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CONTRIBUTION TO INVERSE PROBLEMS AND CONTROLLABILITY ISSUES OF HYPERBOLIC AND PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MODELACIÓN MATEMÁTICA

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RESUMEN DE LA TESIS PARA OPTAR
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POR: ROBERTO ALEJANDRO MORALES PONCE
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El objetivo de esta tesis consiste principalmente en el estudio teórico de algunos resultados de problemas inversos y de controlabilidad en ecuaciones hiperbólicas y parabólicas.

En el Capítulo 1 presentamos una breve introducción de los tópicos tratados en este trabajo. Principalmente, centramos nuestra atención en las definiciones clásicas de controlabilidad y problemas inversos. Posteriormente, indicamos cuáles son los resultados generales obtenidos en esta tesis.

En el Capítulo 2, describimos los resultados de estabilidad obtenida para la reconstrucción de potenciales en un sistema de ecuaciones hiperbólicas acopladas en cascada. Para probar este resultado, nos inspiramos en el método de Bukhgeim-Klibanov combinado con un tipo especial de desigualdades conocidas como estimaciones de Carleman. Estas dos herramientas, junto con el hecho que las ecuaciones del sistema están acopladas en cascada, nos permiten obtener un resultado de estabilidad Lipschitz para la recuperación de todos los potenciales del sistema utilizando mediciones de algunas componentes accesibles de él.

En el Capítulo 3, nos centramos en el estudio de la controlabilidad a cero de una ecuación del calor con condiciones de borde dinámicas. Este problema se puede ver como una ecuación del calor acoplada con una ecuación diferencial ordinaria actuando en un extremo del borde. Nuestros resultados apuntan en dos direcciones. En primer lugar, probamos que este tipo de problemas se puede controlar a cero en una región que está lejos de la interacción entre las dos dinámicas. Usando la dualidad entre observabilidad y controlabilidad, la prueba de este resultado está basado en la construcción de una estimación de Carleman adecuada. En segundo lugar, probamos que una modificación de este tipo de problemas puede ser visto como el problema límite de una familia de problemas parabólicos con coeficientes de difusión discontinuos en donde la difusión es muy alta en una parte del dominio. Adicionalmente, estudiamos el efecto que tiene el control del problema límite en la sucesión de problemas aproximados.

Finalmente, en el Capítulo 4 desarrollamos una manera de obtener una estimación de tipo Carleman para una ecuación del calor con coeficientes de difusión discontinuos. La novedad en esta estrategia están basadas en las ideas del análisis microlocal desarrollado por L. Robbiano y J. Le Rousseau et al. para ecuaciones parabólicas, con la ventaja de que podemos obtener información de la constante de observabilidad.

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The goal of this thesis is the theoretical study of some controllability and inverse problems for hyperbolic and parabolic equations.

In Chapter 1, we present a brief introduction of the general topics of this thesis. We focus on the classical definitions of controllability and inverse problems in partial differential equations. Then, we present the main results of this work.

In Chapter 2, we deal with the potential reconstruction for hyperbolic systems in cascade where measurements of the last component are not available. Roughly speaking, the novelty of this work consists in the Lipschitz stability of this inverse problem from partial measurements of the components of the system. More precisely, we measure all components except the last one. The main tool to achieve this result is a global Carleman estimate for a system of wave equations in cascade where the last component is not accessible.

In Chapter 3, the null controllability of a parabolic equation with dynamic boundary conditions is studied. This problem can be seen as a heat equation with an ordinary differential equation coupling through the boundary. We present our results in two directions. Firstly, we prove that these kind of problems are null-controllable at any time when control acts on a subset which is far from the coupling region. Following the well-known duality between controllability and observability, we prove the associated observability inequality for the adjoint system. Secondly, we prove that a slight modification of this problem can be seen as a limit of a family of parabolic equations with discontinuous diffusion coefficients where the diffusivity is very high in a part of the domain. Additionally, we study the effect of controls for the limit problem in the approximate system.

Finally, in Chapter 4 we develop a suitable Carleman estimate for the heat equation in the presence of an interface. The novelty in this strategy is based on the ideas of microlocal analysis by L. Robbiano and J. Le Rousseau in the context of parabolic equations, with the advantage that we can track the observability constant.

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Contents

Chapter 1

General Introduction

In this chapter, we state the elementary notions concerning inverse problems and controllability issues in partial differential equations (PDE's for short). In order to get a self-contained exposition, we divide this chapter into three sections.

Firstly, in Section 1.1 we restrict our attention to inverse problems for hyperbolic equations and systems. More precisely, we focus on stability results of potential reconstruction for this kind of equations where the observation is on a part of the boundary satisfying suitable geometric and time conditions. Moreover, some other inverse problems concerning hyperbolic equations are considered. In particular, we state results on the stability for some coefficients of an acoustic equation studied by M. Yamamoto and M. Bellassoued.

On the other hand, in Section 1.2 controllability results on parabolic equations are studied. Since the literature is very rich concerning this topic, we reduce our scope to the basic results for the heat equation and their variants. In addition, recent results of controllability for parabolic equations with dynamic boundary conditions are considered.

Finally, in Section 1.3 we state the main theoretical contributions of each topic in this thesis.

1.1 Inverse problems for hyperbolic equations

Intuitively, the observation of an effect in a physic phenomenon may not be sufficient to determine its cause. In fact, if we go inside a room and notice that the temperature is (approximately) uniform, it is difficult for us to know what the distribution of the temperature was four hours ago. Even more, one can think about if there is two different distributions of the temperature which provides the same observation.

The so-called direct problems in PDE's try to describe various physical phenomena such as the propagation of sound, heat, seismic waves, electromagnetic waves, etc. Here, the media properties, the initial state or its conditions on the boundary are assumed to be known.

For example, we can formulate the following direct problem for the acoustic equation:

In the domain $\Omega \subset \mathbb{R}^d$, with $d \geq 1$ with smooth boundary $\partial\Omega$, let $y = y(x,t)$ be a solution of

$$\begin{cases} \rho(x)\partial_t^2 y - c^2(x)\Delta y = f(x,t), & \text{in } \Omega \times (0,T), \\ y(x,0) = y_0,\ \partial_t y(x,0) = y_1, & \text{in } \Omega, \\ y = g, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1.1}$$

Here, $y = y(x,t)$ is the acoustic pressure, $\rho = \rho(x)$ and $c = c(x)$ stand for the density and sound speed of the medium and $f = f(x,t)$ is the source. Under suitable assumptions, system (1.1.1) is well posed, i.e., it has a unique solution and is stable with respect to small perturbations in the data.

Generally speaking, the unknowns in inverse problems include some functions given in the formulation of a direct problem, which are *the solution of our inverse problem*. In order to compute these unknowns, the direct problem is supplied with some additional information about the solution to the direct problem. This one represents *the data of our inverse problem*. For example, in (1.1.1) one can consider the partial data on the flux $\frac{\partial u}{\partial \nu} = h$ on $\partial\Omega \times (0,T)$.

There is not a universal definition for inverse problems. Indeed, given a direct problem (sometimes called forward problem), one can define several inverse problems. For example we say the inverse problem for (1.1.1) is a *source term inverse problem* if it is required to determine the function $f = f(x,t)$. In a similar way, we say that an inverse problem is a *coefficient inverse problem* if it is required to reconstruct the coefficients $c = c(x)$ and/or $\rho = \rho(x)$ in (1.1.1). There exist other classifications based on the additional information, on equations on the structure of the operator, etc. For more details about this topic, we refer to [68].

In contrast to direct problems, inverse problems are ill posed. Mathematically speaking, this means that this kind of problems has no solution in the desired class, or has several solutions, or the solution procedure is unstable, i.e., arbitrarily small errors in the data of the inverse problems may lead to indefinitely large errors in the solutions. For this reason, three questions arise naturally: uniqueness, stability and reconstruction of the coefficients studied. In this thesis, we focus only on uniqueness and stability issues.

Concerning uniqueness, we discuss whether the adopted extra data on the solution can uniquely determine an unknown coefficient or source term. On the other hand, in the stability issue, we are interested on getting the so-called stability estimates. Roughly speaking, these ones determine if it is possible to obtain the norm of the unknown coefficients by partial measurements. Of course, it follows from this that a stability result implies uniqueness. We refer to [16] and [65] for a complete description of these problems.

In general, concerning theoretical methods for coefficient inverse problems, we can consider two types of formulations:

- **Infinitely many measurements by Dirichlet-to-Neumann map:** In this case, the data are all the pairs of Dirichlet boundary inputs and the corresponding Neumann boundary values. For example, given g, we solve (1.1.1) with $f = 0$ in

$\Omega \times (0,T)$, $y(\cdot,0) = \partial_t y(\cdot,0) = 0$ in Ω and $y = g$ on $\partial\Omega \times (0,T)$, we define the map

$$g \mapsto c\frac{\partial y}{\partial \nu}\bigg|_{\partial\Omega\times(0,T)},$$

which is called the Dirichlet-to-Neumann map. Then, in this case, the problem is to determine c from the Dirichlet-to-Neumann map, which means that we have to repeat measurements of $c\frac{\partial y}{\partial \nu}$ on $\partial\Omega \times (0,T)$ after choosing all possible g. For this reason, we say that this is an inverse problem with infinitely many measurements.

- **Finitely many measurements by Carleman estimates:** In contrast to the above formulation, in this one it is sufficient to observe boundary or distributed data of the solution after suitably choosing initial values at finitely many times or a single time. Concerning uniqueness and stability, in 1981 Bukhgeim and Klibanov [26] proposed a fundamental method to obtain uniqueness and stability of the inverse problem based on Global Carleman estimates.

There exists a huge literature on these topics. For more details in other contexts and equations, we refer to [65], [93], [16], and the references therein.

1.1.1 On the well-posedness of the wave equation with potential

In this section, we present the classical inverse problem related to the wave equation. In order to get an idea, let $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a domain with smooth boundary $\partial\Omega$ and $T > 0$. Then, let $u = u(x,t)$ be the solution of the following problem:

$$\begin{cases} \partial_t^2 u - \Delta u + pu = f, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1.2}$$

Here $u = u(x,t)$ denotes the evolution of the amplitude of the waves, $p = p(x,t)$ is a bounded potential and $f = f(x,t)$ denotes a source term acting according to the equation $(1.1.2)_1$. Moreover, (u_0, u_1) denotes the initial state of the waves. In addition, equation $(1.1.2)_3$ is a Dirichlet boundary condition which states that the amplitude of the waves vanishes at the boundary.

The following result asserts that (1.1.2) is well posed in the sense of Hadamard.

Theorem 1.1 *Suppose that $p \in L^\infty(\Omega \times (0,T))$, $f \in L^1(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, the problem (1.1.2) admits a unique (weak) solution satisfying*

$$u \in C^0([0,T];H_0^1(\Omega)), \quad \partial_t u \in C^0([0,T];L^2(\Omega)).$$

Moreover, there exists a constant $C = C(\Omega,T) > 0$ such that the solution u of (1.1.2) satisfies the following estimate:

$$\|u\|_{C^0([0,T];H_0^1(\Omega))} + \|\partial_t u\|_{C^0([0,T];L^2(\Omega))} \leq C\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^1(0,T;L^2(\Omega))}\right). \tag{1.1.3}$$

Notice that Theorem 1.1 gives us the regularity of the solution in the presence of a source term $f \in L^1(0,T;L^2(\Omega))$. Alternatively, one can establish the regularity of the solution u when f belongs in a different functional space:

Theorem 1.2 *Let us assume that $p \in L^\infty(\Omega \times (0,T))$, $f \in W^{1,1}(0,T;H^{-1}(\Omega))$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exists a unique solution u of* (1.1.2) *with the following properties*

$$u \in C^0([0,T];H_0^1(\Omega)), \quad \partial_t u \in C^0([0,T];L^2(\Omega)).$$

Moreover, there exists a positive constant C which depends at least of Ω and $T > 0$ such that the unique solution of (1.1.2) *fulfills*

$$\|u\|_{C^0([0,T];H_0^1(\Omega))} + \|\partial_t u\|_{C^0([0,T];L^2(\Omega))} \leq C \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{W^{1,1}(0,T;L^2(\Omega))} \right). \tag{1.1.4}$$

Let us mention that the inequalities (1.1.3) and (1.1.4) assert the continuous dependence of the solution u with respect to the initial data and source terms, see [80].

We remark that Theorems 1.1 or 1.2 do not provide information about the normal derivative $\partial_\nu u$ of the solution of (1.1.2). To be more precise, if a function belongs to $C^0([0,T];H_0^1(\Omega))$ the normal derivative could not be well defined. However, under the assumptions of Theorem 1.1, we will see that the solution u has an extra regularity. This is given in the following:

Theorem 1.3 (Hidden regularity of wave equation [79]) *Assume the same hypotheses of Theorem 1.1. Then, the solution u of* (1.1.2) *fulfills*

$$\partial_\nu u \in L^2(\partial\Omega \times (0,T)).$$

Furthermore, the application

$$\Lambda : L^1(0,T;L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \mapsto L^2(\partial\Omega \times (0,T))$$

defined by $\Lambda(f, u_0, u_1) = \partial_\nu u$ is well defined and is a linear continuous map, that is to say, there exists a constant $C > 0$ such that

$$\|\partial_\nu u\|_{L^2(\partial\Omega\times(0,T))} \leq C \left(\|f\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right).$$

Remark 1.4 *Under the assumptions of Theorem 1.2 we cannot get any regularity result on the normal derivative $\partial_\nu u$.*

1.1.2 Potential reconstruction for the wave equation

Now we will introduce an inverse problem for (1.1.2). Suppose that the potential p in (1.1.2) is a time-independent function. Then, one can consider the following

Inverse problem: Can we retrieve the potential $p = p(x)$ of (1.1.2) from the knowledge of the flux $\partial_\nu u$ on $\Gamma_0 \times (0,T)$ with $\Gamma_0 \subset \partial\Omega$ or partial measurements of u in $\omega \times (0,T)$ with $\omega \subset \Omega$?

Notice that the above problem is interesting since the partial data is defined only on a part of the boundary. Of course, we point out that this makes sense thanks to the hidden regularity result of the wave equation.

In the following, we are interested in the dependence of the solution u of (1.1.2) with respect to the potential p. For this reason, here and subsequently, $u(p)$ and $u(q)$ stand for the corresponding solution of u associated to the potentials p and q respectively, for fixed initial data (u_0, u_1) and source term f.

Then, thanks to this notation, we can formulate questions related to the above inverse problem in three directions:

- **Uniqueness:** Does the equality
$$\partial_\nu u(p) = \partial_\nu u(q) \text{ on } \Gamma_0 \times (0,T)$$
imply $p = q$ in Ω?
- **Stability:** Is it possible to estimate $\|q - p\|_{L^2(\Omega)}$ or better yet, a stronger norm of $(q - p)$, by a suitable norm of $\partial_\nu u(q) - \partial_\nu u(p)$ in $\Gamma_0 \times (0,T)$?
- **Reconstruction:** Can we find a formula or an algorithm to retrieve the potential p from the knowledge of $\partial_\nu u(p)$ on $\Gamma_0 \times (0,T)$?

Of course, the same three questions can be formulated in the case of partial data of interior observations, i.e., $u(p)$ in $\omega \times (0,T)$, with $\omega \subset \Omega$.

