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Qualitative properties of positive solutions for mixed integro-differential equations

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Abstract

This paper is concerned with the qualitative properties of the solutions of mixed integro-differential equation

$$\begin{cases} (-\Delta)^\alpha_x u + (-\Delta)_y u + u = f(u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^M, \\ u > 0 \;\text{ in } \mathbb{R}^N \times \mathbb{R}^M, \quad \lim_{|(x,y)|\to+\infty} u(x,y) = 0, \end{cases} \tag{0.1}$$

with $N \geq 1$, $M \geq 1$ and $\alpha \in (0,1)$. We study decay and symmetry properties of the solutions to this equation. Difficulties arise due to the mixed character of the integro-differential operators. Here, a crucial role is played by a version of the Hopf's Lemma we prove in our setting. In studying the decay, we construct appropriate super and sub solutions and we use the moving planes method to prove the symmetry properties.

Key words: Integro-differential equation, Hopf's Lemma, Decay, Symmetry.
MSC2010: 35R11, 35B06, 35B40, 35B50.

1 Introduction

The study of qualitative properties of positive solutions to semi-linear elliptic equations in $\mathbb{R}^N$ has been the concern of numerous authors along the last several decades. The asymptotic behavior of the solution at infinity, the actual rate of decay and symmetry properties have been the most studied qualitative properties for these equations. It was the seminal work by Gidas, Ni and Nirenberg [19] that settled these two main qualitative properties for the semi-linear elliptic equation

$$\begin{cases} -\Delta u + u = f(u) \quad \text{in } \mathbb{R}^M, \\ u > 0 \;\text{ in } \mathbb{R}^M, \quad \lim_{|y|\to+\infty} u(y) = 0, \end{cases} \tag{1.1}$$

when the non-linearity is merely Lipschitz continuous, super-linear at the zero, in the sense that

$$f(s) = O(s^p) \quad \text{as } s \to 0, \tag{1.2}$$

for some $p > 1$, and $M \geq 3$. Gidas, Ni and Nirenberg proved that the solutions of (1.1) are radially symmetric and they satisfy the precise decay estimate

$$\lim_{|y|\to+\infty} u(y)e^{|y|}|y|^{\frac{M-1}{2}} = c, \tag{1.3}$$

for certain constant $c > 0$. After this work, many authors extended the results in various directions, generalizing the non-linearity, the elliptic operator or the hypotheses on the solutions. Out of the very many contributions in this direction we mention here only a few: Berestycki and Lions [5], Berestycki and Nirenberg [6], Brock [7], Busca and Felmer [8], Cortázar, Elgueta and Felmer [13], Da Lio and Sirakov [14], Dolbeault and Felmer [16], Gui [20], Kwong [21], Li and Ni [24] and Pacella and Ramaswamy [26].

Recently, much attention has been given to the study of elliptic equations of fractional order. In this direction, Felmer, Quaas and Tan in [17] studied the problem

$$\begin{cases} (-\Delta)^\alpha u + u = f(u) \quad \text{in} \quad \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \qquad \lim_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{1.4}$$

They proved existence and regularity of positive solutions, and also decay and symmetry results. Precisely, it was proved that the solutions u of (1.4) satisfy

$$\frac{c^{-1}}{|x|^{N+2\alpha}} \leq u(x) \leq \frac{c}{|x|^{N+2\alpha}}, \qquad |x| \geq 1, \tag{1.5}$$

for some $c > 1$, when f is superlinear at 0 in the sense that

$$\lim_{s\to 0} \frac{f(s)}{s} = 0.$$

The radial symmetry of the solutions of (1.4) is derived by using the moving planes method in integral form developed in [11, 25], assuming further that $f \in C^1(\mathbb{R})$, it is increasing and there exists $\tau > 0$ such that

$$\lim_{s\to 0} \frac{f'(s)}{s^\tau} = 0. \tag{1.6}$$

This symmetry result was generalized by the authors in [18], using an appropriate truncation argument together with the moving planes method with ideas developed in [23]. We refer to some other papers with more discussions on qualitative properties of solutions to fractional elliptic problems as Cabré and Sire [9], Caffarelli and Silvestre [10], Chen, Li and Ou [11], Barles, Chasseigne, Ciomaga and Imbert [12], Dipierro Palatucci, Valdinoci [15], Li [25], Quaas and Xia [28], Ros-Oton and Serra [29] and Sire and Valdinoci [32].

Both operators, the laplacian and the fractional laplacian, are particular cases of a general class of elliptic operators connected to backward stochastic differential equations associated to Brownian and Levy-Itô processes, see for example Barles, Buckdahn and Pardoux [1], Benth, Karlsen and Reikvam [4] and Pham [27]. Recently, Barles, Chasseigne, Ciomaga and Imbert in [2, 3] and Ciomaga in [12] considered the existence and regularity of solutions for equations involving mixed integro-differential operators belonging to the general class of backward stochastic differential equations mentioned above. A particular case of elliptic integro-differential operator of mixed type is the one considering the laplacian in some of the variables and the fractional laplacian in the others, modeling diffusion sensible to the direction. In view of (1.1) and (1.4) we may write similarly

$$\begin{cases} (-\Delta)_x^\alpha u + (-\Delta)_y u + u = f(u), & (x,y) \in \mathbb{R}^N \times \mathbb{R}^M, \\ u > 0 \ \text{ in } \mathbb{R}^N \times \mathbb{R}^M, & \lim_{|(x,y)|\to+\infty} u(x,y) = 0, \end{cases} \tag{1.7}$$

where $N \geq 1$, $M \geq 1$. The operator $(-\Delta)_y$ denotes the usual laplacian with respect to y, while $(-\Delta)_x^\alpha$ denotes the fractional laplacian of exponent $\alpha \in (0,1)$ with respect to x, i.e.

$$(-\Delta)_x^\alpha u(x,y) = \int_{\mathbb{R}^N} \frac{u(x,y) - u(z,y)}{|x-z|^{N+2\alpha}} dz, \tag{1.8}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. Here the integral is understood in the principal value sense.

In view of the known results on decay and symmetry for solutions of equations (1.1) and (1.4) just described above, it is interesting to ask if these results still hold for solutions of the equation of mixed type (1.7), where the elliptic operator represents diffusion depending on the direction in space. Regarding the asymptotic decay of solution at infinity, the question is interesting since a proper mix of the two variables should be obtained for the decay estimates. The natural way to estimate the decay is through the construction of super and sub solutions involving the fundamental solution of the elliptic operator, which in this case is singular in $\mathbb{R}^N \times \{0\}$. Moreover, the solution of (1.7) cannot be radially symmetric, so this property cannot be used to estimate the decay. On the other hand, regarding radial symmetry, we may still have symmetry in x and y, but the moving planes method would require an adequate version of the Hopf's Lemma, that we prove here.

Our first theorem concerns the decay of solutions for (1.7) with general nonlinearity and it states as follows.

Theorem 1.1 *Let $\alpha \in (0,1)$, $N, M \in \mathbb{N}$, $N \geq 1$ and $M \geq 1$ and let us assume that the function $f : (0,+\infty) \to \mathbb{R}$ is continuous and it satisfies*

$$-\infty < B := \liminf_{v\to 0^+} \frac{f(v)}{v} \leq A := \limsup_{v\to 0^+} \frac{f(v)}{v} < 1. \tag{1.9}$$

Let u be a positive classical solution of problem (1.7), then for any $\epsilon > 0$ small, there exists $C_\epsilon > 1$ such that for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,

$$C_\epsilon^{-1}(1+|x|)^{-N-2\alpha}e^{-\theta_2|y|} \leq u(x,y) \leq C_\epsilon(1+|x|)^{-N-2\alpha}e^{-\theta_1|y|}, \tag{1.10}$$

where

$$\theta_1 = \sqrt{1-A} - \epsilon \quad \textit{and} \quad \theta_2 = \sqrt{1-B} + \epsilon. \tag{1.11}$$

When we compare estimate (1.10) with (1.3) for $N = 0$, we first observe that in ours an exponential decay is obtained, but with a constant C_ϵ depending on ϵ, which is a parameter controlling the rate of exponential decay. This is more clear when $A = B = 0$. On the other hand we are making much more general assumptions on f and, in particular, we are not making any assumption on the radial symmetry of the solution, which is crucial in proving (1.3). We do not know of a decay estimate better than

$$C_\epsilon^{-1}e^{-\theta_2|y|} \leq u(y) \leq C_\epsilon e^{-\theta_1|y|}, \qquad y \in \mathbb{R}^M, \tag{1.12}$$

for solutions of (1.1) under assumption (1.9) for f, and where radial symmetry of the solutions is not available, like in a case where f may depend on y. On the other hand, when $M = 0$, we recover (1.5) from (1.10). For the proof of the decay estimate (1.10) we construct suitable super and sub solutions and we use comparison principle with a version of Hopf's lemma.

When we assume further hypothesis we can get sharper estimates for the decay of the solutions of equation (1.7). Precisely, we have the following result:

Theorem 1.2 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 5$ and the non-linearity $f : (0,+\infty) \to \mathbb{R}$ is non-negative and it satisfies (1.2). Let u be a positive classical solution of (1.7), then there exists a constant $c > 1$ such that for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,*

$$\frac{1}{c}\rho(x,y) \leq u(x,y) \leq c\rho(x,y)(1+|y|)^{\frac{1}{2}}, \tag{1.13}$$

where the function ρ is defined as

$$\rho(x,y) = \min\{\frac{1}{(1+|x|)^{N+2\alpha}}\, ,\, e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}\, ,\, \frac{e^{-|y|}|y|^{1-\frac{M}{2}}}{(1+|x|)^{N+2\alpha}}\}. \tag{1.14}$$

We notice that this theorem gives the expected exponential decay for positive solutions, as suggested by (1.3), assuming the dimension of the space satisfies $M \geq 5$. Moreover, it gives the expected polynomial correction for the lower bound with a gap in the power for the upper bound. This theorem is proved under the assumption (1.2) on the non-linearity, constructing super and sub solutions devised upon the fundamental solution of $(-\Delta)^{\alpha}_x + (-\Delta)_y + id$. In our argument, a crucial role is played by the estimate already obtained in Theorem 1.1. Since the fundamental solution of $(-\Delta)^{\alpha}_x + (-\Delta)_y + id$ has $\mathbb{R}^N \times \{0\}$ as singular set, we cannot use the method in [19] in order to derive our estimate. Moreover, some other arguments in [19] cannot be used either because the solutions of (1.7) are not radial, since the differential operator is not radially invariant and there are no solutions depending only on one of the x or y variables, as can be seen from (1.13),

Even though solutions of (1.7) are not radially symmetric, we can prove partial symmetry in each of the variables x and y and this is the content of our third theorem.

Theorem 1.3 *Assume that $\alpha \in (0, 1)$, $N \geq 1$, $M \geq 1$ and the function $f : (0, +\infty) \to \mathbb{R}$ is locally Lipschitz and it satisfies (1.9). Moreover, we assume that f also satisfies*

(F) *there exist $u_0 > 0$, $\gamma > \frac{N}{N+M} \cdot \frac{2\alpha}{N+2\alpha}$ and $\bar{c} > 0$ such that*

$$\frac{f(v) - f(u)}{v - u} \leq \bar{c} v^{\gamma} \quad \text{for all} \quad 0 < u < v < u_0. \tag{1.15}$$

Then, every positive classical solution u of equation (1.7) satisfies

$$u(x, y) = u(r, s)$$

and $u(r, s)$ is strictly decreasing in r and s, where $r = |x|$ and $s = |y|$.

When $N = 0$, we see that assumption (F) implies $\gamma > 0$ and (1.15) coincides with the assumption considered in [23]. When $M = 0$, assumption (F) implies that $\gamma > \frac{2\alpha}{N+2\alpha}$ and it coincides with the assumption considered in [18], when the solutions is assumed to decay as a power $N + 2\alpha$ at infinity. We remark that the operator $(-\Delta)^{\alpha}_x + (-\Delta)_y$ is a combination of two operators with different differential orders in $x-$variable and $y-$variable, and this produced a combined polynomial-exponential decay and does not allow for radial symmetry, but only partial symmetry as stated in Theorem 1.3.

