Induced Heavy Moving Averages

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This paper presents the induced heavy ordered weighted moving average (IHOWMA) operator. It is an aggregation operator that uses the main characteristics of three well-known techniques: the moving average, induced operator, and heavy aggregation operator. This operator provides a parameterized family of aggregation operators that include the minimum, the maximum, and total operator as special cases. It can be used in a selection process, considering that not all decision makers have the same knowledge and expectations of the future. The main properties of this operator are studied including a wide range of families of IHOWMA operators, such as the heavy ordered weighted moving average operator and uncertain induced heavy ordered weighted moving average operator. The IHOWMA operator is also extended using generalized and quasi-arithmetic means. An example in an investment selection process is also presented.

1. INTRODUCTION

In problems related to the economy, which has high uncertainty and volatility, inclusion of knowledge and expectations of the decision maker about future scenarios is an important way to reduce risk related to the decision. A common aggregation method is the ordered weighted averaging (OWA) operator introduced by Yager.[1] The OWA operator has been used in many applications[2–4] and extended under a wide range of frameworks.[5,6]

This paper focuses on the induced OWA (IOWA), heavy OWA (HOWA), and moving average (MA). The IOWA operator was introduced by Yager and Filev.[7] In this operator, the reordering step is not developed with values of the arguments. Instead it is induced by another mechanism such that the ordered position of arguments depends upon values of their associated order-inducing variables.[22] This operator was extended by using fuzzy numbers,[8] intuitionistic fuzzy information,[9] and generalized and quasi-arithmetic means.[10]

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The HOWA operator[11] is an operator that allows the weighting vector to range from 1 to ∞ or even from $-\infty$ to ∞. With this characteristic, it is possible to add expectations and knowledge of the decision maker to the aggregation information process. This operator has been studied using fuzzy measures[12] and fuzzy numbers.[13] Extensions have been developed by Merigó et al.[14]

Moving average is an average of some part of the whole sample. With this technique, it is possible to consider historical data to forecast a future value. Making comparisons when modifying the sample is also possible. This method is usually used to solve time series smoothing problems and is very common in economics and statistics.[15]

This paper analyzes the use of moving averages with induced heavy aggregation operators in a selection process. The main advantage of this operator is that it unifies historical data with expectations of the future and knowledge of the decision maker, leading to improvement of the decision process. This operator is called induced heavy ordered weighted moving average (IHOWMA).

The main concepts of this new extension have been developed, along with a range of particular cases including the heavy ordered weighted moving average (HOWMA) operator and the uncertain induced heavy ordered weighted moving average (UIHOWMA) operator, which is used in problems with interval numbers.

A generalization of the IHOWMA operator is presented using generalized and quasi arithmetic means.[10,16,17] The main advantage of this operator is that the same formulation includes arithmetic, geometric, or quadratic aggregations. Several extensions using different types of interval and fuzzy numbers were suggested by Merigó and Casanovas.[16,17]

An application of the new approach in an investment selection process is developed. We use percent of profit information from three types of investment from February 2016 to April 2016. In the problem, we identify interval numbers, which is why the UIHOWMA operator is used to select the most profitable option considering characteristics of the decision maker.

The remainder of the paper is organized as follows. In Section 2, we review moving averages and some aggregation operators. Section 3 introduces the IHOWMA operator, and Section 4 develops the generalized IHOWMA operator. Section 5 explains the steps to use induced heavy aggregation operators in a selection process, and Section 6 presents the use of the IHOWMA operator in an investment selection process. Section 7 summarizes the main conclusions of the paper.

2. PRELIMINARIES

In this section, we briefly review some basic concepts that will be used throughout the paper. We analyze the interval numbers, OWA operator, UOWA operator, heavy aggregation operators, induced aggregation operators, moving averages, and generalized aggregation operators.

2.1. Interval Numbers

Interval numbers, introduced by Moore,[18] is a technique that represents uncertainty in a simple way. They can be expressed in different forms. For example, assume a triplet (a_1, a_2, a_3), where a_1 and a_3 are considered the minimum and the maximum of the interval and a_2 is the value with the highest probability or possibility. In the following sections, we define uncertain aggregation operators as those operators that use interval numbers in the analysis.

To rank alternatives using interval numbers, we establish criteria as follows:[17,19]

(a) For doublet, calculate the arithmetic mean of the interval: $(c_1 + c_2)/2$.
(b) For triples and more, calculate a weighted average that gives more importance to central values, that is, $(c_1 + 2c_2 + c_3)/4$.
(c) For 4-tuples we could calculate: $(c_1 + 2c_2 + 2c_3 + c_4)/6$.
(d) And so on.

