Proceedings of the ASME 2018
Verification and Validation Symposium
VVS2018
May 16-18, 2018, Minneapolis, MN, USA

VVS2018-9318

BAYESIAN FRAMEWORK TO QUANTIFY UNCERTAINTIES IN PIEZOELECTRIC ENERGY HARVESTERS

Patricio Peralta
Department of Mechanical Engineering
Universidad de Chile
Santiago, Chile
Email: patricio.peralta@ug.uchile.cl

Rafael O. Ruiz *
Department of Civil Engineering
Universidad de Chile
Santiago, Chile
Email: rafaelruiz@uchile.cl

Viviana Meruane
Department of Mechanical Engineering
Universidad de Chile
Santiago, Chile
Email: vmeruane@ing.uchile.cl

ABSTRACT

The interest of this work is to describe a framework that allows the use of the well-known dynamic estimators in piezoelectric harvester (deterministic performance estimators) but taking into account the random error associated to the mathematical model and the uncertainties on the model parameters. The framework presented could be employed to perform Posterior Robust Stochastic Analysis, which is the case when the harvester can be tested or it is already installed and the experimental data is available. In particular, it is introduced a procedure to update the electromechanical properties of PEHs based on Bayesian updating techniques. The mean of the updated electromechanical properties are identified adopting a Maximum a Posteriori estimate while the probability density function associated is obtained by applying a Laplaces asymptotic approximation (updated properties could be expressed as a mean value together a band of confidence). The procedure is exemplified using the experimental characterization of 20 PEHs, all of them with same nominal characteristics. Results show the capability of the procedure to update not only the electromechanical properties of each PEH (mandatory information for the prediction of a particular PEH) but also the characteristics of the whole sample of harvesters (mandatory information for design purposes). The results reveal the importance to include the model parameter uncertainties in order to generate robust predictive tools in energy harvesting. In that sense, the present framework constitutes a powerful tool in the robust design and prediction of piezoelectric energy harvesters performance.

INTRODUCTION

The dynamic description of piezoelectric energy harvesters (PEHs) has been widely studied in the last decade. Different deterministic modelling techniques and simplifications have been adopted to describe their electro-mechanical coupling effect in order to increase the accuracy on the output power estimation. Although it is a common practice to use deterministic models to predict the input-output behavior of PEHs, perfect predictions are not expected since these devices are not exempt of uncertainties. The accuracy of the output estimation is affected mainly by three factors: (1) the mathematical model used, (2) the uncertainties on the mathematical model parameters and (3) the uncertainties related to the excitation. These uncertainties should be taken into account in order to generate robust and more plausible predictions. Nevertheless, only a limited attention has been paid in the uncertainty quantification related to model parameters in piezoelectric energy harvesters.

In that sense, the adequate modelling of PEH plays a determinant role for their prediction, optimization and design. Several models have been developed, but in particular two of them have been widely accepted and used in the scientific community: the analytical distributed parameter solution introduced by Erturk and Inman [1,2] and the finite element plate model introduced by De Marqui Junior et al. [3]. These models are particularly interesting since they have been ample tested showing an important level of accuracy.

Although the models available offer a good description of the physics involved in PEH, its accuracy relies in the complete knowledge of the electromechanical properties of the piezoelec-

*Address all correspondence to this author.

tric material together the geometrical characteristics of the harvester. In a recent work, Franco and Varoto [4] presented a design framework for PEHs but taken into account the uncertainties related to the geometrical parameters and the load resistance of the harvester. The results revealed that the incorporation of these uncertainties affects significantly the prediction of the PEH. On the otherhand, Ruiz and Meruane [5] observed that the manufacturers typically report these electromechanical properties with variation close to $\pm 20\%$ of their nominal values. In that sense, the authors proposed a procedure to propagate these variations (all obtained from the literature) in order to identified the expected Frequency Response Function (FRF) and its corresponding confidence interval. The work conducted by Ruiz and Meruane presents one of the first efforts to describe the dynamic behaviour of PEHs assuming the electromechanical characteristic as uncertain parameters. As a result, the authors conclude that in order to obtain more robust predictions is mandatory to account the characteristic of the PEH as uncertain parameters or improve the manufacturing tolerance to decrease the variability in the PEH parameters. In other words, the accuracy in the prediction of the well-known deterministic models will be lost if the lack of information related to the characteristic of the PEH is omitted. However, the study performed by the authors was conducted only through simulation; no experiments were performed at the time. The present study deals with the experimental characterization of PEHs and their model parameter updating in order to improve future predictions.

