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Antithetic random fields applied to mine planning under uncertainty

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ABSTRACT

Traditional practice in mine planning often relies on estimation techniques that fail to account for the intrinsic uncertainty of geology and grades, which may have significant consequences in the mine operation. Dealing with this uncertainty has been a major topic in the last years, where different algorithms and stochastic optimization models have been proposed to tackle this issue. However, the increasing complexity of these stochastic models and the use of several simulations to represent the deposit variability impose a computational challenge in terms of resolution times, making them difficult to apply in large data or complex mining operations. In this paper we explore the antithetic random fields approach as a variance reduction technique, to solve a stochastic short-term mine planning problem, aiming to reduce the number of simulations required to obtain a reliable NPV value. The reliability of the result is measured by the variance of the NPV when the problem is optimized with different sets of realizations. Our results show that this technique produces a significant variance reduction in the inference of the expected NPV value in the stochastic problem for a copper deposit application, generating a lower dispersion with a smaller sample size, compared to traditional simulation techniques.

1. Introduction

Uncertainty has been a major challenge in the mining industry. Almost every decision taken in a mining project has to deal with uncertainty in some way. In the last decades there has been an increasing interest in addressing this uncertainty with new techniques that not only account for a single estimate of the uncertain attributes, but a wide range of possible scenarios of the real deposit through geostatistical simulations. This technique has allowed the planner to make decisions considering how a fixed production plan will respond to different scenarios. But a different question is how to consider this uncertainty during mine planning optimization. This is often addressed with stochastic optimization techniques, which assist the mine planner to decide the best strategy under uncertainty of the parameters such as grade, cost, price, etc. However, the incorporation of uncertainty requires a large number of grade scenarios to represent the true variability of the deposit, which makes the problem prohibitively large to solve efficiently. In this paper we tackle this issue answering the question: Can the number of geostatistical simulations in the optimization problem be reduced, without compromising the representation of the deposit's true variability? To answer this question, we use a variance reduction technique applied to geostatistical simulation, and we prove its effectiveness in a stochastic optimization problem.

In the natural resources industry, one approach to reduce the number of simulations is the selection of a small set of simulations that represent the characteristics of a larger set. The final number of scenarios is fixed, based on the computational runtime of the model studied, and the challenge is finding the best subset from the larger set, which minimize some predefined distance measure between both sets based on a relevant attribute of the simulations. Heitsch and Römisch (2003) applied this approach, focusing their work in a heuristic to find the best subset of realizations from the larger set, where they add or subtract a single scenario sequentially until they achieve a satisfactory result. This was also used in power management problems (Dupačová et al. (2003), Heitsch and Römisch (2009), Gröwe-Kuska et al. (2003)). Armstrong et al. (2013) used this methodology in a stochastic mine planning problem, but the sequential approach proposed by Heitsch and Römisch (2003) did not perform as expected, delivering a subset with a value 50% off from the true optimal for a small case study. They proposed a different approach instead, based on a random search

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[1] Authorship statement: Gonzalo Nelis developed the methodology and case studies in the paper. Julian M. Ortiz proposed the methodology and advised Mr. Nelis during the development of this work. Nelson Morales proposed the stochastic optimization formulation, and provided the corresponding explanations for the optimization model.

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procedure considering a proxy variable for each scenario as the total tonnage of ore for a range of cut-off grades, used to select the smaller set. It was tested in a real deposit with promising results: the value computed using 12 scenarios was 1% off the value computed with 100.

Scheidt and Caers (2009a, b) present a methodology to select realizations from a large set based on a clustering method on a projection over a reduced space, using kernels and specific distances to measure similarity between simulations. The application, requires, however, having access to a large set of simulations of petrophysical properties before selecting the subset to post process for history matching in petroleum applications. Other authors have approached the problem by defining a proxy variable that is linked to the performance response, and devised a ranking of the realizations to then select a small number to characterize the uncertainty in the response (Deutsch and Srinivasan (1996); McLennan and Deutsch (2005)). A typical approach consists on processing only a few key scenarios linked to the percentiles 10, 50 and 90, as a way of representing the expected variability of the response (Deutsch (2007), Pereira et al. (2017)). It should be noted that the proxy variable must have a monotonic behavior with respect to the transfer function.

