Noname manuscript No.
(will be inserted by the editor)

Gradient flows, second order gradient systems and convexity

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the date of receipt and acceptance should be inserted later

Abstract We disclose an interesting connection between the gradient flow of a $\mathscr{C}^2$-smooth function ψ and evanescent orbits of the second order gradient system defined by the square-norm of $\nabla\psi$, under adequate convexity assumption. As a consequence, we obtain the following surprising result for two $\mathscr{C}^2$, convex and bounded from below functions ψ_1, ψ_2: if $||\nabla\psi_1|| = ||\nabla\psi_2||$, then $\psi_1 = \psi_2 + k$, for some $k \in \mathbb{R}$.

Keywords Gradient flows, second order gradient equations, convexity, dynamical systems

Mathematics Subject Classification (2000) 26B25, 34C12, 34D05, 34G20 , 37C10, 49J40

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Research supported by grants BASAL PFB-03, FONDECYT 1171854 (Chile) and MTM2014-59179-C2-1-P (MINECO of Spain and ERDF of EU).

1 Introduction

We are interested in the first order gradient system

$$u'(t) = -\nabla\psi(u(t)), \quad t \geq 0, \tag{DS-1}$$

in comparison with the second order gradient system

$$v''(t) = \nabla V(v(t)), \quad t \geq 0, \tag{DS-2}$$

where $\psi : \mathcal{H} \to \mathbb{R}$ is a $\mathscr{C}^2$ function (respectively, $V : \mathcal{H} \to \mathbb{R}$ is a $\mathscr{C}^1$ function), $\nabla\psi$, ∇V denote the respective gradients and $\mathcal{H}$ stands for a Hilbert space, with inner product $\langle\cdot \mid \cdot\rangle$ and associated norm $\|\cdot\|$. Throughout this work, the functions ψ and V will be linked with the relation

$$V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2, \quad x \in \mathcal{H}. \tag{1}$$

We also consider the set of critical points of ψ (singular set)

$$\mathrm{Crit}_\psi = \{x \in \mathcal{H} \mid \nabla\psi(x) = 0\} = \{x \in \mathcal{H} \mid V(x) = 0\}.$$

When ψ is convex, the set Crit_ψ is convex and consists of all (global) minimizers of ψ. Therefore, in this case the set of critical values $\psi(\mathrm{Crit}_\psi)$ is either empty or singleton. We may also observe that Crit_ψ is also the set minimizers of V. Therefore it is also convex, whenever V is assumed so.

By a *global* solution of (DS-1) (respectively, (DS-2)) we mean a function $u \in \mathscr{C}^1([0, +\infty), H)$ (respectively, $v \in \mathscr{C}^2([0, +\infty), H)$) satisfying (DS-1) (respectively, (DS-2)), for all $t \geq 0$. In both cases, we impose the initial condition

$$u(0) = u_0 \qquad (\text{respectively, } v(0) = u_0) \tag{I$_0$}$$

for some given $u_0 \in \mathcal{H}$. This is very common for (DS-1) to obtain unique solutions, whereas for (DS-2) an additional condition on the initial velocity $v'(0)$ is normally required. We deliberately refrain from doing so, but instead, we require the solutions of (DS-2) to be global on $[0, +\infty)$ and to comply with one of the *asymptotic conditions* given in the following definition.

Definition 1.1 ((weakly) evanescent solution) A global solution v of (DS-2) is called

- *weakly evanescent* (in short, *w-evanescent*) if it satisfies

$$\liminf_{t\to+\infty} \|v'(t)\| = \liminf_{t\to+\infty} V(v(t)) = 0, \tag{w-EV}$$

- *evanescent* if it satisfies

$$\|v'(\cdot)\| \in L^2(0, +\infty) \quad \text{and} \quad V(v(\cdot)) \in L^1(0, +\infty).$$

or equivalently

$$\int_0^{+\infty} \left(\|v'(t)\|^2 + V(v(t))\right) dt < +\infty. \tag{EV}$$

Remark 1.2 (i) Condition (EV) can also be seen as an asymptotic condition on the behavior of v as $t \to +\infty$.
(ii) Any evanescent solution of (DS-2) is also w-evanescent.

It is straightforward to see that any global solution of (DS-1) is also solution of (DS-2). However, this solution might fail to satisfy (EV). To see this, let $n = 1$ and $\psi(x) = -x^2$, for $x \in \mathbb{R}$, and notice that $v(t) = e^{2t}x_0$ is solution of (DS-1) (and consequently of (DS-2)), but (EV) fails, since $v \notin L^2(0, +\infty)$. Conversely, a solution of (DS-2) satisfying (EV) and (I_0) might not be solution of (DS-1) since the system (DS-2)–(EV) does not distinguish between ψ and $-\psi$.

Let us further consider the following two conditions:

$$(C) \quad \inf_{z\in\mathcal{H}} \|\nabla\psi(z)\| = 0 \qquad \text{and} \qquad (C^\star) \quad \psi \text{ is bounded below.}$$

By Ekeland's Variational Principle [15, Corollary 2.3] we deduce $(C^\star) \Longrightarrow (C)$. This latter condition (C) is necessary for the existence of w-evanescent solutions of (DS-2).

A constant function $v = \hat{x}$ is a w-evanescent solution of (DS-2) if and only if $\hat{x} \in \mathrm{Crit}_\psi$, while $\mathrm{Crit}_\psi \neq \emptyset$ clearly implies (C). If in addition ψ is convex, then $(C^\star)$ is also fulfilled. The example of the following convex $\mathscr{C}^2$ function

$$\psi(x) = \begin{cases} -\ln(1-x)\,, & \text{if } x \le 0,\\ \frac{1}{2}x^2 + x\,, & \text{if } x \ge 0. \end{cases} \tag{2}$$

shows that (C) and $(C^\star)$ are not equivalent, besides the fact that ψ convex (in this case, only (C) holds).

Description of the results. In this work we show that if either ψ or V is *convex*, then any solution of (DS-2) satisfying (I_0)–(EV) is also solution of (DS-1)–(I_0), and vice-versa. In particular, the second order system (DS-2) coupled with (I_0)-(EV), is well posed and can be integrated to obtain the first order system (DS-1). An important consequence of this result is an intimate link between convexity properties of ψ and of $\|\nabla\psi\|^2$ (Corollary 3.15):

$$(\|\nabla\psi\|^2 \text{ convex and } \psi \text{ bounded below}) \Longrightarrow \psi \text{ convex.}$$

This leads to the following surprising corollary:

$$\|\nabla\psi_1\| = \|\nabla\psi_2\| \Longrightarrow \psi_1 = \psi_2 + \text{constant},$$

provided that one of the following assumptions is fulfilled:

(a) ψ_1 and ψ_2 are convex and $\inf \|\nabla\psi_1\| = 0$ (Theorem 3.9),
(b) $\|\nabla\psi_1\|^2$ is convex and ψ_1 and ψ_2 are bounded below (Corollary 3.18).

Finally, disclosing the link between (DS-1) and (DS-2) leads to a simple variational principle for the first order gradient system (DS-1) when $\|\nabla\psi\|^2$ is convex and ψ bounded below (Proposition 3.21).

Structure of the manuscript. The remainder is organized as follows. In Section 2 we resume basic properties of the first order system (DS-1) for $\psi \in \mathscr{C}^2(\mathcal{H})$ and for the second order system (DS-2) for $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ that will be used in the sequel. No originality is claimed in Subsection 2.1, as well as in the beginning of Subsection 3.1, where most of the stated properties of the first order system (DS-1) are essentially known. These properties are recalled for completeness, provided eventually short proofs to keep the manuscript self-contained. Subsection 2.2 contains properties of the system (DS-2) with emphasis in Lyapunov functions and in asymptotic behavior of the orbits, while Subsection 2.3 is dedicated in comparing the solutions of these two systems.