DETERMINISTIC MODELS FOR THE FRF PREDICTION

Two different approaches have been widely adopted by researchers to predict the dynamic behavior of PEHs (either in unimorh or bimorph configurations). The first approach corresponds to the Analytical Distributed Parameter Solution (ADPS) proposed by Erturk and Inman [1,2], which consist on a standard modal expansion assuming an Euler-Bernoulli beam model. The approach have been used in several investigation since it was originally validated through rigorous experiments [1-3]. The approach allows the computation of the FRF without a significant computational burden since it relies on an analytical procedure, however it is limited to PEHs with a beam-like geometry. To overcome this disadvantage, a procedure based on Finite Element Method (FEM) had been introduced by De Marqui et al. [3] allowing the dynamical prediction of PEHs with different plate-like geometries. In despite of the procedure adopted (the ADPS or the FEM-based model), it is possible to identify the FRF as a function of the electromechanical and geometrical characteristic of the harvester. In that sense, the FRF of PEHs is express as $H(\boldsymbol{\theta}, \mathbf{g})$, here, $\boldsymbol{\theta}$ corresponds to a vector with the electromechanical properties while $\mathbf{g}$ is another vector with the geometrical characteristics of the harvester, such that:

$$\boldsymbol{\theta} = [Y_s \;\; s_{11}^E \;\; d_{31} \;\; \varepsilon_{33}^T \;\; \zeta \;\; \rho_p \;\; \rho_s] \tag{1}$$

$$\mathbf{g} = [L \;\; b \;\; h_p \;\; h_s] \tag{2}$$

The above variables corresponds to the length L, width b, thickness of the piezoelectric layer h_p, thickness of the substructure h_s, the substructures density ρ_s, the piezoelectric layers density ρ_p, the Young Modulus of the substructure Y_s, the elastic compliance at constant electric field s_{11}^E, the piezoelectric strain constant d_{31}, the permittivity at constant stress ε_{33}^T and the damping ratio ζ. The FRF of interest for this work corresponds to the relation of the output voltage with the acceleration at the base of the PEH, for more detailed information on the full expression of H could be found on [5].

EXPERIMENTAL CHARACTERIZATION OF PEHs

The experimental identification of the FRF in PEHs requires the direct measurement of the base acceleration together to the voltage generated by the piezoelectric layer. As the goal of this research is to study the uncertainties in PEHs, it is required to identify the FRFs of various harvesters with the same nominal characteristics (20 bimorph PEHs are used in this study). The manufacturer reports that electromechanical characteristics indicated in Table 1 could present variations of $\pm 20\%$ of their nominal values [6].

The length, width and total thickness of all tested harvesters are verified with a micrometer (precision of 0.001mm). The mean and standard deviation of those geometric characteristics are presented in Table 2. Here, it is possible to observe small coefficients of variation (lower than 2.6%), indicating an important level of precision in the manufacturing process associated to the external dimensioning of the PEH. The variability found here is in agreement with the values previously reported in the literature, by example in [5]. However, it is notorious that the mean value of the thickness obtained from measurements differs from the nominal value reported in Table 1 (differences close to 10%). This difference is relevant since the thickness belongs to the set of parameters that affects the most the variability of the FRF [5], thus it is expected differences between the recorded and the nominally predicted FRF.

Three analyses are independently conducted in order to identify and rank the sources of uncertainty. These analyses evaluate: (1) the repeatability of the FRF measured, (2) the sensitivity of the mounting process over the FRF, and (3) the FRF of several PEHs with identical nominal values.

Test 1 (repeatability of the FRF): The FRF of each PEH is estimated by performing 100 independent measurements, which

TABLE 1. Nominal Characteristic of the PEHs Tested.