Another approach has been the implementation of variance reduction techniques. These are procedures to reduce the estimate variance without increasing the number of simulations or, conversely, achieving the same estimate variance with fewer simulations tested. This usually requires modifying the simulation algorithm, in order to sample the probability space with fewer runs. A complete review of these techniques can be found in Cheng (1986), James (1985) and Kleijnen et al. (2010). This approach does not require a large number of scenarios in the first place, so they can be used to generate the desired number of scenarios directly, which leads to further reduction of the computational complexity when the simulation algorithm is complex.

A particular methodology using antithetic variates was proposed by Guthke and Bárdossy (2012). Their work was based on an extension of this technique to a sequential simulation algorithm. They showed that this implementation can reduce the estimate variance up to 20% of the conventional Monte Carlo methods in two stochastic hydrogeology applications.

In this paper, we develop an implementation of the antithetic random fields proposed by Guthke and Bárdossy (2012) in a geostatistical simulation algorithm, applied to a mine planning under uncertainty problem.

Mine planning under uncertainty is the process of making the mine planning decision considering uncertainty in the parameters such as prices, grade, etc. A complete review of operations research in mine planning can be found in Newman et al. (2010).

Uncertainty has been incorporated into the mine planning process by different means. Of particular interest in this work is the stochastic programming approach, which takes into account different decision stages in a stochastic program, to obtain a mining sequence considering the flexibility of changing some of these decisions based on new information. Among these works are a multistage approach proposed by Boland et al. (2008) and a two-stage method proposed by Moreno et al. (2017).

In this paper we test the variance reduction technique implementation in a two-stage stochastic programming model that evaluates the timing of the blasthole grade information in short-term mine planning, and its effect on the NPV considering grade uncertainty.

2. Methods

2.1. Antithetic variates technique

The implementation proposed in this paper is an extension of the antithetic variates technique, which is based on generating scenarios with negative correlation (Hammersley and Morton (1956)). Consider the example of estimating the expected value of some transfer function f, $Y = \mathbb{E}[f(X)]$, and two outcomes of a random simulation: $f(X_1) = Y_1$ and $f(X_2) = Y_2$. An unbiased estimator using these two values is:

$$Y_{AV} = \frac{Y_1 + Y_2}{2} \quad \mathrm{Var}[Y_{AV}] = \frac{\sigma^2 + \mathrm{Cov}[Y_1, Y_2]}{2} \tag{1}$$

If both outcomes are independent, the value of the covariance is zero and this estimator is equivalent to the conventional Monte Carlo estimation. However, if the correlation between both values is negative, the covariance is negative and then the variance of the estimator would yield a lower result. This is the basic premise of the antithetic variates method: negatively correlating each scenario to achieve a lower estimation variance in the response. The negative correlation is then achieved by modifying the simulation process used to obtain the sample. It is relevant to notice that the transfer function must be monotonic to preserve the negative correlation of the scenarios and obtain a lower variance in the estimation of the response.

For the current application, we are interested in the expected value of the net present value. The method can be applied in more general cases, to characterize any parameter of the full distribution of possible outcomes (e.g. min, max, quartiles, interquartile range, median, etc.), although the impact of this variance reduction technique for these parameters is not studied in the present work.

To apply this technique in geostatistics, we will use the Sequential Gaussian Simulation Algorithm (Deutsch and Journel (1998)), since the application of antithetic variates in this context is straightforward and will be addressed in the next section.

2.2. Antithetic random fields

In Sequential Gaussian Simulation Algorithm, a standard Gaussian random variable is simulated in a spatial grid, where each node of this grid is visited sequentially in a random order, performing a simple kriging estimation using the previously simulated nodes and the normal scores of the true data to condition its value, and drawing a random number from a Gaussian distribution with mean and variance given by the kriging estimate and variance, respectively, to obtain the final simulated value.

Traditionally, to generate m scenarios, this method must be repeated independently m times, with different random paths and different random numbers for the simulation stage. Our proposal is the application of the methodology proposed by Guthke and Bárdossy (2012), imposing a correlation on these random numbers: the same node will be simulated using negatively correlated Gaussian random numbers in different scenarios with a fixed random path.

This methodology can be applied to an arbitrary number of scenarios, where all of them exhibit negative correlation. For m scenarios with n nodes, m standard Gaussian random vectors of size n are generated and a particular correlation matrix between them is imposed. Since we desire to obtain a constant negative correlation among every scenario, consider matrix (2) as a correlation matrix between these m vectors with pairwise correlation coefficient α.

$$\mathbf{C}_m = \begin{pmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{pmatrix} \tag{2}$$

Since we want the lowest possible correlation coefficient α, and considering that matrix (2) has to be a valid correlation matrix, we get that this value is constrained by relation (3) (Guthke and Bárdossy (2012)):

$$\alpha \geq -\frac{1}{m-1} \tag{3}$$

From Eq. (3), if the number of scenarios we want to correlate increases, α gets closer to zero, since the pairwise correlation must decrease to get a negative correlation among every scenario.

