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Computational Section

Filling the gap in the table of smallest regulators up to degree 7 [☆]

Eduardo Friedman [a,*], Gabriel Ramirez-Raposo [b,*]

[a] *Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile*
[b] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile*

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In 2016 Astudillo, Diaz y Diaz and Friedman published sharp lower bounds for regulators of number fields of all signatures up to degree seven, except for fields of degree seven having five real places. We deal with this signature, proving that the field with the first discriminant has minimal regulator. The new element in the proof is an extension of Pohst's geometric method from the totally real case to fields having one complex place.

1. Introduction

Some thirty years ago, the number fields with smallest discriminant for signatures up to degree seven were all known [Od]. Recently [ADF] the same was established for

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* Corresponding authors.
E-mail addresses: friedman@uchile.cl (E. Friedman), gcramirez@uc.cl (G. Ramirez-Raposo).

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regulators, except that no sharp lower bounds were proved for one signature in degree seven. Here we close that gap.

Theorem. *Let k be a number field of degree seven having five real embeddings. Then its regulator R_k satisfies $R_k \geq R_{k_1} = 2.8846\ldots$, where k_1 is the unique field of discriminant $-2\,306\,599$ in this signature.*

More precisely, except for the three unique fields with discriminants $-2\,306\,599$, $-2\,369\,207$ and $-2\,616\,839$, in this signature all fields satisfy $R_k > 3.2$.

2. Proof

If $z_1, ..., z_n$ are non-zero complex numbers, with $|z_1| \leq \cdots \leq |z_n|$, we let

$$P_n(z_1, ..., z_n) := \prod_{1\leq i<j\leq n} \left|1 - \frac{z_i}{z_j}\right|^2 \qquad (|z_1| \leq |z_2| \leq \cdots \leq |z_n|). \tag{1}$$

The case that interests us is when the z_i are the n conjugates of a unit $\varepsilon \in k$, where $k = \mathbb{Q}(\varepsilon)$ is a field of degree n having only one complex place. Throughout we shall denote the real elements by r_i $(1 \leq i \leq n-2)$ and the complex conjugate pair by $x\mathrm{e}^{i\theta}$ and $x\mathrm{e}^{-i\theta}$ $(\theta \in (0, \pi),\ x > 0)$, arranging them so that

$$0 < |r_1| \leq |r_2| \leq \cdots \leq |r_{n-2}|, \qquad |r_t| \leq x \leq |r_{t+1}|. \tag{2}$$

In this case we can factor P_n as

$$P_n = P_{n-2}(r_1, ..., r_{n-2}) \cdot |1-\mathrm{e}^{-2i\theta}|^2 \cdot \prod_{m=1}^{n-2} |1-c_m\mathrm{e}^{i\theta}|^4, \qquad c_m := \begin{cases} r_m/x & \text{if } m \leq t, \\ x/r_m & \text{if } m > t. \end{cases} \tag{3}$$

Note that $c_m \in [-1, 1]$ and $c_m \neq 0$ $(1 \leq m \leq n-2)$. Since $g(c) := |1 - c\,\mathrm{e}^{i\theta}|^2 = 1 + c^2 - 2c\cos(\theta)$ has no local maximum for $c \in \mathbb{R}$, endpoint checks give

$$\left|1 - c\,\mathrm{e}^{i\theta}\right|^2 \leq \begin{cases} 1 & \text{if } 0 \leq \theta \leq \pi/3 \text{ and } 0 \leq c \leq 1, \\ 1 & \text{if } 2\pi/3 \leq \theta \leq \pi \text{ and } -1 \leq c \leq 0, \\ 2\big(1-\cos(\theta)\big) & \text{if } \pi/3 \leq \theta \leq \pi \text{ and } 0 \leq c \leq 1, \\ 2\big(1+\cos(\theta)\big) & \text{if } 0 \leq \theta \leq 2\pi/3 \text{ and } -1 \leq c \leq 0. \end{cases} \tag{4}$$

We shall also need the inequality

$$\big(1-\cos^2(\theta)\big)^a \big(1-\cos(\theta)\big)^b \leq \frac{2^{2a+b} a^a (a+b)^{a+b}}{(2a+b)^{2a+b}} \qquad (a, b > 0,\ \theta \in \mathbb{R}). \tag{5}$$

To prove it, for $t \in [-1, 1]$, let $g(t) := (1-t^2)^a (1-t)^b$ and use elementary calculus.

