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Implications of Superstatistics for steady-state plasmas

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Abstract. Colissionless plasmas are non-extensive systems which, due to the long-range interaction between their components, are incapable of reaching thermal equilibrium, even in a steady state. These systems cannot, therefore, be described statistically by a single canonical distribution with well-defined inverse temperature $\beta = 1/k_B T$ and are commonly described via two alternative approaches: namely Tsallis statistics and Superstatistics. The use of Superstatistics in describing steady-state plasmas has been proposed by several authors, more recently Ourabah *et al.* In this work we study the consequences of this assumption of Superstatistics for steady-state plasmas. We explicitly show that only the ensembles characterized by the condition $P(\boldsymbol{x}|\rho) = \rho(H(\boldsymbol{x}))$ are consistent with a generalized definition of temperature introduced recently. We show how this formalism is employed in the case of plasma, considering interaction with external electromagnetic fields as well as between particles in the plasma. Our results clearly illustrate why low-energy particles tend to the Maxwellian distribution of velocities, while high-energy particles contribute to the long tails of the velocity distribution.

1. Introduction

In the plasma state, the long-range interactions between different ions and electrons are dominant, and this causes these systems to be found in highly non-equilibrium states. Even when these systems are able to reach a steady state, they do not follow Boltzmann-Gibbs statistics [1]. A pressing question then arises: Do these steady-state plasmas have a well-defined temperature? According to equilibrium Thermodynamics, temperature T is defined through the relation

$$\frac{1}{T} = \frac{dS(E)}{dE}, \tag{1}$$

where $S(E) = k_B \ln \Omega(E)$ is the Boltzmann entropy. For a system in thermal equilibrium at temperature T, the velocity distribution of its components is given by the Maxwell-Boltzmann distribution,

$$P(\boldsymbol{v}|\beta) = \frac{1}{Z(\beta)} \exp\left(-\beta \frac{mv^2}{2}\right) \tag{2}$$

with $\beta = 1/(k_B T)$ the inverse temperature. The equipartition theorem connects the parameter β with the average kinetic energy of the system,

$$\frac{3N}{2} k_B T = \Big\langle K \Big\rangle_\beta = \Big\langle \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 \Big\rangle_\beta. \tag{3}$$

In general, we can classify systems into three classes:

(a) Systems in thermodynamical equilibrium, where β is well-defined,
(b) Systems with non-constant β where it makes sense to introduce a *temperature distribution function*,
(c) Other systems where a temperature cannot even be defined. In this last case, usually transient states, we can only speak of average kinetic energy.

For systems in classes (a) and (b) we will denote the probability of having an inverse temperature β as $P(\beta|S)$, in accordance with the formalism known as Superstatistics, which we will explain next.

2. Superstatistics

An interesting proposal for the treatment of non-equilibrium, steady-state systems was introduced in 2003 by Beck and Cohen, called *Superstatistics* [2, 3]. In this framework, one goes from the canonical ensemble

$$P(\boldsymbol{r}, \boldsymbol{p}|\beta) = \frac{\exp(-\beta H(\boldsymbol{r}, \boldsymbol{p}))}{Z(\beta)}, \tag{4}$$

to a superposition of infinite canonical ensembles at different inverse temperatures β,

$$P(\boldsymbol{r}, \boldsymbol{p}|S) = \int_0^\infty d\beta P(\beta|S) \left[\frac{\exp(-\beta H(\boldsymbol{r}, \boldsymbol{p}))}{Z(\beta)}\right] \tag{5}$$

each of them weighted by its probability of occurrence, $P(\beta|S)$. Two cases are of interest. The trivial case when $P(\beta|S)$ is a Dirac delta function recovers the canonical ensemble, and the case when $P(\beta|S)/Z(\beta)$ is a Gamma distribution

$$P(\beta|k, \theta) = \frac{\exp(-\beta/\theta)\beta^{k-1}}{\Gamma(k)\theta^k}, \tag{6}$$

