{R}}$ or the constitutive function $\Pi = \Pi(\mathbf{T},\mathbf{a},\mathbf{b})$ as shown in the example presented below.

In order to see how the incremental formulation presented above can help us to understand the response of residually stressed bodies, it is necessary to consider some boundary value problems, and as first step we need the expression for $\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}}$. From (26) (see (19)) we have in index notation (Cartesian co-ordinates)

$$\begin{aligned}\frac{\partial\Pi}{\partial T_{kl}} &= \Pi_1\delta_{kl} + \Pi_2 T_{kl} + \Pi_3 T_{km}T_{ml} + \Pi_4 a_k a_l \\ &\quad + \Pi_5(T_{km}a_m a_l + a_k T_{lm}a_m) + \Pi_6 b_k b_l \\ &\quad + \Pi_7(T_{km}b_m b_l + b_k T_{lm}b_m) + \Pi_8(\mathbf{a}\cdot\mathbf{b})(a_k b_l + a_l b_k) \\ &\quad + \Pi_9(\mathbf{a}\cdot\mathbf{b})(a_k T_{lm}b_m + T_{km}b_m a_l + b_k T_{lm}a_m + T_{km}a_m b_l),\end{aligned} \tag{133}$$

where we recall that $\Pi_p = \frac{\partial\Pi}{\partial I_p}$, $p = 1,2,\ldots,9$. The second derivatives $\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}}$ can be found by taking the derivative of the above expression with respect to T_{ij} taking into consideration the expression

$$\begin{aligned}\frac{\partial\Pi_p}{\partial T_{ij}} &= \Pi_{p1}\delta_{ij} + \Pi_{p2}T_{ij} + \Pi_{p3}T_{in}T_{nj} + \Pi_{p4}a_i a_j \\ &\quad + \Pi_{p5}(T_{in}a_n a_j + a_i T_{jn}a_n) + \Pi_{p6}b_i b_j \\ &\quad + \Pi_{p7}(T_{in}b_n b_j + b_i T_{jn}b_n) + \Pi_{p8}(\mathbf{a}\cdot\mathbf{b})(a_i b_j + b_i a_j) \\ &\quad + \Pi_{p9}(\mathbf{a}\cdot\mathbf{b})(a_i T_{jn}b_n + T_{in}b_n a_j + b_i T_{jn}a_n + T_{in}a_n b_j),\end{aligned} \tag{134}$$

where we have used the notation $\Pi_{pq} = \frac{\partial^2\Pi}{\partial I_p\partial I_q}$. The procedure to obtain $\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}}$ is straightforward but since the expression is very long we provide it in Appendix B. In the case of the model (53) where we have $\Pi_{11} = 2c_1$, $\Pi_{uv} = 0$ (when $u \neq 1$ and $v \neq 1$), $\Pi_1 = c_0 + 2c_1 I_1$, $\Pi_w = c_w$, $w = 2,3,\ldots,9$, from (220) (see Appendix B) we obtain that

$$\begin{aligned}\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}} &= 2c_1\delta_{ij}\delta_{kl} + \frac{1}{2}c_2(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \frac{1}{2}c_3(\delta_{ik}T_{jl} + T_{ik}\delta_{lj} + \delta_{kj}T_{il} \\ &\quad + T_{kj}\delta_{li}) + \frac{1}{2}c_5(\delta_{ik}a_j a_l + a_k\delta_{li}a_j + \delta_{kj}a_i a_l + a_k\delta_{lj}a_i) \\ &\quad + \frac{1}{2}c_7(\delta_{ik}b_j b_l + b_k\delta_{li}b_j + \delta_{kj}b_i b_l + b_k\delta_{lj}b_i).\end{aligned} \tag{135}$$

To summarize, in the case of (53) we need to find $\mathbf{T}_{\mathrm{R}}$, $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$ by solving the Eqs. $(12)_{1,2}$ and (131) (see (135))

$$(c_0 + 2c_1 T_{\mathrm{R}_{kk}})\delta_{ij} + c_2 T_{\mathrm{R}_{ij}} + c_3 T_{\mathrm{R}_{ik}}T_{\mathrm{R}_{kj}} + c_4 a_i a_j + c_5(T_{\mathrm{R}_{ik}}a_k a_j + a_i T_{\mathrm{R}_{jk}}a_k)$$

[8] $\det[\mathbb{M}] \neq 0$ is a condition that may not be always satisfied. Even if $\det[\mathbb{M}] = 0$ we can still find c_0, c_4, c_5, c_6, c_7 and c_8, but some of them would not be independent.

[9] For simplicity we do not consider body forces.

[10] Here we assume that ρ is approximately constant as a function of the deformation.

$$+c_6 b_i b_j + c_7(T_{\mathrm{R}_{ik}} b_k b_j + b_i T_{\mathrm{R}_{jk}} b_k) + c_8(\mathbf{a}\cdot\mathbf{b})(a_i b_j + b_i a_j) + c_9(\mathbf{a}\cdot\mathbf{b})(a_i T_{jk} b_k + T_{ik} b_k a_j + b_i T_{jk} a_k + T_{ik} a_k b_j) = 0, \quad (136)$$

$$\frac{\partial T_{\mathrm{R}_{ij}}}{\partial x_j} = 0, \quad \rho \frac{\partial^2 \Delta u_i}{\partial t^2} = \frac{\partial \Delta T_{ij}}{\partial x_j}, \quad (137)$$