2.2. OWA Operator

The OWA operator was introduced by Yager,[1] and it is an aggregation operator that has many applications.[2,16,19] It is defined as follows:

DEFINITION 1. *An OWA operator of dimension n is a mapping* $OWA : R^n \to R$ *with an associated weight vector W of dimension n such that* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0, 1]$, *according to the following formula:*

$$OWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j *is the jth largest element of the collection* a_i.

It is possible to distinguish descending OWA (DOWA) and ascending (AOWA) operators. The difference lies in weights that are related by $w_j = w^*_{n-j+1}$, where w_j is the *j*th weight of the DOWA operator and w^*_{n-j+1} the *j*th weight of the AOWA operator.

2.3. Uncertain OWA Operator

The uncertain OWA (UOWA) operator was introduced by Xu and Da.[20] The main characteristic of this operator is uncertainty; this can be included in the operator by using interval numbers. It can be defined as follows:

DEFINITION 2. *Let Ω be the set of interval numbers. An UOWA operator of dimension n is a mapping* $UOWA : \Omega^n \to \Omega$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *according to the following* *formula:*

$$UOWA\ (\tilde{a}_1, \tilde{a}_2, .., \tilde{a}_3) = \sum_{j=1}^{n} w_j \tilde{b}_j, \tag{2}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$ and the $\tilde{a}_i$ are interval numbers.

As in usual interval numbers, we can distinguish the uncertain maximum, minimum, and average. Additionally, as in the case of the OWA operator, we have descending UOWA and ascending UOWA operators.

2.4. Heavy Aggregation Operators (HOWA, UHOWA, and HWA)

The HOWA operator[11] is an extension of the traditional OWA, where the main characteristic of this new operator is in the weight vector, which is not bounded by the sum of 1, but instead bounded according to available information and expectations and knowledge of the decision maker. It can vary from 1 to n or even from $-\infty$ to ∞. This operator can be defined as follows:

DEFINITION 3. *A HOWA operator is a mapping $HOWA : R^n \rightarrow R$ which is associated with a weight vector w, where $w_j \in [0, 1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, such that*

$$HOWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j\ , \tag{3}$$

where b_j is the jth largest element of the collection a_i.

The characteristics of the HOWA operator are: monotonic and commutative, but it is not bounded by the minimum and the maximum operators. Additionally, similar to OWA, we can distinguish between descending HOWA and ascending HOWA operators.

With the possibility to expand the weighting vector from $-\infty$ to ∞, we can drastically under- or overestimate the results of the HOWA operator, considering new scenarios according to information of the decision maker and some expectations of the future of the case in study.

Yager[11] introduced a characteristic for the HOWA operator, that is related to the weighting vector and noted that this vector does not depend on reordering of the argument and introduced the beta value of the vector W. This beta value can be defined as $\beta\ (W) = (|W| - 1)/(n - 1)$. Note that if $\beta = 1$ we obtain the total operator and if $\beta = 0$ we obtain the usual OWA operator.

It is possible to use interval numbers in arguments of the HOWA operator, resulting in the uncertain heavy OWA (UHOWA) operator. Merigó and Casanovas[21] define it as follows:

DEFINITION 4. *Let Ω be the set of interval numbers. An UHOWA operator of dimension n is a mapping $UHOWA : \Omega^n \rightarrow \Omega$ that has an associated weighting vector W* *of dimension n such that* $w_j \in [0, 1]$ *and* $1 \leq \sum_{j=1}^{n} w_j \leq n$*, then*

$$UHOWA\ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{b}_j\ . \tag{4}$$

Note that the HOWA operator can convert to a heavy moving average (HWA) if $w_j = 1/n$. Merigó and Casanovas[17] define it as follows:

DEFINITION 5. *A HWA operator of dimension n is a mapping* $HWA : R^n \rightarrow R$ *that has an associated weighting vector W of dimension n with* $w_j \in [0, 1]$ *and* $1 \leq \sum_{i=1}^{n} w_i \leq n$*, such that*

$$HWA\ (a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i\ , \tag{5}$$

where a_i *is the ith argument of the aggregation. Also note that it is possible to expand the weighting vector from* $-\infty$ *to* ∞*.*

2.5. Induced Aggregation Operators

The IOWA operator was introduced by Yager and Filev[7] as an extension of the OWA operator. Its main difference is that the reordering step is not developed with values of the arguments a_i. In this case, the reordering step is developed with order inducing variables. The IOWA operator can be defined as follows.