generates the so-called q-canonical ensemble,

$$P(\boldsymbol{r}, \boldsymbol{p}|\beta_0, q) = \frac{1}{\eta(\beta_0, q)} \left[1 - (1-q)\beta_0 H(\boldsymbol{r}, \boldsymbol{p})\right]_+^{\frac{1}{1-q}} \tag{7}$$

also known as Tsallis distributions. In space plasmas, the Kappa distribution [4] is an instance of the q-canonical (Tsallis) distribution and, accordingly, has been derived by the use of Superstatistics [5].

3. The fundamental inverse temperature

A “thermal” system, in the sense that temperature can be defined, is characterized by the condition $P(\boldsymbol{r}, \boldsymbol{p}|S) = \rho(\mathcal{H}(\boldsymbol{r}, \boldsymbol{p}))$. In order to completely describe these systems, it is no longer enough to have a single number β; we need a continuous function. This can be $P(\beta|S)$ in the framework of Superstatistics or the ensemble function $\rho(E)$. There is a more convenient possibility, the fundamental inverse temperature function [6, 7], defined by

Ensemble	$\rho(E)$	$\beta_F(E)$
Canonical	$\exp(-\beta_0 E)/Z(\beta_0)$	β_0
q-canonical	$\frac{1}{\eta(\beta_0,q)}\left[1-(1-q)\beta_0 E\right]_+^{\frac{1}{1-q}}$	$\beta_0/(1-(1-q)\beta_0 E)$
Gaussian	$\frac{1}{Z(a,E_t)}\exp(-a(E-E_t)^2)$	$2a(E-E_t)$

Table 1. Generalized Boltzmann factor $\rho(E)$ and fundamental inverse temperature function $\beta_F(E)$ for the canonical, q-canonical and Gaussian [8, 9] ensembles.

$$\beta_F(E) = -\frac{d}{dE}\ln\rho(E). \tag{8}$$

In the case of Superstatistics, $\beta_F(E)$ corresponds to $\langle\beta\rangle_E$, i.e., the average inverse temperature at fixed energy E.

By employing the statistical identity presented in Ref. [10] as the conjugate variables theorem (CVT),

$$\Big\langle \nabla\cdot\boldsymbol{v}\Big\rangle_S = -\Big\langle \boldsymbol{v}\cdot\nabla\ln\rho(H)\Big\rangle_S, \tag{9}$$

we can obtain the generalization of the equipartition theorem for a steady-state plasma,

$$\Big\langle \beta K\Big\rangle_S = \Big\langle \beta_F(E)K\Big\rangle_S = \frac{3N}{2}, \tag{10}$$

where the role of the parameter β is played by the fundamental inverse temperature function $\beta_F(E)$.

4. Thermodynamical description of plasma

In a plasma we have N particles, each with a mass m_i and a charge q_i, with position and velocity vectors $\boldsymbol{r_i}$ and $\boldsymbol{v_i}$ respectively. The probability density of microstates of the plasma can be written as $P(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_N|t) = P(\boldsymbol{R},\boldsymbol{V}|S)$ such that Vlasov equation holds,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \sum_i \boldsymbol{v_i}\cdot\frac{\partial P}{\partial \boldsymbol{r}_i} + \sum_i \frac{\boldsymbol{F}_i}{m_i}\cdot\frac{\partial P}{\partial \boldsymbol{v}_i} = 0 \tag{11}$$

where $\boldsymbol{F}_i = q_i\Big(\boldsymbol{E}(\boldsymbol{r}_i) + \boldsymbol{v}_i\times\boldsymbol{B}(\boldsymbol{r}_i)\Big)$ is the Lorentz force. Vlasov equation is equivalent to Liouville's theorem of conservation of phase space volume for plasma, and due to the presence of the Lorentz force, must be solved self-consistently with Maxwell's equations,

