

Catch-to-stock dependence: The case of small pelagic fishery with bounded harvesting effort

Erica Cruz-Rivera¹  | Héctor Ramírez C.²  | Olga Vasilieva¹ 

¹Departamento de Matemáticas,
Universidad del Valle, Cali, Colombia

²Departamento de Ingeniería Matemática
and Centro de Modelamiento Matemático
(CNRS UMI 2807), Universidad de Chile,
Santiago, Chile

Correspondence

Erica Cruz-Rivera, Departamento de
Matemáticas, Universidad del Valle,
Calle 13 No. 100-00, Cali 760032,
Colombia.
Email: erica.cruz@correounivalle.edu.co

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Abstract

Biologic characteristics of schooling fish species explain why the rates of harvesting in pelagic fisheries are not proportional to the existent stock size and may exhibit no variation between the periods of fish abundance and scarcity. Therefore, the stock-dependent nonlinearities in catchability must be reflected in the design of flexible fishing policies, which target the sustainable exploitation of this important natural resource. In this study, such nonlinearities are expressed through eventual variability of the “catch-to-stock” parameter that measures the sensitivity of an additional catch yield to marginal changes in the fish-stock level. Using the optimal control modeling framework, we establish that each value of the “catch-to-stock” parameter generates a unique steady-state size of the fish stock and the latter engenders an optimal fishing policy that can be sustained as long as the “catch-to-stock” parameter remains unchanged. We also prove the continuous dependence of the steady-state stock and underlying fishing policy upon the mentioned “catch-to-stock” parameter and then focus on the analysis of the equilibrium responses to changes in this parameter induced by external perturbations.

Recommendations for Resource Managers

- Marginal catches of pelagic fish stocks do not react in a linear way to changes in existing stock level, and the latter is captured in our model by the “catch-to-stock” parameter $\beta \in (0, 1)$. Each observable value of β engenders a unique steady-state stock size that defines



an optimal fishing policy, which can be sustained as long as β remains unchanged.

- The ability of fishery managers to detect variations in the levels of hyperstability expressed by the “catch-to-stock” parameter $\beta \in (0, 1)$ may help them to anticipate new equilibrium responses in stock evolution and to make timely adjustments in the fishing policy.
- Plausible estimations of the “catch-to-stock” parameter $\beta \in (0, 1)$, as well as detection of its possible alterations, can be carried out within the framework of Management Strategy Evaluation (MSE) approach where different data collected inside and outside the fishery are contrasted via the validation of a relatively simple decision-making model (presented in this paper) coupled with other “operation models” of higher complexity.
- If the “catch-to-stock” parameter cannot be reasonably assessed ($\beta \rightarrow 0^+$), the fishery managers may rely upon the lower bound of stationary stock size, which depends on economic and biological factors (such as the present and future economic values of the exploited fish stock, its marginal productivity, and underlying dynamics of biological growth).

KEYWORDS

bioeconomic model, Cobb–Douglas production function, hyperstability, maximum principle, optimal control, pelagic fishery, sustainable fishing policy

1 | INTRODUCTION

Small pelagic fish (such as herring, anchovies, capelin, smelts, sardines, or pilchards) is characterized by high reproduction rate and rather short life-cycle. Additionally, small schooling pelagic species have strong recurrent cycles of fish abundance and scarcity and exhibit a distinctive behavioral pattern that makes them particularly vulnerable to capture. They tend to group in large and dense shoals or schools, which constitute an easy target for commercial fishers. During the periods of declining abundance, the overall number of shoals may decrease, while each shoal maintains the same average size as during the periods of increased abundance. Thus, average biomass density in schools remains almost the same during fish abundance and scarcity periods. Therefore, fishers may achieve an almost constant catch per unit effort (CPUE) anytime.

In simple bioeconomic models (see, e.g., Clark, 1990, 2010; Grafton et al., 2004; Hilborn & Walters, 2001; King, 2013; Quinn & Deriso, 1999, and similar textbooks), it is usually assumed that CPUE (catch or harvest H per unit of fishing effort u) is proportional to the existent stock



biomass x , such that $CPUE = H/u = qx$, where q is a constant known as the *catchability* coefficient (or efficiency of an average unit fishing vessel). Therefore, CPUE data are traditionally used to provide information on the historical and current stock abundances (Fréon, Cury, Shannon, & Roy, 2005; Kleisner, Zeller, Froese, & Pauly, 2013; Maunder & Punt, 2004; Mullon, Fréon, & Cury, 2005)¹.

In contrast, changes in CPUE may not be proportional to the actual population size. In other words, CPUE may decline either faster or slower than fish abundance. The latter can be expressed, according to Harley, Myers, and Dunn (2001), by stock-dependent catchability function $q = q(x) = x^{\beta-1}$ instead of the constant catchability coefficient, where $\beta > 0$ is referred to as *elasticity of catchability* with respect to changes in the stock level. Thus, $\beta > 1$ implies that CPUE declines faster than fish stock x , and this situation is known as “hyperdepletion”; conversely, if $\beta < 1$, CPUE declines slower than x , which results in “hyperstability”².

The case of small pelagic fishery exhibits “hyperstability” (Harley et al., 2001; Hilborn & Walters, 2001; Quinn & Deriso, 1999) with catch rates declining slower ($0 < \beta < 1$) or not declining at all ($\beta \rightarrow 0^+$) despite decreasing stock size. Thus, using the CPUE data in an assessment of current fish abundance, while disregarding stock-dependent nonlinearities in catchability (i.e., assuming $\beta = 1$), can misinform fishery managers by producing too “optimistic” estimates of existing fish stock and increase the risk of fishery overexploitation.

For that reason, different scholars propose an alternative use of a nonlinear harvesting function (or instantaneous harvesting rate) in the form of Cobb–Douglas production function (Bannerot & Austin, 1983; Bjorndal & Conrad, 1987; Da-Rocha, Nøstbakken, & Pérez, 2014; Gajardo, Peña-Torres, & Ramírez, 2011; Hannesson, 1983; Harley et al., 2001; Maroto & Moran, 2008; Maroto, Moran, Sandal, & Steinshamn, 2012; Pascoe & Hillary, 2016; Quinn & Deriso, 1999) when dealing with pelagic fisheries, that is,

$$h(x, u) = q(u, x)ux = u^\alpha x^\beta \quad \text{with} \quad q(u, x) = u^{\alpha-1}x^{\beta-1}, \quad (1)$$

where the nonnegative parameter $\alpha > 0$ expresses, according to Gajardo et al. (2011), how the marginal catch productivity (or yield) responds to changes in fishing effort, whereas the nonnegative parameter $\beta > 0$ has the same meaning as above and measures the sensitivity of catch yields with respect to marginal changes in the fish-stock level³. It is worthwhile to note that Maroto et al. (2012) refer to the parameters α and β in a more general sense and simply call them *elasticities* of fishing effort and stock output, respectively.

Standard Gordon–Schaefer-type fishery models with constant catchability are included in the generalized formula of harvesting (1) when CPUE is assumed proportional to stock abundance ($\beta = 1$) and the marginal catch productivity of an additional fishing effort is supposed constant (i.e., $\alpha = 1$). Under this assumption, the resource is always safe from extinction, even at low biological growth rates of the fish stock and high discount rates (Clark, 1990).

However, empirical evidence suggests that, in case of small pelagic stocks exhibiting hyperstability, such simplification is not plausible and that β should be strictly less than unity (see, e.g., Bannerot & Austin, 1983; Bjorndal, 1987; Harley et al., 2001; Mackinson, Sumaila, & Pitcher, 1997; Opsomer & Conrad, 1994), while for some particular schooling species it may even reach values near zero (Mackinson et al., 1997; Pascoe & Hillary, 2016). In contrast, the fishing effort elasticity α is not significantly different from unity for most fleets (Bjorndal & Conrad, 1987; Bjorndal, 1989; Opsomer & Conrad, 1994; Pascoe & Hillary, 2016; Peña-Torres & Basch, 2000). Nonetheless, Hannesson (1983) points out that $\alpha > 1$ can be expected when there are “few” vessels, while $\alpha < 1$ should imply “many” vessels.



It is worthwhile to point out that Gajardo et al. (2011) had thoroughly studied a continuous-time harvesting model of small pelagic fishery with decreasing returns ($0 < \alpha < 1$) and positive but small values of β coupled by the relationship $\alpha + \beta = 1$ and analyzed the response of a unique steady-state biomass and underlying stationary harvesting when $\beta \rightarrow 0^+$ (loss of sensitivity of the harvesting with respect to variations in stock level). Alternatively, Maroto et al. (2012) analyzed a discrete-time pelagic fishery harvesting model with increasing returns ($\alpha > 1$) and fixed $\beta = 0.562$ and reaffirmed the previous result of Bjørndal and Conrad (1987) regarding the case of constant returns ($\alpha = 1$) with $\beta < 1$ where numerical experiments revealed the existence of optimal steady-state equilibrium, in which the resource can be preserved at high stock levels. Moreover, Stoeven (2014) studied a model of recreational fishing with constant catch-to-effort productivity ($\alpha = 1$) and hyperstable stock ($0 < \beta < 1$), while maximizing a constant elasticity of substitution (CES) utility function⁴ where not only the harvested biomass but also the fishing effort itself constitute the benefits for recreational fishers. This is not the case we are interested in, whereas in commercial fishery models the fishing effort is bounded by the fleet capacity and cannot be maximized without enhancing the costs related to the resource exploitation and affecting the overall economic rent.

In the present paper, we follow the main idea of Gajardo et al. (2011), while assuming *constant* marginal productivity of an additional fishing unit ($\alpha = 1, \beta \geq 0$) in the harvesting function of Cobb–Douglas type. Additionally, we suppose that fishing effort is bounded and provides an explicit form of sustainable fishing policy as a function of stationary stock that, in its turn, continuously depends upon the parameter β . This policy enables the social planner to choose an adequate number of fishing units (vessels) in accordance with the maximum allowable fishing effort. The latter marks the principal difference between our work and that of Gajardo et al. (2011) where the fishing effort (a priori assumed unbounded) has been characterized in terms of the stationary stock and underlying shadow price, both continuously dependent upon β and both essentially dependent on the number of vessels in the fishery.

We expect that our findings may help in the evaluation of existent and future fishing policies. In recent years, Management Strategy Evaluation (MSE) approach has become widely used by decision-makers for investigating trade-offs between different objectives of the yields maximization, costs reduction, and challenges of stock conservation (Maunder et al., 2006; Maunder & Punt, 2013; Punt, 2015; Wilberg et al., 2010). The MSE approach relies on development of several “operating models” that simulate the full management cycle by testing a set of comprehensive management strategies under different scenarios of stock population dynamics, to determine a strategy, which is sufficiently robust and predicts a desirable outcome. In this context, our simple single-stock model along with the proposed fishing policy could be tested within the frameworks of MSE approach with different levels of hyperstability expressed by the catch-to-stock parameter β and different sizes of initial fish stocks. This, apart from providing a theoretical basis, may give additional insights into potential behavior of pelagic fisheries and help in further adapting and improving the fishery management models.