The proof of Theorem 1.3 is based on the moving planes method as developed in [18, 23]. In these arguments, the strong maximum principle plays a crucial role and it is available for the laplacian and for the fractional laplacian. However, in the case of our mixed integro-differential operator some difficulties arise and we overcome them with a version of the Hopf's Lemma.

The rest of the paper is organized as follows. In Section §2, we introduce a version of the Hopf's Lemma and a strong maximum principle. In Section §3, we prove the decay of solutions as in Theorem 1.1 and Theorem 1.2 by constructing suitable super and sub solutions. Section §4 is devoted to prove symmetry results presented in Theorem 1.3.

2 Preliminaries

This section is devoted to study the Strong Maximum Principle for mixed integro-differential operators as in equation (1.7). To this end, we prove first a suitable form of the Hopf's Lemma.

However, before to go to this, we recall some basic properties of the Sobolev embeddings. If we denote the Sobolev spaces

$$H(\mathbb{R}^{N+M}) = \{w \in L^2(\mathbb{R}^{N+M}) | \int_{\mathbb{R}^M}\int_{\mathbb{R}^N}(|\xi_1|^{2\alpha}+|\xi_2|^2+1)|\hat{w}(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2 < \infty\}$$

and

$$H^\alpha(\mathbb{R}^{N+M}) = \{w \in L^2(\mathbb{R}^{N+M}) \mid \int_{\mathbb{R}^{N+M}}(|\xi|^{2\alpha}+1)|\hat{w}(\xi)|^2 d\xi < \infty\},$$

with norms

$$\|w\|_H = (\int_{\mathbb{R}^M}\int_{\mathbb{R}^N}(|\xi_1|^{2\alpha}+|\xi_2|^2+1)|\hat{w}(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2)^{\frac{1}{2}}$$

and

$$\|w\|_{H^\alpha} = (\int_{\mathbb{R}^{N+M}}(|\xi|^{2\alpha}+1)|\hat{w}(\xi)|^2 d\xi)^{\frac{1}{2}},$$

respectively, then it is not difficult to see that the following proposition holds.

Proposition 2.1 *For $\alpha \in (0,1)$, we have that*

$$H(\mathbb{R}^{N+M}) \subset H^\alpha(\mathbb{R}^{N+M}) \subset L^p(\mathbb{R}^{N+M}),$$

where the first inclusion is continuous and the second inclusion is continuous if $1 \le p \le \frac{2(N+M)}{N+M-2\alpha}$. Moreover,

$$H(\mathbb{R}^{N+M}) \subset L^p_{loc}(\mathbb{R}^{N+M})$$

is compact if $1 \le p < \frac{2(N+M)}{N+M-2\alpha}$.

We devote the rest of this section to prove the Strong Maximum Principle in our context and to this end, we start with versions of the Maximum Principle and the Hopf's Lemma. In what follows, given Ω an open subset in $\mathbb{R}^N \times \mathbb{R}^M$, we define its closed cylindrical extension in the direction x as

$$\tilde{\Omega} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M : \ \exists\, x' \in \mathbb{R}^N \text{ s.t. } (x', y) \in \bar{\Omega}\}.$$

Given a function h defined in an appropriate domain, we consider the mixed integro-differential operator

$$\mathcal{L}w(x, y) = (-\Delta)^{\alpha}_x w(x, y) + (-\Delta)_y w(x, y) + h(x, y)w(x, y).$$

Lemma 2.1 *Assume that Ω is an open domain of $\mathbb{R}^N \times \mathbb{R}^M$ and the function $h : \Omega \to \mathbb{R}$ satisfies $h \geq 0$ in Ω. If the function $w \in C(\bar{\Omega}) \cap L^{\infty}(\tilde{\Omega})$ satisfies*

$$\begin{cases} \mathcal{L}w \geq 0 \quad \text{in } \ \Omega, \quad w \geq 0 \quad \text{in } \ \tilde{\Omega} \setminus \Omega, \\ \liminf_{(x,y)\in\Omega, |(x,y)|\to\infty} w(x, y) \geq 0 \end{cases} \tag{2.1}$$

then $w \geq 0$ in $\tilde{\Omega}$.

Proof. If not, we may assume that there exists some $(x_0, y_0) \in \Omega$ such that

$$w(x_0, y_0) = \min_{(x,y)\in\tilde{\Omega}} w(x, y) < 0.$$

Then

$$(-\Delta)^{\alpha}_x w(x_0, y_0) = \int_{\mathbb{R}^N} \frac{w(x_0, y_0) - w(z, y_0)}{|x_0 - z|^{N+2\alpha}} dz < 0$$

and

$$(-\Delta)_y w(x_0, y_0) \leq 0$$

and then, since h is non-negative we have $\mathcal{L}w(x_0, y_0) < 0$, which contradicts (2.1), completing the proof. □

It what follows we prove a version of the Hopf's Lemma and for this purpose we need to give some conditions to the boundary of the domain where the function is defined. We say that the domain $\Omega \subset \mathbb{R}^N \times \mathbb{R}^M$ satisfies *interior cylinder condition at* $(x_0, y_0) \in \partial\Omega$ if there exist $r > 0$ and $\tilde{y} \in \mathbb{R}^M$ such that $O_r = B^N_r(x_0) \times B^M_r(\tilde{y})$ satisfies

$$O_r \subset \Omega \quad \text{and} \quad (x_0, y_0) \in \partial O_r, \tag{2.2}$$

where $B^N_r(x_0) = \{x \in \mathbb{R}^N : |x-x_0| < r\}$ and $B^M_r(\tilde{y}) = \{y \in \mathbb{R}^M : |y-\tilde{y}| < r\}$ and, obviously $|\tilde{y} - y_0| = r$. We define also

$$D = \{(x, y) \in O_r : |x - x_0| < \frac{r}{2}, \ |y - \tilde{y}| > \frac{r}{2}\}. \tag{2.3}$$

Lemma 2.2 *[Hopf's Lemma] Let* Ω *be an open set satisfying interior cylinder condition at* $(x_0, y_0) \in \partial\Omega$*. Assume that* $h \in L^\infty(D)$ *and* $w \in C(\bar{\Omega}) \cap Ł^\infty(\tilde{\Omega})$ *satisfies*

$$\mathcal{L}w \geq 0 \quad \text{in } \Omega$$

and

$$0 = w(x_0, y_0) < w(x, y), \quad \forall (x, y) \in \Omega.$$

Further assume that for $r > 0$ *be given in (2.3) and for any* $(x, y) \in D$ *we have*

$$\int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{w(z, y)}{|x - z|^{N+2\alpha}} dz \geq 0. \tag{2.4}$$

Then

$$\limsup_{s \to 0^+} \frac{w(x_0, y_0) - w(x_0, y_0 + s\tilde{y})}{s} < 0, \tag{2.5}$$

moreover, if the limit exists, then

$$\frac{\partial w}{\partial n}(x_0, y_0) < 0, \tag{2.6}$$

where n *is the unit exterior normal vector of* Ω *at the point* (x_0, y_0)*.*

Proof. Let us define

$$\varphi_M(y) = e^{-\beta|y - \tilde{y}|^2} - e^{-\beta r^2}, \quad y \in \bar{B}_r^M(\tilde{y}), \tag{2.7}$$

where $\beta > 0$ will be chosen later. By direct computation, we have that

$$-\Delta\varphi_M(y) = (2M\beta - 4\beta^2|y - \tilde{y}|^2)e^{-\beta|y - \tilde{y}|^2}. \tag{2.8}$$

Next we consider the function

$$v(x, y) = \varphi_N(x)\varphi_M(y), \quad (x, y) \in \tilde{O}_r,$$

where φ_N is the first eigenfunction of Dirichlet problem

$$\begin{cases} (-\Delta)^\alpha \varphi_N(x) = \lambda_1 \varphi_N(x), & x \in B_{r/2}^N(x_0), \\ \varphi_N(x) = 0, & x \in \mathbb{R}^N \setminus B_{r/2}^N(x_0), \end{cases} \tag{2.9}$$

where φ_N is positive and bounded in $B_{r/2}^N(x_0)$ and the first eigenvalue λ_1, is positive, see Propositions 9 and 4 in [30] and [31], respectively.

For $(x, y) \in D$, by (2.8) and (2.9), we obtain that

$$\begin{aligned} \mathcal{L}v(x, y) &= \varphi_M(y)(-\Delta)^\alpha \varphi_N(x) + \varphi_N(x)(-\Delta\varphi_M(y)) + h(x, y)\varphi_N(x)\varphi_M(y) \\ &= \varphi_N(x)[\lambda_1\varphi_M(y) + (2M\beta - 4\beta^2|y - \tilde{y}|^2)e^{-\beta|y - \tilde{y}|^2} + h(x, y)\varphi_M(y)] \\ &\leq \varphi_N(x)e^{-\beta|y - \tilde{y}|^2}(\lambda_1 + 2M\beta - \beta^2 r^2 + \|h\|_{L^\infty(D)}), \end{aligned}$$

where the last inequality holds by the fact that $0 \leq \varphi_M(y) < e^{-\beta|y-\tilde{y}|^2}$ and $|y-\tilde{y}| > r/2$ in D. Let us choose $\beta > 0$ big enough such that

$$\mathcal{L}v \leq 0 \quad \text{in } D. \tag{2.10}$$

On the other hand, since $\varphi_N(x) = 0$ for $|x - x_0| \geq r/2$ and $\varphi_M(y) = 0$ for $|y - \tilde{y}| = r$, it is obvious that $v = 0$ in $A_1 \cup A_2$ where $A_1 = \{(x, y) \in \tilde{D} : |x - x_0| \geq r/2\}$ and $A_2 = \{(x, y) \in \bar{D} : |y - \tilde{y}| = r\}$. If we define the set $A_3 := \{(x, y) \in \bar{D} : |y - \tilde{y}| = r/2\}$, we see that $\tilde{D} \setminus D = A_1 \cup A_2 \cup A_3$. We also observe that v is a bounded function in $\tilde{O}_r$.

Next we prove (2.5) assuming $h \geq 0$. Defining

$$W(x,y) = \begin{cases} w(x,y), & (x,y) \in \bar{O}_r, \\ 0, & (x,y) \in \tilde{O}_r \setminus \bar{O}_r \end{cases} \tag{2.11}$$

and using (2.4), we have that for any $(x, y) \in D$,

$$\mathcal{L}W(x,y) = \mathcal{L}w(x,y) + \int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{w(z,y)}{|x-z|^{N+2\alpha}} dz \geq 0.$$

Combining with (2.10), we have that, for every $\epsilon > 0$

$$\mathcal{L}(W - \epsilon v) \geq 0 \quad \text{in } D. \tag{2.12}$$

Since v is bounded in $\tilde{O}_r$, the set A_3 is a compact subset of O_r and $w > 0$ in O_r, then there exists $\epsilon > 0$ small such that

$$W = w \geq \epsilon v \quad \text{in } A_3.$$

Since $v = 0$ in $A_1 \cup A_2$, $w \geq 0$ in $\bar{O}_r$ and (2.11), we have $W \geq 0 = \epsilon v$ in $A_1 \cup A_2$. Consequently,

$$W - \epsilon v \geq 0 \quad \text{in } \tilde{D} \setminus D.$$

Then we can use Lemma 2.1, recalling that $h \geq 0$ to obtain that

$$W - \epsilon v \geq 0 \quad \text{in } D.$$

In view of the definition of W, since $D \subset \bar{O}_r$, we find that $w - \epsilon v \geq 0$ in D and noticing that $w(x_0, y_0) = v(x_0, y_0) = 0$ we obtain that

$$\frac{w(x_0,y_0) - w(x_0, y_0 + s\tilde{y})}{s} \leq \epsilon \frac{v(x_0,y_0) - v(x_0, y_0 + s\tilde{y})}{s},$$

for all $s \in (0, r/2)$. Thus, we have

$$\begin{aligned}
\limsup_{s\to 0^+} \frac{w(x_0,y_0)-w(x_0,y_0+s\tilde{y})}{s} &\le \epsilon \lim_{s\to 0^+} \frac{v(x_0,y_0)-v(x_0,y_0+s\tilde{y})}{s} \\
&= \epsilon\varphi_N(x_0) \lim_{s\to 0^+} \frac{\varphi_M(y_0)-\varphi_M(y_0+s\tilde{y})}{s} \\
&= -2\epsilon\beta r^2 e^{-\beta r^2}\varphi_N(x_0) < 0,
\end{aligned}$$

completing the proof of (2.5).