Property	Value
ρ_s	Not Available
Y_s	Not Available
s^E_{11}	$16.4x10^{-12}m^2N^{-1}$
d_{31}	$-320x10^{-12}CN^{-1}$
ε^T_{33}	$4500x(8.854x10^{-12})Fm^{-1}$
ρ_p	$7.4x10^3kgm^{-3}$
L	$60mm$
b	$10mm$
Total Thickness	$0.8mm$

TABLE 2. Geometric Characteristic Measured for all PEHs Studied.

Property	Mean	Coefficient of Variation
L	$60.046mm$	0.1%
b	$10.064mm$	0.1%
Total Thickness	$0.713mm$	2.6%

are then used to obtain the mean values at difference frequencies and their respective standard deviations. By example, Figure 1 presents the FRF of a single PEH, the figure shows the 100 measurements (in grey), while the mean value and the standard deviation for a given frequencies are presented in a black solid line and error bars, respectively. For the sake of brevity, results of others PEHs tested are not presented since they showed a similar trend. The interesting result here is related to the dispersion identified (error bars in Fig.1), where it is observed a small coefficient of variation (CV) for all cases indicating an important level of repeatability. In general, the CV is lower than 2% except for frequencies close to resonance, where it is possible to observe CV up to 6% .

Test 2 (Sensitivity of the Mounting Process): In a previous work [5], it was identified that the FRF (more specifically the fundamental frequency) is sensitive to small changes in the PEH length. The effective length of the PEH is defined as the length of the PEH that is in cantilever, as a result, this length depends on the mounting process and the quality of the fixing constrain. The question that arise here is how different is the FRF when a PEH is installed, uninstalled, and reinstalled again. In particular, the FRF of each PEHs studied is identified after repeating the installation process. The PEH is installed and uninstalled 5 times, after each installation the FRF is identified by using 10 independent measurements. Results are presented in Fig. 2 keeping the same format employed in Fig. 1. Here, the CV has the same trend previously observed, it increases close to resonance and decreases out of it, but its value is considerably higher, reaching CVs up to 18% . Note that these CVs are close to be 3 times the CVs identified as the baseline.

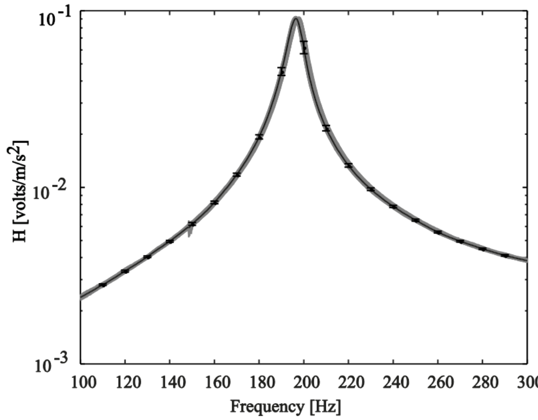

FIGURE 1. Example of repeatibility for a single PEH.

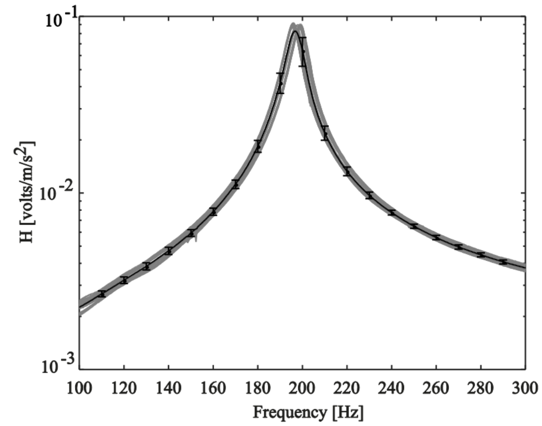

FIGURE 2. Effect of the uncertainty in the mounting process.

