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Existence and nonexistence of positive solutions to some fully nonlinear equation in one dimension

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ABSTRACT

In this paper, we consider the existence (and nonexistence) of solutions to

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(u) \quad \text{in } \mathbf{R}$$

where $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$ denote the Pucci operators with $0 < \lambda \le \Lambda < \infty$, $V(x)$ is a bounded function, $f(s)$ is a continuous function and its typical example is a power-type nonlinearity $f(s) = |s|^{p-1}s$ $(p > 1)$. In particular, we are interested in positive solutions which decay at infinity, and the existence (and nonexistence) of such solutions is proved.

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1. Introduction

In this paper, we study the existence and nonexistence of solutions to the following nonlinear differential equations

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(u) \quad \text{in } \mathbf{R}, \quad u > 0 \quad \text{in } \mathbf{R}, \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1.1}$$

Here V and f are given functions, $0 < \lambda \leq \Lambda < \infty$ constants and $\mathcal{M}^{\pm}_{\lambda,\Lambda}(s)$ the Pucci operators defined by

$$\mathcal{M}^{+}_{\lambda,\Lambda}(s) := \begin{cases} \Lambda s & \text{if } s \geq 0, \\ \lambda s & \text{if } s < 0, \end{cases} \quad \mathcal{M}^{-}_{\lambda,\Lambda}(s) := \begin{cases} \lambda s & \text{if } s \geq 0, \\ \Lambda s & \text{if } s < 0. \end{cases}$$

We remark that when $\lambda = \Lambda$, one has $\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') = \lambda u''$.

One of motivations to study equations like (1.1) is to see to what extent the properties and the results in the semilinear case can be generalized to the fully nonlinear case. When $\lambda = \Lambda$, (1.1) is well studied and it is proved that (1.1) has a solution for various $V(x)$ and $f(s)$ by critical point theory. Here we refer to [11,12] and references therein.

On the other hand, when $\lambda \neq \Lambda$, (1.1) is not studied well. In [7], instead of (1.1), the authors study the existence of positive radial solutions of

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + \gamma u = f(u) \quad \text{in } B_R(0) \subset \mathbf{R}^N, \quad u = 0 \quad \text{on } \partial B_R(0) \tag{1.2}$$

as well as

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + u = u^p \quad \text{in } \mathbf{R}^N.$$

Here $N \geq 3$, $0 \leq \gamma$ and $1 < p < p^{\pm}_{*}$ where $p^{\pm}_{*}$ are critical exponents for $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ (see also [1,3,5,6]). Recently, in [9], the authors show the existence of infinitely many radial solutions of (1.2) when $\gamma = 0$ and $f(s) = |s|^{p-1}s$. Moreover, in [9], the inhomogeneous case is also considered and the existence of infinitely many solutions is shown on a bounded annulus.

In this paper, we aim to treat the inhomogeneous equation on the unbounded domain $\mathbf{R}$. We emphasis that in general the existence of solutions to (1.1) is delicate when the equation is inhomogeneous and the domain is unbounded. Indeed, we shall prove the nonexistence result when $V(x)$ is monotone. See Theorem 1.2 below.

We first deal with the existence result. For $V(x)$, we assume

(V1) $V \in W^{1,\infty}(\mathbf{R})$ and $0 < \inf_{\mathbf{R}} V =: V_0$.
(V2) For a.a. $x \in (-\infty, 0)$ and a.a. $y \in (0, \infty)$, $V'(x) \leq 0 \leq V'(y)$.
(V3) $V(0) \leq V_\infty := \lim_{|x|\to\infty} V(x)$ and there exist $C_0, \xi_0 > 0$ such that

$$(\text{for } \mathcal{M}^{+}_{\lambda,\Lambda}) \quad (0 \leq) V_\infty - V(x) \leq C_0 \exp\left(-2\sqrt{\frac{V_\infty}{\Lambda} + \xi_0}\,|x|\right) \quad \text{for all } x \in \mathbf{R},$$

$$(\text{for } \mathcal{M}^-_{\lambda,\Lambda}) \quad (0 \le) V_\infty - V(x) \le C_0 \exp\left(-2\sqrt{\frac{V_\infty}{\lambda} + \xi_0} |x| \right) \quad \text{for all } x \in \mathbf{R}.$$

Next, for $f(s)$, we suppose the following conditions and an example of $f(s)$ is $f(s) = \sum_{i=1}^k a_i s^{p_i}$ where $0 < a_i$ and $1 < p_i$:

(f1) $f \in C^1(\mathbf{R})$ and $f(s) = 0$ for all $s \le 0$.
(f2) There exists an $\eta_0 > 0$ such that $\lim_{s\to 0} s^{-1-\eta_0} f(s) = 0$.
(f3) As $s \to \infty$,

$$\frac{f(s)}{s} \to \infty \quad \text{and} \quad \frac{f(\theta s)}{f(s)} \to \bar{f}(\theta) \quad \text{in } C_{\mathrm{loc}}((0,1]).$$

(f4) $s \mapsto s^{-1} f(s) : (0,\infty) \to \mathbf{R}$ is strictly increasing.

Remark 1.1. (i) In (f3), it follows that $\bar{f} \in C((0,1])$, $\bar{f}(1) = 1$ and $\bar{f}(\theta) \ge 0$ for $\theta \in (0,1]$. For example, when $f(s) = s^p$ and $f(s) = s\log s$, one sees $\bar{f}(\theta) = \theta^p$ and $\bar{f}(\theta) = \theta$ respectively.

(ii) When $\lambda = \Lambda$, condition (f4) is used to obtain bounded Palais–Smale sequences. The classical condition to obtain bounded Palais–Smale sequences is the Ambrosetti–Rabinowitz condition: $0 < \mu F(s) \le f(s)s$ for some $\mu > 2$ and all $s > 0$ where $F(s) := \int_0^s f(t)\mathrm{d}t$. We remark that (f1)–(f4) do not imply this condition. In fact, consider a function defined by

$$f(s) = \eta(s)s^p + (1 - \eta(s))Cs\log s$$

where $1 < p$, $\eta \in C^\infty([0,\infty), \mathbf{R})$, $\eta'(s) \le 0$ for every $s \in [0,\infty)$, $\eta(s) = 1$ if $0 \le s \le 2$, $\eta(s) = 0$ if $3 \le s$ and $C > 0$ is chosen so that $C\log s \ge s^{p-1}$ in $[2,3]$. It is easily seen that f satisfies (f1)–(f4) with $\bar{f}(\theta) = \theta$ and that $F(s)$ has the growth $s^2\log(s)$ as $s \to \infty$, providing the required counterexample.

Under these conditions, we have

Theorem 1.1. *Under* (V1)–(V3) *and* (f1)–(f4), (1.1) *have a solution.*

Next, we turn to the nonexistence result. In this case, we assume that $V(x)$ is monotone:

(V2') $V'(x) \ge 0$ in $\mathbf{R}$ and

$$\underline{V} = \lim_{x\to-\infty} V(x) < \lim_{x\to\infty} V(x) = \overline{V}.$$

Then we have

Theorem 1.2. *Let* $0 < \lambda \leq \Lambda < \infty$ *and assume* (V1), (V2'), (f1), (f4) *and*

$$\lim_{s \to 0} \frac{f(s)}{s} = 0. \tag{1.3}$$

Then (1.1) *have no solution.*

Remark 1.2. Theorem 1.2 still holds when we replace (V2') by

$$V'(x) \leq 0 \quad \text{in } \mathbf{R}, \quad \overline{V} = \lim_{x \to -\infty} > \lim_{x \to \infty} V(x) = \underline{V}.$$

Here we make some comments on the proofs of Theorems 1.1 and 1.2. First, even though equation (1.1) can be transformed into an equation with variational structure (pointed by Professor Evans), we prefer to use degree theoretic arguments in view of future applications. For Theorem 1.1, we borrow the idea in [4] (cf. [7]). More precisely, we will find a suitable function space X which is a Banach space, and rewrite (1.1) into the equations $(\mathrm{id} - \mathcal{L}^{\pm})(u) = 0$ where $\mathcal{L}^{\pm}(u) := (-\mathcal{M}^{\pm}_{\lambda,\Lambda} + V(x))^{-1} f(u(x))$ for $u \in X$. To find a solution $u \neq 0$, we use the Leray–Schauder degree $\deg_X$ in X and prove that

i) There exists an $r_0 > 0$ such that $\deg_X(\mathrm{id} - \mathcal{L}^{\pm}, B_{r_0}(0), 0) = 1$.
ii) There exists an $r_1 > r_0$ such that $\deg_X(\mathrm{id} - \mathcal{L}^{\pm}, B_{r_1}(0), 0) = 0$.

From i) and ii), we have $\deg_X(\mathrm{id} - \mathcal{L}^{\pm}, A_{r_0,r_1}, 0) \neq 0$ and find a $u_0 \in A_{r_0,r_1}$ so that $(\mathrm{id} - \mathcal{L}^{\pm})(u_0) = 0$ where $A_{r_0,r_1} := \{u \in X \mid r_0 < \|u\|_X < r_1\}$. One of difficulties here is to find a suitable X in order that we can prove the property ii) as well as the map $\mathcal{L}^{\pm} : X \to X$ is compact. A key for proving ii) is a priori estimates of solutions in X. Since we treat the unbounded domain, we need the uniform decay estimates of solutions as well as the uniform L^∞-bounds. This point is different from the bounded domain case and requires delicate arguments. For instance, see Proposition 2.9 below.

We also point out that the argument of Proposition 2.9 is useful to show the nonexistence result namely, Theorem 1.2. Indeed, this case is simpler than Proposition 2.9 and we will prove Theorem 1.2 in section 3.

In Appendix A, we consider (1.1) in the special case when $V(x) \equiv \text{const.} > 0$. In this case, we can prove the unique existence of solutions up to translations. See Proposition 2.1 and Appendix A.

Finally, we would like to make some comments about the higher dimensional version of our problem, that is, the existence and nonexistence of positive solutions to equation

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2 u) + V(x)u = f(u) \quad \text{in } \mathbf{R}^N, \quad u(x) \to 0 \quad \text{as } |x| \to \infty.$$

In the general case this problem may be quite challenging because of the non-divergence form and non-differentiability of the differential operator. In the radial case, the problem may still be hard because of the presence of the singular and non-homogeneous term

$(N-1)u'(r)/r$, which comes from D^2u. In one-dimensional case, we exploit the fact that any function in $\mathcal{S}_t^{\pm}$ has only one maximum point in $\mathbf{R}$ (see Lemma 2.6 for the definition of $\mathcal{S}_t^{\pm}$ and Lemma 2.7 for the monotonicity property). From here we determine the number of changes of concavity of functions in $\mathcal{S}_t^{\pm}$, which is needed for the estimation of energy E_n and to complete the arguments. In the radial case, these monotonicity and concavity properties may become quite subtle due to the presence of the singular term. In the case of $V(r) \equiv$ constant > 0 we refer to [8], where some of these properties are explored in the study of uniqueness of positive solutions. It would be interesting to see whether an ODE approach as in [8] works well in the higher-dimensional case.

2. Proof of Theorem 1.1

Throughout this section, we always assume (f1)–(f4) and (V1)–(V3). We begin with the existence result when $V(x) \equiv$ const. > 0.

Proposition 2.1. *Under* (f1)–(f4), *the equations*

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(u'') + V_\infty u = f(u) \quad \text{in } \mathbf{R}, \quad u > 0 \text{ in } \mathbf{R}, \\ u(x) \to 0 \text{ as } |x| \to \infty, \quad u(0) = \max_{x\in\mathbf{R}} u(x) \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\mathcal{M}^{-}_{\lambda,\Lambda}(u'') + V_\infty u = f(u) \quad \text{in } \mathbf{R}, \quad u > 0 \text{ in } \mathbf{R}, \\ u(x) \to 0 \text{ as } |x| \to \infty, \quad u(0) = \max_{x\in\mathbf{R}} u(x) \end{cases} \tag{2.2}$$

have unique solutions ω_+ and ω_-. Furthermore, there exist $z^{\pm} > 0$, $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} &\omega''_{\pm}(x) < 0 = \omega''_{\pm}(z^{\pm}) < \omega''_{\pm}(y) && \text{for every } x, y \in \mathbf{R} \text{ with } |x| < z^{\pm} < |y|, \\ &\omega_{\pm}(x) \le c_1 \exp(-c_2|x|) && \text{for all } x \in \mathbf{R}. \end{aligned}$$

Finally, if u satisfies

$$\begin{aligned} &-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V_\infty u = f(u) \quad \text{in } \mathbf{R}, \quad u > 0 \text{ in } \mathbf{R}, \\ &u(0) = \max_{\mathbf{R}} u, \quad u(x) \to 0 \text{ if } x \to \infty \text{ or } x \to -\infty, \end{aligned}$$

then $u = \omega_{\pm}$.

We shall prove Proposition 2.1 in Appendix A.

From now on, we may assume $V(0) < V_\infty$ in (V3) and $V(x)$ is not a constant function without loss of generality. Under this additional assumption, we fix an $\eta_1 > 0$ so that

$$\eta_1 < c_2, \quad \Lambda \eta_1^2 \left(1 + \frac{\eta_0}{2}\right)^2 < \frac{V_0}{2}, \tag{2.3}$$

where $V_0 := \inf_{\mathbf{R}} V > 0$, and $\eta_0 > 0$ and $c_2 > 0$ appear in (f2) and Proposition 2.1. We set

$$X_{\eta_1} := \left\{ v \in C(\mathbf{R}) \,\middle|\, \|v\|_{\eta_1} = \sup_{x \in \mathbf{R}} e^{\eta_1 |x|} |v(x)| < \infty \right\}.$$

It is easy to check that $(X_{\eta_1}, \|\cdot\|_{\eta_1})$ is a Banach space.