This paper is organized as follows. In Section 2, we formulate a stylized social planner’s problem and derive its formal solution using the optimal control framework with infinite horizon. Section 3 is divided into two subsections, which jointly constitute the core contribution of our work. Section 3.1 is focused on proving the existence and uniqueness of the stationary level of resource that defines an optimal fishing policy, which is sustainable for each fixed level of hyperstability $\beta \in (0, 1)$ observed in the fishery. Section 3.2 establishes the continuous dependence of the stationary level of resource and underlying fishing policy upon parameter β and presents a thorough analysis of asymptotic behavior of the stationary solution when



$\beta \rightarrow 0^+$, accompanied by bioeconomic interpretations of mathematical findings along with a discussion of results. Finally, Section 4 contains conclusions and final remarks. The mathematical proofs are placed in Appendices A, B, and C.

2 | FORMAL SOLUTION OF SOCIAL PLANNER'S PROBLEM

By “social planner,” we understand a sole owner of a single-species pelagic fishery whose principal goal consists in choosing a single-decision variable (stationary level of fishing effort) that allows one to exploit this natural resource in a *sustainable* way, while maximizing the total discounted value of intertemporal economic rents derived from fishing.

2.1 | Problem formulation

Before proceeding with a mathematical formulation of the problem, let us state that our approach relies on several essential nonmathematical assumptions, which may stand beyond the current practices in some small real-world fisheries, whereas the bigger commercial fisheries should be capable of complying with such assumptions. In more specific terms, our approach requires that fishery managers:

- be aware of the current trends in economic situation; that is, they should have reasonable estimations of the present and future economic values of the asset (exploited fish stock)⁵;
- be able to assess and to monitor the stock size and have a plausible estimation of the natural biological growth of the exploited biomass.⁶;
- be capable of detecting the variability in hyperstability levels (expressed by parameter β in 1) using CPUE records and/or alternative fishery-independent data collections.

Besides these points, we assume that all the vessels participating in the fishery have the same technology and, consequently, the same costs of harvesting. We also suppose that the unit price of harvesting is exogenously fixed by the fishery market.

Our stylized bioeconomic model will be deliberately kept simple enough for focusing more directly on sensitivity analysis of the social planner's optimal choice of fishing effort, and hence of the stationary (sustainable) levels for the harvesting rate and the fish stock, with respect to changes in the parameter β .

Consider a single-species initial stock of pelagic fish $x(0) = x_0 > 0$ and suppose that there are N symmetric and uniformed fishing units (e.g., vessels) that start to harvest simultaneously at $t = 0$. The number of fishing units is defined by the social planner who solely controls the harvesting process. The evolution of fish stock $x(t)$ (or biomass) under exploitation can be described by the following stylized model:

$$\begin{cases} \dot{x}(t) = F(x(t)) - h(t), & t > 0, \\ x(0) = x_0, & x_0 > 0, \end{cases} \quad (2)$$

where $F(x)$ stands for some growth function of biological stock with carrying capacity $K > 0$ such that $F(0) = F(K) = 0$, $F(x) > 0$ for all $x \in (0, K)$ ⁷ and $h(t)$ corresponds to harvesting.



Remark 1 Some examples of $F(x)$ widely used in the literature are the following (see, e.g., Brauer and Castillo-Chavez, 2001):

1. Logistic function: $F_1(x) = ax(1 - x/K)$.
2. Gompertz function: $F_2(x) = \begin{cases} ax \ln(K/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$
3. Smith function: $F_3(x) = \frac{ax(K-x)}{K+bx}$.

All the above functions $F_i(x)$, $i = 1, 2, 3$ are concave (hump-shaped) for $x \in (0, K)$ and satisfy the conditions $F_i(0) = F_i(K) = 0$, $F_i'(0) > 0$, $F_i'(K) < 0$ and $F_i''(x) < 0$ for $x \in (0, K)$. Additionally, a differential equation of the form $\dot{x}(t) = F_i(x(t))$, $i = 1, 2, 3$ with an initial condition $x(0) = x_0$, $x_0 \in [0, K]$ has a unique nonnegative solution $x(t)$, which is bounded for all $t \geq 0$ by the saturation constant K in the sense that $0 \leq x(t) \leq K$. In other words, any trajectory of this equation started in $[0, K]$ remains in $[0, K]$ for all $t \geq 0$.

In accordance with Remark 1, function $F(x)$ is assumed strictly concave and twice differentiable, whereas the harvesting function $h(t)$ in (2) is specified in the form of Cobb-Douglas function, that is,

$$h(t) = Nu^\alpha(t)x^\beta(t) = Nu(t)x^\beta(t) \quad (3)$$

with $\alpha = 1$ and $\beta \geq 0$, where $u: [0, \infty) \mapsto [0, u_{\max}]$ is a piecewise continuous real function that stands for an *admissible fishing effort* (single input or control variable) and includes labor, capital, maintenance, and so forth measured in fixed proportions. Under the above conditions, one can safely assume that the dynamical system (2) with positive initial condition $x_0 > 0$ always has a unique nonnegative solution for a fixed control function $u(t)$.

By defining $\alpha = 1$ in (3), we set constant marginal productivity of an additional fishing effort unit, whereas $\beta \geq 0$ will measure the sensitivity of additional catch yield to marginal changes in the fish-stock level⁸. In other words, lower values of $\beta > 0$ imply less sensitivity of harvest unit to possible variations in fish-stock level and may effectively provoke (*ceteris paribus*) the occurrence of fish extinction. Therefore, we focus in this paper on the case when $\beta \rightarrow 0^+$.

Thus, the social planner's problem consists in choosing each vessel's fishing effort $0 \leq u(\cdot) \leq u_{\max}$, and the corresponding biomass $x = x(\cdot; u)$, solution of (2) and (3), which maximizes the overall discounted value of the fishing rent. To introduce this rent, let us denote by $p > 0$ the unit price of harvesting, and $c \in (0, p)$ the cost of a fishing effort unit, which (according to our standing assumptions) is the same for each vessel. So, for each vessel, the marginal utility, at a given instant $t > 0$, is the same and can be computed as follows:

$$\pi_i(t) = pu(t)x^\beta(t) - cu(t), \quad i = 1, 2, \dots, N.$$

Thus, for the N vessels participating in the fishery and for a given discount rate $r > 0$, the mentioned overall discounted value of the fishing rent is given by

$$J(x, u) = \sum_{i=1}^N \int_0^{\infty} e^{-rt} \pi_i(t) dt = N \int_0^{\infty} e^{-rt} [pu(t)x^\beta(t) - cu(t)] dt. \quad (4)$$



Mathematically speaking, $u(t)$ is a piecewise continuous function (i.e., $u(\cdot) \in PC[0, \infty)$) with bounded range $[0, u_{\max}]$ and the constant value $u_{\max} > 0$ expresses the maximal allowable level of fishing effort. Hence, we define the set of admissible controls in the following way:

$$\mathcal{U} = \{u(\cdot) \in PC[0, \infty) \text{ and } u : [0, \infty) \mapsto [0, u_{\max}]\}. \tag{5}$$

In practice, defining the value of $u_{\max} > 0$ can be a challenging task. Therefore, this issue will be further addressed in the analysis of the models (2)–(4).

It is worthwhile to note that we consider N symmetric fishing units (see (3) and (4)); therefore, any variations in harvesting returns caused by changes in fishing effort $u(\cdot)$ are reflected only at the vessel’s level. This assumption permits us to ignore the effects of congestion in the aggregate fleet.

Summarizing all the above mentioned, we can formulate the infinite horizon problem of optimal control in the following way. For a given initial stock $x_0 \in (0, K]$, find an admissible fishing effort policy $u(\cdot)$ that maximizes the total fishing rent (4) subject to biological stock dynamics (2) with harvesting (3). In mathematical formulation, this problem can be written as

$$\max_{u \in \mathcal{U}} \{J(u, x) : x(\cdot) \text{ solves (2) and (3)}\}. \tag{6}$$

From the mathematical standpoint, problems of the type (6) are usually solved by applying the Pontryagin maximum principle (see, e.g., Grass, Caulkins, Feichtinger, Tragler, & Behrens, 2008; Lenhart & Workman, 2007; Pontryagin, Boltyanskii, Gamkrelidze, & Mishchenko, 1962), which is a necessary condition of optimality to be satisfied by any maximizer of the objective functional (4) over the set of admissible controls (5) and subject to the biological stock dynamics (2) with harvesting function (3).

In the context of our problem, where the right-hand side of the dynamical system (2) with harvesting function (3) and the integrand of the objective functional (4) are differentiable with respect to x and u , the maximum principle can be stated as it is shown in Appendix A.

From its application, if the fishing effort function $u : [0, \infty) \mapsto [0, u_{\max}]$ is a maximizer of the fishing rent (4) and $x = x(\cdot; u(\cdot)) : [0, \infty) \mapsto [0, K]$ corresponds to the fish-stock trajectory under fishing effort $u(\cdot)$, $t \in [0, \infty)$, then there exists a *costate* variable $\lambda(t)$ satisfying the so-called *optimality system*:

$$\dot{x}(t) = F(x(t)) - Nu(t)x^\beta(t), \tag{7a}$$

$$\dot{\lambda}(t) = (r - F'(x(t)))\lambda(t) + \beta Nu(t)x^{\beta-1}(t)(\lambda(t) - p). \tag{7b}$$

Moreover, the optimal control $u(t)$ can be characterized in the form

$$u(t) = \begin{cases} 0 & \text{if } \Psi < 0, \\ u_{\max} & \text{if } \Psi > 0, \\ u^* & \text{if } \Psi = 0, \end{cases} \tag{8}$$



where $u^* \in (0, u_{\max})$ will be introduced later in (13) and Ψ is the so-called *switching function*, which, in this case, is given by

$$\Psi(x(t), \lambda(t)) = N [px^\beta(t) - \lambda(t)x^\beta(t) - c]. \quad (9)$$

It is worthwhile to note that $\lambda(t)$ is usually referred to as *adjoint function* or *costate* associated with optimal control problem (6). This function can be viewed as a time-dependent Lagrange multiplier related to the differential constraint (2) and represents the current-value *shadow price* per unit of stock x , that is, the marginal value of this asset at time $t \in [0, \infty)$. It should be emphasized that shadow price always refers to the asset's value attributed to its future productivity.

Remark 2 Notice that in the case when $\Psi \equiv 0$, there may exist a *singular arc* defined by the relationship

$$x(t) = \left(\frac{c}{p - \lambda(t)} \right)^{1/\beta} \quad (10)$$

or similarly

$$\lambda(t) = p - \frac{c}{x^\beta(t)}. \quad (11)$$

This case is of our main interest and will be thoroughly studied under the assumption $\beta \rightarrow 0^+$.

From the above rationale, it is easy to derive the following outcome. The fishing policy $u(t)$ that maximizes the fishing rent (4) is sustainable for all $t \geq 0$ only if the state and costate trajectories ($x = x(t, u(t))$ and $\lambda(t)$, respectively) corresponding to application of this policy $u(t)$ become invariable as $t \rightarrow \infty$. In mathematical terms, it implies that the optimality system (7) is expected to attain (one of) its stationary state(s) under this policy $u(t)$. In the following subsection, we address the issue of existence of stationary states of the optimality system (7) taking into account the characterization of optimal control (8).

2.2 | Stationary states of the optimality system

Let us denote by (x^*, λ^*) all possible equilibria of the optimality system (7). Let also $\Psi^* = \Psi(x^*, \lambda^*)$, where Ψ was defined in (9).