The main results are resumed in Section 3 and organized as follows: Subsection 3.1 ensembles all results obtained under the driving assumption that ψ convex, while Subsection 3.2 does the same under the assumption V convex. We quote in particular Theorem 3.9 (determination of a convex function by the modulus of its gradient) and its variant Corollary 3.18 which are important consequences of Theorem 3.7 (equivalence of solutions of (DS-1) and (DS-2) if ψ is convex) and Proposition 3.14 respectively. Finally, in Subsection 3.3 we associate to the first order system (DS-1) an alternative variational principle, which is in the spirit of the results of this work.

We assume familiarity with basic properties and characterizations of convex functions. These prerequisites can be found in the classical books [22] or [23].

2 Basic properties of first and second order gradient systems

2.1 First order gradient system: basic properties

In this subsection we recall for completeness basic properties of solutions of the first order gradient system (DS-1), which will be used in the sequel. In this subsection the functions $\psi \in \mathscr{C}^2(\mathcal{H})$ and $V(\cdot)$ given in (1), are not yet assumed to be convex.

Lemma 2.1 (Lyapunov for (DS-1)**)** *Let* $u(\cdot)$ *be a maximal solution of* (DS-1) *defined on* $[0, T_{\max})$ *where* $T_{\max} \in (0, +\infty]$*. Then,*

(i) $\rho(t) := \psi(u(t))$ *is nonincreasing on* $[0, T_{\max})$ *and for every* $T \in (0, T_{\max})$

$$\int_0^T \left\| u'(t) \right\|^2 \, dt = \rho(0) - \rho(T)\,; \tag{3}$$

(ii) $\left\| u'(\cdot) \right\| \in L^2(0, T_{\max})$ *if and only if*

$$\inf_{0 \le t < T_{\max}} \psi(u(t)) > -\infty. \tag{4}$$

Proof. Since $\rho'(t) = \langle \nabla\psi(u(t)) \mid u'(t) \rangle = -\left\| u'(t) \right\|^2 = -\left\| \nabla\psi(u(t)) \right\|^2 \le 0$ we deduce (i). The second assertion follows by taking the limit as $T \to T_{\max}$. □

Remark 2.2 (Strict Lyapunov) Assuming $\psi \in \mathscr{C}^2(\mathcal{H})$ yields that both (DS-1) and the equation $w'(t) = \nabla\psi(w(t))$ admit unique solutions under a given initial condition. A standard argument now shows that if the initial condition is not a singular point (that is, $\nabla\psi(u(0)) \neq 0$), then $\nabla\psi(u(t)) \neq 0$, for every $t > 0$ and ρ is strictly decreasing.

Lemma 2.3 (maximal nonglobal solutions) *If* $u(\cdot)$ *is a maximal solution of* (DS-1) *which is not global* (*i.e.* $T_{\max} < +\infty$)*, then*

$$\inf_{0 \le t < T_{\max}} \psi(u(t)) = \lim_{t \to T_{\max}} \psi(u(t)) = -\infty, \tag{5}$$

and

$$\int_0^{T_{\max}} \left\| u'(t) \right\|^2 \, dt = +\infty. \tag{6}$$

Proof. In view of Lemma 2.1 (i) assertions (5) and (6) are equivalent. Assume now that (6) does not hold. Then the integral

$$\int_0^{T_{\max}} u'(t)dt,$$

converges in $\mathcal{H}$ to the element $u(T_{\max}) - u_0$, where $u(T_{\max}) = \lim_{t\to T_{\max}} u(t)$. Moreover $\nabla\psi(u(T_{\max})) \neq 0$ (*c.f.* Remark 2.2). Considering the Cauchy problem $w'(t) = -\nabla\psi(w(t))$ with initial condition $w(T_{\max}) = u(T_{\max})$, we deduce that the (presumably maximal) solution $u(\cdot)$ can be extended to the right on an interval of the form $[0, T_{\max} + \varepsilon)$ for some $\varepsilon > 0$, which is a contradiction. □

Corollary 2.4 *If ψ is bounded below, then any maximal solution $u(\cdot)$ of* (DS-1) *is global and $\|u'(\cdot)\| \in L^2(0, +\infty)$.*

Proof. If ψ is bounded below, then (5) cannot be satisfied, and the solution u is global. Obviously, (4) is fulfilled yielding $\|u'(\cdot)\| \in L^2(0, +\infty)$. □

Remark 2.5 (grad-coercive functions) A function $\psi \in \mathscr{C}^1(\mathcal{H})$ is called *grad-coercive* if $\|\nabla\psi\|$ is bounded on the sublevel sets $[\psi \le \alpha] := \{x \in \mathcal{H} : \psi(x) \le \alpha\}$, $\alpha \in \psi(\mathcal{H})$.

If ψ is grad-coercive then any maximal solution of (DS-1) is global. Indeed, let $u(\cdot)$ be a maximal solution defined on $[0, T_{\max})$. Since $u(t) \in [\psi \le \psi(u(0))]$, for all $t \in [0, T_{\max})$, the function $\|\nabla\psi(u(\cdot))\|$ is bounded on $[0, T_{\max}]$. Setting $M = \sup_{0\le t<T_{\max}} \|\nabla\psi(u(t))\|$, we obtain

$$\int_0^{T_{\max}} \|u'(t)\|^2 dt = \int_0^{T_{\max}} \|\nabla\psi(u(t))\|^2 dt \le MT_{\max} < +\infty,$$

which contradicts (6). □

Let us observe that ψ can be grad-coercive without being bounded from below. A simple example is the identity function $x \mapsto x$ on $\mathbb{R}$. Similarly, a function which is bounded below is not necessarily grad-coercive, for example the function $x \mapsto \cos(x^2)$.

Remark 2.6 (Relation to other domains) Asymptotic behavior of (DS-1) has been studied by several authors in the framework of analytic geometry (see [18], [24] *e.g.*), in relation to convexity ([7], [13], [14], [19]), to optimization algorithms ([2], [4], [8] *e.g.*) and to PDEs ([12], [16] *e.g.*). Roughly speaking, good asymptotic behavior requires a strong structural assumption (analyticity or convexity), see [1] or [21, p. 12] for classical counterexamples.

2.2 Second order system: properties of evanescent solutions

In this subsection we emphasize properties of (weakly) evanescent solutions of the second order system (DS-2), where $\psi \in \mathscr{C}^2(\mathcal{H})$ and

$$V(x) = \frac{1}{2}||\nabla\psi(x)||^2.$$

Lemma 2.7 (equality of modula) *Let $v(\cdot)$ be a w-evanescent solution of* (DS-2). *Then*

$$\|v'(t)\| = \|\nabla\psi(v(t))\|\,, \quad \textit{for all } t \ge 0\,. \tag{7}$$

Proof. It is easily seen that $I(t) := \left\|v'(t)\right\|^2 - 2V(v(t))$ is a first integral of the system (DS-2), that is, for some $k \in \mathbb{R}$ and all $t \geq 0$ it holds $\left\|u'(t)\right\|^2 = k + 2V(u(t))$. Taking limit inferior as $t \to +\infty$ we infer from (w-EV) that $k = 0$ and the result follows. □

Lemma 2.8 (range of orbits) *If* $\mathrm{Crit}_\psi = \emptyset$*, then the range* $\{v(t)\,;\, t \geq 0\}$ *of any w-evanescent solution* $v(\cdot)$ *of* (DS-2) *cannot be relatively compact.*