Now we focus on the stability problem. To this end, we shall introduce the so-called geometric and time conditions:

- **Geometric condition:** there exists $x_0 \notin \overline{\Omega}$ such that $\Gamma_0 \subset \partial\Omega$ fulfills
$$\{x \in \partial\Omega\,;\, \nu(x) \cdot (x - x_0) \geq 0\} \subset \overline{\Gamma_0}. \tag{1.1.5}$$
- **Time condition:** T is chosen such that $x_0 \notin \overline{\Omega}$ given in the geometric condition satisfies:
$$\sup_{x \in \Omega} |x - x_0| \leq T. \tag{1.1.6}$$

In the case of interior observations, the geometric condition reads as follows: there exists $x_0 \notin \overline{\Omega}$ such that $\omega \subset \Omega$ satisfies
$$\{x \in \partial\Omega\,;\, \nu(x) \cdot (x - x_0) > 0\} \subset \partial\omega \cap \partial\Omega. \tag{1.1.7}$$

Let us emphasize that $\Gamma_0 \subset \partial\Omega$ and $\omega \subset \Omega$ satisfying the geometric condition are not arbitrary subsets. Indeed, in the particular case of Ω being a ball in $\mathbb{R}^2$, the length of Γ_0 is larger than half of the ball. In the same way, ω is a boundary neighborhood of Γ_0, see figure 1.1:

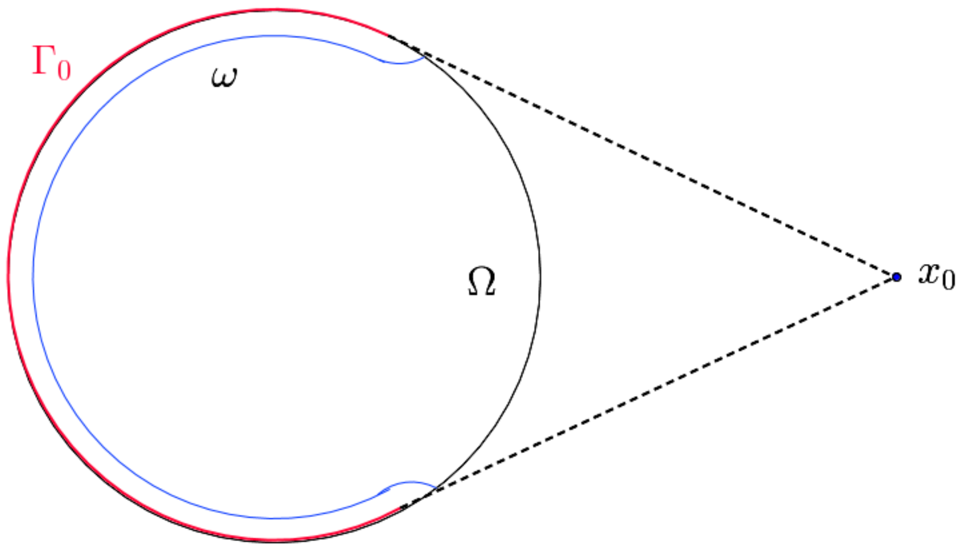

Figure 1.1: Γ_0 and ω satisfying the geometric condition

Roughly speaking, the Geometric and Time conditions assert that, thanks to the Snell law, all rays of geometric optics in Ω, which are simply straight lines reflected on the boundary, should meet the observation region Γ_0 (or ω) at a non-diffractive point in a time less than T.

Let us also introduce the admissible sets of potentials for the above inverse problem. For $m > 0$, we define the set

$$L^\infty_{\leq m}(\Omega) = \{p \in L^\infty(\Omega)\,;\, \|p\|_{L^\infty(\Omega)} \leq m\}.$$

Theorem 1.5 (see [11]) *Let $m > 0$, $K > 0$ and $r > 0$. Let $p \in L^\infty_{\leq m}(\Omega)$. Assume that the solution $u(p)$ of* (1.1.2) *is such that*

$$\|u(p)\|_{H^1(0,T;L^\infty(\Omega))} \leq K,$$

and assume also that the initial datum u_0 satisfies the following positivity condition:

$$\inf_{x\in\Omega} |u_0(x)| \geq r > 0.$$

Additionally, suppose that $\Gamma_0 \subset \partial\Omega$ and $T > 0$ satisfy the geometric and time condition. Then, for all $q \in L^\infty_{\leq m}(\Omega)$, $\partial_\nu u(q) - \partial_\nu u(p)$ belongs to $H^1(0,T;L^2(\Gamma_0))$ and there exists a constant $C = C(m,T,K,r) > 0$ such that for any $q \in L^\infty_{\leq m}(\Omega)$, the following inequalities hold:

$$\|\partial_\nu u(q) - \partial_\nu u(p)\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q-p\|_{L^2(\Omega)}, \tag{1.1.8}$$

and

$$\|q-p\|_{L^2(\Omega)} \leq C\|u(q) - u(p)\|_{H^1(0,T;L^2(\Gamma_0))}. \tag{1.1.9}$$

We emphasize that the estimate (1.1.9) asserts the Lipschitz stability of the inverse problem while (1.1.8) gives the continuous dependence of the normal derivative in the norm $H^1(0,T;L^2(\Gamma_0))$ of the solution with respect to the potentials in the $L^2(\Omega)$-norm.

We remark that the geometric condition is a restrictive assumption on the observations. Without this condition, M. Bellassoued proved in [15] that the same inverse problem has logarithmic stability. More precisely, the author achieved the following result:

Theorem 1.6 (see [15], [16]) *Let $\tilde{\Gamma}$ be an arbitrary subset of $\partial\Omega$. Assume that $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H^2(\Omega)$, and there exists a constant $m_0 > 0$ such that*

$$|u_0(x)| \geq m_0 > 0, \quad \forall x \in \overline{\Omega \setminus \omega}.$$

Then, there exist $T > 0$ sufficiently large and a constant $C > 0$ such that

$$\|p - q\|_{L^2(\Omega)} \leq C \left[\log \left(1 + \frac{C}{\|\partial_\nu (u_p - u_q)\|_{H^1(0,T;L^2(\tilde{\Gamma}))}} \right) \right]^{-1/2}, \tag{1.1.10}$$

for all $p, q \in \Lambda(\Omega)$, where $\Lambda(\Omega)$ is the set of admissible potentials given by

$$\Lambda(\Omega) = \{p \in W^{1,\infty}(\Omega)\,;\, \|p\|_{W^{1,\infty}(\Omega)} \leq M,\, p = p_0 \text{ in } \Omega \setminus \omega\},$$

with $M > 0$ and $p_0 \in C^\infty(\mathbb{R}^n)$ are given arbitrarily. Moreover, the constant C in (1.1.10) is dependent of $\Omega, \omega, T, M, u_0, u_1$, but independent of p, q.

According to the definition of $\Lambda(\Omega)$, we remark that this stability result comes from the fact that p is known in a part of Ω. Let us also mention that the main ingredient of the proof of the above theorem is the Fourier-Bros-Iagolnitzer (FBI) transform. This one is crucially used in order to prove a sharp unique continuation property for hyperbolic equations (see [87], [86]).

The proof of the Theorem 1.5 is based on the Bukhgeim-Klibanov's method and the so-called global Carleman inequalities, which is an interesting result by itself. In order to state this result, we shall introduce some weight functions. For $\beta \in (0, 1)$, we define

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}, \quad \forall (x,t) \in \Omega \times (-T,T). \tag{1.1.11}$$

Then, the Carleman estimate for the wave operator reads as follows:

Theorem 1.7 *(see [11], [96]) Let us assume the geometric and time conditions. Let ψ and φ weight functions defined by (1.1.11). Then, there exist three positive constants C, λ_0 and s_0 such that for all $\lambda \geq \lambda_0$ and $s \geq s_0$, the following inequality holds*

$$\begin{aligned} & s\lambda \int_{-T}^{T} \int_\Omega e^{2s\varphi} \varphi (|\partial_t v|^2 + |\nabla v|^2) dx dt + s^3\lambda^3 \int_{-T}^{T} \int_\Omega e^{2s\varphi} \varphi^3 |v|^2 dx dt \\ & + \int_{-T}^{T} \int_\Omega e^{2s\varphi} |P_1(e^{s\varphi} v)|^2 dx dt \\ \leq & C \int_{-T}^{T} \int_\Omega e^{2s\varphi} |\Box v|^2 dx dt + Cs\lambda \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi} \varphi |\partial_\nu v|^2 d\sigma dt, \end{aligned}$$

for all $v \in L^2(-T,T; H_0^1(\Omega))$ satisfying $\Box v := \partial_t^2 v - \Delta v \in L^2(\Omega \times (-T,T))$ with normal derivative $\partial_\nu v \in L^2(\Gamma_0 \times (-T,T))$, $v(\pm T) = \partial_t v(\pm T) = 0$ in Ω and P_1 defined by

$$P_1 w = \partial_t^2 w - \Delta w + s^2\lambda^2\varphi^2 w(|\partial_t\psi|^2 - |\nabla\psi|^2).$$

Theorem 1.7 ensures the existence of a 2-parameter Carleman estimate (i.e. λ and s) for the wave operator $\Box = \partial_t^2 - \Delta$. As we shall see in Section 2 and 3, often we just need a one-parameter Carleman estimate for our purposes. On the other hand, Carleman estimates provide another interesting result called *Unique Continuation Property* (UCP for short). Then, thanks to Theorem 1.7, we obtain the following:

Corollary 1.8 (UCP for the wave operator) *Suppose that $v \in L^2(-T,T;H_0^1(\Omega))$ is a function which verifies $\Box v = 0$ in $\Omega \times (-T,T)$, $v(\pm T) = \partial_t v(\pm T) = 0$ in Ω, and additionally $\partial_\nu u = 0$ on $\Gamma_0 \times (-T,T)$, where Γ_0 and $T > 0$ satisfy the geometric and time conditions* (1.1.5) *and* (1.1.6)*, respectively. Then, v vanishes in $\Omega \times (-T,T)$.*

1.1.3 Inverse problem for the wave speed of the wave equation

Let Ω be a bounded domain with smooth boundary (C^2 at least) and $T > 0$. Let $u = u(x,t)$ be the solution of

$$\begin{cases} \partial_t^2 u - \operatorname{div}(p(x)\nabla u) = 0, & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = a,\ \partial_t u(\cdot,0) = 0, & \text{in } \Omega, \\ u = b, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1.12}$$

Here, p denotes the bulk-modulus of the acoustic equation considered in a non homogeneous medium. Under smooth assumptions on a, b and p the problem has a unique (weak) solution. We consider the following:

Inverse problem: Determine the coefficient p from the knowledge of partial measurements of u in $\omega \times (0,T)$, with $\omega \subset \Omega$.

In order to formulate the results, we consider $\omega \subset \Omega$ satisfying the geometric condition (1.1.5). In addition, set

$$\mathcal{D} := \sqrt{\sup_{x\in\Omega} |x - x_0|^2 - \inf_{x\in\Omega} |x - x_0|^2}. \tag{1.1.13}$$

Given $\eta \in C^1(\partial\Omega)$ and constants $M_0 \geq 0$, $M_1 > 0$, $0 < \theta_0 \leq 1$ and $\theta_1 > 0$, we define the following admissible sets:

$$\begin{aligned} \mathcal{U}_1 = \Bigg\{ & p \in C^2(\overline{\Omega})\,;\ \|p\|_{C^2(\overline{\Omega})} < M_1,\ \|\nabla p\|_{C(\overline{\Omega})} < M_0, p(x) > \theta_1, \forall x \in \overline{\Omega} \\ & \left| \frac{\nabla p(x) \cdot (x - x_0)}{2p(x)} \right| < 1 - \theta_0, \forall x \in \overline{\Omega \setminus \omega}\,,\ \|u\|_{W^{4,\infty}(\Omega\times(0,T))} < M_1 \Bigg\}, \end{aligned} \tag{1.1.14}$$

and

$$\begin{aligned} \mathcal{U}_2 = \Bigg\{ & p \in C^2(\overline{\Omega})\,;\ \|p\|_{C^2(\overline{\Omega})} < M_1,\ \|\nabla p\|_{C(\overline{\Omega})} < M_0, p(x) > \theta_1, \forall x \in \overline{\Omega} \\ & \frac{\nabla p(x) \cdot (x - x_0)}{2p(x)} < 1 - \theta_0, \forall x \in \overline{\Omega \setminus \omega}\,,\ \|u\|_{W^{4,\infty}(\Omega\times(0,T))} < M_1 \Bigg\}. \end{aligned} \tag{1.1.15}$$

It is possible to replace $<$ and $>$ by $\leq$ and $\geq$ respectively. We choose $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\beta_1 + \frac{M_0\mathcal{D}}{\sqrt{\theta_1}}\sqrt{\beta_1} < \theta_0\theta_1 \tag{1.1.16}$$

and

$$\beta_2 + \frac{M_0\mathcal{D}}{\sqrt{\theta_1}}\sqrt{\beta_2} < \theta_0\theta_1, \quad \theta_1 \inf_{x\in\Omega} |x-x_0|^2 - \beta_2\mathcal{D}^2 > 0. \tag{1.1.17}$$

Now we have all the ingredients to state a stability result for the inverse problem with interior observations:

Theorem 1.9 *Let us consider ω and x_0 satisfying the geometric condition. Let $a \in W^{1,\infty}(\Omega)$ such that*

$$\nabla a(x)\cdot(x-x_0) \neq 0, \quad \forall x \in \overline{\Omega}.$$

Let $k = 1$ or $k = 2$. We choose the observation time $T > 0$ such that

$$T > \frac{1}{\sqrt{\beta_k}}\mathcal{D}. \tag{1.1.18}$$

Then, there exist constants $\kappa \in (0,1)$ and $C > 0$ such that for all $(p,q) \in \mathcal{U}_k$, the associated solutions $u(p)$ and $u(q)$ fulfill the following inequality:

$$\|p-q\|_{L^2(\Omega)} \leq C\left(\sum_{j=2}^{3} \|\partial_t^j(u(p)-u(q))\|_{L^2(\omega\times(0,T))}\right). \tag{1.1.19}$$

The main tool to prove Theorem 1.9 is a Carleman estimate in $H^{-1}(\Omega\times(0,T))$. In order to state this result, let us recall that x_0 is defined by the Geometric condition (1.1.7) and β_1 and β_2 given by (1.1.16) and (1.1.17), respectively. We define the functions $\psi_k = \psi_k(x,t)$ and $\varphi = \varphi_k(x,t)$ by

$$\psi_k(x,t) = |x-x_0|^2 - \beta_k t^2, \tag{1.1.20}$$

and

$$\varphi_k(x,t) = e^{\lambda\psi_k(x,t)}, \quad \lambda > 0. \tag{1.1.21}$$

Now we are ready to state the Carleman estimate with source lying in $H^{-1}(\Omega\times(-T,T))$:

Theorem 1.10 *Let $k \in \{1,2\}$. We assume that $p \in \mathcal{U}_k$, x_0 and ω satisfying* (1.1.7)*. Let $y \in H^1(\Omega\times(-T,T))$ satisfy*

$$\begin{cases} \partial_t^2 y - div(p(x)\nabla y) = \tilde{f} + \partial_t f_0 + \sum_{j=1}^N \partial_j f_j, & in\ \Omega\times(-T_k,T_k), \\ y(\cdot,\pm T_k) = 0, & in\ \Omega, \\ y = 0, & on\ \partial\Omega\times(-T_k,T_k), \end{cases} \tag{1.1.22}$$

where $\tilde{f} \in H^{-1}(\Omega \times (-T_k, T_k))$, $f_j \in L^2(\Omega \times (-T_k, T_k))$, *with* $0 \leq j \leq N$. *Then, there exists a constant* $\mu > 0$ *such that for each* T_k *satisfying*

$$T_k \in \left(\frac{\mathcal{D}}{\sqrt{\beta_k}}, \frac{\mathcal{D}}{\sqrt{\beta_k}} + \mu \right),$$

there exists $\lambda_0 > 0$ *such that there exist constants* $s_0 = s_0(\lambda) > 0$ *and* $C_1 > 0$ *such that*

$$\begin{aligned} s\int_{-T_k}^{T_k}\int_{\Omega} e^{2s\varphi_k}|y|^2 dxdt \leq & C_1 s\int_{-T_k}^{T_k}\int_{\Omega} e^{2s\varphi_k}|f_j|^2 dxdt + C_1\|e^{s\varphi_k}\tilde{f}\|^2_{H^{-1}(\Omega\times(-T_k,T_k))} \\ & + C_1 s\int_{-T_k}^{T_k}\int_{\omega} e^{2s\varphi_k}|y|^2 dxdt, \end{aligned} \tag{1.1.23}$$

for all $s \geq s_0$.