Note that in the case when $\lambda^* \geq p > 0$, there is no positive equilibrium of the optimality system (7). Indeed, in this case, we have that $\Psi^* < 0$ and consequently $u^*(\cdot) = 0$. Then, the only possible equilibrium points of (7) are $x^* = 0$ and K . However, the case $x^* = K$ is discarded due to inequality $F'(K) < 0$, which leads to $r - F'(K) > 0$ and, consequently, to $\lambda^* = 0$ in (7b), obtaining thus a contradiction.

In contrast, the trivial equilibrium $x^* = 0$ (permanent stock extermination) is possible provided that $F'(0) = r$. Nevertheless, this condition is not desirable and it will be avoided in the rest of the paper (see discussion in the beginning of Section 3 and around Theorem 1).



Therefore, in what follows, we consider only the case $\lambda^* < p$, which is reasonable and economically viable. We define the *reference point*

$$\tilde{x} = \left(\frac{c}{p - \lambda^*} \right)^{1/\beta}$$

such that

$$\Psi^* < 0 \Leftrightarrow x^* < \tilde{x} \quad \text{and} \quad \Psi^* > 0 \Leftrightarrow x^* > \tilde{x}.$$

For $\Psi^* < 0$ and $u^* = 0$, there are only two possible equilibria: $x^* = 0$ and K . The first one leads us to trivial steady state $(x^*, \lambda^*) = (0, 0)$, whereas the second one implies satisfying $K < (c/p)^{1/\beta} < 1$, where $c < p$ (meaning that unit fishing cost is less than unit price of fish—otherwise, there is no profit to be gained by fishing), which contradicts the assumption $K > 0$ when β is small enough, and it should be excluded from further analysis.

For $\Psi^* > 0$ and $u^* = u_{\max}$, the analysis is more technical and its details are provided in Appendix B. Summarizing, the cases when $x^* \in [0, \tilde{x}]$ and $x^* = K$ are discarded. However, there may exist $x^* \in (\tilde{x}, K)$ such that

$$\frac{F(x^*)}{(x^*)^\beta} = u_{\max} N$$

and, in this case, λ^* satisfies the relations

$$\lambda^* = \frac{p}{\left[\frac{(r - F'(x^*))x^*}{\beta F(x^*)} + 1 \right]} \quad \text{and} \quad \lambda^* < p.$$

Finally, if $\Psi^* = 0$ for fixed values (x^*, λ^*) , we obtain that the corresponding control is given by

$$u(\cdot) = u^* := \frac{F(x^*)}{N(x^*)^\beta}, \tag{12}$$

which is, in fact, the *singular* equilibrium control for problem (6) when $u^* \in (0, u_{\max}]$. Notice that, in this case, x^* and \tilde{x} coincide. Additionally, since $\lambda^* = p - c(x^*)^{-\beta}$ and we need to ensure that $\dot{\lambda}(t) = 0$ in (7b), we can find the point x^* as a solution of the following algebraic equation:

$$(p(x^*)^\beta - c)(r - F'(x^*)) - \beta c \frac{F(x^*)}{x^*} = 0. \tag{13}$$

Equation (13) allows one to compute the equilibrium level of biomass for the problem (6) provided that its root x^* lays inside the range $(0, K)$. This will be proved in Theorem 1 (Section 3.1), supposing that the corresponding control u^* (calculated by the formula (12)) stays within the range $[0, u_{\max}]$. The latter can be ensured, for instance, for a large enough vessel’s maximal fishing effort u_{\max} or a large enough number of vessels N . Thus, if



$x^* \in (0, K)$ and $u^* \in [0, u_{\max}]$ then the fishing policy $u(t) = u^*$ applied to dynamics (2) would actually keep the biomass at the level x^* when $t \rightarrow \infty$ and thus allow for sustainable fishing using N vessels.

2.3 | Sustainable fishing policies

It is important to point out that Equation (13) can be written as

$$F'(x^*) + \frac{\beta c \frac{F(x^*)}{x^*}}{p[x^*]^\beta - c} = r, \quad (14)$$

which is a variant of the well-known *fundamental equation (or rule) of renewable resource exploitation* (see, e.g., Bjorndal & Gordon, 2007; Clark, 2010; Grafton et al., 2004). The right-hand side of (14) represents the discount rate or the *external rate of return* on assets outside the fishery, whereas its left-hand side represents the instantaneous *internal rate of return* in the fishery. It is worthwhile to point out that the fishery's internal rate of return comes from two sources: The marginal productivity of the stock (expressed by $F'(x^*)$) and the so-called *marginal stock effect* expressed by the second summand (for more details see, e.g., Clark, 2010; Grafton et al., 2004, or similar textbooks).

Thus, Equation (14) states that, at the optimal stock level x^* , the external rate of return must be equal to the instantaneous internal rate of return, and the latter should assure the fishery's profitability and sustainability.

Remark 3 In the case when u^* results greater than u_{\max} , the fishing policy $u(t) = u_{\max}$ applied to dynamics (2) will keep the biomass at some level $x > x^*$. In this case, the social planner will face a loss of potential profits by not having enough capacity of fishing effort at each unit vessel and, therefore, may think of increasing the overall number of vessels from N to $N + \Delta N$ until fulfilling that

$$\frac{F(x^*)}{(N + \Delta N)(x^*)^\beta} \leq u_{\max}.$$

Once the number of vessels is determined as $N + \Delta N$, a sustainable fishing policy defined above should keep the biomass at the level x^* for all $t > 0$.

It is important to point out that before applying the equilibrium (or sustainable) fishing policy u^* defined by (12), some adjustments are required at the beginning. This relates to the initial level $x(0) = x_0$ of the resource.

Indeed, assuming that both initial level of the biomass x_0 and the root of (13), which is now denoted by x^* , are known or at least can be fairly estimated, we have two options:

1. If $x_0 < x^*$, then optimal fishing policy should be

$$u(t) = \begin{cases} 0, & t \in [0, t_*), \\ u^*, & t \geq t_*, \end{cases} \quad (15)$$

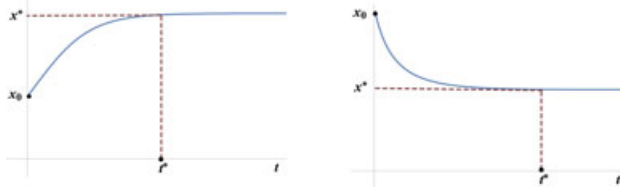


FIGURE 1 Population dynamics trajectories $x(t) = x(t, u(t))$ corresponding to turnpike solutions (15) and (16)

where $t_* > 0$ defines the moment when the resource reaches the level x^* according to its natural growth (see model (2) with $h(t) = 0$), that is, $x^* = x(t_*)$ (see Figure 1, left chart).

2. If $x_0 > x^*$, then optimal fishing policy should be

$$u(t) = \begin{cases} u_{\max}, & t \in [0, t_*), \\ u^*, & t \geq t_*, \end{cases} \tag{16}$$

where $t_* > 0$ has a similar meaning as above, that is, $x^* = x(t_*)$ but the resource level x^* is reached by reducing the fish stock (see Figure 1, right chart).

It is worthwhile to point out that long-term fishing policies of the type (15) and (16) are sometimes referred to as “turnpike solutions” since they seek to approach the steady state x^* as fast as possible. This type of solutions arises often in natural resources sustainable management problems (see, e.g., Clark, 1990, Section 2.5). However, we were not able to prove mathematically that this is the optimal strategy for our problem (6). More details regarding turnpike solutions and their properties can be consulted in the book by Zaslavski (2006).

Fishing policies (15) and (16) may seem rather simple and straightforward at the first glance. However, their practical implementation should require for knowledgeable estimations (or at least some reasonable guesses) regarding the catch-to-stock elasticity parameter β since both u^* and x^* are defined by β (cf. formula (12) with x^* determined as a root of Equation (13)). Additionally, these policies can be maintained as long as β remains unchanged. What kind of alterations in the steady-state biomass x^* and underlying fishing policy u^* can be expected when the fishery exhibits a higher or lower degree of hyperstability due to external factors? What actions should be taken by the fishery’s managers to ensure further sustainable fishing?

The answers to the above questions can be obtained by analyzing the stock equilibrium responses (i.e., fluctuation of the root x^* of stationary equations (13) or (14)) and underlying sustainable fishing policies (expressed by (12)) with respect to changes in β . Additionally, testing the model with different values of β within the frameworks of MSE approach may provide useful insights for better assessment of the available stock size and help in anticipating timely changes in the fishery management.

To illustrate such insights, we should establish the core properties of the stationary equilibrium of the optimality system (7) that engenders the sustainable fishing policy (12), as well as their continuous dependence on the parameter β , and then perform a thorough analysis of the asymptotic behavior of this stationary equilibrium when $\beta \rightarrow 0^+$. The latter corresponds to the “worst scenario” and models the situation when harvesting rate (3) loses sensitivity with respect to current level of stock. The above-mentioned analysis is performed in the following section.



3 | ANALYSIS OF THE STATIONARY EQUILIBRIA

For further analysis, let us impose the condition

$$F'(0) > r \quad (17)$$

that basically refers to expected profitability and economic viability of the fishery. This condition implies that initial marginal productivity of the resource is greater than discount rate r . In other words, the value of this asset (fish stock) grows at a rate greater than that of the capital depreciation. Otherwise, if $F'(0) \leq r$, there would be no economical interest in keeping this asset, whereas full depletion of the resource should allow one to invest the overall harvesting profit elsewhere at the market return rate at least as high as r .

3.1 | Existence and uniqueness of the stationary equilibria

In this subsection, we provide the answer to the following question. Given some sufficiently small β , does the stationary equation (13) have a unique positive solution $x^*(\beta)$ that univocally defines the sustainable fishing effort $u^*(\beta)$ and the underlying shadow price $\lambda^*(\beta)$ by means of the formulas (12) and (11), respectively?

Before proceeding, let us introduce two important quantities:

$$x_r : \text{ such that } F'(x_r) = r, \quad (18)$$

which always exists under the condition (17), and

$$\lambda_r = p - \frac{c}{x_r^\beta}.$$

Notice that x_r is a benchmark that defines the biomass level, at which the marginal productivity of the resource is equal to the rate of capital depreciation and $\lambda_r < p$ stands for its underlying shadow price. In practical terms, x_r marks the threshold of *economically viable resource stock level* below which harvesting cannot be sustainable. The latter is consonant with *fundamental rule of renewable resource exploitation* (see relationship (14) in Section 2.3) in the sense that $x_r < x^*$ because $F'(x)$ is a decreasing function and fulfills $F'(x^*) < F'(x_r) = r$.

It is worth pointing out that the population size x_r is solely defined by the discount rate r and is decreasing with respect to r . Thus, lower values of r induce higher values of x_r , as illustrated in Figure 2. Additionally, if the discounting is ignored ($r = 0$) or is negligible ($r \rightarrow 0^+$), we have $x_r = x_{\text{MSY}}$ where x_{MSY} denotes the unique stock size with maximal resource productivity ($F'(x_{\text{MSY}}) = 0$) that generates the commonly known *maximum sustainable yield (MSY)*⁹. Therefore, $x_r < x_{\text{MSY}}$ for $r > 0$.

The following theorem establishes the existence and uniqueness of a stationary state ($x^*(\beta)$, $\lambda^*(\beta)$) of the optimality system (7) with $u(t)$ replaced by (12) when this stationary state is linked to some particular values of $\beta \in (0, 1)$. It also defines the domain of this stationary state.