Lemma 2.2. *For every $v \in X_{\eta_1}$, the equations*

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(v(x)) \quad \text{in } \mathbf{R}, \quad u \in X_{\eta_1},$$

have unique solutions.

Proof. We prove the claim at the same time for $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$. Let $v \in X_{\eta_1}$. For each $n \in \mathbf{N}$, consider

$$\begin{cases} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(v(x)) & \text{in } (-n,n), \\ u(-n) = 0 = u(n). \end{cases}$$

Then the above equations have a unique solution $u_n \in C^2([-n,n])$ due to (V1). In fact, since $f(s) \geq 0$ by (f1), (f2) and (f4), $u \equiv 0$ is a subsolution of the above equation. In addition, it is easily seen that the function $c(n^2 - x^2)$ with sufficiently large $c > 0$ is a supersolution. Thus, a solution u_n to the above equation exists and it is unique from the standard argument.

Now, the maximum principle yields $u_n \geq 0$ in $[-n,n]$. Moreover, we have

$$\|u_n\|_{L^\infty(-n,n)} \leq V_0^{-1} \|f(v)\|_{L^\infty(\mathbf{R})}.$$

Indeed, let $x_n \in (-n,n)$ be a maximum point of u_n. It follows from the equation and $u_n''(x_n) \leq 0$ that

$$\|u_n\|_{L^\infty(-n,n)} = u_n(x_n) \leq \frac{\|f(v)\|_{L^\infty(\mathbf{R})}}{V(x_n)} \leq \frac{\|f(v)\|_{L^\infty(\mathbf{R})}}{V_0}. \tag{2.4}$$

Next we shall show that there exist $C_3 > 0$ and $\delta_0 > \eta_1$ such that

$$u_n(x) \leq C_3 e^{-\delta_0 |x|} \quad \text{for all } x \in \mathbf{R} \text{ and } n \geq 1. \tag{2.5}$$

To this end, we first notice that (f2) yields

$$f(s) \leq C_4 |s|^{1+\eta_0} \quad \text{for all } |s| \leq \|v\|_{L^\infty}.$$

Hence, by the definition of $\|\cdot\|_{\eta_1}$, we obtain

$$f(v(x)) \le C_4|v(x)|^{1+\eta_0} \le C_4\|v\|_{\eta_1}^{1+\eta_0} e^{-(1+\eta_0)\eta_1|x|} \le C_5 e^{-(1+\eta_0)\eta_1|x|} \tag{2.6}$$

for all $x \in \mathbf{R}$. Recalling (2.3), fix an $R_0 > 0$ so that

$$-\Lambda\eta_1^2\left(1+\frac{\eta_0}{2}\right)^2 + V_0 - C_5 e^{-\eta_0\eta_1 R_0/2} \ge \frac{V_0}{4} > 0. \tag{2.7}$$

We only treat n with $n > R_0$ and set

$$M := 1 + \frac{\|f(v)\|_{L^\infty}}{V_0}, \quad \omega_0(x) := M e^{-(1+\frac{\eta_0}{2})\eta_1(|x|-R_0)}.$$

Noting (2.3), (2.6), (2.7) and

$$\omega_0'' = \left(1+\frac{\eta_0}{2}\right)^2 \eta_1^2\omega_0 \ge 0, \quad M \ge 1,$$

we get the following: for all $R_0 \le |x| \le n$,

$$\begin{aligned}
&-\mathcal{M}^{\pm}_{\lambda,\Lambda}(\omega_0'') + V\omega_0 - f(v) \\
&\ge -\mathcal{M}^{\pm}_{\lambda,\Lambda}(\omega_0'') + V\omega_0 - Mf(v) \\
&\ge \left\{-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V\right\}\omega_0 - MC_5e^{-(1+\eta_0)\eta_1|x|} \\
&\ge \left[\left\{-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V\right\}e^{(1+\frac{\eta_0}{2})\eta_1 R_0} - C_5 e^{-\eta_0\eta_1|x|/2}\right]Me^{-(1+\frac{\eta_0}{2})\eta_1|x|} \\
&\ge \left[-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V_0 - C_5e^{-\eta_0\eta_1R_0/2}\right]Me^{-(1+\frac{\eta_0}{2})\eta_1|x|} \\
&\ge 0.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{M}^{+}_{\lambda,\Lambda}(m_1) - \mathcal{M}^{+}_{\lambda,\Lambda}(m_2) &\ge \mathcal{M}^{-}_{\lambda,\Lambda}(m_1 - m_2), \\
\mathcal{M}^{-}_{\lambda,\Lambda}(m_1) - \mathcal{M}^{-}_{\lambda,\Lambda}(m_2) &\ge \mathcal{M}^{-}_{\lambda,\Lambda}(m_1 - m_2)
\end{aligned} \tag{2.8}$$

for all $m_1, m_2 \in \mathbf{R}$, we have

$$-\mathcal{M}^{-}_{\lambda,\Lambda}(\omega_0'' - u_n'') + V(x)(\omega_0 - u_n) \ge 0$$

for each $R_0 \le |x| \le n$. From (2.4) and the definitions of M and ω_0, we have

$$u_n(\pm R_0) \le M = \omega_0(\pm R_0), \quad 0 = u_n(\pm n) < \omega_0(\pm n).$$

By the comparison principle, we get

$$u_n(x) \le \omega_0(x) \quad \text{for all } R_0 \le |x| \le n.$$

Thus, (2.5) holds with $\delta_0 := (1 + \eta_0/2)\eta_1$.

By the elliptic regularity, one sees that (u_n) is bounded in $C^2_{\mathrm{loc}}(\mathbf{R})$, hence there exists (u_{n_k}) such that $u_{n_k} \to u_0$ in $C^2_{\mathrm{loc}}(\mathbf{R})$, where u_0 satisfies

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u_0'') + V(x)u_0 = f(v(x)) \quad \text{in } \mathbf{R}.$$

Moreover, from (2.5), we obtain

$$u_0(x) \le C_6 e^{-\delta_0|x|} \quad \text{in } \mathbf{R}.$$

Since $\delta_0 > \eta_1$, $u_0 \in X_{\eta_1}$ and the existence of solutions is proved.

For the uniqueness, let $u_1, u_2 \in X_{\eta_1}$ be solutions of

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(v(x)) \quad \text{in } \mathbf{R}$$

and set $w(x) := u_1(x) - u_2(x)$. Then it follows from (2.8) that $\pm w(x)$ satisfy

$$-\mathcal{M}^{-}_{\lambda,\Lambda}(u'') + V(x)u \ge 0 \quad \text{in } \mathbf{R}.$$

Noting that $w(x) \to 0$ as $|x| \to \infty$, combining with the above inequality, $\pm w(x)$ do not have any negative minimum on $\mathbf{R}$. Hence, $w \equiv 0$ and $u_1 \equiv u_2$. Thus we complete the proof. □

Definition 2.3. For $v \in X_{\eta_1}$, we denote by $\mathcal{L}^{\pm}(v)$ the unique solutions of

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(v) \quad \text{in } \mathbf{R}, \quad u \in X_{\eta_1}.$$

Thanks to Lemma 2.2, $\mathcal{L}^{\pm} : X_{\eta_1} \to X_{\eta_1}$. Furthermore,

Lemma 2.4. *The maps $\mathcal{L}^{\pm} : X_{\eta_1} \to X_{\eta_1}$ are compact.*

Proof. Let $(v_n) \subset X_{\eta_1}$ be a bounded sequence and put $u_n = \mathcal{L}^{\pm}(v_n)$. We first show that (u_n) has a convergent subsequence in X_{η_1}. Set

$$M_1 = \sup_{n \ge 1} \|v_n\|_{\eta_1}.$$

Then we have

$$v_n(x) \le M_1 e^{-\eta_1|x|} \quad \text{for all } x \in \mathbf{R},$$

and there exists an $M_2 > 0$ such that

$$\|u_n\|_{L^\infty} \leq \frac{\|f(v_n)\|_{L^\infty}}{V_0} \leq M_2 \quad \text{for all } n \geq 1$$

(see the beginning of proof of Lemma 2.2). Now as in (2.6) and (2.7), choose an $R_2 > 0$ so large that, for $|x| \geq R_2$ we have

$$f(v_n(x)) \leq C_7 e^{-(1+\eta_0)\eta_1|x|}$$

and

$$-\Lambda\eta_1^2\left(1+\frac{\eta_0}{2}\right)^2 + V_0 - C_7 e^{-\eta_0\eta_1|x|/2} \geq \frac{V_0}{4} > 0.$$

For $R > R_2$, set

$$w_R(x) := M_3\left[e^{-(1+\frac{\eta_0}{2})\eta_1(|x|-R_2)} + e^{(1+\frac{\eta_0}{2})\eta_1(|x|-R)}\right]$$

where $M_3 := 1 + M_2$. Since $w_R'' = \left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 w_R \geq 0$, as in the proof of Lemma 2.2, for all $R_2 \leq |x| \leq R$, we get

$$\begin{aligned}
&-\mathcal{M}^{\pm}_{\lambda,\Lambda}(w_R'') + V(x)w_R - f(v_n)\\
&\geq -\mathcal{M}^{\pm}_{\lambda,\Lambda}(w_R'') + V(x)w_R - M_3 f(v_n)\\
&\geq \left\{-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V\right\} M_3\left[e^{-(1+\frac{\eta_0}{2})\eta_1(|x|-R_2)} + e^{(1+\frac{\eta_0}{2})\eta_1(|x|-R)}\right]\\
&\quad -M_3C_7e^{-(1+\eta_0)\eta_1|x|}\\
&\geq \left[-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V_0 - C_7e^{-\eta_0\eta_1R_2/2}\right] M_3e^{-(1+\frac{\eta_0}{2})\eta_1|x|}\\
&\quad + \left[-\Lambda\left(1+\frac{\eta_0}{2}\right)^2\eta_1^2 + V_0\right] e^{(1+\frac{\eta_0}{2})\eta_1(|x|-R)}\\
&\geq 0 = -\mathcal{M}^{\pm}_{\lambda,\Lambda}(u_n'') + V(x)u_n - f(v_n).
\end{aligned}$$

Noting

$$0 \leq u_n(\pm R_2) \leq M_2 \leq w_R(\pm R_2) \quad \text{and} \quad 0 \leq u_n(\pm R) \leq M_2 \leq w_R(\pm R),$$

the comparison principle gives

$$u_n(x) \leq w_R(x) = M_3\left[e^{-(1+\frac{\eta_0}{2})\eta_1(|x|-R_2)} + e^{(1+\frac{\eta_0}{2})\eta_1(|x|-R)}\right],$$

for all $R_2 \leq |x| \leq R$ and $n \geq 1$. Letting $R \to \infty$, we obtain

$$u_n(x) \le C_8 e^{-(1+\frac{\eta_0}{2})\eta_1|x|} \quad \text{for all } x \in \mathbf{R} \text{ and } n \ge 1.$$

Using this exponential decay and the equation, we observe that there exists $C_9 > 0$ such that

$$\|u_n\|_{L^\infty} + \|u_n'\|_{L^\infty} + \|u_n''\|_{L^\infty} \le C_9 \quad \text{for all } n \ge 1.$$

Thus, there exists (u_{n_k}) such that $u_{n_k} \to u_0$ in $C^2_{\mathrm{loc}}(\mathbf{R})$, where u_0 satisfies

$$u_0(x) \le C_8 e^{-(1+\frac{\eta_0}{2})\eta_1|x|} \quad \text{for all } x \in \mathbf{R}.$$

This implies that $u_0 \in X_{\eta_1}$ and $u_{n_k} \to u_0$ in X_{η_1}. Hence, (u_n) is relatively compact in X_{η_1}.

Finally, we prove the continuity of $\mathcal{L}^\pm$. If $v_n \to v_0$ in X_{η_1}, then arguing as in the above, there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u_0$ in $X_{\eta_1} \cap C^2_{\mathrm{loc}}(\mathbf{R})$ where u_0 satisfies

$$-\mathcal{M}^\pm_{\lambda,\Lambda}(u_0'') + V(x)u_0 = f(v_0) \quad \text{in } \mathbf{R}.$$

By Lemma 2.2, u_0 is uniquely determined and does not depend on choices of subsequences. Therefore, it is easily seen that the whole sequence (u_n) converges to u_0 in X_{η_1} and the maps $\mathcal{L}^\pm$ are continuous. □

Using $\mathcal{L}^\pm$, the fact $f(s) \ge 0$ for every $s \in \mathbf{R}$ and the strong maximum principle, we notice that $u \in X_{\eta_1}$ is a solution of (1.1) if and only if $u = \mathcal{L}^\pm(u)$ with $u \ne 0$.

Next, in order to find a nontrivial fixed point of $\mathcal{L}^\pm$ in X_{η_1}, following the idea in [4] (cf. [7]), we shall show that

i) There exists an $r_0 > 0$ such that $\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, B_{r_0}(0), 0) = 1$.
ii) There exists an $r_1 > r_0$ such that $\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, B_{r_1}(0), 0) = 0$.

Here $\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, \Omega, 0)$ stands for the degree of the map $\mathrm{id} - \mathcal{L}^\pm$ in X_{η_1}. From i) and ii), it follows that

$$\mathrm{id} - \mathcal{L}^\pm \ne 0 \quad \text{on } \partial A_{r_0,r_1} \quad \text{and} \quad \deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, A_{r_0,r_1}, 0) = -1$$

where $A_{r_0,r_1} := \{u \in X_{\eta_1} \mid r_0 < \|u\|_{\eta_1} < r_1\}$. Thus, if we can prove i) and ii) we can find a solution of (1.1) in A_{r_0,r_1}.