$$\frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right) = 2c_1 \delta_{ij} \Delta T_{kk} + c_2 \Delta T_{ij} + c_3(\Delta T_{ik} T_{\mathrm{R}_{kj}} + T_{\mathrm{R}_{ik}} \Delta T_{kj}) + c_5(\Delta T_{ik} a_k a_j + a_i \Delta T_{jk} a_k) + c_7(\Delta T_{ik} b_k b_j + b_i \Delta T_{jk} b_k), \quad (138)$$

subject to the boundary conditions

$$T_{\mathrm{R}_{ij}} n_j = 0 \quad \text{on } \partial\kappa_r(\mathcal{B}), \quad \Delta T_{ij} n_j = \Delta\hat{t}_i \quad \text{on } \partial\kappa_r(\mathcal{B})_t, \quad \Delta u_i = \Delta\hat{u}_i \quad \text{on } \partial\kappa_r(\mathcal{B})_u. \quad (139)$$

7.1. Analysis for a cylindrical annulus

In this section we solve (136)–(138) for the special case of a cylindrical annulus, the relevant variables only depend on the radial position and not on time (quasi-static analysis). Recalling the results presented in Section 5.1, here we consider the annulus $r_{\mathrm{i}} \le r \le r_{\mathrm{o}}$, $0 \le \theta \le 2\pi$, $0 \le z \le L$, where the residual stresses are assumed to be of the form $T_{\mathrm{R}_{rr}} = T_{\mathrm{R}_{rr}}(r)$ and $T_{\mathrm{R}_{\theta\theta}} = T_{\mathrm{R}_{\theta\theta}}(r)$ where the rest of the components are assumed to be equal to zero. The equations of equilibrium $(137)_1$ reduces to $T_{\mathrm{R}_{\theta\theta}}(r) = \frac{\mathrm{d}}{\mathrm{d}r}(r T_{\mathrm{R}_{rr}}(r))$ with boundary conditions $T_{\mathrm{R}_{rr}}(r_{\mathrm{i}}) = 0$, $T_{\mathrm{R}_{rr}}(r_{\mathrm{o}}) = 0$. For simplicity we use the solution for $\mathbf{T}_{\mathrm{R}}$ and the expressions for c_k, $k = 1, 2, \ldots, 9$ presented in (54) for the case of a residually stressed tube when the residual stresses are known and depend on the radial position without considering a radial cut for the annulus. In this case $T_{\mathrm{R}_{rr}}(r) = d_0(r - r_{\mathrm{i}})(r - r_{\mathrm{o}})$.

Regarding $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$ let us assume that they are of the form $\Delta T_{rr} = \Delta T_{rr}(r)$, $\Delta T_{\theta\theta} = \Delta T_{\theta\theta}(r)$, $\Delta T_{zz} = \Delta T_{zz}(r)$ and $\Delta T_{ij} = 0$, $i \ne j$, and $\Delta\mathbf{u} = \Delta u_r(r)\mathbf{e}_r$. We assume that the boundary condition $(139)_2$ satisfy

$$\Delta T_r(r_{\mathrm{i}}) = -\Delta P_{\mathrm{i}}, \quad \Delta T_r(r_{\mathrm{o}}) = 0, \quad \Delta T_z(r) = \hat{t}_z(r). \quad (140)$$

We should be able to measure experimentally the radial displacements $\Delta u_r(r_{\mathrm{i}})$ and $\Delta u_r(r_{\mathrm{o}})$. The equation of motion $(137)_2$ becomes

$$\Delta T_{\theta\theta}(r) = \frac{\mathrm{d}}{\mathrm{d}r}[r \Delta T_{rr}(r)]. \quad (141)$$

In the case of (138), from the previous expressions we obtain that

$$\frac{\mathrm{d}\Delta u_r}{\mathrm{d}r} = 2c_1(\Delta T_{rr} + \Delta T_{\theta\theta} + \Delta T_{zz}) + c_2 \Delta T_{rr} + 2c_3 \Delta T_{rr} T_{\mathrm{R}_{rr}}, \quad (142)$$

$$\frac{\Delta u_r}{r} = 2c_1(\Delta T_{rr} + \Delta T_{\theta\theta} + \Delta T_{zz}) + c_2 \Delta T_{\theta\theta} + 2c_3 \Delta T_{\theta\theta} T_{\mathrm{R}_{\theta\theta}} + 2c_5 \Delta T_{\theta\theta} a_2^2 + 2c_7 \Delta T_{\theta\theta} b_2^2, \quad (143)$$

$$0 = 2c_1(\Delta T_{rr} + \Delta T_{\theta\theta} + \Delta T_{zz}) + c_2 \Delta T_{zz} + 2c_5 \Delta T_{zz} a_3^2 + 2c_7 \Delta T_{zz} b_3^2, \quad (144)$$

where $a_2 = b_2 = \cos\beta$ and $a_3 = -b_3 = \sin\beta$. The above system of equations is linear in $\Delta\mathbf{T}$ and $\Delta\mathbf{u}$, and in particular (144) can be used to obtain ΔT_{zz} in terms of ΔT_{rr} and $\Delta T_{\theta\theta}$ as

$$\Delta T_{zz} = -\aleph(\Delta T_{rr} + \Delta T_{\theta\theta}), \quad (145)$$