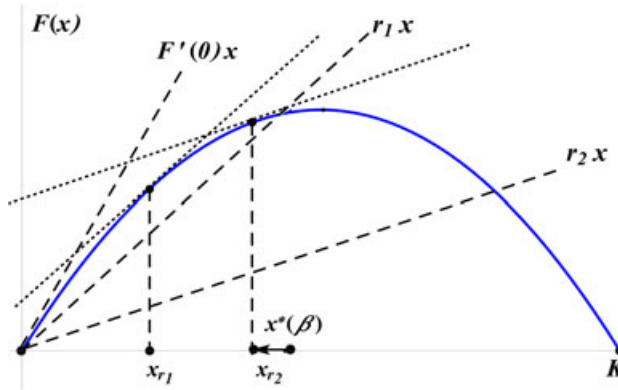


FIGURE 2 Impact of the discount rate r on the level of economically viable resource stock x_r : $F'(0) > r_1 > r_2$ clearly implies that $0 < x_{r_1} < x_{r_2}$

Theorem 1 Let $\beta \in (0, 1)$ and $K > (c/p)^{1/\beta}$. Suppose that condition (17) holds. Then, there exists a unique equilibrium

$$(x^*(\beta), \lambda^*(\beta)) \in (x_r, K) \times (\lambda_r, p)$$

of the optimality system (7), provided that $u(\cdot)$ is constant and equals to

$$u^*(\beta) = \frac{F(x^*(\beta))}{N [x^*(\beta)]^\beta}. \tag{19}$$

Note that $(x^*(\beta), \lambda^*(\beta))$ satisfies the relationship

$$\lambda^*(\beta) = p - \frac{c}{[x^*(\beta)]^\beta}. \tag{20}$$

Proof Formal proof of this theorem is given in Appendix C. □

An immediate consequence of Theorem 1 is the following corollary (its formal proof is also provided in Appendix C).

Corollary 1 Equation (13) has no roots in $[(c/p)^{1/\beta}, x_r]$ when $\beta \in (0, 1)$.

Remark 4 Since we have supposed that $p > c$, it holds that $(c/p)^{1/\beta} \rightarrow 0^+$ when $\beta \rightarrow 0^+$. Notice that, the hypothesis $K > (c/p)^{1/\beta}$ becomes superfluous when $\beta > 0$ is small enough.

The principal outcome of Theorem 1 has a direct interpretation from the bioeconomic standpoint. For each level of hyperstability $\beta \in (0, 1)$ detected in schooling fishery, there is one and only one steady-state level of resource biomass $x^* = x^*(\beta)$ that defines, by means of relationship (19), the steady-state policy $u^*(\beta)$ for sustainable fishing. Additionally, this steady-state level $x^* = x^*(\beta)$ is the unique root of the fundamental equation of renewable resource exploitation (14) with given $\beta \in (0, 1)$ (this issue is directly addressed in the proof of Theorem 1



presented in Appendix C). In other words, $x^* = x^*(\beta)$ is exactly the optimal size of exploited renewable resource that allows, for each fixed $\beta \in (0, 1)$, to keep the fishery's internal rate of return equal to the external rate of return on assets outside the fishery.

In contrast, it is also understood that even plausible estimations of $x^*(\beta)$ and underlying $u^*(\beta)$ for an observable level of hyperstability β are somewhat idealistic, since β may exhibit sudden and unpredictable variations. Therefore, it is important to analyze what kind of changes in $x^*(\beta)$ and $u^*(\beta)$ can be expected when β varies. When fishery managers apply the MSE approach, it seems reasonable to include as one of the objectives to maintain the level of resource at the target population size of $x^*(\beta)$ and to keep the underlying fishing effort as close as possible to $u^*(\beta)$, while testing different operating models with variable values of β .

Another important outcome of Theorem 1 relates to the form of stationary fishing effort $u^*(\beta)$ (see formula (19)) that admits a certain degree of flexibility in decision-making for the fishery's administration. Namely, the decision-makers may choose the number of fishing units, N , in accordance with a maximum allowable level of fishing effort u_{\max} . Note that sustainable fishing effort $u^*(\beta)$ explicitly depends on N , whereas the underlying stationary stock $x^*(\beta)$ defined as the root of Equation (13) or (14) is independent of N . If $u^*(\beta)$ is far below u_{\max} , the number of fishing vessels can be reduced to match the level u_{\max} . Otherwise, if $u^*(\beta) > u_{\max}$, additional vessels can be rented or acquired (see Remark 3).

Finally, the current trends in the economy (expressed in our model by the discount rate r) may also have an impact on the size of sustainable stock $x^*(\beta)$. Under the condition (17), it always holds that $x_r > 0$ and x_r is independent of $\beta \in (0, 1)$. However, when r becomes rather large (i.e., $r \rightarrow F'(0)$ —see Figure 2), the benchmark x_r may become very small (i.e., $x_r \rightarrow 0$) and this is the case when the possibility of fishery depletion and collapse would increase dramatically when $\beta \rightarrow 0^+$. This issue will be further discussed in Section 3.2.

3.2 | Asymptotic analysis when $\beta \rightarrow 0^+$

In this subsection, we analyze the behavior of the unique steady state $(x^*(\beta), \lambda^*(\beta))$ of the optimality system (7), given by Theorem 1, with respect to variations of the parameter $\beta \in (0, 1)$. In particular, we are interested in the case when $\beta \rightarrow 0^+$, which models the loss of sensitivity of the harvesting yield with respect to marginal changes in the level of fish stock. This analysis should provide some useful insights regarding possible alterations of the steady state and underlying fishing policy when β is affected by some exogenous (climatic, environmental, and technological) perturbations. The following theorem establishes the continuity of steady-state solution $(x^*(\beta), \lambda^*(\beta))$ with respect to parameter $\beta \in (0, 1)$.

Theorem 2 *Assume that all hypotheses of Theorem 1 are fulfilled. Then, for $\beta \in (0, 1)$ small enough, there exists a continuously differentiable mapping $\beta \mapsto (x^*(\beta), \lambda^*(\beta))$ defined as the unique solution of the system (7) provided that (19) holds. Moreover, we have that*

$$\lim_{\beta \rightarrow 0^+} x^*(\beta) = x_r, \quad \lim_{\beta \rightarrow 0^+} \lambda^*(\beta) = p - c \quad (21)$$

and, consequently,

$$\lim_{\beta \rightarrow 0^+} u(x^*(\beta), \lambda^*(\beta)) = \frac{F(x_r)}{N}.$$



Proof Formal proof of this theorem is given in Appendix C. □

In virtue of Theorem 1, the steady state of the optimality system (7) is unique for each β ; therefore, Theorem 2 implies that $(x(\beta), \lambda(\beta)) = (x^*(\beta), \lambda^*(\beta))$ for β small enough. So, hereafter we use the notation $(x(\beta), \lambda(\beta))$ for steady states of the optimality system (7) without any confusion.

Remark 5 Continuous dependence of the steady-state biomass $x(\beta)$ upon catch-to-stock sensitivity parameter β combined with the left-side relationship in (21) and the fact that $x(\beta) > x_r$ for all $\beta \in (0, 1)$ established in Appendix C (see proof of Theorem 1) indicates how this unique steady state reacts to changes in β . In particular, if the observable value of β increases (decreases) from β_1 to β_2 , then the steady-state biomass also increases (decreases) from $x^*(\beta_1)$ to $x^*(\beta_2)$ when both β_1 and β_2 are relatively small.

When $\beta \rightarrow 0^+$ and harvesting function (3) becomes (almost) independent¹⁰ of the current level of biomass x , the optimal control problem (6) with $\alpha = 1, \beta = 0$ has a turnpike solution (Clark, 1990, Chapter 4) and underlying trajectories of the optimality system are approaching the values $x = x_r$ and $\lambda = p - c$ as fast as possible. A similar outcome was obtained by Gajardo et al. (2011), while considering the harvesting function (3) of Cobb–Douglas type with $\alpha + \beta = 1$ and analyzing the behavior of steady-state solution $(x(\beta), \lambda(\beta))$ with $\alpha \rightarrow 1^-$ and $\beta \rightarrow 0^+$.

In contrast, the limiting case of the fundamental equation of renewable resource exploitation (14) when $\beta \rightarrow 0^+$ brings us to the so-called “golden rule of capital investment” (Clark, 2010, p. 29), that is,

$$F'(x^*) = r = F'(x_r).$$

The above relationship implies that the marginal productivity of the resource at x^* constitutes the unique source of the instantaneous internal rate of return in the fishery and that the marginal stock effect (expressed by the second summand in the right-hand side of (14)) vanishes as $\beta \rightarrow 0^+$. In other words, the current profit becomes independent of the current level of resource x and, as a result, the optimal stationary biomass equilibrium $x(\beta)$ tends to x_r (see Figure 2). Theorem 2 has established that a unique steady state $(x(\beta), \lambda(\beta))$ of the optimality system (7) continuously depends on the parameter β and has finite limits when $\beta \rightarrow 0^+$. The following proposition will be helpful for understanding the rate of changes in the asymptotic behavior of the steady-state solution $(x(\beta), \lambda(\beta))$ induced by the loss of sensitivity of additional catch yield with respect to x in the harvesting function.

Proposition 1 *The limits of the derivatives of the stationary states $(x(\beta), \lambda(\beta))$ with respect to β are given by*

$$\begin{aligned} \lim_{\beta \rightarrow 0^+} \frac{dx(\beta)}{d\beta} &= - \frac{cF(x_r)}{F''(x_r)(p - c)x_r} > 0, \\ \lim_{\beta \rightarrow 0^+} \frac{d\lambda(\beta)}{d\beta} &= c \ln(x_r). \end{aligned} \tag{22}$$



In particular, for $\beta > 0$ small enough¹¹, we have:

1. $\frac{dx}{d\beta} > 0$ that is, $x(\beta)$ decreases as β decreases.
2.
 - (a) If $\ln(x_r) > 0$, then $\frac{d\lambda}{d\beta} > 0$, that is, $\lambda(\beta)$ decreases, as β decreases.
 - (b) If $\ln(x_r) < 0$, then $\frac{d\lambda}{d\beta} < 0$, that is, $\lambda(\beta)$ increases, as β decreases.

Proof Formal proof of this proposition is given in Appendix C. □

The meaning of relationship (22) is in line with the results of Theorems 1 and 2. Effectively, from the mathematical standpoint, relationship (22) implies that $x(\beta)$ approaches the benchmark x_r always from above when $\beta \rightarrow 0^+$ (see Figure 2). For β small enough, a lower value of β should induce a lower stationary value of $x(\beta)$ and, consequently, a lower internal return in the fishery. The latter follows from the fundamental rule (14) where the marginal stock effect (second summand in the left-hand side) becomes smaller as $\beta \rightarrow 0^+$, whereas greater values of β should increase the marginal stock productivity and future profits to be obtained by keeping an additional unit of x at sea.

In contrast, Part 2 of Proposition 1 indicates *how* the stationary solution $\lambda(\beta)$ asymptotically approaches its limit $(p - c)$ stated in Theorem 2 when $\beta \rightarrow 0^+$. Namely, if $x_r > 1$, that is, x_r is far from the origin meaning that $F'(0)$ is far above r , then $\lambda(\beta)$ approaches its limit $(p - c)$ from above when $\beta \rightarrow 0^+$. Otherwise, if $x_r < 1$ (i.e., x_r is close to the origin meaning that $F'(0)$ is just slightly above r), then $\lambda(\beta)$ approaches its limit $(p - c)$ from below when $\beta \rightarrow 0^+$.