Proof. Let $v(\cdot)$ be a w-evanescent solution of (DS-2). If $\{v(t)\,;\, t \geq 0\}$ were relatively compact, then there would exist $z_0 \in K$ such that $V(z_0) = \min_{x \in K} V(x)$. Since $V(z_0) \geq 0$ we deduce by (w-EV) that $V(z_0) = 0$, which yields $\mathrm{Crit}_\psi \neq \emptyset$. □

The following proposition assembles properties of the evanescent solutions of (DS-2):

Proposition 2.9 (Properties of evanescent solutions) *Let* $v(\cdot)$ *be a w-evanescent solution of* (DS-2)*. Then:*

(i) $\lim_{t \longrightarrow +\infty} \psi(v(t)) \in \mathbb{R}$ *and*

$$|\psi(v(0)) - \psi(v(t))| \leq \int_0^t \left\|v'(s)\right\|^2 ds\,, \quad \textit{for all } t \geq 0\,. \tag{8}$$

(ii) *If* ψ *is coercive* (*i.e.,* $[\psi \leq \alpha]$ *is bounded, for all* $\alpha \in \psi(\mathcal{H})$)*, then* $v(\cdot)$ *is bounded.*

(iii) *If* $\left\|\nabla^2 \psi(v(\cdot)\right\|$ *is bounded, then* $\lim_{t\to+\infty} \left\|v'(t)\right\| = \lim_{t\to+\infty} V(v(t)) = 0.$

(iv) *The function*

$$\phi(t) := v'(t) + \sigma \nabla \psi(v(t)), \quad \sigma \in \{-1, 1\}$$

satisfies

$$\phi'(t) = \sigma \nabla^2 \psi(v(t)) \phi(t).$$

Proof. Set $r(t) := \psi(v(t))$, $t \geq 0$. Then $r'(t) = \langle v'(t) \mid \nabla \psi(u(t))\rangle$. By Cauchy-Schwarz inequality and Lemma 2.7 we get $|r'(t)| \leq \left\|v'(t)\right\| \|\nabla \psi(v(t))\| = \left\|v'(t)\right\|^2$. Using (EV) we conclude, after integration, that for every $t > 0$

$$|r(t) - r(0)| \leq \int_0^t \left\|v'(s)\right\|^2 ds \leq \int_0^{+\infty} \left\|v'(s)\right\|^2 ds < +\infty\,.$$

Therefore, the limit $\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} \psi(v(t))$ exists and (i) holds. We deduce easily that the range $\{r(t) : t \geq 0\}$ is bounded, yielding $v(t) \in [\psi \leq \eta]$, for some $\eta > 0$ and all $t \geq 0$. Therefore (ii) holds. Differentiating the function $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ and substituting $x = v(t)$ we deduce

$$\|\nabla V(v(t))\| \leq \left\|\nabla^2 \psi(v(t))\right\| \|\nabla \psi(v(t))\|\,. \tag{9}$$

On the other hand,

$$\left|\frac{d}{dt}\left[V(v(t))\right]\right| = \left|\langle \nabla V(v(t)) \mid v'(t)\rangle\right| \leq \|\nabla V(v(t))\| \left\|v'(t)\right\| \tag{10}$$

Combining (9) with (10) and recalling (7) and the definition of V we get

$$\left| \frac{d}{dt} [V(v(t))] \right| \leq 2 \left\| \nabla^2 \psi(v(t)) \right\| V(v(t)). \tag{11}$$

Since $v(\cdot)$ is evanescent, $V(v(\cdot)) \in L^1(0,+\infty)$, while $\left\| \nabla^2 \psi(v(\cdot)) \right\|$ is bounded by assumption. We deduce from (11) that $\frac{d}{dt}[V(v(\cdot))] \in L^1(0,+\infty)$. Therefore the limit $\lim_{t\to+\infty} V(v(t))$ exists (and necessarily equals zero, since $V(v(\cdot)) \in L^1(0,+\infty)$). Thus (iii) holds. Finally, (iv) follows from direct calculation, using (DS-2) and (1). □

The following proposition will be used in the sequel.

Proposition 2.10 (Further asymptotic properties of evanescent solutions) *Let $v(\cdot)$ be an evanescent solution of* (DS-2) *where V is given by* (1)*. Then*

$$\frac{||v(t) - v(0)||}{t}, \ \frac{\|v(t)\|}{\sqrt{t^2+1}} \in L^2(0,+\infty) \ ; \quad \lim_{t\to+\infty} \frac{\|v(t)\|}{\sqrt{t}} = 0 \tag{12}$$

and for every $t \geq 0$ it holds

$$\int_0^t \frac{\|v(t) - v(0)\|^2}{t^2} \, dt \ \leq \ 4 \int_0^t \|v'(t)\|^2 \, dt \, . \tag{13}$$

Proof. (i) Set $w(t) = v(t) - v(0)$, $t \geq 0$ (therefore $\lim_{t\to 0^+} \frac{w(t)}{t} = v'(0)$). Integrating by parts and using Cauchy-Schwarz inequality we obtain for every $t > 0$

$$\int_0^t \frac{\|w(s)\|^2}{s^2} \, ds = -\frac{\|w(t)\|^2}{t} + 2 \int_0^t \frac{\langle w(s) \mid w'(s) \rangle}{s} \, ds \, \leq \, 2 \int_0^t \frac{\langle w(s) \mid w'(s) \rangle}{s} \, ds$$
$$\leq 2 \left(\int_0^t \frac{\|w(s)\|^2}{s^2} \, ds \right)^{1/2} \left(\int_0^t \left\| w'(s) \right\|^2 \, ds \right)^{1/2},$$

yielding

$$\int_0^t \frac{\|w(s)\|^2}{s^2} \, ds \leq 4 \int_0^t \left\| w'(s) \right\|^2 \, ds = 4 \int_0^t \left\| v'(s) \right\|^2 \, ds \, .$$

Therefore (13) follows. In particular, since $v(\cdot)$ is evanescent solution, we conclude that $(t^{-1} \|w(t)\| \in L^2(0,+\infty)$ (hence a fortiori, $(t^2+1)^{-1/2} \|w(t)\| \in L^2(0,+\infty)$). Since $(t^2+1)^{-1/2} \in L^2(0,+\infty)$, we deduce easily that $(t^2+1)^{-1/2} \|v(t)\| \in L^2(0,+\infty)$.

(ii) Fix $t_0 > 0$. Then for all $t > t_0$ we have

$$\int_{t_0}^t \frac{\|v(s)\|^2}{s^2} \, ds = -\frac{\|v(t)\|^2}{t} + \frac{\|v(t_0)\|^2}{t_0} + 2 \int_{t_0}^t \frac{\langle v(s) \mid v'(s) \rangle}{s} \, ds.$$

Both integrals in the above expression converge as $t \to +\infty$, yielding that $\lim_{t\to+\infty} \frac{\|v(t)\|^2}{t}$ also exists. This limit is necessarily zero since $t^{-1} \|v(t)\| \in L^2(t_0,+\infty)$. □

2.3 Comparison of solutions of (DS-1) and (DS-2).

We now focus attention upon comparison between solutions the first order system (DS-1) and (weakly) evanescent solutions of the second order gradient system (DS-2), where $\psi \in \mathscr{C}^2(\mathcal{H})$ and V is given by (1).

The following result states that each solution $u(\cdot)$ of (DS-1) is also an evanescent solution of (DS-2) unless $\lim_{t\to+\infty} \psi(u(t)) = -\infty$. As underlined in the introduction, the inverse is more complicated: in general, evanescent solutions of (DS-2) are not necessarily solutions of (DS-1). Surprisingly, under a convexity assumption on either ψ or V, evanescent solutions of (DS-2) are also solutions of (DS-1).