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{12}$$

$$\nabla\times\boldsymbol{B} = \mu_0\Big(\boldsymbol{J} + \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\Big) = \mu_0\boldsymbol{J}, \tag{13}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{14}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} = 0. \tag{15}$$

Here $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ are the charge and current densities, respectively, which in a steady state can be expressed as expectations over the ensemble as

$$\rho = \rho(\boldsymbol{r}) = \langle \sum_i q_i \delta(\boldsymbol{r}_i - \boldsymbol{r}) \rangle_S$$
$$\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{r}) = \langle \sum_i q_i \boldsymbol{v}_i \delta(\boldsymbol{r}_i - \boldsymbol{r}) \rangle_S. \tag{16}$$

We see then that, in a steady state, $P(\boldsymbol{R}, \boldsymbol{V}|S)$ is uniquely determined either by $\{\Phi(\boldsymbol{r}), \boldsymbol{A}(\boldsymbol{r})\}$ or $\{\rho(\boldsymbol{r}), \boldsymbol{J}(\boldsymbol{r})\}$. The non-relativistic Hamiltonian of the system is $\mathcal{H} = \sum_i \mathcal{H}_i$, with

$$\mathcal{H}_i(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2m_i}(\boldsymbol{p} - q_i \boldsymbol{A}(\boldsymbol{r}))^2 + q_i \Phi(\boldsymbol{r}), \tag{17}$$

and where the momentum $\boldsymbol{p}_i = m_i \boldsymbol{v}_i + q_i \boldsymbol{A}(\boldsymbol{r}_i)$. We can write the energy for the i-th particle in terms of its position and velocity as

$$\mathcal{E}_i(\boldsymbol{r}, \boldsymbol{v}) = \frac{1}{2} m_i \boldsymbol{v}^2 + q_i \Phi(\boldsymbol{r}), \tag{18}$$

which does not depend on the vector potential $\boldsymbol{A}$. In order to define temperature, the probability density $P(\boldsymbol{R}, \boldsymbol{V}|S)$ must have the form

$$P(\boldsymbol{R}, \boldsymbol{V}|S) = \rho(\mathcal{E}) = \rho\Big(\sum_i \mathcal{E}_i(\boldsymbol{r}_i, \boldsymbol{v}_i)\Big), \tag{19}$$
$$\mathcal{E}_i(\boldsymbol{r}, \boldsymbol{v}) = \frac{1}{2} m_i \boldsymbol{v}^2 + q_i \Phi(\boldsymbol{r}). \tag{20}$$

We can verify that this *Ansatz* automatically solves the steady-state Vlasov equation,

$$\sum_i \boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i} \ln P(\boldsymbol{R}, \boldsymbol{V}|S) + \frac{\boldsymbol{F}_i}{m_i} \cdot \frac{\partial}{\partial \boldsymbol{v}_i} \ln P(\boldsymbol{R}, \boldsymbol{V}|S) =$$
$$\rho'(\mathcal{E}) \sum_i \left[\boldsymbol{v}_i \cdot q_i \nabla \Phi(\boldsymbol{r}_i) + \frac{q_i}{m_i} \left(-\nabla \Phi(\boldsymbol{r}_i) + \boldsymbol{v}_i \times \boldsymbol{B}(\boldsymbol{r}_i) \right) \cdot m_i \boldsymbol{v}_i \right] = 0. \tag{21}$$

By imposing consistency with Gauss law we find that any such "thermal" plasma must be found in an ensemble $\rho(\mathcal{H})$ for which

$$-\epsilon_0 \nabla^2 \Phi(\boldsymbol{r}) = \int_0^\infty dK \cdot K^{3N/2} \int d\boldsymbol{R} \left[\sum_{i=1}^N q_i \delta(\boldsymbol{r}_i - \boldsymbol{r}) \right] \rho \left(K + \sum_i q_i \Phi(\boldsymbol{r}_i) \right), \tag{22}$$

holds. There cannot be any net current inside such a plasma, because by requiring consistency with Ampère's law,