The case for general h can be done simply by replacing h by h^+. In fact, since $w > 0$ in Ω, we have

$$(-\Delta)_x^\alpha w(x,y) + (-\Delta)_y w(x,y) + h^+(x,y)w(x,y) \ge 0, \quad (x,y) \in \Omega$$

and similarly we obtain that

$$(-\Delta)_x^\alpha v(x,y) + (-\Delta)_y v(x,y) + h^+(x,y)v(x,y) \le 0, \quad (x,y) \in D,$$

so we may proceed as before to get (2.5) and the proof is complete. □

In order to state the Strong Maximum Principle to be used in our moving planes procedure, it is convenient to consider property (P):

(P) We say that a function $w : \tilde{\Omega} \to \mathbb{R}$ satisfies property (P) if whenever $(x_0, y_0) \in \Omega$ such that

$$0 = w(x_0, y_0) = \inf_{(x,y)\in\Omega} w(x,y),$$

then

$$w(x, y_0) \equiv 0, \quad \forall x \in \mathbb{R}^N.$$

The following lemma is in preparation of the strong maximum principle.

Lemma 2.3 *Let Ω be an open set in $\mathbb{R}^N \times \mathbb{R}^M$ and w have property (P). We denote*

$$\Omega_0 = \{(x,y) \in \Omega : w(x,y) = \inf_{\Omega} w = 0\}. \tag{2.13}$$

If $\emptyset \ne \Omega_0 \subsetneqq \Omega$, then $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at any point $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$.

Proof. Since $\emptyset \ne \Omega_0 \subsetneqq \Omega$, we have that $\emptyset \ne \partial\Omega_0 \cap \Omega \subset \partial(\Omega \setminus \Omega_0)$. For any $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$, let us denote $r = \frac{1}{4} dist((x_0, y_0), \partial\Omega)$ and let $\tilde{y} \in \mathbb{R}^M$ such that $(x_0, \tilde{y}) \in \Omega \setminus \Omega_0$ and $|\tilde{y} - y_0| = r$. Since w has property (P), then $w = 0$ in $\tilde{\Omega}_0$, where $\tilde{\Omega}_0$ is the extension of Ω_0 in x-direction and as $\Omega \setminus \Omega_0$ is open, we have that $B_r^N(x_0) \times B_r^M(\tilde{y}) \subset \Omega \setminus \Omega_0$. Therefore, $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$. □

Theorem 2.1 *[Strong Maximum Principle] Let Ω be an open set of $\mathbb{R}^N \times \mathbb{R}^M$, the function $h \in L^\infty_{loc}(\Omega)$ and $w \in C(\bar{\Omega}) \cap L^\infty(\tilde{\Omega})$ has the property (P) satisfying*

$$\mathcal{L}w \geq 0 \quad \text{in } \Omega \quad \text{and} \quad w \geq 0 \quad \text{in } \Omega. \tag{2.14}$$

Assume that $\Omega_0 \neq \emptyset$ defined by (2.13) and there exists some $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$ such that (2.4) holds in corresponding D.

Then w must be 0 in $\tilde{\Omega}$.

Proof. Assume that $\Omega_0 \neq \Omega$. By Lemma 2.3, $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$ and then $w(x_0, y_0) = 0$ by $w \in C(\bar{\Omega})$ and the definition of Ω_0. Furthermore, we observe that $\bar{D}$ is compact in Ω and then $h \in L^\infty(\bar{D})$. Using Lemma 2.2, we obtain (2.5), which is impossible by the fact of $w(x_0, y_0) = \inf_\Omega w = 0$. Therefore, $\Omega_0 = \Omega$, i.e. $w \equiv 0$ in Ω. Since w has property (P), then $w \equiv 0$ in $\tilde{\Omega}$. $\square$

3 Decay estimate

3.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 on decay estimates for positive classical solutions of equation (1.7). The main work is to construct appropriate super and sub solutions and then the decay estimate is derived by Lemma 2.1.

Before proving Theorem 1.1, we introduce some computations gathered in the next proposition. For $\alpha \in (0, 1)$ and $\mu > 0$, we define the function $\psi_\mu : \mathbb{R}^N \to \mathbb{R}$ as follows

$$\psi_\mu(x) = \begin{cases} \mu^{-N-2\alpha}, & |x| < \mu, \\ |x|^{-N-2\alpha}, & |x| \geq \mu. \end{cases} \tag{3.1}$$

Proposition 3.1 *For any $\mu > 0$, there exists $R_0 > 3\mu$ and $c > 0$, independent of μ, such that*

$$-c\mu^{-2\alpha}\psi_\mu(x) \leq (-\Delta)^\alpha \psi_\mu(x) \leq -c^{-1}\mu^{-2\alpha}\psi_\mu(x), \quad x \in B^c_{R_0}. \tag{3.2}$$

Proof. We consider along the proof that $\mu > 0$ and $x \in \mathbb{R}^N$ satisfies $|x| > 3\mu$. We define

$$A(\mu, x, z) = \frac{\psi_\mu(x+z) + \psi_\mu(x-z) - 2\psi_\mu(x)}{|z|^{N+2\alpha}}, \qquad z \in \mathbb{R}^N$$

and we observe that

$$(-\Delta)^\alpha \psi_\mu(x) = -\frac{1}{2}\int_{\mathbb{R}^N} A(\mu, x, z)dz. \tag{3.3}$$

Now we compute the integral above by decomposing the domain in various pieces. First we consider the integral over $B_{\frac{|x|}{3}}(0)$. We observe that $|x \pm z| \geq \mu$ for all $z \in B_{\frac{|x|}{3}}(0)$, then by (3.1) we obtain

$$\begin{aligned} |\int_{B_{\frac{|x|}{3}}(0)} A(\mu, x, z)dz| &= |\int_{B_{\frac{|x|}{3}}(0)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} - 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz| \\ &= |x|^{-N-4\alpha}|\int_{B_{\frac{1}{3}}(0)} \frac{|z+e_x|^{-N-2\alpha} + |z-e_x|^{-N-2\alpha} - 2}{|z|^{N+2\alpha}}dz| \\ &\leq c_1|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(0)} \frac{|z|^2}{|z|^{N+2\alpha}}dz \leq c_2|x|^{-N-4\alpha}, \end{aligned} \tag{3.4}$$

where $e_x = \frac{x}{|x|}$ and $c_1, c_2 > 0$ are independent of μ. Next we consider the integral over $B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)$. We observe that for all $z \in B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)$ we have $|x+z| \geq |x-z| \geq \mu$ and then we obtain

$$\begin{aligned} &\int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} A(\mu, x, z)dz \\ = &\int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} - 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz \\ = &|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z+e_x|^{-N-2\alpha} + |z-e_x|^{-N-2\alpha} - 2}{|z|^{N+2\alpha}}dz \\ \leq &c_3|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} |z-e_x|^{-N-2\alpha}dz \leq c_4\mu^{-2\alpha}|x|^{-N-2\alpha}, \end{aligned}$$

where the first inequality holds since $|z+e_x| \geq |z-e_x|$ for $z \in B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)$ and $|z| \geq \frac{2}{3}$ for $z \in B_{\frac{1}{3}}(e_x)$. For the inequality on the other side, we obtain

$$\begin{aligned} &\int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} A(\mu, x, z)dz \\ = &|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z+e_x|^{-N-2\alpha} + |z-e_x|^{-N-2\alpha} - 2}{|z|^{N+2\alpha}}dz \end{aligned}$$

$$\geq \ |x|^{-N-4\alpha}\Big(\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z-e_x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz - \int_{B_{\frac{1}{3}}(e_x)} \frac{2}{|z|^{N+2\alpha}}dz\Big)$$

$$\geq \ c_5|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} |z-e_x|^{-N-2\alpha}dz - c_6|x|^{-N-4\alpha}$$

$$\geq \ c_7\mu^{-2\alpha}|x|^{-N-2\alpha} - c_8|x|^{-N-4\alpha},$$

where the second inequality holds by $|z| \leq \frac{4}{3}$ for $z \in B_{\frac{1}{3}}(e_x)$. Consequently,

$$c_7\mu^{-2\alpha}|x|^{-N-2\alpha} - c_8|x|^{-N-4\alpha} \leq \int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} A(\mu,x,z)dz \leq c_4\mu^{-2\alpha}|x|^{-N-2\alpha}, \tag{3.5}$$

where the constants $c_4, c_7, c_8 > 0$ are independent of μ. The estimate for the integral over $B_{\frac{|x|}{3}}(-x) \setminus B_\mu(-x)$ is similar.

Next we consider the integral over $B_\mu(x)$. We observe that, for $z \in B_\mu(x)$ we have since $|x+z| > \mu > |x-z|$ and $|z| \geq |x| - \mu \geq \frac{2|x|}{3}$, thus

$$\int_{B_\mu(x)} A(\mu,x,z)dz = \int_{B_\mu(x)} \frac{|x+z|^{-N-2\alpha} + \mu^{-N-2\alpha} - 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz$$

$$\leq 2\int_{B_\mu(x)} \frac{\mu^{-N-2\alpha}}{|z|^{N+2\alpha}}dz \leq c_9\mu^{-2\alpha}(|x|-\mu)^{-N-2\alpha} \leq c_{10}\mu^{-2\alpha}|x|^{-N-2\alpha}$$

and, for the other inequality

$$\int_{B_\mu(x)} A(\mu,x,z)dz \ \geq \ \int_{B_\mu(x)} \frac{-2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz$$

$$\geq \ -c_{11}\mu^N|x|^{-N-2\alpha}(|x|-\mu)^{-N-2\alpha} \geq -c_{12}|x|^{-N-4\alpha},$$

where c_9, c_{10}, c_{11} and c_{12} are positive constant independent of μ. Therefore,

$$-c_{12}|x|^{-N-4\alpha} \leq \int_{B_\mu(x)} A(\mu,x,z)dz \leq c_{10}\mu^{-2\alpha}|x|^{-N-2\alpha}. \tag{3.6}$$

The integral over $B_\mu(-x)$ is exactly the same. Finally, we consider the complementary integral over $D(x) = \mathbb{R}^N \setminus (B_{\frac{|x|}{3}}(0) \cup B_{\frac{|x|}{3}}(x) \cup B_{\frac{|x|}{3}}(-x))$. For $|x| > 3\mu$ and $z \in D(x)$, we have that $|x \pm z| \geq \frac{|x|}{3}$, thus

$$|\int_{D(x)} A(\mu,x,z)dz| \ \leq \ \int_{D(x)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} + 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz$$

$$\leq \ c_{13}|x|^{-N-2\alpha}\int_{\mathbb{R}^N\setminus B_{\frac{|x|}{3}}(0)} \frac{1}{|z|^{N+2\alpha}}dz$$

$$\leq \ c_{14}|x|^{-N-4\alpha}, \tag{3.7}$$

where $c_{13} > 0$ and $c_{14} > 0$ are independent of μ. Therefore, by (3.4)-(3.7), there exist $c_{15}, c_{16} > 1$ independent of μ such that

$$c_{15}^{-1}\mu^{-2\alpha}|x|^{-N-2\alpha} - c_{15}|x|^{-N-4\alpha} \leq \int_{\mathbb{R}^N} A(\mu, x, z)dz$$
$$\leq c_{16}\mu^{-2\alpha}|x|^{-N-2\alpha} + c_{16}|x|^{-N-4\alpha} \leq c_{15}\mu^{-2\alpha}|x|^{-N-2\alpha},$$

where we used that $|x| > 3\mu$. Choosing $R_0 > 3\mu$ such that $c_{15}^{-1}\mu^{-2\alpha} - c_{15}|x|^{-2\alpha} \geq \frac{1}{2}c_{15}^{-1}\mu^{-2\alpha}$ for $|x| \geq R_0$, together with (3.3), we obtain (3.2). □

In what follows we provide a proof of our first theorem on the decay of the positive solutions of our equation.

