Minerals Engineering 128 (2018) 284–287

Contents lists available at ScienceDirect

Minerals Engineering

journal homepage: www.elsevier.com/locate/mineng

Short communication

Analytical solutions for the flow depth of steady laminar, Bingham plastic tailings down wide channels

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ARTICLE INFO

Keywords:
Tailings
Laminar flow
Bingham plastic
Analytical solution

ABSTRACT

At mid to high concentrations, fine mine tailings are non-Newtonian, and their rheology is commonly expressed as Bingham plastic. Discharges of such fine materials in tailing storage facilities form shallow channels. In this short communication, exact analytic expressions to relate the volume flow per unit width and the flow depth are derived for a Bingham plastic in terms of a newly-defined dimensionless parameter. Simplified approximations, valid for the near-plug and quasi-Newtonian fluid limits are proposed.

1. Introduction

Among the new trends in global mining is the need to reduce water consumption in arid areas given its increasing scarcity or cost, when transported from remote locations. This reality concurs with progressive decrease in ore grades (Ihle and Kracht, 2018), and consequently the need to process finer ores for optimal liberation. Low water content tailings —either thickened or paste—, can behave as non-segregating during transport (Simms, 2017). While they are discharged, they can form channels if they are thickened (Blight, 2010) or flow forming sheet-like structures if they are paste tailings (Robinsky, 1999). In this context, their high concentration nature makes them non-Newtonian, and it is common to assume that they flow as Bingham plastics (Huang and García, 1997; Sofrá and Boger, 2002; Blight, 2010). Both in the case of the self-organized flow after discharge in thickened tailings storage facilities or when highly viscous paste slurries are disposed as cones, the corresponding flow is often laminar throughout their whole trajectory or evolves from turbulent to laminar (Henriquez and Simms, 2009). As such discharges are unconfined, flow prediction requires setting a relationship between the mean flow velocity and the flow depth, given the particular rheological behavior of the slurry, in this case Bingham plastic. To this purpose, a number of authors have proposed models for various fluid types and channel geometries. In particular, Haldenwang (2003) and Alderman and Haldenwang (2007) present complete reviews of empirical and semi-empirical models, Javadi et al. (2015) analyze a new Reynolds number with previous experimental data, while a more recent account for developments in friction factor models in closed conduits is given in Carravetta et al. (2015). Once the model is set, either flow-depth or rheology is obtained by solving non-linear equations (except on the Newtonian case) or an inverse problem on the rheological parameters if they are unknown.

In the present short communication, explicit analytical and semi-analytical solutions for the relation between volume flow per unit width and the laminar discharge flow of a wide Bingham plastic channel is given, thus complementing a previous work for pipe flows (Ihle and Tamburrino, 2012), and serving as an alternative approach for Bingham rheological parameter determination.

2. Problem description

We consider the problem of a steady, uniform flow of a Bingham plastic down an inclined plane, assumed as an infinitely wide channel. Transient flow features such as the flow of the front of the discharge (commonly described using lubrication theory in Newtonian or non-Newtonian fluids as in, *e.g.*, Benjamin, 1957; Yih, 1963; Lister, 1992; Balmforth et al., 2006) or the presence of roll waves (Tamburrino and Ihle, 2013) are not considered herein. The shear rate-shear stress relationship of a Bingham plastic is given by (Liu and Mei, 1989):

$$\eta\frac{\partial U}{\partial z} = \begin{cases} 0 & \text{if } |\tau| < \tau_0 \\ \tau - \tau_0 \mathrm{sgn}\left(\frac{\partial U}{\partial z}\right) & \text{if } |\tau| \geqslant \tau_0, \end{cases} \quad (1)$$

where U, η, τ and τ_0 are the main component of velocity, Bingham viscosity, shear stress and yield stress, respectively. The function sgn is