Then, to impose this correlation matrix to m standard Gaussian random vectors of size n, the following algorithm must be followed:

1. The correlation coefficient α is calculated based on relation (3) and the correlation matrix (2) is generated.
2. A matrix $\mathbf{B}$ is found such that $\mathbf{B}\mathbf{B}^{\intercal} = \mathbf{C}_m$. This could be achieved using a Cholesky decomposition technique.
3. Then, for each node to simulate $i \leq n$:
   - A tuple of m standard Gaussian random numbers is generated, $\mathbf{g}_m^i$.
   - A vector $\mathbf{z}_m^i = \mathbf{B}\mathbf{g}_m^i$ is calculated
4. Finally, a matrix of Gaussian random numbers is constructed, $\mathbf{R} = (\mathbf{z}_m^i)^{\intercal}$ where each row i is the vector $\mathbf{z}_m^i$ transposed. This matrix has n rows, one for each node to simulate, and m columns, one for each scenario, and the correlation matrix between the columns is $\mathbf{C}_m$.

Then, to get a negative correlation among m scenarios, the following Antithetic Sequential Gaussian Simulation algorithm must be implemented:

For each m-tuple of scenarios:

1. **Random Path:** A random path is generated.
2. **Random Numbers:** The matrix of random numbers $\mathbf{R}$ is generated based on the previous algorithm.
3. For each scenario s of this m-tuple:
   (a) **Simple Kriging:** Visit each node i from scenario s according to the random path and perform a simple kriging estimation using nearby data and any previously simulated nodes.
   (b) **Simulate Value:** Assign the value of this node as:

$$Y(x_i) = Y^{KS}(x_i) + \sigma^{KS}(x_i)\mathbf{R}_{i,s} \tag{4}$$

The main difference between the conventional sequential Gaussian simulation algorithm and this antithetic variant is that the random numbers are calculated beforehand. Also, the random path is fixed for each m-tuple since the same node must be simulated in the same order to achieve the negative correlation between scenarios.

2.3. Short-term stochastic mine planning problem

To evaluate the impact of the antithetic random fields in mine planning we will use a two-stage stochastic model proposed in Nelis and Morales (2017) that aims to address the short-term mine planning problem evaluating the impact on the economic value of getting the blasthole grade information in advance. In a two-stage stochastic optimization problem, decision variables are split into two groups. The first group, or first stage variables, is made of variables that need to be equal in all possible scenarios. The second group, resource variables, may depend on the scenario, i.e., they need to be compatible with first stage decisions, but can adapt depending on actual values of the uncertain parameters.

In the setting that we propose, we have a set of periods $\mathscr{T}$, and consider a time-period $t^* \in \mathscr{T}$ which defines the moment at which information about the BH arrived, which we consider the moment at which the actual grades of the blocks are known. Decisions up to that point can only use average information for the uncertain grades, but after that point they can adapt to them. This is exemplified in Fig. 1, where on the left, we have depicted a conceptual scheduling up to period $t^* = 4$. After that period, multiple possible schedules are possible (as many as grade scenarios are considered).

Formally, the problem is described as follows: let $\mathscr{B}$ be the set of mining blocks, $\mathscr{D}$ the set of destinations for each block $b \in \mathscr{B}$, $\mathscr{T}$ the set of periods and $\mathscr{S}$ the set of blasthole grade scenarios (with $|\mathscr{S}|$ the cardinality of such set). As it was stated before, t^* is the period when the blasthole information is available, with $\mathscr{T}^*_{\ominus}$ as the set of periods until t^*, and $\mathscr{T}^*_{\oplus}$ as the set of periods after t^*. Let v_{btds} be the profit for sending block b to destination d at period t in scenario s, and $\bar{v}_{btd}$ be the average profit across scenarios. The objective is to maximize the schedule's NPV and the decisions variables are separated according to the blasthole information arrival: the first stage decision variable, y_{btd}, equals 1 if block $b \in \mathscr{B}$ is sent to destination $d \in \mathscr{D}$ at period $t \in \mathscr{T}^*_{\ominus}$ and 0 otherwise; and the second stage variable, x_{btds}, equals 1 if block $b \in \mathscr{B}$ is sent to destination $d \in \mathscr{D}$ at period $t \in \mathscr{T}^*_{\oplus}$ in scenario $s \in \mathscr{S}$, and 0 otherwise. Then, a short version of the two-stage stochastic model is defined as:

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$$\max \sum_{b\in\mathscr{B}}\sum_{d\in\mathscr{D}}\sum_{t\in\mathscr{T}^*_{\ominus}} y_{btd}\,\bar{v}_{btd} + \frac{1}{|\mathscr{S}|}\sum_{s\in\mathscr{S}}\sum_{b\in\mathscr{B}}\sum_{d\in\mathscr{D}}\sum_{t\in\mathscr{T}^*_{\oplus}} x_{btds}\,v_{btds} \tag{5}$$

$$\text{such that } \sum_{d\in\mathscr{D}}\sum_{t\in\mathscr{T}^*_{\ominus}} y_{btd} + \sum_{d\in\mathscr{D}}\sum_{t\in\mathscr{T}^*_{\ominus}} x_{btds} = 1 \qquad \forall\, s \in \mathscr{S},\, b \in \mathscr{B} \tag{6}$$

$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{h} \quad \forall\, s \in \mathscr{S} \tag{7}$$

Eq. (5) is the objective function, which maximizes the expected NPV, Eq. (6) states that each block can only be extracted once, and Eq. (7) represents the rest of constraints for a scheduling problem such as capacity, precedence or blending, with $\mathbf{A}$, $\mathbf{B}$ matrices to form the constraints, $\mathbf{h}$ the corresponding right-side vector and $\mathbf{x}$, $\mathbf{y}$ the vectors of decision variables. A complete version of the model can be found in Nelis and Morales (2017).

The results obtained by the model depends on the set of grade scenarios, $\mathscr{S}$. On one hand, we need as many scenarios as possible to obtain a reliable NPV result that considers the true variability of the deposit but, on the other hand, each scenario adds variables and constraints to the problem, making it harder to solve. Therefore, this is a suitable optimization problem to test the proposed variance reduction technique.

2.4. Convergence analysis

Comparison between antithetic and conventional simulation methodologies is based on analyzing the variance between solutions achieved in the optimization problem. Recall that each optimization problem is solved using a number $|\mathscr{S}|$ of realizations (see Section 2.3). The optimized NPV in problem (5) represents the expected value of the NPVs obtained over different realizations. Our aim is to optimize such NPV with the smallest number of realizations for a given precision. If a number of scenarios $|\mathscr{S}|$ is sufficient, the NPV obtained should converge to the same value for any set of $|\mathscr{S}|$ realizations, or equivalently, the NPV's obtained using different sets of $|\mathscr{S}|$ scenarios should have a low variance around the true optimum value. Therefore, the dispersion of the result over different sets of $|\mathscr{S}|$ scenarios is a measure of the convergence in the stochastic problem.

Thus, the procedure to compare the use of conventional and antithetic realizations is:

1. A large number of realizations is generated with each method:
   (a) For the conventional realizations, 600 scenarios are generated.
   (b) For the antithetic realizations, the same number of scenarios (600) is generated in sets of m realizations. Firstly, 300 sets of $m = 2$ antithetic random fields are generated, which are noted ARF2. Secondly, 60 sets of $m = 10$ antithetic random fields are generated, noted ARF10.
2. The dispersion resulting from solving problem (5) as a function of the type of simulation and their degree of correlation is analyzed. Problem (5) is solved 30 times using samples of size $|\mathscr{S}| = 2,\ 10$ and 20 realizations. Depending on how the realizations are generated (conventional, ARF2 or ARF10) individual realizations are selected within each sample without replacement. This is explained next:
   (a) For the conventional realizations, 30 sets of $|\mathscr{S}| = 2$ randomly selected realizations are first used. The process is repeated with