Proof of Theorem 1.1. By definition of A and B in (1.9), for any $\epsilon > 0$, there exits $\delta_\epsilon > 0$ such that

$$(B - \epsilon^2)t \leq f(t) \leq (A + \epsilon^2)t, \quad \forall\, t \in (0, \delta_\epsilon). \tag{3.8}$$

Since u is a positive solution of (1.7) vanishing at infinity, there exists $R_\epsilon > 0$ such that $0 < u(x, y) < \delta_\epsilon$ for any $(x, y) \in B^c_{R_\epsilon}$. Therefore,

$$(-\Delta)^\alpha_x u + (-\Delta)_y u + (1 - A - \epsilon^2)u \leq 0 \quad \text{in } B^c_{R_\epsilon} \tag{3.9}$$

and

$$(-\Delta)^\alpha_x u + (-\Delta)_y u + (1 - B + \epsilon^2)u \geq 0 \quad \text{in } B^c_{R_\epsilon}. \tag{3.10}$$

Next we define the function $\phi_\nu : \mathbb{R}^M \to \mathbb{R}$ as $\phi_\nu(y) = e^{-\nu|y|}$, where $\nu > 0$ and we find that for $y \in \mathbb{R}^M \setminus \{0\}$,

$$-\Delta\phi_\nu(y) = \nu\left(\frac{M-1}{|y|} - \nu\right)\phi_\nu(y). \tag{3.11}$$

Step 1. *There exists $C(\epsilon) > 1$ such that*

$$u(x, y) \leq C(\epsilon)e^{-\theta_1|y|}, \qquad (x, y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{3.12}$$

To prove (3.12) we let $U_1(x, y) = \phi_{\theta_1}(y)$, for $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$ and then, by (3.11), we have

$$\begin{aligned}&(-\Delta)^\alpha_x U_1 + (-\Delta)_y U_1 + (1 - A - \epsilon^2)U_1\\ &= \left[\theta_1\left(\frac{M-1}{|y|} - \theta_1\right) + 1 - A - \epsilon^2\right]U_1 \geq 0,\end{aligned} \tag{3.13}$$

if $\epsilon \leq \sqrt{1-A}$. By definition of U_1 and ϕ_{θ_1} we have that $U_1 = 1$ in $\mathbb{R}^N \times \{0\}$ and $U_1 \geq e^{-\theta_1 R_\epsilon}$ in $\bar{B}_{R_\epsilon}$ and, since u is bounded, there exists $\rho_1 > 0$ depending on ϵ, such that

$$W_1 = \rho_1 U_1 - u \geq 0 \quad \text{in } \bar{B}_{R_\epsilon} \cup (\mathbb{R}^N \times \{0\}).$$

Combining (3.9) with (3.13), we obtain

$$(-\Delta)_x^\alpha W_1 + (-\Delta)_y W_1 + (1 - A - \epsilon^2) W_1 \geq 0 \quad \text{in } \bar{B}_{R_\epsilon}^c \cap (\mathbb{R}^N \times \{0\})^c.$$

By Lemma 2.1, this implies that $W_1 \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^M$ and then

$$u(x,y) \leq \rho_1 U_1(x,y) = \rho_1 \phi_{\theta_1}(y) = \rho_1 e^{-\theta_1 |y|}, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{3.14}$$

Step 2. *There exists $C(\epsilon) > 1$ such that*

$$u(x,y) \leq C(\epsilon)|x|^{-N-2\alpha}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{3.15}$$

Let c and R_0 be as in Proposition 3.1 $\mu = (c/(2\epsilon\sqrt{(1-A)} - 2\epsilon^2))^{\frac{1}{2\alpha}}$ and consider the function $U_2(x,y) = \psi_\mu(x)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. Then, by (3.2), we have for all $(x,y) \in (B_{R_0}^N(0))^c \times \mathbb{R}^M$ that

$$\begin{aligned}&(-\Delta)_x^\alpha U_2 + (-\Delta)_y U_2 + (1 - A - \epsilon^2) U_2 \\ &\geq (-c\mu^{-2\alpha} + 1 - A - \epsilon^2) U_2 \geq 0\end{aligned} \tag{3.16}$$

for $0 < \epsilon < \sqrt{1-A}$. Let us denote $W_2 = \rho_2 U_2 - u$, where $\rho_2 > 0$ is such that

$$W_2 \geq \rho_2 (R_0 + R_\epsilon)^{-N-2\alpha} - u \geq 0 \quad \text{in } \bar{B}_{R_\epsilon} \cup (\overline{B_{R_0}^N(0)} \times \mathbb{R}^M).$$

Combining (3.9) with (3.16), we obtain that

$$(-\Delta)_x^\alpha W_2 + (-\Delta)_y W_2 + (1 - A - \epsilon^2) W_2 \geq 0 \quad \text{in } \bar{B}_{R_\epsilon}^c \cap (\overline{B_{R_0}^N(0)} \times \mathbb{R}^M)^c.$$

By Lemma 2.1, we have that $W_2 = \rho_2 U_2 - u \geq 0$ in $R^N \times \mathbb{R}^M$ and then, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,

$$u(x,y) \leq \rho_2 U_2(x,y) = \rho_2 \psi_\mu(x) \leq \rho_2 |x|^{-N-2\alpha}.$$

Step 3. *There exists $C(\epsilon) > 1$ such that*

$$u(x,y) \leq C(\epsilon)|x|^{-N-2\alpha} e^{-\theta_1 |y|}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{3.17}$$

Let us consider the function $V(x,y) = \psi_\mu(x)\phi_{\theta_1}(y)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, with μ as defined above. From (3.2) and (3.11), we have that

$$\begin{aligned}&(-\Delta)_x^\alpha V + (-\Delta)_y V + (1 - A - \epsilon^2) V \\ &\geq \left[-c\mu^{-2\alpha} + \theta_1 \left(\frac{M-1}{|y|} - \theta_1\right) + 1 - A - \epsilon^2\right] V \geq 0,\end{aligned} \tag{3.18}$$

for $(x,y) \in (B_{R_0}^N(0))^c \times (\mathbb{R}^M \setminus \{0\})$ and assuming that $0 < \epsilon < \sqrt{1-A}$. Since u, V are bounded in $\bar{B}_{R_\epsilon}$ and V is positive, there is $\bar{\rho}_1 > 0$ large such that

$$\bar{\rho}_1 V - u \geq 0 \quad \text{in } \bar{B}_{R_\epsilon}.$$

By (3.12) and (3.14), we may choose $\bar\rho_2 > 0$ such that

$$\begin{aligned}\bar\rho_2 V - u &\geq \bar\rho_2 R_0^{-N-2\alpha}\phi_{\theta_1}(y) - u \geq 0 \quad \text{in} \quad \overline{B^N_{R_0}(0)} \times \mathbb{R}^M \quad \text{and} \\ \bar\rho_2 V - u &\geq \bar\rho_2 \psi_\mu(x) - u \geq 0 \quad \text{in} \quad \mathbb{R}^N \times \{0\}.\end{aligned}$$

Taking $\bar\rho = \max\{\bar\rho_1, \bar\rho_2\}$, defining $W = \bar\rho V - u$ and combining (3.9) with (3.18), we have that

$$W \geq 0 \quad \text{in} \quad \bar B_{R_\epsilon} \cup (\overline{B^N_{R_0}(0)} \times \mathbb{R}^M) \cup (\mathbb{R}^N \times \{0\}) \quad \text{and}$$

$$(-\Delta)^\alpha_x W + (-\Delta)_y W + (1 - A - \epsilon^2) W \geq 0 \quad \text{in} \quad \bar B^c_{R_\epsilon} \cap ((B^N_{R_0}(0))^c \times (\mathbb{R}^M \setminus \{0\})).$$

Then, by Lemma 2.1, we have that $\bar\rho V - u \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^M$. Thus, there exists $C(\epsilon) > 1$ such that

$$u(x,y) \leq C(\epsilon)\psi_\mu(x)\phi_{\theta_1}(y) \leq C(\epsilon)|x|^{-N-2\alpha}e^{-\theta_1|y|}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M.$$

Step 4. *There exists $C_1(\epsilon) > 0$ and $R > 0$ such that*

$$u(x,y) \geq C_1(\epsilon)e^{-\theta_2|y|}, \qquad (x,y) \in \overline{B^N_R(0)} \times \mathbb{R}^M. \tag{3.19}$$

Let R_0 be as in Proposition 3.1 and let $R > R_0$ such that $\lambda_1 < \epsilon^2$, where λ_1 is the first eigenvalue of the fractional Dirichlet problem (2.9) with $x_0 = 0$ and $r = 4R$. Let φ_N be the first eigenfunction of (2.9) and define $V_1(x,y) = \varphi_N(x)\phi_{\theta_2}(y)$ for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. From (2.9) and (3.11), for $(x,y) \in B^N_{2R}(0) \times (B^M_{R_1}(0))^c$ with $R_1 = \frac{M-1}{\epsilon}$, we have

$$\begin{aligned} &(-\Delta)^\alpha_x V_1 + (-\Delta)_y V_1 + (1 - B + \epsilon^2)V_1 \\ &= \left[\lambda_1 + \theta_2\left(\frac{M-1}{|y|} - \theta_2\right) + 1 - B + \epsilon^2\right] V_1 \\ &\leq [\epsilon^2 + \theta_2(\epsilon - \theta_2) + 1 - B + \epsilon^2]V_1 \leq 0, \end{aligned} \tag{3.20}$$

if $\epsilon < \sqrt{1-B}$. Let us define $w_1 = u - r_1 V_1$, where $r_1 > 0$ is such that

$$w_1 \geq 0 \quad \text{in } \overline{B}_{R_\epsilon} \cup (\overline{B^N_{2R}(0) \times B^M_{R_1}(0)})$$

and observe that $w_1 \geq 0$ in $(B^N_{2R}(0))^c \times \mathbb{R}^M$ since $V_1 = 0$. Combining (3.10) with (3.20), we obtain that

$$(-\Delta)^\alpha_x w_1 + (-\Delta)_y w_1 + (1 - B + \epsilon^2) w_1 \geq 0 \quad \text{in} \quad (B^N_{2R}(0) \times (B^M_{R_1}(0))^c) \cap B^c_{R_\epsilon}$$

and then, by Lemma 2.1, we have that

$$w_1 = u - r_1 V_1 \geq 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^M.$$

Since φ_N is classical solution of (2.9) with $r = 4R$ and $x_0 = 0$ then $\varphi_N(x)$ is positive in $\overline{B_R^N(0)} \subset \mathbb{R}^N$, we can finally choose $C_1(\epsilon) > 0$ such that

$$u(x,y) \geq r_1\varphi_N(x)\phi_{\theta_2}(y) \geq C_1(\epsilon)e^{-\theta_2|y|}, \quad \forall (x,y) \in \overline{B_R^N(0)} \times \mathbb{R}^M. \tag{3.21}$$

Step 5. *There exists $C_1(\epsilon) > 0$ such that, for R and R_1 as in Step 4,*

$$u(x,y) \geq C_1|x|^{-N-2\alpha}, \qquad (x,y) \in (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}. \tag{3.22}$$

