DYNAMICS OF A PLANAR COULOMB GAS

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Abstract. We study the long-time behavior of the dynamics of interacting planar Brownian particles, confined by an external field and subject to a singular pair repulsion. The invariant law is an exchangeable Boltzmann – Gibbs measure. For a special inverse temperature, it matches the Coulomb gas known as the complex Ginibre ensemble. The difficulty comes from the interaction which is not convex, in contrast with the case of one-dimensional log-gases associated with the Dyson Brownian Motion. Despite the fact that the invariant law is neither product nor log-concave, we show that the system is well-posed for any inverse temperature and that Poincaré inequalities are available. Moreover the second moment dynamics turns out to be a nice Cox – Ingersoll – Ross process, in which the dependency over the number of particles leads to identify two natural regimes related to the behavior of the noise and the speed of the dynamics.

Contents

1. Introduction and statement of the results

1.1. **The model and its well-posedness.** This work is concerned with the dynamics of $N \geq 2$ particles at positions $x_1, \ldots, x_N$ in $\mathbb{R}^d$, $d \geq 1$, confined by an external field and experiencing a singular pair repulsion. The configuration space that we are interested in is the open subset $D \subset (\mathbb{R}^d)^N$ defined by

$$D := (\mathbb{R}^d)^N \setminus \cup_{i \neq j}\{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N : x_i = x_j\} \tag{1.1}$$

where i, j run over $\{1, \ldots, N\}$. The boundary of D in the compactification of $(\mathbb{R}^d)^N$ is

$$\partial D := \{\infty\} \cup \cup_{i \neq j}\{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N : x_i = x_j\}.$$

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The vector $x = (x_1, \ldots, x_N) \in D$ encodes the position of the N particles, and the energy $H(x)$ of this configuration is modeled by

$$H(x) := \frac{1}{N}\sum_{i=1}^{N} V(x_i) + \frac{1}{2N^2}\sum_{1\leq i\neq j\leq N} W(x_i - x_j) =: H_V(x) + H_W(x). \tag{1.2}$$

Here, $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is an external *confinement potential* such that $V(z) \to +\infty$ as $z \to \infty$, and $W : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a *pair or two-body interaction potential* such that $W(z) = W(-z)$ and $W(z) \to +\infty$ as $z \to 0$ (singularity). Unless otherwise stated, we consider particles in $\mathbb{R}^2 \equiv \mathbb{C}$, with quadratic confinement and Coulomb repulsion, namely:

$$d = 2, \quad V(z) = |z|^2, \quad W(z) = \log\frac{1}{|z|^2}. \tag{1.3}$$

Here $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^2$ (modulus of the complex number z). With this notation, we study the system of N interacting particles in $\mathbb{R}^2$ modeled by a diffusion process $X^N = (X_t^N)_{t\geq 0}$ on D, solution of the stochastic differential equation

$$\mathrm{d}X_t^N = \sqrt{2\frac{\alpha_N}{\beta_N}}\mathrm{d}B_t^N - \alpha_N \nabla H(X_t^N)\,\mathrm{d}t, \tag{1.4}$$

for any choice of speed $\alpha_N > 0$ and inverse temperature $\beta_N > 0$; here $(B_t^N)_{t\geq 0}$ is a standard Brownian motion of $(\mathbb{R}^2)^N$. In other words, letting $X_t^N = (X_t^{i,N})_{1\leq i\leq N}$ and $B_t^N = (B_t^{i,N})_{1\leq i\leq N}$ denote the components of X_t^N and B_t^N,

$$\mathrm{d}X_t^{i,N} = \sqrt{2\frac{\alpha_N}{\beta_N}}\mathrm{d}B_t^{i,N} - \frac{\alpha_N}{N}\nabla V(X_t^{i,N})\,\mathrm{d}t - \frac{\alpha_N}{N^2}\sum_{j\neq i}\nabla W(X_t^{i,N} - X_t^{j,N})\,\mathrm{d}t, \quad 1 \leq i \leq N.$$

Since $V(z) = |z|^2$ and $W(z) = -2\log|z|$ we have more explicitly

$$\mathrm{d}X_t^{i,N} = \sqrt{2\frac{\alpha_N}{\beta_N}}\mathrm{d}B_t^{i,N} - 2\frac{\alpha_N}{N}X_t^{i,N}\,\mathrm{d}t - 2\frac{\alpha_N}{N^2}\sum_{j\neq i}\frac{X_t^{j,N} - X_t^{i,N}}{|X_t^{i,N} - X_t^{j,N}|^2}\,\mathrm{d}t, \quad 1 \leq i \leq N. \tag{1.5}$$

To lightweight the notations, we will very often drop the notation N in the superscript, writing in particular X_t, B_t, X_t^i, and B_t^i instead of X_t^N, B_t^N, $X_t^{i,N}$ and $B_t^{i,N}$ respectively. We shall see later that the cases $\beta_N = N$ and $\beta_N = N^2$ are particularly interesting, the latter being related to the complex Ginibre Ensemble in random matrix theory.

Global pathwise well posedness of a solution X to the stochastic differential equation (1.5) is not automatically granted since W is singular. Nevertheless, the set D is path-connected (see Lemma 3.1) and, given an initial condition X_0 in D, one can resort to classic stochastic differential equations properties to define, in a unique pathwise way, the process X^N up to the explosion time

$$T_{\partial D} := \sup_{\varepsilon>0} T_\varepsilon \in [0, +\infty]. \tag{1.6}$$

Here,

$$T_\varepsilon = \inf\left\{t \geq 0 : \max_{1\leq i\leq N}|X_t^i| \geq \varepsilon^{-1} \text{ or } \min_{1\leq i\leq N}|X_t^i - X_t^j| \leq \varepsilon\right\}$$

is the first exit time of a typical compact set in D. Then, one can show that explosion never occurs:

Theorem 1.1 (Global well posedness and absence of explosion)**.** *For any $X_0 = x \in D$, pathwise uniqueness and strong existence on $[0, +\infty)$ hold for the stochastic differential equation (1.5) on $[0, +\infty)$, and we have $T_{\partial D} = +\infty$ a.s.*

The absence of explosion provided by Theorem 1.1 is remarkably independent of the choice of the inverse temperature, and this is in contrast with the behavior of the Dyson Brownian motion associated with the one-dimensional log-gas, see for instance [48]. The proof of Theorem 1.1 is given in Section 3. It uses the fact that W is the fundamental solution of the Poisson – Laplace equation. The main idea is similar to the one used for other singular repulsion models, such as in [48], or for vortices such as in [31], but the result ultimately relies on quite specific properties of our model (1.5). Note also that our particles will never collide and in particular never collide at the same time, in contrast with for instance the singular attractive model studied in [34] – see also [19] for the control of explosion using the Fukushima technology.

Hence there exists a unique Markov process $X = (X_t)_{t\geq 0}$ solution of (1.4). Its infinitesimal generator L is given for a smooth enough $f : D \to \mathbb{R}$ by

$$Lf = \frac{\alpha_N}{\beta_N}\Delta f - \alpha_N \nabla H \cdot \nabla f. \tag{1.7}$$

Here Δ and ∇ are understood in $(\mathbb{R}^2)^N \equiv \mathbb{R}^{2N}$ and $u \cdot v = \langle u, v\rangle$ denotes the Euclidean scalar product. By symmetry of the evolution, the law of X_t is exchangeable for every $t \geq 0$, as soon as it is exchangeable for $t = 0$. Recall that the law of a random vector is exchangeable when it is invariant by any permutation of the coordinates of the vector. It is then natural to encode the particle system with its empirical measure

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

1.2. **Second moment dynamics.** Theorem 1.2 gives the evolution of the second moment

$$H_V(X_t) = \frac{1}{N}\sum_{i=1}^N |X_t^i|^2 = \int_{\mathbb{R}^2} |x|^2 \,\mu_t^N(\mathrm{d}x)$$

of μ_t^N. This evolution depends on the choices for α_N and β_N, for which meaningful choices are discussed in Section 1.4. We let W_1 denote the (Kantorovich –) Wasserstein transportation distance of order one defined by $W_1(\mu,\nu) = \inf\,\{\mathbb{E}[|X-Y|] : X \sim \mu, Y \sim \nu\}$ for every probability measures μ and ν on $\mathbb{R}$ with finite first moment.

Theorem 1.2 (Second moment dynamics)**.** *The process $(H_V(X_t))_{t\geq 0}$ is an ergodic Markov process, equal in law to the Cox – Ingersoll – Ross process $(R_t)_{t\geq 0}$ given by the unique solution in $[0,\infty)$ of the stochastic differential equation*

$$\mathrm{d}R_t = \sqrt{\frac{8\alpha_N}{N\beta_N}R_t}\,\mathrm{d}b_t + 4\frac{\alpha_N}{N}\left[\frac{N}{\beta_N} + \frac{N-1}{2N} - R_t\right]\mathrm{d}t, \tag{1.8}$$

where $(b_t)_{t\geq 0}$ is a real standard Brownian motion. In particular, its invariant distribution is the Gamma law Γ_N on $\mathbb{R}_+$ with shape parameter $N + \frac{N-1}{2N}\beta_N$ and scale parameter β_N, and density with respect to the Lebesgue measure on $\mathbb{R}$ given by

$$r \in \mathbb{R} \mapsto \gamma_N(r) := \frac{\beta_N^{N+\frac{\beta_N(N-1)}{2N}}}{\Gamma(N+\frac{\beta_N(N-1)}{2N})} r^{(N-1)(1+\frac{\beta_N}{2N})}\mathrm{e}^{-r\beta_N}\mathbf{1}_{r\geq 0}.$$

Moreover, for any $t \geq 0$ we have

$$W_1(\mathrm{Law}(H_V(X_t)),\Gamma_N) \leq \mathrm{e}^{-4\frac{\alpha_N}{N}t}\; W_1(\mathrm{Law}(H_V(X_0)),\Gamma_N). \tag{1.9}$$

Furthermore for any $x \in D$ and $t \geq 0$, we have

$$\mathbb{E}[H_V(X_t) \mid X_0 = x] = H_V(x)\,\mathrm{e}^{-4\alpha_N t/N} + \Big(\frac{1}{2} + \frac{N}{\beta_N} - \frac{1}{2N}\Big)\Big(1 - \mathrm{e}^{-4\alpha_N t/N}\Big). \tag{1.10}$$

In particular, as $t \to \infty$, the left-hand sides in (1.9) *and* (1.10) *converge to* 0 *and* $1/2 + N/\beta_N - 1/(2N)$ *respectively with a speed independent of* N *as soon as* α_N *is linear in* N.

A Cox – Ingersoll – Ross (CIR) process also naturally arises as the dynamics of the second empirical moment of the vortex system studied in [30]. Theorem 1.2 is proved in Section 4.