Lemma 1. *Assume $\theta \in \mathbb{R}$ and $-1 \le c_m \le 1$ for $1 \le m \le r$. Let d_+ be the number of c_m with $c_m > 0$, let d_- be the number of c_m with $c_m < 0$, and define*

$$B_r = B_r(\theta, c_1, ..., c_r) := |1 - \mathrm{e}^{-2i\theta}|^2 \prod_{m=1}^{r} |1 - c_m \mathrm{e}^{i\theta}|^4. \tag{6}$$

Then, letting $a := 1 + 2\min(d_+, d_-)$, $b := 2|d_+ - d_-|$ and $f := 2\max(d_+, d_-)$,

$$B_r \le \max\Big(\frac{4^{2a+b}a^a(a+b)^{a+b}}{(2a+b)^{2a+b}}, \frac{4^{2+f}(1+f)^{1+f}}{(2+f)^{2+f}}\Big). \tag{7}$$

Proof. Replacing θ by $-\theta$ if necessary, we can assume $0 \le \theta \le \pi$. We shall first show that if $\pi/3 \le \theta \le 2\pi/3$, then B_r is bounded by the first element inside the max in (7). Say $d_+ \ge d_-$, so that $a = 1 + 2d_-$ and $b = 2(d_+ - d_-)$. Then, using (4),

$$\begin{aligned}
B_r &= 4\big(1 - \cos^2(\theta)\big)\Big(\prod_{m=1}^{r} |(1 - c_m e^{i\theta})|^2\Big)^2 \\
&\le 4\big(1 - \cos^2(\theta)\big)\Big(\prod_{\substack{m \\ c_m > 0}} 2\big(1 - \cos(\theta)\big)\Big)^2 \Big(\prod_{\substack{m \\ c_m < 0}} 2\big(1 + \cos(\theta)\big)\Big)^2 \\
&= 2^{2+2(d_+ + d_-)}\big(1 - \cos^2(\theta)\big)\big(1 - \cos(\theta)\big)^{2d_+}\big(1 + \cos(\theta)\big)^{2d_-} \\
&= 2^{2a+b}\big(1 - \cos^2(\theta)\big)^{1+2d_-}\big(1 - \cos(\theta)\big)^{2(d_+ - d_-)} \\
&= 2^{2a+b}\big(1 - \cos^2(\theta)\big)^a\big(1 - \cos(\theta)\big)^b \le \frac{2^{2(2a+b)}a^a(a+b)^{a+b}}{(2a+b)^{2a+b}} \qquad \big(\text{see } (5)\big),
\end{aligned}$$

proving (7) in this case. If $d_+ < d_-$, just write $\theta = \pi - \theta'$ to reduce to the previous case. If $0 \le \theta < \pi/3$, or $2\pi/3 < \theta \le \pi$, a similar argument proves (7), but now the upper bound involves f in (7) rather than a and b. □

Lemma 2 *(Pohst). For $\alpha, \beta \in [-1, 1]$, the following hold.*

(*i*) *if* $\alpha \ge 0$, then $(1 - \alpha)(1 - \alpha\beta) \le 1$.

(*ii*) $(1 - \alpha)(1 - \beta)(1 - \alpha\beta) \le 2$.

(*iii*) *if* $|\alpha| \le |\beta|$ and $\beta \ne 0$, then $\big(1 - \alpha\big)\big(1 - \beta\big)\big(1 - (\alpha/\beta)\big) \le 2$.

Proof. Inequalities (*i*) and (*ii*) [Po, p. 468] can be proved by checking for critical points and the boundary. The last one follows from (*ii*), on replacing α by α/β. □

Lemma 3. *Suppose $n = 7$ and $c_1 > 0$ in (3), then $P_7 < \mathrm{e}^{12} < 162755$.*

Proof. We begin with (3),

$$P_7 = B_5 P_5 = B_5(\theta, c_1, ..., c_5) P_5(r_1, ..., r_5) \qquad \big(\text{see (1) and (6)}\big). \tag{8}$$

There are 16 possibilities for the signs of $c_2, ..., c_5$, which we divide into three cases:

(1) Three of the c_m are of one sign and two have the opposite sign ($1 \le m \le 5$). Hence, in the notation of Lemma 1, $a = 5$, $b = 2$ and $f = 6$.
(2) One of the c_m is of one sign and four have the opposite sign. Hence $a = 3$, $b = 6$ and $f = 8$.
(3) All of the c_m are positive.