Lemma 2.11 (Characterization of w-evanescent/evanescent solutions) *Let $u(\cdot)$ be a global solution of* (DS-1). *Then,*

(i) *u is a w-evanescent solution of* (DS-2) *if and only if*

$$\inf_{t\geq 0} \|\nabla\psi(u(t))\| = \inf_{z\in\mathcal{H}} \|\nabla\psi(z)\| = 0. \tag{14}$$

(ii) *u is a evanescent solution of* (DS-2) *if and only if*

$$\inf_{t\geq 0} \psi(u(t)) > -\infty. \tag{15}$$

Proof. Let $u(\cdot)$ be a global solution of (DS-1). This is obviously also a global solution of (DS-2) and satisfies $\left\|u'(t)\right\|^2 = 2V(u(t))$. Let us first assume that (14) holds. If $\nabla\psi(u(0)) = 0$, then $u(t) = u(0)$ for all $t \geq 0$ and $u(\cdot)$ is trivially w-evanescent. If $\nabla\psi(u(0)) \neq 0$, then $V(u(t)) = \frac{1}{2}\|\nabla\psi(u(t))\|^2 \neq 0$ for all $t \geq 0$ (*c.f.* Remark 2.2), hence for every $s \geq 0$ it holds

$$\inf_{t\geq 0} V(u(t)) = \inf_{t\geq s} V(u(t)) = 0\,,$$

yielding again that $u(\cdot)$ is a w-evanescent solution of (DS-2). The converse is obvious, hence (i) is established.

Assertion (ii) follows directly from Lemma 2.1 (ii). □

Combining Lemma 2.3 and Lemma 2.11 we deduce the following corollary.

Corollary 2.12 *Any bounded maximal solution of* (DS-1) *is an evanescent solution of* (DS-2).

Finally, combining Corollary 2.4 with Lemma 2.11 we obtain the following result.

Proposition 2.13 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded from below and $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$. For every $x_0 \in \mathcal{H}$,* (DS-2) *has at least one evanescent solution satisfying $v(0) = x_0$, which coincides with the unique global solution of the first order gradient system* (DS-1).

3 Main results

This section contains the main results of the manuscript, which are presented in three subsections. Before we proceed, let us first recall the following continuous form of the classical Opial's lemma [20] that will be used in the sequel. (See also [3, Lemma 17.2.5 (p. 704)] for a proof.)

Lemma 3.1 (Opial type lemma) *Let S be a nonempty subset of a Hilbert space H and $w : [0, +\infty) \to \mathcal{H}$ be a map. Assume that for every $z \in S$, the limit $\lim_{t \longrightarrow +\infty} \|w(t) - z\|$ exists and is finite and that all weak sequential limits of $w(\cdot)$, as $t \to +\infty$ belong to S. Then $w(t)$ converges weakly to a point of S as $t \to +\infty$.*

3.1 The case ψ convex.

Throughout this subsection we shall assume that the function $\psi \in \mathscr{C}^2(\mathcal{H})$ is convex and V is given by (1). We shall be interested in comparing the solutions of (DS-1) and (DS-2). The following result is essentially known (see for instance [9, Thm 3.1-3.2] for a proof in a more general context of multivalued evolution equations). It is recalled (together with a short proof) for completeness.

Proposition 3.2 (Lyapunov functions for (DS-1)**)** *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex. Then for every initial condition $x_0 \in \mathcal{H}$, the unique maximal solution $u(\cdot)$ of* (DS-1) *satisfying* (I_0) *is global. Moreover,*

(i) *$\rho(t) = \psi(u(t))$ is convex, nonincreasing and*

$$\inf_{t \geq 0} \psi(u(t)) = \lim_{t \to +\infty} \psi(u(t)) = \inf_{z \in \mathcal{H}} \psi(z). \tag{16}$$

(ii) *For every $y \in \mathcal{H}$ and $t > 0$ it holds*

$$\left\|u'(t)\right\| \leq \|\nabla\psi(y)\| + \frac{1}{t}\,\|u(0) - y\|.$$

(iii) *$t \mapsto \left\|u'(t)\right\| = \|\nabla\psi(u(t))\|$ is nonincreasing and*

$$\lim_{t \to +\infty} \left\|u'(t)\right\| = \inf_{z \in \mathcal{H}} \|\nabla\psi(z)\|\,. \tag{17}$$

(iv) *$\|u(\cdot) - \hat{x}\|$ is nonincreasing, for every $\hat{x} \in \mathrm{Crit}_{\psi}$.*

Proof. By Lemma 2.1 (i), $\rho(\cdot)$ is nonincreasing. Differentiating twice, evoking (DS-1) and the positive semi-definiteness of the Hessian $\nabla^2\psi(x)$ for all $x \in \mathcal{H}$ we deduce

$$\forall t \geq 0,\;\; \rho''(t) \;=\; -\frac{d}{dt}\left(\left\|u'(t)\right\|^2\right) = \langle \nabla^2\psi(u(t))\nabla\psi(u(t)) \mid \nabla\psi(u(t))\rangle \;\geq\; 0 \tag{18}$$

yielding convexity of ρ. Assume now that $u(\cdot)$ is a maximal solution of (DS-1) which is not global, that is, it is defined on $[0, T_{\max})$ with $T_{\max} < +\infty$. It follows from (18) that $\sup_{0 \leq t < T_{\max}} \left\|u'(t)\right\| < +\infty$, whence $u(\cdot)$ is Lipschitz continuous on $[0, T_{\max})$. Thus, by the Lipschitz-Cauchy theorem, the (maximal) solution $u(\cdot)$ admits a right extension on an interval of the form $[0, T_{\max} + \varepsilon)$ for some $\varepsilon > 0$ which is a contradiction. Therefore

$T_{\max} = +\infty$ and $u(\cdot)$ is global. We now prove (16). Assume towards a contradiction that for some $y_\star \in \mathcal{H}$ we have

$$\alpha := \inf_{t\geq 0} \psi(u(t)) - \psi(y_\star) \,>\, 0. \tag{19}$$

Setting $K(t) = \frac{1}{2}\left\|u(t) - y_\star\right\|^2$ we deduce using convexity of ψ and (19) that

$$K'(t) = \langle u'(t) \mid u(t) - y_\star\rangle = \langle \nabla\psi(u(t)) \mid y_\star - u(t)\rangle \leq \psi(y_\star) - \psi(u(t)) \leq -\alpha < 0\,,$$

yielding $\lim_{t\to+\infty} K(t) = -\infty$, a contradiction. Thus (i) holds. Assertion (ii) is [9, Theorem 3.2], while (iii) follows directly from (DS-1), convexity of $\rho(\cdot)$ and (ii). To establish (iv) it suffices to prove that the function $h(t) = \dfrac{1}{2}\left\|u(t) - \hat{x}\right\|^2$ has nonpositive derivative, for all $t \geq 0$. This can be proved by a straightforward calculation, evoking convexity of ψ and the fact that $\hat{x} \in \mathrm{Crit}_\psi$. □

Remark 3.3 Under the assumptions of Proposition 3.2, for $y \in \mathcal{H}$, we set

$$E_y(t) := \frac{1}{2}\left\|u(t) - y\right\|^2 + \int_0^t \left(\psi(u(s)) - \psi(y)\right) ds \quad \text{(energy function)}$$

By convexity of ψ we deduce

$$E_y'(t) = \langle \nabla\psi(u(t)) \mid y - u(t)\rangle + \psi(u(t)) - \psi(y) \leq 0,$$

that is, $E_y(\cdot)$ is nonincreasing on $[0, +\infty)$. □

The following proposition relates with the critical points of ψ.

Proposition 3.4 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex and $u(\cdot)$ a global solution of* (DS-1).