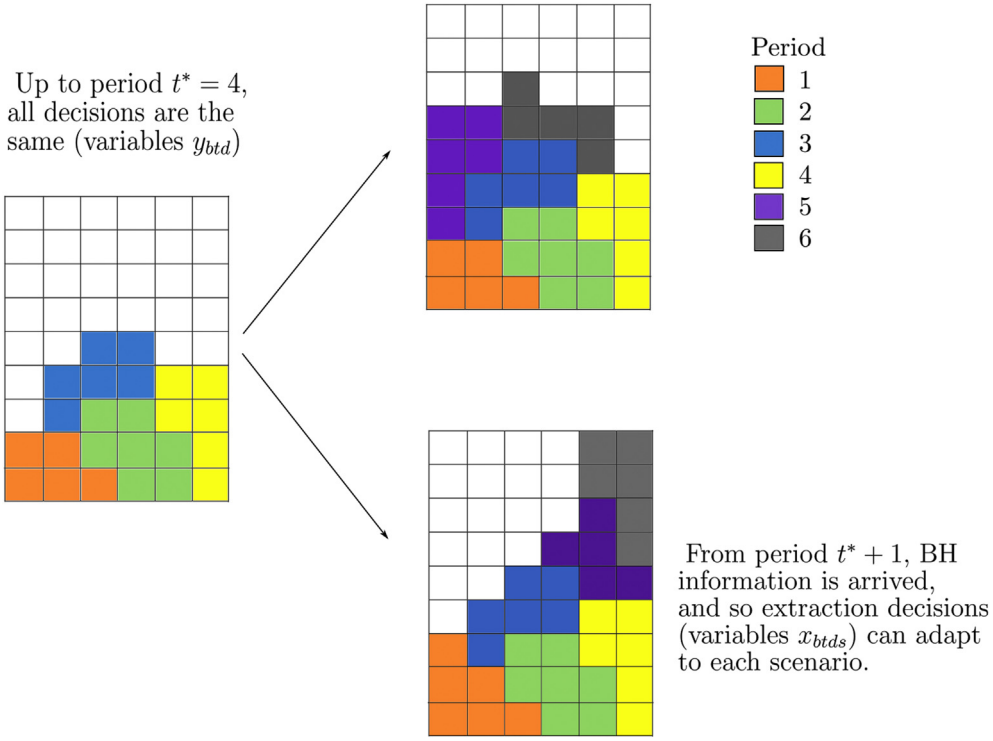

Fig. 1. Conceptual representation of an adaptive schedule for $t^* = 4$. To keep the example simple, we have colored the blocks by extraction period and not destination.

30 sets of $|\mathscr{S}| = 10$ and then with sets of $|\mathscr{S}| = 20$ scenarios.

(b) For the case of ARF2, 30 sets of paired realizations are selected for the case of samples of size $|\mathscr{S}| = 2$. For samples of size $|\mathscr{S}| = 10$, each one of the 30 cases optimized is made of 5 sets of ARF2. Similarly, for samples of size $|\mathscr{S}| = 20$, each one of the 30 cases is made of 10 sets of ARF2.

(c) For the case of ARF10, 30 pairs of realizations belonging to the same set of ARF10 are selected for the case of samples of size $|\mathscr{S}| = 2$. For samples of size 10, each one of the 30 cases optimized is made of one set of ARF10. Similarly, for samples of size 20, each one of the 30 cases is made of 2 sets of ARF10.

3. A statistical analysis is performed over the 30 optimal values of the objective function obtained in each case. The mean value is an estimator of the true objective function value, while the variance is a measure of the precision of this estimation. Therefore, the performance of the antithetic random fields technique can be measured in terms of the variance obtained in this procedure compared to the variance of conventional scenarios.

Table 1 shows a summary of how the cases were generated. Sample size $|\mathscr{S}|$ represents how many realizations are used to solve each one of the 30 instances of problem (5). The selection of realizations for each sample size $|\mathscr{S}|$ depends on how the realizations are generated (conventional, ARF2 or ARF10), as the table indicates.

Table 1
Cases for convergence analysis.

Sample Size $\|\mathscr{S}\|$	ARF2	ARF10	Conventional	Total number of realizations used
2	1 set	2 realizations randomly obtained from 1 set	2 realizations selected randomly	60
10	5 sets	1 set	10 realizations selected randomly	300
20	10 sets	2 sets	20 realizations selected randomly	600

3. Results

3.1. Conditional and non-conditional simulation

The results of this variance reduction technique with and without conditional data are presented, to verify the effect of correlating the random numbers among different scenarios. The non-conditional simulation is performed in a regular grid with 1 m separation between nodes, in a domain of $250 \times 250 \times 10m^3$ and a spherical variogram model. For the conditional simulation, real data from a copper deposit located in Chile was used, with a Gaussian transformation prior to the simulation process. Some basic statistics of the copper content are shown in Table 2.

Fig. 2 presents a plan view of the simulated domain without conditional data. Zones with high value in one scenario present a low simulated value in the other, showing the effect of the negative correlation in the random numbers. The large scale structures presented in both simulations are exactly the same since the random path and the variogram model are unaltered. On the other hand, Fig. 3 presents a plan view of the simulated deposit with conditional data. The effect of these data is clear: both scenarios present zones with similar values and structures, in response to the hard data used in the simulation, specially in the center of the deposit where the drillhole information is denser.