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https://doi.org/10.1016/j.mineng.2018.09.010
Received 27 June 2018; Received in revised form 31 August 2018; Accepted 6 September 2018
Available online 11 September 2018

defined as $\mathrm{sgn}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. This model has been used to predict transient gold tailing discharges, where film flow assumptions can give an accurate description of the front advance of thin sheets on slopes (Liu and Mei, 1989; Henriquez and Simms, 2009). Given this type of fluid has a yield stress, this kind of flow can have sections where the shear stress is below τ_0 (Liu and Mei, 1989). The resulting, so-called plug flow, affects the mean velocity profile and therefore the total flow depth given a volume flow per unit width q_0, affecting the dynamical balance in the system. The critical depth h_p of a completely plugged layer corresponds to (e.g Griffiths, 2000):

$$h_p = \frac{\tau_0}{\rho_m g\sin\theta}, \tag{2}$$

where g is the magnitude of the gravity acceleration vector, ρ_m the density of the solid-liquid mixture and θ the slope of the plane. This condition is often used to estimate required tailing deposit capacities in tailing storage facilities (Robinsky, 1999), and is independent of viscosity given there is no shear within this layer.

3. Governing equations

Under the set of hypotheses denoted above, the flow is assumed slender and two-dimensional. The corresponding momentum equation on an infinite plane of slope θ is given by:

$$\rho_m\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U}\right) = -\nabla p + \frac{\partial\tau}{\partial z}\hat{\imath} + g(\sin\theta\hat{\imath} + \cos\theta\hat{\boldsymbol{k}}), \tag{3}$$

where $\mathbf{U} = U\hat{\imath} + W\hat{\boldsymbol{k}}$ is the velocity vector, p the pressure and $(\hat{\imath}, \hat{\boldsymbol{k}})$ are unit vectors aligned with the (x, z) axes shown in Fig. 1. The corresponding boundary conditions are $\tau(z = 0) = \rho_m gH\sin\theta$ and $\tau(z = H) = 0$ (Huang and García, 1997). The continuity of the shear stress and the existence of a yield stress implies that at some point there exists a critical height z^* where the shear stress is lower than τ_0. This point is denoted as $z^* = (1-\lambda)H$, with $\lambda < 1$. It is noted that when $\lambda \ll 1$, the effect of the yield stress is negligible, and the fluid is quasi-Newtonian. On the other hand, the case $\lambda \lessapprox 1$ corresponds to the situation when most of the flow is moving as a solid. If $\lambda = 1$, $H = h_p$ and the column remains stagnant. Under the latter assumptions, the velocity field $\mathbf{U} = U(z)$, corresponding to the solution of (3), reads:

$$u(z) = \begin{cases} \frac{U_p z}{(1-\lambda)H}\left[2-\frac{z}{(1-\lambda)H}\right] & \text{if } 0 \leqslant z \leqslant (1-\lambda)H \\ U_p & \text{otherwise,} \end{cases} \tag{4}$$

with

$$U_p = \frac{\rho_m g(1-\lambda)^2 H^2\sin\theta}{2\eta}. \tag{5}$$

Integrating the velocity profile, the corresponding average velocity is given by (e.g. Bird et al., 1987):

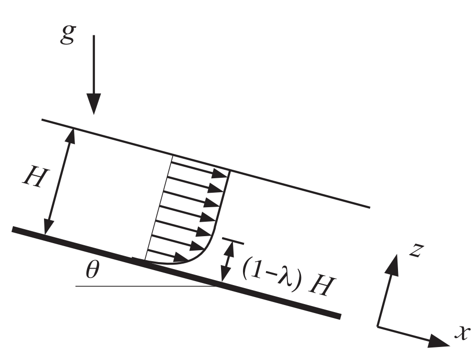

Fig. 1. Schematic of the problem. Plug flow exists for $(1-\lambda)H \leqslant z^* \leqslant H$.

$$\overline{U} = \frac{\tau_w H}{3\eta}\left[1-\frac{3}{2}\frac{\tau_0}{\tau_w} + \frac{1}{2}\left(\frac{\tau_0}{\tau_w}\right)^3\right], \tag{6}$$

where $\tau_w = \rho_m gH\sin\theta$, is the shear stress at the bottom of the layer. This relation is similar to that found in other works for rectangular channels and laminar flow (see Alderman and Haldenwang, 2007).