(i) *If $\mathrm{Crit}_\psi \neq \emptyset$, then $\lim_{t\to+\infty} \left\|u'(t)\right\| = 0$ and there exists $\hat{x}_\star \in \mathrm{Crit}_\psi$ such that $u(t) \underset{t\to+\infty}{\rightharpoonup} \hat{x}_\star$ (weakly). Moreover, $\rho_*(t) := \psi(u(t)) - \psi(\hat{x}_\star) \in L^1(0, +\infty)$ and*

$$\int_0^{+\infty} \left(\psi(u(s)) - \psi(\hat{x}_\star)\right) ds \leq \frac{1}{2}\|u(0) - \hat{x}_\star\|^2. \tag{20}$$

(ii) *If $\mathrm{Crit}_\psi = \emptyset$, then $\lim_{t\to+\infty} \|u(t)\| = +\infty$.*

(iii) *$u(\cdot)$ is bounded if and only if $\mathrm{Crit}_\psi \neq \emptyset$.*

Proof. (i). Let us assume $\mathrm{Crit}_\psi \neq \emptyset$. It follows that $\inf_{z\in H} \|\nabla\psi(z)\| = 0$ and by (17) $\lim_{t\to+\infty} \left\|u'(t)\right\| = 0$. To prove the weak convergence of $u(\cdot)$ to some $\hat{x}_\star \in \mathrm{Crit}_\psi$, we shall use Lemma 3.1. To this end, we set $S = \mathrm{Crit}_\psi$ and recall that for every $\hat{x} \in \mathrm{Crit}_\psi$, the function $\|u(\cdot) - \hat{x}\|$ is positive and nonincreasing (Proposition 3.2 (iv)). Therefore, it has a finite limit. To conclude, we need to ensure that for every $y \in \mathcal{H}$ such that $u(t_n) \underset{n\to+\infty}{\rightharpoonup} y$ (weakly) as $t_n \to +\infty$, it holds $y \in \mathrm{Crit}_\psi$. Since the convex function ψ is weakly lower semicontinuous (see [22, Ch. 3] *e.g.*), we get

$$\psi(y) \leq \liminf_{n\to+\infty} \psi(u(t_n)) = \lim_{n\to+\infty} \psi(u(t_n)) = \inf_{z\in\mathcal{H}} \psi(z),$$

yielding that y is a global minimizer of ψ. Therefore $y \in \mathrm{Crit}_\psi$ and the assertion follows.

Let us prove inequality (20). Let $y \in \mathrm{Crit}_\psi$. Since the function

$$E_y(t) = \frac{1}{2}\|u(t) - y\|^2 + \int_0^t (\psi(u(s)) - \psi(y))\, ds$$

is nonincreasing (*c.f.* Remark 3.3) we get

$$\int_0^t (\psi(u(s)) - \psi(y))\, ds \leq \frac{1}{2}\|u(0)\|^2 - \frac{1}{2}\|u(t)\|^2 - \langle u(0) - u(t) \mid y\rangle.$$

Taking the limit superior as $t \to +\infty$ yields

$$\int_0^{+\infty} (\psi(u(s)) - \psi(y))\, ds \leq \frac{1}{2}\|u(0)\|^2 - \frac{1}{2}\|\hat{x}_\star\|^2 - \langle u(0) - \hat{x}_\star \mid y\rangle \tag{21}$$

since $\liminf_{t \longrightarrow +\infty} \|u(t)\| \geq \|\hat{x}_\star\|$ and $\lim_{t\to+\infty} \langle u(0) - u(t) \mid y\rangle = \langle u(0) - \hat{x}_\star \mid y\rangle$. The result follows by taking $y = \hat{x}_\star$.

(ii). The function ψ, being weakly lower semicontinuous, attains its minimum on the (weakly compact) closed ball $\bar{B}_R$ of radius $R > 0$ centered at the origin. Since ψ does not have any global minimizer, we deduce

$$\min_{x \in \bar{B}_R} \psi(x) > \inf_{x \in \mathcal{H}} \psi(x).$$

The result follows from the fact that $\lim_{t\to+\infty} \psi(u(t)) = \inf_{x\in\mathcal{H}} \psi(x)$.

(iii). This is a consequence of the two last assertions. □

Remark 3.5 Inequality (20) can be slightly improved. More precisely, we have

$$\int_0^{+\infty} (\psi(u(s)) - \psi(\hat{x}_\star))\, ds \leq \frac{1}{2}\|v\|^2 + \langle v \mid \hat{x}_\star\rangle - \sigma_{\mathrm{Crit}_\psi}(v), \text{ with } v = u(0) - \hat{x}_\star. \tag{22}$$

where $\sigma_{\mathrm{Crit}_\psi}$ denotes the support function of the convex set Crit_ψ. This inequality can be deduce from inequality (21) given hereafter by taking the infimum on $y \in \mathrm{Crit}_\psi$. □

The following result will play a key role in the sequel.

Proposition 3.6 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex and V as in* (1). *Then any w-evanescent solution of* (DS-2) *is also a (global) solution of the gradient system* (DS-1).

Proof. Let $v(\cdot)$ be a w-evanescent solution of (DS-2) and set $\phi(t) = v'(t) + \nabla\psi(v(t))$. Then for all $t \geq 0$, it holds $\|\phi(t)\| \leq \|v'(t)\| + \|\nabla\psi(v(t))\|$. By Lemma 2.7 we deduce

$$\|\phi(t)\| \leq 2\|\nabla\psi(v(t))\| = V(v(t)),$$

whence $\liminf_{t\to+\infty} \|\phi(t)\| = 0$, since $v(\cdot)$ is a w-evanescent solution. We also know that

$$\phi'(t) = \nabla^2\psi(v(t))\phi(t)$$

(see Proposition 2.9 (iv)). Thus,

$$\frac{d}{dt}\left(\|\phi(t)\|^2\right) = 2\langle\phi(t) \mid \nabla^2\psi(v(t))\phi(t)\rangle \geq 0,$$

since ψ is convex. Hence $\|\phi\|^2$ is increasing. Therefore, since $\liminf_{t\to+\infty} \|\phi(t)\| = 0$ we deduce $\phi = 0$, which yields that $v(\cdot)$ is solution of the first order gradient system (DS-1). □

We are now ready to state our main results.

Theorem 3.7 (second-order gradient system; ψ convex) *If $\psi \in \mathscr{C}^2(\mathcal{H})$ is convex,* (DS-2) *has a w-evanescent solution $v(\cdot)$ satisfying* (I_0) *if and only if (C) holds. Then $v(\cdot)$ is unique and is also the unique solution of the first order system* (DS-1) *that satisfies* (I_0)*. Moreover,*

(i) *v is evanescent if and only if ψ is bounded below*

(ii) *v is bounded if and only if* $\mathrm{Crit}_\psi \neq \emptyset$*.*

Proof. As already mentioned in the introduction, condition (C) is necessary for the existence of a w-evanescent solution of (DS-2). Conversely, suppose that (C) is fulfilled. Then there exists a unique global solution $u(\cdot$ of (DS-1) satisfying $u(0) = u_0 \in \mathcal{H}$ (*c.f.* Proposition 3.2). Condition (14) is fulfilled, thanks to (17) and (C). Thus, in view of Lemma 2.11, $u(\cdot)$ is a w-evanescent solution of (DS-2) satisfying (I_0). Uniqueness is straightforward from Proposition 3.6. Indeed, any w-evanescent solution of (DS-2) which satisfies (I_0) is necessarily the unique global solution of (DS-1) under the same initial condition (I_0). Finally, combining (15) with (16) we deduce that this solution is evanescent if and only if ψ is bounded below. From Proposition 3.2, we also deduce that this solution is bounded if and only if $\mathrm{Crit}_\psi \neq \emptyset$. □

To illustrate Theorem 3.7 consider the convex $\mathscr{C}^2$ function ψ given in (2). Recall that ψ satisfies (C) but not $(C^\star)$. The first-order system $u'(t) = -\psi'(u(t))$, $u(0) = 0$ has the unique solution $u(t) = 1 - \sqrt{1+2t}$, $t \geq 0$, which is also the unique w-evanescent solution of (DS-2) (*c.f.* Theorem 3.7). Clearly this solution is not evanescent (ψ is not bounded from below).