Table 2
Drillhole dataset 1.

Data points	2376	
Average	1.05	%
Minimum	0.12	%
Maximum	7.24	%
Standard deviation	0.64	%

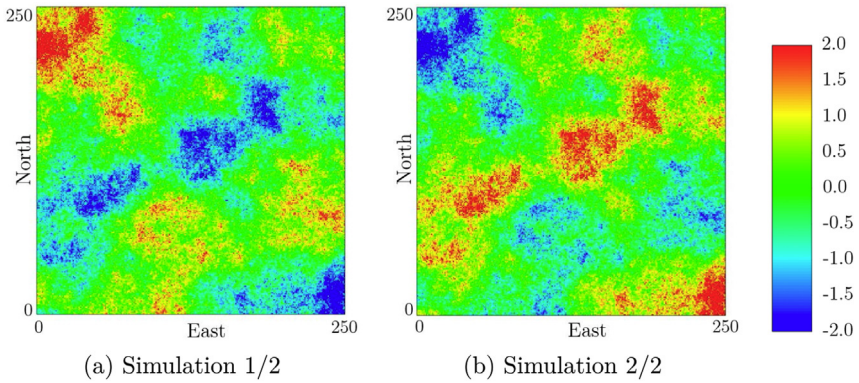

Fig. 2. Pair of antithetic simulations ($\alpha = -1$).

However, around the corners of the images, reversed structures can be seen since the conditioning effect is weaker with fewer data points.

More information can be gathered calculating the experimental correlation coefficient of every scenario. Fig. 4 shows the average of 100 pairs of conventional and antithetic simulations obtained with different number of scenarios simultaneously correlated (m). For the simulation without conditional data, the correlation coefficient is close to zero for the conventional case since every scenario is independent, while the antithetic simulations present a lower correlation coefficient that match the value obtained by Equation (3), showing that the Gaussian sequential algorithm preserves the random number correlation. The behavior seen for the conditional case is similar: the antithetic random fields present a lower correlation coefficient than the conventional simulations, and this coefficient increases with the number of scenarios correlated simultaneously. However, the value of the coefficient is positive even when the random numbers exhibit negative correlation, showing the effect of the conditional data across every scenario. Even with the effect of these data, the scenarios generated by the antithetic random fields technique still show a lower correlation than conventional technique, proving that the proposed methodology can be used with conditional data as well.

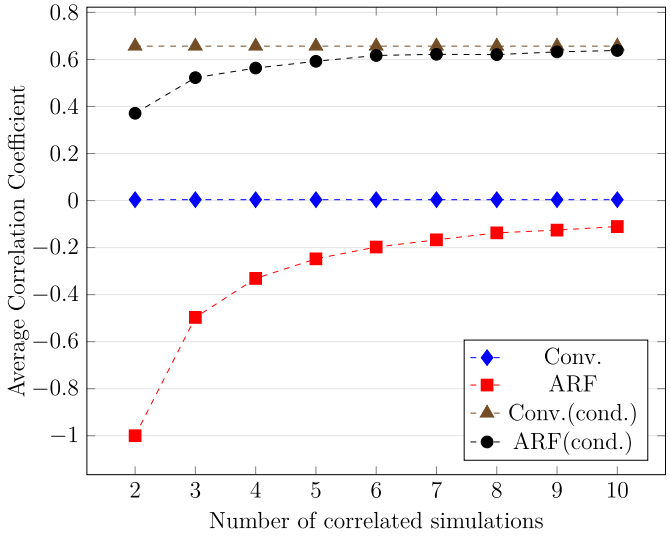

Fig. 4. Comparison of correlation coefficient for conventional and antithetic random fields.

3.2. Short-term stochastic mine planning

3.2.1. Case study

The model described in Section 2.3 was used to study the convergence rate of the proposed variance reduction technique. Since the model is suitable for the short-term scheduling problem, a single bench of a real deposit was used, consisting of 1547 blocks with a total tonnage 9.57 Mton at an average copper grade of 0.33%. The main parameters for the schedule are shown in Table 3. For this deposit, the methodology described in Section 2.4 was implemented.

3.2.2. Scheduling results

The results are presented in Fig. 5. The middle line inside the box represents the average value of 30 instances calculated for each type of simulation, the box size represents two times the standard deviation of these instances and the upper and lower whiskers represent the maximum and minimum values.