Test 3 (Uncertainties between nominally identical PEHs): The FRF of each single PEH is identified through 20 independent measurements and the results are presented in Fig. 3. Following the previous figures legend, the black solid line represents the expected value for a given frequency while each grey solid line represents a single measurement of the FRF. Since 20 PEHs are tested and each one is tested 20 times, the total number of solid grey lines is 400. Here, the error bars represent the standard

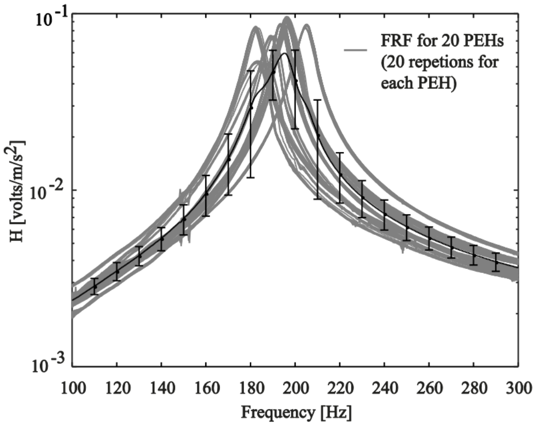

FIGURE 3. Measurement of the FRF of 20 PEH, all of them with identical nominal characteristics.

deviation of the measurement for a given excitation frequency (as it is used in previous cases). Results are very interesting since the CVs observed are considerably greater than previous cases (at least 4 times higher than test 2), reaching in some cases values up to 60%.

Ultimately, the CVs identified in test 3 corresponds to variations introduced by the combination of: noise, clapping condition and variations in the geometric parameters together with variations presented in the electromechanical properties of each PEH. In that sense, the effect of uncertainties associated to the geometrical characteristics and the electromechanical properties of the harvesters could be estimated by subtracting the CV identified in test 2 to the CV identified in test 3. However, the geometric parameters of the PEHs tested were fully identified in Table 2, where it is possible to observe coefficient of variations lower than 2.6%, suggesting that the uncertainties identified can be attributed primarily to the electromechanical characteristics of the PEH.

PROBABILISTIC PREDICTION USING NOMINAL VALUES

The procedure presented by Ruiz and Meruane [5] is employed here to quantify the expected uncertainties in the FRF. The authors used the well-known prediction model proposed by Erturk and Inman in 2008 [1,2] to estimate the variability of the FRF due to variations in the geometric parameters and in the electromechanical properties of the harvester. Next, a brief description of this procedure is offered. First, it is necessary to establish a deterministic model to identify the FRF of the PEH. As it was mentioned before, the ADPS is used for this study. The goal is to estimate the variation of the FRF due to variations on the vector $\boldsymbol{\theta}$. In that sense, the model parameter vector $\boldsymbol{\theta}$ can

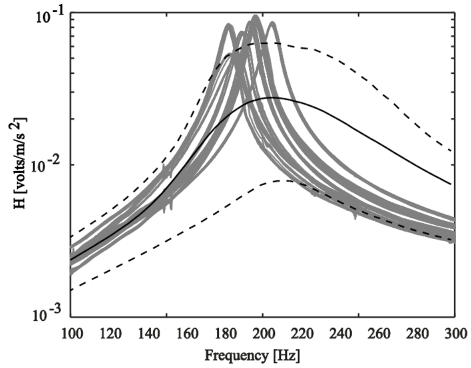

FIGURE 4. Comparison between a prior probabilistic prediction and measurements

be modelled as a random vector defined by a multivariate PDF identified as $p(\boldsymbol{\theta})$. These uncertainties are propagated such that the expected value of the FRF can be estimated by solving the following probabilistic integral:

$$E[H] = \int_{\Theta} H(\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{3}$$

Additionally, it is possible to identified the probability (P_o) that H exceed a certain value $H_{threshold}$:

$$P(H > H_{threshold}) = P_o \tag{4}$$

There are different procedures to solve Eqs.(3) and (4), being the stochastic simulation one of the most commonly procedure adopted, which is the scheme used in the present work, specifically the Monte Carlo Importance Sampling technique. A more completed discussion on how to solve these equations can be found in [5,7].

The probabilistic model for the model parameters $p(\boldsymbol{\theta})$ is defined by using the values reported in [5,6] for the geometry and variations of $\pm 20\%$ for the electromechanical properties of the harvester (as it is reported by the manufacturer). All variables are assumed to be Gaussian.