First we show i), namely,

Lemma 2.5. *There exists an $r_0 > 0$ such that $\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, B_{r_0}(0), 0) = 1$.*

Proof. It suffices to prove that there exists an $r_0 > 0$ such that $(\mathrm{id} - \beta\mathcal{L}^{\pm})(u) \neq 0$ for all $u \in \partial B_{r_0}(0)$ and all $\beta \in [0,1]$ since the homotopy invariance gives

$$\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^{\pm}, B_{r_0}(0), 0) = \deg_{X_{\eta_1}}(\mathrm{id}, B_{r_0}(0), 0) = 1.$$

We first notice that for $\beta > 0$, the equations $u = \beta\mathcal{L}^{\pm}(u)$ are equivalent to

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = \beta f(u) \quad \text{in } \mathbf{R}$$

for $u \in X_{\eta_1}$. If $u \in X_{\eta_1} \setminus \{0\}$ satisfies $u = \beta\mathcal{L}^{\pm}(u)$ with $\beta > 0$, then the fact that $f(s) \geq 0$ for all $s \in \mathbf{R}$ yields $u > 0$ in $\mathbf{R}$. Since $u(x) \to 0$ as $|x| \to \infty$, let $x_0 \in \mathbf{R}$ be a maximum point of u. As in the proof of Lemma 2.2, from $\beta \in [0,1]$ and $f(s) > 0$ for $s > 0$ due to (f4), we get

$$0 < V_0 \leq V(x_0) \leq \frac{\beta f(u(x_0))}{u(x_0)} \leq \frac{f(u(x_0))}{u(x_0)}.$$

By (f2), we may find a $\delta_1 > 0$, which is independent of β and u, so that

$$\delta_1 \leq u(x_0) = \|u\|_{L^\infty} \leq \|u\|_{\eta_1}$$

for all $u \in X_{\eta_1} \setminus \{0\}$ and $\beta \in (0,1]$ with $u = \beta\mathcal{L}^{\pm}(u)$. Therefore, selecting an $r_0 \in (0, \delta_1)$, we see that

$$(\mathrm{id} - \beta\mathcal{L}^{\pm})(u) \neq 0$$

for all $u \in \partial B_{r_0}(0)$ and for all $\beta \in (0,1]$. Thus the lemma holds. □

To show ii), we need some preparations. From (V3), we may select a $\kappa_0 > 0$ so that

$$[-3\kappa_0, 3\kappa_0] \subset [V_\infty - V > 0] := \{x \in \mathbf{R} \mid V_\infty - V(x) > 0\}. \tag{2.9}$$

Next choose a $\varphi_0 \in C_0^\infty(\mathbf{R})$ satisfying

$$\begin{aligned} &\varphi_0(-x) = \varphi_0(x), \quad 0 \leq \varphi_0 \leq 1 \quad \text{in } \mathbf{R}, \quad \varphi_0'(x) \leq 0 \quad \text{in } [0,\infty), \\ &\varphi_0(x) = 1 \quad \text{if } 0 \leq x \leq \kappa_0, \quad \varphi_0(x) = 0 \quad \text{if } 2\kappa_0 \leq x. \end{aligned} \tag{2.10}$$

Then we first prove

Lemma 2.6. *There exists a $\tilde{t} = \tilde{t}(f, V_\infty) > 0$ such that*

$$\frac{\kappa_0^2}{4\Lambda} t \leq \|u\|_{L^\infty([-\kappa_0,\kappa_0])} \leq \|u\|_{X_{\eta_1}} \quad \text{for each } t \geq \tilde{t} \quad \text{and} \quad u \in \mathcal{S}_t^{\pm} \tag{2.11}$$

where

$$\mathcal{S}_t^{\pm} := \{u \in X_{\eta_1} \mid -\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(u) + t\varphi_0\}.$$

Proof. By (f2) and (f3), there exists a $c(f, V_\infty) > 0$ such that

$$\inf_{0 \le s} \left(\frac{f(s)}{s} - V_\infty \right) s \ge -c(f, V_\infty).$$

Choose a $\tilde{t} = \tilde{t}(f, V_\infty) > 0$ so that if $t \ge \tilde{t}$, then $-c(f, V_\infty) + t \ge t/2$. For this $\tilde{t}$, we shall prove that (2.11) holds.

Let $t \ge \tilde{t}$ and $u \in \mathcal{S}_t^{\pm}$. Since $t > 0$, we have $u \not\equiv 0$. Thus $u > 0$ in $\mathbf{R}$ due to $f(s) \ge 0$ in $\mathbf{R}$ and the strong maximum principle. Hence, (V3) yields

$$\begin{aligned} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') = f(u) + t\varphi_0 - V(x)u &= \left(\frac{f(u)}{u} - V(x) \right) u + t\varphi_0 \\ &\ge \left(\frac{f(u)}{u} - V_\infty \right) u + t\varphi_0 \ge -c(f, V_\infty) + t\varphi_0 \quad \text{in } \mathbf{R}. \end{aligned}$$

By the definition of φ_0, we see

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') \ge \frac{t}{2} \quad \text{in } [-\kappa_0, \kappa_0],$$

which implies

$$u'' \le -\frac{t}{2\Lambda} \quad \text{in } [-\kappa_0, \kappa_0].$$

Integrating the inequality over $[x, y] \subset [-\kappa_0, \kappa_0]$, one has

$$u'(y) \le u'(x) - \frac{t}{2\Lambda}(y - x) \quad \text{for } -\kappa_0 \le x \le y \le \kappa_0. \tag{2.12}$$

Now we divide our arguments into two cases:

Case 1. There exists an $x_0 \in [-\kappa_0, 0]$ such that $u'(x_0) \le 0$.

Case 2. $u' > 0$ in $[-\kappa_0, 0]$.

In Case 1, we put $x = x_0$ and integrate (2.12) in y over $[x_0, \kappa_0]$ to obtain

$$u(\kappa_0) \le u(x_0) + u'(x_0)(\kappa_0 - x_0) - \frac{t}{4\Lambda}(\kappa_0 - x_0)^2 \le u(x_0) - \frac{t}{4\Lambda}(\kappa_0 - x_0)^2.$$

Hence,

$$\|u\|_{L^\infty([-\kappa_0,\kappa_0])} \ge u(x_0) \ge u(\kappa_0) + \frac{t}{4\Lambda}(\kappa_0 - x_0)^2 \ge \frac{t}{4\Lambda}(\kappa_0 - x_0)^2 \ge \frac{\kappa_0^2}{4\Lambda} t.$$

Thus (2.11) holds.

In Case 2, putting $y = 0$ in (2.12), it follows that

$$-\frac{t}{2\Lambda}x \le u'(x) \quad \text{for every } x \in [-\kappa_0, 0].$$

Integrating this inequality over $[-\kappa_0, 0]$, we obtain

$$\frac{\kappa_0^2}{4\Lambda}t \le u(0) - u(-\kappa_0) < u(0) \le \|u\|_{L^\infty([-\kappa_0,\kappa_0])}.$$

Thus (2.11) holds and we complete the proof. □

Next, we shall prove some properties of elements in $\mathcal{S}_t^\pm$.

Lemma 2.7. *Let $t \ge 0$ and $u \in \mathcal{S}_t^\pm \setminus \{0\}$. Then either*
(i) *There exists an $x_0 \in \mathbf{R}$ such that $u'(y) < 0 < u'(x)$ for all $x < x_0 < y$*
or else
(ii) *There are $y_0 < 0 < z_0$ such that $u'(y_0) = 0 = u'(z_0)$ and $u'(x) \ne 0$ if $x \ne y_0, z_0$.*
In particular, every $u \in \mathcal{S}_t^\pm \setminus \{0\}$ has only one maximum point in $\mathbf{R}$.

Proof. For $u \in \mathcal{S}_t^\pm \setminus \{0\}$, it suffices to prove the following claim:

Claim. If $u'(z_0) = 0$ holds for some $z_0 \ge 0$, then $u'(x) < 0$ for every $x > z_0$. Similarly, if $u'(y_0) = 0$ holds for $y_0 \le 0$, then $u'(x) > 0$ for all $x < y_0$. In particular, each $u \in \mathcal{S}_t^\pm \setminus \{0\}$ has at most one critical point in $[0, \infty]$ (resp. $(-\infty, 0]$).

We first remark that since $u(-x)$ satisfies the same type of equation by (2.10) and (V1)–(V3), it is enough to prove the first assertion. To this end, suppose that $z_0 \ge 0$ satisfies $u'(z_0) = 0$ and set, for all $x \in \mathbf{R}$,

$$\tilde{u}(x) = u(z_0 + |x|), \quad \tilde{V}(x) = V(z_0 + |x|) \quad \text{and} \quad \tilde{\varphi}_0(x) = \varphi_0(z_0 + |x|).$$

Then, since $u'(z_0) = 0$ and $z_0 \ge 0$, $\tilde{u} \in C^1(\mathbf{R}) \cap C^2(\mathbf{R} \setminus \{0\})$, $\tilde{u}(x) \to 0$ as $|x| \to \infty$. Moreover $\tilde{u}$, $\tilde{V}$ and $\tilde{\varphi}_0$ are even and $\tilde{V}'(x) \ge 0$, $\tilde{\varphi}_0'(x) \le 0$ for a.a. $x \ge 0$ and

$$-\mathcal{M}_{\lambda,\Lambda}^\pm(\tilde{u}'') + \tilde{V}\tilde{u} = f(\tilde{u}) + t\tilde{\varphi}_0 \quad \text{in} \quad \mathbf{R} \setminus \{0\}.$$

Furthermore, by the differential equations and $u \in C^2(\mathbf{R})$, we have $\tilde{u} \in C^2(\mathbf{R})$ and the equation above is satisfied in $\mathbf{R}$.

We shall prove Claim by the moving plane method. For $\lambda > 0$, define $x_\lambda = 2\lambda - x$, $\Sigma_\lambda = \{x \in (0, \infty) \mid \lambda < x\}$ and

$$u_\lambda(x) = \tilde{u}(x_\lambda) - \tilde{u}(x), \quad \varphi_\lambda(x) = \tilde{\varphi}_0(x_\lambda) - \tilde{\varphi}_0(x).$$

Since

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(\tilde{u}'')(x_\lambda) + \tilde{V}(x_\lambda)\tilde{u}(x_\lambda) = f(\tilde{u}(x_\lambda)) + t\tilde{\varphi}_0(x_\lambda)$$
$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(\tilde{u}'')(x) + \tilde{V}(x)\tilde{u}(x) = f(\tilde{u}(x)) + t\tilde{\varphi}_0(x),$$

we have

$$\begin{aligned}&-\left(\mathcal{M}^{\pm}_{\lambda,\Lambda}(\tilde{u}'')(x_\lambda) - \mathcal{M}^{\pm}_{\lambda,\Lambda}(\tilde{u}'')(x)\right) + \left(\tilde{V}(x_\lambda) - \tilde{V}(x)\right)\tilde{u}(x_\lambda) + \tilde{V}(x)u_\lambda \\ &= f(\tilde{u}(x_\lambda)) - f(\tilde{u}(x)) + t(\tilde{\varphi}_0(x_\lambda) - \tilde{\varphi}_0(x)).\end{aligned}$$

Noting (2.8), $|x_\lambda| \le |x|$, $\tilde{V}(x_\lambda) \le \tilde{V}(x)$ and $\tilde{\varphi}_0(x) \le \tilde{\varphi}_0(x_\lambda)$ for all $x \in \Sigma_\lambda$, we have

$$-\mathcal{M}^{-}_{\lambda,\Lambda}(u_\lambda'') + \tilde{V}(x)u_\lambda \ge f(\tilde{u}(x_\lambda)) - f(\tilde{u}(x)) \quad \text{in} \quad \Sigma_\lambda.$$

Moreover, from

$$\begin{aligned}f(\tilde{u}(x_\lambda)) - f(\tilde{u}(x)) &= \int_0^1 f'(\tilde{u}(x) + \theta u_\lambda(x))\mathrm{d}\theta u_\lambda(x) \\ &=: g_\lambda(x)u_\lambda(x),\end{aligned}$$

we have

$$-\mathcal{M}^{-}_{\lambda,\Lambda}(u_\lambda'') + (\tilde{V}(x) - g_\lambda(x))u_\lambda(x) \ge 0 \quad \text{in} \quad \Sigma_\lambda.$$

Since $\tilde{u} > 0$ in $\mathbf{R}$, $f'(0) = 0$ by (f1) and $\tilde{u}(x) \to 0$ as $x \to \infty$, it is not difficult to see that the strong maximum principle implies that for all λ sufficiently large,

$$u_\lambda > 0 \quad \text{in} \quad \Sigma_\lambda, \quad u_\lambda'(\lambda) = -2\tilde{u}'(\lambda) = -2u'(z_0 + \lambda) > 0. \tag{2.13}$$

Next, set

$$\lambda_* = \inf\{\lambda > 0 \,|\, u_{\tilde{\lambda}} > 0 \quad \text{in} \quad \Sigma_{\tilde{\lambda}} \quad \text{for all} \quad \tilde{\lambda} > \lambda\}.$$

From the above observation, we have $0 \le \lambda_* < \infty$. In addition, notice that if $u_\lambda \ge 0$ in Σ_λ, then

$$\begin{aligned}0 &\le -\mathcal{M}^{-}_{\lambda,\Lambda}(u_\lambda'') + \left(\tilde{V}(x) - g_\lambda(x)\right)u_\lambda(x) \\ &\le -\mathcal{M}^{-}_{\lambda,\Lambda}(u_\lambda'') + \left(\tilde{V}(x) - g_\lambda(x)\right)_+ u_\lambda(x) \quad \text{in} \quad \Sigma_\lambda.\end{aligned}$$

In particular, since $u_{\lambda_*} \ge 0$ in Σ_{λ_*}, the strong maximum principle yields either

(i) $u_{\lambda_*} > 0$ in Σ_{λ_*}, $u'_{\lambda_*}(\lambda_*) > 0$
 or else
(ii) $u_{\lambda_*} \equiv 0$ in Σ_{λ_*}.