Here, it is worthwhile to recall that $\lambda(\beta)$ refers to the current shadow value of the steady-state fish stock $x(\beta)$ attributed to its future productivity. Thus, if $x(\beta) \rightarrow x_r$ and x_r is relatively large (cf. Case 2(a) in Proposition 1), then $\lambda(\beta) > p - c$ and approaches the value $(p - c)$ from above. This implies that for larger x_r , the future value of a unit of fish at sea always remains above $(p - c)$. In contrast, if $x(\beta) \rightarrow x_r$ and x_r is rather small (cf. Case 2(b) in Proposition 1), then $\lambda(\beta) < p - c$ meaning that $\lambda(\beta)$ approaches the value $(p - c)$ from below and the future value of a unit of fish at sea always remains below $(p - c)$. Figure 3 illustrates the behavior of steady state $\lambda(\beta)$ when $\beta \rightarrow 0^+$ as described in Proposition 1.

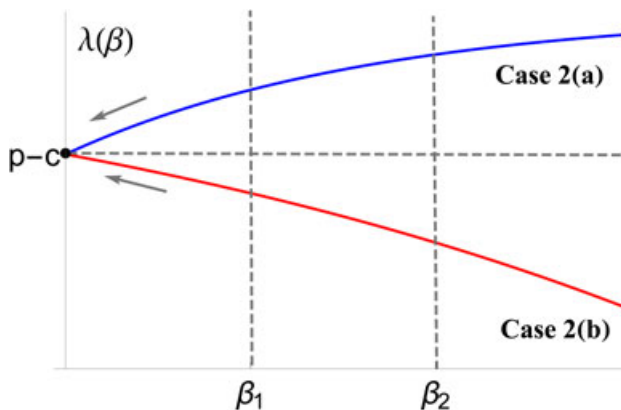


FIGURE 3 Schematic behavior of the steady state $\lambda(\beta)$ for large values of x_r (Case 2(a), r is relatively small) and small of x_r (Case 2(b), r is relatively large), as described in Proposition 1



From the above argument, we may conclude that if x_r is relatively large (Case 2(a) in Proposition 1), then the social planner’s valuation of marginal investment in the fish stock x is high since $\lambda(\beta) > p - c$ when $\beta \rightarrow 0^+$. Alternatively, if x_r is rather small (Case 2(b) in Proposition 1), then the social planner’s valuation of the marginal investment in the fish stock x is low since $\lambda(\beta) < p - c$ when $\beta \rightarrow 0^+$. This conclusion is in line with the result of Gajardo et al. (2011, cf. Proposition 3 in that work) but the present interpretation is more straightforward since it only involves the value of benchmark x_r (economically viable fish-stock size under present discount rate $r > 0$) and does not engage the number of fishing units N .

The following proposition describes the marginal effect on the stationary solution $u(\beta)$ expressed by (19) as $\beta \rightarrow 0^+$.

Proposition 2 *Let $u(\beta)$ be given by (19). Then, we have that*

$$\lim_{\beta \rightarrow 0} \frac{du(\beta)}{d\beta} = -\frac{F(x_r)}{N} \left[\ln(x_r) + \frac{rc}{F''(x_r)(p - c)x_r} \right]. \tag{23}$$

In particular, for $\beta > 0$ small enough and if $\ln(x_r) < 0$, then $\frac{du(\beta)}{d\beta} > 0$ and $\frac{d\lambda(\beta)}{d\beta} < 0$.

Proof Formal proof of this proposition is given in Appendix C. □

It has been mentioned already that $\ln(x_r) < 0$ implies that x_r is close to the origin meaning that r is rather high and stays just slightly below $F'(0)$. In this case, Proposition 2 points out that $\frac{du(\beta)}{d\beta} > 0$ and $u(\beta)$ decreases as $\beta \rightarrow 0^+$. In other words, lower values of β should induce the choice of a lower stationary fishing effort $u(\beta)$.

In contrast, when r is very small (close to zero) the term within square brackets in (23) should be positive meaning that $\frac{du(\beta)}{d\beta} < 0$ and that stationary fishing effort $u(\beta)$ will increase when $\beta \rightarrow 0^+$, whereas $x(\beta)$ and $\lambda(\beta)$ will decline and approach their limit values (x_r and $p - c$, respectively) from above. This result is consistent with Theorem 2 of this paper and stays in line with similar outcomes obtained by Gajardo et al. (2011, cf. Proposition 7 in that work).

Remark 6 However, for intermediate values of $r > 0$ (neither very large nor very small) there is no certainty regarding the behavior of the steady-state fishing effort $u(\beta)$ with respect to changes in β . If the observable value of β increases from β_1 to β_2 , the steady-state effort $u(\beta_1)$ may either increase or decrease to $u(\beta_2)$ even if both β_1 and β_2 are relatively small. The final outcome should principally depend on the proximity of β_1 and β_2 to r .

Now, let us recall that the steady-state harvesting is given by

$$h(\beta) := h(x(\beta), u(\beta)) = Nu(\beta)[x(\beta)]^\beta$$

according to (3). In the following proposition, we study the behavior of $h(\beta)$ and its marginal value $\frac{\partial h(x(\beta), u(\beta))}{\partial x}$ at the equilibrium when $\beta \rightarrow 0^+$. For simplicity, let us denote

$$h_x(\beta) := \frac{\partial h(x(\beta), u(\beta))}{\partial x}.$$



Proposition 3 For $\beta > 0$ small enough, the steady-state harvesting $h(\beta)$ and the marginal steady-state harvesting $h_x(\beta)$ decrease as $\beta \rightarrow 0^+$. More precisely,

$$\lim_{\beta \rightarrow 0^+} \frac{dh(\beta)}{d\beta} > 0, \quad \lim_{\beta \rightarrow 0^+} \frac{dh_x(\beta)}{d\beta} > 0.$$

Proof Formal proof of this proposition is given in Appendix C. □

Proposition 3 basically states that any decrease in β will always induce a reduction of steady-state levels of the biomass $x(\beta)$, of the corresponding harvesting $h(\beta)$, and equilibrium marginal harvesting $h_x(\beta)$. In other words, $x(\beta)$, $h(\beta)$, and $h_x(\beta)$ exhibit strong monotonicity with respect to β .

However, the equilibrium response of marginal harvesting productivity

$$h_u(\beta) := \frac{\partial h(u(\beta), x(\beta))}{\partial u}$$

to changes in β does not follow the same pattern and the underlying reason is disclosed in the following statement.

Proposition 4 The marginal harvesting productivity $h_u(\beta)$ of fishing effort at the steady state has the same monotonicity properties as the shadow price $\lambda(\beta)$ when β varies and it holds that

$$\lim_{\beta \rightarrow 0} \frac{dh_u(\beta)}{d\beta} = N \ln(x_r). \quad (24)$$

Proof Formal proof of this proposition is given in Appendix C. □

From (24) it is clear that for $\beta \rightarrow 0^+$ the marginal harvesting productivity $h_u(\beta)$ at equilibrium increases when r is (relatively) large and stays just slightly below $F'(0)$ since in this case we may have $x_r < 1$ and $\ln(x_r) < 0$. Otherwise, that is, when $x_r > 1$ and r is (relatively) small, $h_u(\beta)$ should decrease when $\beta \rightarrow 0^+$. This outcome agrees with variability of the steady-state shadow price $\lambda(\beta)$ (see Part 2 of Proposition 1 and the relationship (24)). Here, again it is worthwhile to recall the meaning of $\lambda(\beta)$ that refers to future discounted value of the current steady-state stock $x(\beta)$. Apparently, if r is large and x_r is small, we have that $\lambda(\beta) < p - c$ and the shadow price $\lambda(\beta)$ approaches $(p - c)$ by increasing from below when $\beta \rightarrow 0^+$ (see Case 2 (b) displayed in Figure 3). This induces an increase in the marginal harvesting productivity $h_u(\beta)$ at equilibrium as $\beta \rightarrow 0^+$. Alternatively, if r is small and x_r is large, we have that $\lambda(\beta) > p - c$ so that $\lambda(\beta)$ approaches $(p - c)$ by descending from above when $\beta \rightarrow 0^+$ and this induces a decrease in the marginal harvesting productivity $h_u(\beta)$ at equilibrium as $\beta \rightarrow 0^+$ (see Case 2(a) displayed in Figure 3).

The above argument reaffirms the result of Gajardo et al. (2011, cf. Proposition 6 in that work) regarding the monotonicity of both $\lambda(\beta)$ and $h_u(\beta)$ when $\beta \rightarrow 0^+$ but the present interpretation is more straightforward since it directly deals with the benchmark x_r of economically viable fish-stock size under present discount rate $r > 0$ and does not involve the number of fishing units N .



TABLE 1 Summary of dependencies with respect to β

Parametric situation	$\frac{dx}{d\beta}$	$\frac{d\lambda}{d\beta}$	$\frac{du}{d\beta}$	$\frac{dh}{d\beta}$	$\frac{d}{d\beta} \left(\frac{\partial h}{\partial x} \right)$	$\frac{d}{d\beta} \left(\frac{\partial h}{\partial u} \right)$
$x_r > 1$	+	+	?	+	+	+
$x_r < 1$	+	-	+	+	+	-

Note. +: increasing; -: decreasing.

Finally, Table 1 provides a summary of the main derived results, which are consistent with those obtained by Gajardo et al. (2011, cf. Table 1 in that work) and fills some gaps left in their work. Namely, the dependency of steady-state shadow price $\lambda(\beta)$ with respect to β has been established, and it was shown that steady-state harvesting effort $u(\beta)$ should decrease as $\beta \rightarrow 0^+$ when r is relatively large ($x_r < 1$). The latter clearly indicates an increased probability of fishery collapse and local resource extermination caused by the loss of sensitivity ($\beta \rightarrow 0^+$) of additional catch yield with respect to marginal changes in the fish-stock level.

4 | CONCLUSIONS

During both abundance and scarcity periods, small pelagic fish species tend to form big shoals, which provide easy targets for large natural predators and fishing nets. The latter implies that marginal catch productivity of an additional fishing effort unit may exhibit constant responses regardless of the fish-stock variability. Empirical evidence on small pelagic fish stocks also suggests that harvesting yields may exhibit reduced sensitivity with respect to variations in the stock levels.

In this context, the stylized bioeconomic model (2) with harvesting defined by (3) proposed in this paper ($\alpha = 1$ and small enough $\beta \in (0, 1)$) seems reasonable despite its simplicity and allows one to focus on the dependence of the stationary solutions upon the nonlinear catch-to-stock parameter $0 < \beta < 1$, as well as to analyze the equilibrium responses to changes in this parameter induced by some exogenous (climatic, environmental, technological) perturbations.