DEFINITION 6. *An IOWA operator of dimension n is an application* $IOWA : R^n \times R^n \rightarrow R$ *that has a weighting vector associated, W of dimension n where the sum of the weights is 1 and* $w_j \in [0, 1]$*, where an induced set of ordering variables is included* (u_i) *such that the formula is*

$$IOWA\ (\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j, \tag{6}$$

where b_j *is the* a_i *value of the OWA pair* $\langle u_i, a_i \rangle$ *having the jth largest* u_i*.* u_i *is the order inducing variable, and* a_i *is the argument variable.*

From a generalized perspective of the reordering step, we can distinguish between the descending IOWA (DIOWA) operator and the ascending IOWA (AIOWA) operator. The weight of these operators is related to $w_j = w^*_{n-j+1}$, where w_j is the *j*th weight of the DIOWA and w^*_{n-j+1} the *j*th weight of the AIOWA operator.

The IOWA operator is an averaging operator. This is reflected by the fact that the operator is monotonic, commutative, bounded, and idempotent, both for the DIOWA and the AIOWA operator. Note that the OWA operator is obtained when $u_i = a_i$, for all *i*.[7,22]

2.6. Moving Averages

A moving average is a usual average that moves toward some part of the sample. This method has been used in economics and statistics to solve time series smoothing problems.[15] The moving average, according to Kenney and Keeping,[23] can be defined as follows:

DEFINITION 7. *Moving averages are defined as a sequence given* $\{a_i\}_{i=1}^{N}$, *where a moving average n is a new sequence* $\{s_i\}_{i=1}^{N-n+1}$ *defined from* a_i *taking the arithmetic mean of the sequence of n terms, such that*

$$s_i = \frac{1}{n} \sum_{j=i}^{i+n-1} a_j, \tag{7}$$

The usual moving average can be extended by using weighted averages, obtaining the weighted moving average (WMA). Merigó and Yager[24] defined it as follows:

DEFINITION 8. *A WMA of dimension m is a mapping* $WMA : R^m \rightarrow R$ *that has an associated weighting vector W of dimension m with* $W = \sum_{i=1+t}^{m+t} w_i = 1$ *and* $w_i \in [0, 1]$, *such that*

$$WMA\ (a_{1+t}, a_{2+t}, \ldots, a_{m+t}) = \sum_{i=1+t}^{m+t} w_i a_i\ , \tag{8}$$

where a_i *is the ith argument, m is the total number of arguments considered from the whole sample, and t indicates the movement performed in the average from the initial analysis. Note that if* $w_i = 1/m$ *for all i, the WMA becomes the MA aggregation.*

The moving average can also be combined with the OWA operator generating the ordered weighted moving average (OWMA). Merigó and Yager[24] defined it as follows:

DEFINITION 9. *An OWMA of dimension m is a mapping* $OWMA : R^m \rightarrow R$ *that has an associated weighting vector W of dimension m with* $W = \sum_{j=1+t}^{m+t} w_j = 1$ *and* $w_j \in [0, 1]$, *such that*

$$OWMA\ (a_{1+t}, a_{2+t}, \ldots, a_{m+t}) = \sum_{j=1+t}^{m+t} w_j b_j\ , \tag{9}$$

where b_j *is the jth largest argument of the* a_i, *m is the total number of arguments considered from the whole sample, and t indicates the movement in the average from the initial analysis.*

The OWMA can also be extended using the IOWA operator and its characteristic of an induced weighted vector resulting in the IOWMA operator. Merigó and Yager[24] defined it as follows:

DEFINITION 10. *An IOWMA of dimension m is a mapping* $IOWMA : R^M \times R^M \rightarrow R$ *that has an associated weighting vector W of dimension m with* $W = \sum_{j=1+t}^{m+t} w_j = 1$ *and* $w_j \in [0, 1]$, *such that*

$$IOWMA\ (\langle u_{1+t}, a_{1+t}\rangle, \langle u_{2+t}, a_{2+t}\rangle, \ldots, \langle u_{m+t}, a_{m+t}\rangle) = \sum_{j=1+t}^{m+t} w_j b_j, \qquad (10)$$

where b_j *is the* a_i *value of the IOWMA pair* u_i, a_i *having the jth largest* u_i, u_i *is the order inducing variable,* a_i *is the argument variable, m is the total number of arguments considered from the whole sample, and t indicates movement in the average from the initial analysis.*

2.7. Generalized Aggregation Operators

Generalized aggregation operators are those that provide a general formulation, including a wide range of particular cases, such as the generalized and quasi-arithmetic mean. Note that we focus on the quasi-arithmetic mean because it includes the generalized as a particular case.[10,25,26] The weighted quasi-arithmetic mean (Quasi-WA) is defined by Merigó and Yager[24] as follows:

DEFINITION 11. *A Quasi-WA operator of dimension n is a mapping* $Quasi - WA :$ $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n with* $\sum_{i=1}^{n} w_i = 1$ *and* $w_i \in [0, 1]$, *such that*

$$Quasi - WA\ (a_1, a_2, \ldots, a_n) = g^{-1}\left(\sum_{i=1}^{n} w_i g(a_i)\right), \qquad (11)$$

where g(b) is a strictly continuous monotone function. Note that if $w_i = 1/n$ *for all i, the QWA becomes the simple quasi-arithmetic mean. Moreover, if* $g(a) = a^{\lambda}$, *the QWA becomes the weighted generalized mean.*

The QWA operator can be combined with the OWA operator to obtain the ordered weighted quasi-arithmetic mean (Quasi-OWA). This operator is defined by Fodor et al.[27] as follows:.

DEFINITION 12. *A Quasi-OWA operator of dimension n is a mapping* $Quasi - OWA : R^n \rightarrow R$ *that has an associated weighting vector W of dimension n with* $\sum_{j=1}^{n} g(b_j) = 1$ *and* $w_j \in [0, 1]$, *and such that*

$$Quasi - OWA\ (a_1, a_2, \ldots, a_3) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \qquad (12)$$

where b_j is the jth largest of the a_i and g(b) is a strictly continuous monotone function.

Note that if we induce the reordering step of the weighting vector we obtain the Quasi-IOWA operator. Merigó and Gil-Lafuente[10] defined it as follows:

DEFINITION 13. *A Quasi-IOWA operator of dimension n is a mapping $Quasi - IOWA : R^n x\ R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $W = \sum_{j=1}^{n} w_j = 1$, such that*

$$Quasi - IOWA\left(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle\right) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_j\right)\right), \tag{13}$$

where b_j is the a_i value of the Quasi-IOWA pair u_i, a_i having the jth largest u_i, u_i is the order inducing variable, a_i is the argument, and g(b) is a strictly continuous monotonic function.

Also the IOWMA operator can be generalized, and similar to the other operators the Quasi-IOWMA is presented because it includes the generalized IOWMA as a particular case.

DEFINITION 14. *An induced order weighted quasi-arithmetic moving average (Quasi-IOWMA) of dimension m is a mapping $Quasi - IOWMA : R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension m with $W = \sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0, 1]$, such that*

$$Quasi - IOWMA\left(\langle u_{1+t}, a_{1+t}\rangle, \ldots, \langle u_{n+t}, a_{n+t}\rangle\right) = g^{-1}\left(\sum_{j=1+t}^{m+t} w_j g\left(b_j\right)\right), \tag{14}$$

where b_j is the a_i value of the IOWMA pair u_i, a_i having the jth largest u_i, u_i is the order inducing variable, a_i is the argument variable, m is the total number of arguments considered from the whole sample, t indicates the movement in the average from the initial analysis, and g(b) is a strictly continuous monotone function.

It is important to note that if $w_i = 1/m$ for all *i,* the Quasi-IOWMA operator becomes the Quasi-MA operator. Furthermore, if $g(a) = a^{\lambda}$, the Quasi-IOWMA becomes the generalized weighted moving average.

3. INDUCED HEAVY MOVING AVERAGES

3.1. Theoretical Foundations

The IHOWMA operator is an extension of the OWA operator and combines characteristics of the IOWA operator and the HOWA operator with the moving average. This operator uses a moving average with an associated weighting vector that ranges from $-\infty$ to ∞; it also uses order-inducing variables in the reordering of information. It can be defined as follows.

DEFINITION 15. *A IHOWMA operator is defined as a given sequence* $\{a_i\}_{i=1}^{N}$, *where a new sequence* $\{s_i\}_{i=1}^{N-m+1}$ *is multiplied by a heavy weighting vector, such that*

$$IHOWMA\ (\langle u_{1+t}, a_{1+t}\rangle, \langle u_{2+t}, a_{2+t}\rangle, \ldots, \langle u_{n+t}, a_{m+t}\rangle) = \sum_{j=1+t}^{m+t} w_j b_j\ , \qquad (15)$$

where b_j *is jth element that has the largest value of* u_i, u_i *is the order inducing variable, and W is an associated weighting vector of dimension m with* $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$. *Observe that we can also expand the weighting vector from* $-\infty$ *to* ∞. *Thus, the weighting vector w becomes* $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

Note that by allowing the weighting vector to range from $-\infty$ to ∞ it is possible to under- or overestimate the result based on the information available and knowledge of the decision maker. Because of the reordering step, similar to the OWA operator, it is possible to distinguish between descending and ascending IHOWMA operators.