1.3. **Invariant probability measure and long-time behavior.** Despite the repulsive interaction, the confinement is strong enough to give rise to an equilibrium. Namely, the Markov process $(X_t)_{t\geq 0}$ admits a unique invariant probability measure which is reversible. It is the Boltzmann – Gibbs measure P^N on $D \subset (\mathbb{R}^2)^N$ with density

$$\frac{\mathrm{d}P^N(x_1,\dots,x_N)}{\mathrm{d}x_1\cdots\mathrm{d}x_N} = \frac{\mathrm{e}^{-\beta_N H(x_1,\dots,x_N)}}{Z_N} = \frac{\mathrm{e}^{-\frac{\beta_N}{N}\sum_{i=1}^N |x_i|^2}}{Z_N}\prod_{1\leq i<j\leq N}|x_i - x_j|^{\frac{2\beta_N}{N^2}} \tag{1.11}$$

where

$$Z_N := \int_D \mathrm{e}^{-\beta_N H(x_1,\dots,x_N)}\mathrm{d}x_1\cdots\mathrm{d}x_N$$

is a normalizing constant known as the partition function. Such a Boltzmann – Gibbs measure with a Coulomb interaction is called a Coulomb gas. Actually $H(x) \to +\infty$ when $x \to \partial D$ and $\mathrm{e}^{-\beta H}$ is Lebesgue integrable on D for any $\beta > 0$, see Lemma 3.2. Moreover the density of P^N does not vanish on D. One can extend it on $(\mathbb{R}^2)^N$ by zero, seeing P^N as a probability measure on $(\mathbb{R}^2)^N$. Since the domain D and the function H are both invariant by permutation of the N particles, the law P^N is exchangeable. The behavior of P^N relies crucially on the "inverse temperature" β_N. The choice $\beta_N = N^2$ gives a determinantal structure to P^N which is known in this case as the complex Ginibre ensemble in random matrix theory. As we will see in Section 1.4.1, there is another interesting regime which is $\beta_N = N$.

In Theorem 1.3 below we quantify the long time behavior of our Markov process X^N via a Poincaré inequality for its invariant measure P^N. Recall that if S is an open subset of $\mathbb{R}^n$ and $\mathcal{F}$ is a class of smooth functions on S, then a probability measure μ on S satisfies a Poincaré inequality on $\mathcal{F}$ with constant $c > 0$ if for every $f \in \mathcal{F}$,

$$\mathrm{Var}_\mu(f) := \mathbb{E}_\mu(f^2) - (\mathbb{E}_\mu f)^2 \leq c\,\mathbb{E}_\mu(|\nabla f|^2) \quad \text{where} \quad \mathbb{E}_\mu(f) := \int f\,\mathrm{d}\mu, \tag{1.12}$$

see [49] for instance. If f is the density with respect to μ of a probability measure ν then the quantity $\mathrm{Var}_\mu(f) = \mathbb{E}_\mu(|f-1|^2)$ is nothing else but the chi-square divergence $\chi(\nu\|\mu)$.

Theorem 1.3 (Poincaré inequality)**.** *Let $\mathcal{F}$ be the set of $\mathcal{C}^\infty$ functions $f : D \to \mathbb{R}$ with compact support in D, in the sense that the closure of $\{x \in D : f(x) \neq 0\}$ is compact and is included in D. Then for any N, the probability measure P^N on $(\mathbb{R}^2)^N$ satisfies a Poincaré inequality on $\mathcal{F}$ with a constant which may depend on N.*

By (1.7), the invariance of P^N gives

$$-\mathbb{E}_{P^N}(fLf) = \frac{\alpha_N}{\beta_N}\mathbb{E}_{P^N}(|\nabla f|^2)$$

where $|\nabla f|^2 = \sum_{i=1}^n |\nabla_{x_i} f|^2$ for $\nabla f = (\nabla_{x_i} f)_{1\leq i\leq N}$ in $(\mathbb{R}^2)^N$. Let P_t^N be the law of X_t^N in $(\mathbb{R}^2)^N$. Up to determining a dense class of test functions stable by the dynamics, it is classical, see [49, Sec. 3.2] or [5], that the Poincaré inequality (1.12) for P^N with constant $c = c_N$ imply the exponential convergence of P_t^N to P^N, namely

$$\chi(P_t^N\|P^N) \leq \mathrm{e}^{-\frac{2t}{c_N}\frac{\alpha_N}{\beta_N}}\chi(P_0^N\|P^N).$$

More precisely, provided we already know that P_t^N has a smooth density f_t^N, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}_{P^N}(f_t^N) = \frac{\mathrm{d}}{\mathrm{d}t}\int (f_t^N)^2\,\mathrm{d}P^N = 2\int f_t^N L f_t^N\,\mathrm{d}P^N \leq -2\frac{\alpha_N}{\beta_N c_N}\mathrm{Var}_{P^N}(f_t^N).$$

Theorem 1.3 is proved in Section 5. Poincaré inequalities can classically be proved by spectral decomposition, tensorization, convexity, perturbation, or Lipschitz deformation arguments, see [5]. None of these approaches seem to be available for P^N.

Remark 1.4 (Eigenvector). *It turns out that H_V is up to an additive constant an eigenvector of L. Namely, from (2.7) we get*

$$LU = -\frac{4\alpha_N}{N}U \quad \text{where} \quad U := H_V - \frac{N}{\beta_N} - \frac{N-1}{2N}.$$

This fact is the key of the proof of Theorem 1.2. However, due to the varying sign of U, we do not know how to use U with the Lyapunov method to get a Poincaré inequality.

Remark 1.5 (Tensorization). *The invariant measure P^N of X is not product, in contrast for instance with the case of vortex models with constant intensity studied in* [31].

Remark 1.6 (Convexity). *Neither the domain D nor the energy $H : D \to \mathbb{R}$ are convex, see Proposition 5.1, and thus the law P^N is not log-concave. Remarkably, for one-dimensional log-gases, one can order the particles, which has the effect of producing a convex domain instead of D on which H is convex, and in this case P^N satisfies in fact a logarithmic Sobolev inequality which is stronger, see for instance the forthcoming book* [27] *and also* [21] *for the optimal Poincaré constant. Here $d = 2$ and the one dimensional trick is not available.*

Remark 1.7 (Lipschitz deformation). *The law P^N is not a Lipschitz deformation of the Gaussian law on $\mathcal{M}_N(\mathbb{C})$. Actually, the map which to $M \in \mathcal{M}_N(\mathbb{C})$ associates its eigenvalues in $\mathbb{C}$ is not Lipschitz. To see it take $M, M' \in \mathcal{M}_n(\mathbb{C})$ with $M_{j,j+1} = 1$ for $j = 1, \ldots, n-1$ and $M_{jk} = 0$ otherwise, and $(M' - M)_{jk} = \varepsilon$ if $(j,k) = (n,1)$ and $(M' - M)_{jk} = 0$ otherwise. Then the eigenvalues of $M' - M$ are*

$$\{\varepsilon^{1/n} e^{2ik\pi/n} : 0 \le k \le n-1\},$$

while the Hilbert – Schmidt norm and operator norm of $M' - M$ are both equal to ε. Note that in contrast, this map is Lipschitz for Hermitian matrices and more generally for normal matrices; this statement is known as the Hoffman – Wielandt inequality [37].

The proof of Theorem 1.3 is based on a Lyapunov function and as usual this does not provide in general a good dependence on N. Of course it is natural to ask about the dependence in N and in α_N and β_N of the best constant in Theorem 1.3 and, specifically, if convergence to equilibrium can be expected to hold at a rate that does not depend on N, as in [44]. Theorem 1.8 below and the previous Theorem 1.2 and Remark 1.4 constitute steps in that direction.

Theorem 1.8 (Uniform Poincaré inequality for the one particle marginal). *If $\beta_N = N^2$ then the one-particle marginal law $P^{1,N}$ of P^N on $\mathbb{R}^2$ satisfies a Poincaré inequality, with a constant which does not depend on N. In particular, the smallest (i.e. best) constant for P^N is bounded below uniformly in N.*

Theorem 1.8 is proved in Section 6.

Although the measure P^N is not product, at least in the regime $\beta_N = N^2$ a product structure arises asymptotically as N goes to infinity. More precisely, for $k \le N$, let $P^{k,N}$ be the k-th dimensional marginal distribution of the exchangeable probability measure P^N, as in (1.16); then, in the regime $\beta_N = N^2$, we have

$$P^{k,N} - (P^{1,N})^{\otimes k} \to 0, \qquad N \to \infty \tag{1.13}$$

weakly with respect to continuous bounded functions. It follows from Theorem 1.9 below.

Theorem 1.9 (Chaoticity)**.** *Let $\beta_N = N^2$ and let μ_∞ be the uniform distribution on the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ with density $\varphi_\infty(z) = \pi^{-1}\mathbf{1}_{|z|\leq 1}$. For every fixed $k \geq 1$,*

$$P^{k,N} \to \mu_\infty^{\otimes k}, \qquad N \to \infty$$

weakly with respect to continuous and bounded functions. Moreover, denoting $\varphi^{k,N}$ the density of the marginal distribution $P^{k,N}$, as defined in (1.16)*, we have*

$$\varphi^{1,N} \to \varphi_\infty \quad \textit{and} \quad \varphi^{2,N} \to \varphi_\infty^{\otimes 2}, \qquad N \to \infty$$

uniformly on compact subsets of respectively

$$\{z \in \mathbb{C} : |z| \neq 1\} \quad \textit{and} \quad \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \neq 1, |z_2| \neq 1, z_1 \neq z_2\}.$$

Theorem 1.9 is proved in Section 7. Note that the convergence of $\varphi^{1,N}$ cannot hold uniformly on arbitrary compact sets of $\mathbb{C}$ since the pointwise limit is not continuous on the unit circle. Moreover the convergence of $\varphi^{2,N}$ cannot hold on $\{(z, z) : z \in \mathbb{C}, |z| < 1\}$ since, by (1.16), $\varphi^{2,N}(z, z) = 0$ for any $N \geq 2$ and $z \in \mathbb{C}$ while $\varphi^{1,N}(z)\varphi^{1,N}(z) \to_N 1/\pi^2 \neq 0$ when $|z| < 1$, and this phenomenon is due to the singularity of the interaction.

The case $\beta_N = N^2$ is related to random matrix theory, see Section 1.4.1. To our knowledge, Theorem 1.8 and Theorem 1.9 have not appeared previously in this domain.

1.4. **Comments and open problems.**

1.4.1. *Inverse temperature.* Following [23], there are two natural regimes $\beta_N = N$ and $\beta_N = N^2$.