The comparison between measurements and predictions are presented in Figure 4. There, it is possible to observe the measurements in grey, the dotted lines correspond to the exceedance probability equal to 90% $(P_o = 0.9)$ and 10% $(Po = 0.1)$, while the black solid line corresponds to the expected value estimated by the predictor. Here, it is important to recall Eq.(4) which defines the exceedance probability, in that sense, the area between the dotted lines represents an interval confidence equal to 80%.

The first interesting observation is that the measurements lie inside the interval confidence area, which shows the capability of the numerical procedure to estimate the possible location of the actual FRF. However, the fundamental frequency estimated by the predictor is higher than the mean fundamental frequency observed, the prediction estimates 200 Hz while the fundamental frequency observed has a mean of 195 Hz. This bias identified between mean values of prediction and measurement indicates that the nominal values employed in the predictions are not completely accurate. Additionally, it is observed that the dispersion used in the predictive model is considerable greater than the actual dispersion of the set of PEHs.

BAYESIAN APPROACH TO UPDATE MODEL PARAMETERS

The following section presents a Bayesian updating method in order to identify the model parameters of a PEH. The idea behind the method is to incorporate the information gained in the experimental test to decrease the level of uncertainty associated to the main characteristics of the PEH. Based on the experimental results, it is proposed to incorporate an additive prediction error such that the estimation of the FRF can be obtained computing the following equation:

$$h = H(\boldsymbol{\theta}) + e \tag{5}$$

where e is an error defined by a Gaussian PDF with zero mean and a specific standard deviation σ_e. Under this assumption, the real system FRF (h) is defined by the following PDF:

$$p(y \mid \boldsymbol{\theta}) = \frac{1}{\sigma_e\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma_e^2}(h - H(\boldsymbol{\theta}))^2\right] \tag{6}$$

Note that Eq.6 represents the real PDF of the FRF when the model parameters $\boldsymbol{\theta}$ are known. If the information about model parameters is incomplete (model parameters defined by a PDF $p(\boldsymbol{\theta})$), then it is possible to propagate the uncertainties to compute the expected value of the real FRF as:

$$E[h] = \int H(\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{7}$$

Here, $p(\boldsymbol{\theta})$ receive the name of prior PDF, since it has the initial information of the model parameters (information that usually is given by the manufacturer, leading to important variations in the estimation of the FRF). In that sense, Eq.7 and Eq.3 are essentially the same. The important aspect of the Bayesian approach is that the prior PDF $p(\boldsymbol{\theta})$ could be updated by using experimental data, then, it is obtained a posterior PDF denoted as $p(\boldsymbol{\theta} \mid D)$. Where the letter D stands to explicitly indicate that the PDF has been updated by the experimental data D. In this case, Eq.7 is re-wrote as:

$$E[h \mid D] = \int H(\boldsymbol{\theta}) p(\boldsymbol{\theta} \mid D) d\boldsymbol{\theta} \tag{8}$$

The procedure to identify the posterior PDF $p(\boldsymbol{\theta} \mid D)$ is presented in the following paragraphs. If the concept presented in Eq.6 is extended for multiple outputs, such that $\mathbf{h} = \{h_1\ h_2\ ...\ h_M\}$, then the output probability model for M-measurements is given by:

$$p(\mathbf{h} \mid \boldsymbol{\theta}) = \prod_{m=1}^{M} p(h_m \mid \boldsymbol{\theta}) \tag{9}$$

Note that Eq.9 exhibits an independence between the errors of different pairs of outputs. In other words, the prediction error at certain point does not affect the prediction error at other points. Since the experimental data D is available, the Bayes Theorem can be applied here to update the probability model $p(\boldsymbol{\theta})$, such that:

$$p(\boldsymbol{\theta} \mid D) = \frac{p(D \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})}{\int p(D \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}} \tag{10}$$

Since Eq.9 is based on measurements, the $p(\mathbf{h} \mid \boldsymbol{\theta})$ can be interpreted as $p(D \mid \boldsymbol{\theta})$, which Eq.10 is expressed as:

$$p(\boldsymbol{\theta} \mid D) = \frac{p(\mathbf{h} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})}{\int p(\mathbf{h} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}} \tag{11}$$

Here, $p(\mathbf{h} \mid \boldsymbol{\theta})$ is the likelihood function which gives the probability to have the data D given the model parameters $\boldsymbol{\theta}$. On the other hand, the denominator of Eq.11 is a normalizing constant which is also called the evidence of the system.