where we have defined

$$\aleph = \frac{2c_1}{(2c_1 + c_2 + 2c_5 a_3^2 + 2c_7 b_3^2)}, \quad (146)$$

and where we have assumed that $2c_1 + c_2 + 2c_5 a_3^2 + 2c_7 b_3^2 \ne 0$. From (143) we have $\Delta u_r = r[2c_1(\Delta T_{rr} + \Delta T_{\theta\theta} + \Delta T_{zz}) + c_2 \Delta T_{\theta\theta} + 2c_3 \Delta T_{\theta\theta} T_{\mathrm{R}_{\theta\theta}} + 2c_5 \Delta T_{\theta\theta} a_2^2 + 2c_7 \Delta T_{\theta\theta} b_2^2]$, therefore, replacing this in (142), and considering (141) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{ r\left[2c_1(1-\aleph)\Delta T_{rr} + \ell_2 \frac{\mathrm{d}}{\mathrm{d}r}(r\Delta T_{rr})\right]\right\} = \ell_1 \Delta T_{rr} + 2c_1(1-\aleph)\frac{\mathrm{d}}{\mathrm{d}r}(r\Delta T_{rr}), \quad (147)$$

where we have used the definitions

$$\ell_1 = 2c_1(1-\aleph) + c_2 + 2c_3 T_{\mathrm{R}_{rr}}, \quad \ell_2 = 2c_1(1-\aleph) + c_2 + 2c_3 T_{\mathrm{R}_{\theta\theta}} + 2c_5 a_2^2 + 2c_7 b_2^2. \quad (148)$$

Eq. (147) is solved using the finite element method using the program Comsol 3.4 [49]. The expressions for $T_{\mathrm{R}_{rr}}$, $T_{\mathrm{R}_{\theta\theta}}$ and c_0, c_4 and c_8 are taken from (54), (55), and from (60)–(62) we have: $c_0(r) = -2c_1 I_{\mathrm{R}_1} - T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}})$, $c_4(r) = \frac{1}{4}\{T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}})\csc^2\beta - 4T_{\mathrm{R}_{\theta\theta}}[c_5 + c_9\cos(2\beta)] + (T_{\mathrm{R}_{rr}} - T_{\mathrm{R}_{\theta\theta}})[c_2 + c_3(T_{\mathrm{R}_{rr}} + T_{\mathrm{R}_{\theta\theta}})]\sec^2\beta\}$and $c_8(r) = -\frac{1}{4}\{4T_{\mathrm{R}_{\theta\theta}}[c_5 + c_9\cos(2\beta)] + 4T_{\mathrm{R}_{rr}}(c_2 + c_3 T_{\mathrm{R}_{rr}})\cot(2\beta)\csc(2\beta) + T_{\mathrm{R}_{\theta\theta}}(c_2 + c_3 T_{\mathrm{R}_{\theta\theta}})\sec^2\beta\}\sec(2\beta)$. We assume that c_1, c_2, c_3, $c_5 = c_7$ and c_9 are constant, and use the values presented in Section 5.3 and shown in Table 1.

In Fig. 7 results are presented for the components of the incremental stress $\Delta\mathbf{T}$ and Δu_r (the incremental radial displacement), for different values for the constant d_0 (see (54)), in terms of the dimensionless radius $\bar{r} = r/r_{\mathrm{i}}$, for the case $\Delta P_{\mathrm{i}} = 10^{-3}$.

The behaviour of the solid for the case $d_0 = 0$ would be that of a linearized anisotropic solid whose response has two preferred directions. It is interesting to compare the results for ΔT_{rr} and $\Delta T_{\theta\theta}$ for the other values for d_0. For ΔT_{rr} we observe that the component of the incremental stress increases near the middle of the annulus as d_0 increases. The same happens with regard to $\Delta T_{\theta\theta}(r_{\mathrm{i}})$ as d_0 increases, which is the opposite of the tendency observed in the full nonlinear problem in Section 5.3. On the other hand, the behaviour of ΔT_{zz} is similar (in a qualitative manner) with the behaviour of T_{zz} for the nonlinear problem (see Section 5.3). Regarding the behaviour of Δu_r we see that as d_0 increases Δu_r, which is positive, decreases globally on the surface of the annulus, i.e., as the residual stresses increases, it would be more difficult to deform the annulus, which would be in keeping with our expectations, since the distribution of residual stresses is such that $T_{\mathrm{R}_{rr}}$ is negative for the whole annulus, while $T_{\mathrm{R}_{\theta\theta}}$ is negative for some interval of the form $r_{\mathrm{i}} \le r \le r^*$, i.e., in order to deform the annulus the incremental load ΔP_{i} first must overcome such initial stresses, and that is the reason Δu_r decreases as d_0 increases.