The characteristics of the IHOWMA operator are as follows:

(a) It is monotonic because if $a_i \geq d_i$, for all i, then $IHOWMA\,(a_1, \ldots, a_n) \geq IHOWMA(d_1, \ldots, d_n)$.
(b) It is commutative because any permutation of the argument has the same evaluation.
(c) It can be bounded if the weight vector ranges from 1 to ∞, and it can consider that if the weight vector ranges from $-\infty$ to ∞, the IHOWMA operator is not bounded.
(d) The characteristic of the beta value explained in Definition 3 (HOWA operator) also applies to the IHOWMA operator.

When a problem is based on interval numbers, the uncertain IHOWMA (UI-HOWMA) operator is obtained. This operator can be defined as follows:

DEFINITION 16. *Let* Ω *be the set of interval numbers. An UIHOWMA operator of dimension n is a mapping* $UIHOWMA : \Omega^n \times \Omega^n \rightarrow \Omega$ *that acts on the sequence* $\{s_i\}_{i=1}^{N-n+1}$, *which is multiplied by a heavy weighting vector, according to*

$$UIHOWA\ (\langle(u_{1+t}, \tilde{a}_{1+t}\rangle, \langle u_{2+t}, \tilde{a}_{2+t}\rangle, \ldots, \langle u_{n+t}, \tilde{a}_{m+t}\rangle) = \sum_{j=1+t}^{m+t} w_j \tilde{b}_j\ , \qquad (16)$$

where $\tilde{b}_j$ *is the jth element that has the largest value* u_i, u_i *is the order inducing variable,* $\tilde{a}_i$ *are interval numbers, and W is an associated weighting vector of dimension m with* $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$.

Note that if the weighting vector is not induced, then the IHOWMA operator becomes the HOWMA operator, such that

DEFINITION 17. *A HOWMA operator is defined as a given sequence* $\{a_i\}_{i=1}^{N}$, *where a new sequence* $\{s_i\}_{i=1}^{N-n+1}$ *is multiplied by a heavy weighting vector, such that*

$$HOWMA\ (s_i) = \sum_{j=1+t}^{m+t} w_j b_j\ , \tag{17}$$

where b_j *is the jth largest element of the collection* $a_1, a_2, \ldots, a_n$ *and W is an associated weighting vector of dimension m with* $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$. *Observe that we can also expand the weighting vector from* $-\infty$ *to* ∞. *Thus, the weighting vector w becomes* $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

3.2. Families of the IHOWMA Operator

In this section, different families of the IHOWMA operator are presented. We demonstrate a wide range of particular cases that can be used in the IHOWMA operator, leading to different results. These particular cases are found by analyzing the coefficient β, so a wide range of particular cases are presented by giving different values and interpretations of the β value. By choosing different values for the weight vector, we obtain different types of aggregation operators. Some of the families of the IHOWMA operator are as follows:

(a) If $\beta = 0$ the IHOWMA operator becomes the IOWMA operator.
(b) When $w_n = 1,\ w_j = 0$, for all $j \neq n$ and $\beta = 0$, we obtain the minimum IOWMA. If $w_1 = 1,\ w_j = 0$, for all $j \neq n$ and $\beta = 0$, we obtain the maximum IOWMA. Finally, the total operator is obtained when $\beta = 1$.
(c) The IOWMA operator becomes the MA aggregation when $w_i = 1/m$ for all *i*.
(d) The arithmetic mean IHOWMA: If $w_j = 1/n$ for all *j* and $p_i = 1/n$ for all *i*.
(e) The median IHOWMA: if *n* is odd we assign $w_{(n+1)/2} = 1$ and $w_{j*} = 0$ for all others. If *n* is even we assign, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j*} = 0$ for all others.
(f) *The weighted median IHOWMA*: Select the argument b_k that has the *k*th largest argument such that the sum of the weights from 1 to *k* is equal to or higher than 0.5 and the sum of the weights from 1 to *k* – 1 is less than 0.5.
(g) *The general olympic-IHOWMA operator*: if $w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j*} = 1/(n-2k)$, where $k < n/2$.
(h) *The centered-IHOWMA*: if it is symmetric, it strongly decays from the center to the maximum and the minimum, and it is inclusive.

To understand this approach, we present a simple numerical example of the UIHOWMA operator.