The parameter λ could be also written as the ratio between the depth given by (2) and the flow depth:

$$\lambda = \frac{h_p}{H} = \frac{\tau_0}{H\rho_m g\sin\theta}. \tag{7}$$

From the right side of (7), also $\lambda = \tau_0/\tau_w$. Assuming the inflow q_0 is known, an integral volume conservation statement can be imposed to obtain an equation for the flow depth H, $q_0 = \int_0^H U(z')\mathrm{d}z'$. Using (7) in the latter expression yields the nondimensional equation

$$\mathscr{N}-(h-1)^2(2h + 1) = 0, \tag{8}$$

where $h = H\rho_m g\sin\theta/\tau_0$ and

$$\mathscr{N} = \frac{6\eta q_0(\rho_m g\sin\theta)^2}{\tau_0^3} \tag{9}$$

is a dimensionless control parameter, which is related to the non-Newtonian characteristic of the fluid, where $\mathscr{N} \ll 1$ implies a strong influence of the yield stress, and $\mathscr{N} \gg 1$ a quasi-Newtonian fluid behavior. In particular, defining the Bingham number as $B = \frac{\tau_0}{\eta\overline{U}/H}$, interpreted as a dimensionless yield stress (Balmforth et al., 2006), it is straightforward to obtain that:

$$\mathscr{N} = \frac{6h^2}{B}. \tag{10}$$

Given flow and fluid properties, $\mathscr{N}$ is always positive. If $\mathscr{N} \leq 1$, (8) has three real roots. The solution for the present problem pose no ambiguity, as one root is negative and one of the positive solutions is lower than one, which is not compatible with the requirement that flow depth must exceed h_p implying, in dimensionless terms, that $h \geqslant 1$. Thus, for $0 \leqslant \mathscr{N} \leq 1$, the only meaningful solution is the largest root of (8):

$$h(0 \leqslant \mathscr{N} < 1) = \frac{1}{2} + \cos\left[\frac{1}{3}\arccos(2\mathscr{N}-1)\right] \tag{11}$$

$$\approx 1 + \frac{\sqrt{3}}{3}\mathscr{N}^{1/2}-\frac{\mathscr{N}}{9} + \frac{5\sqrt{3}}{162}\mathscr{N}^{3/2}-\frac{8}{243}\mathscr{N}^2 + \mathscr{O}(\mathscr{N}^{5/2}) \tag{12}$$

The argument of the second term on the right hand side of (11) cannot be reduced further using trigonometric identities (Abramowitz and Stegun, 1965). The expression (12) shows the first terms of the corresponding Taylor expansion around $\mathscr{N} = 0$. This is plotted in Fig. 2, where it is shown that for $\mathscr{N} \ll 1$, $h-1 \sim \mathscr{N}^{1/2}$. In dimensional terms, this means that

$$H_{\mathscr{N}\ll 1} \sim h_p + \left(\frac{2\eta q_0}{\tau_0}\right)^{1/2}. \tag{13}$$

Here, by virtue of (10) $\mathscr{N} \ll 1$ implies $B \gg 1$ or, in other words, a very high dimensionless yield stress. The last result is expressed in terms of B as $h \approx \left(1-\sqrt{\frac{6}{B}}\right)^{-1}$. The second term on the right hand side of (13) naturally suggests itself as a flow length scale provided $[2\eta q_0(\rho_m g\sin\theta)^2]^{1/3} \ll \tau_0$. It is independent of the slope, indicating that for this, near-plugging regime, the sheared part of the velocity field strongly obeys a purely yield-viscous stress balance.

If $\mathscr{N} \geqslant 1$, there is one real root of (8) whose algebraic expression is given by

$$h(\mathscr{N} \geqslant 1) = \frac{1}{2^{2/3}}\left(h_+ + h_- + \frac{1}{2^{1/3}}\right) \tag{14}$$