1.2 Controllability issues in PDE's

Control theory is the area of mathematics concerning dynamical systems whose behavior can be changed by means of controls applied through actuators. This is also a rich interdisciplinary branch of mathematics, with applications in areas such as biology, chemistry, engineering, economics and seismic prospection. For more details about this theory, we refer to the books [79], [91] and the reviews [48], [22] and the references therein.

Roughly speaking, a control system can be written in the following abstract form

$$\begin{cases} \dfrac{dy}{dt} = L(y,u), \quad 0 < t < T, \\ y(0) = y_0. \end{cases} \tag{1.2.1}$$

where $y \in Y$ and $u \in \mathcal{U}_{ad}$. Here y is the *state*, the unknown of the problem that we want to control, y_0 is the initial state, u is the control, the variable that can be chosen appropriately to act on the system and $\mathcal{U}_{ad}$, and Y stand for the set of admissible controls and the state space, respectively.

Given a control system like (1.2.1), we can formulate the so-called *controllabiliy problem*, which can be stated as follows:

Controllability problem: find a control $u \in \mathcal{U}_{ad}$ such that the associated state behaves in a appropriate manner in a given final time $T > 0$.

We distinguish four different notions of controllability. We say that the system (1.2.1) is approximately controllable if, for any initial state y_0, it is possible to steer the solution to a state arbitrarily close to any given target (in an appropriate topology). The exact controlability of (1.2.1) asserts that the system can be driven from any initial data to a prescribed target. On the other hand, we say that the system (1.2.1) has the null-controllability property if, for any initial data, the solution can be driven to zero. Finally, another interesting concept of controllability is the exact controllability to trajectories, which means that it is possible to steer the state of the system to join a control-free prescribed trajectory, i.e., a given solution of the system without control.

From a mathematical viewpoint, the literature is very rich on controllability problems, see for instance [92], [32],[41],[5] and the references therein. The control theory started to be developed in the beginning of the 1960's for finite dimensional systems. The linear case of this problem is by now completely understood thanks to the Kalman rank condition, and moreover the four notions of controllability introduced before are equivalent. Furthermore, the case of nonlinear finite dimensional systems has been intensely studied in the last two decades and there are many powerful sufficient conditions for local and global controllability, see [32].

Nevertheless, in the context of PDE's the situation is more delicate, even in the linear case. The main reason is that a linear PDE governing the evolution of a process may be of hyperbolic type (wave equation, Maxwell equations), of dispersive type (plate equation, Schrödinger equations, Korteweg-de Vries equation), or of parabolic type (heat equation, Stokes equation). Each equation induces specific properties on the trajectories: propagation of singularities with finite velocity for hyperbolic equations, infinite speed propagation property together with a weak (resp. strong) smoothing effect for dispersive (resp. parabolic) equations, and time irreversibility for parabolic equations.

Accordingly to the above description of the evolution of a linear PDE, we cannot expect equivalence between the different notions of controllability in general. For instance, the regularizing effect of the heat equation asserts that the associated solution of a L^2 initial state is a smooth function. Thus, it is difficult to ensure exact controllability for the heat equation when the control acts in a small part of the domain. On the other hand, the location and the duration of the control play an important role in the controllability of the wave equation. This role may be completely hidden in the finite dimension setting.

Generally, the study of controllability of linear PDE's is equivalent to a suitable observability inequality for the adjoint problem. This means that we need full knowledge of the solution of the adjoint problem at a given time using only local measurements of it. Nevertheless, we emphasize that the proof of such inequalities are a challenging issue and requires tools such as Ingham inequalities [64] [72], multiplier methods [71] [58], [79], [84], microlocal analysis [10], [77] [27], or Carleman estimates [61], [50], [48], [45].

1.2.1 Classical results on controllability of parabolic equations

In this subsection, we follow the presentation given in [83]. Let Ω be a bounded open set with boundary of class C^2 and $\omega \subset \Omega$ a non-empty open subset of Ω. Given $T > 0$ we consider the following non-homogeneous heat equation:

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega f, & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.2.2}$$

In (1.2.2), $u = u(x,t)$ is the state and $f = f(x,t)$ is the control function with a support localized in ω.

Theorem 1.11 *For any $f \in L^2((0,T) \times \omega)$ and $u_0 \in L^2(\Omega)$ problem* (1.2.2) *has a unique* *weak solution* $u \in C([0,T]; L^2(\Omega))$ *given by the variation of constants formula*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\chi_\omega f(s)ds$$

where $(S(t))_{t\geq 0}$ *is the semigroup of constractions generated by* $-\Delta$ *in* $L^2(\Omega)$.

Moreover, if $f \in W^{1,1}(0,T; L^2(\omega))$ *and* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$*, problem* (1.2.2) *has a classical solution*

$$u \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$$

and (1.2.2) *is verified in* $L^2(\Omega)$ *for all* $t > 0$.

From the fact that $u \in L^\infty(0,T; L^2(\Omega))$, it follows that $u \in L^2(0,T; H_0^1(\Omega))$. Consequently, whenever $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T; L^2(\omega))$ the solution verifies

$$u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$$

and we have the following energy estimate:

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|_{L^2(\Omega)} dt \leq C \int_0^t \|f\|^2_{L^2(\Omega)} dt + C\|u_0\|^2_{L^2(\Omega)}.$$

Now, we focus on the null-controllability results on the heat equation with Dirichlet boundary conditions where $\omega \subset \mathbb{R}$ is arbitrary. More precisely, we wonder if for each $T > 0$ and $u_0 \in L^2(\Omega)$ in (1.2.2), there exists a control $f \in L^2(\omega \times (0,T))$ such that

$$u(T) = 0, \quad \text{ in } \Omega.$$

First, we notice that one of the most important properties of the heat equation is its regularizing effect. When $\Omega \setminus \omega \neq \emptyset$, the solutions of (1.2.2) belong to $C^\infty(\Omega \setminus \omega)$ at time $t = T$. Hence, the restriction of the elements of $R(T; u_0)$ to $\Omega \setminus \omega$ are smooth functions. Then, unless the trivial case $\omega = \Omega$, that is to say, when the control function acts on the entire domain Ω, exact controllability may not hold. In this sense, the notion of exact controllability is not very relevant for the heat equation. This is due to its strong time irreversibility of the system under consideration.

Moreover, it is not difficult to see that if null controllability holds, then any initial data may be let to any final state of the form $S(T)v_0$ with $v_0 \in L^2(\Omega)$, i.e., to the range of the semigroup in time $t = T$. Null controllability implies approximate controllability. Indeed, this is a consequence that the eigenfunctions of the laplacian operator belong to $S(T)[L^2(\Omega)]$. Then we deduce that $R(T; u_0)$ is dense in $L^2(\Omega)$, which is the definition of approximate controllability.

On the other hand, approximate controllability together with uniform estimates on the approximate controls as ε goes to zero may lead to null controllability properties. More precisely, given u^1, we have that $u^1 \in R(T; u_0)$ if and only if there exists a sequence of controls $(f_\varepsilon)_{\varepsilon>0}$ such that

$$\|u(T,\cdot) - u_1\|_{L^2(\Omega)} \leq \varepsilon$$

and $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T) \times \omega)$. Indeed, in this case any weak limit in $L^2(\omega \times (0,T))$ of the sequence of controls $(f_\varepsilon)_{\varepsilon>0}$ gives an exact control which makes that $u(\cdot,T) = u_1$ in Ω.

1.2.2 Null controllability of the heat equation for parabolic equations with dynamic boundary conditions

In this part, we follow the presentation of [81]. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $N \geq 2$ and $T > 0$. Let $y = y(x,t)$ be the solution of the following problem

$$\begin{cases} \partial_t y - d\Delta y + a(x,t)y = \chi_\omega v(x,t), & \text{in } \Omega \times (0,T), \\ \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + d\partial_\nu y + b(x,t)y_\Gamma = 0, & \text{on } \Gamma \times (0,T), \\ y_\Gamma(x,t) = y|_\Gamma(x,t), & \text{on } \Gamma \times (0,T), \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}), & \text{in } \Omega \times \Gamma. \end{cases} \tag{1.2.3}$$

Here, $\omega \subset\subset \Omega$ is an arbitrary nonempty open subset of Ω, $y_0 \in L^2(\Omega)$ and $y_{0,\Gamma} \in L^2(\Gamma)$ are the initial data, the constants δ, d are positive, $a \in L^\infty(\Omega \times (0,T))$ and $b \in L^\infty(\Gamma \times (0,T))$. In addition, $y|_\Gamma$ denotes the trace of a function $y : \Omega \to \mathbb{R}$, ν is the outer unit normal field, $\partial_\nu y = (\nu \cdot \nabla y)|_\Gamma$ stands for the normal derivative at Γ, and Δ_Γ denotes the Laplace-Beltrami operator on Γ.

Then, the main question is: find a control $v \in L^2(\omega \times (0,T))$ such that the solution y of (1.2.3) satisfies

$$y(\cdot, T) = 0, \text{ in } \overline{\Omega}.$$

Following the classical equivalence between controllability and observability, we introduce the following adjoint system:

$$\begin{cases} -\partial_t \varphi - d\Delta \varphi + a(x,t)\varphi = 0, & \text{in } \Omega \times (0,T), \\ -\partial_t \varphi_\Gamma - \delta\Delta_\Gamma \varphi + d\partial_\nu \varphi + b(x,t)\varphi_\Gamma = 0, & \text{on } \Gamma \times (0,T), \\ \varphi_\Gamma(x,t) = \gamma_\Gamma(x,t), & \text{on } \Gamma \times (0,T), \\ (\varphi(\cdot,T), \varphi(\cdot,T)) = (\varphi_T, \varphi_{T,\Gamma}), & \text{in } \Omega \times \Gamma. \end{cases} \tag{1.2.4}$$

Proposition 1.12 *There is a constant $C > 0$ such that for all $(\varphi_T, \varphi_{T,\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ the mild solution $(\varphi, \varphi_\Gamma)$ of the backward problem (1.2.4) satisfies*

$$\|\varphi(\cdot,0)\|^2 + \|\varphi_\Gamma(\cdot,0)\|^2 \leq C \int_0^T \int_\omega |\varphi|^2 dxdt. \tag{1.2.5}$$

Given $R > 0$, the constant $C = C(R)$ can be chosen independently of a, b with

$$\|a\|_\infty, \|b\|_\infty \leq R$$

.

To prove Proposition 1.12, the authors prove a Carleman estimate for the problem (1.2.4). Let us emphasize that weights appearing in such estimate are the same in [48] for the case Dirichlet boundary conditions and in the classical text of A. V. Fursikov and O. Yu. Imanuvilov [50] for mixed boundary conditions. Of course, such functions are based on an auxiliary function η^0 whose existence is guaranteed in the following result:

Lemma 1.13 *Given a nonempty open set $\omega' \subset\subset \Omega$, there is a function $\eta^0 \in C^2(\overline{\Omega})$ such that*

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0, \text{ on } \Gamma, \quad |\nabla \eta^0| > 0, \text{ in } \overline{\Omega \setminus \omega'}.$$

We emphasize that, since $|\nabla \eta^0|^2 = |\nabla_\Gamma \eta^0|^2 + |\partial_\nu \eta^0|^2$ on Γ, the function η^0 in the above lemma satisfies

$$\nabla_\Gamma \eta^0 = 0, \quad |\nabla \eta^0| = |\partial_\nu \eta^0|, \quad \partial_\nu \eta^0 \leq -c < 0, \text{ on } \Gamma,$$

for some constant $c > 0$. Now let us define the Carleman weight functions. For $\lambda, m > 1$, we set

$$\begin{aligned}
\alpha(x,t) =& (t(T-t))^{-1} \left(e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \right), \\
\xi(x,t) =& (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))},
\end{aligned}$$

for $x \in \overline{\Omega}$ and $t \in (0,T)$. Notice that α and ξ are smooth and strictly positive on $\overline{\Omega} \times (0,T)$ and blow up as $t \to 0$ and as $t \to T$. Moreover, such functions are constant on the boundary Γ so that

$$\nabla_\Gamma \alpha = 0, \quad \text{and } \nabla_\Gamma \xi = 0, \quad \text{on } \Gamma.$$

Lemma 1.14 *Let $T > 0$, $\omega \subset\subset \Omega$ be a nonempty and open subset of Ω, $d, \delta > 0$, $a \in L^\infty(\Omega \times (0,T))$ and $b \in L^\infty(\Gamma_T)$. Let $\omega' \subset\subset \omega$. Define η^0, α and ξ as above with respect to ω'. Then, there exist constants $C > 0$, $\lambda_1 \geq 1$ and $s_1 \geq 1$ such that*

$$\begin{aligned}
& s^{-1} \int_0^T \int_\Omega e^{-2s\alpha} \xi^{-1} (|\partial_t \varphi|^2 + |\Delta \varphi|^2) dx dt + s^{-1} \int_0^T \int_\Gamma e^{-2s\alpha} \xi^{-1} (|\partial_t \varphi_\Gamma|^2 + |\Delta_\Gamma \varphi|^2) dS dt \\
& + s\lambda^2 \int_0^T \int_\Omega e^{-2s\alpha} \xi |\nabla \varphi|^2 dx dt + s\lambda \int_0^T \int_\Gamma e^{-2s\alpha} \xi |\nabla_\Gamma \varphi_\Gamma|^2 dS dt \\
& + s^3 \lambda^4 \int_0^T \int_\Omega e^{-2s\alpha} \xi^3 |\varphi|^2 dx dt + s^3 \lambda^3 \int_0^T \int_\Gamma e^{-2s\alpha} \xi^3 |\varphi_\Gamma|^2 dS dt \\
& + s\lambda \int_0^T \int_\Gamma e^{-2s\alpha} \xi |\partial_\nu \varphi|^2 dS dt \\
\leq & C s^3 \lambda^4 \int_0^T \int_\omega e^{-2s\alpha} \xi^3 |\varphi|^2 dx dt + C \int_0^T \int_\Omega e^{-2s\alpha} |\partial_t \varphi + d\Delta \varphi - a\varphi|^2 dx dt \\
& + C \int_0^T \int_\Gamma e^{-2s\alpha} |\partial_t \varphi_\Gamma + \delta \Delta_\Gamma \varphi_\Gamma - d\partial_\nu \varphi - b\varphi_\Gamma|^2 dS dt,
\end{aligned} \tag{1.2.6}$$

for all $\lambda \geq \lambda_1$, and for all

$$(\varphi, \varphi_T) \in H^1(0,T; L^2(\Omega) \times L^2(\Gamma)) \cap L^2(0,T; \mathbb{H}^2),$$

where $\mathbb{H}^2 = \{(y, y_\Gamma) \in H^2(\Omega) \times H^2(\Gamma) \text{ with } y|_\Gamma = y_\Gamma\}$. Furthermore, given $R > 0$, the constant $C = C(R)$ can be chosen independently of all a, b with

$$\|a\|_\infty, \|b\|_\infty \leq R.$$

1.3 Main results of the thesis

In this section we briefly introduce the problems and results obtained in this thesis. The main topics covered here are:

• **Potential reconstruction for a class of hyperbolic systems in cascade:** In Chapter 2, we analyze the simultaneous reconstruction of each potentials $q_1, \dots, q_n$ defined in $\Omega \subset \mathbb{R}^N$, $N \geq 1$ in a linear hyperbolic system of the form

$$\begin{cases} \Box u_1 + q_1 u_1 = a_1 u_2 + g_1, & \text{in } \Omega \times (0,T), \\ \Box u_2 + q_2 u_2 = a_2 u_3 + g_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box u_{n-1} + q_{n-1} u_{n-1} = a_{n-1} u_n + g_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box u_n + q_n u_n = g_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k u_j(0) = u_j^k, \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ u_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.3.1}$$

from a reduced number of controls of $(u_j)_{j \in \mathcal{I}}$ with $j \in \mathcal{I} \subset \{1, \dots, n\}$. Inspired in the Bukhgeim-Klibanov method, we prove a Lipschitz stability result for these coefficients of the form:

$$\sum_{j=1}^{n} \|q_j - \tilde{q}_j\|^2_{L^2(\Omega)} \leq C \sum_{j=1}^{n-2} \|u_j - \tilde{u}_j\|^2_{H^3(0,T;L^2(\omega))} + C\|u_{n-1} - \tilde{u}_{n-1}\|^2_{H^4(0,T;L^2(\omega))}, \tag{1.3.2}$$

where $\tilde{u}_j$ with $j = 1, \dots, n$ is the solution of (1.3.1) associated to $(\tilde{q}_1, \dots, \tilde{q}_n)$ (source and initial conditions are fixed), C is a positive constant independent of these potentials and $\omega \subset \Omega$. We point out that in (1.3.2), measurements of $u_n - \tilde{u}_n$ do not appear in our results. This means that we can reconstruct the potentials $q_1, \dots, q_n$ of (1.3.1) without any knowledge of the last component of the system.

The main ingredient to prove this result is a Carleman estimate for problems having the form:

$$\begin{cases} \Box v_1 + r_1 v_1 = v_2 + h_1, & \text{in } \Omega \times (-T,T), \\ \Box v_2 + r_2 v_2 = v_3 + h_2, & \text{in } \Omega \times (-T,T), \\ \quad \vdots & \quad \vdots \\ \Box v_{n-1} + r_{n-1} v_{n-1} = v_n + h_{n-1}, & \text{in } \Omega \times (-T,T), \\ \Box v_n + r_n v_n = h_n, & \text{in } \Omega \times (-T,T), \\ \partial_t^k v_j(\pm T) = 0, \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ v_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (-T,T). \end{cases} \tag{1.3.3}$$

More precisely, using the abbreviation

$$\begin{aligned} I(\alpha, v, \Omega) =& s^\alpha \int_\Omega e^{2s\varphi(0)} (s^2 |v(0)|^2 + |\partial_t v(0)|^2 + |\nabla v(0)|^2) dx \\ &+ s^{\alpha+1} \int_{-T}^{T} \int_\Omega e^{2s\varphi} (s^2 |v|^2 + |\partial_t v|^2 + |\nabla v|^2) dx dt, \end{aligned}$$

with

$$\psi(x,t) = |x-x_0|^2 - \beta t^2 + C_0, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}, \quad \forall (x,t) \in \Omega \times (-T,T),$$

and suitable assumptions on $x_0 \in \mathbb{R}^N$, $\alpha, \beta, C_0, \lambda > 0$ and $T > 0$ we get

$$\begin{aligned}
&\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\
\leq& C_2 s^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^2|v_j|^2 + |\partial_t v_j|^2\big)dxdt + C_2 s^{\alpha} \sum_{j=1}^{n-2} \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|h_j|^2 dxdt \\
&+ C_2 s^3 \int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^5|v_{n-1}|^2 + s^3|\partial_t v_{n-1}|^2 + |\partial_t^2 v_{n-1}|^2\big)dxdt \\
&+ C_2 \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(s^3|h_{n-1}|^2 + |h_n|^2 + s|\partial_t h_{n-1}|^2 + |\partial_t h_n|^2\big)dxdt.
\end{aligned} \tag{1.3.4}$$

for all $s \geq s_0 > 0$, where $v_1, \ldots, v_n$ is a solution of (1.3.3) and C being a positive constant. The results of Chapter 2 are based on the article [29] in collaboration with Nicolás Carreño y Axel Osses.

• **Controllability properties of a class of heat equation with dynamic boundary conditions:** In Chapter 3, the null controllability of a suitable class of 1-D parabolic equations with dynamic boundary conditions is studied. The prototype of such problems is

$$\begin{cases}
\partial_t u(x,t) - \partial_x^2 u(x,t) = \chi_\omega(x) v(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\
(u(x,0), u_\Gamma(0)) = (u_0(x), u_{\Gamma,0}), & \forall x \in \Omega_L, \\
u_\Gamma(t) = u(0,t), & \forall t \in (0,T), \\
u(-L_1, t) = 0, & \forall t \in (0,T), \\
u_\Gamma'(t) + \partial_x u(0,t) = 0, & \forall t \in (0,T).
\end{cases} \tag{1.3.5}$$

Here $\Omega_L = (-L_1, 0) \subset \mathbb{R}$ with $L_1 > 0$ and $\omega \subset \Omega_L$. In other words, the goal is to steer the state u of (1.3.5) to a null final target by a suitable choice of the control function, i.e., given $(u_0, u_{0,\Gamma}) \in L^2(\Omega_L) \times \mathbb{R}$ and $T > 0$ we want to find a control $v \in L^2(\omega \times (0,T))$ such that the associated solution of (1.3.5) satisfies

$$u(x,T) = 0, \quad \forall x \in \overline{\Omega_L}.$$

This means that the first equation is controlled directly by the action of v, while the ODE at $x = 0$ is being controlled indirectly through the coupling.

Concerning this question, our results provide that (1.3.5) is null-controllable at any time $T > 0$ with $\omega = (-L_1, -a)$, with $a > 0$. However, some discussions are presented in the case $\omega \subset\subset \Omega_L$.

Following the duality between controllability and observability, the proof of this result consists in obtaining an observability estimate of the form

$$\|z(\cdot,0)\|^2_{L^2(\Omega_L)} + |z_\Gamma(0)|^2 \leq C \int_0^T \int_\omega |z|^2 dxdt, \tag{1.3.6}$$

for each $(z_T, z_{T,\Gamma}) \in L^2(\Omega_L) \times \mathbb{R}$, where (z, z_Γ) is the solution of the adjoint system

$$\begin{cases} \partial_t z(x,t) + \partial_x^2 z(x,t) = 0, & \forall (x,t) \in \Omega_L \times (0,T), \\ (z(x,T), z_\Gamma(T)) = (z_T(x), z_{\Gamma,T}), & \forall x \in \Omega_L, \\ z(-L_1,t) = 0, & \forall t \in (0,T), \\ z_\Gamma'(t) - \partial_x z(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{1.3.7}$$

In order to prove (1.3.6) we use a Carleman estimate of the form

$$\begin{aligned} & s^3 \int_0^T \int_{\Omega_L} e^{-2s\varphi}(t(T-t))^{-3\alpha}|y|^2 dxdt + s\int_0^T \int_{\Omega_L} e^{-2s\varphi}(t(T-t))^{-\alpha}|\partial_x y|^2 dxdt \\ & + s^{-1}\int_0^T \int_{\Omega_L} e^{-2s\varphi}(t(T-t))^{\alpha}(|\partial_x^2 y|^2 + |\partial_t y|^2)dxdt \\ & + s^3 \int_0^T e^{-2s\varphi(0,t)}(t(T-t))^{-3\alpha}|y_\Gamma(t)|^2 dt \\ & + s\int_0^T e^{-2s\varphi(0,t)}(t(T-t))^{-\alpha}|\partial_x y(0,t)|^2 dt + \int_0^T e^{-2s\varphi(0,t)}|y_\Gamma'(t)|^2 dt \\ \leq & C\int_0^T \int_{\Omega_L} e^{-2s\varphi}|\partial_t y + \partial_x^2 y|^2 dxdt + C\int_0^T e^{-2s\varphi(0,t)}|y_\Gamma'(t) - \partial_x y(0,t)|^2 dt \\ & + Cs\int_0^T e^{-2s\varphi(-L_1,t)}(t(T-t))^{-\alpha}|\partial_x y(-L_1,t)|^2 dt, \end{aligned} \tag{1.3.8}$$

for all $s \geq s_1 \geq 1$ and for all (y, y_Γ) smooth enough, where $\varphi = \theta(t)\psi(x)$ and

$$\begin{aligned} \theta(t) =& (t(T-t))^{-\alpha}, \quad \forall t \in (0,T), \\ \psi(x) =& -\frac{1}{4L_1}x^2 + x + 2L_1, \quad \forall x \in \Omega_L. \end{aligned}$$

On the other hand, we prove that a similar problem to (1.3.5) appears as limit of a sequence of solutions for parabolic problems with discontinuous diffusion coefficients. In this context, functional setting of both problems play an important role.

In addition, according to the above result of convergence, questions arise naturally. One of them is: can we employ the limit control of the problem to drive the solutions of the approximate system too? Under suitable assumptions of the initial conditions, we prove that the last system is approximately controllable at any time $T > 0$.

These results are based on a joint work with Jéremi Dardé and Sylvain Ervedoza.

• **Controllability of a 1-D heat equation with discontinuous diffusion coefficients:** In Chapter 4, we study controllability properties of the following class of problems:

$$\begin{cases} \partial_t u(x,t) - \partial_x(\sigma(x)\partial_x u(x,t)) = 0, & \forall (x,t) \in \Omega \times (0,T), \\ u(x,0) = u^0(x), & \forall x \in \Omega, \\ u(-L_1,t) = v(t), & \forall t \in (0,T), \\ \partial_x u(L_2,t) = 0, & \forall t \in (0,T). \end{cases} \tag{1.3.9}$$

Here, $\Omega = (-L_1, L_2) \subset \mathbb{R}$ with $L_1, L_2 > 0$, $T > 0$ and σ is given by

$$\sigma(x) = \begin{cases} \sigma_1^2, & \forall x \in (-L_1, 0), \\ \sigma_2^2, & \forall x \in (0, L_2). \end{cases}$$

Moreover, the control $v = v(t)$ acts only in the left-hand side of the domain. Then, once again, in order to obtain the null controllability of such systems, we look for a Carleman estimate of the form

$$\begin{aligned} & s^3 \int_0^T \int_\Omega \rho |z|^2 dxdt + s \int_0^T \int_\Omega \mu |\partial_x z|^2 dxdt \\ \leq & C \int_0^T \int_\Omega \nu |\partial_t \pm \partial_x(\sigma \partial_x z)|^2 dxdt + Cs \int_0^T \mu(t, -L_1) |\partial_x z(t, -L_1)|^2 dt, \end{aligned}$$

for some positive functions ρ, μ and ν. In order to prove it, we use similar arguments based on [86] and [75] and suitable localization in time functions. In spite of microlocal techniques, this choice allows us to keep tracking of the observability constant.

Chapter 2

Potential reconstruction for a class of hyperbolic systems in cascade

In this chapter we present new results concerning the potential reconstruction of wave systems in cascade when some components of this ones (that is to say, some variables of the system) are not available to get partial measurements. We adapt the Bukhgeim-Klibanov method to the case of hyperbolic systems and we use Carleman estimates for the scalar wave equation to achieve a new Carleman estimates for a hyperbolic system in cascade with missing components. Let us mention that the main results of this chapter were published in [29] in collaboration with Nicolás Carreño and Axel Osses.

The outline of this chapter is as follows. In Section 2.1, we introduce the basic notation, the inverse problem that we will consider along this chapter. Additionally, we give a literature discussion about this subject and we state the main result obtained, i.e. the Theorem 2.1. In Section 2.2, we adapt the Carleman estimate for the scalar wave equation to deduce a new Carleman inequality to our problem (see Theorem 2.8). In Section 2.3, we modify the Bukhgeim-Klibanov's method to proof of the Theorem 2.1.

2.1 General Setting

In this section, we devote to introduce the main results about an inverse problem for a hyperbolic system in cascade. Let us start giving basic notations. Let Ω be a smooth open set in $\mathbb{R}^d$ with boundary $\partial\Omega$, $d \geq 1$ and $T > 0$.

Before going further, let us mention that the results available in this section could be formulated under weak smoothness assumptions of the boundary of Ω. Indeed, for instance we can take Ω be a bounded, connected and open subset of $\mathbb{R}^d$ with boundary of class C^4, but the goal of these hypothesis is to simplify the presentation.

Then, according to the notation previously introduced, let us consider the following coupled hyperbolic system in cascade:

$$\begin{cases} \Box u_1 + q_1 u_1 = a_1 u_2 + g_1, & \text{in } \Omega \times (0,T), \\ \Box u_2 + q_2 u_2 = a_2 u_3 + g_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box u_{n-1} + q_{n-1} u_{n-1} = a_{n-1} u_n + g_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box u_n + q_n u_n = g_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k u_j(0) = u_j^k,\ k = 0,1,\ j = 1,\dots,n, & \text{in } \Omega, \\ u_j = 0,\ j = 1,\dots,n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.1.1}$$

Here, $\Box := \partial_t^2 - \Delta$ is the D'Alembertian operator, a_j are non-zero constants, $u_j^k \in L^2(\Omega)$ are the initial conditions and $q_j \in L^2(\Omega)$ are the potentials and $g_j \in L^2(\Omega \times (0,T))$ are the source terms, for every $k = 0,1$ and $j = 1,\dots,n$.

Suppose, for instance, that $g_j \in L^1(0,T; L^2(\Omega))$, $u_j^0 \in H_0^1(\Omega)$ and $u_j^1 \in L^2(\Omega)$, $j = 1,\dots,n$. Then, according to the results presented in Section 1.1.1 on the well-posedness of the scalar wave equation, it is not difficult to deduce that the system (2.1.1) is well posed in the sense of Hadamard and moreover the normal derivative of each component u_j with $1 \le j \le n$ belongs in $L^2(\partial\Omega \times (0,T))$. We recall that in the case of source terms g_j lying in $W^{1,1}(0,T; H^{-1}(\Omega))$ with $1 \le j \le n$, we still have the same well-posedness result but we do not have any regularity result on the normal derivative on each component.