To prove this, we define $V_2(x,y) = \psi_\mu(x)\eta_M(y)$ for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, where η_M is the solution of

$$\begin{cases} -\Delta\eta_M(y) = \bar{\lambda}_1\eta_M(y), & y \in B_{R_2}^M(0), \\ \eta_M(y) = 0, & y \in (B_{R_2}^M(0))^c, \end{cases} \tag{3.23}$$

with $R_2 > R_1$ such that $\bar{\lambda}_1 < \epsilon^2$. Here $\mu = [c(1 - B + 2\epsilon^2)]^{\frac{-1}{2\alpha}}$ with c as in Proposition 3.1 and ψ_μ defined in (3.1). By (3.2) and (3.23), for $(x,y) \in ((B_R^N(0))^c \times \mathbb{R}^M) \cap (\mathbb{R}^N \times B_{R_2}^M(0))$, we have that

$$\begin{aligned} &(-\Delta)_x^\alpha V_2 + (-\Delta)_y V_2 + (1 - B + \epsilon^2)V_2 \\ &\leq (-c^{-1}\mu^{-2\alpha} + \bar{\lambda}_1 + 1 - B + \epsilon^2)V_2 = 0. \end{aligned} \tag{3.24}$$

Let $w_2 = u - r_2V_2$, with $r_2 > 0$ such that

$$w_2 \geq 0 \quad \text{in } \bar{B}_{R_\epsilon} \cup \overline{(B_R^N(0)} \times \mathbb{R}^M) \cup (\mathbb{R}^N \times (B_{R_2}^M(0))^c).$$

Combining (3.10) with (3.24), we obtain that

$$(-\Delta)_x^\alpha w_2 + (-\Delta)_y w_2 + (1 - B + \epsilon^2)w_2 \geq 0$$

in $B_{R_\epsilon}^c \cap ((B_R^N(0))^c \times \mathbb{R}^M) \cap (\mathbb{R}^N \times B_{R_2}^M(0))$. By Lemma 2.1, we have then

$$w_2 = u - r_2V_2 \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^M.$$

Since η_M is positive in $\overline{B_{R_1}^M(0)} \subset B_{R_2}^M(0)$, there exists $C_1(\epsilon) > 0$ such that for any $(x,y) \in (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}$, we have that

$$u(x,y) \geq r_2\psi_\mu(x)\eta_M(y) \geq C_1(\epsilon)|x|^{-N-2\alpha}.$$

Step 6. *There exist $C_1(\epsilon) > 0$ such that, for R as in Step 4,*

$$u(x,y) \geq C_1(\epsilon)|x|^{-N-2\alpha}e^{-\theta_2|y|}, \qquad (x,y) \in (B_R^N(0))^c \times \mathbb{R}^M. \tag{3.25}$$

To prove this we let $\tilde{V}(x,y) = \psi_\mu(x)\phi_{\theta_2}(y)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$ with μ as defined above. Using (3.2) and (3.11), for $(x,y) \in (B_R^N(0))^c \times (B_{R_1}^M(0))^c$ with $R_1 = \frac{M-1}{\epsilon}$, we have that

$$\begin{aligned}
&(-\Delta)_x^\alpha \tilde{V} + (-\Delta)_y \tilde{V} + (1 - B + \epsilon^2)\tilde{V} \\
&\le \left[-c^{-1}\mu^{-2\alpha} + \theta_2 \left(\frac{M-1}{|y|} - \theta_2 \right) + 1 - B + \epsilon^2 \right] \tilde{V} \\
&\le [\theta_2(\epsilon - \theta_2) + 1 - B + \epsilon^2]\tilde{V} \le 0, \qquad (3.26)
\end{aligned}$$

if $0 < \epsilon < \sqrt{1-B}$. Since u is positive and V is bounded in $\overline{B}_{R_\epsilon}$, we can choose $\tilde{r}_1 > 0$ such that

$$u - \tilde{r}_1 V \ge 0 \quad \text{in} \ \ \overline{B}_{R_\epsilon}.$$

Since ψ_μ is bounded in $\overline{B_R^N(0)}$, using (3.21), there exists $\tilde{r}_2 > 0$ such that

$$u - \tilde{r}_2 V \ge u - \tilde{r}_2 c_1 e^{-\theta_2 |y|} \ge 0 \quad \text{in} \ \ \overline{B_R^N(0)} \times \mathbb{R}^M,$$

and by (3.22), there exists $\tilde{r}_3 > 0$ such that

$$u - \tilde{r}_3 V \ge u - \tilde{r}_3 |x|^{-N-2\alpha} \ge 0 \quad \text{in} \ \ (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}.$$

Taking $\tilde{r} = \min\{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3\}$ and combining (3.10) with (3.26), we obtain that

$$\begin{aligned}
w = u - \tilde{r}V \ge 0 \quad &\text{in} \ \ \bar{B}_{R_\epsilon} \cup (\overline{B_R^N(0)} \times \mathbb{R}^M) \cup ((B_R^N(0))^c \times \overline{B_{R_1}^M(0)}) \ \ \text{and} \\
(-\Delta)_x^\alpha w + (-\Delta)_y w + (1 - B + \epsilon)w \ge 0 \quad &\text{in} \ \ \bar{B}_{R_\epsilon}^c \cap ((B_R^N(0))^c \times (B_{R_1}^M(0))^c).
\end{aligned}$$

Thus Lemma 2.1, we have that $w \ge 0$ in $\mathbb{R}^N \times \mathbb{R}^M$ and then (3.25) holds.

Finally, Step 1 $-$ Step 6 completes the proof. □

3.2 Proof of Theorem 1.2

This subsection is devoted to prove Theorem 1.2. Our proof is based on the fundamental solution of the mixed integro-differential operator. We first study the fundamental solution $\mathcal{K}$ for

$$(-\Delta)_x^\alpha u + (-\Delta)_y u + u = 0 \quad \text{in} \quad \mathbb{R}^N \times (\mathbb{R}^M \setminus \{0\}),$$

which can be characterized by

$$\mathcal{K}(x,y) = \int_0^\infty e^{-t}\mathcal{H}(x,y,t)dt, \qquad (3.27)$$

where

$$\mathcal{H}(x,y,t)=\int_{\mathbb{R}^M}\int_{\mathbb{R}^N} e^{-2\pi i(x,y)\cdot(\xi_1,\xi_2)-t(|\xi_1|^{2\alpha}+|\xi_2|^2)}d\xi_1 d\xi_2. \tag{3.28}$$

In fact, for $\phi\in\mathcal{S}$, we have that

$$\begin{aligned}\langle\mathcal{K},\phi\rangle &= \int_{\mathbb{R}^{N+M}}\int_0^\infty\int_{\mathbb{R}^{N+M}} e^{-2\pi i(x,y)\cdot(\xi_1,\xi_2)-t(|\xi_1|^{2\alpha}+|\xi_2|^2+1)}\phi(x,y)d\xi_1 d\xi_2 dt dx dy\\ &= \int_{\mathbb{R}^{N+M}}\left[\int_0^\infty e^{-t(|\xi_1|^{2\alpha}+|\xi_2|^2+1)}dt\int_{\mathbb{R}^{N+M}} e^{-2\pi i(x,y)\cdot(\xi_1,\xi_2)}\phi(x,y)dxdy\right]d\xi_1 d\xi_2\\ &= \int_{\mathbb{R}^{N+M}}\left[\frac{1}{|\xi_1|^{2\alpha}+|\xi_2|^2+1}\int_{\mathbb{R}^{N+M}} e^{-2\pi i(x,y)\cdot(\xi_1,\xi_2)}\phi(x,y)dxdy\right]d\xi_1 d\xi_2\\ &= \left\langle\frac{1}{|\xi_1|^{2\alpha}+|\xi_2|^2+1},\mathcal{F}\phi\right\rangle.\end{aligned}$$

Next we want to find some properties of $\mathcal{H}$. To this end, we consider

$$\mathcal{H}_\alpha(x,t)=\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi_1-t|\xi_1|^{2\alpha}}d\xi_1 \quad\text{and}\quad \mathcal{H}_1(y,t)=\int_{\mathbb{R}^M} e^{-2\pi i y\cdot\xi_2-t|\xi_2|^2}d\xi_2.$$

It is well known that the function $\mathcal{H}_\alpha$ has the following properties:

$$\mathcal{H}_\alpha(x,t)=t^{-\frac{N}{2\alpha}}\mathcal{H}_\alpha(t^{-\frac{1}{2\alpha}}x,1) \quad\text{and}\quad \lim_{|x|\to\infty}|x|^{N+2\alpha}\mathcal{H}_\alpha(x,1)=C,$$

where $C>0$, which imply that there exists $c_1>0$ and $c_2>$ such that

$$c_1\min\{t^{-\frac{N}{2\alpha}},t|x|^{-N-2\alpha}\}\le\mathcal{H}_\alpha(x,t)\le c_2\min\{t^{-\frac{N}{2\alpha}},t|x|^{-N-2\alpha}\}, \tag{3.29}$$

see [22, 17]. By the definition of $\mathcal{H}$, we have that

$$\mathcal{H}(x,y,t)=\mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t). \tag{3.30}$$

Since we have

$$\mathcal{H}_1(y,t)=(4\pi t)^{-\frac{M}{2}}e^{-\frac{|y|^2}{4t}}, \tag{3.31}$$

see [22], together with (3.27)-(3.30), for $|y|>2$,

$$\begin{aligned}\mathcal{K}(x,y) &= \int_0^\infty e^{-t}\mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t)dt\\ &\ge c_1\int_0^\infty e^{-t}\min\{t^{-\frac{N}{2\alpha}},t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}}e^{-\frac{|y|^2}{4t}}dt\\ &\ge c_1\int_{\frac{|y|}{2}}^{\frac{|y|}{2}+1} e^{-t}\min\{t^{-\frac{N}{2\alpha}},t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}}e^{-\frac{|y|^2}{4t}}dt\\ &\ge c_3\min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}},|x|^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\},\end{aligned}$$

for some $c_3 > 0$. On the other hand, since for $n \geq 3$ we have

$$\int_0^\infty e^{-t}(4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}} dt \leq c_4 e^{-|y|}|y|^{2-n}(1+|y|)^{\frac{n-3}{2}}$$

with $c_4 > 0$ (see [22]), for $M \geq 5$ we have that

$$\begin{aligned}
\mathcal{K}(x,y) &= \int_0^\infty e^{-t}\mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t)dt \\
\leq\ & c_2 \int_0^\infty e^{-t} \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt \\
\leq\ & c_5 \min\{\int_0^\infty e^{-t}(4\pi t)^{-\frac{N}{2\alpha}-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt, |x|^{-N-2\alpha}\int_0^\infty e^{-t}(4\pi t)^{1-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt\} \\
\leq\ & c_6 \min\{e^{-|y|}|y|^{2-\frac{N}{\alpha}-M}(1+|y|)^{\frac{N}{2\alpha}+\frac{M}{2}-\frac{3}{2}}, |x|^{-N-2\alpha}e^{-|y|}|y|^{4-M}(1+|y|)^{\frac{M-5}{2}}\}
\end{aligned}$$

Therefore, for $N \geq 1$ and $M \geq 5$, there exist $c_8 > c_7 > 0$ such that

$$c_7\rho(x,y) \leq \mathcal{K}(x,y) \leq c_8\rho(x,y)|y|^{\frac{1}{2}}, \quad (x,y) \in \mathbb{R}^N \times (B_2^M(0))^c, \tag{3.32}$$

where $\rho(x,y)$ is defined in (1.14). In what follows, we construct super and sub-solutions to obtain the decay estimate given in Theorem 1.2.

Proof of Theorem 1.2. By the estimate in Theorem 1.1, we observe that, for constants $c_{10} > c_9 > 0$ such that

$$c_9(1+|x|)^{-N-2\alpha} \leq u(x,y) \leq c_{10}(1+|x|)^{-N-2\alpha}, \quad (x,y) \in \mathbb{R}^N \times B_2^M(0),$$

so we only need to prove (1.13) holds for $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$.