Next we prove that if $\mu > 0$ and $u_\mu > 0$ in Σ_μ hold, then there exists an $\epsilon_\mu > 0$ such that $u_{\tilde{\mu}} > 0$ in $\Sigma_{\tilde{\mu}}$ provided $|\mu - \tilde{\mu}| < \epsilon_\mu$. To see this, we remark that

$$u_{\tilde{\mu}} \to u_\mu \quad \text{in} \quad C^1_{\mathrm{loc}}([\mu, \infty)).$$

Since $u'_\mu(\mu) > 0$ holds due to $u_\mu > 0$ in Σ_μ and the strong maximum principle, for sufficiently small ϵ_μ, we observe that $|\mu - \tilde{\mu}| < \epsilon_\mu$ implies $u_{\tilde{\mu}} > 0$ in $(\tilde{\mu}, R_\epsilon)$ where $R_\epsilon > 0$ is chosen so that $x \geq R_\epsilon$ implies $g_{\tilde{\mu}}(x) \leq V_0/2$.

From

$$-\mathcal{M}^-_{\lambda,\Lambda}(u''_{\tilde{\mu}}) + (V - V_0/2)u_{\tilde{\mu}} \geq 0 \quad \text{in} \quad [R_\epsilon, \infty), \quad u_{\tilde{\mu}}(x) \to 0 \quad \text{as } x \to \infty$$

and the strong maximum principle, $u_{\tilde{\mu}}$ cannot take a non-positive minimum. Hence $|\mu - \tilde{\mu}| < \epsilon_\mu$ implies $u_{\tilde{\mu}} > 0$ in $\Sigma_{\tilde{\mu}}$ and $u'_{\tilde{\mu}}(\tilde{\mu}) > 0$.

By this claim we see that if $u_{\lambda_*} > 0$ in Σ_{λ_*}, then $\lambda_* = 0$. Thus, $\lambda_* = 0$ holds provided (i) occurs. Moreover, we also see from (2.13) that $\tilde{u}'(x) < 0$ for all $x > 0$.

On the other hand, let us consider the case $\lambda_* > 0$ and $u_{\lambda_*} \equiv 0$ in Σ_{λ_*}. In this case, we notice that $-2\tilde{u}'(\lambda) = u'_\lambda(\lambda) > 0$ for all $\lambda > \lambda_*$ and $\tilde{u}(2\lambda_* - x) = \tilde{u}(x)$ for all $x \geq \lambda_*$. Since $\tilde{u}'(0) = 0$, we have $\tilde{u}'(2\lambda_*) = 0$, which is a contradiction. Hence, (ii) only occurs when $\lambda_* = 0$ and it follows from (2.13) that $\tilde{u}'(x) < 0$ for all $x > 0$.

By the above observations, we obtain $\lambda_* = 0$ and $\tilde{u}'(x) < 0$ for all $x > 0$, which implies $u'(x) < 0$ for all $x > z_0$. Thus we complete the proof. □

Lemma 2.8. *There exists an $M_0 > 0$ such that*

$$\|u\|_{L^\infty(\mathbf{R})} \leq M_0 \quad \text{for each } u \in \mathcal{S}^\pm_t \quad \text{and} \quad t \geq 0.$$

Proof. We argue by contradiction and suppose that there are $(s_n) \subset [0, \infty)$ and $u_n \in \mathcal{S}^\pm_{s_n}$ such that $\tau_n := \|u_n\|_{L^\infty(\mathbf{R})} \to \infty$. Thanks to Lemma 2.7, let $(x_n) \subset \mathbf{R}$ be a unique maximum point of (u_n) and set

$$v_n(x) := \frac{1}{\tau_n} u_n\left(x_n + \sqrt{\frac{\tau_n}{f(\tau_n)}}x\right), \quad \varphi_n(x) := \varphi_0\left(x_n + \sqrt{\frac{\tau_n}{f(\tau_n)}}x\right),$$
$$V_n(x) := V\left(x_n + \sqrt{\frac{\tau_n}{f(\tau_n)}}x\right).$$

Then v_n satisfies

$$v_n(x) \leq v_n(0) = 1, \quad -\mathcal{M}^\pm_{\lambda,\Lambda}(v''_n) + \frac{\tau_n}{f(\tau_n)} V_n v_n = \frac{f(\tau_n v_n)}{f(\tau_n)} + \frac{s_n}{f(\tau_n)}\varphi_n \quad \text{in } \mathbf{R}.$$

Recalling Lemma 2.6, we have

$$\frac{\kappa_0^2}{4\Lambda}s_n - \frac{\kappa_0^2}{4\Lambda}\tilde{t} \le \|u_n\|_{L^\infty(\mathbf{R})} = \tau_n \quad \text{for all } n.$$

Hence, by (f3), $s_n/f(\tau_n) \to 0$ as $n \to \infty$. Moreover, noting that $f(s)$ is increasing in $[0,\infty)$ by (f4), it follows from $v_n(x) \le 1$ that

$$0 \le \frac{f(\tau_n v_n)}{f(\tau_n)} \le 1 \quad \text{in } \mathbf{R}.$$

Noting $v_n'(0) = 0$ and $\tau_n/f(\tau_n) \to 0$, we may extract a subsequence (still denoted by (n)) such that

$$v_n \to v_0 \quad \in C^1_{\mathrm{loc}}(\mathbf{R}), \quad 0 \le v_0 \le 1 \quad \text{in } \mathbf{R}, \quad v_0(0) = 1, \quad v_0'(0) = 0.$$

Furthermore, by $0 \le v_n \le 1$ and (f3), we have

$$\frac{f(\tau_n v_n)}{f(\tau_n)} \to \bar{f}(v_0) \quad \text{in } C_{\mathrm{loc}}([v_0 > 0]).$$

Since $0 \in [v_0 > 0]$, let I be a component of $[v_0 > 0]$ satisfying $0 \in I$. Then we have

$$v_n \to v_0 \quad \text{in } C^2_{\mathrm{loc}}(I), \quad -\mathcal{M}^\pm_{\lambda,\Lambda}(v_0'') = \bar{f}(v_0) \quad \text{in } I.$$

When $I = (-c_2, c_1)$ and $c_1 < \infty$, since $v_0'(0) = 0$, $v_0(0) = 1$, $\bar{f} \ge 0$, $\bar{f}(1) = 1 > 0$ and $v_0(c_1) = 0$, we observe that $v_0'(c_1) < 0$, however, this contradicts $0 \le v_0 \le 1$ in $\mathbf{R}$.

On the other hand, if $I = (-c_2, \infty)$, then by $v_0'(0) = 0$, $v_0(0) = 1$ and $-\mathcal{M}^\pm_{\lambda,\Lambda}(v_0'') = \bar{f}(v_0)$, we observe that v_0 must hit a zero at some $x_0 > 0$ with $v_0'(x_0) < 0$, however this contradicts $0 \le v_0 \le 1$ again. Hence, Lemma 2.8 holds and we complete the proof. □

The next proposition is a key in order to prove $\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^\pm, B_{r_1}(0), 0) = 0$ for some $r_1 > r_0$.

Proposition 2.9. *There exists an $M_1 > 0$ such that*

$$\|u\|_{X_{\eta_1}} \le M_1 \quad \text{for each } u \in \mathcal{S}^\pm_t \quad \text{and} \quad t \ge 0.$$

Assuming Proposition 2.9, we first prove Theorem 1.1. Before the proof, we remark that for every $t \ge 0$ and $v \in X_{\eta_1}$, the equations

$$-\mathcal{M}^\pm_{\lambda,\Lambda}(u'') + V(x)u = f(v) + t\varphi_0 \quad \text{in } \mathbf{R}$$

have unique solutions in X_{η_1}. Indeed, we may prove this claim in a similar way to the proof of Lemma 2.2 thanks to $\varphi_0 \in C_0^\infty(\mathbf{R})$. Thus, we denote by $F^\pm(t,v)$ these unique solutions. Furthermore, we may show that the maps $(t,v) \mapsto F^{\pm}(t,v) : [0,\infty) \times X_{\eta_1} \to X_{\eta_1}$ are compact as in Lemma 2.4.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Choose $r_1 := M_1 + \frac{\kappa_0^2}{4\Lambda}\tilde{t} + r_0$ where M_1, $\tilde{t}$ and r_0 appear in Proposition 2.9 and Lemmas 2.6 and 2.5. We first claim that

$$u - F^{\pm}(t_1, u) \neq 0 \quad \text{in } \overline{B_{r_1}} \tag{2.14}$$

where $t_1 > \tilde{t}$ is chosen so that $\frac{\kappa_0^2}{4\Lambda} t_1 > r_1$. Indeed, let $u \in \overline{B_{r_1}}$ satisfy $u - F^{\pm}(t_1, u) = 0$. Noting $u \in \mathcal{S}^{\pm}_{t_1}$ and $t_1 > \tilde{t}$, Lemma 2.6 gives a contradiction:

$$r_1 \geq \|u\|_{X_{\eta_1}} \geq \|u\|_{L^\infty(\mathbf{R})} \geq \frac{\kappa_0^2}{4\Lambda} t_1 > r_1.$$

Hence, (2.14) holds.

Since Proposition 2.9 and the choice of r_1 imply

$$u - F^{\pm}(t,u) \neq 0 \quad \text{on } \partial B_{r_1} \quad \text{for every } t \geq 0,$$

it is easily seen from (2.14) and the homotopy invariance of degree that

$$\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^{\pm}, B_{r_1}(0), 0) = 0.$$

Combining this with Lemma 2.5, we obtain

$$\deg_{X_{\eta_1}}(\mathrm{id} - \mathcal{L}^{\pm}, A_{r_0,r_1}, 0) = -1$$

and solutions of (1.1) in A_{r_0,r_1}. This completes the proof. $\square$

Before proceeding to the proof of Proposition 2.9, we remark the following fact on the function $g_\infty(s) := f(s) - V_\infty s$, which will be used below.

Fact. *There exists a unique $s_\infty > 0$ such that*

$$g_\infty(s) < 0 = g_\infty(s_\infty) < g_\infty(t) \quad \textit{for all } 0 < s < s_\infty < t. \tag{2.15}$$

This fact follows from (f1)–(f4). In fact, for sufficiently small $s > 0$, by (f2), we get $g_\infty(s) < 0$. On the other hand, (f3) yields $g_\infty(s) \to \infty$ as $s \to \infty$, hence, there exists an $s_\infty > 0$ so that $g_\infty(s_\infty) = 0$. Moreover, from

$$g_\infty(s) = s\left(\frac{f(s)}{s} - V_\infty\right)$$

and (f4), we see that (2.15) holds.

Now we prove Proposition 2.9.

Proof of Proposition 2.9. We argue indirectly and suppose that there exists $(s_n, u_n) \in [0,\infty) \times X_{\eta_1}$ such that $u_n \in \mathcal{S}^{\pm}_{s_n}$ and $\|u_n\|_{X_{\eta_1}} \to \infty$. Remark that u_n satisfies

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u_n'') + V u_n = f(u_n) + s_n \varphi_0 \quad \text{in } \mathbf{R}.$$

By Lemma 2.7, u_n has only one maximum point and denote it by x_n. Our first aim is to show

$$(x_n) \text{ is bounded.} \tag{2.16}$$

To prove (2.16), suppose that $x_n \to \infty$. We may assume $3\kappa_0 < x_n$. Setting

$$v_n(x) := u_n(x + x_n), \quad V_n(x) := V(x + x_n), \quad \varphi_n(x) := \varphi_0(x + x_n),$$

we see $\varphi_n \equiv 0$ in $[0,\infty)$ thanks to $3\kappa_0 < x_n$. Furthermore, by Lemma 2.7, we have

$$\begin{aligned} &-\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_n'') + V_n v_n = f(v_n) + s_n \varphi_n \quad \text{in } \mathbf{R}, \quad v_n(0) = \max_{\mathbf{R}} v_n > 0, \\ &v_n'(y) \le 0 \le v_n'(x) \quad \text{for } x < 0 < y, \quad V_n \to V_\infty, \quad s_n \varphi_n \to 0 \quad \text{in } C_{\mathrm{loc}}(\mathbf{R}). \end{aligned}$$

In the sequel, we divide our arguments into several steps.

Step 1. *One has*

$$v_n \to \omega_{\pm} \quad \text{strongly in } C^2_{\mathrm{loc}}(\mathbf{R}) \tag{2.17}$$

where $\omega_{\pm}$ are unique solutions of (2.1) and (2.2) (see Proposition 2.1).

We first notice that (v_n) is bounded in $L^\infty(\mathbf{R})$ due to Lemma 2.8. Combining with $V_n \to V_\infty$ and $s_n \varphi_n \to 0$ in $C_{\mathrm{loc}}(\mathbf{R})$, we may extract a subsequence (still denoted by (n)) so that

$$\begin{aligned} &v_n \to v_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}), \quad -\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_0'') + V_\infty v_0 = f(v_0) \quad \text{in } \mathbf{R}, \\ &v_0(0) = \max_{\mathbf{R}} v_0, \quad 0 \le v_0 \quad \text{in } \mathbf{R}, \quad v_0'(y) \le 0 \le v_0'(x) \quad \text{for } x < 0 < y. \end{aligned}$$

By $v_n(0) = \max_{\mathbf{R}} v_n$, we have $v_n''(0) \le 0$. Since $v_n(0) > 0$ and $\varphi_n \equiv 0$ on $[0,\infty)$, we get

$$f(v_n(0)) = -\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_n''(0)) + V_n(0) v_n(0) \ge V_n(0) v_n(0),$$

which implies

$$V_n(0) \le \frac{f(v_n(0))}{v_n(0)}.$$

By $V_n(0) \to V_\infty$ and (f2), we may find a $\delta_0 > 0$ so that $v_n(0) \geq \delta_0$ for all n. Thus $v_0(0) \geq \delta_0$ and $v_0 > 0$ in $\mathbf{R}$. Now from $v_0'(y) \leq 0$ in $[0, \infty)$, one has

$$v_{0,\infty} := \lim_{y \to \infty} v_0(y) \geq 0.$$

Since $-\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_0'') = g_\infty(v_0)$ in $\mathbf{R}$, it follows from (2.15) that

$$\text{either } v_{0,\infty} = 0 \text{ or } v_{0,\infty} = s_\infty > 0.$$

If $v_{0,\infty} = 0$, then by Proposition 2.1, we have $v_0 = \omega_\pm$ and Step 1 holds.