- **Random matrix theory regime:** $\beta_N = N^2$**.** This is natural from the point of view of random matrices. Namely let M be a random $N \times N$ complex matrix with independent and identically distributed Gaussian entries on $\mathbb{C}$ with mean 0 and variance $1/N$ with density $z \in \mathbb{C} \mapsto \pi^{-1}N\exp(-N|z|^2)$. The variance scaling is chosen so that by the law of large numbers, asymptotically as $N \to \infty$, the rows and the columns of M are stabilized: they have unit norm and are orthogonal. The density of the random matrix M is proportional to

$$M \mapsto \prod_{1\leq j,k\leq N} \exp\Big(-N|M_{jk}|^2\Big) = \exp\left(-N\mathrm{Tr}(MM^*)\right).$$

The spectral change of variables $M = U(D + N)U^*$, which is the Schur unitary decomposition, gives that the joint law of the eigenvalues of M has density

$$\varphi^{N,N}(z_1, \ldots, z_n) := \frac{N^{\frac{N(N+1)}{2}}}{1!2!\cdots N!} \frac{\mathrm{e}^{-\sum_{i=1}^N N|z_i|^2}}{\pi^N} \prod_{i<j} |z_i - z_j|^2 \tag{1.14}$$

with respect to the Lebesgue measure on $\mathbb{C}^N$. This law is usually referred to as the "complex Ginibre Ensemble", see [35, 32, 42, 12]. This matches P^N with (1.3) with $\beta_N = N^2$ so that the density of P^N on $(\mathbb{R}^2)^N = \mathbb{C}^N$ can be written as

$$\frac{\mathrm{d}P^N(z_1, \ldots, z_N)}{\mathrm{d}z_1 \cdots \mathrm{d}z_N} = \varphi^{N,N}(z_1, \ldots, z_N). \tag{1.15}$$

It is a well known fact – see [45, p. 271], [39, p. 150], or [32, 38] – that for every $1 \leq k \leq N$, the k-th dimensional marginal distribution $P^{k,N}$ of P^N has density

$$\begin{aligned}\varphi^{k,N}(z_1, \ldots, z_k) &= \int_{\mathbb{C}^{N-k}} \varphi^{N,N}(z_1, \ldots, z_N)\,\mathrm{d}z_{k+1} \cdots \mathrm{d}z_N \\ &= \frac{(N-k)!}{N!} \frac{\mathrm{e}^{-N(|z_1|^2+\cdots+|z_k|^2)}}{\pi^k N^{-k}} \det\left[(\mathrm{e}_N(Nz_i\overline{z}_j))_{1\leq i,j\leq k}\right],\end{aligned} \tag{1.16}$$

where $\mathrm{e}_N(w) := \sum_{\ell=0}^{N-1} w^\ell/\ell!$ is the truncated exponential series. The energy H is a quadratic functional of the empirical measure $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ of the particles:

$$H(x_1,\dots,x_N) = \int V(x)\,\mu^N(\mathrm{d}x) + \frac{1}{2}\iint_{\neq} W(x-y)\,\mu^N(\mathrm{d}x)\mu^N(\mathrm{d}y) \tag{1.17}$$
$$=: \mathcal{E}_{\neq}(\mu^N)$$

where "$\neq$" indicates integration outside the diagonal. Since $\beta_N \gg N$ as $N\to\infty$, under $(P^N)_N$, the sequence of empirical measures $(\mu^N)_N$ satisfies a large deviation principle with speed $(\beta_N)_N$ and good rate function $\mathcal{E} - \inf\mathcal{E}$ where $\mathcal{E}$ is given for nice probability measures μ on $\mathbb{R}^2$ by

$$\mathcal{E}(\mu) := \int V(x)\,\mu(\mathrm{d}x) + \frac{1}{2}\iint W(x-y)\,\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

See for instance [47, 36, 23] and references therein. The functional $\mathcal{E}$ is strictly convex where it is finite, lower semi-continuous with compact level sets, and it achieves its global minimum for a unique probability measure μ_∞ on $\mathbb{R}^2$, which is the uniform distribution on the unit disc with density $z\in\mathbb{C}\mapsto \pi^{-1}\mathbf{1}_{|z|\leq 1}$. From the large deviation principle it follows that almost surely

$$\mu^N \underset{N\to\infty}{\longrightarrow} \mu_\infty := \arg\inf\mathcal{E} \tag{1.18}$$

weakly, regardless of the way we put $(P^N)_N$ in the same probability space.

- **Crossover regime:** $\beta_N = N$**.** In this case P^N has density proportional to

$$(x_1,\dots,x_N)\in D \mapsto \mathrm{e}^{-\sum_{i=1}^N |x_i|^2} \prod_{1\leq i<j\leq N} |x_i - x_j|^{\frac{2}{N}}.$$

We do not have a determinantal formula as in (1.16), and this gas is not associated with a standard random matrix ensemble. It is a two dimensional analogue of the one dimensional gas studied in [1] leading to a Gauss-Wigner crossover. Following [23], we can expect that under $(P^N)_N$ the sequence of empirical measures $(\mu^N)_N$ satisfies a large deviation principle with speed $(\beta_N)_N$ and rate function $\widetilde{\mathcal{E}} - \inf\widetilde{\mathcal{E}}$; here $\widetilde{\mathcal{E}}$ is given for every probability measure μ on $\mathbb{R}^2$ by

$$\widetilde{\mathcal{E}}(\mu) := \mathcal{E}(\mu) - \mathcal{S}(\mu) \quad \text{where} \quad \mathcal{S}(\mu) := -\int \frac{\mathrm{d}\mu}{\mathrm{d}x}\log\frac{\mathrm{d}\mu}{\mathrm{d}x}\,\mathrm{d}x$$

when μ is absolutely continuous with respect to the Lebesgue measure, while $\mathcal{S}(\mu) := +\infty$ otherwise. $\mathcal{S}$ is the so-called Boltzmann – Shannon entropy. The minimizer of $\widetilde{\mathcal{E}}$ is no longer compactly supported but can still be characterized by Euler – Lagrange equations, and is a crossover between the uniform law on the disc and the standard Gaussian law on $\mathbb{R}^2$. See [23] for the link with Sanov's large deviation principle.

1.4.2. *Dyson Brownian Motion.* If we start with an $N\times N$ random matrix

$$M_t = (M_t^{j,k})_{1\leq j,k\leq N}$$

with i.i.d. entries following the diffusion $\mathrm{d}M_t^{j,k} = \mathrm{d}B_t^{j,k} - N\nabla V(M_t^{j,k})\,\mathrm{d}t$ then the eigenvalues in $\mathbb{C}=\mathbb{R}^2$ of M_t will not match our diffusion X solution of (1.4). This is due to the fact that M_t is not a normal matrix in the sense that $M_tM_t^* \neq M_t^*M_t$ with probability one as soon as M_t has a density. In fact the Schur unitary decomposition of M_t writes $M_t = U_tT_tU_t^*$ where U_t is unitary and $T_t = D_t + N_t$ is upper triangular, D_t is diagonal, and N_t is nilpotent. The dynamics of D_t is perturbed by N_t. The dynamics (1.4) is not the analogue of the Dyson Brownian motion, the process of the eigenvalues associated with the Gaussian Unitary Ensemble, the one-dimensional log-gas studied in [2, 43]. We refer to [7, 13] and references therein for more information on this topic.

1.4.3. *Initial conditions.* In the case of the one-dimensional log-gas known as the Dyson Brownian Motion, the stochastic differential equation still admits a unique strong solution when the particles coincide initially. This is proved in [2, Prop. 4.3.5] by crucially using the ordered particle system. Unfortunately, it does not seem possible to extend such an argument to higher dimensions. But it is likely that at least weak well-posedness should still hold for our model.

1.4.4. *Arbitrary dimension, confinement, and interaction.* As in [23], many aspects should remain valid in arbitrary dimension $d \geq 2$, with a Coulomb repulsion and a more general confinement V. For instance, by analogy with the case without interaction studied in [49, Th. 2.2.19], it is natural to expect that Theorem 1.1 remains valid beyond the quadratic confinement case, for example in the quadratic "dispersive" case $V(x) = -|x|^2$, and in confined cases for which $V(x) \to +\infty$ as $x \to \infty$ with polynomial growth. Nevertheless, our choice is to entirely devote the present article to the two-dimensional quadratic confinement case: this model is probably the richest in structure, notably due to its link with the Ginibre Coulomb gas, which is a remarkable exactly solvable model.

The model with non-singular interaction has extensively been studied in arbitrary dimension, in relation with McKean – Vlasov equations, see [44, 46, 50] and references therein. The model in dimension $d = 1$ with logarithmic singular interaction has also extensively been studied, see for instance [20, 10, 14, 29, 43] and references therein. See also [6].

1.4.5. *Logarithmic Sobolev inequality and other functional inequalities.* It is natural to ask whether P^N satisfies a logarithmic Sobolev inequality, which is stronger than the Poincaré inequality with half the same constant, see [3, 5]. Indeed, for P^N, a Lyapunov approach is probably usable by following the lines of [17, Proof of Prop. 3.5], see also [18], but there are technical problems due to the shape of D which comes from the singularity of the interaction. Observe that the one-particle marginal $P^{1,N}$ satisfies indeed a logarithmic Sobolev inequality with a constant uniform in N, as mentioned in Remark 6.2 after the proof of Theorem 1.8.

Still about functional inequalities, the study of concentration of measure for Coulomb gases in relation with Coulomb transport inequalities is considered in the recent work [24].

1.4.6. *Mean-field limit.* In the regime $\beta_N = N^2$, by (1.18) the empirical measure μ^N under P^N tends to μ_∞ as $N \to \infty$. More generally, when the law of X_0 is exchangeable and for general β_N, one can ask about the behavior of the empirical measure of the particles $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ as $N \to \infty$ and as $t \to \infty$. This corresponds to study the following scheme:

$$P_t^N \underset{t\to\infty}{\longrightarrow} P^N \quad \text{and} \quad \begin{array}{ccc} \mu_t^N & \underset{t\to\infty}{\longrightarrow} & \mu^N \\ \downarrow & & \downarrow \\ \mu_t & \underset{t\to\infty}{\longrightarrow} & \mu_\infty \end{array}$$

for a suitable deterministic limit μ_∞.

At fixed N, the limit $\lim_{t\to\infty} P_t^N = P^N$, valid for an arbitrary initial condition $X_0 = x$, corresponds to the ergodicity phenomenon for the Markov process X, quantified by the Poincaré inequality of Theorem 1.3. By the mean-field structure of (1.5) and (1.7), it is natural to expect that if

$$\sigma := \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \in [0, +\infty)$$

then the sequence $((\mu_t^N)_{t\geq 0})_N$ converges, as a continuous process with values in the space of probability measures in $\mathbb{R}^2$, to a solution of the following McKean – Vlasov partial differential equation with singular interaction:

$$\partial_t \mu_t = \sigma \Delta \mu_t + \nabla \cdot ((\nabla V + \nabla W * \mu_t)\mu_t). \tag{1.19}$$

The convergence of $((\mu_t^N)_{t\geq 0})_N$ can be thought of as a sort of law of large numbers. This is well understood in the one-dimensional case with logarithmic interaction, see for instance [48, 20], using tightness and characterization of the limiting laws. However the uniqueness arguments used in one-dimension are no longer valid for our model, and different ideas need to be developed, see [22]. We also refer to [26] and references therein for the analysis of similar evolution equations without noise and confinement.

Theorem 1.2 suggests to take $\alpha_N = N$. Let us comment on the couple of special cases already considered in our large deviation principle analysis of $(P^N)_N$: $\beta_N = N^2$ and $\beta_N = N$, when $\alpha_N = N$.

- **Random matrix theory regime with vanishing noise: $\alpha_N = N$ and $\beta_N = N^2$.** In this case $\sigma = 0$ and the limiting McKean – Vlasov equation (1.19) does not have a diffusive part. Since $\alpha_N = N$ we have a constant speed for the second moment evolution. Since $\beta_N = N^2$ we have explicit determinantal formulas for P^N from the complex Ginibre Ensemble (1.16). The absence of diffusion implies that if we start from an initial state μ_0 which is supported in a line, then μ_t will still be supported in this line for any $t \in [0, \infty)$, and will thus never converge as $t \to \infty$ to the uniform distribution on the unit disc of the complex plane. In particular, the long time equilibrium depends clearly on the initial condition.
- **Crossover regime with non-vanishing noise: $\alpha_N = N$ and $\beta_N = N$.** In this case $\sigma = 1$ and the McKean – Vlasov equation (1.19) has a diffusive term. This regime is also considered in [15, 16] for instance, see also [33]. The Keller – Segel model studied in [34, 19] is the analogue with an attractive interaction instead of repulsive.

2. Useful formulas

In this section we gather several useful formulas related to the energy $H = H_V + H_W$ and the operator L, defined in (1.2) and (1.7) respectively. Recall that $V(z) = |z|^2$ and $W(z) = -2\log|z|$ on $\mathbb{R}^2 \setminus \{0\}$, giving

$$\nabla V(z) = 2z \quad \text{and} \quad \nabla W(z) = -\frac{2z}{|z|^2}.$$

Moreover we let $|x|^2 = \sum_{i=1}^N |x_i|^2$ for $x = (x_1, \ldots, x_N) \in (\mathbb{R}^2)^N$.