Example 1. Assume the following income for three different companies shown in Table I in an aggregation process:

Assume an ordered inducing variable $U = (C_1 = (10, 15, 5)\ ,\ C_2 = (5, 10, 15),\ C_3 = (15, 10, 5))$, and the following weighting vector $W = (0.3, 0.3, 0.4)$. If we want to forecast the income for April based on the information below using the IHOWMA operator, the results will be

$$C_1 = (0.3 * 70{,}000)\ + (0.3 * 80{,}000) + \ (0.4 * 83{,}000) = \ 78{,}200$$

Table I. Available data

	January	February	March
C_1	80,000	90,000	87,000
C_2	70,000	85,000	91,000
C_2	83,000	92,000	86,000

$$C_2 = (0.3 * 92{,}000) + (0.3 * 85{,}000) + (0.4 * 90{,}000) = 89{,}100$$

$$C_3 = (0.3 * 87{,}000) + (0.3 * 91{,}000) + (0.4 * 86{,}000) = 87{,}800$$

We see from this information that the company with better forecast income will be $C_2 > C_3 > C_1$.

4. GENERALIZED IHOWA OPERATOR

The IHOWMA operator provides a wide range of cases using generalized and quasi-arithmetic means.[24,28,29] In this section, we present the Quasi-HOWMA operator and the Quasi-IHOWMA operator because they include the generalized mean as a particular case.

DEFINITION 18. *A Quasi-HOWMA operator of dimension n is a mapping HOWMA* : $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n that is defined by a given sequence* $\{a_i\}_{i=1}^{N}$, *where you obtain a new sequence* $\{s_i\}_{i=1}^{N-n+1}$, *which is multiplied by a heavy weighting vector, such that*

$$Quasi - HOWMA\ (s_i) = g^{-1} \sum_{j=1+t}^{m+t} w_j g\left(b_j\right), \tag{18}$$

where $g(a_i)$ *is a strictly continuous monotonic function and* b_j *is the jth largest element of the collection* $a_1, a_2, \ldots, a_n$, *W is an associated weighting vector of dimension m with* $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$, *and* $g(b_j)$ *is a strictly continuous monotone function.*

DEFINITION 19. *A Quasi-IHOWMA operator is defined as a given sequence* $\{a_i\}_{i=1}^{N}$, *where you obtain a new sequence* $\{s_i\}_{i=1}^{N-m+1}$ *that is multiplied by a heavy weighting vector, such that*

$$Quasi - IHOWMA\left(\langle u_1, a_{1+t}\rangle, \langle u_2, a_{2+t}\rangle, \ldots, \langle u_n, a_{m+t}\rangle\right) = g^{-1} \sum_{j=1+t}^{m+t} w_j g\left(b_j\right), \tag{19}$$

where $g(b_j)$ *is a continuous monotonic function and* b_j *is jth element that has the largest value of* u_i, u_i *is the order inducing variable, and W is an associated*

Table II. Families of generalized IHOWMA operators

Particular case	Quasi-IHOWMA	Quasi-HOWMA
$u_i = \frac{1}{n}$, for all i	Quasi-arithmetic heavy moving average (Quasi-HOWMA)	Quasi-arithmetic heavy moving average (Quasi-HMA)
$g\ (b) = b^{\lambda}$	Generalized IHOWMA	Generalized HOWMA
$g\ (b) = b$	IHOWMA	HOWMA
$g\ (b) = b^2$	Heavy ordered weighted moving quadratic average (IHOWMQA)	Heavy ordered weighted moving quadratic average (HOWMQA)
$g(b) \to b^{\lambda},\ for\ \lambda \to 0$	Heavy ordered weighted moving geometric average (IHOWMGA)	Heavy ordered weighted moving geometric average (HOWMGA)
$g\ (b) = b^{-1}$	Heavy ordered weighted moving harmonic average (IHOWMHA)	Heavy ordered weighted moving harmonic average (HOWMHA)
$g\ (b) = b^3$	Heavy ordered weighted moving cubic average (IHOWMCA)	Heavy ordered weighted moving cubic average (HOWMCA)
$g(b) \to b^{\lambda}$, for $\lambda \to \infty$	Heavy maximum	Heavy maximum
$g(b) \to b^{\lambda}$, for $\lambda \to \infty$	Heavy minimum	Heavy minimum

weighting vector of dimension m with $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$. *Observe that we can also expand the weighting vector from* $-\infty$ *to* ∞. *Thus, the weighting vector w becomes* $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

Note that if interval numbers are used, the Quasi-UIHOWMA operator is obtained. It can be defined as follows:

Definition 20. *Let* Ω *be the set of interval numbers. A Quasi-UIHOWMA operator of dimension n is a mapping* $Quasi - UIHOWMA : \Omega^n \times \Omega^n \to \Omega$ *that acts on the sequence* $\{s_i\}_{i=1}^{N-n+1}$, *which is multiplied by a heavy weighting vector, according to*

$$Quasi - IHOWA(\langle\langle u_{1+t}, \tilde{a}_{1+t}\rangle, \langle u_{2+t}, \tilde{a}_{2+t}\rangle, \ldots, \langle u_{n+t}, \tilde{a}_{m+t}\rangle) = \sum_{j=1+t}^{m+t} w_j g\left(\tilde{b}_j\right) \ , \tag{20}$$

where $\tilde{b}_j$ *is the jth element that has the largest value* u_i, u_i *is the order inducing variable,* $\tilde{a}_i$ *are interval numbers, W is an associated weighting vector of dimension m with* $W : 1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ *and* $w_i \in [0, 1]$, *and* $g(\tilde{b}_j)$ *is a strict, continuous monotonic function.*

In Table II, we briefly present some of the main particular cases of the Quasi-IHOWMA operator and Quasi-HOWMA operator.

The same families of the generalized IHOWMA operator apply to the UI-HOWMA operator. The difference is that now these aggregation operators address interval numbers.

5. INVESTMENT SELECTION WITH INDUCED HEAVY MOVING AVERAGES

In an investment selection process, aversion to loss and the risk of each choice play an important role for the investor, so including these characteristic in the selection process will generate the best investment according to their needs, skills, and knowledge of the financial market.[30–32]

The use of information aggregation operators can integrate all of these characteristics in the selection process, such that use of UIHOWMA operators helps decision makers to select an appropriate choice taking in account personal characteristics and expectations for the future of the financial market.

To use UIHOWMA operators in the selection process, there are a number of steps to obtain the best results. These steps are as follows:

Step 1. Identify alternatives available, in this case, different types of investment (A_1, A_2, A_3).

Step 2. Choose the number of historical months to consider, for this case three months.

Step 3. Analyze the data using the opinion of several experts and form collective results that summarize information given by the experts.

Step 4. Calculate the weighting vector $W = (w_{1+t}, .., w_{m+t})$

Step 5. Calculate the order-inducing variable $U = (u_{1+t}, \dots, u_{m+t})$

Step 6. Use the IHOWMA (or UIHOWMA) operator for each of the criteria (c_1, c_2, c_3) for alternatives identified in Step 1.

Step 7. Adopt a decision based on results found using different aggregation operators. In the case of uncertain aggregation operators, use the ranking methods presented in Section 2.1.

6. SELECTION PROCESS FOR THREE INVESTMENT OPTIONS

In this section, we investigate a simple numerical example to understand the approach suggested in this article. The example shows a multiperson decision-making problem in the selection of investments.

Step 1. We consider the following alternatives for investment

a_1 *Tbills*$_{MEX}$

a_2 *Sight Savings*

a_3 *Investment fund*

Step 2. The historical profit information for each alternative is presented as a 3-tuplet because each investment option has a minimum, median, and maximum percent of profit according to the quantity of the investment.

Step 3. Since the historical data are imprecise, three experts evaluate the data and provide their own opinion (see Tables III–V). Assume that opinions of the three experts are equally important.

Table III. Historical profit of the investments: Expert 1

Month	$Tbills_{MEX}$	*Sight saving*	*Investment fund*
April 2016	(0.39,0.41,0.46)	(0.41,0.43,0.48)	(0.30,0.47,0.64)
March 2016	(0.34,0.38,0.42)	(0.24,0.26,0.28)	(0.21,0.34,0.50)
February 2016	(0.30,0.34,0.38)	(0.15,0.19,0.24)	(0.13,0.33,0.42)

Table IV. Historical profit of the investments: Expert 2

Month	$Tbills_{MEX}$	*Sight saving*	*Investment fund*
April 2016	(0.37,0.42,0.45)	(0.42,0.47,0.49)	(0.26,0.51,0.59)
March 2016	(0.35,0.36,0.40)	(0.23,0.24,0.29)	(0.20,0.36,0.51)
February 2016	(0.29,0.32,0.37)	(0.16,0.20,0.25)	(0.14,0.30,0.40)

Table V. Historical profit of the investments: Expert 3

Month	$Tbills_{MEX}$	*Sight saving*	*Investment fund*
April 2016	(0.35,0.40,0.44)	(0.40,0.45,0.46)	(0.25,0.52,0.60)
March 2016	(0.36,0.37,0.41)	(0.22,0.25,0.27)	(0.19,0.35,0.52)
February 2016	(0.31,0.33,0.36)	(0.17,0.21,0.26)	(0.18,0.30,0.38)

Table VI. Historical profit of the investments: Collective results

Month	$Tbills_{MEX}$	*Sight saving*	*Investment fund*
April 2016	(0.37,0.41,0.45)	(0.42,0.45,0.48)	(0.27,0.50,0.61)
March 2016	(0.35,0.37,0.41)	(0.23,0.25,0.28)	(0.20,0.35,0.51)
February 2016	(0.30,0.33,0.37)	(0.16,0.20,0.25)	(0.15,0.31,0.40)

Information from the experts is aggregated using the arithmetic mean ($W =$ 1/3, 1/3, 1/3) forming the collective results shown in Table VI.