Now we assume $v_{0,\infty} = s_\infty$. By $v_0' \leq 0$ in $[0, \infty)$ and (2.15), we have $v_0 \geq s_\infty$ in $[0, \infty)$ and

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_0'') = g_\infty(v_0) \geq 0 \quad \text{on } [0, \infty).$$

Moreover, if $v_0(0) > s_\infty$, then the strict inequality holds at $x = 0$. However, this contradicts facts $v_0'(0) = 0 > v_0''(0)$, $v_0''(x) \leq 0$ for $x \in [0, \infty)$ and $v_0(x) \to s_\infty$ as $x \to \infty$. Thus we get $v_0 \equiv s_\infty$ in $[0, \infty)$.

Next, we put

$$\begin{aligned} E_{n,+}(x) &:= \frac{\Lambda}{2}(v_n'(x))^2 + F(v_n(x)) - \frac{V_n}{2} v_n^2 \qquad \text{for } \mathcal{M}^+_{\lambda,\Lambda}, \\ E_{n,-}(x) &:= \frac{\lambda}{2}(v_n'(x))^2 + F(v_n(x)) - \frac{V_n}{2} v_n^2 \qquad \text{for } \mathcal{M}^-_{\lambda,\Lambda}. \end{aligned}$$

We also put $h_n(x) := V_n(x) - f(v_n(x))/v_n(x)$. Recalling $V_n(x) = V(x + x_n)$ and $x_n \to \infty$, we may assume that $V_n'(x) \geq 0$ in $(0, \infty)$. Notice also that v_n is strictly decreasing in $(0, \infty)$ by Lemma 2.7. Hence (f4) yields that $h_n(x)$ is strictly increasing in $[0, \infty)$. Since $v_n''(0) \leq 0$ and $\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_n'') = v_n h_n$ in $[0, \infty)$, we see $h_n(0) \leq 0$. Noting $h_n(x) \to V_\infty > 0$ as $x \to \infty$, there exists a unique $z_n^\pm \geq 0$ such that $h_n(z_n^\pm) = 0$. Therefore, one has

$$v_n''(x) < 0 < v_n''(y) \quad \text{for } 0 \leq x < z_n^\pm < y.$$

Moreover, taking a subsequence if necessary, we may assume $v_n(z_n^\pm) \to \tilde{s} \geq 0$ since $v_n(z_n^\pm)$ is bounded. Noting $V_n(z_n^\pm) \to V_\infty$ as $n \to \infty$ and letting $n \to \infty$ in $h_n(z_n^\pm) = 0$, it follows from (f2) that

$$\tilde{s} > 0 \quad \text{and} \quad V_\infty = \frac{f(\tilde{s})}{\tilde{s}}.$$

Thus by (2.15), we obtain $\tilde{s} = s_\infty$ and $v_n(z_n^\pm) \to s_\infty$. Recalling $v_n''(x) \geq 0$ for $x \geq z_n^\pm$, $V_n' \geq 0$ in $[0, \infty)$ and $\varphi_n \equiv 0$ in $[0, \infty)$, we have

$$E'_{n,+}(x) = v'_n(x)\left(\Lambda v''_n + f(v_n) - V_n v_n\right) - \frac{V'_n}{2}v_n^2 = -\frac{V'_n}{2}v_n^2 \le 0 \quad \text{in } [z_n^+,\infty),$$
$$E'_{n,-}(x) = -\frac{V'_n}{2}v_n^2 \le 0 \quad \text{in } [z_n^-,\infty).$$

Thanks to $E_{n,\pm}(x) \to 0$ as $x \to \infty$, one sees $E_{n,\pm}(z_n^\pm) \ge 0$. Since it follows from (2.15) that

$$V_n(z_n^\pm) \to V_\infty, \quad G_\infty(s_\infty) = \int_0^{s_\infty} g_\infty(s)\mathrm{d}s = \min_{[0,\infty)} G(s) < 0,$$

we obtain

$$(v'_n(z_n^\pm))^2 \ge \frac{2}{\Lambda}\left\{\frac{V_n(z_n^\pm)}{2}v_n^2(z_n^\pm) - F(v_n(z_n^\pm))\right\} \to -\frac{2}{\Lambda}G_\infty(s_\infty) > 0$$

By the fact that (v''_n) is bounded in $[-1,\infty)$, we may find a $\delta_1, \delta_2 > 0$ so that

$$|v'_n(x)| \ge \delta_1 > 0 \quad \text{in } [z_n^\pm - \delta_2, z_n^\pm + \delta_2].$$

Due to this and the fact $v'_n(0) = 0$, shrinking $\delta_2 > 0$ if necessary, we may assume $z_n^\pm \ge \delta_2 > 0$ for any n. Furthermore, by $v_n(z_n^\pm) \to s_\infty$ and $v'_n \le 0$ in $[0,\infty)$, we obtain

$$v_n(0) \ge v_n(z_n^\pm - \delta_2) = v_n(z_n^\pm) - \int_{z_n^\pm-\delta_2}^{z_n^\pm} v'_n(x)\mathrm{d}x \ge v_n(z_n^\pm) + \delta_1\delta_2 \to s_\infty + \delta_1\delta_2.$$

However, this contradicts $v_0 \equiv s_\infty$ in $[0,\infty)$. Thus $v_{0,\infty} = 0$ and Step 1 holds.

To proceed further, we need some preparations. First, combining the monotonicity of v_n with (2.17), we can prove that

$$v_n \to \omega_\pm \quad \text{strongly in } L^\infty(\mathbf{R}). \tag{2.18}$$

Moreover, we may also derive the uniform exponential decay in $[0,\infty)$:

$$v_n(x) + |v'_n(x)| \le c_3 \exp(-c_4 x) \quad \text{for all } x \ge 0 \quad \text{and} \quad n \ge 1 \tag{2.19}$$

where $c_3, c_4 > 0$ do not depend on n. Indeed, by (2.18), $\varphi_n \equiv 0$ in $(0,\infty)$, (V1) and (f2), we may find an $R_0 > 0$, which is independent of n, such that for all $n \ge 1$, $v_n(R_0) < 1$ and

$$0 = -\mathcal{M}^\pm_{\lambda,\Lambda}(v''_n) + \left(V_n - \frac{f(v_n)}{v_n}\right)v_n \ge -\mathcal{M}^\pm_{\lambda,\Lambda}(v''_n) + \frac{V_0}{2}v_n \quad \text{in } (R_0,\infty).$$

On the other hand, the function

$$w_0(x) := \exp\left(-\sqrt{\frac{V_0}{2\Lambda}}(x - R_0)\right)$$

satisfies

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(w_0'') + \frac{V_0}{2}w_0 \geq 0 \quad \text{in } (0,\infty), \quad w_0(R_0) = 1.$$

It follows from (2.8) that for every $n \geq 1$,

$$-\mathcal{M}^{-}_{\lambda,\Lambda}(w_0'' - v_n'') + \frac{V_0}{2}(w_0 - v_n) \geq 0 \quad \text{in } (R_0,\infty),$$
$$(w_0 - v_n)(R_0) > 0, \quad (w_0 - v_n)(x) \to 0 \quad \text{as } x \to \infty.$$

Therefore, the comparison theorem yields $v_n \leq w_0$ in $[R_0,\infty)$. Using the differential equation, we also get the uniform decay for v_n'', v_n', hence, (2.19) holds.

Using the same notation $z_n^{\pm}$ to the above, namely, unique points satisfying $v_n''(z_n^{\pm}) = 0$ and $z_n^{\pm} \geq 0$, we claim that $z_n^{\pm} \to z^{\pm}$ where $z^{\pm}$ are unique points satisfying $z^{\pm} > 0$ and $\omega_{\pm}''(z^{\pm}) = 0$. In fact, the unique existence of $z^{\pm}$ is ensured by Proposition 2.1. Furthermore, by (2.18), (V1), (f2), $\varphi_n \equiv 0$ in $[0,\infty)$ and $\omega_{\pm}(x) \to 0$ as $|x| \to \infty$, there exist n_0 and $R_0 > 0$ such that if $n \geq n_0$ and $x \geq R_0$, then

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(v_n'') = V_n(x)v_n(x) - f(v_n(x)) > 0,$$

which yields $z_n^{\pm} \leq R_0$. Moreover, by $\omega_{\pm}''(0) < 0$, we also observe that $z_n^{\pm}$ never approaches to 0. Thus, by the uniqueness of $z^{\pm}$, we have $z_n^{\pm} \to z^{\pm}$ and we may assume $z_n^{\pm} > 0$.

Next, since v_n is strictly increasing in $(-\infty, 0]$ and strictly decreasing in $[0,\infty)$, let $y_n^{\pm}(s)$ and $z_n^{\pm}(s)$ be inverse functions of v_n satisfying $y_n^{\pm}(s) \leq 0 \leq z_n^{\pm}(s)$ for $0 < s \leq v_n(0)$. In particular, we have

$$y_n^{\pm}, z_n^{\pm} \in C((0, v_n(0)], \mathbf{R}), \quad v_n(y_n^{\pm}(s)) = s = v_n(z_n^{\pm}(s)) \quad \text{for } 0 < s \leq v_n(0).$$

Moreover, $y_n^{\pm}, z_n^{\pm}$ are smooth except for at most two points $s = v_n(0)$ and $s = v_n(y)$ where $v_n'(y) = 0$ and $y \neq 0$. Set $y_n^{\pm} := y_n^{\pm}(v_n(z_n^{\pm}))$, namely, $y_n^{\pm} < 0$ and $v_n(y_n^{\pm}) = v_n(z_n^{\pm})$ hold. Moreover, by $\omega_{\pm}(-x) = \omega_{\pm}(x)$, $y_n^{\pm} \to -z^{\pm}$ as $n \to \infty$. Next, set

$$E_{n,\infty,+}(x) := \frac{\Lambda}{2}(v_n'(x))^2 + F(v_n(x)) - \frac{V_\infty}{2}v_n^2(x) \qquad \text{for } \mathcal{M}^{+}_{\lambda,\Lambda},$$
$$E_{n,\infty,-}(x) := \frac{\lambda}{2}(v_n'(x))^2 + F(v_n(x)) - \frac{V_\infty}{2}v_n^2(x) \qquad \text{for } \mathcal{M}^{-}_{\lambda,\Lambda}.$$

Remark that $E_{n,\infty,\pm}(x) \to 0$ as $|x| \to \infty$. Then we shall prove

Step 2. *We have*

$$0 \le -E_{n,\infty,+}(z_n^+) \le c \exp\left(-2x_n\sqrt{\frac{V_\infty}{\Lambda}+\xi_0}\right),$$

$$0 \le -E_{n,\infty,-}(z_n^-) \le c \exp\left(-2x_n\sqrt{\frac{V_\infty}{\lambda}+\xi_0}\right)$$

where $c>0$ is independent of n and $\xi_0>0$ the constant in (V3).

First we notice that

$$\begin{aligned} &E'_{n,\infty,+}(x) = v'_n(x)(\Lambda v''_n + f(v_n) - V_\infty v_n) \\ &= \begin{cases} v'_n\left[(V_n - V_\infty)v_n - s_n\varphi_n\right] & \text{if } v''_n(x) \ge 0, \\ v'_n\left[\frac{\Lambda}{\lambda}(V_n v_n - f(v_n) - s_n\varphi_n) + f(v_n) - V_\infty v_n\right] & \text{if } v''_n(x) < 0 \end{cases} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &E'_{n,\infty,-}(x) = v'_n(x)(\lambda v''_n + f(v_n) - V_\infty v_n) \\ &= \begin{cases} v'_n\left[(V_n - V_\infty)v_n - s_n\varphi_n\right] & \text{if } v''_n(x) \ge 0, \\ v'_n\left[\frac{\lambda}{\Lambda}(V_n v_n - f(v_n) - s_n\varphi_n) + f(v_n) - V_\infty v_n\right] & \text{if } v''_n(x) < 0. \end{cases} \end{aligned} \tag{2.21}$$

Since $v''_n > 0$ in $(z_n^\pm, \infty)$, $v'_n \le 0$ in $[0,\infty)$ and $\varphi_n \equiv 0$ in $[0,\infty)$, we get

$$(E_{n,\infty,\pm})'(x) = v'_n(V_n - V_\infty)v_n \ge 0 \quad \text{in } (z_n^\pm, \infty).$$

Hence, (V3), $z_n^\pm \to z^\pm > 0$ and (2.19) give

$$\begin{aligned} 0 \le -E_{n,\infty,\pm}(z_n^\pm) &= \int_{z_n^\pm}^\infty (E_{n,\infty,\pm})'(x)\mathrm{d}x = \int_{z_n^\pm}^\infty (-v'_n)v_n(V_\infty - V_n)\mathrm{d}x \\ &\le \begin{cases} c\exp\left(-2\sqrt{\frac{V_\infty}{\Lambda}+\xi_0}(x_n + z_n^+)\right) & (\text{for } \mathcal{M}^+_{\lambda,\Lambda}) \\ c\exp\left(-2\sqrt{\frac{V_\infty}{\lambda}+\xi_0}(x_n + z_n^-)\right) & (\text{for } \mathcal{M}^-_{\lambda,\Lambda}) \end{cases} \\ &\le \begin{cases} c\exp\left(-2x_n\sqrt{\frac{V_\infty}{\Lambda}+\xi_0}\right) & (\text{for } \mathcal{M}^+_{\lambda,\Lambda}) \\ c\exp\left(-2x_n\sqrt{\frac{V_\infty}{\lambda}+\xi_0}\right) & (\text{for } \mathcal{M}^-_{\lambda,\Lambda}). \end{cases} \end{aligned}$$

Hence, Step 2 holds.