It can be seen that the mean converges to a value of 19.4 MUS$ for the largest sample size ($|\mathscr{S}| = 20$) for every simulation type. Smaller sample sizes ($|\mathscr{S}| = 2$ and $|\mathscr{S}| = 10$) show a more erratic behavior, especially for the sample size of $|\mathscr{S}| = 2$ where the mean value is always

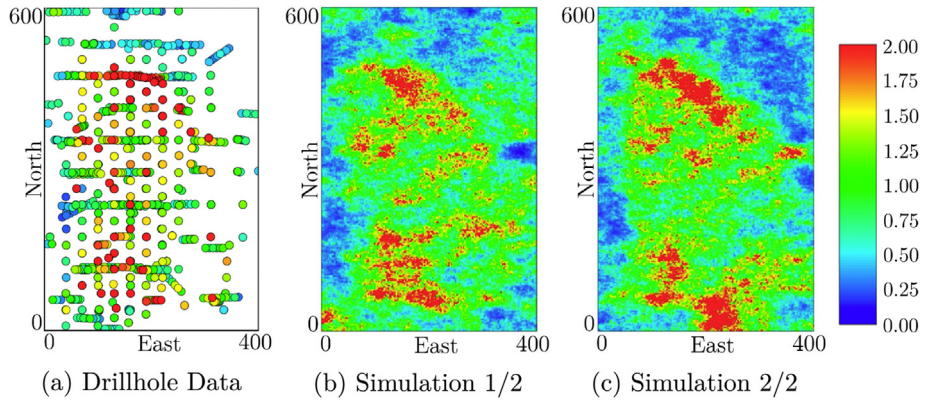

Fig. 3. Pair of antithetic simulations ($\alpha = -1$) and plan view of the conditional data.

Table 3
Scheduling parameters.

Price	2.5	US$/lb
Recovery	85	%
Mining Cost	1.5	US$/ton
Processing Cost	10	US$/ton
Selling Cost	0.3	US$/lb
Discount Rate	10	%
Mining Capacity	2.1	MTon/month
Processing Capacity	1.2	MTon/month

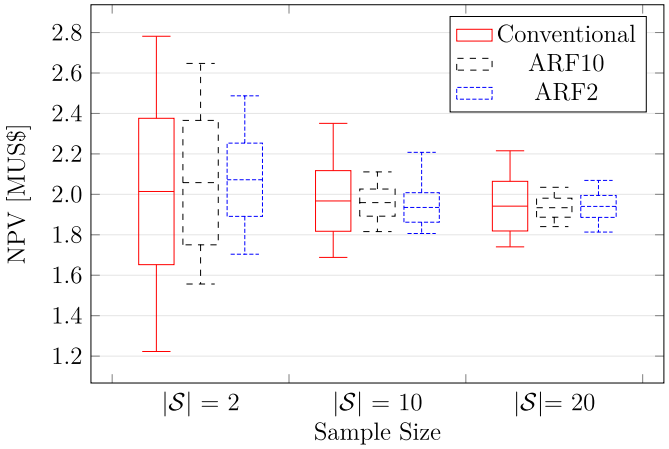

Fig. 5. Convergence of the value for the stochastic problem.

higher than 20 MUS$. Since the real value of the objective function is unknown for this type of problem, we should expect that every type of simulation converges to the same value. For this specific case, this holds true for the largest sample size ($|\mathcal{S}| = 20$), where the mean values across every type of simulation have a maximum difference of 0.4%. For the smaller sample sizes, this difference is larger (2.9% for $|\mathcal{S}| = 2$ and 1.7% for $|\mathcal{S}| = 10$), but not significant for this case study.

Since every type of simulation converges to a similar value, the standard deviation is a measure of the precision of this estimation. As it was expected, the larger the sample size $|\mathcal{S}|$, the lower the standard deviation for every case. However, for the same sample size, conventional simulation always produces the highest dispersion. This indicates that the proposed algorithm achieves a variance reduction for this optimization problem. The magnitude of this reduction is variable, as it can be seen in Table 4. Specifically, with a sample size $|\mathcal{S}| = 2$, ARF2 achieves a reduction of 50% from the conventional standard deviation. For $|\mathcal{S}| = 10$, ARF10 achieves a reduction of 55%, and for $|\mathcal{S}| = 20$, the reduction is 62%.