The most probable values for the model parameters can be calculated by adopting a Maximum a Posteriori estimator (MAP), where the goal is to identify the maximum of the numerator in Eq.11 (please refers to [8] for further details):

$$\boldsymbol{\theta}_{MAP} = \arg\max[\log(p(\mathbf{h} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}))] \tag{12}$$

Different methods to solve Eq.11 can be implemented. By example, methods based on stochastic simulations require generating samples from $p(\boldsymbol{\theta} \mid D)$, which is computational challenging since occupies a small and non-trivial-shape region in the model parameter space [8]. On the other hand, the Laplace method of asymptotic approximation could be used as long as the log of likelihood function be twice differentiable and significant peaked (condition attained for M large) around a global maximum. The approximation behind this method relies in the assumption that the numerator of Eq.11 is a Gaussian function centered at $\boldsymbol{\theta}_{MAP}$, then, the covariance of the posterior model parameter is equal to the inverse of the Hessian matrix of $-\log(p(\mathbf{h} \mid \boldsymbol{\theta})p(\boldsymbol{\theta}))$ evaluated at $\boldsymbol{\theta}_{MAP}$.

Method Applied to a PEH

The procedure presented in previous section is applied to one of the PEH used in the test. It is important to note that the idea is to present a procedure to identify the actual parameters of a single PEH by taking advantage of the information gained in the experimental test. In order to update the probability model for the model parameter vector, it necessary to define first the model parameters that will be considered uncertain. In general, the magnitudes related to geometric characteristics can be easily obtained with an important degree of certainty. As it was discussed previously in Tables 1 and 2, the variability of the geometrical characteristics are small (lower than 2.6%) compared with the variability associated with the electromechanical properties of the harvester. In that sense, the geometrical characteristics of the harvester can be considered known parameters (or free of uncertainties). Now, the model parameter vector $\boldsymbol{\theta}$ contains only the parameters that are considered uncertain.

The prior probability model $p(\boldsymbol{\theta})$ is defined then as an uncorrelated Gaussian distribution where the mean corresponds to the nominal values (presented in Table 1), and coefficient of variations are equal to 20% for each parameters. It is important to remark that the information of this variability was directly given by the manufacturer [6]. The experimental set of data to perform the updating consist in a set of 20 independent measurement of the same PEH.

The results are presented in Table 3, where the nominal values are compared to the prior and posterior mean values. The coefficient of variation are also shown in parenthesis. Note that the prior and posterior models are Gaussian distributions with different means and different coefficient of variations while the additive prediction error is assumed as a Gaussian distribution with zero mean and coefficient of variation equal to 4% (which is chosen based on the information obtained in test 1).

Here, it is shown the differences between the prior (which is defined having not previous information of the system) and posterior distribution (established by including the information coming from measurements). Note that the mean values of both distributions are similar but the variances present important differences. The posterior distribution corresponds to a narrower distribution compared with the prior. This behavior is expected since the inclusion of additional information (measurements) tends to decrease the uncertainties.

TABLE 3. Prior and Posterior PDF employed in the Bayesian procedure.

$\boldsymbol{\theta}$	Prior	Posterior
ρ_s	$7.4x10^3\,kgm^{-3}(20\%)$	$8.4x10^3\,kgm^{-3}(15\%)$
Y_s	$6.1x10^{10}\,Pa(20\%)$	$6.1x10^{10}\,Pa(18\%)$
s_{11}^E	$16.4x10^{-12}\,m^2/N(20\%)$	$14.1x10^{-12}\,m^2/N(8\%)$
d_{31}	$-320x10^{-12}\,C/N(20\%)$	$-246x10^{-12}\,C/N(5\%)$
ε_{33}^T	$4500\varepsilon_o\,F/m(20\%)$	$3500\varepsilon_o\,F/m(6\%)$
ρ_p	$7.4x10^3\,kgm^{-3}(20\%)$	$7.3x10^3\,kgm^{-3}(15\%)$
ζ	$0.017(20\%)$	$0.015(3\%)$
ε_o	$= 8.854x10^{-12}$	