Step 3. *One has*

$$E_{n,\infty,\pm}(z_n^\pm) \le E_{n,\infty,\pm}(y_n^\pm).$$

Recalling $y_n^\pm \to -z^\pm$ and $x_n \to \infty$, we notice that for each $s \in [v_n(z_n^\pm), v_n(0)]$, (V2) and $y_n^\pm(s) \le z_n^\pm(s)$ imply $V_n(y_n^\pm(s)) \le V_n(z_n^\pm(s))$. Moreover, we may assume $\varphi_n \equiv 0$ in $[y_n^\pm, \infty)$. Hence, noting

$$\begin{aligned} &\mathcal{M}^\pm_{\lambda,\Lambda}(v_n'') = V_n v_n - f(v_n) \quad \text{in } [y_n^\pm, z_n^\pm], \quad v_n''(x) < 0 \quad \text{in } [0, z_n^\pm), \\ &v_n(y_n^\pm(s)) = s = v_n(z_n^\pm(s)) \quad \text{for } s \in [v_n(z_n^\pm), v_n(0)], \end{aligned}$$

we obtain

$$v_n''(x) < 0 \quad \text{for all } x \in (y_n^\pm, z_n^\pm).$$

From this it follows that $v_n'(z) < 0 < v_n'(y)$ for $y_n^\pm \le y < 0 < z \le z_n^\pm$ and $y_n^\pm, z_n^\pm \in C^1([v_n(z_n^\pm), v_n(0)))$. Thus we see from (2.20), $v_n(z_n^+) = v_n(y_n^+)$, the monotonicity of V_n and the change of variables $s = v_n(x)$ that

$$\begin{aligned} E_{n,\infty,+}(0) - E_{n,\infty,+}(y_n^+) &= \int_{y_n^+}^{0} (E_{n,\infty,+})'(x)\mathrm{d}x \\ &= \int_{y_n^+}^{0} \left[\frac{\Lambda}{\lambda} \{V_n v_n - f(v_n)\} + f(v_n) - V_\infty v_n \right] v_n' \mathrm{d}x \\ &= \int_{v_n(y_n^+)}^{v_n(0)} \left[\frac{\Lambda}{\lambda} \left\{V_n(y_n^+(s))s - f(s)\right\} + f(s) - V_\infty s \right] \mathrm{d}s \\ &\le \int_{v_n(z_n^+)}^{v_n(0)} \left[\frac{\Lambda}{\lambda} \left\{V_n(z_n^+(s))s - f(s)\right\} + f(s) - V_\infty s \right] \mathrm{d}s \\ &= -\int_{0}^{z_n^+} \left[\frac{\Lambda}{\lambda} \{V_n(x)v_n - f(v_n)\} + f(v_n) - V_\infty v_n \right] v_n' \mathrm{d}x \\ &= -\int_{0}^{z_n^+} (E_{n,\infty,+})'(x)\mathrm{d}x = E_{n,\infty,+}(0) - E_{n,\infty,+}(z_n^+) \end{aligned}$$

Hence, $E_{n,\infty,+}(z_n^+) \le E_{n,\infty,+}(y_n^+)$. In a similar way, we can prove $E_{n,\infty,-}(z_n^-) \le E_{n,\infty,-}(y_n^-)$ and Step 3 holds.

In what follows, we derive the estimates for $E_{n,\infty,\pm}(y_n^\pm)$. First we prove

Step 4. *$E'_{n,\infty,\pm}(x) \le 0$ in $(-\infty, y_n^\pm)$ for sufficiently large n.*

For $E_{n,\infty,+}$, by $v_n' \ge 0$ in $(-\infty, y_n^+)$ and (2.20), if $x < y_n^+$ and $v_n''(x) \ge 0$, then we have

$$(E_{n,\infty,+})'(x) = v_n' \{(V_n - V_\infty)v_n - s_n\varphi_n\} \le 0.$$

On the other hand, if $x < y_n^+$ and $v_n''(x) < 0$, then $\lambda \le \Lambda$ gives

$$\begin{aligned}(E_{n,\infty,+})'(x) &= v_n'(\Lambda v_n'' + f(v_n) - V_\infty v_n)\\ &= v_n' \{(\Lambda - \lambda)v_n'' + (V_n - V_\infty)v_n - s_n\varphi_n\} \le 0.\end{aligned}$$

Hence, $(E_{n,\infty,+})'(x) \le 0$ in $(-\infty, y_n^+)$.

For $\mathcal{M}^-_{\lambda,\Lambda}$, if $x < y_n^-$ and $v_n''(x) \ge 0$, then we have

$$(E_{n,\infty,-})'(x) = v_n' \{(V_n - V_\infty)v_n - s_n\varphi_n\} \le 0.$$

On the other hand, we consider the case $v_n''(x) \le 0$ and $x < y_n^-$. We first remark that for sufficiently large n, we have $(E_{n,\infty,-})'(x) \le 0$ provided $x \in (-\infty, 3\kappa_0 - x_n]$ and $v_n''(x) \le 0$. In fact, it follows from (2.18), (f2), $\omega_-(x) \to 0$ as $|x| \to \infty$ and (V1) that one can find n_0 and $R_0 \ge 0$ so that

$$f(v_n) - V_\infty v_n \le 0 \quad \text{for each } n \ge n_0 \quad \text{and} \quad x \le -R_0.$$

Since we may assume $3\kappa_0 - x_n \le -R_0$ for $n \ge n_0$ due to $x_n \to \infty$, the condition $v_n''(x) \le 0$ and $x \le 3\kappa_0 - x_n$ give

$$(E_{n,\infty,-})'(x) = v_n'(x) \{\lambda v_n''(x) + f(v_n) - V_\infty v_n\} \le 0.$$

Therefore, we only consider in $[3\kappa_0 - x_n, y_n^-]$ and remark that $\varphi_n \equiv 0$ on the interval.

Next, we shall show that $f(v_n(x)) - V_\infty v_n(x) \le 0$ when $v_n''(x) \le 0$ and $x \in [3\kappa_0 - x_n, y_n^-]$. Noting $v_n(y_n^-) = v_n(z_n^-)$, $v_n(x) \le v_n(y_n^-)$ for $x \in [3\kappa_0 - x_n, y_n^-]$ and

$$v_n''(z_n^-) = 0 = V_n(z_n^-)v_n(z_n^-) - f(v_n(z_n^-)),$$

we infer from (V2), (f4) and $v_n'(x) \ge 0$ in $[3\kappa_0 - x_n, y_n^-]$ that

$$0 = V_n(z_n^-) - \frac{f(v_n(y_n^-))}{v_n(y_n^-)} \le V_\infty - \frac{f(v_n(x))}{v_n(x)} \quad \text{for all } x \in [3\kappa_0 - x_n, y_n^-].$$

Thus $f(v_n(x)) - V_\infty v_n(x) \le 0$ in $[3\kappa_0 - x_n, y_n^-]$. Therefore, when $x \in [3\kappa_0 - x_n, y_n^-]$ and $v_n''(x) \le 0$, it follows from (2.21) that

$$(E_{n,\infty,-})'(x) \le \lambda v_n'(x)v_n''(x) \le 0.$$

Hence, Step 4 holds.

Step 5. *One has*

$$E_{n,\infty,\pm}(y_n^\pm) \le \int_{-3\kappa_0-x_n}^{-2\kappa_0-x_n} v_n' v_n (V_n - V_\infty)\mathrm{d}x.$$

By Step 4, we have $(E_{n,\infty,\pm})'(x) \le 0$ in $(-\infty, y_n^\pm)$. Since $E_{n,\infty,\pm}(x) \to 0$ as $x \to -\infty$, we obtain

$$E_{n,\infty,\pm}(y_n^\pm) = \int_{-\infty}^{y_n^\pm} (E_{n,\infty,\pm})'(x)\mathrm{d}x \le \int_{-3\kappa_0-x_n}^{-2\kappa_0-x_n} (E_{n,\infty,\pm})'(x)\mathrm{d}x \tag{2.22}$$

Recalling (2.18), (V1), (f2), $x_n \to \infty$ and $\varphi_n \equiv 0$ in $[-3\kappa_0 - x_n, -2\kappa_0 - x_n]$, we may assume that

$$\mathcal{M}^\pm_{\lambda,\Lambda}(v_n'') = V_n v_n - f(v_n) \ge 0 \quad \text{in } [-3\kappa_0 - x_n, -2\kappa_0 - x_n].$$

Hence, $v_n''(x) \ge 0$ in $[-3\kappa_0 - x_n, -2\kappa_0 - x_n]$ and

$$(E_{n,\infty,\pm})'(x) = v_n'(V_n - V_\infty)v_n \quad \text{in } [-3\kappa_0 - x_n, -2\kappa_0 - x_n].$$

Thus it is easily seen from (2.22) that Step 5 holds.

Step 6. *There exists a $c > 0$, which is independent of n, such that*

$$\begin{aligned} \min\{v_n(x), v_n'(x)\} &\ge c\exp\left(-|x|\sqrt{\tfrac{V_\infty}{\Lambda}}\right) \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda}, \\ \min\{v_n(x), v_n'(x)\} &\ge c\exp\left(-|x|\sqrt{\tfrac{V_\infty}{\lambda}}\right) \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda} \end{aligned} \tag{2.23}$$

for all $x \le -2\kappa_0 - x_n$ and sufficiently large n.

Set

$$\psi_+(x) := c\exp\left(-|x|\sqrt{\frac{V_\infty}{\Lambda}}\right), \quad \psi_-(x) := c\exp\left(-|x|\sqrt{\frac{V_\infty}{\lambda}}\right)$$

where $c > 0$ is chosen so that

$$\inf_{n\ge1} v_n(0) \ge c.$$

Notice that this is possible from (2.18). Then for each n, it follows that $v_n(0) \ge \psi_\pm(0)$ and in $(-\infty, 0)$,

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(\psi''_{\pm}) + V_\infty \psi_{\pm} = 0 \le f(v_n) + s_n \varphi_n = -\mathcal{M}^{\pm}_{\lambda,\Lambda}(v''_n) + V_n v_n \le -\mathcal{M}^{\pm}_{\lambda,\Lambda}(v''_n) + V_\infty v_n.$$

By (2.8), one has

$$0 \le -\mathcal{M}^{-}_{\lambda,\Lambda}(v''_n - \psi''_{\pm}) + V_\infty(v_n - \psi_{\pm}) \quad \text{in } (-\infty, 0),$$
$$(v_n - \psi_{\pm})(0) \ge 0, \quad (v_n - \psi_{\pm})(x) \to 0 \quad \text{as } x \to -\infty.$$

Therefore, we get $\psi_{\pm}(x) \le v_n(x)$ in $(-\infty, 0)$ and (2.23) holds for v_n.

For v'_n, by $\varphi_n \equiv 0$ in $(-\infty, -2\kappa_0 - x_n)$ and (2.18), there exists a $c > 0$ such that

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(v''_n) = V_n v_n - f(v_n) \ge c v_n \quad \text{in } (-\infty, -2\kappa_0 - x_n)$$

for all sufficiently large n. Hence, (2.23) holds for v''_n. Noting

$$v'_n(x) = \int_{-\infty}^{x} v''_n(y) \mathrm{d}y,$$

(2.23) holds.

Step 7. *Conclusion (Completion of the proof for (2.16)).*

We first notice that by the choice of $\kappa_0 > 0$, one has

$$\min_{[-3\kappa_0 - x_n, -2\kappa_0 - x_n]} (V_\infty - V_n) = \min_{[-3\kappa_0, -2\kappa_0]} (V_\infty - V(x)) > 0. \tag{2.24}$$

By Step 6, we observe that for $x \in [-3\kappa_0 - x_n, -2\kappa_0 - x_n]$

$$v'_n(x) v_n(x) \ge \begin{cases} c \exp\left(-2(3\kappa_0 + x_n)\sqrt{\dfrac{V_\infty}{\Lambda}}\right) & (\text{for } \mathcal{M}^{+}_{\lambda,\Lambda}), \\ c \exp\left(-2(3\kappa_0 + x_n)\sqrt{\dfrac{V_\infty}{\lambda}}\right) & (\text{for } \mathcal{M}^{-}_{\lambda,\Lambda}). \end{cases} \tag{2.25}$$

Therefore, using (2.24), (2.25) and Step 5, we obtain

$$-E_{n,\infty,+}(y_n^+) \ge c \exp\left(-2x_n \sqrt{\frac{V_\infty}{\Lambda}}\right) \qquad (\text{for } \mathcal{M}^{+}_{\lambda,\Lambda}),$$
$$-E_{n,\infty,-}(y_n^-) \ge c \exp\left(-2x_n \sqrt{\frac{V_\infty}{\lambda}}\right) \qquad (\text{for } \mathcal{M}^{-}_{\lambda,\Lambda})$$

for some $c > 0$. However, by Steps 2 and 3, we have a contradiction. Hence, we may find an $M_2 > 0$ so that $x_n \le M_2$.

For the lower bound of (x_n), by introducing $\tilde{u}_n(x) := u_n(-x)$, we can reduce the case into the case $x_n \to \infty$. Thus (2.16) holds.