An immediate consequence of Theorem 3.7 and Proposition 3.2 is the following result.

Corollary 3.8 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex, assume (C) holds and let $v(\cdot)$ be a w-evanescent solution of* (DS-2)*. Then $v(\cdot)$ satisfies the properties stated in Proposition 3.2 and Proposition 3.4.*

We are ready to state the following surprising consequence.

Theorem 3.9 (determination via modulus of gradient) *Let $\psi_1, \psi_2 \in C^2(\mathcal{H})$ be convex and assume*

– $\|\nabla\psi_1(z)\| = \|\nabla\psi_2(z)\|$ *for all $z \in \mathcal{H}$;*

– $\inf_{z\in\mathcal{H}} \|\nabla\psi_1(z)\| = 0$ *(this assumption holds whenever ψ_1 or ψ_2 is bounded below).*

Then, $\psi_1 = \psi_2 + c$ for some constant $c \in \mathbb{R}$.

Proof. Let ψ_1 and ψ_2 be two convex functions of class $\mathscr{C}^2$ satisfying $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ and $\inf_{z\in\mathcal{H}} \|\nabla\psi_1(z)\| = 0$. Let $x \in \mathcal{H}$ an arbitrary point and let $v(\cdot)$ be the unique evanescent solution of the system

$$v''(t) = \nabla V(v(t)) \text{ for } t \geq 0,\ v(0) = x,$$

with $V = \frac{1}{2}\|\nabla\psi_1\|^2 = \frac{1}{2}\|\nabla\psi_2\|^2$ (*c.f.* Theorem 3.7). Then $v(\cdot)$ is also solution of the first order systems

$$v'(t) = -\nabla\psi_1(v(t)), \quad v(0) = x\,,$$

and

$$v'(t) = -\nabla\psi_2(v(t)), \quad v(0) = x.$$

Hence $\nabla\psi_1(x) = \nabla\psi_2(x)$. Since x is arbitrary, the result follows. □

Remark 3.10 In [6] it has been shown that a continuous (respectively, smooth) convex 1-coercive function can be determined (up to a constant) by knowing its subgradients (respectively, gradients) in specific points of its domain (namely, the ones that correspond to strongly exposed points of the epigraph). Theorem 3.9 asserts that a knowledge of the modulus of the gradient (rather than the gradient itself) suffices to determine a $\mathscr{C}^2$ convex function, provided it is bounded from below. This latter assumption is important, if we think of the example of the functions $\psi_1(x) = x$ and $\psi_2(x) = -x$. Still the result is rather unexpected.

A direct consequence of Theorem 3.9 is the following.

Corollary 3.11 (Eikonal equation - I) *Let $f \in \mathscr{C}^1(\mathcal{H})$ be nonnegative. Then, the eikonal equation*

$$\|\nabla\psi\|^2 = f, \tag{23}$$

has at most one (up to a constant) convex, bounded below solution in $\mathscr{C}^2(\mathcal{H})$.

3.2 The case V convex.

In this subsection the driving assumption is the convexity of the function $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$, where $\psi \in \mathscr{C}^2(\mathcal{H})$. The focus is again the comparison of the solutions of the systems (DS-1) and (DS-2).

The following result reveals a characteristic property of the solutions of (DS-2), which is reminiscent to an analogous property for the orbits of the first order system with convex potential.

Proposition 3.12 (Contraction of solutions of (DS-2)**)** *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ and assume that $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ is convex. If v_1 and v_2 are two evanescent solutions of equation* (DS-2)*, then the function*

$$q(t) := \|v_1(t) - v_2(t)\|^2$$

is convex and nonincreasing on $[0, +\infty)$. In particular if $v_1(0) = v_2(0)$, then $v_1 = v_2$.

Proof. It suffices to prove that q is convex and nonincreasing. Differentiating twice and evoking monotonicity of ∇V (see [22, Ch. 2] *e.g.*) we get

$$q''(t) = \langle v_1''(t) - v_2''(t) \mid v_1(t) - v_2(t)\rangle + \left\|v_1'(t) - v_2'(t)\right\|^2$$

$$= \langle \nabla V(v_1(t)) - \nabla V(v_2(t)) \mid v_1(t) - v_2(t)\rangle + \left\|v_1'(t) - v_2'(t)\right\|^2 \geq 0,$$

which yields convexity of q. Let us prove that q is decreasing. By Proposition 2.10, we have

$$\int_0^\infty \frac{q(t)}{t^2+1} dt = \frac{1}{2}\int_0^\infty \frac{\|v_2(t) - v_1(t)\|^2}{t^2+1} dt < +\infty.$$

Suppose that there exists $t_0 > 0$ such that $q'(t_0) > 0$. Since q is convex, we would have

$$q(t) \geq q'(t_0)(t - t_0) + q(t_0), \quad \text{for all } t \geq t_0\,,$$

yielding

$$\int_0^\infty \frac{q(t)}{t^2+1} dt = +\infty\,, \quad \text{a contradiction.}$$

Hence, q is decreasing and the result follows. □

Lemma 3.13 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$, assume that $V(\cdot)$ given by* (1) *is convex and let $v(\cdot)$ be an evanescent solution of* (DS-2)*. If $\mathrm{Crit}_\psi \neq \emptyset$, then*

(i) *$h(t) := \|v(t) - \hat{x}\|$ is nonincreasing, for every $\hat{x} \in \mathrm{Crit}_\psi$;*
(ii) *$v(\cdot)$ is bounded;*
(iii) *There exists $\hat{x}_\star \in \mathrm{Crit}_\psi$ such that $v(t) \underset{t\to+\infty}{\rightharpoonup} \hat{x}_\star$ (weakly).*

Proof. Let $v(\cdot)$ be a evanescent solution of (DS-2) and $\hat{x} \in \mathrm{Crit}_\psi$. Applying Proposition 3.12 for $u_1(t) = v(t)$ and $u_2(t) = \hat{x}$ for $t \geq 0$, we get (i). Since $\mathrm{Crit}_\psi \neq \emptyset$, (ii) is a direct consequence of (i), while (iii) can be proved in a similar way as in Proposition 3.4, using Lemma 3.1 and convexity of V. The details are left to the reader. □

Proposition 3.14 (second-order gradient system; V convex) *Let $V(\cdot)$ be convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded from below. Then* (DS-2) *has a unique evanescent solution satisfying* (I_0)*, which is also the unique solution of* (DS-1) *that satisfies* (I_0).

Proof. From Corollary 2.4 and Cauchy-Lipschitz there exists a unique global solution of (DS-1) satisfying the initial condition (I_0). According to Lemma 2.11 this solution is also an evanescent solution of (DS-2). Uniqueness follows from Proposition 3.12. □

We obtain the following consequence.