Our primary goal was to propose a sustainable fishing policy as a function of stationary stock and capable of responding in an adequate way to variations of the catch-to-stock parameter $0 < \beta < 1$ that characterizes the level of hyperstability of pelagic stock. The underlying results (formulated by Theorems 1 and 2) revealed that sustainable fishing is possible as long as the equilibrium biomass level stays above the benchmark x_r defined by (18). This benchmark is independent of the hyperstability parameter β and serves as an inferior bound of the equilibrium biomass, $x^*(\beta)$, when the harvesting rate becomes insensitive to variations in the stock level ($\beta \rightarrow 0^+$). The actual size of x_r essentially depends on the discount rate $r > 0$ and its proximity (closeness or farness) to initial marginal productivity of the resource $F'(0)$ (see relationship (17)). The wider is the gap between $F'(0)$ and r , the higher is the benchmark x_r , and the lower is the probability of fishing collapse and resource extermination. If the gap between $F'(0)$ and r is narrow, the benchmark value x_r may become very small and reach “dangerous level” ($0 < x_r < 1$). The latter, aggravated by sensitivity loss ($\beta \rightarrow 0^+$) may result in a drastic reduction of the stationary stock with an increased risk of fishing collapse.

Generally speaking, the value of hyperstability parameter β summarizes the combined effect of climatic, environmental, and technological factors on the interaction between the size of pelagic fish stock and the harvesting yields per unit of fishing effort. Strong recurrent cycles of fish abundance and scarcity are typical for small pelagic species, and a schooling fishery may



also experience variations in the value of parameter β caused by climatic changes, contamination or pollution, stock relocation, and fishermen's capacity to search for fish shoals.

Pelagic fisheries that intend to operate in a sustainable way must have capacities to detect any observable alterations in the value of β and act accordingly. Detection of variability in the value of β is a challenging task and its accomplishment may require additional painstaking work referring to the model's validation under possible alterations of β . As pointed out by Lehuta, Girardin, Mahévas, Travers-Trolet, and Vermard (2016), such validation can be carried out by applying the MSE approach consisting in "dynamic coupling" between the studied model of the type (2) and an "operating model" of higher complexity^{1,2}. Under such approach, the CPUE records and other internal data of the fishery are contrasted with fishery-independent empirical data collected from different sources to ensure the ability of both models for capturing the observable patterns and to obtain a value of β that renders a better match.

Once a change in the value of β is detected, the fishery managers may assess the corresponding steady-state stock size $x^*(\beta)$ as a unique root of the fundamental equation of natural resource exploitation (14), and then adjust, according to (19), the optimal fishing policy $u^*(\beta)$ that can be sustained as long as β remains unchanged. Therefore, timely detection of observable alterations in the value of β is a fundamental issue for design of flexible policies targeting sustainable fishing. It may not only provide useful insights into potential behavior of fisheries but also serve as a theoretical basis, upon which new (or adapted) applied models can be tried on real fisheries.

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ORCID

Erica Cruz-Rivera  <http://orcid.org/0000-0003-2207-7706>

Héctor Ramírez C.  <http://orcid.org/0000-0002-6239-1890>

Olga Vasilieva  <http://orcid.org/0000-0002-1886-2528>

END NOTES

¹ Nonetheless, there are other and more sophisticated approaches for assessment of existent fish stocks where CPUE time-series are amended by the fishery-independent data collected from different sources, such as ecosystem surveys, acoustic or aerial monitoring, age and length structure of harvested biomass, climatic records, and so forth (Maunder et al., 2006; Maunder & Punt, 2013; Wilberg, Thorson, Linton, & Berkson, 2010).

² For more details regarding "hyperdepletion" and "hyperstability" situations, we invite the reader to revise additional literature (e.g., Harley et al., 2001; Hilborn & Walters, 2001; Pascoe & Hillary, 2016; Quinn & Deriso, 1999).

³ It is worth noting that, in the case of constant unit cost of fishing effort, lower values of β express less sensitivity of unit harvest cost with respect to variations in the fish-stock level, and $\beta = 0$ implies that harvesting rate h is stock-independent.

⁴ In economics, the CES utility function describes changes in consumers' preferences between differentiated goods; see original definition of CES utility function in Arrow, Chenery, Minhas, and Solow (1961).



- ⁵ In mathematical terms, the difference between present and future asset's values is determined by the discount rate (r), which will be introduced further in this section.
- ⁶ Using mathematical terminology, this requires to have fair guesses regarding the form of the population dynamics function $F(x)$ that appears in the model (2) together with reasonable estimations of its underlying parameters.
- ⁷ Here, we consider an *unscaled* model and the biological stock available for harvesting is measured in mass units (e.g., tonnes). Here, K expresses the maximum level of biological stock, which is supportable by the environment in the absence of harvesting. It is natural to suppose that $x(t)$ is bounded from above for all $t \geq 0$ and K can be viewed as its upper boundary (saturation constant).
- ⁸ It is worthwhile to point out that *normalized forward sensitivity index* of harvesting function (3) with respect to current level of biomass x is $\Upsilon_x^h = \beta$. The latter implies that if x is decreased by any degree, the harvesting h will be reduced only by a β -fraction of that degree. Formal definition and more details regarding the concept of normalized forward sensitivity index can be consulted in the original work of Chitnis, Hyman, and Cushing (2008).
- ⁹ This stays in line with the results of Pascoe and Hillary (2016) obtained for hyperstable pelagic fisheries ($\beta = 0$) where discounting was ignored ($r = 0$).
- ¹⁰ It should be pointed out that some authors deliberately suppose that harvesting function is independent of the current level of biomass, see, for example, Maroto and Moran (2008).
- ¹¹ By saying "small enough" here, as well as further on, we naturally refer to all $\beta \in (0, \beta^*)$, where β^* has the same meaning as in Lemma C1 presented in Appendix C and in the proof of Theorem 2.
- ¹² The task of detecting variability of β can be combined, within the frameworks of the MSE approach, with validation of other parameters of the model whose values may alter the steady-state level of resource $x^*(\beta)$, such as r , p , c , and the form of the function $F(x)$ describing the growth of biological stock (cf. Equation (13) or (14) that engenders $x^*(\beta)$ as a unique root).
- ¹³ It is important to point out that the transversality condition written in the form (A2) usually holds for $r > 0$ whenever the state and costate variables (x and λ , respectively) are well behaved at infinity. However, there are cases when adjoint function λ could exhibit some pathological asymptotic behaviors at infinity, making unclear the validity of this transversality condition. Such cases do not arise when both state and control variables are bounded. For a complete review regarding the behavior of adjoint variables at infinity and formal proofs of equivalency between different types of transversality conditions, we suggest the reader to review the works of Aseev and Kryazhimskiy (2004, 2004) and Michel (1982).
- ¹⁴ Here, we have applied the fundamental theorem of calculus to a strictly decreasing function $F'(x)$: $x F'(x) < \int_0^x F'(s) ds = F(x) - F(0) = F(x)$.

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APPENDIX A: APPLICATION OF THE PRONTRYAGIN MAXIMUM PRINCIPLE TO PROBLEM (6)

Let $u : [0, \infty) \mapsto [0, u_{\max}]$ be a fishing effort function, which maximizes the fishing rent (4) and $x = x(\cdot; u(\cdot)) : [0, \infty) \mapsto [0, K]$ be the corresponding fish-stock trajectory. Then, the Pontryagin maximum principle (Grass et al., 2008; Pontryagin et al., 1962) established that u is also a maximizer of the *current-value Hamiltonian function* defined by

$$H(x, u, \lambda) = N(pux^\beta - cu) + \lambda(F(x) - Nux^\beta), \tag{A1}$$

in the sense that $H(x, u, \lambda)$ attains its maximum in the same $u(t)$, that is,

$$H(x, u, \lambda) = \max_{v \in [0, u_{\max}]} H(x, v, \lambda), \text{ almost for all } t > 0.$$

Additionally, the maximum principle states that an absolutely continuous function $\lambda : [0, \infty) \mapsto \mathbb{R}$ (differentiable almost everywhere on \mathbb{R}) that appears in (A1) solves the so-called “adjoint dynamical system”

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial H(x, u, \lambda)}{\partial x}, \quad t > 0$$

with *transversality condition*¹³ given by

$$\lim_{t \rightarrow \infty} x(t)\lambda(t)e^{-rt} = 0. \tag{A2}$$

Thus, by applying the maximum principle, the optimal control problem (6) of maximization of the functional (4) can be reduced to “pointwise” maximization of the Hamiltonian (A1) with respect to control variable of fishing effort $u \in \mathcal{U}$ for each $t \in [0, \infty)$. The latter is usually done by requiring that both state $x(t)$ and *costate* $\lambda(t)$ satisfy the so-called *optimality system* in the maximizer u :

$$\dot{x}(t) = F(x(t)) - Nu(t)x^\beta(t), \tag{A3a}$$



$$\dot{\lambda}(t) = (r - F'(x(t)))\lambda(t) + \beta Nu(t)x^{\beta-1}(t)(\lambda(t) - p) \quad (\text{A3b})$$

with initial and transversality conditions given by

$$x(0) = x_0, \quad \lim_{t \rightarrow \infty} x(t)\lambda(t)e^{-rt} = 0.$$

Since the Hamiltonian (A1) is a scalar function of three arguments ($H: [0, K] \times [0, u_{\max}] \times \mathbb{R} \mapsto \mathbb{R}$), and its maximization with respect to the control variable $u \in [0, u_{\max}]$ (bounded fishing effort) is pointwise (i.e., at each $t \in [0, \infty)$), standard optimization results can be applied. Namely, the maximum of $H(x, u, \lambda)$ with respect to u and for each $t \in [0, \infty)$ is attained either in the interior critical point $u^* \in (0, u_{\max})$, where $\frac{\partial H}{\partial u} = 0$, or at one of the endpoints, that is, $u = 0$ or $u = u_{\max}$.

In contrast, it is easy to note that the Hamiltonian (A1) is *linear* in u , and the standard procedure in this case (see, e.g., Clark, 1990, 2010; Grass, Caulkins, Feichtinger, Tragler, & Behrens, 2008; Lenhart & Workman, 2007) consists in defining the *switching function*

$$\Psi(x(t), \lambda(t)) = \frac{\partial H}{\partial u}(x(t), u(t), \lambda(t)) = N(px^{\beta}(t) - \lambda(t)x^{\beta}(t) - c)$$

that identifies in which $t \in [0, \infty)$ the Hamiltonian is decreasing ($\Psi < 0$), increasing ($\Psi > 0$) or invariable ($\Psi = 0$) with respect to u . Thus, the optimal control $u(t)$ can be written in the form

$$u(t) = \begin{cases} 0 & \text{if } \Psi < 0, \\ u_{\max} & \text{if } \Psi > 0, \\ u^* & \text{if } \Psi = 0, \end{cases} \quad (\text{A4})$$

where $u^* \in (0, u_{\max})$ was introduced in (12).

Furthermore, using the structure (A4) of u , the state/costate optimality system (A3), or (7) in the main body of the paper, can be written as

$$\begin{cases} \dot{x}(t) = \sigma_1(x(t), \lambda(t)), \\ \dot{\lambda}(t) = \sigma_2(x(t), \lambda(t)), \end{cases}$$

where

$$\sigma_1(x, \lambda) = \begin{cases} F(x) & \text{if } \Psi < 0, \\ F(x) - Nu_{\max}x^{\beta} & \text{if } \Psi > 0, \\ F(x) - Nu^*x^{\beta} & \text{if } \Psi = 0, \end{cases}$$

$$\sigma_2(x, \lambda) = \begin{cases} \lambda(r - F'(x)) & \text{if } \Psi < 0, \\ \lambda(r - F'(x)) - \beta Nu_{\max}x^{\beta-1}(p - \lambda) & \text{if } \Psi > 0, \\ \lambda(r - F'(x)) - \beta Nu^*x^{\beta-1}(p - \lambda) & \text{if } \Psi = 0. \end{cases}$$



APPENDIX B: DETAILED ANALYSIS OF STATIONARY EQUILIBRIUM POINTS

For $\Psi^* > 0$ and $u^*(\cdot) = u_{\max}$, we have that any fixed point (x^*, λ^*) of the optimality system (7) must satisfy

$$F(x^*) - Nu_{\max}(x^*)^\beta = 0, \quad x^* \in [0, K], \quad \lambda^*(r - F'(x^*)) - \beta Nu_{\max}(x^*)^{\beta-1}(p - \lambda^*) = 0,$$

and it is easy to see that $x^* = K$ does not comply with these equations. Also, $x^* = 0 < \tilde{x}$, so it is excluded. However, there may exist $x^* \in (0, K)$ such that

$$\frac{F(x^*)}{(x^*)^\beta} = u_{\max}N, \quad \tilde{x} < x^* < K. \tag{B1}$$

Let us characterize (x^*, λ^*) in this case.