8. Further remarks

In the present paper we have studied how residual stresses in elastic bodies can be modelled, when considering some classes of constitutive equations where the strain is given as a function of the stress, namely $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$. It has been found that the definition of residual stresses in this case is very simple and appealing from the physical point of view, since if for a reference configuration we have residual stresses $\mathbf{T}_{\mathrm{R}}$, we simply require that such residual stresses satisfy the equilibrium equation $\operatorname{div}\mathbf{T}_{\mathrm{R}} = \mathbf{0}$, the traction free boundary condition $\mathbf{T}_{\mathrm{R}}\mathbf{n} = \mathbf{0}$, plus the additional condition $\mathbf{0} = \mathfrak{h}(\mathbf{T}_{\mathrm{R}})$. Using such theory we have studied in detail the problem of residual stresses in a cylindrical annulus, assuming that all the non-zero components of the stress depend only on the radial position. The residual stresses can be assumed to be known (up to the value of some constants), and from $\mathbf{0} = \mathfrak{h}(\mathbf{T}_{\mathrm{R}})$ we have obtained the expressions for some of the material parameters for a simple constitutive model $\mathfrak{h}(\mathbf{T})$ (see Section 5.1). Alternatively, we could assume we have additional information from the residually stressed body, when this is cut, assuming that the cut will release the residual stresses. In such a case denoting $\mathbf{u}_{\mathrm{C}}$ as the displacement field due to the body being cut, the additional condition $\boldsymbol{\varepsilon}_{\mathrm{C}} = \mathfrak{h}(\mathbf{0})$ (where $\boldsymbol{\varepsilon}_{\mathrm{C}}$ is the strain field calculated with $\mathbf{u}_{\mathrm{C}}$) can be used to obtain additional restrictions on the material parameters for the model. In the case of the cylindrical annulus that has been studied in Section 5.2, we obtained an equation (see Eq. (79)) that can be used either to find one of the material parameters in terms of the radial component of the residual stress, or to find the components of the stress if the material parameters are known. We have presented two ways to assess the appropriateness of the different assumptions for our model, one is to study the same body under the influence of stresses that are of magnitude equal or greater than the residual stresses, investigating how the body behaves for different values of the parameters. Another method is to apply small external loads, assuming that such loads will generate small stresses (in comparison to the residual stresses) in the body, and using this

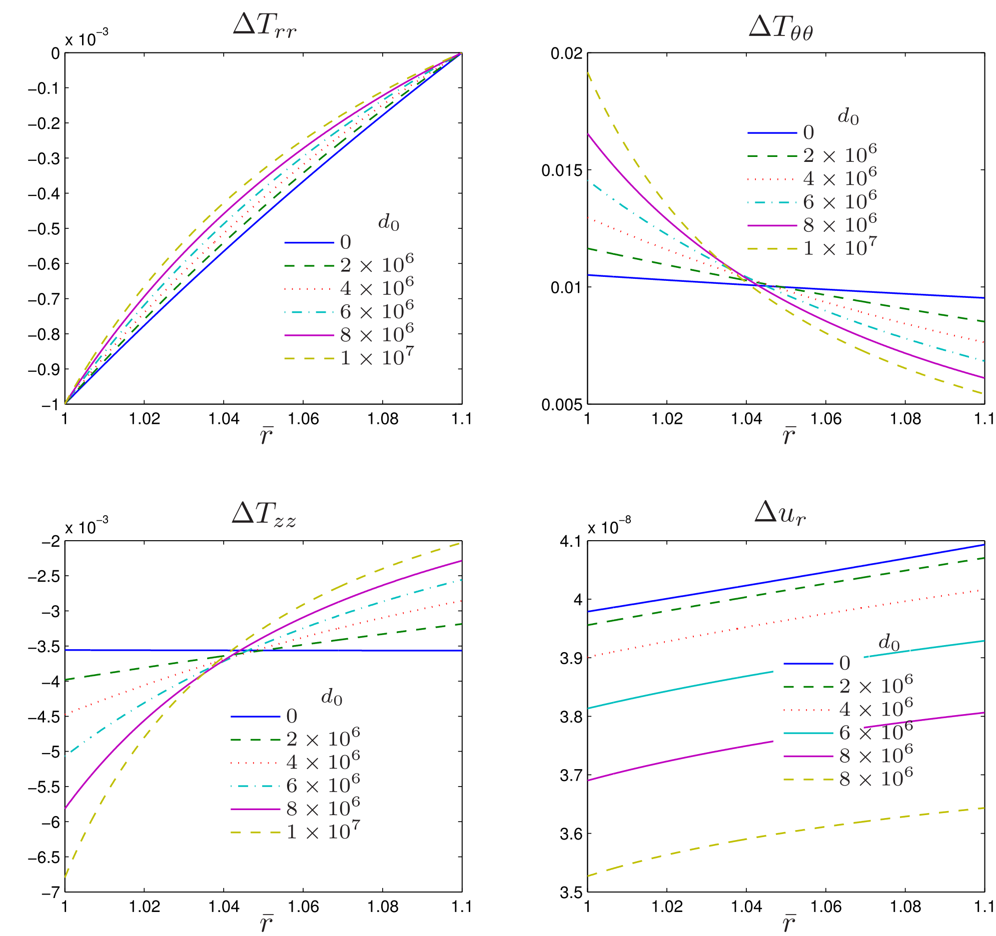

Fig. 7. Results for the different components of the incremental stress tensor $\Delta\mathbf{T}$ and the incremental radial displacement Δu_r, for different values for d_0 (see (54)) and $\Delta P_{\rm i} = 10^{-3}$.

fact to linearize the equations. In the case of the annulus, we have studied these two methods, in Section 5.3 we studied the same residually stressed annulus analysed in Section 5.1, now under the influence of external loads, and we obtained results by exploring the influence of the residual stresses on the stress $\mathbf{T}$ and the displacement $\mathbf{u}$. In Section 7 the same annulus is studied, but now considering incremental equations under the assumption that the external loads are very small. This second approach can be very important in its own right, since from the practical point of view, for a residually stressed body we can assume some expression for $\mathbf{T}_{\rm R}$ such that $\operatorname{div}\mathbf{T}_{\rm R} = \mathbf{0}$ and $\mathbf{T}_{\rm R}\mathbf{n} = \mathbf{0}$ are satisfied, obtaining restrictions on $\mathfrak{h}(\mathbf{T})$, and thereafter, we can apply many different external small loads to the same body, obtaining the incremental stresses and in particular the incremental displacements, which we can compare against experimental results, thereby assessing if the expressions for the residual stress and the constitutive equations are valid.