Step 1: *Lower bound.* Let $\tilde{u} = \mathcal{K} * \chi_{B_1^N(0)\times B_1^M(0)}$, where $\chi_{B_1^N(0)\times B_1^M(0)}$ is the characteristic function of $B_1^N(0) \times B_1^M(0)$. By (3.32), we have that

$$\tilde{u}(x,y) \geq c_{11} \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}, (1+|x|)^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\}, \tag{3.33}$$

for all $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$, where $c_{11} > 0$. By definition of $\tilde{u}$, we have

$$(-\Delta)_x^\alpha \tilde{u} + (-\Delta)_y \tilde{u} + \tilde{u} = 0 \quad \text{in } \mathbb{R}^N \times (\mathbb{R}^M \setminus \{0\}) \setminus (B_1^N(0) \times B_1^M(0))$$

and, by (3.32) and Theorem 1.1, there exists $c_{12} > 0$ such that $u \geq c_{11}\tilde{u}$ in $\mathbb{R}^N \times \{y \in \mathbb{R}^M : |y| = 2\}$. Since f is nonnegative, we use the Comparison Principle to obtain that, for any $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$

$$u(x,y) \geq c_{11}\tilde{u}(x,y) \geq c_{12} \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}, (1+|x|)^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\}.$$

Step 2: *Upper bound.* For $y \in \mathbb{R}^M$ with $|y| \geq 2$, there exists $1 \leq i \leq M$ such that $|y_i| > 1$, we may assume that $y_1 > 1$. Let $\bar{u}(x,y) = \mathcal{K}(x,y)(1 - |y_1|^{-1})$, then by direct computation

$$\begin{aligned} (-\Delta)_y \bar{u} &= (1 - |y_1|^{-1})(-\Delta)_y \mathcal{K} - 2y_1^{-2}\partial_{y_1}\mathcal{K} + 2\mathcal{K}y_1^{-3} \\ &\geq (-\Delta)_y \mathcal{K}(1 - |y_1|^{-1}) + 2\mathcal{K}y_1^{-3}, \end{aligned}$$

where the last inequality holds since $y_1 > 0$ and $\partial_{y_1}\mathcal{K} < 0$. Therefore, by (3.32), we have that for $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$,

$$\begin{aligned} & (-\Delta)_x^\alpha \bar{u}(x,y) + (-\Delta)_y \bar{u}(x,y) + \bar{u}(x,y) \\ \geq\ & [(-\Delta)_x^\alpha \mathcal{K} + (-\Delta)_y \mathcal{K} + \mathcal{K}](1 - |y_1|^{-1}) + 2\mathcal{K}(x,y)y_1^{-3} \geq 2\mathcal{K}(x,y)|y|^{-3} \\ \geq\ & 2c_8 \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}-3}, |x|^{-N-2\alpha}e^{-|y|}|y|^{-\frac{M}{2}-2}\}. \end{aligned} \tag{3.34}$$

Since $f(u) = O(u^p)$ near $u = 0$ for some $p > 1$, by Theorem 1.1 with $\epsilon = \frac{p-1}{4p}$, we have that

$$(-\Delta)_x^\alpha u + (-\Delta)_y u + u) = f(u) \leq c_{13}(1 + |x|)^{-(N+2\alpha)p}e^{-\frac{3p+1}{4}|y|},$$

where $c_{13} > 0$. We notice that $\frac{3p+1}{4} > 1$. By definition of $\bar{u}$, (3.32) and Theorem 1.1 with $\epsilon = \frac{p-1}{4p}$, there exists $c_{14} > 0$ such that $u \leq c_{14}\bar{u}$ in $\mathbb{R}^N \times \{y \in \mathbb{R}^M : |y| = 2\}$. By Comparison Principle, we have that

$$\begin{aligned} u(x,y) &\leq c_{14}\bar{u}(x,y) \leq c_{14}\mathcal{K}(x,y) \\ &\leq c_{15} \min\{e^{-|y|}|y|^{\frac{1}{2}-\frac{N}{2\alpha}-\frac{M}{2}}, (1 + |x|)^{-N-2\alpha}e^{-|y|}|y|^{\frac{3}{2}-\frac{M}{2}}\} \end{aligned}$$

for all $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$ and some $c_{15} > 0$. This complete the proof.

□

4 Symmetry results

In this section, we prove Theorem 1.3 by moving planes method. Let u be a classical positive solution of (1.7) and consider first the y-direction. Let

$$\Sigma_\lambda^{y_1} = \{(x, y_1, y') \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{M-1} \mid y_1 > \lambda\},$$

$$T_\lambda^{y_1} = \{(x, y_1, y') \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{M-1} \mid y_1 = \lambda\}$$

and $u_\lambda(x, y_1, y') = u(x, 2\lambda - y_1, y')$ for $\lambda \in \mathbb{R}$. We introduce a preliminary inequality which plays a crucial role in the procedure of moving planes.

Lemma 4.1 *Under the assumptions of Theorem 1.3, for any $\lambda \in \mathbb{R}$, there exists $c_1 > 0$, independent of λ, such that*

$$\begin{aligned} & c_1\Big(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy\Big)^{\frac{N+M-2\alpha}{N+M}} \\ \leq & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy < \infty. \end{aligned}$$

Proof. First we show that the integrals are finite. We observe that u_λ satisfies the same equation (1.7) as u in $\Sigma_\lambda^{y_1}$. Taking $(u_\lambda - u)^+$ as test function in the equations for u and u_λ, subtracting and integrating in $\Sigma_\lambda^{y_1}$, we find

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy \\ = & \int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy. \end{aligned} \tag{4.1}$$

Now we only need to prove that

$$\int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy < +\infty. \tag{4.2}$$

In fact, for any given $\lambda \in \mathbb{R}$, using (1.10), we choose $R > 1$ such that

$$0 < u_\lambda(x,y) \leq C_\epsilon (1 + |x|)^{-N-2\alpha} e^{-\theta_1 |y_\lambda|} < s_0, \quad \forall (x,y) \in B_R^c,$$

where $y_\lambda = (2\lambda - y_1, y')$ for $y = (y_1, y') \in \mathbb{R}^M$ and s_0 is from (F).

If $u_\lambda(x,y) > u(x,y)$ for some $(x,y) \in \Sigma_\lambda^{y_1} \cap B_R^c$, we have $0 < u(x,y) < u_\lambda(x,y) < s_0$. Using (1.15) with $v = u_\lambda(x,y)$, then

$$\frac{f(u_\lambda(x,y)) - f(u(x,y))}{u_\lambda(x,y) - u(x,y)} \leq \bar{c} u_\lambda^\gamma(x,y),$$

then

$$(f(u_\lambda(x,y)) - f(u(x,y)))^+ (u_\lambda(x,y) - u(x,y))^+ \leq \bar{c} u_\lambda^{\gamma+2}(x,y).$$

The inequality above is obvious if $u_\lambda(x,y) \leq u(x,y)$ for some $(x,y) \in \Sigma_\lambda^{y_1} \cap B_R^c$. Then

$$(f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ \leq \bar{c} u_\lambda^{\gamma+2} \quad \text{in} \quad \Sigma_\lambda^{y_1} \cap B_R^c.$$

Therefore,

$$\begin{aligned}
&\int_{\Sigma_\lambda^{y_1}\cap B_R^c} (f(u_\lambda)-f(u))^+(u_\lambda-u)^+dxdy \\
\le\ & \bar{c}\int_{\Sigma_\lambda^{y_1}\cap B_R^c} u_\lambda^{\gamma+2}(x,y)dxdy \\
\le\ & \bar{c}C_\epsilon\int_{\Sigma_\lambda^{y_1}} (1+|x|)^{-(N+2\alpha)(\gamma+2)}e^{-(\gamma+2)\theta_1|y_\lambda|}dxdy \\
\le\ & \bar{c}C_\epsilon\int_{\mathbb{R}^N} (1+|x|)^{-(N+2\alpha)(\gamma+2)}dx\int_{\mathbb{R}^M} e^{-(\gamma+2)\theta_1|y|}dy<+\infty,
\end{aligned}$$

where the last inequality holds by $\gamma>\frac{2\alpha N}{(N+M)(N+2\alpha)}$. Since u and u_λ are bounded and f is locally Lipschitz, we have

$$\int_{\Sigma_\lambda^{y_1}\cap B_R} (f(u_\lambda)-f(u))^+(u_\lambda-u)^+dxdy<+\infty.$$

Therefore, (4.2) holds. Together with (4.1), we have the second inequality in the result.

Next we show that the first inequality holds in Lemma 4.1. Let us denote

$$w(x,y)=\begin{cases}(u_\lambda-u)^+(x,y), & (x,y)\in\Sigma_\lambda^{y_1},\\ (u_\lambda-u)^-(x,y), & (x,y)\in(\Sigma_\lambda^{y_1})^c\end{cases} \tag{4.3}$$

and

$$\mathrm{supp}(w)=\overline{\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^M \mid w(x,y)\neq 0\}},$$

where $(u_\lambda-u)^+(x,y)=\max\{(u_\lambda-u)(x,y),\ 0\}$, $(u_\lambda-u)^-(x,y)=\min\{(u_\lambda-u)(x,y),\ 0\}$. We observe that $w(x,y_1,y')=-w(x,2\lambda-y_1,y')$ for $(x,y_1,y')\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^{M-1}$ and

$$w=u_\lambda-u \quad \text{in } \ \mathrm{supp}(w). \tag{4.4}$$

It is obvious that for $(x,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$, $\{z\in\mathbb{R}^N|\ (z,y)\in(\Sigma_\lambda^{y_1})^c\}=\emptyset$ and

$$\begin{aligned}
\mathbb{R}^N \ =\ & \{z\in\mathbb{R}^N|\ (z,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)\}\cup\\
& \{z\in\mathbb{R}^N|\ (z,y)\in\Sigma_\lambda^{y_1}\cap(\mathrm{supp}(w))^c\}\cup\{z\in\mathbb{R}^N|\ (z,y)\in(\Sigma_\lambda^{y_1})^c\}.
\end{aligned}$$

Combining with (4.4), then for $(x,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$,

$$(-\Delta)_x^\alpha w(x,y)-(-\Delta)_x^\alpha(u_\lambda-u)(x,y)=\int_{\mathbb{R}^N}\frac{(u_\lambda-u)(z,y)-w(z,y)}{|x-z|^{N+2\alpha}}dz$$

$$=\int_{\{z\in\mathbb{R}^N:(z,y)\in\Sigma_\lambda^{y_1}\cap(\mathrm{supp}(w))^c\}}\frac{(u_\lambda-u)(z,y)}{|x-z|^{N+2\alpha}}dz\le 0, \tag{4.5}$$

where the last inequality holds by $u_\lambda - u \le 0$ in $\Sigma_\lambda^{y_1} \cap (\text{supp}(w))^c$. On one hand, from (4.5) and $w = (u_\lambda - u)^+ > 0$ in $\Sigma_\lambda^{y_1} \cap \text{supp}(w)$, we have that

$$\int_{\Sigma_\lambda^{y_1} \cap \text{supp}(w)} (-\Delta)_x^\alpha w \, w dxdy \le \int_{\Sigma_\lambda^{y_1} \cap \text{supp}(w)} (-\Delta)_x^\alpha (u_\lambda - u)(u_\lambda - u)^+ dxdy. \tag{4.6}$$

On the other hand, we know that $w(x,y) = (u_\lambda - u)(x,y)$ and $(-\Delta)_y w(x,y) = (-\Delta)_y (u_\lambda - u)(x,y)$ for $(x,y) \in \Sigma_\lambda^{y_1} \cap \text{supp}(w)$. Together with (4.6), then

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1} \cap \text{supp}(w)} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w] \, w dxdy \\ \le & \int_{\Sigma_\lambda^{y_1} \cap \text{supp}(w)} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy \end{aligned}$$

and then by the fact of $w = (u_\lambda - u)^+ = 0$ in $\Sigma_\lambda^{y_1} \cap (\text{supp}(w))^c$, we have that

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w] \, w dxdy \\ \le & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+. \end{aligned} \tag{4.7}$$

By the definition of w, we have that

$$\int_{\mathbb{R}^{N+M}} |w|^2 dxdy = 2 \int_{\Sigma_\lambda^{y_1}} |w|^2 dxdy,$$

$$\int_{\mathbb{R}^{N+M}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy = 2 \int_{\Sigma_\lambda^{y_1}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy,$$

$$\int_{\mathbb{R}^{N+M}} (-\Delta)_y w \, w dxdy = 2 \int_{\Sigma_\lambda^{y_1}} (-\Delta)_y w \, w dxdy,$$

$$\int_{\mathbb{R}^{N+M}} (-\Delta)_x^\alpha w \, w dxdy = 2 \int_{\Sigma_\lambda^{y_1}} (-\Delta)_x^\alpha w \, w dxdy,$$

then, together with Proposition 2.1, we obtain that

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w] \, w dxdy \\ = & \frac{1}{2} \int_{\mathbb{R}^{N+M}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w] \, w dxdy \\ \ge & c_3 (\int_{\mathbb{R}^{N+M}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\ = & c_3 (2 \int_{\Sigma_\lambda^{y_1}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}}, \end{aligned} \tag{4.8}$$

for some $c_3 > 0$. Combining (4.7) with (4.8), by $w = (u_\lambda - u)^+$ in $\Sigma_\lambda^{y_1}$, we get the first inequality in Lemma 4.1. The proof is complete. $\square$

Lemma 4.2 *Under the assumptions of Theorem 1.3, for any $\lambda \in \mathbb{R}$, there exists $c_4 > 0$ independent of λ such that*

$$\begin{aligned} & c_4\Big(\int_{\Sigma_\lambda^{x_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy\Big)^{\frac{N+M-2\alpha}{N+M}} \\ \le & \int_{\Sigma_\lambda^{x_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy < \infty, \end{aligned}$$

where $\Sigma_\lambda^{x_1} = \{(x_1, x', y) \in \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^M \mid x_1 > \lambda\}$.