Gradient. By (1.2), for any $x \in D$ and $i \in \{1, \ldots, N\}$,

$$\nabla_{x_i} H_V(x) = \frac{1}{N}\nabla V(x_i) = \frac{2}{N}x_i \tag{2.1}$$

and

$$\nabla_{x_i} H_W(x) = \frac{1}{N^2}\sum_{j\neq i}\nabla W(x_i - x_j) = -\frac{2}{N^2}\sum_{j\neq i}\frac{x_i - x_j}{|x_i - x_j|^2}. \tag{2.2}$$

Hessian. By (2.1)-(2.2), for any $x \in D$ and $i, j \in \{1, \ldots, N\}$,

$$\nabla^2_{x_i,x_j} H_V(x) = \begin{cases} \frac{1}{N}\nabla^2 V(x_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\nabla^2_{x_i,x_j} H_W(x) = \begin{cases} +\frac{1}{N^2}\sum_{k\neq i}\nabla^2 W(x_i - x_k) & \text{if } i = j \\ -\frac{1}{N^2}\nabla^2 W(x_i - x_j) & \text{if } i \neq j. \end{cases}$$

This gives

$$\nabla^2 H_V = \frac{2}{N} I_{2N} \qquad \text{and} \qquad \nabla^2 H_W = \frac{1}{N^2} A \tag{2.3}$$

where I_{2N} is the $2N \times 2N$ identity matrix and A is a $N \times N$ bloc matrix with diagonal and off-diagonal 2×2 blocs

$$A_{i,i} = \sum_{k \neq i} \nabla^2 W(x_i - x_k), \quad A_{i,j} = -\nabla^2 W(x_i - x_j), \quad i \neq j.$$

Operator. The generator L defined in (1.7) on functions $f : D \to \mathbb{R}$ is given by

$$\begin{aligned} Lf(x) &= \frac{\alpha_N}{\beta_N} \Delta f(x) - \frac{\alpha_N}{N} \sum_{i=1}^N \nabla V(x_i) \cdot \nabla_{x_i} f(x) - \frac{\alpha_N}{N^2} \sum_{1 \le i \neq j \le N} \nabla W(x_i - x_j) \cdot \nabla_{x_i} f(x) \\ &= \frac{\alpha_N}{\beta_N} \Delta f(x) - 2\frac{\alpha_N}{N} \sum_{i=1}^N x_i \cdot \nabla_{x_i} f(x) + 2\frac{\alpha_N}{N^2} \sum_{1 \le i \neq j \le N} \frac{(x_i - x_j) \cdot \nabla_{x_i} f(x)}{|x_i - x_j|^2}. \quad (2.4) \end{aligned}$$

Let us compute now LH_V, LH_W, and LH. First of all, since ∇W is odd, we get by symmetrization from (2.1)-(2.2) that

$$\begin{aligned} |\nabla H|^2(x) &= \frac{1}{N^2} \sum_{i=1}^N \left| \nabla V(x_i) \right|^2 + \frac{1}{N^4} \sum_{i=1}^N \Big| \sum_{j \neq i} \nabla W(x_i - x_j) \Big|^2 \\ &\quad + \frac{1}{N^3} \sum_{i \neq j} (\nabla V(x_i) - \nabla V(x_j)) \cdot \nabla W(x_i - x_j) \\ &= \frac{4}{N^2} |x|^2 + \frac{4}{N^4} \sum_{i=1}^N \Big| \sum_{j \neq i} \frac{x_i - x_j}{|x_i - x_j|^2} \Big|^2 - 4\frac{N-1}{N^2}. \quad (2.5) \end{aligned}$$

Moreover, from (2.3) and $\Delta W = 0$ on D, we get

$$\Delta H_W(x) = 0 \quad \text{and} \quad \Delta H(x) = \Delta H_V(x) = \sum_{i=1}^N \frac{1}{N} \Delta V(x_i) = 4. \quad (2.6)$$

By (2.1)-(2.2) and by symmetry we also have

$$\begin{aligned} LH_V(x) &= \frac{\alpha_N}{\beta_N} \Delta H_V(x) - \alpha_N \nabla H(x) \cdot \nabla H_V(x) \\ &= 4\frac{\alpha_N}{\beta_N} - \frac{4\alpha_N}{N^2} |x|^2 + \frac{4\alpha_N}{N^3} \sum_{i=1}^N \sum_{j \neq i} \frac{x_i - x_j}{|x_i - x_j|^2} \cdot x_i \\ &= 4\frac{\alpha_N}{\beta_N} - \frac{4\alpha_N}{N^2} |x|^2 + \frac{2\alpha_N}{N^3} \sum_{1 \le i \neq j \le N} \frac{x_i - x_j}{|x_i - x_j|^2} \cdot (x_i - x_j) \\ &= 4\frac{\alpha_N}{\beta_N} + 2\alpha_N \frac{N-1}{N^2} - \frac{4\alpha_N}{N} H_V(x) \quad (2.7) \end{aligned}$$

and likewise

$$\begin{aligned} LH_W(x) &= \frac{\alpha_N}{\beta_N} \Delta H_W(x) - \alpha_N \nabla H(x) \cdot \nabla H_W(x) \\ &= 2\alpha_N \frac{N-1}{N^2} - 4\frac{\alpha_N}{N^4} \sum_{i=1}^N \Big| \sum_{j \neq i} \frac{x_i - x_j}{|x_i - x_j|^2} \Big|^2 \quad (2.8) \end{aligned}$$

From (2.7) and (2.8) we finally get

$$\begin{aligned} LH(x) &= LH_V(x) + LH_W(x) \\ &= 4\frac{\alpha_N}{\beta_N} + 4\alpha_N \Big(\frac{N-1}{N^2} - \frac{1}{N} H_V(x) - \frac{1}{N^2} \sum_{i=1}^N \Big| \frac{1}{N} \sum_{j \neq i} \frac{x_i - x_j}{|x_i - x_j|^2} \Big|^2 \Big). \quad (2.9) \end{aligned}$$

Note that the fact that the singular repulsion potential W is the fundamental solution of the diffusion part Δ simplifies the expression of LH, in contrast with the situation in dimension 1 studied in [48, p. 559], see also [43].

3. Proof of Theorem 1.1

Lemma 3.1 (Connectivity). *The set D defined by (1.1) is path-connected in $(\mathbb{R}^2)^N$.*

Proof of Lemma 3.1. It suffices to show that for any $x := (x_1, \dots, x_N) \in D$ and $y := (y_1, \dots, y_N) \in D$, there exists a continuous map $\gamma := (\gamma_1, \dots, \gamma_N) : [0,1] \mapsto D$ such that $\gamma(0) = x$ and $\gamma(1) = y$, which must be understood as the position in time of N moving particles in space. This corresponds to move a cloud of N distinct and distinguishable particles into another cloud of N distinct and distinguishable particles. Let us proceed by induction on N. The property is immediate for $N = 1$. Suppose that $N \geq 1$ and assume that one has already constructed $t \in [0,1] \mapsto (\gamma_1(t), \dots, \gamma_N(t))$. One can first construct γ_{N+1} in such a way that $\{t \in [0,1] : \gamma_{N+1}(t) \in \{\gamma_1(t), \dots, \gamma_N(t)\}\}$ is a finite set. Second, one may modify the path γ_{N+1}, locally at the intersection times by varying the speed, in order to make this set empty. This is possible since $d = 2$, and possibly impossible if $d = 1$ since a particle cannot bypass another one. □

Lemma 3.2 (Coercivity). *For any fixed N, we have $H \geq 0$,*

$$\lim_{x \to \partial D} H(x) = +\infty,$$

and $\mathrm{e}^{-\beta H}$ is Lebesgue integrable on D for any $\beta > 0$.

Proof of Lemma 3.2. Let $x = (x_1, \dots, x_N)$ in D. Then

$$\frac{1}{2}\sum_{i \neq j} |x_i - x_j|^2 = \frac{1}{2}\sum_{i,j=1}^{N} |x_i - x_j|^2 = N\sum_{i=1}^{N} |x_i|^2 - \Big|\sum_{i=1}^{N} x_i\Big|^2 \leq N|x|^2$$

so for $u_{ij} = |x_i - x_j|^2$ it holds

$$2N^2\, H(x) = N|x|^2 + N|x|^2 - \sum_{i \neq j} \log u_{ij} \geq N|x|^2 + \sum_{i \neq j}\Big(\frac{u_{ij}}{2} - \log u_{ij}\Big).$$

But $u/2 - \log u \geq 1 - \log(2) \geq 1/4$ for all $u > 0$, so

$$H(x) \geq \frac{|x|^2}{2N} + \frac{1}{2N^2}\frac{N(N-1)}{4} \geq \frac{|x|^2}{2N} + \frac{1}{16}\cdot \tag{3.1}$$

In particular $H \geq 0$ and $\mathrm{e}^{-\beta H}$ is Lebesgue integrable on D for any $\beta > 0$.

We now prove that $H(x) \to +\infty$ as $x \to \partial D$. It suffices to show that for any $R > 0$ there exists $A > 0$ and $\varepsilon > 0$ such that $H(x) \geq R$ as soon as $\max_{1 \leq i \leq N} |x_i| \geq A$ or $\min_{1 \leq i \neq j \leq N} |x_i - x_j| \leq \varepsilon$. First, let us fix $R > 0$. Then, by (3.1), $H(x) \geq R$ as soon as $|x|^2 \geq 2NR$, giving such an A.

Then, for $\varepsilon > 0$ to be chosen later, assume that for some $i \neq j$ we have $|x_i - x_j| \leq \varepsilon$. Then, by definition of $H(x)$,

$$N^2 H(x) \geq 2\log\frac{1}{|x_i - x_j|} + \sum_{\substack{1 \leq k \neq l \leq N \\ \{k,l\} \neq \{i,j\}}} \log\frac{1}{|x_k - x_l|}.$$

We can assume that $\max_{1 \leq r \leq N} |x_r| \leq A$ otherwise we have already seen that $H(x) \geq R$. Hence, for any (k,l) with $k \neq l$ we have

$$\log\frac{1}{|x_k - x_l|} \geq -\log(1 + |x_k|) - \log(1 + |x_l|) \geq -2\log(1 + A)$$

using the inequality $|a-b| \leq (1+|a|)(1+|b|)$ for $a,b \in \mathbb{C}$. As a consequence

$$N^2 H(x) \geq -2\log\varepsilon - 2N^2\log(1+A),$$

which is $\geq R$ for a small enough ε. □

In the sequel we use the notation $\mathbb{P}_x = \mathbb{P}[\,\cdot \mid X_0 = x]$ and $\mathbb{E}_x = \mathbb{E}[\,\cdot \mid X_0 = x]$.