Acknowledgement

R. Bustamante would like to express his gratitude for the financial support provided by FONDECYT (Chile) under grant no. 1160030. K.R. Rajagopal thanks the National Science Foundation and the Office of Naval Research for support of this work.

Appendix A

The variables l_k presented in (126) are defined as

$$l_1 = (T_{{\rm R}_{11}}a_1 + T_{{\rm R}_{12}}a_2 + T_{{\rm R}_{13}}a_3)a_1, \quad l_2 = (T_{{\rm R}_{12}}a_1 + T_{{\rm R}_{22}}a_2 + T_{{\rm R}_{23}}a_3)a_2, \tag{149}$$

$$l_3 = (T_{{\rm R}_{13}}a_1 + T_{{\rm R}_{23}}a_2 + T_{{\rm R}_{33}}a_3)a_3, \tag{150}$$

$$l_4 = (T_{{\rm R}_{11}}a_1 + T_{{\rm R}_{12}}a_2 + T_{{\rm R}_{13}}a_3)a_2 + (T_{{\rm R}_{12}}a_1 + T_{{\rm R}_{22}}a_2 + T_{{\rm R}_{23}}a_3)a_1, \tag{151}$$

$$l_5 = (T_{{\rm R}_{11}}a_1 + T_{{\rm R}_{12}}a_2 + T_{{\rm R}_{13}}a_3)a_2 + (T_{{\rm R}_{13}}a_1 + T_{{\rm R}_{23}}a_2 + T_{{\rm R}_{33}}a_3)a_1, \tag{152}$$

$$l_6 = (T_{{\rm R}_{12}}a_1 + T_{{\rm R}_{22}}a_2 + T_{{\rm R}_{23}}a_3)a_3 + (T_{{\rm R}_{13}}a_1 + T_{{\rm R}_{23}}a_2 + T_{{\rm R}_{33}}a_3)a_2. \tag{153}$$

Also w_k are defined through

$$w_1 = (T_{{\rm R}_{11}}b_1 + T_{{\rm R}_{12}}b_2 + T_{{\rm R}_{13}}b_3)b_1, \quad w_2 = (T_{{\rm R}_{12}}b_1 + T_{{\rm R}_{22}}b_2 + T_{{\rm R}_{23}}b_3)b_2, \tag{154}$$

$$w_3 = (T_{{\rm R}_{13}}b_1 + T_{{\rm R}_{23}}b_2 + T_{{\rm R}_{33}}b_3)b_3, \tag{155}$$

$$w_4 = (T_{{\rm R}_{11}}b_1 + T_{{\rm R}_{12}}b_2 + T_{{\rm R}_{13}}b_3)b_2 + (T_{{\rm R}_{12}}b_1 + T_{{\rm R}_{22}}b_2 + T_{{\rm R}_{23}}b_3)b_1, \tag{156}$$

$$w_5 = (T_{{\rm R}_{11}}b_1 + T_{{\rm R}_{12}}b_2 + T_{{\rm R}_{13}}b_3)b_2 + (T_{{\rm R}_{13}}b_1 + T_{{\rm R}_{23}}b_2 + T_{{\rm R}_{33}}b_3)b_1, \tag{157}$$

$$w_6 = (T_{{\rm R}_{12}}b_1 + T_{{\rm R}_{22}}b_2 + T_{{\rm R}_{23}}b_3)b_3 + (T_{{\rm R}_{13}}b_1 + T_{{\rm R}_{23}}b_2 + T_{{\rm R}_{33}}b_3)b_2, \tag{158}$$

and r_k are defined through

$$r_1 = -(c_1 I_{{\rm R}_1} + c_2 T_{{\rm R}_{11}} + c_3 j_1 + 2c_9 z_1), \quad r_2 = -(c_1 I_{{\rm R}_1} + c_2 T_{{\rm R}_{22}} + c_3 j_2 + 2c_9 z_2), \tag{159}$$

$$r_3 = -(c_1 I_{{\rm R}_1} + c_2 T_{{\rm R}_{33}} + c_3 j_3 + 2c_9 z_3), \quad r_4 = -(c_2 T_{{\rm R}_{12}} + c_3 j_4 + c_9 z_4), \tag{160}$$

$$r_5 = -(c_2 T_{{\rm R}_{13}} + c_3 j_5 + c_9 z_5), \quad r_6 = -(c_2 T_{{\rm R}_{23}} + c_3 j_6 + c_9 z_6), \tag{161}$$

where

$$j_1 = T_{R_{11}}^2 + T_{R_{12}}^2 + T_{R_{13}}^2, \quad j_2 = T_{R_{12}}^2 + T_{R_{22}}^2 + T_{R_{23}}^2, \quad j_3 = T_{R_{13}}^2 + T_{R_{23}}^2 + T_{R_{33}}^2, \tag{162}$$

$$j_4 = T_{R_{11}}T_{R_{12}} + T_{R_{12}}T_{R_{22}} + T_{R_{13}}T_{R_{23}}, \quad j_5 = T_{R_{11}}T_{R_{13}} + T_{R_{12}}T_{R_{23}} + T_{R_{13}}T_{R_{33}}, \tag{163}$$

$$j_6 = T_{R_{12}}T_{R_{13}} + T_{R_{22}}T_{R_{23}} + T_{R_{23}}T_{R_{33}}, \tag{164}$$