We finally derive a contradiction in order to complete the proof of Proposition 2.9. By (2.16), we may assume $x_n \to x_0$. Next, from Lemmas 2.6 and 2.8, we observe that if $s_n \geq \tilde{t}(f, V_\infty)$, then

$$\frac{\kappa_0^2}{4\Lambda} s_n \leq \|u_n\|_{L^\infty(\mathbf{R})} \leq M_0.$$

Therefore, (s_n) is also bounded and assume that $s_n \to s_0$. Thus from the equation, we also get $u_n \to u_0$ in $C^2_{\mathrm{loc}}(\mathbf{R})$,

$$\begin{aligned} &-\mathcal{M}^\pm_{\lambda,\Lambda}(u_0'') + V u_0 = f(u_0) + s_0\varphi_0 \quad \text{in } \mathbf{R}, \quad u_0(x_0) = \max_{\mathbf{R}} u_0, \\ &u_0'(y) \leq 0 \leq u_0'(x) \quad \text{for } x \leq x_0 \leq y, \quad u_0 \geq 0 \text{ in } \mathbf{R}. \end{aligned} \tag{2.26}$$

If $u_0(x_0) = 0$, namely $u_0 \equiv 0$, then by the monotonicity of u_n ($u_n'(y) \leq 0 \leq u_n'(x)$ for $x \leq x_n \leq y$), we choose an $R_0 > 3\kappa_0$ so that

$$V u_n - f(u_n) \geq \frac{V_0}{2} u_n$$

for all $|x| \geq R_0$ and sufficiently large n. Therefore, we have

$$\mathcal{M}^\pm_{\lambda,\Lambda}(u_n'') = V u_n - f(u_n) \geq \frac{V_0}{2} u_n \quad \text{for every } |x| \geq R_0.$$

Hence, we may derive the uniform exponential decay:

$$u_n(x) \leq c \exp\left(-|x| \sqrt{\frac{V_0}{2\Lambda}} \right)$$

for all $x \in \mathbf{R}$ and n. By the definition of X_{η_1} and (2.3), this asserts that (u_n) is bounded in X_{η_1}, however, this contradicts $\|u_n\|_{X_{\eta_1}} \to \infty$.

Next we consider the case $u_0(x_0) > 0$ and shall show that $\lim_{|x|\to\infty} u_0(x) = 0$. If this is true, then as in the above, we can derive a uniform exponential decay and get a contradiction. Set $u_\infty := \lim_{x\to\infty} u_0(x)$. Since u_0 is a bounded solution of (2.26), we have

$$\lim_{x\to\infty} \mathcal{M}^\pm_{\lambda,\Lambda}(u_0'')(x) = \lim_{x\to\infty} \left(V u_0 - f(u_0) - s_0\varphi_0 \right) = V_\infty u_\infty - f(u_\infty).$$

Thus by (2.15), either $u_\infty = 0$ or else $u_\infty = s_\infty$. Let us assume $u_\infty = s_\infty$. From (2.26), we get $u_0 \geq s_\infty$ in $[x_0, \infty)$. Since $V_\infty s - f(s) < 0$ for $s > s_\infty$, one sees that

$$\mathcal{M}^\pm_{\lambda,\Lambda}(u_0'') = V u_0 - f(u_0) - s_0\varphi_0 \leq V_\infty u_0 - f(u_0) - s_0\varphi_0 \leq 0 \quad \text{in } (x_0, \infty).$$

Since $u_0'(x_0) = 0$ and $u_0(x) \to s_\infty$ as $x \to \infty$, we conclude that $V \equiv V_\infty$, $u_0 \equiv s_\infty$ and $s_0\varphi_0 \equiv 0$ in $[x_0, \infty)$. Hence, by (2.9) and $V \equiv V_\infty$ in $[x_0, \infty)$, we see $3\kappa_0 \le x_0$. Thus we may assume $2\kappa_0 \le x_n$. Now set

$$E_{n,+}(x) := \frac{\Lambda}{2}(u_n'(x))^2 + F(u_n(x)) - \frac{V(x)}{2}u_n^2(x) \qquad \text{for } \mathcal{M}^+_{\lambda,\Lambda},$$
$$E_{n,-}(x) := \frac{\lambda}{2}(u_n'(x))^2 + F(u_n(x)) - \frac{V(x)}{2}u_n^2(x) \qquad \text{for } \mathcal{M}^-_{\lambda,\Lambda}.$$

Since $x_n \ge 2\kappa_0$, $V'(x) \ge 0$ in $[x_n, \infty)$ and $u_n(x) \to 0$ as $|x| \to \infty$, arguing as in the case to v_n above, we may find unique $z_n^\pm \ge x_n$ such that

$$u_n''(x) < 0 = u_n''(z_n^\pm) < u_n''(y) \quad \text{for } x_n \le x < z_n^\pm < y.$$

Therefore,

$$(E_{n,\pm})'(x) = -\frac{V'}{2}u_n^2 \le 0 \quad \text{in } [z_n^\pm, \infty), \quad \lim_{x\to\infty} E_{n,\pm}(x) = 0, \quad E_{n,\pm}(z_n^\pm) \ge 0.$$

By $u_n''(z_n^\pm) = 0$, one has

$$V_0 \le V(z_n^\pm) = \frac{f(u_n(z_n^\pm))}{u_n(z_n^\pm)}.$$

From (f2), we may find a $\delta_0 > 0$ so that $\delta_0 \le u_n(z_n^\pm)$. Recalling $x_0 \le \liminf_{n\to\infty} z_n^\pm$, $V \equiv V_\infty$ in $[x_0, \infty)$ and (2.15), we obtain

$$u_n(z_n^\pm) \to s_\infty,$$
$$F(u_n(z_n^\pm)) - \frac{V(z_n^\pm)}{2}u_n(z_n^\pm)^2 \to F(s_\infty) - \frac{V_\infty}{2}s_\infty^2 = G_\infty(s_\infty) < 0.$$

Combining with $E_{n,\pm}(z_n^\pm) \ge 0$, we may find a $\delta_1 > 0$ so that

$$(u_n'(z_n^\pm))^2 \ge \delta_1.$$

Noting that $u_n'(x_n) = 0$ and (u_n'') is bounded in $L^\infty(\mathbf{R})$, we have $0 < \delta_2 \le z_n^\pm - x_n$ for some $\delta_2 > 0$, and

$$(u_n'(x))^2 \ge \delta_3^2 > 0 \quad \text{in } [z_n^\pm - \delta_4, z_n^\pm]$$

for some $\delta_3, \delta_4 > 0$ with $\delta_4 \le \delta_2$. Thus

$$u_n(x_n) \ge u_n(z_n^\pm - \delta_4) \ge u_n(z_n^\pm) + \delta_3\delta_4 \to s_\infty + \delta_3\delta_4.$$

This contradicts $u_n(x_n) \to u_0(x_0) = s_\infty$. Hence, $u_\infty = 0$.

For $\lim_{x\to-\infty} u_0(x) = 0$, by introducing $v_n(x) = u_n(-x)$ and $v_0(x) = u_0(-x)$, we can reduce into the former case and get $\lim_{x\to-\infty} u_0(x) = 0$. Now we complete the proof of Proposition 2.9. □

3. Non-existence theorem

In this section we prove Theorem 1.2 that asserts that the equation (1.1) does not have a solution when V is monotone. The argument below is similar to that of Proposition 2.9.

Proof of Theorem 1.2. Let us suppose for contradiction that u is a positive solution of (1.1) and let x_0 be a maximum point of u. Noting that V is non-decreasing and that the argument in Lemma 2.7 works under (f1) and (1.3), if $\bar{x}$ satisfies $u'(\bar{x}) = 0$ then $u'(x) < 0$ for all $x \in (\bar{x}, \infty)$. Thus x_0 is the unique critical point of u.

To proceed further, we make some preparations. Since u is strictly decreasing in (x_0, ∞) and V non-decreasing in $\mathbf{R}$, by (f4), we see that the function

$$h(x) := V(x) - \frac{f(u(x))}{u(x)} : [x_0, \infty) \to \mathbf{R}$$

is strictly increasing. Moreover, since $u''(x_0) \leq 0$ and $\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') = uh(x)$ in $[x_0, \infty)$, we have $h(x_0) \leq 0$. Hence, by $h(x) \to \overline{V} > 0$ as $x \to \infty$ thanks to (V2'), there is a unique $z^{\pm} \geq x_0$ such that $h(z^{\pm}) = 0$. In particular, since $u''(y) > 0 = u''(z_{\pm})$ if $z^{\pm} < y$, we have $x_0 < z^{\pm}$ due to $u'(x_0) = 0$ and $u(x) \to 0$ as $x \to \infty$. Remark also that $u''(x) < 0$ for all $x_0 \leq x < z^{\pm}$.

Recalling that u is strictly increasing in $(-\infty, x_0)$ and decreasing in (x_0, ∞), u has two inverse functions $y^{\pm}(s)$ and $z^{\pm}(s)$ satisfying $y^{\pm}(s) < x_0 < z^{\pm}(s)$ for $0 < s < u(x_0)$. Next we define $y^{\pm} = y^{\pm}(u(z^{\pm}))$ and

$$H_+(x) = \frac{\Lambda}{2}(u'(x))^2 + F(u(x)) - \frac{V(z^+)}{2}u^2(x) \qquad \text{for } \mathcal{M}^+_{\lambda,\Lambda},$$
$$H_-(x) = \frac{\lambda}{2}(u'(x))^2 + F(u(x)) - \frac{V(z^-)}{2}u^2(x) \qquad \text{for } \mathcal{M}^-_{\lambda,\Lambda}.$$

Notice that $y^{\pm} < z^{\pm}$ follow from $x_0 < z^{\pm}$. To complete the proof of Theorem 1.2, we proceed in various steps.

Step 1. *$H_{\pm}(z^{\pm}) \geq 0$ and if $V(z^{\pm}) < \overline{V}$, then $H_{\pm}(z^{\pm}) > 0$.*

We start with

$$H'_{\pm}(x) = u'(x)\left(V(x) - V(z^{\pm})\right)u(x) \text{ if } u''(x) \geq 0, \tag{3.1}$$

$$H'_{+}(x) = u'(x)\left(\frac{\Lambda}{\lambda}h(x)u(x) + f(u(x)) - V(z^+)u(x)\right) \text{ if } u''(x) \leq 0, \tag{3.2}$$

$$H'_-(x) = u'(x)\left(\frac{\lambda}{\Lambda}h(x)u(x) + f(u(x)) - V(z^-)u(x)\right) \text{ if } u''(x) \le 0. \tag{3.3}$$

Noticing that $u' < 0 \le V'$ and $0 \le u''$ in $(z^\pm, \infty)$, from (3.1) we have $H'_\pm(x) \le 0$ in $(z^\pm, \infty)$. In case $V(z^\pm) < \overline{V}$ we additionally have $H'_\pm(x) \not\equiv 0$. From $H_\pm(x) \to 0$ as $|x| \to \infty$ we conclude that Step 1 holds.

Step 2. *$H_\pm(z^\pm) \le H_\pm(y^\pm)$ and if $V \not\equiv$ const. in $[y^\pm, z^\pm]$, then $H_\pm(z^\pm) < H_\pm(y^\pm)$.*

We use arguments similar to those of Step 3 of the proof of Proposition 2.9. Noting that $x_0 < z^\pm$, $u''(z^\pm(s)) < 0$ for every $s \in (u(z^\pm), u(x_0))$, $u''(z^\pm) = 0 = h(z^\pm)$ and V is non-decreasing, we observe that for each $s \in (u(z^\pm), u(x_0))$,

$$h(y^\pm(s)) = V(y^\pm(s)) - \frac{f(s)}{s} \le V(z^\pm(s)) - \frac{f(s)}{s} = h(z^\pm(s)) < h(z^\pm) = 0. \tag{3.4}$$

From $\mathcal{M}^\pm_{\lambda,\Lambda}(u'') = u(x)h(x)$ it follows that $u''(x) < 0$ in $(y^\pm, z^\pm)$. Hence, by (3.2), (3.4) and changing variables $s = u(x)$, we have

$$\begin{aligned} H_+(x_0) - H_+(y^+) &= \int_{y^+}^{x_0} \left[\frac{\Lambda}{\lambda}h(x)u(x) + f(u(x)) - V(z^+)u(x)\right] u'(x)\mathrm{d}x \\ &= \int_{u(y^+)}^{u(x_0)} \left[\frac{\Lambda}{\lambda}h(y^+(s))s + f(s) - V(z^+)s\right] \mathrm{d}s \\ &\le \int_{u(z^+)}^{u(x_0)} \left[\frac{\Lambda}{\lambda}h(z^+(s))s + f(s) - V(z^+)s\right] \mathrm{d}s \\ &= -\int_{x_0}^{z^+} \left[\frac{\Lambda}{\lambda}h(x)u(x) + f(u(x)) - V(z^+)u(x)\right] u'(x)\mathrm{d}x \\ &= -\int_{x_0}^{z^+} H'_+(x)\mathrm{d}x = H_+(x_0) - H_+(z^+). \end{aligned}$$

Thus $H_+(z^+) \le H_+(y^+)$ and if $V \not\equiv$ const. in $[y^+, z^+]$, then we have $V(y^+) < V(z^+)$ and $h(y^\pm(s)) < h(z^\pm(s))$ for s close to $u(z^\pm)$. Hence, in this case, $H_+(z^+) < H_+(y^+)$ holds. Using (3.3) instead of (3.2), the case of H^- is treated in a similar way.

Step 3. *$H'_\pm(x) \le 0$ in $(-\infty, y^\pm)$ and if $V \not\equiv$ const. in $(-\infty, y^\pm)$, then $H'_\pm \not\equiv 0$.*

First we consider H_+. We observe that $u' > 0$ and $V'(x) \ge 0$ in $(-\infty, y^+)$, so that when $u''(x) \ge 0$, (3.1) implies

$$H'_+(x) = u'(x)(V(x) - V(z^+))u(x) \le 0. \tag{3.5}$$

On the other hand, if $u''(x) < 0$, then recalling that $\lambda \le \Lambda$, we have

$$\begin{aligned} H'_+(x) &= u'(\Lambda u'' + f(u) - V(z^\pm)u) \\ &= u' \left\{ (\Lambda - \lambda)u'' + (V(x) - V(z^+))u \right\} \le 0. \end{aligned} \tag{3.6}$$

Hence, $H'_+ \le 0$ in $(-\infty, y^+)$. If $V \not\equiv$ const. in $(-\infty, y^+)$, we may find $x_1 < y^+$ such that $V(x_1) < V(y^+) \le V(z^+)$. Then, from (3.5) or (3.6), we have $H'_+(x_1) < 0$.