First, note that the function $s(x) = \frac{F(x)}{x^\beta}$, $x \in [0, K]$, is concave for $\beta > 0$ small enough (see Figure B1). In fact,

$$s''(x) = \frac{1}{x^{\beta+2}} [x^2 F''(x) + \beta(-2x F'(x) + (\beta + 1)F(x))],$$

and by using the concavity of F together with the boundedness of F and F' on $[0, K]$, it follows that $s''(x) < 0$, that is, $s(x)$ is strictly concave, for $\beta > 0$ small enough.

Consequently, depending on the value Nu_{\max} , there may exist two, one, or none solutions of (B1). If such solution x^* exists, then (x^*, λ^*) should satisfy Equation (8), that is,

$$\lambda^*(r - F'(x^*)) + \beta \frac{F(x^*)}{x^*} (\lambda^* - p) = 0, \quad \tilde{x} < x^* < K.$$

This directly leads to

$$\lambda^* = \frac{p}{\left[\frac{(r - F'(x^*))x^*}{\beta F(x^*)} + 1 \right]}.$$

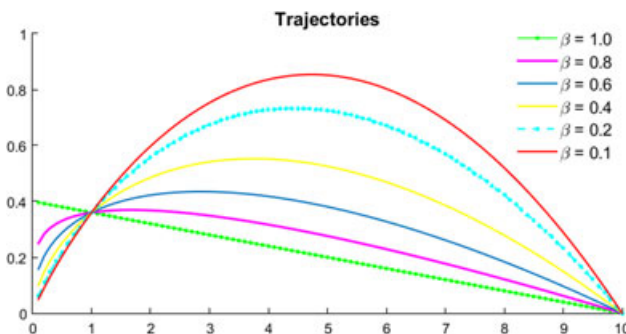


FIGURE B1 Function $s(x)$ with $F(x) = F_1(x)$ (logistic function, see Remark 1)



Notice that $\lambda^* < p$. Indeed, since $\tilde{x} < x^* < K$, that is,

$$\left(\frac{c}{p - \lambda^*} \right)^{1/\beta} < x^* < K,$$

we immediately obtain that

$$\lambda^* < p - \frac{c}{(x^*)^\beta} < p - \frac{c}{K^\beta} < p.$$

APPENDIX C: FORMAL PROOFS

Proof of Theorem 1 *Let us consider the singular equilibrium control $u(\cdot) = u^*$ given by (19).*

First, we show the existence of $(x^(\beta), \lambda^*(\beta))$ for a specified $\beta \in (0, 1)$. To do that, we define the function g inspired by Equation (13)*

$$g(x) = (px^\beta - c)(r - F'(x)) - \beta c \frac{F(x)}{x}.$$

In view of condition (17) and due to continuity and concavity of F , there is a unique $x_r \in (0, K)$ such that $F'(x_r) = r$. Let us show that $g(x_r) < 0$ and $g(K) > 0$. Indeed, we first have

$$g(x_r) = (px_r^\beta - c)(r - F'(x_r)) - \beta c \frac{F(x_r)}{x_r} = -\beta c \frac{F(x_r)}{x_r} < 0.$$

By hypothesis, we have that $F(K) = 0$ and $F'(K) < 0$. So, it trivially holds that $r - F'(K) > 0$. This together with the hypothesis $K > (c/p)^{1/\beta}$ results in

$$g(K) = (pK^\beta - c)(r - F'(K)) - \beta c \frac{F(K)}{K} = (pK^\beta - c)(r - F'(K)) > 0.$$

Since g is a continuous function for all $x \in [x_r, K]$, and it holds that $g(x_r) < 0$ and $g(K) > 0$, then by virtue of the intermediate value theorem, there exists a fixed $\beta \in (0, 1)$ at least one $x^(\beta) \in (x_r, K)$ such that $g(x^*(\beta)) = 0$.*

It is worthwhile to note that (11) uniquely defines $\lambda^(\beta)$ in terms of $x^*(\beta)$ as it appears in (20) and thus provides a steady-state solution $(x^*(\beta), \lambda^*(\beta))$ of the optimality system (7) for a given $\beta \in (0, 1)$. Moreover, since $x^*(\beta) \in (x_r, K)$, then it holds that*

$$\lambda_r = p - \frac{c}{x_r^\beta} < p - \frac{c}{[x^*(\beta)]^\beta} < p - \frac{c}{K^\beta} < p,$$

which implies that $\lambda^(\beta) \in (\lambda_r, p)$.*



It only remains to prove that this equilibrium $x^*(\beta)$ is unique. This task is not straightforward and requires for more detailed analysis.

First, relationship $g(x^*(\beta)) = 0$ implies that

$$F'(x^*(\beta)) - r = \frac{\beta c}{c - p[x^*(\beta)]^\beta} \cdot \frac{F(x^*(\beta))}{x^*}$$

and let us consider three auxiliary functions

$$\begin{aligned} \xi(x) &= \frac{\beta c}{c - px^\beta}, & G(x) &= \frac{F(x)}{x}, \\ \phi(x) &= \frac{\beta c}{c - px^\beta} \cdot \frac{F(x)}{x} = \xi(x)G(x). \end{aligned} \tag{C1}$$

Function $\xi(x)$ is defined for all $x > (c/p)^{1/\beta}$, and it is negative and strictly increasing on (x_r, K) since

$$\xi'(x) = \frac{pc\beta^2x^{\beta-1}}{(c - px^\beta)^2} > 0 \quad \text{and} \quad \xi(x) \leq \xi(K) = \frac{\beta c}{c - pK^\beta} < 0$$

for all $x \in (x_r, K)$. In contrast, function $G(x)$ is positive and strictly decreasing for all $x \in (x_r, K)$. It is easy to see that $G(x) > 0$ when $x \in (x_r, K)$. To prove that $G(x)$ is strictly decreasing, let us recall that $F''(x) < 0$ meaning that $F'(x)$ is a strictly decreasing function for all $x \in [0, K]$. Therefore, the relationship

$$G'(x) = \frac{x F'(x) - F(x)}{x^2} < 0$$

is true for all $x \in [0, K]$ and it also holds for $x \in (x_r, K)^{14}$.

Thus, we can now conclude that function $\phi(x)$ is negative and strictly increasing within (x_r, K) since

$$\phi'(x) = \xi'(x)G(x) + \xi(x)G'(x) > 0.$$

Now, suppose there are two roots $x_1^*(\beta)$ and $x_2^*(\beta)$ of (13) for the same fixed $\beta \in (0, 1)$ such that $x_r < x_1^*(\beta) \leq x_2^*(\beta) < K$. Then, we have $\phi(x_1^*(\beta)) \leq \phi(x_2^*(\beta))$ and it follows that

$$F'(x_1^*(\beta)) - r = \phi(x_1^*(\beta)) \leq \phi(x_2^*(\beta)) = F'(x_2^*(\beta)) - r.$$

Consequently, $F'(x_1^*(\beta)) \leq F'(x_2^*(\beta))$. However, this implies that $x_1^*(\beta) \geq x_2^*(\beta)$, because $F'(x)$ is a decreasing function. In other words, $x_1^*(\beta) = x_2^*(\beta)$, and so there is only one root $x^*(\beta)$ of (13), as desired. □



Proof of Corollary 1 Let us suppose the contrary, that is, that Equation (13) has a strictly positive root $x^*(\beta)$ such that $0 < (c/p)^{1/\beta} < x^*(\beta) < x_r$. Recall that function $\phi(x)$ defined by (C1) is continuous as well as negative and strictly increasing for all $x > (c/p)^{1/\beta}$. Therefore, $x^*(\beta) < x_r$ implies that

$$\phi(x^*(\beta)) = F'(x^*(\beta)) - r = F'(x^*(\beta)) - F'(x_r) < \phi(x_r) < 0.$$

In contrast, F' is a decreasing function for all $x \in [0, K]$ and $F'(x^*) - F'(x_r) < 0$ implies $x^*(\beta) > x_r$, which is a contradiction.

Let us state now an auxiliary result, which will be helpful for further asymptotic analysis of the optimality system (7) when $\beta \rightarrow 0^+$. \square

Lemma C1 There exists $0 < \beta^* < 1$ such that for all $\beta \in (0, \beta^*)$ and $x \in [x_r, K]$, we have

$$-c\beta \left[A(x) - F'(x) + \beta \frac{F(x)}{x} \right] + \lambda x^\beta [xF''(x) - A(x)] < 0,$$

where $A(x) = r - F'(x)$.

Proof Recall that $F'(\cdot)$ is a strictly decreasing function. Therefore, $r = F'(x_r) > F'(x)$ meaning that $r > F'(x)$ for $x \in [x_r, K]$. So we can formally write

$$\begin{aligned} A(x) - F'(x) + \frac{\beta F(x)}{x} &= r - F'(x) - F'(x) + \frac{\beta F(x)}{x} \\ &= r - 2F'(x) + \frac{\beta F(x)}{x} \\ &> r - 2r + \frac{\beta F(x)}{x} \\ &> -r, \end{aligned}$$

and

$$-c\beta \left[A(x) - F'(x) + \beta \frac{F(x)}{x} \right] < -c\beta(-r) = c\beta r.$$

In contrast,

$$\lambda x^\beta [xF''(x) - A(x)] \leq \lambda x^\beta \max_{x \in [x_r, K]} \{xF''(x) - A(x)\} < 0,$$

because we have that $F''(x) < 0$ for $x \in [0, K]$ and $A(x) > 0$ for $x \in (x_r, K)$. Then, we also have that

$$xF''(x) - A(x) < 0, \quad x \in [x_r, K].$$



Now, if we assume that

$$x_r < x < K \quad \text{and} \quad \lambda_r < \lambda < p,$$

then we have that

$$x_r^\beta < x^\beta < K^\beta \quad \text{and} \quad \lambda_r x_r^\beta < \lambda x^\beta < pK^\beta.$$

From these facts, we conclude that

$$\begin{aligned} \lambda x^\beta (xF''(x) - A(x)) &\leq \lambda x^\beta \max_{x \in [x_r, K]} \{xF''(x) + F'(x) - r\} \\ &\leq \lambda_r x_r^\beta \max_{x \in [x_r, K]} \{xF''(x) + F'(x) - r\} < 0. \end{aligned}$$

Putting together previous estimates, we see that

$$\begin{aligned} -c\beta \left[A(x) - F'(x) + \frac{\beta F(x)}{x} \right] + \lambda x^\beta [xF''(x) - (r - F'(x))] \\ < c\beta r + \lambda_r x_r^\beta \max_{x \in [x_r, K]} \{xF''(x) + F'(x) - r\}. \end{aligned}$$

So, we may choose some (sufficiently small) $\beta^* > 0$ such that for $0 < \beta < \beta^*$

$$\beta cr + x_r^\beta \lambda_r \max_{x \in [x_r, K]} \{xF''(x) - (r - F'(x))\} < 0.$$

The latter implies that for all $\beta \in (0, \beta^*)$ it holds that

$$-c\beta \left[A(x) - F'(x) + \frac{\beta F(x)}{x} \right] + \lambda x^\beta [xF''(x) - A(x)] < 0$$

as desired. □

Proof of Theorem 2 First, let us consider some fixed $\beta \in (0, 1)$. According to Theorem 1, there exists only one solution $(x^*(\beta), \lambda^*(\beta))$ of the following auxiliary system:

$$v(\beta, x, \lambda) := (p - \lambda)x^\beta - c = 0, \tag{C2a}$$

$$\eta(\beta, x, \lambda) := \beta(p - \lambda)F(x) - x\lambda(r - F'(x)) = 0, \tag{C2b}$$

In the system (C2), the first equation (C2a) corresponds to relationship (11), whereas the second equation (C2b) is obtained by replacing u by the expression (19) in (7b) of the optimality system and then multiplying by x . Since the role of β is clear from the context, we will just write (x^*, λ^*) instead of $(x^*(\beta), \lambda^*(\beta))$ further on.