with

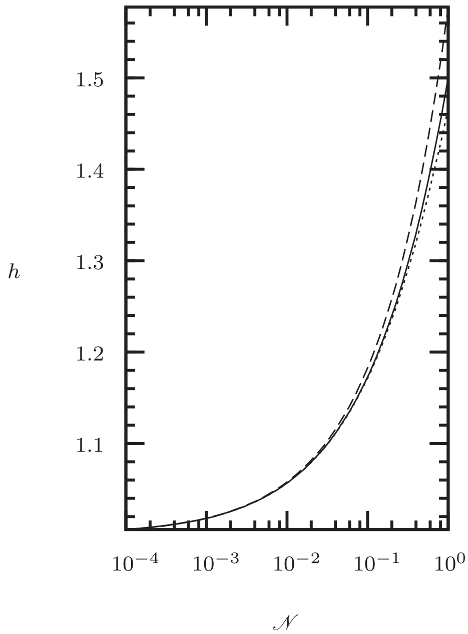

Fig. 2. Flow depth out of (8) for $10^{-4} \leqslant \mathcal{N} \leq 1$. The solid line corresponds to (11). The dashed and dotted lines represents the first two and three terms of the expansion (12), respectively.

$$h_{\pm} = \left[\mathcal{N} \pm \mathcal{N}^{1/2}(\mathcal{N}-1)^{1/2} - \frac{1}{2}\right]^{1/3}. \tag{15}$$

A plot of the solution is depicted by the solid line of Fig. 3. It is noted that $h = (\mathcal{N}/2)^{1/3}$ is an asymptote for large values of $\mathcal{N}$. This means, in dimensional terms, that

$$H_{\mathcal{N}\gg 1} \sim \left(\frac{3\eta q_0}{\rho_m g \sin\theta}\right)^{1/3}, \tag{16}$$

which is independent of the yield stress and, moreover, corresponds to the asymptotic solution for large values of time for the low-Reynolds-number Newtonian fluid flow (Lister, 1992). Expressing the dimensionless result in terms of the Bingham number yields $h \approx 3/B$

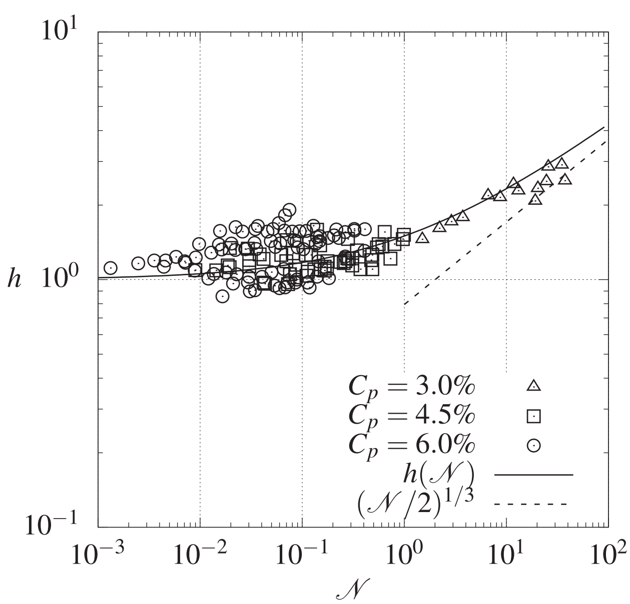

Fig. 3. Experimental data by Haldenwang and Slatter (2006) using bentonite. The solid line represents the solution of (8), as (11) for $\mathcal{N} < 1$ and (14) for $\mathcal{N} \geqslant 1$.

provided $B \ll 1$.

The solution (14) can be considered as a length scale of a thin Bingham sheet flow and it is meaningful as long as the flow is laminar. When otherwise, $\tau \gg \tau_0$ and the effect of viscosity becomes unimportant. In this case, the weight is no longer balanced by viscosity, but rather by inertial forces, given by the left hand side of (3).

Defining a Froude number $\mathrm{Fr} = \overline{U}/\sqrt{gH}$ and a Reynolds number $\mathrm{Re} = \rho_m \overline{U} H/\eta$, from (6), it follows that

$$\mathrm{Fr}^2 = \frac{\mathrm{Re}\sin\theta}{3}\left[1 - \frac{3}{2}\frac{1}{h} + \frac{1}{2}\frac{1}{h^3}\right]. \tag{17}$$