Proof. The proof proceeds similarly to the proof of Lemma 4.1, the only difference is to show (4.5) with $(x, y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)$. It is obvious that

$$\begin{aligned} \mathbb{R}^N \;\; = \;\; & \{z \in \mathbb{R}^N |\ (z, y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)\} \cup \\ & \{z \in \mathbb{R}^N |\ (z, y) \in \Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c\} \cup \\ & \{z \in \mathbb{R}^N |\ (z, y) \in (\Sigma_\lambda^{x_1})^c \cap (\mathrm{supp}(w))^c\} \cup \\ & \{z \in \mathbb{R}^N |\ (z, y) \in (\Sigma_\lambda^{x_1})^c \cap \mathrm{supp}(w)\} \end{aligned}$$

and $w = u_\lambda - u$ in $\mathrm{supp}(w)$, then for $(x, y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)$,

$$\begin{aligned} & (-\Delta)_x^\alpha w(x, y) - (-\Delta)_x^\alpha (u_\lambda - u)(x, y) = \int_{\mathbb{R}^N} \frac{(u_\lambda - u)(z, y) - w(z, y)}{|x - z|^{N+2\alpha}} dz \\ = & \int_{\{z \in \mathbb{R}^N |\ (z,y) \in \Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c\}} \Big(\frac{1}{|x - z|^{N+2\alpha}} - \frac{1}{|x - z_\lambda|^{N+2\alpha}}\Big)(u_\lambda - u)(z, y) dz \\ \le & \ 0, \end{aligned}$$

where $z_\lambda = (2\lambda - z_1, z')$ for $z = (z_1, z') \in \mathbb{R}^N$ and the last inequality holds by $u_\lambda - u \le 0$ in $\Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c$. $\square$

Theorem 4.1 *Under the assumptions of Theorem 1.3, for $x \in \mathbb{R}^N$, we have*

$$u(x, y) = u(x, |y|)$$

and u is strictly decreasing in y-direction.

Proof. We divide the proof into three steps.
Step 1: $\lambda_0 := \sup\{\lambda \mid u_\lambda \le u \text{ in } \Sigma_\lambda^{y_1}\}$ is finite. Since u decays at infinity, we observe that the set $\{\lambda \mid u_\lambda \le u \text{ in } \Sigma_\lambda^{y_1}\}$ is nonempty. Using $(u_\lambda - u)^+$ as a

test function in the equation for u and u_λ, by (1.15) and Hölder inequality, for λ big (negative), we find that

$$\begin{aligned}
&\int_{\Sigma_\lambda^{y_1}}[(-\Delta)_x^\alpha(u_\lambda-u)+(-\Delta)_y(u_\lambda-u)+(u_\lambda-u)](u_\lambda-u)^+dxdy\\
&=\int_{\Sigma_\lambda^{y_1}}(f(u_\lambda)-f(u))(u_\lambda-u)^+dxdy\\
&=\int_{\Sigma_\lambda^{y_1}}\frac{f(u_\lambda)-f(u)}{u_\lambda-u}[(u_\lambda-u)^+]^2dxdy\le\bar c\int_{\Sigma_\lambda^{y_1}}u_\lambda^\gamma[(u_\lambda-u)^+]^2dxdy\\
&\le c_5\int_{\Sigma_\lambda^{y_1}}(1+|x|)^{-\gamma(N+2\alpha)}e^{-\gamma\theta_1|y_\lambda|}[(u_\lambda-u)^+]^2dxdy\\
&\le c_5(\int_{\Sigma_\lambda^{y_1}}(1+|x|)^{-a}e^{-b|y_\lambda|}dxdy)^{\frac{2\alpha}{N+M}}(\int_{\Sigma_\lambda^{y_1}}|(u_\lambda-u)^+|^{\frac{2(N+M)}{N+M-2\alpha}}dxdy)^{\frac{N+M-2\alpha}{N+M}},
\end{aligned}$$

where $a=\frac{\gamma(N+2\alpha)(N+M)}{2\alpha}$ and $b=\frac{\theta_1\gamma(N+M)}{2\alpha}$. Since $\gamma>\frac{2\alpha N}{(N+2\alpha)(N+M)}$, we have that $a>N$. Then we can choose $R>0$ such that for all $\lambda<-R$,

$$c_5(\int_{\Sigma_\lambda^{y_1}}(1+|x|)^{-a}e^{-b|y_\lambda|}dxdy)^{\frac{2\alpha}{N+M}}\le\frac{1}{4}.$$

By Lemma 4.1, we obtain that

$$\int_{\Sigma_\lambda^{y_1}}|(u_\lambda-u)^+|^{\frac{2(N+M)}{N+M-2\alpha}}dxdy=0,\qquad\forall\ \lambda<-R.$$

Thus $u_\lambda\le u$ in $\Sigma_\lambda^{y_1}$ for all $\lambda<-R$ and then conclude that $\lambda_0\ge-R$. On the other hand, since u decays at infinity, then there exist $\lambda_1\in\mathbb{R}$ and $(x,y)\in\Sigma_\lambda^{y_1}$ such that $u(x,y)<u_{\lambda_1}(x,y)$. Hence λ_0 is finite.

Step 2: $u\equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$. Assuming the contrary, we have that $u\not\equiv u_{\lambda_0}$ and $u\ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, in this case the following claim holds.

Claim 1. *If $u\not\equiv u_{\lambda_0}$ and $u\ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then $u>u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.*

Let us assume, for the moment, that Claim 1 is true, then for any given $\lambda\in(\lambda_0,\lambda_0+\epsilon)$, where $\epsilon>0$ is chosen later. Let $P=(0,\cdots,\lambda,\cdots,0)\in T_\lambda^{y_1}$ and $B(P,R)$ be the ball centered at P and with radius $R>1$ to be chosen later. Define $B_1=\Sigma_\lambda^{y_1}\cap B(P,R)$ and let us consider $(u_\lambda-u)^+$ test function in the equation for u and u_λ in $\Sigma_\lambda^{y_1}$, then from Lemma 4.1 we obtain

$$\begin{aligned}
&(\int_{\Sigma_\lambda^{y_1}}|(u_\lambda-u)^+|^{\frac{2(N+M)}{N+M-2\alpha}}dxdy)^{\frac{N+M-2\alpha}{N+M}}\\
\le\ & c_6\int_{\Sigma_\lambda^{y_1}}[(-\Delta)_x^\alpha(u_\lambda-u)+(-\Delta)_y(u_\lambda-u)+(u_\lambda-u)](u_\lambda-u)^+dxdy
\end{aligned}$$

$$= \quad c_6 \int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy. \tag{4.9}$$

We estimate the integral on the right. Proceeding as in Step 1, we can choose $R > 1$ big enough such that

$$c_7 \Big(\int_{\Sigma_\lambda^{y_1} \setminus B_1} (1 + |x|)^{-a} e^{-b|y_\lambda|} dxdy \Big)^{\frac{2\alpha}{N+M}} \leq \frac{1}{4}$$

for some $c_7 > 0$, where $a = \frac{\gamma(N+2\alpha)(N+M)}{2\alpha}$ and $b = \frac{\theta_1 \gamma(N+M)}{2\alpha}$. Then

$$\begin{aligned}
&\int_{\Sigma_\lambda^{y_1} \setminus B_1} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy \leq \bar{c} \int_{\Sigma_\lambda^{y_1} \setminus B_1} u_\lambda^\gamma |(u_\lambda - u)^+|^2 dxdy \\
&\leq c_7 \Big(\int_{\Sigma_\lambda^{y_1} \setminus B_1} (1 + |x|)^{-a} e^{-b|y_\lambda|} dxdy \Big)^{\frac{2\alpha}{N+M}} \Big(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy \Big)^{\frac{N+M-2\alpha}{N+M}} \\
&\leq \frac{1}{4} \Big(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy \Big)^{\frac{N+M-2\alpha}{N+M}}.
\end{aligned} \tag{4.10}$$

Now using Claim 1, we choose $\epsilon > 0$ such that $c_8 |B_1 \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N+M}} < 1/4$, for some $c_8 > 0$. Since f is locally Lipschitz, using Hölder inequality, we have

$$\begin{aligned}
&\int_{B_1} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy \leq c_{46} \int_{B_1} |(u_\lambda - u)^+|^2 \chi_{\mathrm{supp}(u_\lambda - u)^+} dxdy \\
&= c_8 |B_1 \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N+M}} \Big(\int_{B_1} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy \Big)^{\frac{N+M-2\alpha}{N+M}} \\
&\leq \frac{1}{4} \Big(\int_{B_1} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy \Big)^{\frac{N+M-2\alpha}{N+M}}.
\end{aligned} \tag{4.11}$$

From (4.9), (4.10) and (4.11), it follows that $(u_\lambda - u)^+ = 0$ in $\Sigma_\lambda^{y_1}$. Then $u_\lambda \leq u$ in $\Sigma_\lambda^{y_1}$ for $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, which contradicts the definition of λ_0. As a consequence, we have $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.

In order to complete Step 2, we only need to prove Claim 1.

Proof of Claim 1. By contradiction, if there exists $(\bar{x}, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}$ such that $u(\bar{x}, \bar{y}) = u_{\lambda_0}(\bar{x}, \bar{y})$, then

$$\begin{aligned}
&(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) + (-\Delta)_y (u - u_{\lambda_0})(\bar{x}, \bar{y}) + (u - u_{\lambda_0})(\bar{x}, \bar{y}) \\
&= f(u(\bar{x}, \bar{y})) - f(u_{\lambda_0}(\bar{x}, \bar{y})) = 0.
\end{aligned}$$

Since $(u - u_{\lambda_0})(\bar{x}, \bar{y}) = \min_{\Sigma_{\lambda_0}^{y_1}} (u - u_{\lambda_0}) = 0$, we have $(-\Delta)_y (u - u_{\lambda_0})(\bar{x}, \bar{y}) \leq 0$, then

$$(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) \geq 0. \tag{4.12}$$

The other side, we observe that $\{z \in \mathbb{R}^N | \ (z, \bar{y}) \in (\Sigma_{\lambda_0}^{y_1})^c\} = \emptyset$ when $(\bar{x}, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}$. By $u(\bar{x}, \bar{y}) = u_{\lambda_0}(\bar{x}, \bar{y})$ and then

$$(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) = -\int_{\mathbb{R}^N} \frac{(u - u_{\lambda_0})(z, \bar{y})}{|\bar{x} - z|^{N+2\alpha}} dz$$

$$= -\int_{\{z \in \mathbb{R}^N | \ (z, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}\}} \frac{(u - u_{\lambda_0})(z, \bar{y})}{|\bar{x} - z|^{N+2\alpha}} dz \leq 0, \tag{4.13}$$

where the last inequality holds by $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.