We now proceed to apply the Implicit Function Theorem to the auxiliary system (C2). A direct computation shows that at the point (β, x^*, λ^*) , we have that

$$\begin{aligned}\frac{\partial v}{\partial x}(\beta, x^*, \lambda^*) &= \frac{c\beta}{x^*}, \\ \frac{\partial v}{\partial \lambda}(\beta, x^*, \lambda^*) &= -(x^*)^\beta, \\ \frac{\partial \eta}{\partial x}(\beta, x^*, \lambda^*) &= \frac{c\beta}{(x^*)^\beta} F'(x^*) - \lambda^*(r - F'(x^*) - x^*F''(x^*)), \\ \frac{\partial \eta}{\partial \lambda}(\beta, x^*, \lambda^*) &= -\beta F(x^*) - x^*(r - F'(x^*)).\end{aligned}$$

In other words, we have that the Jacobian matrix $M(\beta)$ of (v, η) evaluated at the point (β, x^*, λ^*) is given by

$$\begin{aligned}M(\beta) &= \begin{bmatrix} \frac{\partial v}{\partial x}(\beta, x^*, \lambda^*) & \frac{\partial v}{\partial \lambda}(\beta, x^*, \lambda^*) \\ \frac{\partial \eta}{\partial x}(\beta, x^*, \lambda^*) & \frac{\partial \eta}{\partial \lambda}(\beta, x^*, \lambda^*) \end{bmatrix} \\ &= \begin{bmatrix} \frac{c\beta}{x^*} & -(x^*)^\beta \\ \frac{c\beta}{[x^*]^\beta} F'(x^*) - \lambda^*(A(x^*) - x^*F''(x^*)) & -\beta F(x^*) - x^*A(x^*) \end{bmatrix},\end{aligned}\tag{C3}$$

where $A(x^*) = r - F'(x^*)$ is defined in Lemma C1 formulated and proved above. Recall that $A(x^*) > 0$ since $r = F'(x_r) > F'(x^*)$ for $x_r < x^*$, where $F'(\cdot)$ is a decreasing function. Now, a direct computation shows that

$$\det(M(\beta)) = -c\beta \left[A(x^*) - F'(x^*) + \beta \frac{F(x^*)}{x^*} \right] + \lambda^*(x^*)^\beta [x^*F''(x^*) - A(x^*)].\tag{C4}$$

So, applying Lemma C1, there exists some $\beta^* \in (0, 1)$ such that for all $\beta \in (0, \beta^*)$ it holds that

$$\det(M(\beta)) < 0.$$

Therefore, in virtue of the Implicit Function Theorem, there exists a continuously differentiable (C^1) mapping $\beta \mapsto (x(\beta), \lambda(\beta))$ defined for all $\beta \in (0, \beta^*)$ (small enough) such that $(x(\beta), \lambda(\beta))$ is a C^1 solution of the system (C2).

Now, let us establish the convergence of the equilibrium states $(x(\beta), \lambda(\beta))$ parameterized by β when $\beta \rightarrow 0^+$. It follows from Theorem 1 that both $x(\cdot)$ and $\lambda(\cdot)$ belong to the compact set $[x_r, K] \times [\lambda_r, p]$. To prove the convergence of $x(\beta)$ and $\lambda(\beta)$, when $\beta \rightarrow 0^+$, it suffices to prove that $(x_r, p - c)$ is the limit point for any convergent sequence $\{(x(\beta_k), \lambda(\beta_k))\}$.

Let $\{\beta_k\}$ be a sequence converging to 0 from above, and such that $(x(\beta_k), \lambda(\beta_k)) \rightarrow (\bar{x}, \bar{\lambda}) \in [x_r, K] \times [\lambda_r, p]$. Since we have that



$$\lambda(\beta) = p - \frac{c}{[x(\beta)]^\beta},$$

then we conclude that $\tilde{\lambda} = p - c$. In other words,

$$\lim_{\beta \rightarrow 0^+} \lambda(\beta) = p - c.$$

Now, from Equation (C2b) we get that

$$\tilde{\lambda}(r - F'(\tilde{x})) = 0,$$

which implies that $F'(\tilde{x}) = r$. From the strict concavity of F , the only element satisfying this relationship is x_r ; therefore, $\tilde{x} = x_r$, that is,

$$\lim_{\beta \rightarrow 0^+} x(\beta) = x_r.$$

Finally, when $u = u(x(\beta), \lambda(\beta))$ is given by (19), it is easy to see that

$$\lim_{\beta \rightarrow 0^+} u(x(\beta), \lambda(\beta)) = \frac{F(x_r)}{N}.$$

Proof of Proposition 1 Recall that Theorem 2 establishes the existence of a differentiable curve $\beta \mapsto (x(\beta), \lambda(\beta))$ satisfying the system (C2). To calculate the derivatives $\frac{dx(\beta)}{d\beta}$ and $\frac{d\lambda(\beta)}{d\beta}$, we then apply the Implicit Function Theorem:

$$\begin{aligned} \frac{dx(\beta)}{d\beta} &= - \frac{[x(\beta)]^\beta (p - \lambda(\beta))}{\det(M(\beta))} \times [F(x(\beta)) - \ln(x(\beta))(\beta F(x(\beta)) + x(\beta)(r - F'(x(\beta))))], \\ \frac{d\lambda(\beta)}{d\beta} &= \frac{[x(\beta)]^{\beta-1} (p - \lambda(\beta))}{\det(M(\beta))} \times [x(\beta)\ln(x(\beta))[\beta(p - \lambda(\beta))F'(x(\beta)) - \lambda(\beta)(r - F'(x(\beta))) \\ &\quad + x(\beta)\lambda(\beta)F''(x(\beta))] - \beta(p - \lambda(\beta))F(x(\beta))], \end{aligned}$$

where $M(\beta)$ was defined in (C3) and $\det(M(\beta))$ was computed in (C4). When $\beta \rightarrow 0^+$ we have that

$$x(\beta) \rightarrow x_r, \quad \lambda(\beta) \rightarrow p - c, \quad \det(M(\beta)) \rightarrow (p - c)x_r F''(x_r).$$

Therefore, we conclude that

$$\begin{aligned} \lim_{\beta \rightarrow 0^+} \frac{dx(\beta)}{d\beta} &= - \frac{cF(x_r)}{x_r F''(x_r)(p - c)} > 0, \\ \lim_{\beta \rightarrow 0^+} \frac{d\lambda(\beta)}{d\beta} &= c \ln(x_r). \end{aligned}$$



Proof of Proposition 2 We note that $u(\beta) = \frac{F(x(\beta))}{N[x(\beta)]^\beta}$ according to (19). Therefore,

$$\frac{du(\beta)}{d\beta} = \frac{1}{N} \left[\frac{F'(x(\beta))}{[x(\beta)]^\beta} \cdot \frac{dx(\beta)}{d\beta} - F(x(\beta)) \left(\frac{\ln(x(\beta))}{[x(\beta)]^\beta} + \frac{\beta}{[x(\beta)]^{\beta+1}} \cdot \frac{dx(\beta)}{d\beta} \right) \right].$$

Taking the limit of the above expression when $\beta \rightarrow 0^+$, we get that the relationship (23) holds. If we assume $\ln(x_r) < 0$, then we directly obtain that $\frac{du(\beta)}{d\beta} > 0$ due to strict concavity of F and having in mind that $p > c$. Additionally, by Proposition 1, we also have in this case that $\frac{d\lambda(\beta)}{d\beta} < 0$.

Proof of Proposition 3 Theorem 2 establishes that $\lim_{\beta \rightarrow 0^+} x(\beta) = x_r$ and this limit $x_r > 0$ is only reachable from above when $\beta \rightarrow 0^+$ since we have that $\frac{dx(\beta)}{d\beta} > 0$ if β is small enough (cf. Proposition 1). In contrast, let us recall that $x(\beta)$ is an equilibrium state; therefore, $h(\beta) = F(x(\beta))$ and we can calculate

$$\lim_{\beta \rightarrow 0^+} \frac{dh(\beta)}{d\beta} = \lim_{\beta \rightarrow 0^+} F'(x(\beta)) \frac{dx(\beta)}{d\beta} = -\frac{crF(x_r)}{x_r F''(x_r)(p-c)} > 0,$$

Here, we used the relationships $F'(x_r) = r > 0$ and (22) together with strict concavity of F . Further, we have that

$$h_x(\beta) = \frac{\partial}{\partial x} [Nu(\beta)[x(\beta)]^\beta] = \beta Nu(\beta)[x(\beta)]^{\beta-1}$$

and, consequently,

$$\begin{aligned} \frac{dh_x(\beta)}{d\beta} &= Nu(\beta)[x(\beta)]^{\beta-1} + \beta N[x(\beta)]^{\beta-1} \cdot \frac{du(\beta)}{d\beta} + \beta Nu(\beta) \left([x(\beta)]^{\beta-1} \ln(x(\beta)) \right. \\ &\quad \left. + (\beta-1)[x(\beta)]^{\beta-2} \cdot \frac{dx(\beta)}{d\beta} \right). \end{aligned}$$

Using the fact that $Nu(\beta)[x(\beta)]^\beta = F(x(\beta))$ and passing to the limit when $\beta \rightarrow 0^+$, we conclude that

$$\lim_{\beta \rightarrow 0^+} \frac{dh_x(\beta)}{d\beta} = \frac{F(x_r)}{x_r} > 0.$$

Proof of Proposition 4 The proof of this proposition is immediate and follows from the formula (10) together with the result of Proposition 1:

$$\frac{dh_u(\beta)}{d\beta} = \frac{d}{d\beta} \left[\frac{cN}{p - \lambda(\beta)} \right] = \frac{cN}{[p - \lambda(\beta)]^2} \cdot \frac{d\lambda}{d\beta}.$$

Passing to limit when $\beta \rightarrow 0^+$ in the above relationship, we directly arrive to (24).