$$z_1 = (\mathbf{a}\cdot\mathbf{b})[a_1(b_1T_{R_{11}} + b_2T_{R_{12}} + b_3T_{R_{13}}) + b_1(a_1T_{R_{11}} + a_2T_{R_{12}} + a_3T_{R_{13}})], \tag{165}$$

$$z_2 = (\mathbf{a}\cdot\mathbf{b})[a_2(b_1T_{R_{12}} + b_2T_{R_{22}} + b_3T_{R_{23}}) + b_2(a_1T_{R_{12}} + a_2T_{R_{22}} + a_3T_{R_{23}})], \tag{166}$$

$$z_3 = (\mathbf{a}\cdot\mathbf{b})[a_3(b_1T_{R_{13}} + b_2T_{R_{23}} + b_3T_{R_{33}}) + b_3(a_1T_{R_{13}} + a_2T_{R_{23}} + a_3T_{R_{33}})], \tag{167}$$

$$z_4 = (\mathbf{a}\cdot\mathbf{b})[a_1(b_1T_{R_{12}} + b_2T_{R_{22}} + b_3T_{R_{23}}) + a_2(b_1T_{R_{11}} + b_2T_{R_{12}} + b_3T_{R_{13}}) + b_1(a_1T_{R_{12}} + a_2T_{R_{22}} + a_3T_{R_{23}}) + b_2(a_1T_{R_{11}} + a_2T_{R_{12}} + a_3T_{R_{13}})], \tag{168}$$

$$z_5 = (\mathbf{a}\cdot\mathbf{b})[a_1(b_1T_{R_{13}} + b_2T_{R_{23}} + b_3T_{R_{33}}) + a_3(b_1T_{R_{11}} + b_2T_{R_{12}} + b_3T_{R_{13}}) + b_1(a_1T_{R_{13}} + a_2T_{R_{23}} + a_3T_{R_{33}}) + b_3(a_1T_{R_{11}} + a_2T_{R_{12}} + a_3T_{R_{13}})], \tag{169}$$

$$z_6 = (\mathbf{a}\cdot\mathbf{b})[a_2(b_1T_{R_{13}} + b_2T_{R_{23}} + b_3T_{R_{33}}) + a_3(b_1T_{R_{12}} + b_2T_{R_{22}} + b_3T_{R_{23}}) + b_2(a_1T_{R_{13}} + a_2T_{R_{23}} + a_3T_{R_{33}}) + b_3(a_1T_{R_{12}} + a_2T_{R_{22}} + a_3T_{R_{23}})]. \tag{170}$$

Appendix B

In this section we document the explicit expression for the second derivative of Π, which is used in Section 7:

$$\begin{aligned}\frac{\partial^2\Pi}{\partial T_{ij}\partial T_{kl}} = {}& \Pi_{11}\mathscr{A}^{(1)}_{ijkl} + \Pi_{12}\mathscr{A}^{(2)}_{ijkl} + \Pi_{13}\mathscr{A}^{(3)}_{ijkl} \\ &+\Pi_{14}\mathscr{A}^{(4)}_{ijkl} + \Pi_{15}\mathscr{A}^{(5)}_{ijkl} + \Pi_{16}\mathscr{A}^{(6)}_{ijkl} \\ &+\Pi_{17}\mathscr{A}^{(7)}_{ijkl} + \Pi_{18}\mathscr{A}^{(8)}_{ijkl} + \Pi_{19}\mathscr{A}^{(9)}_{ijkl} \\ &+\Pi_{22}\mathscr{A}^{(10)}_{ijkl} + \Pi_{23}\mathscr{A}^{(11)}_{ijkl} + \Pi_{24}\mathscr{A}^{(12)}_{ijkl} \\ &+\Pi_{25}\mathscr{A}^{(13)}_{ijkl} + \Pi_{26}\mathscr{A}^{(14)}_{ijkl} + \Pi_{27}\mathscr{A}^{(15)}_{ijkl} \\ &+\Pi_{28}\mathscr{A}^{(16)}_{ijkl} + \Pi_{29}\mathscr{A}^{(17)}_{ijkl} + \Pi_{33}\mathscr{A}^{(18)}_{ijkl} \\ &+\Pi_{34}\mathscr{A}^{(19)}_{ijkl} + \Pi_{35}\mathscr{A}^{(20)}_{ijkl} + \Pi_{36}\mathscr{A}^{(21)}_{ijkl} \\ &+\Pi_{37}\mathscr{A}^{(22)}_{ijkl} + \Pi_{38}\mathscr{A}^{(23)}_{ijkl} + \Pi_{39}\mathscr{A}^{(24)}_{ijkl} \\ &+\Pi_{44}\mathscr{A}^{(25)}_{ijkl} + \Pi_{45}\mathscr{A}^{(26)}_{ijkl} + \Pi_{46}\mathscr{A}^{(27)}_{ijkl} \\ &+\Pi_{47}\mathscr{A}^{(28)}_{ijkl} + \Pi_{48}\mathscr{A}^{(29)}_{ijkl} + \Pi_{49}\mathscr{A}^{(30)}_{ijkl} \\ &+\Pi_{55}\mathscr{A}^{(31)}_{ijkl} + \Pi_{56}\mathscr{A}^{(32)}_{ijkl} + \Pi_{57}\mathscr{A}^{(33)}_{ijkl} \\ &+\Pi_{58}\mathscr{A}^{(34)}_{ijkl} + \Pi_{59}\mathscr{A}^{(35)}_{ijkl} + \Pi_{66}\mathscr{A}^{(36)}_{ijkl} \\ &+\Pi_{67}\mathscr{A}^{(37)}_{ijkl} + \Pi_{68}\mathscr{A}^{(38)}_{ijkl} + \Pi_{69}\mathscr{A}^{(39)}_{ijkl} \\ &+\Pi_{77}\mathscr{A}^{(40)}_{ijkl} + \Pi_{78}\mathscr{A}^{(41)}_{ijkl} + \Pi_{79}\mathscr{A}^{(42)}_{ijkl} \\ &+\Pi_{88}\mathscr{A}^{(43)}_{ijkl} + \Pi_{89}\mathscr{A}^{(44)}_{ijkl} + \Pi_{99}\mathscr{A}^{(45)}_{ijkl} \\ &+\Pi_{2}\mathscr{C}^{(1)}_{ijkl} + \Pi_{3}\mathscr{C}^{(2)}_{ijkl} + \Pi_{5}\mathscr{C}^{(3)}_{ijkl} \\ &+\Pi_{7}\mathscr{C}^{(1)}_{ijkl},\end{aligned} \tag{171}$$