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_{i=1} q_i \int d\boldsymbol{R} \delta(\boldsymbol{r}_i - \boldsymbol{r}) \int d\boldsymbol{V} \cdot \boldsymbol{v}_i \rho \left(\sum_i \frac{1}{2} m_i v_i^2 + \sum_i q_i \Phi(\boldsymbol{r}_i) \right) = 0, \tag{23}$$

which is zero by symmetry. For a given electrostatic potential $\Phi(\boldsymbol{r})$, the steady-state ensemble must be a solution of

$$-\epsilon_0 \nabla^2 \Phi(\boldsymbol{r}) = \int_0^\infty dK \cdot K^{3N/2} \int d\boldsymbol{R} \left[\sum_{i=1}^N q_i \delta(\boldsymbol{r}_i - \boldsymbol{r}) \right] \rho \left(K + \sum_i q_i \Phi(\boldsymbol{r}_i) \right) \tag{24}$$

and this constrains the form of the fundamental inverse temperature compatible with Φ. By defining $\rho_1(\epsilon)$ such that

$$\frac{\partial}{\partial \epsilon_i} \ln \rho(\epsilon) = \frac{\partial}{\partial \epsilon_i} \ln \rho_1(\epsilon_i), \tag{25}$$

we can greatly simplify our consistency equation and write it as

$$-\epsilon_0 \nabla^2 \Phi(\boldsymbol{r}) = \sum_{i=1}^{N} q_i \rho_1(q_i \Phi(\boldsymbol{r})). \tag{26}$$

We can also write the single-particle distributions of position and velocity in terms of ρ_1. They are given by

$$P_i(\boldsymbol{r}|S) = \rho_1(q_i \Phi(\boldsymbol{r})), \tag{27}$$

$$P_i(\boldsymbol{v}|S) = \rho_1\Big(\frac{m_i \boldsymbol{v}^2}{2}\Big). \tag{28}$$

In the same way, the single-particle fundamental inverse temperature is then

$$\beta_{1F}(\mathcal{E}_i) = -\frac{\partial}{\partial \mathcal{E}_i} \ln \rho_1(\mathcal{E}_i). \tag{29}$$

An interesting consequence of this is that ions or electrons in different ranges of energy will be seen as different populations described by different velocity distributions. This is commonly observed in plasmas. Furthermore, different spatial regions can appear as having different temperatures.

Assuming $\mathcal{E}_i \ll \mathcal{E}$ we can approximate the effective single-particle temperature by using the following Taylor expansion,

$$\beta_{1F}(\mathcal{E}_i) \approx \Big\langle \hat{\beta}_F(\mathcal{E} + \mathcal{E}_i) \Big\rangle_S = \Big\langle \hat{\beta}_F(\mathcal{E}) \Big\rangle_S + \mathcal{E}_i \Big\langle \hat{\beta}'_F(\mathcal{E}) \Big\rangle_S + \frac{1}{2}\mathcal{E}_i^2 \Big\langle \hat{\beta}''_F(\mathcal{E}) \Big\rangle_S + \ldots \tag{30}$$

This means that, in a first approximation, the single-particle fundamental temperature is a constant, equal to the average fundamental temperature of the system, therefore the particles are approximately Maxwellian for low kinetic energies, and can have other, more "exotic" distributions for higher energies.

5. Conclusions

In this work we have shown that, for any steady-state plasma in which $P(\boldsymbol{R}, \boldsymbol{V}|S) = \rho(\mathcal{H})$, we can provide a "fundamental temperature" that completely describes the system in Superstatistics. A thermal plasma in this sense is fully described by either its single-particle ensemble function ρ_1 or its single-particle fundamental inverse temperature β_{1F} . These plasmas can naturally have a segregation in their energy distributions between slow (approximately Maxwellian) and fast particles.

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