In case (1), Lemma 1 gives $B_5 < 4842.63$ and Pohst's inequality [Po, Satz IV] gives $P_5 \le 16$. Now (8) yields $P_7 < 77483$, proving the lemma in case (1). In case (2), which includes 5 possible sign patterns for $c_2, ..., c_5$, Lemma 1 only gives $B_5 < 40624$, but we will improve Pohst's inequality [Po, Satz IV] to $P_5 \le 4$, which will imply the lemma in this case. Following Pohst [Po, p. 467], for $1 \le i, \ell, \ell' \le 4$ let

$$x_i := \frac{r_i}{r_{i+1}}, \quad y_{\ell,\ell'} := 1 - \prod_{i=\ell}^{\ell'} x_i = 1 - \frac{r_\ell}{r_{\ell'}}, \quad A = \prod_{1 \le \ell \le \ell' \le 4} y_{\ell,\ell'} = \sqrt{P_5(r_1, ..., r_5)}.$$

Note that $A = A(x_1, ..., x_4)$, $-1 \le x_i \le 1$, $0 \le y_{\ell,\ell'} \le 2$ and that the signs of the x_i's are determined from those of the c_m's and vice-versa, as we assumed $c_1 > 0$.
Subcase (2a): $\operatorname{sign}(c_2, ..., c_5) = (+,+,+,-)$ or $(-,-,-,-)$. Then $\operatorname{sign}(x_1, ..., x_4) = (+,+,+,-)$ or $(-,+,+,+)$. Since $A(x_1, x_2, x_3, x_4) = A(x_4, x_3, x_2, x_1)$, in this subcase it suffices to consider $\operatorname{sign}(x_1, ..., x_4) = (+,+,+,-)$. Now,

$$\begin{aligned} A &= y_{1,1} y_{2,2} y_{3,3} y_{4,4} y_{1,2} y_{2,3} y_{3,4} y_{1,3} y_{2,4} y_{1,4} \\ &= (y_{1,1} y_{2,2} y_{1,2})(y_{3,3} y_{3,4})(y_{2,3} y_{2,4})(y_{1,3} y_{1,4})(y_{4,4}). \end{aligned}$$

Since $x_1, x_2, x_1 x_2 \ge 0$, we have trivially that $y_{1,1} y_{2,2} y_{1,2} \le 1$. By Lemma 2 ($i$), using $x_3, x_2 x_3, x_1 x_2 x_3 \ge 0$, we have $y_{3,3} y_{3,4} \le 1$, $y_{2,3} y_{2,4} \le 1$ and $y_{1,3} y_{1,4} \le 1$. Finally $y_{4,4} \le 2$, and so $A \le 2$ in subcase (2a).
Subcase (2b): $\operatorname{sign}(c_2, ..., c_5) = (+,+,-,+)$ or $(-,+,+,+)$. Then $\operatorname{sign}(x_1, ..., x_4) = (+,+,-,-)$ or $(-,-,+,+)$. Again, we may assume $\operatorname{sign}(x_1, ..., x_4) = (+,+,-,-)$. Grouping differently, $A = (y_{1,1} y_{1,4} y_{2,4})(y_{2,2} y_{2,3})(y_{1,2} y_{1,3})(y_{3,3} y_{4,4} y_{3,4})$. Trivially we have $y_{1,1} y_{1,4} y_{2,4} \le 1$. By Lemma 2 ($i$), since $x_2, x_1 x_2 \ge 0$, we have $y_{2,2} y_{2,3} \le 1$ and $y_{1,2} y_{1,3} \le 1$. By Lemma 2 (ii), $y_{3,3} y_{4,4} y_{3,4} \le 2$, and so again $A \le 2$.
Subcase (2c): $\operatorname{sign}(c_2, ..., c_5) = (+,-,+,+)$. Then $\operatorname{sign}(x_1, ..., x_4) = (+,-,-,+)$. Write $A = (y_{1,3} y_{1,4} y_{2,4})(y_{1,1} y_{1,2})(y_{4,4} y_{3,4})(y_{2,2} y_{3,3} y_{2,3})$. Again trivially, $y_{1,3} y_{1,4} y_{2,4} \le 1$. By Lemma 2 ($i$), since $x_1, x_4 \ge 0$, we have $y_{1,1} y_{1,2} \le 1$ and $y_{4,4} y_{3,4} \le 1$. Finally, by Lemma 2 (ii), we have $y_{2,2} y_{3,3} y_{2,3} \le 2$. Hence $A \le 2$, finishing case (2).