Corollary 3.15 (convexity criterium) *Let $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ be convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded below. Then, ψ is convex.*

Proof. Fix $z_1, z_2 \in \mathcal{H}$ and denote by $u_1(\cdot)$ and $u_2(\cdot)$ solutions of (DS-1) with z_1 and z_2 as initial data. Since u_1 and u_2 are also evanescent solutions of (DS-2), we know that the function

$$q(t) = \frac{1}{2} \|u_1(t) - u_2(t)\|^2 , \quad \text{for } t \geq 0$$

is decreasing (*c.f.* Proposition 3.12). Thus,

$$0 \geq q'(t) = -\langle \nabla\psi(u_1(t)) - \nabla\psi(u_1(t)) \mid u_1(t) - u_2(t)\rangle,$$

or equivalently,

$$\langle \nabla\psi(u_1(t)) - \nabla\psi(u_2(t)) \mid u_1(t) - u_2(t)\rangle \geq 0\,.$$

Taking the limit $t \to 0$ we deduce that

$$\langle \nabla\psi(z_1) - \nabla\psi(z_2) \mid z_1 - z_2\rangle \geq 0,$$

which yields that ψ is convex (see [22, Ch. 2] *e.g.*). □

Remark 3.16 Corollary 3.15 is false if ψ is not supposed bounded below: Indeed, let $\psi(x) = x^3$. Then $V(x) = \frac{1}{2}|\psi'(x)|^2 = \frac{9}{2}x^4$ is convex, but ψ is not. Another two dimensional example is $\psi(x_1, x_2) = x_1^4 - x_2^2$.

Remark 3.17 If $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ is convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ is bounded from below, then combining Corollary 3.15 with Theorem 3.7 we deduce that every evanescent solution $u(\cdot)$ of (DS-2) satisfies the assertions of Corollary 3.8 (since ψ is convex). In particular, $u(\cdot)$ is bounded if and only if $\mathrm{Crit}_\psi \neq \emptyset$.

The following result is a direct consequence of Theorem 3.9 and Corollary 3.15.

Corollary 3.18 *Let $\psi_1, \psi_2 \in \mathscr{C}^2(\mathcal{H})$ be bounded below satisfying*

$$\|\nabla\psi_1(x)\| = \|\nabla\psi_2(x)\| , \quad \textit{for all } x \in \mathcal{H}.$$

Then, if $V(x) = \frac{1}{2}\|\nabla\psi_1(x)\|^2 \ \big(= \frac{1}{2}\|\nabla\psi_2(x)\|^2\big)$ is convex, we deduce that both ψ_1 and ψ_2 are convex and equal (up to a constant).

Let us illustrate Corollary 3.18 for the case where ψ_1, ψ_2 are the quadratic forms

$$\psi_1(x) = \frac{1}{2}\langle x \mid A_1 x\rangle, \quad \text{and} \quad \psi_2(x) = \frac{1}{2}\langle x \mid A_2 x\rangle,$$

where A_i is a symmetric linear bounded operator, for $i \in \{1, 2\}$. One can easily verify that ψ_i is bounded below if and only if A_i is positive semidefinite. In the latter case identity $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ means that $\|A_1 x\| = \|A_2 x\|$ for all $x \in \mathcal{H}$, yielding $A_1^2 = A_2^2$. Thus, $A_1 = A_2$ (since A_1 and A_2 are positive semidefinite) and $\psi_1 = \psi_2$. This is in accordance with Corollary 3.18. This example also shows the importance of the assumption that ψ_1 and ψ_2 are bounded below. Indeed, if $A_2 = -A_1 \neq 0$, then $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ and $\psi_1 - \psi_2$ is not constant.

A direct consequence of Corollary 3.18 is the following result.

Corollary 3.19 (Eikonal equation - II) *Let $f \in \mathscr{C}^1(\mathcal{H})$ be nonnegative and convex. Then, the eikonal equation*

$$\|\nabla\psi\|^2 = f, \tag{24}$$

has at most one bounded below solution in $\mathscr{C}^2(\mathcal{H})$ up to an additive constant. In addition, this solution is convex.

3.3 An alternative variational principle for (DS-1)

In [10]–[11], Brézis and Ekeland proved the following variational characterization when ψ is a proper, convex and lower semicontinuous functional defined on a Hilbert space $\mathcal{H}$. In this case (DS-1) becomes

$$u'(t) \underset{a.e}{\in} -\partial f(u(t)), \quad t \geq 0.$$

If $u(\cdot)$ is an absolutely continuous solution of the above differential inclusion on $[0,T]$ for some $T > 0$, with initial condition (I_0), then $u(\cdot)$ is the unique minimizer of the functional

$$\mathscr{J}(u) = \int_0^T \big(\psi(u(t)) + \psi^\star(-u'(t))\big)\, dt + \frac{1}{2}\,\|u(T)\|^2,$$

where $\psi^\star$ designates the Legendre conjugate of ψ. We also refer to [5] and [17] for extensions of this variational principle.

We now present an alternative variational principle for the first order gradient system (DS-1). The formulation is based on the connection with the second order system (DS-2). This latter can be seen as the Euler-Lagrange equation associated to a conventional functional. More precisely, for any real number $T > 0$, we consider the functional

$$J(T;w) = \int_0^T \left(\frac{1}{2}\left\|w'(t)\right\|^2 + \frac{1}{2}\|\nabla\psi(w(t))\|^2\right) dt + \psi(w(T)).$$

We state the following

Proposition 3.20 (Variational formulation) *Let $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ be convex, $\psi \in \mathscr{C}^2(\mathcal{H})$ bounded below and $T > 0$. Then $u \in \mathscr{C}^0([0,T];\mathcal{H}) \cap \mathscr{C}^1((0,T);\mathcal{H})$ is a solution of* (DS-1) *on* $[0,T]$ *if and only if*

$$J(T;u) \leq J(T;w), \tag{25}$$

for all $w \in \mathscr{C}^0([0,T];\mathcal{H}) \cap \mathscr{C}^1((0,T);\mathcal{H})$ satisfying $w(0) = u(0)$.

Proof. In view of Corollary 3.15, ψ is convex. Let $u(\cdot)$ solution of (DS-1) on $[0,T]$ and $w \in C^1([0,T],\mathcal{H})$ such that $w(0) = u(0)$. Set $h = w - u$. Then,

$$J(T;w) - J(T;u) = \int_0^T \left(\langle u'(t) \mid h'(t)\rangle + \frac{1}{2}\left\|h'(t)\right\|^2 + V(u(t)+h(t)) - V(u(t))\right) dt$$
$$+ \quad \psi(u(T)+h(T)) - \psi(u(T)).$$

Using convexity of ψ and V we deduce

$$J(T;w) - J(T;u) \geq \int_0^T \big(\langle u'(t) \mid h'(t)\rangle + \langle \nabla V(u(t)) \mid h(t)\rangle\big)\, dt$$
$$+\langle \nabla\psi(u(T)) \mid (h(T)\rangle,$$

Integrating by parts yields

$$J(T;w) - J(T;u) \quad \geq$$
$$\int_0^T \langle -u''(t) + \nabla V(u(t)) \mid h(t)\rangle\, dt + \langle u'(T) + \nabla\psi(u(T)) \mid h(T)\rangle.$$

Since $u(\cdot)$ is solution of (DS-1), it is also solution of (DS-2), therefore

$$\int_0^T \langle -u''(t) + \nabla V(u(t)) \mid h(t)\rangle \, dt + \langle u'(T) + \nabla\psi(u(T)) \mid h(T)\rangle = 0\,,$$

yielding $J(T;w) \geq J(T;u)$.

Conversely, let $u \in C^1([0,T],\mathcal{H})$. Assume that $J(T;w) \geq J(T;u)$ for all $w \in C^1([0,T],\mathcal{H})$ such that $w(0) = u(0)$. By a conventional argument, we know that u is of class $\mathscr{C}^2$. Moreover, u satisfies the Euler-Lagrange equation $u''(t) = \nabla V(u(t))$ and the *transversality* condition $u'(T) + \nabla\psi(u(T)) = 0$. Set $\phi(t) = u'(t) + \nabla\psi(u(t))$ for $t \geq 0$. Therefore ϕ is a solution of the linear differential equation $\phi'(t) = \nabla^2\psi(u(t))\phi(t)$ (see Proposition 2.9) with $\phi(T) = 0$. We infer that ϕ is the trivial solution $\phi = 0$, which yields that u is solution of (DS-1) on $[0,T]$. This ends the proof. □

We now consider the functional

$$J^\star_\infty(w) = \int_0^{+\infty} \left(\frac{1}{2}\left\|w'(t)\right\|^2 + \frac{1}{2}\|\nabla\psi(w(t))\|^2\right) dt.$$

We are ready to state the main result of this subsection.