Steps 4 and 5. Note that $n = 3$ because the decision maker believes that these are the months that still hold important information, the ordered inducing variables $U = (A_1 = (5, 10, 20), A_2 = (10, 5, 20), A_3 = (5, 20, 10))$ and the weighted vector is $W = (0.30, 0.35, 0.40) = 1.05$ because the future scenario of the investment is optimistic.

Step 6. Using the above information, the uncertain moving average (UMA), uncertain heavy moving average (UHMA) operator, uncertain heavy ordered weighted moving average (UHOWMA) operator, and UIHOWMA operator are applied to generate various selection scenarios (see Table VII).

Step 7. Unify the results of interval numbers according to the procedure explained in Section 2.1. (see Table VIII). Rank alternatives according to the results (see Table IX).

Note that by using the UIHOWMA operator, the order of the alternatives changes because of inclusion of the induced weighted vector instead of the regular one. In this sense, the ranking of alternatives includes more information about the

Table VII. Results using aggregation operators

Operator	*Tbills*$_{MEX}$	*Sight saving*	*Investment fund*
UMA	(0.34,0.37,0.41)	(0.27,0.30,0.34)	(0.21,0.39,0.51)
UHMA	(0.36,0.39,0.43)	(0.28,0.32,0.35)	(0.22,0.41,0.53)
UHOWMA	(0.35,0.38,0.43)	(0.27,0.30,0.34)	(0.21,0.40,0.52)
UIHOWMA	(0.36,0.39,0.43)	(0.29,0.32,0.36)	(0.22,0.41,0.54)

Abbreviations: UMA, the uncertain moving average operator: UHMA, uncertain heavy moving average operator; UHOWMA, uncertain heavy ordered weighted moving average operator; UIHOWMA, uncertain induced heavy ordered weighted moving average operator.

Table VIII. Unification of the interval numbers

Operator	*Tbills*$_{MEX}$	*Sight saving*	*Investment fund*
UMA	0.3725	0.3016	0.3716
UHMA	0.3911	0.3167	0.39025
UHOWMA	0.3872	0.3043	0.3813
UIHOWMA	0.3950	0.3192	0.3961

Abbreviations: UMA, the uncertain moving average operator: UHMA, uncertain heavy moving average operator; UHOWMA, uncertain heavy ordered weighted moving average operator; UIHOWMA, uncertain induced heavy ordered weighted moving average operator.

Table IX. Ranking of the alternatives

Operator	Ranking
UMA	*Tbills*$_{MEX}$ > *Investment fund* > *Sight saving*
UHMA	*Tbills*$_{MEX}$ > *Investment fund* > *Sight saving*
UHOWMA	*Tbills*$_{MEX}$ > *Investment fund* > *Sight saving*
UIHOWMA	*Investment fund* >*Tbills*$_{MEX}$ > *Sight saving*

Abbreviations: UMA, the uncertain moving average operator: UHMA, uncertain heavy moving average operator; UHOWMA, uncertain heavy ordered weighted moving average operator; UIHOWMA, uncertain induced heavy ordered weighted moving average operator.

decision maker instead of only historical data. With the use of the UIHOWMA operator, knowledge, expectations, and characteristics of the decision maker can be added making results more complex and specialized.

7. CONCLUSIONS

This paper has introduced a new extension of the OWA operator called the IHOWMA operator. This operator uses the main characteristics of three techniques: the moving average, IOWA operator, and heavy OWA (HOWMA) operator. This operator uses historical information and combines it with a weighted vector, reflecting the expectations of the decision maker. Note that the reordering step in the process is induced.

We analyze this new operator by providing its definition and studying some of its main properties. We developed a wide range of families of the IHOWMA operator, such as the arithmetic mean, median, weighted median, olympic mean, and generalized IHOWMA. Other interesting cases were described including the HOWMA and UIHOWMA, where interval numbers are used.

An application of the new approach to the selection process has also been developed. We observe that using the UIHOWMA operator instead of other operators leads to different results about the expectation of the future of alternatives. Using the UIHOWMA operator, the decision maker will have a more complete vision of situations.

In future research, we expect to develop new extensions of the OWA operator[33] by considering a probabilistic heavy ordered weighted moving average operator including characteristics such as fuzzy sets,[34,35] distance measure,[36–38] or the use of expertons in a group decision-making problem.[39]

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