Proof of Theorem 1.1. We first construct the process X starting in D up to its explosion time. Given an initial condition $x \in D$, for each $\varepsilon \in (0, \min_{1\leq i\leq N}|x^i - x^j|)$ we consider a smooth function W^ε on $\mathbb{R}^2$ coinciding with W on $\{z \in \mathbb{R}^2 : |z| \geq \varepsilon\}$ and we set

$$H^\varepsilon = H_V + H_{W^\varepsilon}.$$

Given a Brownian motion B in a fixed probability space we let X^ε denote the unique pathwise solution to the stochastic differential equation

$$\mathrm{d}X_t^\varepsilon = \sqrt{2\frac{\alpha_N}{\beta_N}}\mathrm{d}B_t - \alpha_N \nabla H^\varepsilon(X_t^\varepsilon)\,\mathrm{d}t, \qquad X_0^\varepsilon = x. \tag{3.2}$$

Notice that for $\varepsilon' \in (0,\varepsilon]$, the processes X^ε and $X^{\varepsilon'}$ coincide up to the stopping time

$$T^{\varepsilon,\varepsilon'} = \inf\{s \geq 0 : \min_{i\neq j} |(X_s^{\varepsilon'})^i - (X_s^{\varepsilon'})^j| \leq \varepsilon\}.$$

For each $\varepsilon \in (0, \min_{1\leq i\leq N}|x^i - x^j|)$ we can thus unambiguously define a stopping time T^ε and a process X on $[0,T^\varepsilon]$, setting $T^\varepsilon = T^{\varepsilon,\varepsilon'}$ and $X = X^{\varepsilon'}$ for any $\varepsilon' \in (0,\varepsilon)$. By continuity, we have $T^{\varepsilon'} > T^\varepsilon$ a.s., and so X is uniquely defined up to the stopping time $T_{\partial D}$ defined in (1.6). On the other hand, the process X satisfies equation (1.5) on each interval $[0,T^\varepsilon)$ and hence on $[0,T_{\partial D})$ too. Thus, we just have to prove that $T_{\partial D} = \infty$ a.s.

Given $R > 0$, define the stopping times

$$T'_R := \inf\{t \geq 0 : H(X_t) > R\} \in [0,\infty], \quad \text{and} \quad T' := \lim_{R\to\infty} T'_R = \sup_{R>0} T'_R \in [0,\infty].$$

Lemma 3.2 gives $\{T' = \infty\} \subset \{T_{\partial D} = \infty\}$: indeed on $\{T' = \infty\}$, for every $t \geq 0$ we have $\sup_{s\in[0,t]} H(X_s) < \infty$; by Lemma 3.2 this means that $T_{\partial D} = \infty$.

Let us now show that $\mathbb{P}_x(T' = \infty) = 1$. Thanks to (2.9), we have $LH \leq c$ on D for $c = 4\alpha_N(1/\beta_N + 1/N)$. Moreover, given $R \geq 1$ and proceeding as in the end of the proof of Lemma 3.2 we can choose $\varepsilon < e^{-CRN^2\log N}$ for a numerical constant C such that the function H (respectively LH) coincides with H^ε (respectively LH^ε) along the trajectory of X on the interval $[0,T'_R]$; we can therefore apply the Itô formula to $X_{t\wedge T'_R}$ and H^ε to get that

$$\mathbb{E}_x(H(X_{t\wedge T'_R})) - H(x) = \mathbb{E}_x\left(\int_0^{t\wedge T'_R} LH(X_s)\,\mathrm{d}s\right) \leq \mathbb{E}_x\left(\int_0^{t\wedge T'_R} c\,\mathrm{d}s\right) \leq ct, \tag{3.3}$$

for each $t \geq 0$. In particular

$$\sup_{R>0} \mathbb{E}_x(H(X_{t\wedge T'_R})) < \infty.$$

On the other hand, since H is everywhere nonnegative by Lemma 3.2, we have

$$R\,\mathbf{1}_{T'_R\leq t} \leq H(X_{t\wedge T'_R}),$$

from which it follows that

$$\mathbb{P}_x(T'_R \leq t) \leq \frac{1}{R}\sup_{R>0}\mathbb{E}_x(H(X_{t\wedge T'_R})).$$

Finally $\mathbb{P}_x(T' \leq t) = \lim_{R\to\infty}\mathbb{P}_x(T'_R \leq t) = 0$, for any $t \geq 0$, and thus $\mathbb{P}_x(T' = \infty) = 1$. □

Note that our proof of non-explosion notably differs from the one of [48] and [43]: we deal with ∂D at once, instead of handling separately ∞ and $x_i \neq x_j$, thanks to the geometric Lemma 3.2.

Remark 3.3.

a) *From the previous proof we see that the process X and the process X^ε as in* (3.2) *coincide up to the stopping time T^ε. Moreover, $T^\varepsilon \to \infty$ a.s. as $\varepsilon \to 0$. This readily implies that $X^\varepsilon \to X$ a.s. uniformly on each finite time interval $[0,T]$ and, in particular that $\mathrm{Law}(X^\varepsilon) \to \mathrm{Law}(X)$ in $C([0,T],(\mathbb{R}^2)^N)$.*

b) *Since H is bounded from below and LH is bounded from above, letting $R \to \infty$ in the first equality in* (3.3) *and using twice Fatou's Lemma we get that*

$$\mathbb{E}_x(H(X_t)) - H(x) \leq \mathbb{E}_x\Big(\int_0^t LH(X_s)\,\mathrm{d}s\Big)$$

with both sides finite, for all $t \geq 0$.

4. Proof of Theorem 1.2

Proof of Theorem 1.2. By the Itô formula and (2.7), $N^{-1}|X_t|^2 = H_V(X_t)$ evolves according to the stochastic differential equation

$$\begin{aligned} \mathrm{d}H_V(X_t) &= LH_V(X_t)\mathrm{d}t + \sqrt{2\frac{\alpha_N}{\beta_N}}\nabla H_V(X_t)\,\mathrm{d}B_t \\ &= \Big(4\frac{\alpha_N}{\beta_N} + 2\alpha_N\frac{N-1}{N^2} - \frac{4\alpha_N}{N}H_V(X_t)\Big)\mathrm{d}t + \sqrt{2\frac{\alpha_N}{\beta_N}}\frac{2}{N}X_t\,\mathrm{d}B_t. \end{aligned} \tag{4.1}$$

The process $H_V(X_t)$ thus satisfies, until the first time it hits 0, the stochastic differential equation (1.8) with the Brownian motion b_t defined by $\mathrm{d}b_t = \frac{X_t\cdot \mathrm{d}B_t}{|X_t|}$. Standard properties of the CIR process (see [25]) and the fact that $4\frac{\alpha_N}{\beta_N}+2\alpha_N\frac{N-1}{N^2} \geq 4\frac{\alpha_N}{N\beta_N}$, imply this stopping time is ∞ a.s. Pathwise uniqueness for (1.8) ensures that the law of $H_V(X_t)$ is the same as for the CIR process (in particular, its invariant distribution is given in [25]).

Ergodicity of the solution R to (1.8) is proved in [41] by a non quantitative approach. Let us prove the long time convergence bound (1.9) in Wasserstein-1 distance. By standard arguments, it is enough to show that for any pair (R_t^x, R_t^y) of solutions to (1.8) driven by the same (fixed) Brownian motion b_t, and such that $(R_0^x, R_0^y) = (x,y)$, one has

$$\mathbb{E}[|R_t^x - R_t^y|] \leq \mathrm{e}^{-4\frac{\alpha_N}{N}t}|x-y|.$$

This can be done adapting classical uniqueness argument for square root diffusions found in [40]. Indeed, consider the function

$$x \in \mathbb{R}_+ \mapsto \rho(x) := \sqrt{\frac{8\alpha_N}{N\beta_N}x}$$

and the sequence $\{a_\ell\}_{\ell\geq 1}$ defined as

$$a_0 = 1 \quad \text{and} \quad a_\ell = a_{\ell-1}\mathrm{e}^{-\ell\frac{8\alpha_N}{N\beta_N}},\ \ell \geq 1.$$

Note that $a_\ell \searrow 0$ and $\int_{a_\ell}^{a_{\ell-1}} \rho(z)^{-2}\mathrm{d}z = \ell$. For each $\ell \geq 1$, let moreover $z \mapsto \psi_\ell(z)$ be a non-negative continuous function supported on $(a_\ell, a_{\ell-1})$ such that $\int_{a_\ell}^{a_{\ell-1}} \psi_\ell(z)\mathrm{d}z = 1$ and $0 \leq \psi_\ell(z) \leq 2\ell^{-1}\rho(z)^{-2}$ for $a_\ell < z < a_{\ell-1}$. Consider also the even non-negative and twice continuously differentiable function ϕ_ℓ defined by

$$\phi_\ell(x) = \int_0^{|x|} dy \int_0^y \psi_\ell(z)\mathrm{d}z, \quad x \in \mathbb{R}$$

For all $x \in \mathbb{R}$ it satisfies : $\phi_\ell(x) \nearrow |x|$, $\phi_\ell'(x) \to \mathrm{sign}(x)$ as $\ell \to \infty$, $0 \leq \phi_\ell'(x)x \leq |x|$ and $0 \leq \phi_\ell''(x)\frac{8\alpha_N}{N\beta_N}|x| \leq 2\ell^{-1}$. Applying the Itô formula to ϕ_ℓ and $\zeta_t := R_t^x - R_t^y$ we get

$$\phi_\ell(\zeta_t) = M_t^\ell - 4\frac{\alpha_N}{N}\int_0^t \phi_\ell'(\zeta_s)\zeta_s\,\mathrm{d}s + 4\frac{\alpha_N}{N\beta_N}\int_0^t \phi_\ell''(\zeta_s)\zeta_s\,\mathrm{d}s$$

for some martingale M_t^ℓ. Taking expectation, letting $\ell \to \infty$ and applying Gronwall's lemma, the desired inequality is obtained. Assertion (1.10) follows from (4.1), noting that the function $f(t) = \mathbb{E}[H_V(X_t) \mid X_0 = x]$ solves

$$f(t) = f(0) + \int_0^t \Big(4\frac{\alpha_N}{\beta_N} + 2\alpha_N \frac{N-1}{N^2} - \frac{4\alpha_N}{N} f(s)\Big)\,\mathrm{d}s$$

for all $t > 0$, and integrating this equation. □

5. Proof of Theorem 1.3

Proposition 5.1 (Lack of convexity)**.** *The set D defined by* (1.1) *is not convex. Moreover, the Hessian matrix of the function H is not always positive definite on D.*

Proof of Proposition 5.1. The set D is not convex since $0 \in [-x, x] \cap D^c$ for any $x \in D$.

The convexity of H could be studied using a bloc version of the Ghershgorin theorem, see [28], if W were convex. Unfortunately it turns out that W is nowhere convex. More precisely, setting $z = (a, b)^\top \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we get

$$W(z) = -\log(|z|^2) = -\log(a^2 + b^2)$$

and

$$\nabla W(z) = -2\frac{z}{|z|^2} = -2\frac{(a,b)^\top}{a^2+b^2} \quad \text{and} \quad \nabla^2 W(z) = 2\frac{\begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}}{(a^2+b^2)^2}.$$

Thus

$$\mathrm{Tr}(\nabla^2 W(z)) = 0 \quad \text{and} \quad \det(\nabla^2 W(z)) = -\frac{4}{|z|^4}.$$

Consequently the two eigenvalues $\lambda_\pm(z)$ of $\nabla^2 W(z)$ satisfy

$$\lambda_-(z) = -\lambda_+(z) = -\frac{2}{a^2+b^2} = -\frac{2}{|z|^2} \underset{z\to 0}{\longrightarrow} -\infty,$$

and have respective eigenvectors $(-b, a)$ and (a, b). In particular W is not convex.

Now, by (2.3), if we fix $x_1, \ldots, x_{N-1}$ and let x_N tend to x_1, then $\nabla^2 W(x_1 - x_j)$ will remain bounded for any $j \in \{2, \ldots, N-1\}$ while the smallest eigenvalue of $\nabla^2 W(x_1 - x_N)$ blows down to $-\infty$. Therefore $\nabla^2_{x_1,x_1} H(x_1, \ldots, x_N)$ and thus $\nabla^2 H(x_1, \ldots, x_N)$ is not positive definite for such points.