Let us point out that the hyperbolic and parabolic systems play an important role in mathematical models which come from biological, chemical, engineering, mechanical and medical applications. Nevertheless, some components of such models are not accessible in practice. Motivated for this kind of limitations, some natural questions arise: Can we observe such systems from incomplete measurements? Can we retrieve information of the inaccessible components of such systems from information of the accesible ones? These questions has been studied recently by several authors for different kind of PDE models, see for instance [1],[18] and [6] and the bibliographic discusion below.

In this chapter, we are interested in the following inverse problem associated to the system (2.1.1):

Inverse problem: Is it possible to retrieve the potentials $q_1,\dots,q_n$ in system (2.1.1) from incomplete data, that is to say, from a reduced number of measurements of the solution, saying (u_j), with $j \in \mathcal{I} \subset \{1,\dots,n\}$?

We point out that our goal is the study of dependence of the solutions $u_1,\dots,u_n$ with respect to the potentials $q_1,\dots,q_n$. Then, in order to understand this we shall write $u_j[Q]$ where $Q = (q_1,\dots,q_n)$ and $1 \le j \le n$. For simplicity, we ignore for instance the dependence of the solutions with respect to the initial conditions and source terms.

Concerning to the inverse problem previously stated, we can formulate questions in three directions: Let $Q = (q_1,\dots,q_n)$ and $\tilde{Q} = (\tilde{q}_1,\dots,\tilde{q}_n)$ be two sets of potentials for the system (2.1.1),

- **Uniqueness:** Suppose that the available measurements of the system coincide in a part of the domain $\omega \subset \Omega$ for two sets of potentials, i.e.,

$$u_j[Q] = u_j[\tilde{Q}], \quad \text{in } \omega \times (0,T),$$

for each $j \in \mathcal{I}$. Then, can we conclude that $Q = \tilde{Q}$ in Ω?

- **Stability:** is it possible to estimate

$$\|Q - \tilde{Q}\|_{(L^2(\Omega))^n} := \sum_{j=1}^{n} \|q_j - \tilde{q}_j\|_{L^2(\Omega)}$$

or better, a strong norm of $Q - \tilde{Q}$ by a suitable norm of $u_j[Q] - u_j[\tilde{Q}]$, $j \in \mathcal{I}$ in $\omega \times (0,T)$?

- **Reconstruction:** Can we find a formula or an algorithm to rebuild the potentials $Q = (q_1, \ldots, q_n)$ from the knowledge of $u_j[Q]$ with $j \in \mathcal{I}$?

In this chapter, we are interested in the stability of the potentials in terms of the observations of the solution of system (2.1.1). In particular, we restrict our attention in the case when the last component of the system is missing.

2.1.1 Literature review

Before to state the main results of this topic, let us briefly discuss the available literature about inverse problems of wave equation and systems and their relation with exact controllability.

In 1992, Bukhgeim and Klibanov dealt for the first time with uniqueness issues in inverse problems for the wave equation in [69] using local Carleman estimates. Then, the first results about the stability of inverse problems for hyperbolic equations were obtained using local Carleman estimates (see e.g.[97], [61] and [65]). Concerning other inverse problems for the wave equation with a single observation, we refer to [63], [62], [70], and [11] and the references therein. In these articles, the authors consider the case of interior or Dirichlet boundary data observation satisfying stronger geometric conditions and they use global Carleman estimates.

For an arbitrary set of observation, we refer to [14] and [15] for logarithmic stability results. Roughly speaking, these results are connected with stability results of elliptic thanks to the Fourier-Bros-Iagolnitzer (FBI) transform. Let us also mention the work [66] where the authors proved the uniqueness of the inverse problem of recovering a spatial component of the source term of the wave equation from the final observation data.

However, to the best of the author's knowledge, there exist few works concerning inverse problems for coupled parabolic or hyperbolic systems with incomplete measurements of their components. In the recent work [6], the authors study the reconstruction of the spatial distribution of external forces only from data of one component of a 2 coupled hyperbolic system in cascade. The proof is based on an observability property of such system, following the approach of [96].

Similar inverse problems for linear and semilinear parabolic systems like reaction-diffusion systems has been studied in [34], [18], [17], [35] and [33]. In these articles, the authors deal with identification and stability of the inverse problem of recovering parameters and initial conditions of such systems from a finite number of measurements of one component using appropriate Carleman estimates for parabolic equations.

Furthermore, hyperbolic-parabolic systems are considered in [51] with different kinds of observations. Another relevant work is [56] for the Stokes system, where the authors give a reconstruction algorithm for a source of the form $F(x,t) = f(x)\sigma(t)$ from incomplete velocity measurements.

Exact controllability properties of hyperbolic systems with a reduced number of controls has been extensively studied and there exist many works published on this topic. In [1], a strategy called Two-Level Energy method is developed to prove positive results in the case of wave-type systems (see also [2], [3], [7] and the references therein). Moreover these results allow to deduce null-controllability results for the heat or the Schrödinger equations satisfying the geometric control condition using the transmutation method.

Furthermore, the literature is also very rich concerning controllability results for coupled parabolic systems with a reduced numbers of controls in the one or multidimensional setting. We refer to the survey article [8], [45], and the references therein.

Coupled systems are also connected with insensitizing control problems, notion introduced by Lions in [79]. Indeed, these problems are equivalent to the null-controllability of a cascade system. We reference to [36], [90], and [4] for some results about this subject in the case of wave-type equations, [23], [28] and [37] in the case of parabolic equations and systems.

2.1.2 Main result

Now, we will state a Lipschitz stability result for system (2.1.1), from observations in all the components of the system except the last one. In order to state this result, we shall introduce some geometrical and time conditions which are classical in the context of control and inverse problems for hyperbolic equations. Specifically, let $x_0 \notin \overline{\Omega}$, $\Gamma_0 \subset \partial\Omega$ and $T > 0$.

- **Geometric condition:** x_0 and Γ_0 satisfy the following inclusion:

$$\{x \in \partial\Omega\,;\ (x - x_0)\cdot\nu(x) \geq 0\} \subset \Gamma_0 \subset \partial\Omega. \tag{2.1.2}$$

- **Time condition:** There exists $\beta \in (0,1)$ such that

$$\sup_{x\in\Omega} |x - x_0| < \sqrt{\beta}. \tag{2.1.3}$$

We emphasize that the Geometric condition given above is the same in 1.1.1. However, the Time condition is slightly different from the mentioned in the above chapter. This change is just for technical reasons to state our results in a simple way.

Now, let us introduce the admissible set of the unknown potentials. For a positive number m, we define the set

$$L^\infty_{\leq m}(\Omega) = \{p \in L^\infty(\Omega)\,;\ \|p\|_{L^\infty(\Omega)} \leq m\}.$$

Now we have all the ingredients to state the main result of this chapter:

Theorem 2.1 *Suppose that $\Gamma_0 \subset \partial\Omega$, $T > 0$ and $x_0 \notin \overline{\Omega}$ satisfy the geometric and time conditions* (2.1.2) *and* (2.1.3)*. Let $\omega \subset \Omega$ such that $\overline{\Gamma_0} \subset \partial\omega \cap \partial\Omega$. Let $(u_1, \dots, u_n)$ and $(\tilde{u}_1, \dots, \tilde{u}_n)$ be the solutions of the system* (2.1.1) *associated to the potentials $q_1, \dots, q_n \in L^\infty_{\leq m}(\Omega)$ and $\tilde{q}_1, \dots, \tilde{q}_n \in L^\infty_{\leq m}(\Omega)$, respectively, with $m > 0$. Assume that there exists a constant $c > 0$ such that*

$$\|\tilde{u}_j(0)\|^2_{L^2(\Omega)} \geq c, \quad \forall j = 1, 2, \dots, n. \tag{2.1.4}$$

Furthermore, suppose that

$$\begin{cases} u_j, \tilde{u}_j \in H^3(0,T; H^2(\Omega) \cap H^1_0(\Omega) \cap L^\infty(\Omega)), & j = 1, \dots, n, \\ u_{n-1}, \tilde{u}_{n-1} \in H^4(0,T; H^2(\Omega) \cap H^1_0(\Omega) \cap L^\infty(\Omega)). \end{cases}$$

Then, there exists a constant $C = C(\beta, c, T, \Omega, \omega)$ such that

$$\sum_{j=1}^{n} \|q_j - \tilde{q}_j\|^2_{L^2(\Omega)} \leq C \sum_{j=1}^{n-2} \|u_j - \tilde{u}_j\|^2_{H^3(0,T;L^2(\omega))} + C\|u_{n-1} - \tilde{u}_{n-1}\|^2_{H^4(0,T;L^2(\omega))}. \tag{2.1.5}$$

Remark 2.2 *Let us emphasize that inequality* (2.1.5) *establishes the Lipschitz stability of the hyperbolic system* (2.1.1) *with incomplete measurements in the sense that u_n is missing. Moreover, notice that the estimate* (2.1.5) *does not depend on the observations of the gradients.*

Remark 2.3 *Theorem 2.1 is also valid if we suppose that the coupling coefficients a_j are not constants satisfying*

$$a_j(x) \geq c > 0, \qquad \text{in } \omega',$$

where $\omega' \subset \Omega$ such that $\overline{\Gamma_0} \subset \partial\omega'$ and $\omega' \cap \omega \neq \emptyset$. In other words, the inequality (2.1.5) *holds if the coupled and the observations regions of each components meet.*

As we said before, the main tool of the proof of Theorem 2.1 is a Carleman estimate for a hyperbolic system in cascade where we do not have access to the observations associated to the last component. This inequality depends on a suitable Carleman estimate for the scalar wave equation in the spirit of the work of Imanuvilov and Yamamoto [62] (see also [12]).

2.2 Carleman estimates

The goal of this section is to prove a Carleman estimate for a system of wave equations in cascade. In order to do that, our starting point is a suitable Carleman estimate for the scalar wave equation. Nevertheless, before doing that, we will give some technical results.

2.2.1 Technical results

We start with the following

Lemma 2.4 *Let $z \in L^2(-T,T;H_0^1(\Omega))$ be a function such that $\Box z+pz \in L^2(\Omega\times(-T,T))$, $\partial_\nu z \in L^2(\partial\Omega\times(-T,T))$ and $z(\pm T)=0$ in Ω, with $p \in L^\infty(\Omega)$. Let $\gamma \in \mathbb{R}$. Let $\omega_1,\omega_2 \subset \Omega$ be two open sets such that $\overline{\omega_1} \subset \omega_2$.*

a) If $\tilde{\varphi} \in C^1([-T,T]\times\overline{\Omega})$, then there exists a constant $C>0$ such that

$$\begin{aligned}\int_{-T}^{T}\int_{\omega_1} e^{2s\tilde{\varphi}}|\nabla z|^2 dxdt \leq& C s^{\max\{2,\gamma\}}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|z|^2 dxdt + C\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\partial_t z|^2 dxdt\\ &+ Cs^{-\gamma}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\Box z+pz|^2 dxdt,\end{aligned} \tag{2.2.1}$$

for all $s \geq 1$.

b) If the function $\tilde{\varphi} \in C^1([-T,T];C^2(\overline{\Omega}))$ satisfies

$$\inf_{x\in\overline{\Omega}} |\nabla\tilde{\varphi}(t)| \geq c_0 > 0, \quad \forall t \in [-T,T],$$

then, there exist two positive constants C and s_0 independent of s such that

$$\begin{aligned}&s^2\int_{-T}^{T}\int_{\omega_1} e^{2s\tilde{\varphi}}|z|^2 dxdt + \int_{-T}^{T}\int_{\omega_1} e^{2s\tilde{\varphi}}|\partial_t z|^2 dxdt\\ \leq& C s^{\max\{0,\gamma-2\}}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\nabla z|^2 dxdt + Cs^{-\gamma}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\Box z+pz|^2 dxdt,\end{aligned} \tag{2.2.2}$$

for all $s \geq s_0$.

Remark 2.5 *The principal significance of part a) on Lemma 2.4 is that it allows to drop the local term of the gradient. This fact plays an important role in some steps of the proof of the Carleman estimate for the wave system in cascade in Subsection 2.2.*

Proof of Lemma 2.4. Let us consider a function $\xi \in C^\infty(\Omega,\mathbb{R})$ such that

$$\begin{cases} 0 \leq \xi \leq 1, & \text{in } \Omega,\\ \xi \equiv 1, & \text{in } \omega_1,\\ \xi \equiv 0, & \text{in } \Omega\setminus\overline{\omega_2}.\end{cases}$$

Additionally, we suppose that ξ has the form $\xi = e^{\phi}$ in $\omega_2 \setminus \overline{\omega_1}$, for some smooth function ϕ. We have the following identity:

$$\begin{aligned}&\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\partial_t^2 z dxdt - \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\Delta z dxdt + \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi p|z|^2 dxdt\\ =&\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z(\Box z+pz) dxdt.\end{aligned} \tag{2.2.3}$$

Integration by parts yields

$$\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\partial_t^2 z dxdt = -2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\partial_t\tilde{\varphi}\xi z\partial_t z dxdt - \int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\partial_t z|^2 dxdt, \tag{2.2.4}$$

and

$$
\begin{aligned}
-\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\Delta z dxdt =& 2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\nabla\tilde{\varphi}\cdot\nabla z dxdt + \int_{-T}^{T}\int_{\omega_2} e^{2s\varphi} z\nabla\xi\cdot\nabla z dxdt \\
&+ \int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\nabla z|^2 dxdt.
\end{aligned}
\tag{2.2.5}
$$

Substituting (2.2.4) and (2.2.5) into (2.2.3), we have

$$
\begin{aligned}
&\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\nabla z|^2 dxdt \\
=&2s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t\tilde{\varphi} z\partial_t z dxdt + \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi|\partial_t z|^2 dxdt + \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi p|z|^2 dxdt \\
&+ \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z(\Box z + pz) dxdt - 2s\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi z\nabla\tilde{\varphi}\cdot\nabla z dxdt \\
&- \int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}} z\nabla\xi\cdot\nabla z dxdt = J_1 + J_2.
\end{aligned}
\tag{2.2.6}
$$

Here, J_1 is the sum of the first four terms of (2.2.6) and J_2 is the sum of the fifth and sixth terms of the same equation. Straightforward computations show that

$$
\begin{aligned}
|J_1| \leq& 2\|\xi\|_{C^0(\overline{\omega_2})}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\partial_t z|^2 dxdt + \frac{1}{3}s^{-\gamma}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|\Box z + pz|^2 dxdt \\
&+ \Lambda\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|z|^2 dxdt,
\end{aligned}
\tag{2.2.7}
$$

where Λ is defined by

$$
\Lambda = \left(\|\partial_t\tilde{\varphi}\|^2_{C^0(\overline{(\omega_2\times(-T,T))})}s^2 + \frac{3}{4}\|\xi\|_{C^0(\overline{\omega_2})}s^{\gamma} + \|p\|_{L^\infty(\omega_2)}\right)\|\xi\|_{C^0(\overline{\omega_2})}
$$

and

$$
\begin{aligned}
|J_2| \leq& \left(3\|\nabla\tilde{\varphi}\|^2_{L^\infty(\omega_2\times(-T,T))}s^2 + \frac{3}{4}\|\nabla\phi\|_{L^\infty(\omega_2\setminus\overline{\omega_1})}\right)\|\xi\|_{L^\infty(\omega_2)}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}|z|^2 dxdt \\
&+ \frac{2}{3}\int_{-T}^{T}\int_{\omega_2} e^{2s\tilde{\varphi}}\xi|\nabla z|^2 dxdt
\end{aligned}
\tag{2.2.8}
$$

Combining (2.2.7), (2.2.8) with (2.2.6) we obtain (2.2.1), which completes the part a) of Lemma 2.4. The rest of the proof runs as before but additionally we have to estimate the local term $|z|^2$ by using the weighted Poincaré inequality (see [11], Lemma 2.4). $\square$

Now, we introduce the classical Carleman weights for the scalar wave equation. Suppose that Γ_0, x_0 and $T > 0$ satisfy the Geometric and Time condition (2.1.2) and (2.1.3). Let $\beta \in (0, T)$. For $(x, t) \in \Omega \times (-T, T)$, we define the following functions:

$$
\psi(x,t) = |x - x_0|^2 - \beta t^2 + C_0, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}, \tag{2.2.9}
$$

where $\lambda > 0$ and $C_0 > 0$ is chosen such that $\psi \geq 0$ (and therefore $\varphi \geq 1$) in $\Omega \times (-T, T)$.