Corollary 3.21 *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded below,* $\mathrm{Crit}_\psi \neq \emptyset$ *and $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ be convex. Then, $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ is a global solution of* (DS-1) *if and only if*

$$J^\star_\infty(u) \leq J^\star_\infty(w), \tag{26}$$

for any bounded function $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ with $w(0) = u(0)$.

Proof. Let us first assume that $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ is a global solution of (DS-1). (In particular this yields that u is an evanescent solution of (DS-2) (see Corollary 2.12) and $J^\star_\infty(u) < +\infty$.) We now define for $z \in \mathscr{C}^1([0,T];\mathcal{H})$ and $T > 0$

$$J^\star(T;z) := \int_0^T \left(\frac{1}{2}\left\|z'(t)\right\|^2 + \frac{1}{2}\|\nabla\psi(z(t))\|^2\right) dt.$$

Let $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ be a bounded function satisfying $w(0) = u(0)$ and let us set $h = w - u$. Following the proof of inequality (25), we deduce

$$J^\star(T;w) \geq J^\star(T;u) + \langle u'(T) \mid h(T)\rangle. \tag{27}$$

Notice that u is bounded and $\lim_{T\to+\infty}\left\|u'(T)\right\| = 0$ (Proposition 3.4). Thus $h = w-u$ is also bounded. Taking the limit as $T \to +\infty$ yields

$$J^\star_\infty(w) \geq J^\star_\infty(u). \tag{28}$$

Conversely, suppose that $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ satisfying (28) for any bounded function $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ with $w(0) = u(0)$. Let $\hat{x} \in \mathrm{Crit}_\psi \neq \emptyset$ and consider the function

$$w_0(t) = e^{-t}(u(0) - \hat{x}) + \hat{x}.$$

Set $R = \|u(0) - \hat{x}\|$. Obviously, $w_0(t) \in B(\hat{x},R)$ for all $t \geq 0$ and

$$V(w_0(t)) = V(w_0(t)) - V(\hat{x}) \leq \sup_{B(\hat{x},R)}\|\nabla V\|\,\|w(t) - \hat{x}\| = \sup_{B(\hat{x},R)}\|\nabla V\|\,\|w(0) - \hat{x}\|\,e^{-t}.$$

It follows that $J^\star_\infty(w_0) < +\infty$. Thus, $J^\star_\infty(u) < +\infty$.

Consider now an arbitrary real number $T > 0$ and let $h \in \mathscr{C}^1([0, +\infty); \mathcal{H})$ having a compact support included in $[0, T]$. Then,

$$J^\star_\infty(u + h) - J^\star_\infty(u) = J^\star(T; u + h) - J^\star(T; u).$$

Thus,

$$J^\star(T, u + h) \geq J^\star(T, u).$$

From the latter we deduce that u satisfies Euler-Lagrange equation $u''(t) = \nabla V(u(t))$ on $(0, T)$. Since $T > 0$ is arbitrary, u is a global solution of (DS-2) on $[0, +\infty)$. Since $J^\star_\infty(u) < +\infty$, it is also an evanescent solution. In view of Proposition 3.14, u is also solution of (DS-1). □

Acknowledgment. This work has been completed during mutual visits of the first and the third author to the University of Chile and the University of Versailles SQY respectively. The authors wish to thank the host institutions for hospitality.

References

1. P.-A. Absil, R. Mahony, B. Andrews, Convergence of the iterates of descent methods for analytic cost functions, *SIAM J. Optim.* **16** (2005), 531–547.
2. F. Alvarez, J. Bolte, O. Brahic, Hessian Riemannian gradient flows in convex programming. *SIAM J. Control Optim.* **43** (2004), 477–501.
3. H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV spaces: Applications to PDEs and optimization*, MOS-SIAM Series on Optimization **17** (Second Edition), Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, Mathematical Optimization Society (2014).
4. H. Attouch, J. Peypouquet, P. Redont, A dynamical approach to an inertial forward-backward algorithm for convex minimization, *SIAM J. Optim.* **24** (2014), 232–256.
5. G. Auchmuty, Variational principles for finite dimensional initial value problems. In *Control methods in PDE-dynamical systems*, pp. 45–56, Contemp. Math., 426, Amer. Math. Soc., Providence, RI, 2007.
6. M. Bachir, A. Daniilidis, J.-P. Penot, Lower subdifferentiability and Integration, *Set-Valued Analysis* **10** (2002), 89–108.
7. J.-B. Baillon Un exemple concernant le comportement asymptotique de la solution du problème $du/dt + \partial\varphi(u) \ni 0$, *J. Funct. Anal.* **28** (1978), 369–376.
8. H. Bauschke, J. Bolte, M. Teboulle, A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications, *Math. Oper. Res.* **42** (2017), 330–348.
9. H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). American Elsevier Publishing Co., Inc., New York, 1973. vi+183 pp.
10. H. Brezis, I. Ekeland, Un principe variationnel associé à certaines équations paraboliques. Le cas dépendant du temps. (French) *C. R. Acad. Sci. Paris Sér.* A-B **282** (1976), no. 20, Ai, A1197–A1198.
11. H. Brezis, I. Ekeland, Un principe variationnel associé à certaines équations paraboliques. Le cas indépendant du temps. (French) *C. R. Acad. Sci. Paris Sér.* A-B **282** (1976), no. 17, Aii, A971–A974.
12. R. Chill, A. Haraux, M. Jendoubi, Applications of the Łojasiewicz-Simon gradient inequality to gradient-like evolution equations, *Anal. Appl. (Singap.)* **7** (2009), 351–372.
13. A. Daniilidis, O. Ley, S. Sabourau, Asymptotic behaviour of self-contracted planar curves and gradient orbits of convex functions, *J. Math. Pures Appl.* **94** (2010), 183–199.
14. A. Daniilidis, G. David, E. Durand, A. Lemenant, Rectifiability of self-contracted curves in the Euclidean space and applications, *J. Geom. Anal.* **25** (2015), 1211–1239.

15. I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.
16. A. Haraux, M. Jendoubi, The Łojasiewicz gradient inequality in the infinite-dimensional Hilbert space framework, *J. Funct. Anal.* **260** (2011), 2826–2842.
17. N. Ghoussoub, L. Tzou, A variational principle for gradient flows. *Math. Ann.* **330** (2004), no. 3, 519–549.
18. K. Kurdyka, On gradients of functions definable in o-minimal structures, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 769–783.
19. P. Manselli, C. Pucci, Maximum lenght of steepest curves for quasiconvex functions, *Geometriae Dedicata* **38** (1991), 211–227.
20. Z. Opial, Weal convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.
21. J. Palis, W. De Melo, *Geometric theory of dynamical systems. An introduction,* (Translated from the Portuguese by A. K. Manning), Springer-Verlag, New York-Berlin, 1982.
22. R. Phelps, *Convex functions, monotone operators and differentiability.* (2nd eds). Lecture Notes in Mathematics **1364**, Springer-Verlag, Berlin, 1993.
23. T. Rockafellar, *Convex analysis.* (Reprint of the 1970 original.) Princeton University Press, Princeton, NJ, 1997.
24. F. Sanz, Non-oscillating solutions of analytic gradient vector fields, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 1045–1067.