Note however that we may also use (2.3) to get that $\nabla^2 H(x_1, \ldots, x_N)$ is positive definite at points of D for which all the differences $x_i - x_j$ are large enough. □

The following Lemma is the gradient version of Lemma 3.2.

Lemma 5.2 (Gradient coercivity)**.** *For any N and $x = (x_1, \ldots, x_N)$ in D we have*

$$|\nabla H(x)|^2 \geq \frac{4}{N^2}|x|^2 + \frac{4}{N^4}\sum_{i\neq j}\frac{1}{|x_i - x_j|^2} - 4\frac{N-1}{N^2}.$$

In particular

$$\lim_{x\to\partial D} |\nabla H(x)| = +\infty.$$

Proof of Lemma 5.2. This is a consequence of (2.5) and the fact that for any N and any distinct $x_1, \ldots, x_N \in \mathbb{R}^2$,

$$S_N := \sum_{i=1}^N \Big|\sum_{j\neq i}\frac{x_i - x_j}{|x_i - x_j|^2}\Big|^2 - \sum_{i=1}^N\sum_{j\neq i}\frac{1}{|x_i - x_j|^2} \geq 0. \tag{5.1}$$

For the proof of (5.1), we first observe that

$$S_2 = 0$$

and we now consider $N \geq 3$ for which

$$S_N = 2\sum_{i=1}^N \sum_{\substack{1\leq j<k\leq N\\ j,k\neq i}} \frac{(x_i-x_j)\cdot(x_i-x_k)}{|x_i-x_j|^2|x_i-x_k|^2}.$$

Decomposing

$$\sum_{i=1}^N \sum_{\substack{1\leq j<k\leq N\\ j,k\neq i}} \cdot = \sum_{1\leq i<j<k\leq N} \cdot + \sum_{1\leq j<i<k\leq N} \cdot + \sum_{1\leq j<k<i\leq N} \cdot$$

and letting $I=j, J=i$ and $K=k$ in the second sum on the right-hand side and $I=j, J=k$ and $K=i$ in the third sum, we see that $S_N/2$ is equal to

$$\sum_{1\leq i<j<k\leq N} \frac{|x_j-x_k|^2(x_i-x_j)\cdot(x_i-x_k)}{|x_i-x_j|^2|x_j-x_k|^2|x_k-x_i|^2} + \frac{|x_k-x_i|^2(x_j-x_i)\cdot(x_j-x_k)}{|x_i-x_j|^2|x_j-x_k|^2|x_k-x_i|^2} + \frac{|x_i-x_j|^2(x_k-x_i)\cdot(x_k-x_j)}{|x_i-x_j|^2|x_j-x_k|^2|x_k-x_i|^2}.$$

But

$$|x_j-x_k|^2 = |x_i-x_j|^2 + |x_i-x_k|^2 - 2\,(x_i-x_j)\cdot(x_i-x_k)$$

so

$$S_N = 4\sum_{1\leq i<j<k\leq N} \frac{|x_i-x_j|^2|x_i-x_k|^2 - (x_i-x_j)\cdot(x_i-x_k)^2}{|x_i-x_j|^2|x_j-x_k|^2|x_k-x_i|^2}.$$

Hence $S_N \geq 0$ by the Schwarz inequality. This shows also that equality is achieved when x_i-x_j and x_i-x_k are parallel for any i,j,k for instance when $x_i=(i,0)$ for any i, thanks to the equality case in the Schwarz inequality. Let us observe from the proof that the same bound would hold in any Hilbert space. □

The following lemma is the counterpart on H_W of Theorem 1.2 for H_V. It is likely that the bounds in the lemma are not optimal, as we would expect bounds independent of N. This is probably due to our use of the bound (5.1). The lemma is not used but has its own interest as we see that the particular speed $\alpha_N = N$ naturally appears in the upper bounds, as in Theorem 1.2.

Lemma 5.3 (Energy evolution)**.** *For every $x \in D$ and $t \geq 0$, let us define*

$$\eta_x(t) := \frac{2N}{N-1}\mathbb{E}_x[H_W(X_t)] \quad \textit{where} \quad H_W(x) := \frac{1}{2N^2}\sum_{i\neq j} W(x_i-x_j).$$

Then, for every $x \in D$ and $t \geq 0$,

$$\eta_x(t) \leq -\log\Big(\mathrm{e}^{-\eta_x(0)-4\alpha_N t/N} + \frac{2}{N}(1-\mathrm{e}^{-4\alpha_N t/N})\Big)$$

and in particular

$$\eta_x(t) \leq \log\frac{N}{2(1-\mathrm{e}^{-4\alpha_N t/N})} \quad \textit{and} \quad \eta_x(t) \leq \max\Big(\eta_x(0), \log\frac{N}{2}\Big).$$

Proof of Lemma 5.3. Taking expectation to the first line in equation (4.1) and subtracting the obtained identity from the inequality in Remark 3.3 b), we get

$$\mathbb{E}_x(H_W(X_t)) - H_W(x) \leq \mathbb{E}_x\Big(\int_0^t LH_W(X_s)\,\mathrm{d}s\Big)$$

for all $t \geq 0$. But from (2.8) and (5.1) we get

$$\begin{aligned}LH_W(x) &= 2\frac{\alpha_N}{N^2}(N-1) - 4\frac{\alpha_N}{N^4}\sum_{i=1}^N\Big|\sum_{j\neq i}\frac{x_i-x_j}{|x_i-x_j|^2}\Big|^2\\ &\leq 2\frac{\alpha_N}{N^2}(N-1) - 4\frac{\alpha_N}{N^4}\sum_{i\neq j}\frac{1}{|x_i-x_j|^2}\\ &= 4\frac{\alpha_N}{N^2}\frac{N-1}{N}\left[\frac{N}{2} - \frac{1}{N(N-1)}\sum_{i\neq j}\frac{1}{|x_i-x_j|^2}\right].\end{aligned}$$

On the other hand, by the Jensen inequality,

$$H_W(x) = \frac{N-1}{2N}\frac{1}{N(N-1)}\sum_{i\neq j}\log\frac{1}{|x_i-x_j|^2} \leq \frac{N-1}{2N}\log\left(\frac{1}{N(N-1)}\sum_{i\neq j}\frac{1}{|x_i-x_j|^2}\right).$$

Therefore, we get

$$LH_W(x) \leq 4\frac{\alpha_N}{N^2}\frac{N-1}{N}\left[\frac{N}{2} - \mathrm{e}^{\frac{2N}{N-1}H_W(x)}\right].$$

Using again the Jensen inequality, it follows that

$$\begin{aligned}\eta_x(t) &\leq \eta_x(0) + \frac{2N}{N-1}\int_0^t \mathbb{E}_x LH_W(X_s)\,\mathrm{d}s\\ &\leq \eta_x(0) + \frac{8\alpha_N}{N^2}\int_0^t\left[\frac{N}{2} - \mathbb{E}_x\big[\mathrm{e}^{\frac{2N}{N-1}H_W(X_s)}\big]\right]\mathrm{d}s\\ &\leq \eta_x(0) + \frac{8\alpha_N}{N^2}\int_0^t\left[\frac{N}{2} - e^{\eta_x(s)}\right]\mathrm{d}s.\end{aligned}$$

Therefore

$$\mathrm{e}^{-\eta_x(t)} \geq \mathrm{e}^{-\eta_x(0)-4\alpha_N t/N} + \frac{2}{N}(1-\mathrm{e}^{-4\alpha_N t/N}) \geq \min\{\frac{2}{N}, \mathrm{e}^{-\eta_x(0)}\}$$

by time integration for the first bound and then, for the second bound, by writing the obtained expression as the interpolation between $2/N$ and $\mathrm{e}^{-\eta_x(0)}$. Dropping the $\mathrm{e}^{-\eta_x(0)-4\alpha_N t/N}$ term gives the second upper bound in the lemma. □

Proof of Theorem 1.3. In order to prove that P^N satisfies a Poincaré inequality, we follow the approach developed in [4] based on a Lyapunov function together with a local Poincaré inequality (see also the proof of [17, Th. 1.1]). This approach amounts to find a positive $\mathcal{C}^2$ function ϕ on D, a compact set $K \subset D$ and positive constants c, c', such that on D

$$L\phi \leq -c\phi + c'\mathbf{1}_K.$$

Such a ϕ is called a Lyapunov function. Indeed, for a centered $f \in \mathcal{F}$ this gives

$$\int f^2\,\mathrm{d}P^N \leq \int_K \frac{c'}{c\phi}f^2\,\mathrm{d}P^N + \int -\frac{L\phi}{c\phi}f^2\mathrm{d}P^N.$$

The first term of the right-hand side can be controlled using a local Poincaré inequality, in other words a Poincaré inequality on every ball included in D, by comparison to the uniform measure. The second one can be handled using an integration by parts which is allowed since $f \in \mathcal{F}$. See [4] and [17] for the details.

For our model P^N we take the $\mathcal{C}^\infty$ function

$$\phi = \mathrm{e}^{\gamma H}$$

for some $\gamma > 0$. This function is larger than or equal to 1 by Lemma 3.2, and the probability measure P^N has a smooth positive density on D, which provides a local Poincaré constant that may depend on N however.

Let us check that ϕ is a Lyapunov function. To this end, let us show that there exist constants $c, c'' > 0$ and a compact set $K \subset D$ such that, on D,

$$\frac{L\phi}{\phi} \le -c + c''\mathbf{1}_K.$$

Indeed, since ϕ is positive and bounded on the compact set K, this gives, on D,

$$L\phi \le -c\phi + c'' \sup_{x\in K} |\phi(x)|\mathbf{1}_K = -c\phi + c'\mathbf{1}_K.$$

In order to compute $L\phi/\phi$, we observe that

$$\nabla\phi = \gamma\phi\nabla H \quad \text{and} \quad \Delta\phi = \gamma^2\phi|\nabla H|^2 + \gamma\phi\Delta H.$$

Therefore, by (1.7),

$$\frac{\beta_N}{\alpha_N\gamma}\frac{L\phi}{\phi} = \frac{\Delta\phi - \beta_N\nabla H\cdot\nabla\phi}{\gamma\phi} = \Delta H + (\gamma - \beta_N)|\nabla H|^2.$$

Now $\Delta H = 4$ on D by (2.6). Moreover by Lemma 5.2, for $\gamma < \beta_N$ there exists a compact set $K \subset D$ such that

$$(\beta_N - \gamma) \inf_{(x_1,\dots,x_N)\in K^c} |\nabla H|^2(x_1,\dots,x_N) > 5.$$

One can take for instance

$$K = \{x \in (\mathbb{R}^2)^N : |x| \le R \text{ and } \min_{i\neq j} |x_i - x_j| \ge \varepsilon\}$$

for $R > 0$ large enough and $\varepsilon > 0$ small enough.