Next we consider H_-. We have $u' > 0$ for $x < y^-$, hence if $u''(x) \ge 0$, then we have

$$H'_-(x) = u'(x) \left\{ V(x) - V(z^-) \right\} u(x) \le 0. \tag{3.7}$$

On the other hand, assume that $u''(x) < 0$. Since $u' > 0$ in $(-\infty, y^-)$, we get $u(x) < u(y^-) = u(z^-)$. Therefore, by the definition of z^- and (f4), we find

$$0 = h(z^-) = V(z^-) - \frac{f(u(z^-))}{u(z^-)} < V(z^-) - \frac{f(u(x))}{u(x)},$$

which yields $f(u(x)) - V(z^-)u(x) < 0$. Thus, from (3.3) and monotonicity of V, it follows that

$$\begin{aligned} H'_-(x) &= u'(x) \left[\frac{\lambda}{\Lambda} h(x)u(x) + f(u(x)) - V(z^-)u(x) \right] \\ &= u'(x) \left[\frac{\lambda}{\Lambda} \left\{ V(x) - V(z^-) \right\} u(x) \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\Lambda}\right) \left\{ f(u(x)) - V(z^-)u(x) \right\} \right] \le 0. \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we get $H'_-(x) \le 0$ in $(-\infty, y^-)$. Moreover, it is easily seen that when $V \not\equiv$ const. in $(-\infty, y^-)$, $H'_- \not\equiv 0$ holds.

Step 4. *Conclusion.*

By Steps 1–3, we get

$$0 \le H_\pm(z^\pm) \le H_\pm(y^\pm) = \int_{-\infty}^{y^\pm} H'_\pm(x) \mathrm{d}x \le 0. \tag{3.9}$$

However, $\underline{V} < \overline{V}$, so we have $V \not\equiv$ const. in either $(-\infty, y^\pm)$ or $(y^\pm, z^\pm)$ or $(z^\pm, \infty)$. Consequently, at least one inequality in (3.9) is strict, providing a contradiction and completing the proof. □

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Appendix A. Proof of Proposition 2.1

Here we prove Proposition 2.1.

Proof of Proposition 2.1. We first prove the existence of solutions. For $\alpha > 0$, we consider

$$-u'' = \Lambda^{-1} g_\infty(u) \quad \text{in } \mathbf{R}, \quad (u'(0), u(0)) = (0, \alpha), \tag{A.1}$$

$$-u'' = \lambda^{-1} g_\infty(u) \quad \text{in } \mathbf{R}, \quad (u'(0), u(0)) = (0, \alpha) \tag{A.2}$$

where $g_\infty(s) := f(s) - V_\infty s$, and we write $u_{\Lambda,\alpha}$ and $u_{\lambda,\alpha}$ for unique solutions of (A.1) and (A.2). By (f1)–(f4), it is well known that there exists an $\alpha_0 > 0$ so that $u_{\Lambda,\alpha}(x)$ hits zero at some point $x_\alpha > 0$ ($u_{\Lambda,\alpha}(x_\alpha) = 0$) if $\alpha > \alpha_0$, u_{Λ,α_0} is a positive solution of (A.1) and $u_{\Lambda,\alpha_0}(x) \to 0$ as $|x| \to \infty$, and $u_{\Lambda,\alpha}(x)$ a positive periodic solution of (A.1) when $\alpha < \alpha_0$. The number $\alpha_0 > 0$ is characterized by

$$G_\infty(\alpha_0) = 0 \tag{A.3}$$

and (A.3) has a unique positive solution due to (2.15) (or (f1)–(f4)). For instance, see [2,10]. Therefore, (A.3) yields $\alpha_0 > s_\infty$. The same statement holds for $u_{\lambda,\alpha}$.

For $\mu > 0$, set

$$E[u,\mu](x) := \frac{1}{2}(u'(x))^2 + \mu^{-1} G_\infty(u(x)).$$

Then it is easily seen that

$$\frac{\mathrm{d}}{\mathrm{d}x} E[u_{\Lambda,\alpha}, \Lambda](x) \equiv 0 \equiv \frac{\mathrm{d}}{\mathrm{d}x} E[u_{\lambda,\alpha}, \lambda](x) \quad \text{in } \mathbf{R}.$$

In particular, since $u_{\Lambda,\alpha_0}(x)$, $u'_{\Lambda,\alpha_0}(x)$, $u_{\lambda,\alpha_0}(x)$, $u'_{\lambda,\alpha_0}(x) \to 0$ as $x \to \infty$, we have $E[u_{\Lambda,\alpha_0}, \Lambda] \equiv 0 \equiv E[u_{\lambda,\alpha_0}, \lambda]$ in $\mathbf{R}$.

Since $u_{\Lambda,\alpha_0}(0) = \alpha_0 > s_\infty$ and $u_{\Lambda,\alpha_0}(x) \to 0$ as $x \to \infty$, we may choose $x_\Lambda > 0$ so that $u_{\Lambda,\alpha_0}(x_\Lambda) = s_\infty$ and $u_{\Lambda,\alpha_0}(x) < s_\infty$ for every $x > x_\Lambda$. Recalling $G_\infty(s_\infty) < 0$ and $E[u_{\Lambda,\alpha_0}, \Lambda](x_\Lambda) = 0$, we obtain $u'_{\Lambda,\alpha_0}(x_\Lambda) < 0$ and

$$\begin{aligned}\lambda^{-1} \min_{\mathbf{R}} G_\infty &< \frac{1}{2}(u'_{\Lambda,\alpha_0}(x_\Lambda))^2 + \lambda^{-1} G_\infty(u_{\Lambda,\alpha_0}(x_\Lambda)) \\ &= E[u_{\Lambda,\alpha_0}, \lambda](x_\Lambda) < E[u_{\Lambda,\alpha_0}, \Lambda](x_\Lambda) = 0.\end{aligned}$$

By (A.3) and (2.15), the equation

$$\lambda^{-1} G_\infty(s) = E[u_{\Lambda,\alpha_0}, \lambda](x_\Lambda) \in \left(\lambda^{-1} \min_{\mathbf{R}} G_\infty, 0\right) \tag{A.4}$$

has two solutions $0 < s_1 < s_\infty < s_2 < \alpha_0$.

Now we consider $u_{\lambda,s_2}(x)$. Since u_{λ,s_2} is periodic and $E[u_{\lambda,s_2}, \lambda](x) = E[u_{\lambda,s_2}, \lambda](0) = \lambda^{-1} G_\infty(s_2) < 0$ in $\mathbf{R}$, we observe that

$$\max_{\mathbf{R}} u_{\lambda,s_2} = s_2 > s_\infty > s_1 = \min_{\mathbf{R}} u_{\lambda,s_2}.$$

Hence, we may select a $y_1 > 0$ so that $u_{\lambda,s_2}(x) > s_\infty = u_{\lambda,s_2}(y_1)$ for each $x \in [0, y_1)$. From the choice of y_1 and (A.4), it follows that

$$u_{\lambda,s_2}(y_1) = s_\infty = u_{\Lambda,\alpha_0}(x_\Lambda), \quad E[u_{\lambda,s_2}, \lambda](y_1) = \lambda^{-1} G_\infty(s_2) = E[u_{\Lambda,\alpha_0}, \lambda](x_\Lambda),$$

which implies $|u'_{\lambda,s_2}(y_1)| = |u'_{\Lambda,\alpha_0}(x_\Lambda)|$. By $u'_{\lambda,s_2}(y_1), u'_{\Lambda,\alpha_0}(x_\Lambda) \le 0$ due to the definition of y_1 and x_Λ, we obtain $u'_{\lambda,s_2}(y_1) = u'_{\Lambda,\alpha_0}(x_\Lambda)$. Thus, set

$$u(x) := \begin{cases} u_{\lambda,s_2}(x) & \text{if } 0 \le x \le y_1, \\ u_{\Lambda,\alpha_0}(x - y_1 + x_\Lambda) & \text{if } y_1 < x, \end{cases}$$

and $u(x) := u(-x)$ for $x < 0$, it is easily seen that $u \in C^1(\mathbf{R})$ and u satisfies $-\mathcal{M}^+_{\lambda,\Lambda}(u'') = g_\infty(u)$ in $\mathbf{R} \setminus \{\pm y_1\}$. In addition, from the definition of u, it follows that

$$\lim_{h \downarrow 0} \frac{u'(y_1 + h) - u'(y_1)}{h} = 0 = \lim_{h \uparrow 0} \frac{u'(y_1 + h) - u'(y_1)}{h}.$$

Hence, $u \in C^2(\mathbf{R})$ and u is a solution of (2.1). Moreover, we observe that $u''(x) < 0 < u''(y)$ for all $|x| < y_1$ and $|y| > y_1$. Further, it is known that u_{Λ,α_0} decays exponentially, so does u.

For (2.2), we start with u_{λ,α_0} instead of u_{Λ,α_0}. Then we choose an $x_\lambda > 0$ so that $u_{\lambda,\alpha_0}(x) < s_\infty = u_{\lambda,\alpha_0}(x_\lambda)$ for every $x \in (x_\lambda, \infty)$. In this case, instead of (A.4), we consider the equation

$$\Lambda^{-1}G_\infty(s) = E[u_{\lambda,\alpha_0},\Lambda](x_\lambda) > E[u_{\lambda,\alpha_0},\lambda](x_\lambda) = 0$$

and this equation has only one solution $s_1 > \alpha_0$ due to (2.15). Let us consider u_{Λ,s_1}. By $s_1 > \alpha_0$, we may find a $z_1 > 0$ satisfying $u_{\Lambda,s_1}(z_1) = 0$. Thus, choose a $y_1 > 0$ so that $u_{\Lambda,s_1}(x) > s_\infty = u_{\Lambda,s_1}(y_1)$ for all $x \in [0, y_1)$ and set

$$u(x) := \begin{cases} u_{\Lambda,s_1}(x) & \text{if } 0 \le x \le y_1, \\ u_{\lambda,\alpha_0}(x - y_1 + x_\lambda) & \text{if } y_1 < x. \end{cases}$$

Then as in the above, we can check that $u \in C^2(\mathbf{R})$ is a solution of (2.2), decays exponentially and $u''(x) < 0 < u''(y)$ for each $|x| < y_1$ and $|y| > y_1$.

Next, we prove the uniqueness of solutions of (2.1) and (2.2). Let u_1 be a solution of (2.1) constructed in the above and u any solution of (2.1). By the sign property of g_∞, we deduce that $u(0) \ge s_\infty$. Moreover, notice that

$$-\mathcal{M}^+_{\lambda,\Lambda}(u'') = g_\infty(u) \quad \text{in } \mathbf{R} \quad \Leftrightarrow \quad -u'' = (\mathcal{M}^+_{\lambda,\Lambda})^{-1}(g_\infty(u)) \quad \text{in } \mathbf{R}$$

where $(\mathcal{M}^+_{\lambda,\Lambda})^{-1}(s) = \Lambda^{-1}s$ if $s \ge 0$ and $(\mathcal{M}^+_{\lambda,\Lambda})^{-1}(s) = \lambda^{-1}s$ if $s < 0$. Since $(\mathcal{M}^+_{\lambda,\Lambda})^{-1}$ and f are locally Lipschitz continuous, the initial value problem

$$-u'' = (\mathcal{M}^+_{\lambda,\Lambda})^{-1}(g_\infty(u)) \quad \text{in } \mathbf{R}, \quad (u'(z), u(z)) = (\alpha_1, \alpha_2) \tag{A.5}$$

has a unique solution u_{z,α_1,α_2} for every $z, \alpha_1 \in \mathbf{R}$ and $\alpha_2 > 0$. Since $g_\infty(s_\infty) = 0$, if $u(0) = s_\infty$, then we infer that $u \equiv u_{0,0,s_\infty} \equiv s_\infty$, which contradicts $u(x) \to 0$ as $x \to \infty$. Hence, $u(0) > s_\infty$.

Now choose $z_\Lambda > 0$ so that $u(z_\Lambda) = s_\infty > u(x)$ for all $x > z_\Lambda$. Then u satisfies

$$-u'' = \Lambda^{-1}g_\infty(u) \quad \text{in } (z_\Lambda, \infty).$$

Noting $E[u,\Lambda](x) \equiv 0$ in $[z_\Lambda, \infty)$ and $u(z_\Lambda) = s_\infty$, we have

$$(u'(z_\Lambda), u(z_\Lambda)) = (u'_{\Lambda,\alpha_0}(x_\Lambda), u_{\Lambda,\alpha_0}(x_\Lambda)) = (u'_1(y_1), u_1(y_1)).$$

Thus it is easily seen from the construction of u_1 and the unique solvability of the initial value problem for (A.5) with $z = z_\Lambda$ that $u(x) = u_1(x + y_1 - z_\Lambda)$ in $\mathbf{R}$. Noting that

$$u_1(0) = \max_{\mathbf{R}} u_1 > u_1(x) \quad \text{for } x \ne 0, \quad u(0) = \max_{\mathbf{R}} u,$$

we deduce that $y_1 = z_\Lambda$ and $u_1 \equiv u$. Hence, the uniqueness of solutions of (2.1) holds. Similarly, we can prove the uniqueness of solutions of (2.2).

Remark that the above argument can be applied to conclude $u \equiv u_1$ if u satisfies $-\mathcal{M}^\pm_{\lambda,\Lambda}(u'') = g_\infty(u)$ in $\mathbf{R}$ with $u(0) = \max_{\mathbf{R}} u$, $u > 0$ in $\mathbf{R}$ and $u(x) \to 0$ as either $x \to \infty$ or $x \to -\infty$. Thus we complete the proof. □

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