Combining (4.12) with (4.13), we obtain that $(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) = 0$ and then from (4.13), we have that

$$u(z, \bar{y}) = u_{\lambda_0}(z, \bar{y}), \quad \forall z \in \mathbb{R}^N, \tag{4.14}$$

this means that $u - u_{\lambda_0}$ has property (P) and by $u \not\equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$ we have

$$(\bar{x}, \bar{y}) \in (\Sigma_{\lambda_0}^{y_1})_0 := \{(x, y) \in \Sigma_{\lambda_0}^{y_1} \mid (u - u_{\lambda_0})(x, y) = \inf_{\Sigma_{\lambda_0}^{y_1}} (u - u_{\lambda_0}) = 0\} \subsetneqq \Sigma_{\lambda_0}^{y_1}.$$

Moreover, by Proposition 2.3 with $\Omega = \Sigma_{\lambda_0}^{y_1}$, we observe that $\Sigma_{\lambda_0}^{y_1} \setminus (\Sigma_{\lambda_0}^{y_1})_0$ satisfies interior cylinder condition at point $(x_0, y_0) \in \partial(\Sigma_{\lambda_0}^{y_1})_0 \cap \Sigma_{\lambda_0}^{y_1}$. Then there exist $r > 0$ small and $\tilde{y} \in \mathbb{R}^M$ such that

$$O_r := B_r^N(x_0) \times B_r^M(\tilde{y}) \subset \Sigma_{\lambda_0}^{y_1} \setminus (\Sigma_{\lambda_0}^{y_1})_0 \quad \text{and} \quad (x_0, y_0) \in \partial O_r.$$

Let D be defined by (2.3). Since $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then for any $(x, y) \in D$, we have

$$\int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{(u - u_{\lambda_0})(z, y)}{|x - z|^{N+2\alpha}} dz \geq 0.$$

Finally, it is obvious that

$$(-\Delta)_x^\alpha (u - u_{\lambda_0}) + (-\Delta)_y (u - u_{\lambda_0}) + h(u - u_{\lambda_0}) = 0 \quad \text{in } \Sigma_{\lambda_0}^{y_1},$$

where $h = 1 - \frac{f(u) - f(u_{\lambda_0})}{u - u_{\lambda_0}} \in L_{loc}^\infty(\Sigma_{\lambda_0}^{y_1})$. Then we use Theorem 2.1 to obtain

$$u \equiv u_{\lambda_0} \quad \text{in } \tilde{\Sigma}_{\lambda_0}^{y_1},$$

which contradicts the condition of $u \not\equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then we obtain the results in Claim 1.

Step 3. By translation, we may say that $\lambda_0 = 0$. Repeating the argument from the other side, we find that u is symmetric about y_1-axis. Using the same argument in any y-direction, we conclude that

$$u(x, y) = u(x, |y|), \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^M.$$

Finally, we prove that $u(x,|y|)$ is strictly decreasing in $|y|>0$. Indeed, for any given $y_1<\widetilde{y}_1<0$ and letting $\lambda=\frac{y_1+\widetilde{y}_1}{2}$. Then, as proved above we have

$$u>u_\lambda \quad \text{in} \quad \Sigma_\lambda^{y_1}.$$

For any given $x\in\mathbb{R}^N$, we observe that $(x,\widetilde{y}_1,0,\cdots,0)\in\Sigma_\lambda^{y_1}$, then

$$u(x,\widetilde{y}_1,0,\cdots,0)>u_\lambda(x,\widetilde{y}_1,0,\cdots,0)=u(x,y_1,0,\cdots,0).$$

Using the result of $u(x,y)=u(x,|y|)$ for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^M$ and $|\widetilde{y}_1|<|y_1|$, we conclude monotonicity of u respect to y. This completes the proof. $\square$

Next we study the symmetry result in x-direction.

Theorem 4.2 *Under the assumptions of Theorem 1.3, for $y\in\mathbb{R}^M$, we have*

$$u(x,y)=u(|x|,y)$$

and u is strictly decreasing in x-direction.

Proof. The proof of this theorem goes like the one for Theorem 4.1. The only place where there is a difference is in the following property: *if $u\not\equiv u_{\lambda_0}$ and $u\ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, then $u>u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$.* By contradiction, if there exists $(\bar{x},\bar{y})\in\Sigma_{\lambda_0}^{x_1}$ such that $u(\bar{x},\bar{y})=u_{\lambda_0}(\bar{x},\bar{y})$, then

$$\begin{aligned}&(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y})+(-\Delta)_y(u-u_{\lambda_0})(\bar{x},\bar{y})+(u-u_{\lambda_0})(\bar{x},\bar{y})\\&=f(u(\bar{x},\bar{y}))-f(u_{\lambda_0}(\bar{x},\bar{y}))=0.\end{aligned}$$

Since $u\ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, we have $(u-u_{\lambda_0})(\bar{x},\bar{y})=\min_{\Sigma_{\lambda_0}^{x_1}}(u-u_{\lambda_0})=0$ and $(-\Delta)_y(u-u_{\lambda_0})(\bar{x},\bar{y})\le 0$ and then

$$(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y})\ge 0.$$

The other side, by direct computation, we have that

$$\begin{aligned}&(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y})=\int_{\mathbb{R}^N}\frac{(u_{\lambda_0}-u)(z,\bar{y})}{|\bar{x}-z|^{N+2\alpha}}dz\\&=\int_{\{z\in\mathbb{R}^N|\ (z,\bar{y})\in\Sigma_{\lambda_0}^{x_1}\}}\Big(\frac{1}{|\bar{x}-z|^{N+2\alpha}}-\frac{1}{|\bar{x}-z_{\lambda_0}|^{N+2\alpha}}\Big)(u_{\lambda_0}-u)(z,\bar{y})dz\le 0,\end{aligned}$$

where $z_{\lambda_0}=(2\lambda_0-z_1,z')$ for $z=(z_1,z')\in\mathbb{R}^N$ and the last inequality holds by $u\ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$. Therefore,

$$u(z,\bar{y})=u_{\lambda_0}(z,\bar{y}),\quad \forall z\in\mathbb{R}^N, \tag{4.15}$$

this means that $u-u_{\lambda_0}$ has property (P) and by $u\neq u_{\lambda_0}$ in $\Sigma^{x_1}_{\lambda_0}$ we have

$$(\bar{x},\bar{y})\in(\Sigma^{x_1}_{\lambda_0})_0:=\{(x,y)\in\Sigma^{x_1}_{\lambda_0}\mid(u-u_{\lambda_0})(x,y)=\inf_{\Sigma^{x_1}_{\lambda_0}}(u-u_{\lambda_0})=0\}\subsetneqq\Sigma^{x_1}_{\lambda_0}.$$

Moreover, by Proposition 2.3, we observe that $\Sigma^{x_1}_{\lambda_0}\setminus(\Sigma^{x_1}_{\lambda_0})_0$ satisfies interior cylinder condition at point $(x_0,y_0)\in\partial(\Sigma^{x_1}_{\lambda_0})_0\cap\Sigma^{x_1}_{\lambda_0}$. Then there exist $r_1>0$ and $\tilde{y}\in\mathbb{R}^M$ such that for all $r\in(0,r_1]$,

$$O_r:=B^N_r(x_0)\times B^M_r(\tilde{y})\subset\Sigma^{x_1}_{\lambda_0}\setminus(\Sigma^{x_1}_{\lambda_0})_0\quad\text{and}\quad(x_0,y_0)\in\partial O_r.$$

Next we show that there exists some $r\in(0,r_1]$ such that for any $(x,y)\in D$,

$$\int_{\mathbb{R}^N\setminus B^N_r(x_0)}\frac{(u-u_{\lambda_0})(z,y)}{|x-z|^{N+2\alpha}}dz\geq0,\tag{4.16}$$

where D is defined by (2.3). Indeed, since $u\not\equiv u_{\lambda_0}$ and $u\geq u_{\lambda_0}$ in $\Sigma^{x_1}_{\lambda_0}$, then for $(x,y)\in D\subset\Sigma^{x_1}_{\lambda_0}$, we have that

$$\int_{\mathbb{R}^N}\frac{(u-u_{\lambda_0})(z,y)}{|x-z|^{N+2\alpha}}dz>0.$$

Let us define

$$r(x,y)=\sup\{r\in(0,r_1]:\int_{\mathbb{R}^N\setminus B^N_r(x_0)}\frac{(u-u_{\lambda_0})(z,y)}{|x-z|^{N+2\alpha}}dz\geq0\}.\tag{4.17}$$

Let $r_m=\inf_{(x,y)\in D}r(x,y)$, it is obvious that $r_m\in[0,r_1]$. Now we prove that $r_m>0$. By contradiction, if $r_m=0$, then there exist a sequence $(x_n,y_n)\in D$ and $(\tilde{x},\tilde{y})\in\bar{D}$ such that $(x_n,y_n)\to(\tilde{x},\tilde{y})$ and $r(x_n,y_n)\to0$, as $n\to+\infty$. Since $r(x,y)$ is continuous, then $r(\tilde{x},\tilde{y})=0$. If $(\tilde{x},\tilde{y})\in\bar{D}\setminus(\Sigma^{x_1}_{\lambda_0})_0$, i.e. $u(\tilde{x},\tilde{y})>u_{\lambda_0}(\tilde{x},\tilde{y})$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{(u-u_{\lambda_0})(z,\tilde{y})}{|\tilde{x}-z|^{N+2\alpha}}dz\\=&\int_{\{z\in\mathbb{R}^N|\ (z,\tilde{y})\in\Sigma^{x_1}_{\lambda_0}\}}(u-u_{\lambda_0})(z,\tilde{y})(\frac{1}{|\tilde{x}-z|^{N+2\alpha}}-\frac{1}{|\tilde{x}-z_{\lambda_0}|^{N+2\alpha}})dz>0.\end{aligned}$$

By the continuity of the integration and (4.17), we obtain that $r(\tilde{x},\tilde{y})>0$, which is impossible.

Then $(\tilde{x},\tilde{y})\in\bar{D}\cap(\Sigma^{x_1}_{\lambda_0})_0$, i.e. $u(\tilde{x},\tilde{y})=u_{\lambda_0}(\tilde{x},\tilde{y})$. Since the function $u-u_{\lambda_0}$ has property (P), then for any $\tilde{r}>0$,

$$\int_{\mathbb{R}^N\setminus B^N_{\tilde{r}}(x_0)}\frac{(u-u_{\lambda_0})(z,\tilde{y})}{|\tilde{x}-z|^{N+2\alpha}}dz=0.$$

Combining with (4.17), we obtain that $r(\tilde{x}, \tilde{y}) = r_1 > 0$, which contradicts $r(\tilde{x}, \tilde{y}) = 0$. As a consequence, we have that $0 < r_m \le r_1$. Taking $r = r_m$, then (4.16) holds for any $(x, y) \in D$. Finally, it is obvious that

$$(-\Delta)_x^\alpha (u - u_{\lambda_0}) + (-\Delta)_y (u - u_{\lambda_0}) + h(u - u_{\lambda_0}) = 0 \quad \text{in } \Sigma_{\lambda_0}^{x_1},$$

where $h = 1 - \frac{f(u) - f(u_{\lambda_0})}{u - u_{\lambda_0}} \in L^\infty_{loc}(\Sigma_{\lambda_0}^{x_1})$. Then we use Theorem 2.1 to obtain that

$$u \equiv u_{\lambda_0} \quad \text{in } \tilde{\Sigma}_{\lambda_0}^{x_1},$$

which contradicts the condition of $u \neq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$. Then $u > u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, to complete the proof. □

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