For brevity, we shall use the following notation

$$
\begin{aligned}
I(\alpha, v, \Omega) =& s^\alpha \int_\Omega e^{2s\varphi(0)} \big(s^2|v(0)|^2 + |\partial_t v(0)|^2 + |\nabla v(0)|^2\big) dx \\
&+ s^{\alpha+1} \int_{-T}^{T} \int_\Omega e^{2s\varphi} \big(s^2|v|^2 + |\partial_t v|^2 + |\nabla v|^2\big) dx dt
\end{aligned}
$$

In the remainder of this section, C denotes a generic positive constant which depends at least on Γ_0, T and x_0 and may change from line to line.

Proposition 2.6 *Assume that Γ_0, T and x_0 satisfy the Geometric condition and Time condition* (2.1.2) *and* (2.1.3) *and let $p \in L^\infty_{\leq m}(\Omega)$ with $m > 0$. Let us consider the Carleman weight functions defined in* (2.2.9). *Let $\omega_0 \subset \Omega$ be an open subset such that $\overline{\Gamma_0} \subset \partial\omega_0 \cap \partial\Omega$. Then, there exist two positive constants $C_1 = C_1(\Gamma_0, T, x_0, \omega_2)$ and $s_0 = s_0(\Gamma_0, T, x_0, \omega_2)$ independent of s such that for all $s \geq s_0$, we have*

$$
I(0, v, \Omega) \leq C_1 \int_{-T}^{T} \int_\Omega e^{2s\varphi} |\Box v + pv|^2 dx dt + C_1 s \int_{-T}^{T} \int_{\omega_0} e^{2s\varphi} \big(s^2|v|^2 + |\partial_t v|^2\big) dx dt, \tag{2.2.10}
$$

for all $v \in L^2(-T, T; H^1_0(\Omega))$ such that $\Box v + pv \in L^2(\Omega \times (-T, T))$, $\partial_\nu v \in L^2(\partial\Omega \times (-T, T))$ and $v(\pm T) = \partial_t v(\pm T) = 0$ in Ω.

Remark 2.7 *In contrast to the Theorem 2.5 in [11] in the case of the Carleman estimate of the scalar wave equation with a single boundary observation, we emphasize that Proposition 2.6 requires the assumptions $z(\pm T) = \partial_t z(\pm T) = 0$ in Ω. This point becomes important if we want to eliminate more components in the inequality* (2.1.5) *of Theorem 2.1.*

Let us point out that the proof of the Proposition 2.6 is straightforward and many of the ingredients of the proof are already available in the literature (see for instance [62] and [12]). Nevertheless, for our purposes, it is convenient to write the Carleman estimate for wave equation under the form of Proposition 2.6. For the sake of completeness, we will give the proof of this result.

Proof of Proposition 2.4. For $s \geq 1$, let us define

$$
E_s(t) = \frac{1}{2} \int_\Omega e^{2s\varphi(t)} \big(|\partial_t v(t)|^2 + |\nabla v(t)|^2\big) dx, \quad \forall t \in (-T, T).
$$

Differentiation with respect to t and integration by parts in space yields

$$
\begin{aligned}
\frac{dE_s}{dt}(t) =& s \int_\Omega e^{2s\varphi(t)} \partial_t \varphi(t) \left(|\partial_t v(t)|^2 + |\nabla v(t)|^2\right) dx + \int_\Omega e^{2s\varphi(t)} \partial_t v(t) \Box v(t) dx \\
&- 2s \int_\Omega e^{2s\varphi(t)} \partial_t v(t) \nabla v(t) \cdot \nabla \varphi(t) dx, \quad \forall t \in (-T, T).
\end{aligned}
$$

After integration on $(-T,0)$ in time we obtain

$$\begin{aligned}E_s(0) =& s\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}\partial_t\varphi\left(|\partial_t v|^2+|\nabla v|^2\right)dxdt + \int_{-T}^{0}\int_{\Omega} e^{2s\varphi}\partial_t v\Box v dxdt\\ &- 2s\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}\partial_t v\nabla v\cdot\nabla\varphi dxdt,\end{aligned}$$

where we have used $v(-T)=\partial_t v(-T)=0$ in Ω. Applying Young's inequality and the weighted Poincaré inequality to v (see [11], Lemma 2.4) we obtain

$$\begin{aligned}&\int_{\Omega} e^{2s\varphi(0)}\big(s^2|v(0)|^2+|\partial_t v(0)|^2+|\nabla v(0)|^2\big)dx\\ \le& Cs\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(s^2|v|^2+|\partial_t v|^2+|\nabla v|^2\big)dxdt + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\Box v+pv|^2 dxdt, \quad \forall s\ge s_0.\end{aligned} \tag{2.2.11}$$

On the other hand, let us recall the classical Carleman estimate for the wave equation with $\lambda=\lambda_0$ fixed applied to v:

$$\begin{aligned}&s\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(s^2|v|^2+|\partial_t v|^2+|\nabla v|^2)dxdt\\ \le& C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\Box v+pv|^2dxdt + Cs\int_{-T}^{T}\int_{\Gamma_0} e^{s\varphi}|\partial_\nu v|^2 d\sigma dt.\end{aligned} \tag{2.2.12}$$

Let us consider an open subset $\omega_0'\subset\omega_0$ such that $\overline{\omega_0'}\subset\omega$ and $\partial\omega_0'\cap\partial\Omega\subset\partial\omega_0\cap\partial\Omega$. Consider the function $\eta\in C^\infty(\overline{\Omega},\mathbb{R})$ satisfying

$$\begin{cases}0\le\eta\le 1, & \text{in } \Omega,\\ \eta\equiv 1, & \text{in } \Omega\setminus\overline{\omega_0'},\\ \eta=\partial_\nu\eta\equiv 0, & \text{on } \Gamma_0.\end{cases}$$

Replacing v by ηv in (2.2.12), we have

$$\begin{aligned}&s\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(s^2|v|^2+|\partial_t v|^2+|\nabla v|^2\big)dxdt\\ \le& C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\Box v+pv|^2dxdt + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(|v|^2+|\nabla v|^2\big)dxdt\\ &+ s\int_{-T}^{T}\int_{\omega_0'} e^{2s\varphi}\big(s^2|v|^2+|\partial_t v|^2+|\nabla v|^2\big)dxdt,\end{aligned} \tag{2.2.13}$$

where we have used that $\Box(\eta v)=\eta\Box v-\Delta\eta v-2\nabla\eta\cdot\nabla v$ in $\Omega\times(-T,T)$ and $\nabla\eta\equiv 0$ in $\Omega\setminus\overline{\omega_1}$. Notice that the second term of the right-hand side of (2.2.13) can be absorbed taking s large enough. Finally, combining the previous estimate obtained with (2.2.11) and applying the estimate (2.2.1) with $\tilde{\varphi}=\varphi$, $\omega_1=\omega_0'$, $\omega_2=\omega_0$ and $\gamma=1$, the proof of (2.2.10) is complete. $\square$

2.2.2 A new Carleman estimate for a hyperbolic system

The aim of this section is to prove a Carleman estimate for a wave-type system with potentials. In order to formulate our result, let us consider the following system:

$$\begin{cases} \Box v_1 + r_1 v_1 = v_2 + h_1, & \text{in } \Omega \times (-T,T), \\ \Box v_2 + r_2 v_2 = v_3 + h_2, & \text{in } \Omega \times (-T,T), \\ \qquad \vdots & \qquad \vdots \\ \Box v_{n-1} + r_{n-1} v_{n-1} = v_n + h_{n-1}, & \text{in } \Omega \times (-T,T), \\ \Box v_n + r_n v_n = h_n, & \text{in } \Omega \times (-T,T), \\ \partial_t^k v_j(\pm T) = 0, \ k = 0,1, \ j = 1,\dots,n, & \text{in } \Omega, \\ v_j = 0, \ j = 1,\dots,n, & \text{on } \partial\Omega \times (-T,T). \end{cases} \tag{2.2.14}$$

Here, $r_j \in L^\infty(\Omega)$ are the potentials and $h_j \in L^2(\Omega \times (-T,T))$ are the source terms, for each $j = 1,\dots,n$.

Now, we are in position to state the Carleman estimate for system (2.2.14), which is one of the main results of this article:

Theorem 2.8 *Let us consider the Carleman weights defined in* (2.2.9)*, where* $\Gamma_0 \subset \partial\Omega$*,* $T > 0$ *and* $x_0 \notin \overline{\Omega}$ *satisfy the geometric and time conditions* (2.1.2) *and* (2.1.3)*. For* $m > 0$*, suppose that* $r_j \in L^\infty_{\leq m}(\Omega)$ *,* $j = 1,\dots,n$*, and let* $\omega \subset \Omega$ *be an open set such that* $\overline{\Gamma_0} \subset \partial\omega \cap \partial\Omega$*. In addition, consider* $h_j \in L^2(\Omega \times (-T,T))$ *for each* $j = 1,\dots,n-2$ *and* $h_{n-1}, h_n \in H^1(-T,T; L^2(\Omega))$ *such that*

$$\begin{cases} v_j \in H^1\left(-T,T; H^2(\Omega) \cap H^1_0(\Omega)\right), \ j = 1,\dots,n, \\ v_{n-1} \in H^2(-T,T; H^2(\Omega) \cap H^1_0(\Omega)). \end{cases}$$

Furthermore, we choose $1 < \alpha < 2$*. Then, there exist two positive constants* C_2 *and* s_0 *which depends at least on* $\Gamma_0, \Omega, \omega, T, x_0) > 0$ *such that for all* $s \geq s_0$*, the solution* $(v_1,\dots,v_n)$ *of system* (2.2.14) *satisfies*

$$\begin{aligned} &\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\ \leq & C_2 s^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\omega} e^{2s\varphi} \left(s^2 |v_j|^2 + |\partial_t v_j|^2\right) dxdt + C_2 s^\alpha \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |h_j|^2 dxdt \\ & + C_2 s^3 \int_{-T}^{T} \int_{\omega} e^{2s\varphi} \left(s^5 |v_{n-1}|^2 + s^3 |\partial_t v_{n-1}|^2 + |\partial_t^2 v_{n-1}|^2\right) dxdt \\ & + C_2 \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(s^3 |h_{n-1}|^2 + |h_n|^2 + s|\partial_t h_{n-1}|^2 + |\partial_t h_n|^2\right) dxdt. \end{aligned} \tag{2.2.15}$$

Remark 2.9 *We emphasize that the Carleman estimate* (2.2.15) *depends only on* h_n *and* $\partial_t h_n$ *in the last component.*

Proof of Theorem 2.8. Let ω_1 and ω_2 be two subsets of ω be two open sets such that $\overline{\Gamma_0} \subset \partial\omega_j \cap \partial\Omega$ for each $j = 1,2$ and $\overline{\omega_1} \subset \omega_2$ and $\overline{\omega_2} \subset \omega$.

We start applying the Carleman inequality of Proposition 2.6 to $v_1, \ldots, v_n$ in system (2.2.14) with $\omega_0 = \omega_1$. We have:

$$\begin{aligned}
&\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\
\leq & Cs^\alpha \sum_{j=2}^{n} \int_{-T}^{T}\int_\Omega e^{2s\varphi}|v_j|^2 dxdt + Cs^\alpha \sum_{j=1}^{n-1} \int_{-T}^{T}\int_\Omega e^{2s\varphi}|h_j|^2 dxdt + C\int_{-T}^{T}\int_\Omega e^{2s\varphi}|h_n|^2 dxdt \\
& + Cs^{\alpha+1}\sum_{j=1}^{n-1}\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}(s^2|v_j|^2 + |\partial_t v_j|^2)dxdt + Cs\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}(s^2|v_n|^2 + |\partial_t v_n|^2)dxdt.
\end{aligned}$$

Note that the first term of the right-hand side of the inequality above can be absorbed by taking s large enough since $1 < \alpha < 2$. Therefore, we can rewrite this inequality as follows:

$$\begin{aligned}
&\sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\
\leq & Cs^\alpha \sum_{j=1}^{n-1} \int_{-T}^{T}\int_\Omega e^{2s\varphi}|h_j|^2 dxdt + C\int_{-T}^{T}\int_\Omega e^{2s\varphi}|h_n|^2 dxdt \\
& + Cs^{\alpha+1}\sum_{j=1}^{n-1}\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\left(s^2|v_j|^2 + |\partial_t v_j|^2\right)dxdt \\
& + Cs\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\left(s^2|v_n|^2 + |\partial_t v_n|^2\right)dxdt.
\end{aligned} \tag{2.2.16}$$

Now we are going to estimate the local term of v_n and $\partial_t v_n$ in (2.2.16). To do this, we consider a cut-off function $\xi \in C^\infty(\Omega, \mathbb{R})$ such that

$$\begin{cases} 0 \leq \xi \leq 1 & \text{in } \Omega, \\ \xi \equiv 1 & \text{in } \omega_1, \\ \xi \equiv 0 & \text{in } \Omega \setminus \overline{\omega_2}. \end{cases}$$

Using the equation of v_{n-1} in (2.2.14), we see that:

$$\begin{aligned}
s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|v_n|^2 dxdt =& s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi v_n\left(\Box v_{n-1} + r_{n-1}v_{n-1}\right)dxdt \\
& - s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi v_n h_{n-1} dxdt.
\end{aligned} \tag{2.2.17}$$

Let us estimate each term of the equation above. First, by Young's inequality for every $\delta > 0$, there exists a constant $C = C(\delta)$ such that

$$s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi h_{n-1}v_n dxdt \leq \delta s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|v_n|^2 dxdt + Cs^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|h_{n-1}|^2 dxdt, \tag{2.2.18}$$

On the other hand, integration by parts yields

$$\begin{aligned}
&s^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t^2 v_{n-1}v_n dxdt\\
=&s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi(4s^2|\partial_t\varphi|^2+2s\partial_t^2\varphi)v_{n-1}v_n dxdt+4s^4\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t\varphi v_{n-1}\partial_t v_n dxdt\\
&+s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi v_{n-1}\partial_t^2 v_n dxdt,
\end{aligned} \tag{2.2.19}$$

and

$$\begin{aligned}
&-s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\Delta v_{n-1}v_n dxdt\\
=&2s^4\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\nabla\varphi\,(v_n\nabla v_{n-1}-v_{n-1}\nabla v_n)dxdt-s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi v_{n-1}\Delta v_n dxdt\\
&+s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\nabla\xi\,(v_n\nabla v_{n-1}-v_{n-1}\nabla v_n)dxdt.
\end{aligned} \tag{2.2.20}$$