It is relevant to notice that the simulation type that achieves the highest reduction depends on the sample size, $|\mathcal{S}|$. For $|\mathcal{S}| = 2$, the highest reduction is achieved by ARF2. Moreover, the standard deviation obtained for ARF10 and conventional are similar. This is explained by the antithetic random fields algorithm and the sample size used: for a sample size of $|\mathcal{S}| = 2$, each instance using ARF2 is generated with one complete tuple, while for ARF10 each instance is generated using 2 random elements of a larger tuple. Therefore, the correlation coefficient between these two scenarios is lower for ARF2 compared to ARF10, which is reflected in the lower standard deviation of the optimization problem. These results indicate that the variability of the optimal values obtained in the optimization problems is reduced when realizations are generated maximizing their negative correlation.

Table 4
Standard deviation for different cases.

Sample size $\lvert\mathcal{S}\rvert$	ARF2	ARF10	Conventional
2	1 810 605	3076 323	3 618 671
10	729 879	671 042	1 499 136
20	543 159	469 921	1 227 579

In the case of $|\mathcal{S}| = 10$ and $|\mathcal{S}| = 20$ using ARF2, each sample is made of 5 and 10 independently generated sets of realizations, with no control of the correlation among different sets. Using ARF2 still improves the result as compared to using conventional realizations. However, given that realizations from different sets of ARF2 must be combined to generate the sample of size $|\mathcal{S}| = 10$ or $|\mathcal{S}| = 20$, the reduction is not as good as if the realizations were generated in a single set of antithetic random fields.

Considering different sample sizes, using ARF10 with a sample size of $|\mathcal{S}| = 10$ achieves a lower variance than conventional simulations with a sample size of $|\mathcal{S}| = 20$. Therefore, fewer antithetic simulations are needed to achieve the same precision as conventional simulations. Since the stochastic optimization problems in mine planning are highly demanding in computational resources, using fewer scenarios to represent the grade variability of the deposit could lead to being able to solve more complex problems in terms of deposit size, number of periods or destinations, making this kind of models useful in real-life applications.

This kind of analysis raises the question of the minimum number of scenarios necessary to achieve a good representation of the deposit. As it can be seen in Fig. 5, the difference between the maximum and minimum value for a given sample size can be significant. For our largest sample size, this difference in our antithetic simulations approach is 11% for ARF10 and 14% for ARF2, and up to 26% for the conventional algorithm, which is significant. For this reason, using a single instance for this particular deposit and optimization model could lead to a result with a high error. Although this conclusion is specific for this case study, this phenomenon can occur in other models and deposits. Therefore, an in-depth analysis of the number of scenarios considered is recommended.

A possible extension of the antithetic simulation methodology is taking advantage of the fixed path for each tuple. For a fixed path and neighborhood, the kriging weights remain the same for each scenario, reducing the computational cost of the simulation algorithm. The use of a single fixed path is often discouraged since it can lead to artifacts and poor reproduction of the covariance, but recent research has shown that this effect can be reduced using a larger neighborhood and a multi-grid approach (Nussbaumer et al. (2018)). Another option to reduce the occurrence of artifacts is using different tuples to obtain a single instance. In our case study, the use of multiple tuples of ARF2 to generate a sample size of $|\mathcal{S}| = 10$ and $|\mathcal{S}| = 20$ still achieved a variance reduction, with each tuple using a different path.

4. Conclusions

An application of a variance reduction technique for a geostatistical simulation framework is presented. This application was successful in generating scenarios with negative correlation in non-conditional simulations, as well as scenarios with lower correlation coefficient than the conventional algorithm in conditional simulation. This led to a significant reduction of variance in the estimation of the expected NPV value for a short-term stochastic scheduling problem, compared to the conventional simulation algorithm. This variance reduction technique could be used to solve very costly computational models with fewer scenarios, although the magnitude of this scenario reduction depends on the optimization model and the deposit variability.

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Appendix A. Computer Code Availability

The proposed algorithm is implemented in `sgsim_arf.for` (74.4 KB), available in https://github.com/gnelis/sgsim_antithetic since March 2018. This implementation is based on the open source code `sgsim. for` from GSLIB (Deutsch and Journel (1998), www.gslib.com), available in FORTRAN77 and FORTRAN90. The code developer was Gonzalo Nelis (address: Av. Tupper, 2007, Santiago, Chile. Contact Number: +56-9-88397760. E-mail: gnelis@delphoslab.cl). The code requires a FORTRAN compiler, and it was tested using Windows 10. Everyone is granted permission to copy, modify and redistribute this code, but under the condition that the original `sgsim. for` copyright is preserved.

Appendix B. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.cageo.2018.09.003.

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