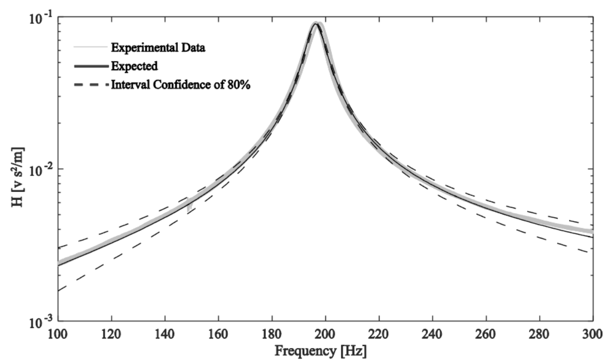

FIGURE 5. FRF obtained by the implementation of the Bayesian procedure.

After the identification of the posterior probability density function (values presented in Table 3), it is possible to propagate these uncertainties in order to study the new dispersion obtained in the estimation of the FRF. The propagation is made by solving Eq.8 adopting a Monte Carlo Approach, results are presented in Fig.5, where it is observed the reduction on the confidence interval (dotted lines) due to the updating of the model uncertainties. Additionally, it is observed a good agreement between the expected value of the FRF and the measurements obtained previously, indicating the versatility and the of this procedure to improve the prediction of the FRF in PEHs. Although the results presented here corresponds only to the updating characteristics of a single PEH, the procedure could be expanded to identify the characteristics not only of a single harvester but to the whole set of PEHs.

CONCLUSIONS

A framework to propagate uncertainties in piezoelectric energy harvesters was presented. The framework presents a series of significant advantages since: (1) it is compatible with any well-known energy harvester performance predictor (deterministic models), (2) it is independent of the number of piezoelectric and substructure layers, (3) it allows to define expected values as well as confidence intervals for the FRF associated to the output voltage, and (4) it allows to update the characteristics of the PEH based on experimental data. An extensive experimental characterization is also presented by testing 20 PEHs with identical nominal characteristics. The experimental measurements served as data to exemplified the framework presented. Results reveal the real necessity (validated from the experimental point of view) to incorporate the uncertainties in the prediction of the FRF. Additionally, the framework proposed was exemplified by updating the electromechanical characteristics of a single PEH increasing the accuracy in the estimation of its FRF.

ACKNOWLEDGMENT

This research is supported by the Comision Nacional de Investigacion Cientifica y Tecnologica de Chile through the project CONICYT/FONDECYT/3160491.

REFERENCES

[1] A. Erturk and D. J. Inman, A distributed parameter electromechanical model for cantilevered piezoelectric energy harvesters, J. Vib. Acoust. Trans. ASME, vol. 130, no. 4, 2008.

[2] A. Erturk and D. J. Inman, An experimentally validated bimorph cantilever model for piezoelectric energy harvesting from base excitations, Smart Mater. Struct., vol. 18, no. 2, p. 025009, 2009.

[3] C. D. M. Junior, A. Erturk, and D. J. Inman, An electromechanical finite element model for piezoelectric energy harvester plates, J. Sound Vib., vol. 327, no. 1, pp. 925, 2009.

[4] V. R. Franco and P. S. Varoto, Parameter uncertainties in the design and optimization of cantilever piezoelectric energy harvesters, Mech. Syst. Signal Process., vol. 93, pp. 593609, 2017.

[5] R. O. Ruiz and V. Meruane, Uncertainties propagation and global sensitivity analysis of the frequency response function of piezoelectric energy harvesters, Smart Mater. Struct., vol. 26, no. 6, p. 065003, 2017.

[6] Sinocera 2016 http://sinocera.net/

[7] R. D. Rosenkrantz, ET Jaynes: Papers on probability, statistics and statistical physics, vol. 158. Springer Science and Business Media, 2012.

[8] J. L. Beck and A. A. Taflanidis, Prior and Posterior Robust Stochastic Predictions for Dynamical Systems Using Probability Logic, Int. J. Uncertain. Quantif., vol. 3, no. 4, pp. 271288, 2013.