where we recall that $\Pi_{ij} = \frac{\partial^2\Pi}{\partial I_i\partial I_j}$ and where we have defined

$$\mathscr{A}^{(1)}_{ijkl} = \delta_{ij}\delta_{kl}, \tag{172}$$

$$\mathscr{A}^{(2)}_{ijkl} = T_{ij}\delta_{kl} + \delta_{ij}T_{kl}, \tag{173}$$

$$\mathscr{A}^{(3)}_{ijkl} = T_{in}T_{nj}\delta_{kl} + \delta_{ij}T_{kn}T_{nl}, \tag{174}$$

$$\mathscr{A}^{(4)}_{ijkl} = a_ia_j\delta_{kl} + \delta_{ij}a_ka_l, \tag{175}$$

$$\mathscr{A}^{(5)}_{ijkl} = (T_{in}a_na_j + a_iT_{jn}a_n)\delta_{kl} + \delta_{ij}(T_{kn}a_na_l + a_kT_{ln}a_n), \tag{176}$$

$$\mathscr{A}^{(6)}_{ijkl} = b_ib_j\delta_{kl} + \delta_{ij}b_kb_l, \tag{177}$$

$$\mathscr{A}^{(7)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)\delta_{kl} + \delta_{ij}(T_{kn}b_nb_l + b_kT_{ln}b_n), \tag{178}$$

$$\mathscr{A}^{(8)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)\delta_{kl} + \delta_{ij}(a_kb_l + a_lb_k)], \tag{179}$$

$$\mathscr{A}^{(9)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[\delta_{ij}(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jn}b_n + T_{in}b_na_j + b_iT_{jn}a_n + T_{in}a_nb_j)\delta_{kl}], \tag{180}$$

$$\mathscr{A}^{(10)}_{ijkl} = T_{ij}T_{kl}, \tag{181}$$

$$\mathscr{A}^{(11)}_{ijkl} = T_{in}T_{nj}T_{kl} + T_{ij}T_{kn}T_{nl}, \tag{182}$$

$$\mathscr{A}^{(12)}_{ijkl} = a_ia_jT_{kl} + T_{ij}a_ka_l, \tag{183}$$

$$\mathscr{A}^{(13)}_{ijkl} = (T_{in}a_na_j + a_iT_{jn}a_n)T_{kl} + T_{ij}(T_{kn}a_na_l + a_kT_{ln}a_n), \tag{184}$$

$$\mathscr{A}^{(14)}_{ijkl} = b_ib_jT_{kl} + T_{ij}b_kb_l, \tag{185}$$

$$\mathscr{A}^{(15)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)T_{kl} + T_{ij}(T_{kn}b_nb_l + b_kT_{ln}b_n)], \tag{186}$$

$$\mathscr{A}^{(16)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)T_{kl} + T_{ij}(a_kb_l + a_lb_k)], \tag{187}$$

$$\mathscr{A}^{(17)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[T_{ij}(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jn}b_n + T_{in}b_na_j + b_iT_{jn}a_n + T_{in}a_nb_j)T_{kl}], \tag{188}$$

$$\mathscr{A}^{(18)}_{ijkl} = T_{in}T_{nj}T_{km}T_{ml}, \tag{189}$$

$$\mathscr{A}^{(19)}_{ijkl} = a_ia_jT_{kn}T_{nl} + T_{in}T_{nj}a_ka_l, \tag{190}$$

$$\mathscr{A}^{(20)}_{ijkl} = (T_{in}a_na_j + a_iT_{jn}a_n)T_{km}T_{ml} + T_{in}T_{nj}(T_{km}a_ma_l + a_kT_{lm}a_m), \tag{191}$$

$$\mathscr{A}^{(21)}_{ijkl} = b_ib_jT_{kn}T_{nl} + T_{in}T_{nj}b_kb_l, \tag{192}$$

$$\mathscr{A}^{(22)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)T_{km}T_{ml} + T_{in}T_{nj}(T_{km}b_mb_l + b_kT_{lm}b_m)], \tag{193}$$

$$\mathscr{A}^{(23)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)T_{km}T_{ml} + T_{in}T_{nj}(a_kb_l + a_lb_k)], \tag{194}$$

$$\mathscr{A}^{(24)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[T_{im}T_{mj}(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jm}b_m + T_{im}b_ma_j + b_iT_{jm}a_m + T_{im}a_mb_j)T_{kn}T_{nl}], \tag{195}$$