By (2.2.19), (2.2.20) and $\Box v_n+r_nv_n=h_n$, we have

$$\begin{aligned}
&s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi v_n\,(\Box v_{n-1}+r_{n-1}v_{n-1})dxdt\\
\le&C\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\big(s^3|h_{n-1}|^2+|h_n|^2\big)dxdt\\
&+Cs^5\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\,\big(s^2|v_{n-1}|^2+|\partial_t v_{n-1}|^2+|\nabla v_{n-1}|^2\big)dxdt+\delta I(0,v_n,\omega_2),
\end{aligned} \tag{2.2.21}$$

for every $\delta>0$, where we have used the Young inequality. Moreover, by part a) of Lemma 2.4 applied to v_{n-1}, ω_2 and ω with $\gamma=3$ one has

$$\begin{aligned}
s^5\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\nabla v_{n-1}|^2dxdt\le&Cs^2\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(|v_n|^2+|h_{n-1}|^2\big)dxdt\\
&+Cs^5\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^3|v_{n-1}|^2+|\partial_t v_{n-1}|^2\big)dxdt.
\end{aligned} \tag{2.2.22}$$

Substituting (2.2.22) into (2.2.21) and substituting the obtained estimate into (2.2.18), we conclude that

$$\begin{aligned}
&s^3\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}|v_n|^2dxdt\\
\le&C\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^3|h_{n-1}|^2+|h_n|^2\big)dxdt\\
&+Cs^8\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|v_{n-1}|^2dxdt+Cs^5\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|\partial_t v_{n-1}|^2dxdt+\delta I(0,v_n,\omega),
\end{aligned} \tag{2.2.23}$$

for every $\delta > 0$, where we have included the integral term of $|v_n|^2$ which has in front s^2 in $\delta I(0, v_n, \omega)$ by taking s large enough. In the same manner, we can estimate the local term of $\partial_t v_n$. In fact, let us consider the function ξ defined above. Then,

$$\begin{aligned} &s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\partial_t v_n|^2 dxdt \\ =&-s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n\partial_t h_{n-1}dxdt + s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n(\Box\partial_t v_{n-1} + r_{n-1}\partial_t v_{n-1})dxdt. \end{aligned} \tag{2.2.24}$$

First, notice that for all $\delta > 0$, there is a positive constant $C = C(\delta)$ such that

$$\begin{aligned} -s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n\partial_t h_{n-1}dxdt \leq &\delta s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\partial_t v_n|^2 dxdt \\ &+ C(\delta)s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi|\partial_t h_{n-1}|^2 dxdt. \end{aligned} \tag{2.2.25}$$

On the other hand,

$$\begin{aligned} &s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n(\Box\partial_t v_{n-1} + r_{n-1}\partial_t v_{n-1})dxdt \\ =&s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n(\partial_t^3 v_{n-1} - \Delta\partial_t v_{n-1} + r_{n-1}\partial_t v_{n-1})dxdt \\ =&J_1 + J_2 + J_3. \end{aligned} \tag{2.2.26}$$

Then, we compute the terms J_k, with $k = 1, 2, 3$. We start with J_1. Integration by parts yields

$$J_1 = -2s^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t\varphi\partial_t v_n\partial_t^2 v_{n-1}dxdt - s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t^2 v_n\partial_t^2 v_{n-1}dxdt. \tag{2.2.27}$$

The last term can be estimated by using integration by parts again. After straightforward computations, we get

$$\begin{aligned} &-s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t^2 v_n\partial_t^2 v_{n-1}dxdt \\ =&-2s^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi(2s|\partial_t\varphi|^2 + \partial_t^2\varphi)\partial_t v_{n-1}\partial_t v_n dxdt + s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\xi\partial_t^3 v_n\partial_t v_{n-1}dxdt. \end{aligned} \tag{2.2.28}$$

Substituting (2.2.28) into (2.2.27) we deduce that

$$\begin{aligned} J_1 =&-4s^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t\varphi\partial_t v_n\partial_t^2 v_{n-1}dxdt \\ &-2s^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi(2s|\partial_t\varphi|^2 + \partial_t^2\varphi)\partial_t v_{n-1}\partial_t v_n dxdt + s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t^3 v_n\partial_t v_{n-1}dxdt. \end{aligned} \tag{2.2.29}$$

Now, we estimate J_2. To do this, we proceed integrating by parts in space:

$$\begin{aligned} J_2 =& s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}(2s\xi\partial_t v_n\nabla\varphi + \partial_t v_n\nabla\xi)\nabla\partial_t v_{n-1}dxdt \\ &+ s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\nabla\partial_t v_{n-1}\cdot\nabla\partial_t v_n dxdt. \end{aligned} \tag{2.2.30}$$

Moreover, the second term of the right-hand side of (2.2.30) can be estimated as follows: first we integrate by parts in space to get

$$\begin{aligned} & s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\nabla\partial_t v_{n-1}\cdot\nabla\partial_t v_n dxdt \\ =& - s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}(2s\xi\nabla\varphi + \nabla\xi)\nabla\partial_t v_n\partial_t v_{n-1}dxdt \\ & - s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\Delta\partial_t v_n\partial_t v_{n-1}dxdt. \end{aligned}$$

However, notice that the term $\nabla\partial_t v_n$ cannot be absorbed by using the classical Carleman approach. To solve this, we integrate by parts in time the first term of the right-hand side of the above equation and therefore we obtain

$$\begin{aligned} & s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\nabla\partial_t v_{n-1}\cdot\nabla\partial_t v_n dxdt \\ =& s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\partial_t\left(e^{2s\varphi}(2s\xi\nabla\varphi + \nabla\xi)\right)\nabla v_n\partial_t v_{n-1}dxdt \\ & + s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}[(2s\xi\nabla\varphi + \nabla\xi)\cdot\nabla v_{n-1}]\,\partial_t^2 v_{n-1}dxdt \\ & - s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\Delta\partial_t v_n\partial_t v_{n-1}dxdt. \end{aligned} \tag{2.2.31}$$

Substituting (2.2.31) into (2.2.30) we get the following estimate for J_2:

$$\begin{aligned} J_2 =& s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}(2s\xi\partial_t v_n\nabla\varphi + \partial_t v_n\nabla\xi)\cdot\nabla\partial_t v_{n-1}dxdt \\ & + s\int_{-T}^{T}\int_{\omega_2} \partial_t\left[e^{2s\varphi}(2s\xi\nabla\varphi + \nabla\xi)\right]\cdot\nabla v_n\partial_t v_{n-1}dxdt \\ & + s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}(2s\xi\nabla\varphi + \nabla\xi)\nabla v_{n-1}\partial_t^2 v_{n-1}dxdt \\ & - s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\Delta\partial_t v_n\partial_t v_{n-1}dxdt. \end{aligned} \tag{2.2.32}$$

On the other hand, J_3 can be written as follows:

$$
\begin{aligned}
J_3 =& s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\xi r_{n-1}\partial_t v_{n-1}\partial_t v_n dxdt\\
=& s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\xi r_{n}\partial_t v_{n-1}\partial_t v_n dxdt + s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\xi (r_{n-1}-r_n)\partial_t v_{n-1}\partial_t v_n dxdt.
\end{aligned}
\tag{2.2.33}
$$

Thus, substituting (2.2.29) (2.2.32) and (2.2.33) into (2.2.26) and using Young's inequality and the fact that $\partial_t^3 v_n - \Delta\partial_t v_n + r_n\partial_t v_n = \partial_t h_n$ we deduce that for all $\delta > 0$, there exists a positive constant $C = C(\delta)$ such that

$$
\begin{aligned}
& s\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\xi\partial_t v_n(\Box\partial_t v_{n-1} + r_{n-1}\partial_t v_{n-1})dxdt\\
=& C(\delta)\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\big(s^{2+\gamma^*}|\partial_t v_{n-1}|^2 + |\nabla v_{n-1}|^2 + |\partial_t^2 v_{n-1}|^2\big)dxdt\\
& + C(\delta)s^{-\gamma^*}\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\partial_t h_n|^2 dxdt + \delta s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}|\partial_t v_n|^2 + |\nabla v_n|^2 dxdt.
\end{aligned}
\tag{2.2.34}
$$

for every $\gamma^* > 0$. Furthermore, the local term ∇v_n in $\omega_1\times(-T,T)$ can be estimated by using the technical lemmas introduced in the above section and the weighted Poincaré inequality as follows:

$$
\begin{aligned}
& s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\nabla v_{n-1}|^2 dxdt\\
\le& Cs^3\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(s^2|v_{n-1}|^2 + |\partial_t v_{n-1}|^2)dxdt\\
& + Cs^2\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(|v_n|^2 + |h_{n-1}|^2)dxdt.
\end{aligned}
\tag{2.2.35}
$$

Substituting (2.2.35) into (2.2.34) we have

$$
\begin{aligned}
& s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}\xi\partial_t v_n(\Box\partial_t v_{n-1} + r_{n-1}\partial_t v_{n-1})dxdt\\
\le& C(\delta)s^3\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}\big(s^{2+\gamma^*}|\partial_t v_{n-1}|^2 + |\partial_t^2 v_{n-1}|^2\big)dxdt\\
& + C(\delta)s^{-\gamma^*}\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|\partial_t h_{n-1}|^2 dxdt + \delta\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(|\partial_t v_n|^2 + |\nabla v_n|^2)dxdt\\
& + Cs^2\int_{-T}^{T}\int_{\omega} e^{2s\varphi}|h_{n-1}|^2 dxdt + Cs^2\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|v_n|^2 dxdt.
\end{aligned}
\tag{2.2.36}
$$

for each $\delta > 0$ and $\gamma^* > 0$. Finally substituting (2.2.25) (2.2.36) into (2.2.24) and using $\xi = 1$ in ω_0 we have

$$
\begin{aligned}
& s\int_{-T}^{T}\int_{\omega_1} e^{2s\varphi}|\partial_t v_n|^2 dxdt \\
\leq & C(\delta)s^3\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(s^2|v_{n-1}|^2+s^{2+\gamma^*}|\partial_t v_{n-1}|^2|\partial_t^2 v_{n-1}|^2)dxdt \\
& + C(\delta)\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(s^2|h_{n-1}|^2+s|\partial_t h_{n-1}|^2+s^{-\gamma^*}|\partial_t h_n|^2)dxdt \\
& + \delta s\int_{-T}^{T}\int_{\omega} e^{2s\varphi}(dxdt|\partial_t v_n|^2+|\nabla v_n|^2).
\end{aligned}
\tag{2.2.37}
$$

Finally, by substituting (2.2.23) and (2.2.37) into (2.2.16), by taking the Carleman parameter $s \geq 1$ large enough and by choosing $\delta > 0$ sufficiently small, we obtain

$$
\begin{aligned}
& \sum_{j=1}^{n-1} I(\alpha, v_j, \Omega) + I(0, v_n, \Omega) \\
\leq & Cs^\alpha \sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|h_j|^2 dxdt + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(s^3|h_{n-1}|^2+|h_n|^2\big)dxdt \\
& + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\big(s|\partial_t h_{n-1}|^2+|\partial_t h_n|^2\big)dxdt \\
& + Cs^{\alpha+1}\sum_{j=1}^{n-2}\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^2|v_j|^2+|\partial_t v_j|^2\big)dxdt \\
& + Cs^3\int_{-T}^{T}\int_{\omega} e^{2s\varphi}\big(s^5|v_{n-1}|^2+s^3|\partial_t v_{n-1}|^2+|\partial_t^2 v_{n-1}|^2\big)dxdt,
\end{aligned}
$$

which completes the proof of Theorem 2.8. □

2.3 Proof of Theorem 2.1

The plan of the proof of Theorem 2.1 contains three parts:

Step 1 In the same spirit of the Bukhgeim-Klibanov method, we rewrite appropriately system (2.1.1) to apply the estimate (2.2.15) in Theorem 2.8.

Step 2 After applying the Carleman estimate of Theorem 2.8 to the new system, we estimate the residual and source terms.

Step 3 We conclude the proof gathering the estimates of the previous steps and eliminating the small order terms.

- **Step 1: Setting**

For each $j = 1, \dots, n$, let us denote by $y_j = u_j - \tilde{u}_j$, $p_j = q_j$, $f_j = q_j - \tilde{q}_j$ and $R_j = \tilde{u}_j$.

Then, following this notation, $y_1, \dots, y_n$ solves:

$$\begin{cases} \Box y_1 + p_1 y_1 = y_2 + f_1 R_1, & \text{in } \Omega \times (0,T), \\ \Box y_2 + p_2 y_2 = y_2 + f_2 R_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box y_{n-1} + p_{n-1} y_{n-1} = y_n + f_{n-1} R_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box y_n + p_n y_n = f_n R_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k y(0) = 0, \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ y_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.3.1}$$

For each $j = 1, \dots, n$, we set $w_j = \partial_t^2 y_j$. Then, the new variables solve the following system:

$$\begin{cases} \Box w_1 + p_1 w_1 = w_2 + f_1 \partial_t^2 R_1, & \text{in } \Omega \times (0,T), \\ \Box w_2 + p_2 w_2 = w_3 + f_2 \partial_t^2 R_2, & \text{in } \Omega \times (0,T), \\ \quad \vdots & \quad \vdots \\ \Box w_{n-1} + p_{n-1} w_{n-1} = w_n + f_{n-1} \partial_t^2 R_{n-1}, & \text{in } \Omega \times (0,T), \\ \Box w_n + p_n w_n = f_n \partial_t^2 R_n, & \text{in } \Omega \times (0,T), \\ \partial_t^k w_j(0) = f_j \partial_t^k R_j(0), \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ w_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.3.2}$$

Now, we want to apply Theorem 2.8 to a suitable system. In order to do that, we extend system (2.3.2) in an even way, setting $w_j(x,t) = w_j(x,-t)$ for all $(x,t) \in \Omega \times (-T,0)$. We also extend the functions R_j, $\partial_t R_j$ and $\partial_t^2 R_j$ in an even way and keep the same notations for the new system.

To be able to apply the Carleman estimate (2.2.15), the functions w_j must satisfy $\partial_t^k w(\pm T) = 0$ in Ω, for $k = 0,1$. However, this condition does not hold. To avoid this difficulty, we consider a cut-off function $\theta \in C_c^\infty((-T,T),\mathbb{R})$ defined as follows:

$$\begin{cases} 0 \le \theta \le 1, & \text{in } (-T,T), \\ \theta \equiv 1, & \text{in } (-T+\tau, T-\tau). \end{cases}$$

According to the definition of θ, it is clear that $z_j = \theta w_j$, for $j = 1, \dots, n$, solves

$$\begin{cases} \Box z_1 + p_1 z_1 = z_2 + F_1, & \text{in } \Omega \times (-T,T), \\ \Box z_2 + p_2 z_2 = z_3 + F_2, & \text{in } \Omega \times (-T,T), \\ \quad \vdots & \quad \vdots \\ \Box z_{n-1} + p_{n-1} z_{n-1} = z_n + F_{n-1}, & \text{in } \Omega \times (-T,T), \\ \Box z_n + p_n z_n = F_n, & \text{in } \Omega \times (-T,T), \\ \partial_t^k z(0) = f_j \partial_t^k R(0), \ k = 0,1, \ j = 1, \dots, n, & \text{in } \Omega, \\ \partial_t^k z(\pm T) = 0, \ k = 0,1,2, \ j = 1, \dots, n, & \text{in } \Omega, \\ z_j = 0, \ j = 1, \dots, n, & \text{on } \partial\Omega \times (-T,T). \end{cases} \tag{2.3.3}$$

Here, the functions F_j are defined by

$$F_j = \theta f_j \partial_t^2 R_j + \partial_t^2 \theta w_j + 2\partial_t \theta \partial_t w_j, \qquad \text{in } \Omega \times (-T, T),$$

for each $j = 1, \dots, n$.