Then $\beta_N/(\alpha_N\gamma)L\phi/\phi \le -1$ on $D \setminus K$ and the Poincaré inequality is proved. Note that we can take $\gamma = 1$ if $\beta_N \ge N$. □

Remark 5.4 (Poincaré inequality for P^2). *Let us give an alternative direct proof of the Poincaré inequality for the probability measure P^2. Consider indeed the change of variable $(u, v) = ((x_1 + x_2)/2, (x_1 - x_2)/2)$ on $\mathbb{R}^2 \times \mathbb{R}^2$, which has the advantage to decouple the variables (this miracle is available only in the two particle case $N = 2$). Letting $\beta = \beta_2$, we get a probability density function on $\mathbb{R}^2 \times \mathbb{R}^2$ proportional to*

$$(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathrm{e}^{-\beta|u|^2-\beta|v|^2-+\beta/2\log|v|} = \mathrm{e}^{-\beta|u|^2}|v|^{\beta/2}\mathrm{e}^{-\beta|v|^2}.$$

This probability measure is the tensor product of the Gaussian measure, which satisfies a Poincaré inequality, and of the measure μ with density

$$\frac{\mathrm{e}^{-\beta\Psi(v)}}{Z} \quad \textit{with} \quad \Psi(v) = |v|^2 - \frac{1}{2}\log|v|.$$

The measure μ is not log-concave at all (singularity at zero notably) but $\Psi(v)$ is a convex function of the norm $r = |v|$. Hence [9, Th. 1] *ensures that μ satisfies a Poincaré inequality, and then so does our product measure by tensorization.*

Note that one can prove Poincaré for μ by using a Lyapunov function as in the proof of Theorem 1.3, instead of [9, Th. 1]*: namely if $L' := \Delta - \beta\nabla\Psi\cdot\nabla$ in dimension two and $\phi = \mathrm{e}^{\beta\Psi}/2$, then*

$$\frac{L'\phi}{\phi} = \frac{\beta}{2}\Delta\Psi - \frac{\beta^2}{4}|\nabla\Psi|^2, \quad \nabla\Psi(v) = 2v - \frac{v}{2|v|^2}, \quad \Delta\Psi = 4$$

for $v \ne 0$ (recall that $\log|v|$ is harmonic in dimension two). Therefore

$$\frac{L'\phi}{\phi} = 2\beta^2 - \beta^2\Big(|x| - \frac{1}{4|x|}\Big)^2 \le -c + c'\mathbf{1}_K$$

for the compact set

$$K := \{x \in \mathbb{R}^2 : r \le |x| \le R\}$$

with $0 < r < R$ well chosen.

6. Proof of Theorem 1.8

Recall that if μ^N is the random empirical measure under P^N then for any continuous and bounded test function $f:\mathbb{R}^2\to\mathbb{R}$, using exchangeability and (1.15),

$$\begin{aligned}\mathbb{E}_{P^N}\int_{\mathbb{R}^2} f(x)\,\mu^N(\mathrm{d}x) &= \int_{(\mathbb{R}^2)^N}\left(\frac{1}{N}\sum_{k=1}^N f(x_k)\right)\varphi^{N,N}(x_1,\dots,x_N)\mathrm{d}x_1\cdots\mathrm{d}x_N\\ &= \int_{\mathbb{R}^2} f(x)\,\varphi^{1,N}(x)\,\mathrm{d}x = \mathbb{E}_{P^{1,N}}(f)\end{aligned}$$

where $P^{1,N}$ is the 1-dimensional marginal of P^N. By Theorem 1.9, as $N\to\infty$, the density $\varphi^{1,N}$ of $P^{1,N}$ tends to the density of the uniform distribution μ_∞ on the unit disc of $\mathbb{R}^2$. The probability measure μ_∞ satisfies a Poincaré inequality for the Euclidean gradient, since for instance it is a Lipschitz contraction of the standard Gaussian on $\mathbb{R}^2$. Unfortunately, the convergence of densities above is not enough to deduce that $P^{1,N}$ satisfies a Poincaré inequality (uniformly in N or not).

Proof of Theorem 1.8. The idea is to view $P^{1,N}$ as a Boltzmann – Gibbs measure and to use some hidden convexity. Namely, from (1.16) its density is given on $\mathbb{R}^2$ by

$$\varphi^{1,N}(x)=\frac{\mathrm{e}^{-N|x|^2-\psi(\sqrt{N}x)}}{\pi}\quad\text{with}\quad \psi(x):=-\log\sum_{\ell=0}^{N-1}\frac{|x|^{2\ell}}{\ell!}. \tag{6.1}$$

If we now write $f(x)=|x|^2+\psi(x)=g(r^2)$ with $r=|x|$ and $g(t)=t-\log\sum_{\ell=0}^{N-1}t^\ell/\ell!$ then

$$\nabla f(x)=2g'(r^2)x\quad\text{and}\quad\nabla^2 f(x)=4g''(r^2)x\otimes x+2g'(r^2)I_2$$

and

$$g'(t)=\frac{\frac{t^{N-1}}{(N-1)!}}{\sum_{\ell=0}^{N-1}\frac{t^\ell}{\ell!}}\geq 0\quad\text{and}\quad g''(t)=\frac{t^{N-2}}{(N-2)!}\frac{\left(\sum_{\ell=0}^{N-1}\frac{t^\ell}{\ell!}-\frac{t}{N-1}\sum_{\ell=0}^{N-2}\frac{t^\ell}{\ell!}\right)}{\left(\sum_{\ell=0}^{N-1}\frac{t^\ell}{\ell!}\right)^2}\geq 0.$$

It follows that f is convex (note that its Hessian vanishes at the origin), and in other words $P^{1,N}$ is log-concave. Therefore, according to a criterion stated in [8, Th. 1.2] and essentially due to Kannan, Lovász and Simonovits, it suffices to show that the second moment of $P^{1,N}$ is uniformly bounded in N.

But, using the density (6.1) of $P^{1,N}$, this moment is

$$\int_{\mathbb{R}^2}|x|^2P^{1,N}(\mathrm{d}x)=\sum_{\ell=0}^{N-1}\frac{N^\ell}{\ell!}\int_0^\infty r^{2(\ell+1)}2r\mathrm{e}^{-Nr^2}\,dr=\sum_{\ell=0}^{N-1}\frac{N^\ell}{\ell!}\frac{(\ell+1)!}{N^{\ell+2}}=\frac{N+1}{2N}\leq\frac{1}{2}\cdot \tag{6.2}$$

This concludes the argument thanks to the Bobkov criterion. □

With $\beta_N=N^2$ and since $P^{1,N}=\mathbb{E}\mu^N$, (6.2) is consistent with (1.10) since in this case

$$\lim_{t\to\infty}\mathbb{E}[H_V(X_t)\,|\,X_0=x]=\frac{1}{2}+\frac{1}{N}-\frac{1}{2N}=\frac{N+1}{2N}\cdot$$

Note also that, by (6.2), the second moment of $P^{1,N}=\mathbb{E}\mu^N$ tends to $1/2$ as $N\to\infty$; this turns out to be the second moment of its weak limit μ_∞ since

$$\int_{\mathbb{R}^2}|x|^2\,\mu_\infty(\mathrm{d}x)=\frac{2\pi}{\pi}\int_0^1 r^3\mathrm{d}r=\frac{1}{2}\cdot$$

Observe finally that a bound on the second moment of $P^{1,N}=\mathbb{E}\mu^N$ can be obtained as follows. Let M be a $N\times N$ random matrix with i.i.d. entries of Gaussian law $\mathcal{N}(0,\frac{1}{2N}I_2)$

(in other words an element of the Complex Ginibre Ensemble); then, by Weyl's inequality [37, Th. 3.3.13] on the eigenvalues,

$$\int_{\mathbb{R}^2}|x|^2\,\mathbb{E}\mu^N(\mathrm{d}x)=\frac{1}{N}\mathbb{E}\sum_{k=1}^N|\lambda_k(M)|^2\leq\frac{1}{N}\mathbb{E}\sum_{k=1}^N\lambda_k(MM^*)=\frac{1}{N}\mathbb{E}\mathrm{Tr}(MM^*)=1.$$

Remark 6.1 (Poincaré via spherical symmetry)**.** *The probability measure $P^{1,N}$ is also spherically symmetric, or rotationally invariant, as in Bobkov* [9] *(see also* [11]*). Namely, in the notation $f(x)=g(r^2)$ with $r=|x|$ for the "potential" of the density of $P^{1,N}$, as in the proof of Theorem 1.8, let $h(r)=g(r^2)$. Then*

$$f(x)=h(r),\quad\nabla f(x)=h'(r)\frac{x}{|x|}\quad\text{and}\quad\nabla^2 f(x)=h''(r)\frac{x\otimes x}{|x|^2}+h'(r)\frac{\begin{pmatrix}x_2^2&-x_1x_2\\-x_1x_2&x_1^2\end{pmatrix}}{|x|^3}.$$

The matrix on the right-hand side has non-negative trace and null determinant, and is thus positive semi-definite (it is the Hessian of the norm $x\mapsto|x|=r$). Moreover

$$h'(t)=2g'(t^2)t,\quad h''(t)=4g''(t)t^2+2g'(t^2)\geq 0.$$

It follows that $P^{1,N}$ is a spherically symmetric probability measure on $\mathbb{R}^2$, and its density is a log-concave function of the norm (and it vanishes at the origin). Now according to [9, Th. 1]*, it follows that the probability measure $P^{1,N}$ satisfies a Poincaré inequality with a constant which depends only on the second moment, which again is bounded in N.*

Remark 6.2 (Logarithmic Sobolev inequality)**.** *According to Bobkov's result* [8, Th. 1.3]*, we even get for $P^{1,N}$ a logarithmic Sobolev inequality with a uniform constant in N provided that $P^{1,N}$ has a sub-Gaussian tail uniformly in N (which is stronger than the second moment control). This is indeed the case. Namely, if $Z\sim P^{1,N}$, then for any real $R\geq 0$,*

$$\mathbb{P}(Z\geq R)=\int_0^{2\pi}\int_R^{+\infty}\frac{\mathrm{e}^{-Nr^2}}{\pi}\sum_{\ell=0}^{N-1}\frac{(Nr^2)^\ell}{\ell!}\,r\mathrm{d}\theta\mathrm{d}r=\frac{1}{N}\int_{NR^2}^{+\infty}\mathrm{e}^{-s}\sum_{\ell=0}^{N-1}\frac{s^\ell}{\ell!}\,\mathrm{d}s.$$

Moreover

$$\frac{1}{N}\sum_{\ell=0}^{N-1}\frac{s^\ell}{\ell!}\leq\frac{s^N}{N!}\leq 2^N\mathrm{e}^{\frac{1}{2}s}$$

for $s\geq N$. Hence, for $R\geq 2$,

$$\mathbb{P}(Z\geq R)\leq\int_{NR^2}^{+\infty}2^N\mathrm{e}^{-\frac{1}{2}s}\,\mathrm{d}s=2^{N+1}\mathrm{e}^{-\frac{1}{2}NR^2}\leq 4\mathrm{e}^{-\frac{1}{2}R^2}.$$

7. Proof of Theorem 1.9

Proof of the first part of Theorem 1.9. It is a consequence of (1.18) and of the following theorem. Indeed, by Lebesgue's dominated convergence, (1.18) implies that $\mathbb{E}F(\mu^N)$ tends to $F(\mu_\infty)$ for every continuous and bounded function $F:\mathcal{P}(E)\to\mathbb{R}$. In other words, (i) holds in Theorem 7.1, whence (ii), which is exactly the first part of Theorem 1.9.