$$\mathscr{A}^{(25)}_{ijkl} = a_ia_ja_ka_l, \tag{196}$$

$$\mathscr{A}^{(26)}_{ijkl} = (T_{in}a_na_j + a_iT_{jn}a_n)a_ka_l + a_ia_j(T_{kn}a_na_l + a_kT_{ln}a_n), \tag{197}$$

$$\mathscr{A}^{(27)}_{ijkl} = b_ib_ja_ka_l + a_ia_jb_kb_l, \tag{198}$$

$$\mathscr{A}^{(28)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)a_ka_l + a_ia_j(T_{kn}b_nb_l + b_kT_{ln}b_n), \tag{199}$$

$$\mathscr{A}^{(29)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)a_ka_l + a_ia_j(a_ka_l + a_la_k)], \tag{200}$$

$$\mathscr{A}^{(30)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[a_ia_j(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jn}b_n + T_{in}b_na_j + b_iT_{jn}a_n + T_{in}a_nb_j)a_ka_l], \tag{201}$$

$$\mathscr{A}^{(31)}_{ijkl} = (T_{in}a_na_j + a_iT_{jn}a_n)(T_{km}a_ma_l + a_kT_{lm}a_m), \tag{202}$$

$$\mathscr{A}^{(32)}_{ijkl} = b_ib_j(T_{km}a_na_l + a_kT_{ln}a_n) + (T_{in}a_na_j + a_iT_{jn}a_n)b_kb_l, \tag{203}$$

$$\mathscr{A}^{(33)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)(T_{km}a_ma_l + a_kT_{lm}a_m) + (T_{in}a_na_j + a_iT_{jn}a_n)(T_{km}b_mb_l + b_kT_{lm}b_m), \quad (204)$$

$$\mathscr{A}^{(34)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)(T_{km}a_ma_l + a_kT_{lm}a_m) + (T_{in}a_na_j + a_iT_{jn}a_n)(a_kb_l + a_lb_k)], \quad (205)$$

$$\mathscr{A}^{(35)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(T_{im}a_ma_j + a_iT_{jm}a_m)(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jm}b_m + T_{im}b_ma_j + b_iT_{jm}a_m + T_{im}a_mb_j)(T_{kn}a_na_l + a_kT_{ln}a_n)], \quad (206)$$

$$\mathscr{A}^{(36)}_{ijkl} = b_ib_jb_kb_l, \quad (207)$$

$$\mathscr{A}^{(37)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)b_kb_l + b_ib_j(T_{kn}b_nb_l + b_kT_{ln}b_n)], \quad (208)$$

$$\mathscr{A}^{(38)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)b_kb_l + b_ib_j(a_kb_l + a_lb_k)], \quad (209)$$

$$\mathscr{A}^{(39)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[b_ib_j(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jn}b_n + T_{in}b_na_j + b_iT_{jn}a_n + T_{in}a_nb_j)b_kb_l], \quad (210)$$

$$\mathscr{A}^{(40)}_{ijkl} = (T_{in}b_nb_j + b_iT_{jn}b_n)(T_{km}b_mb_l + b_kT_{lm}b_m), \quad (211)$$

$$\mathscr{A}^{(41)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + b_ia_j)(T_{kn}b_nb_l + b_kT_{ln}b_n) + (T_{in}b_nb_j + b_iT_{jn}b_n)(a_kb_l + a_lb_k)], \quad (212)$$

$$\mathscr{A}^{(42)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(T_{im}b_mb_j + b_iT_{jm}b_m)(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jm}b_m + T_{im}b_ma_j + b_iT_{jm}a_m + T_{im}a_mb_j)(T_{kn}b_nb_l + b_kT_{ln}a_n)], \quad (213)$$

$$\mathscr{A}^{(43)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})^2(a_ib_j + b_ia_j)(a_kb_l + a_lb_k), \quad (214)$$

$$\mathscr{A}^{(44)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})[(a_ib_j + a_jb_i)(a_kT_{ln}b_n + T_{kn}b_na_l + b_kT_{ln}a_n + T_{kn}a_nb_l) + (a_iT_{jn}b_nT_{in}b_na_j + b_iT_{jn}a_n + T_{in}a_nb_j)(a_kb_l + a_lb_k)], \quad (215)$$

$$\mathscr{A}^{(45)}_{ijkl} = (\mathbf{a}\cdot\mathbf{b})^2(a_iT_{jm}b_m + T_{im}b_ma_j + b_iT_{jm}a_m + T_{im}a_mb_j)(a_kT_{ln}b_n + T_{kn}b_na_lb_kT_{ln}a_n + T_{kn}a_na_l), \quad (216)$$

$$\mathscr{C}^{(1)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}), \quad (217)$$

$$\mathscr{C}^{(2)}_{ijkl} = \frac{1}{2}(\delta_{ik}T_{jl} + T_{ik}\delta_{lj} + \delta_{kj}T_{il} + T_{kj}\delta_{li}), \quad (218)$$

$$\mathscr{C}^{(3)}_{ijkl} = \frac{1}{2}(\delta_{ik}a_ja_l + a_k\delta_{li}a_j + \delta_{kj}a_ia_l + a_k\delta_{lj}a_i), \quad (219)$$

$$\mathscr{C}^{(4)}_{ijkl} = \frac{1}{2}(\delta_{ik}b_jb_l + b_k\delta_{li}b_j + \delta_{kj}b_ib_l + b_k\delta_{lj}b_i). \quad (220)$$

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