Identifying the two aforementioned cases, when $\mathcal{N} \gg 1$

$$\mathrm{Fr}^2 \sim \frac{\mathrm{Re}\sin\theta}{3}, \tag{18}$$

implying, again, that the yield stress effect vanishes and therefore the result is that of a purely Newtonian fluid. In particular, in this limiting condition, the corresponding friction coefficient f corresponds to $f = 24/\mathrm{Re}$, as expected for an infinitely wide, Newtonian fluid laminar channel (Chow, 1959). When, on the other hand, $\mathcal{N} \ll 1$,

$$\mathrm{Fr}^2 \sim \frac{8\sin\theta}{f_0}\left(1 - \sqrt{\frac{16}{f_0 \mathrm{Re}}}\right), \tag{19}$$

with $f_0 = 8\tau_0/\rho_m \overline{U}^2$. For large f_0 and finite Reynolds number, which is the case of this laminar flow, $\mathrm{Fr} \sim (8\sin\theta/f_0)^{1/2}$, corresponding to a statement of balance between the bottom shear stress exerted due to the slope and the yield one.

Fig. 3 shows a comparison between the obtained curve $h(\mathcal{N})$ and laboratory channel data from Haldenwang and Slatter (2006). In such work, a large laminar and turbulent channel flow database was obtained using 75, 150 and 300 mm width flumes transporting Bingham and Herschel-Bulkley fluids. For the purposes of the present work, data for only laminar flows of Bingham fluid were considered. Additionally, to minimize lateral wall effect artifacts only 150 mm and 300 mm channel flow data have been considered. Turbulent-laminar transitions were estimated using Hanks (1963) criterion. As flumes were not infinite in width, corresponding Reynolds and Hedström numbers required to assess transition to turbulence were computed using the hydraulic radius rather than flow depth. Fig. 3 shows reasonable fit for $h(\mathcal{N})$, with some scattering in the range $10^{-2} \lesssim \mathcal{N} \lesssim 1$. In this case, it is interesting to note that the present analytic solution, given by (11), is in most of the cases below than experimental data. The scattering may be explained by lateral plug flow effects along with possible overpredictions of the yield stress when computed from Bingham plastic flow curve extrapolation (Coussot, 1994). In contrast, for $\mathcal{N} \to 0^+$, where the flow is predominantly of plug-type (Eq. (13)), and for large values of $\mathcal{N}$, where the effect of the yield stress is negligible (Eq. (16)), experimental results show better agreement with the analytical curve.

Due to their simplicity, present results can be used to formulate a simple least-square problem to determine both the Bingham viscosity and yield stress in a simple wide flume configuration, as an alternative to other flume-based approaches (Coussot, 2005; Uhlherr et al., 2000). An advantage of the present set of analytic solutions is that they don't rely on adjustable parameters (except, of course, the rheological parameters of interest).

4. Conclusions

Explicit, analytical solutions of the uniform laminar flow of an infinitely wide Bingham fluid moving downslope in steady state have been identified for the first time. In particular, two asymptotic regimes have been found in terms of an external dimensionless parameter $\mathcal{N} = 6\eta q_0(\rho_m g\sin\theta)^2/\tau_0^3$, namely one where most of the fluid layer is unsheared and an opposite instance where most of the fluid layer is sheared. In the former case, corresponding to the limit of small $\mathcal{N}$, the sheared layer depth scales with $(2\eta q_0/\tau_0)^{1/2}$, whereas the unsheared region corresponds to $h_p = \tau_0/\rho_m g\sin\theta$. In the limit of large $\mathscr{N}$, corresponding to the quasi-Newtonian fluid regime, the flow depth becomes independent of the yield stress, τ_0, as expected, and corresponds to $(3\eta q_0/\rho_m g\sin\theta)^{1/3}$. Reasonable agreement between the dimensionless flow depth $h = H/h_p$, expressed as a function of $\mathscr{N}$, was found with previously reported experimental data within several decades.

Acknowledgements

The authors gratefully acknowledge support from the Department of Mining Engineering of University of Chile. They also thanks the Chilean Commission for Science and Technology through Fondecyt Projects 1160971 and 1161751, and INNOVA CORFO Project CSIRO Chile 10CEII-9007. The authors would also like to thank Mr. Luis Navarro for useful discussions regarding the preparation of the present work and Mr. Germán Ibarra for data processing and Prof. Yarko Niño for valuable comments to this manuscript.

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