Theorem 7.1 (Characterizations of chaoticity)**.** *Let E be a Polish space and $\mathcal{P}(E)$ be the Polish space of Borel probability measures on E endowed with the weak convergence topology. Let μ be an element of $\mathcal{P}(E)$ and let $(P^N)_N$ a sequence of exchangeable probability measures on E^N. Let us define the random empirical measure*

$$\mu^N=\frac{1}{N}\sum_{i=1}^N\delta_{X_i}$$

where $(X_1,\ldots,X_N)$ has law P^N. Then the following properties are equivalent:

(i) *the law of μ^N converges to δ_μ weakly in $\mathcal{P}(\mathcal{P}(E))$;*

(ii) *for any fixed $k \leq N$ the k-th dimensional marginal distribution $P^{k,N}$ of P^N converges weakly in $\mathcal{P}(E^k)$ to the product probability measure $\mu^{\otimes k}$;*
(iii) *the 2-nd dimensional marginal $P^{2,N}$ of P^N converges to $\mu^{\otimes 2}$ weakly in $\mathcal{P}(E^2)$.*

Proof of Theorem 7.1. Theorem 7.1 is stated for instance in [51, p. 260], [46, Prop. 4.2] and [50, Prop. 2.2], but with a sketchy proof that (iii) implies (i). For the reader's convenience, we detail this proof when E satisfies the following property : there exists a countable subset D of the set $C_b(E)$ of continuous and bounded functions $E \to \mathbb{R}$, such that for $(\mu_n)_n, \mu$ in $\mathcal{P}(E)$, it holds $\int \phi\, d\mu_n \to \int \phi\, d\mu$ for any ϕ in $C_b(E)$ as soon as it holds for any ϕ in D. For instance this property holds when E is the Euclidean space.

Since $\mathcal{P}(\mathcal{P}(E))$ is metrizable, it is enough to check that for any sequence $(N_k)_k$ there exists a subsequence $(N_{k_j})_j$ such that the law of $\mu^{N_{k_j}}$ converges to δ_μ. But, by expanding the square, exchangeability and (iii),

$$\mathbb{E}\left(\Big|\int_E \phi\, d\mu^{N_k} - \int_E \phi\, d\mu\Big|^2\right) \to 0, \qquad k \to +\infty$$

for any ϕ in $C_b(E)$ and hence in D. Hence for any such ϕ there exists a subsequence still denoted $(N_{k_j})_j$ such that $\int \phi\, d\mu^{N_{k_j}} \to \int \phi\, d\mu$ almost surely. Now, by a diagonal extraction argument, we can build another subsequence $(N_{k_j})_j$ such that, almost surely, $\int \phi\, d\mu^{N_{k_j}} \to \int \phi\, d\mu$ for any ϕ in D. By definition of D, this implies that, almost surely, $\mu^{N_{k_j}}$ converges to μ in the metric space $\mathcal{P}(E)$. It follows that the law of $\mu^{N_{k_j}}$ converges to δ_μ by the Lebesgue dominated convergence theorem. Hence (i) since $\mathcal{P}(\mathcal{P}(E))$ is metrizable. ☐

Proof of the second part of Theorem 1.9. We first describe the behavior of the one-marginal density function $\varphi^{1,N}$. From (1.16) it is given by

$$\varphi^{1,N}(z) = \frac{\mathrm{e}^{-N|z|^2}}{\pi}\mathrm{e}_N(N|z|^2), \quad z \in \mathbb{C},$$

where $\mathrm{e}_N(w) := \sum_{\ell=0}^{N-1} w^\ell/\ell!$ is the truncated exponential series. Then, pointwise in $\mathbb{C}$,

$$\varphi^{1,N}(z) \to \frac{1}{\pi}\Big(\mathbf{1}_{|z|<1} + \frac{1}{2}\mathbf{1}_{|z|=1}\Big), \qquad N \to \infty. \tag{7.1}$$

Namely, by rotational invariance, it suffices to consider the case $z = r > 0$. Next, if $Y_1, \ldots, Y_N$ are i.i.d. random variables following the Poisson distribution of mean r^2, then

$$\mathrm{e}^{-Nr^2}\mathrm{e}_N(Nr^2) = \mathbb{P}(Y_1 + \cdots + Y_N < N) = \mathbb{P}\left(\frac{Y_1 + \cdots + Y_N}{N} < 1\right).$$

Now, as $N \to \infty$, $\frac{Y_1+\cdots+Y_N}{N} \to r^2$ almost surely by the law of large numbers, and thus the right-hand side above tends to 0 if $r > 1$ and to 1 if $r < 1$. In other words

$$\mathrm{e}^{-Nr^2}\mathrm{e}_N(Nr^2) \to \mathbf{1}_{r<1}$$

provided $r \neq 1$. For $r = 1$ by the central limit theorem we get

$$\mathbb{P}\left(\frac{Y_1 + \cdots + Y_N}{N} < 1\right) = \mathbb{P}\left(\frac{Y_1 + \cdots + Y_N - N}{\sqrt{N}} < 0\right) \to \frac{1}{2}.$$

In fact, the convergence in (7.1) holds uniformly on compact sets outside the unit circle $|z| = 1$, as shown in Lemma 7.2 below. It cannot hold uniformly on arbitrary compact sets of $\mathbb{C}$ since the pointwise limit is not continuous on the unit circle.

We now turn to the two-marginal density function $\varphi^{2,N}$. By (1.16) it is given by

$$\begin{aligned}\varphi^{2,N}(z_1, z_2) &= \frac{N}{N-1}\frac{\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}}{\pi^2}(\mathrm{e}_N(N|z_1|^2)\mathrm{e}_N(N|z_2|^2) - |\mathrm{e}_N(Nz_1\overline{z}_2)|^2)\\ &= \frac{N}{N-1}\varphi^{1,N}(z_1)\varphi^{1,N}(z_2) - \frac{N}{N-1}\frac{\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}}{\pi^2}|\mathrm{e}_N(Nz_1\overline{z}_2)|^2 \end{aligned} \tag{7.2}$$

for every $z_1, z_2 \in \mathbb{C}$.

It follows that for any $N \geq 2$ and $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}\Delta_N(z_1,z_2) &:= \varphi^{2,N}(z_1,z_2) - \varphi^{1,N}(z_1)\varphi^{1,N}(z_2) \\ &= \frac{1}{N-1}\varphi^{1,N}(z_1)\varphi^{1,N}(z_2) - \frac{N}{N-1}\frac{\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}}{\pi^2}|\mathrm{e}_N(Nz_1\overline{z}_2)|^2. \end{aligned} \tag{7.3}$$

In particular, using $\varphi^{2,N} \geq 0$ for the lower bound,

$$-\varphi^{1,N}(z_1)\varphi^{1,N}(z_2) \leq \Delta_N(z_1,z_2) \leq \frac{1}{N-1}\varphi^{1,N}(z_1)\varphi^{1,N}(z_2).$$

From this and Lemma 7.2 we first deduce that for any compact subset K of $\{z \in \mathbb{C} : |z| > 1\}$

$$\lim_{N\to\infty} \sup_{\substack{z_1\in\mathbb{C}\\ z_2\in K}} |\Delta_N(z_1,z_2)| = \lim_{N\to\infty} \sup_{\substack{z_1\in K\\ z_2\in\mathbb{C}}} |\Delta_N(z_1,z_2)| = 0.$$

To conclude the proof of Theorem 1.9 it remains to show that $\Delta_N(z_1,z_2) \to 0$ as $N \to \infty$ when z_1 and z_2 are in compact subsets of $|z_1| < 1, |z_2| < 1$. In this case $|z_1\overline{z}_2| \leq 1$, and Lemma 7.2 gives

$$|\mathrm{e}_N(Nz_1\overline{z}_2)|^2 \leq 2\mathrm{e}^{2N\mathfrak{Re}(z_1\overline{z}_2)} + 2r_N^2(z_1\overline{z}_2).$$

Next, using the elementary identity $2\mathfrak{Re}(z_1\overline{z}_2) = |z_1|^2 + |z_2|^2 - |z_1 - z_2|^2$, we get

$$\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}|\mathrm{e}_N(Nz_1\overline{z}_2)|^2 \leq 2\mathrm{e}^{-N|z_1-z_2|^2} + 2\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}r_N^2(z_1\overline{z}_2). \tag{7.4}$$

Since $|z_1\overline{z}_2| \leq 1$, the formula for r_N in Lemma 7.2 gives

$$\mathrm{e}^{-N(|z_1|^2+|z_2|^2)}r_N^2(z_1\overline{z}_2) \leq \mathrm{e}^{-N(|z_1|^2+|z_2|^2-2-\log|z_1|^2-\log|z_2|^2)}\frac{(N+1)^2}{2\pi N}.$$

Using (7.3), (7.4) and the bounds $\varphi^{1,N} \leq 1/\pi$ and $u - 1 - \log u > 0$ for $0 < u < 1$, it follows that $\Delta_N(z_1,z_2)$ tends to 0 as $N \to \infty$ uniformly in z_1, z_2 on compact subsets of

$$\{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, z_1 \neq z_2\}.$$

This achieves the proof of Theorem 1.9.

Lemma 7.2 (Exponential series)**.** *Let* $\mathrm{e}_N(w) := \sum_{\ell=0}^{N-1} w^\ell/\ell!$ *denote the truncated exponential series. For every* $N \geq 1$ *and* $z \in \mathbb{C}$,

$$|\mathrm{e}_N(Nz) - \mathrm{e}^{Nz}\mathbf{1}_{|z|\leq 1}| \leq r_N(z)$$

where

$$r_N(z) := \frac{\mathrm{e}^N}{\sqrt{2\pi N}}|z|^N\left(\frac{N+1}{N(1-|z|)+1}\mathbf{1}_{|z|\leq 1} + \frac{N}{N(|z|-1)+1}\mathbf{1}_{|z|>1}\right).$$

In particular, for any compact subset $K \subset \mathbb{C} \setminus \{z \in \mathbb{C} : |z| = 1\}$,

$$\lim_{N\to\infty}\sup_{z\in K}\left|\varphi^{1,N}(z) - \frac{\mathbf{1}_{|z|\leq 1}}{\pi}\right| = \pi^{-1}\lim_{N\to\infty}\sup_{z\in K}\left|\mathrm{e}^{-N|z|^2}\mathrm{e}_N(N|z|^2) - \mathbf{1}_{|z|\leq 1}\right| = 0.$$

Proof of Lemma 7.2. As in Mehta [45, Ch. 15], for every $N \geq 1$, $z \in \mathbb{C}$, if $|z| \leq N$ then

$$|\mathrm{e}^z - \mathrm{e}_N(z)| = \left|\sum_{\ell=N}^{\infty}\frac{z^\ell}{\ell!}\right| \leq \frac{|z|^N}{N!}\sum_{\ell=0}^{\infty}\frac{|z|^\ell}{(N+1)^\ell} = \frac{|z|^N}{N!}\frac{N+1}{N+1-|z|},$$

while if $|z| > N$ then

$$|\mathrm{e}_N(z)| \leq \sum_{\ell=0}^{N-1}\frac{|z|^\ell}{\ell!} \leq \frac{|z|^{N-1}}{(N-1)!}\sum_{\ell=0}^{N-1}\frac{(N-1)^\ell}{|z|^\ell} \leq \frac{|z|^{N-1}}{(N-1)!}\frac{|z|}{|z|-N+1}.$$

Therefore, for every $N \geq 1$ and $z \in \mathbb{C}$,

$$|\mathrm{e}_N(Nz) - \mathrm{e}^{Nz}\mathbf{1}_{|z|\leq 1}| \leq \frac{N^N}{N!}\Big(|z|^N \frac{N+1}{N+1-|Nz|}\mathbf{1}_{|z|\leq 1} + |z|^{N-1}\frac{|Nz|}{|Nz|-N+1}\mathbf{1}_{|z|>1}\Big).$$

It remains to use the Stirling bound $\sqrt{2\pi N}N^N \leq N!\mathrm{e}^N$ to get the first result. □

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