In case (3) we have $c_m > 0$, and so $r_m > 0$ for $m = 1, \dots, 5$. We shall need

$$R_{\ell,\ell'} := (1 + c_\ell)(1 + c_{\ell'})\big(1 - (r_\ell/r_{\ell'})\big) \le 2 \qquad \big(\ell < \ell'\big). \tag{9}$$

To prove (9), we consider three possibilities according to the position of t in (2). If $\ell' \le t$, then by (3), $c_\ell = r_\ell/x,\ c_{\ell'} = r_{\ell'}/x$. Hence $|c_\ell| \le |c_{\ell'}|$ and so Lemma 2 (*iii*) yields (9) (on setting $\alpha := -c_\ell,\ \beta := -c_{\ell'}$). Similarly, if $\ell > t$, $c_\ell = x/r_\ell,\ c_{\ell'} = x/r_{\ell'}$, so $|c_{\ell'}| \le |c_\ell|$ and Lemma 2 (*iii*) yields (9) (with $\alpha := -c_{\ell'},\ \beta := -c_\ell$). Lastly, if $\ell \le t < \ell'$, then $c_\ell = r_\ell/x,\ c_{\ell'} = x/r_{\ell'}$. Now (9) follows from Lemma 2 (*ii*).

In case (3), using $0 \le 1 - \dfrac{r_\ell}{r_{\ell'}} \le 1\ \ (\ell < \ell')$ and (9), we obtain $P_7 \le 2^{12}$ from

$$\begin{aligned}\sqrt{P_7} &= |1 - e^{-2i\theta}| \ \cdot \prod_{1\le\ell<\ell'\le5}\Big(1 - \frac{r_\ell}{r_{\ell'}}\Big) \ \cdot\ \prod_{m=1}^{5}|1 - c_m e^{i\theta}|^2 \\ &\le 2\prod_{1\le\ell<\ell'\le5}\Big(1 - \frac{r_\ell}{r_{\ell'}}\Big)\ \cdot\ \prod_{m=1}^{5}(1 + c_m)^2 = 2R_{1,2}R_{2,3}R_{3,4}R_{4,5}R_{1,5}\Big(1 - \frac{r_1}{r_3}\Big)\Big(1 - \frac{r_1}{r_4}\Big) \\ &\quad \cdot \Big(1 - \frac{r_2}{r_4}\Big)\Big(1 - \frac{r_2}{r_5}\Big)\Big(1 - \frac{r_3}{r_5}\Big) \le 2R_{1,2}R_{2,3}R_{3,4}R_{4,5}R_{1,5} \le 2^6. \qquad \square\end{aligned}$$

We can now prove the Theorem in the Introduction, which we do not repeat here. Suppose $R_k \le 3.2$, and let ε yield the positive minimum value of the Euclidean length m_k (see [ADF, eq. (1)]) on the units of k. Then $k = \mathbb{Q}(\varepsilon)$. Let $r_1, \dots r_5$ be the five real conjugates of ε, ordered so that $|r_1| \le \cdots \le |r_5|$, and let $xe^{\pm i\theta}$ be the two complex conjugates $\big(x > 0,\ \theta \in (0, \pi)\big)$. Replacing ε by $-\varepsilon$ if necessary, we may assume that $r_1 > 0$, so $c_1 > 0$ with notation as in (3). Using the value $\gamma_5 = \sqrt[5]{8}$ for Hermite's constant in dimension 5, we find $m_k(\varepsilon) \le \big(3.2\sqrt{6}\big)^{1/5}\sqrt{\gamma_5} < 1.85847$ [ADF, eq. (5)]. A short calculation using Lemma 3 now yields $\log|D_k| < 31.492$ (*cf.* the proof of [ADF, eq. (4)]).

This range of discriminant can be easily handled by the method of [ADF]. Namely, Table 2, Lemmas 4 and 5 in [ADF] can be used to show that $R_k > 3.2$ in the range $3\,030\,000 \le |D_k| \le e^{31.492}$. Thus $|D_k| < 3\,030\,000$. We conclude the proof by examining the regulators of the seven fields in the range $|D_k| < 3\,030\,000$ [DyD].

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