• **Step 2: Applying Carleman estimate for hyperbolic systems** In this step, we denote by C a generic positive constant which depends at least of Γ_0, m, T, ω and x_0 and may change from line to line.

Applying the Carleman estimate of Theorem 2.8 to the system (2.3.3) with $v_j = z_j$, $r_j = p_j$ and $h_j = F_j$, we see that

$$\begin{aligned}
&\sum_{j=1}^{n-1} I(\alpha, z_j, \Omega) + I(0, z_n, \Omega) \\
\leq & Cs \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |F_j|^2 dxdt + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \big(s^3 |F_{n-1}|^2 + |F_n|^2\big) dxdt \\
&+ C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \big(s |\partial_t F_{n-1}|^2 + |\partial_t F_n|^2\big) dxdt \\
&+ Cs^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \big(s^2 |z_j|^2 + |\partial_t z_j|^2\big) dxdt \\
&+ Cs^3 \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \big(s^5 |z_{n-1}|^2 + s^3 |\partial_t z_{n-1}|^2 + |\partial_t^2 z_{n-1}|^2\big) dxdt.
\end{aligned} \tag{2.3.4}$$

Note that the assumption (2.1.4) implies

$$c < |R_j(0)|^2, \quad \forall j = 1, 2, \dots, n.$$

Then, the following estimate holds:

$$c \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 dx \leq \int_{\Omega} e^{2s\varphi(0)} |f_j R_j(0)|^2 dx = \int_{\Omega} e^{2s\varphi(0)} |z_j(0)|^2 dx. \tag{2.3.5}$$

Hence, summing (2.3.5) over j, we deduce that

$$\frac{1}{c} \sum_{j=1}^{n-1} I(\alpha, z_j) + \frac{1}{c} I(0, z_n) \geq s^{\alpha+2} \sum_{j=1}^{n-1} \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 dx + s^2 \int_{\Omega} e^{2s\varphi(0)} |f_n|^2 dx.$$

Now we estimate the global terms of F_j and its derivatives, for each $j = 1, \dots, n$. By definition,

$$\begin{aligned}
&\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|F_j|^2 dxdt \\
\leq& 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\theta f_j \partial_t^2 R_j|^2 dxdt + 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt \\
\leq& C\int_{\Omega} e^{2s\varphi(0)}|f_j|^2 dx + 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt.
\end{aligned} \tag{2.3.6}$$

Now, we focus our attention on estimating the global term of $2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j$, for $j = 1, \ldots, n$. Notice that if the Time condition (2.1.3) holds, the Carleman weight ψ defined in (2.2.9) satisfies

$$\psi(x, \pm T) = |x - x_0|^2 - \beta T^2 + C_0 < C_0, \quad \text{ in } \Omega.$$

Then, we choose $\tau > 0$ such that

$$\psi(x,t) \leq C_0, \quad \text{ in } \Omega \times ([-T, -T+\tau] \cup [T-\tau, T]),$$

and therefore,

$$\varphi(x,t) = e^{\lambda C_0} < e^{\lambda\psi(x,0)} = \varphi(x,0), \quad \text{ in } \Omega \times ([-T, -T+\tau] \cup [T-\tau, T]).$$

Since the derivatives of θ vanish in $[-T+\tau, T-\tau]$ we see that

$$\begin{aligned}
&\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\partial_t\theta\partial_t w_j + \partial_t^2\theta w_j|^2 dxdt \\
\leq& C\left(\int_{-T}^{-T+\tau} + \int_{T-\tau}^{T}\right) e^{2se^{\lambda C_0}} \int_{\Omega} \left(|\partial_t w_j|^2 + |w_j|^2\right) dxdt.
\end{aligned}$$

Now, we will estimate the last term of the above inequality. To do this, we will use the following energy estimates of (2.3.2):

$$\int_{\Omega} |\partial_t w_j(t)|^2 dx + \int_{\Omega} |\nabla w_j(t)|^2 dx \leq C\int_{\Omega} |f_j(t)|^2 dx + C\int_{\Omega} |w_{j+1}(t)|^2 dx, \quad \forall t \in (-T, T),$$

for each $j = 1, \ldots, n$ and

$$\int_{\Omega} |\partial_t w_n(t)|^2 dx + \int_{\Omega} |\nabla w_n(t)|^2 dx \leq C\int_{\Omega} |f_n|^2 dx, \forall t \in (-T, T),$$

where have used that $R_j \in H^2(-T, T; L^\infty(\Omega))$ for each $j = 1, \ldots, n$. Integrating on $(-T, -T+\tau) \cup (T-\tau, T)$ the estimate above and using the Poincaré inequality to w_j, we see that

$$\begin{aligned}
&\left(\int_{-T}^{-T+\tau} + \int_{T-\tau}^{T}\right) e^{2se^{\lambda C_0}} \int_{\Omega} \left(|\partial_t w_j|^2 + |w_j|^2\right) dxdt \\
\leq& \int_{\Omega} e^{2s\varphi(0)}|f_j|^2 dx + Ce^{se^{\lambda C_0}} \int_{-T}^{T}\int_{\Omega} |w_{j+1}|^2 dxdt,
\end{aligned}$$

for each $j = 1, \ldots, n-1$. Furthermore, due to the structure in cascade of system (2.3.3), we obtain

$$e^{se^{\lambda C_0}} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |w_{j+1}|^2 dxdt \leq C e^{se^{\lambda C_0}} \sum_{i=j+1}^{n} \int_{\Omega} |f_i|^2 dx \leq C \sum_{i=j+1}^{n} \int_{\Omega} e^{2s\varphi(0)} |f_i|^2 dx,$$

for each $j = 2, \ldots, n-1$. Therefore, for every $j = 1, \ldots, n$, we deduce that

$$\int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |2\partial_t \theta \partial_t w_j + \partial_t^2 \theta w_j|^2 dxdt \leq C \sum_{i=j}^{n} \int_{\Omega} e^{2s\varphi(0)} |f_i|^2 dx. \tag{2.3.7}$$

Substituting (2.3.7) in (2.3.6), we see that

$$\begin{aligned} & s^{\alpha} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |F_j|^2 dxdt + \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \big(s^3 |F_{n-1}|^2 + |F_n|^2 \big) dxdt \\ & \leq C s^{\alpha} \sum_{j=1}^{n-2} \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 dx + C \int_{\Omega} e^{2s\varphi(0)} \left(s^3 |f_{n-1}|^2 + |f_n|^2 \right) dx. \end{aligned} \tag{2.3.8}$$

In the same manner we can see that

$$\int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \big(s |\partial_t F_{n-1}|^2 + |\partial_t F_n|^2 \big) dxdt \leq C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \big(s |f_{n-1}|^2 + |f_n|^2 \big) dxdt. \tag{2.3.9}$$

Thus, substituting (2.3.8) and (2.3.9) into (2.3.4), and taking s large enough, we have

$$\begin{aligned} & s^{\alpha+2} \sum_{j=1}^{n-1} \int_{\Omega} e^{2s\varphi(0)} |f_j|^2 dx + s^2 \int_{\Omega} |f_n|^2 dx \\ \leq & C s^{\alpha+1} \sum_{j=1}^{n-2} \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \big(s^2 |z_j|^2 + |\partial_t z_j|^2 \big) dxdt \\ & + C s^3 \int_{-T}^{T} \int_{\omega_2} e^{2s\varphi} \big(s^5 |z_{n-1}|^2 + s^3 |\partial_t z_{n-1}|^2 + |\partial_t^2 z_{n-1}|^2 \big) dxdt. \end{aligned} \tag{2.3.10}$$

• **Step 3: Last arrangements and conclusion**

From (2.3.10), we fix the parameter s and put it into the constant C:

$$\begin{aligned} \sum_{j=1}^{n} \int_{\Omega} |f_j|^2 dx \leq & C \sum_{j=1}^{n} \int_{-T}^{T} \int_{\omega_\omega} e^{2s\varphi} (|z_j|^2 + |\partial_t z_j|^2) dxdt \\ & + C \int_{-T}^{T} \int_{\omega} \left(|z_{n-1}|^2 + |\partial_t z_{n-1}|^2 + |\partial_t^2 z_{n-1}|^2 \right) dxdt, \end{aligned}$$

where we have used that the Carleman weights defined in (2.2.9) are bounded. Moreover, by definition of each z_j we see that

$$\sum_{j=1}^{n} \|f_j\|_{L^2(\Omega)}^2 \leq C \sum_{j=1}^{n-1} \|y_j\|_{H^3(-T,T;L^2(\omega))}^2 + \|y_n\|_{H^4(-T,T;L^2(\omega))}^2. \tag{2.3.11}$$

Finally, replacing $f_j = q_j - \tilde{q}_j$ and $y_j = u_j - \tilde{u}_j$ by (2.3.11) for each $j = 1, \ldots, n$, we conclude the proof of Theorem 2.1. □

Chapter 3

Controllability properties of a class of heat equations with dynamic boundary conditions

3.1 Introduction and main results

In this chapter, the null controllability property for a suitable class of parabolic equations with dynamic boundary conditions is studied. In order to state the main results of this article, we shall introduce some notation. Let $\Omega_L = (-L_1, 0)$ be a bounded interval with $L_1 > 0$, $\omega \subset \Omega_L$, and $T > 0$. Then, let us consider $(u, u_\Gamma) \in L^2(\Omega_L \times (0,T)) \times L^2(0,T)$ be a solution of

$$\begin{cases} \partial_t u(x,t) - \partial_x^2 u(x,t) = \chi_\omega(x) v(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ (u(x,0), u_\Gamma(0)) = (u_0(x), u_{\Gamma,0}), & \forall x \in \Omega_L, \\ u_\Gamma(t) = u(0,t), & \forall t \in (0,T), \\ u(-L_1, t) = 0, & \forall t \in (0,T), \\ u_\Gamma'(t) + \partial_x u(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.1.1}$$

Here, the pair (u, u_Γ) stands for the state of (3.1.1), $(u_0, u_{0,\Gamma}) \in L^2(\Omega_L) \times \mathbb{R}$ is the initial state and $v \in L^2(\omega \times (0,T))$ denotes the control acting on $\omega \subset \Omega_L$. Notice that (3.1.1) can be treated as a coupled system of dynamic equations for u and u_Γ with side condition $u\big|_\Gamma = u_\Gamma$ at the boundary $x = 0$. Moreover, if u is smooth enough, then $u_\Gamma'(t) = \partial_t u(0,t)$.

In this chapter, we analyze the **null controllability** of (3.1.1), i.e., given any data $(u_0, u_{0,\Gamma}) \in L^2(\Omega_L) \times \mathbb{R}$ and $T > 0$, we want to find a control $v \in L^2(\omega \times (0,T))$ such that the associated solution satisfies

$$u(x,T) = 0, \quad \forall x \in \overline{\Omega_L}.$$

In other words, the goal is to steer the state of (3.1.1) to a null final target by a suitable choice of the control function. In addition, we point out that in our model the control is applied in a (small) subset of Ω_L. This means that the first equation is controlled directly by the action of v, while the ODE at $x = 0$ is being controlled indirectly, through the coupling.

Sometimes, boundary conditions like $(3.1.1)_5$ are called of Wentzell type and in the unidimensional case has the general form $\partial_t u - a\Delta u + b\partial_\nu u = 0$ with $a \geq 0$ and $b \in \mathbb{R}$. In our case, dynamic surface diffusion contributions are neglected (i.e. $a = 0$).

Physically, equation $(3.1.1)_5$ can be viewed as a transport equation acting on a neighborhood of the boundary at $x = 0$. Then, the unidirectionally *heat wave* travels into the region Ω_L and this wave lives only for an infinitesimally short time. Of course, once the *heat wave* is inside the region, diffusion is the primary process. For a complete description of physical interpretation for Wentzell boundary conditions in linear and nonlinear models we refer to [31], [57], [53].

Parabolic models with general Wentzell boundary conditions were introduced in the context of the heat equation by A. Favini, G. Goldstein, J. Goldstein and S. Romanelli [44] and subsequently have been intensely studied in the last two decades by many authors, see for instance [54] [94], [95], [55], [52] and the references therein.

The study of controllability properties of parabolic equations are well-known in the case of Dirichlet and Neumann boundary conditions (see for instance [50],[78],[48]), as well as for Robin or Fourier boundary conditions [38], [46], [47]. Moreover, controllability properties of parabolic equations with discontinuous diffusion coefficients are recently studied in [39], [19], [20], [22], [21] and [82].

However, to the best of our knowledge, there are a few work concerning controllability for parabolic equations with dynamic boundary conditions. In particular, optimal control problem and approximate controllability have been considered in [59] and [13] in the case of global controls, i.e, $\omega = \Omega$. Moreover, in [73] the authors studied approximate controllability of a one-dimensional heat equation with dynamical boundary conditions by using the theory of *one-sided coupled* operator matrices developed by K.-J. Engel in [40].

Recently, in [81] null controllability for parabolic equations with dynamic boundary conditions with surface diffusion (i.e. $a > 0$) is studied. In particular, the authors consider Generalized Wentzell conditions with surface diffusion acting on the boundary. In this case, the result was proved by applying Carleman estimates for the homogeneous dual problem. The authors used the weight functions defined in the work of O. Yu Imanuvilov et al [62] (see also [61] and [48]). In this context, the novelty relies in the fact that several boundary terms appears in the deduction of the Carleman estimate. Some of these enter in the final estimate, a few cancel, and others can be controlled using the smoothing effect of the surface diffusion. Let us also mention the papers [60] and [24] where the authors studied the local null controllability of some two-dimensional fluid-structure interaction problems where parabolic equations are coupled with some dynamics (typically an ODE) in a part of the boundary. Again, the main ingredient in the proof of this result is a suitable Carleman estimate, which will be applied to the corresponding adjoint system.

In our case, we obtain a null controllability result for (3.1.1) when the control region is far from the right-hand side of the interval Ω_L even if initial data u_0 and $u_{\Gamma,0}$ are not related. More precisely, the main result of this paper is the following

Theorem 3.1 *Suppose that $\omega = (-L_1, -a)$ with $a > 0$. Then, the system defined by* (3.1.1) *is null-controllable at time $T > 0$.*

Using the well-known equivalence between null controllability and observability (see e.g. [48], [79]), the proof of Theorem 3.1 consists in obtaining a suitable observability inequality for the corresponding adjoint system. This will be done by obtaining an auxiliary Carleman inequality.

In order to state the second main result of this paper, let $\Omega_R = (0, 1)$ and set $\Omega = (-L_1, 1)$. From now on, we shall use the following notation: for a function $h : \Omega \to \mathbb{R}$, h_L and h_R stand for the restriction of Ω_L and Ω_R, respectively.

Now, let $(u, u_\Gamma) \in L^2(\Omega_L \times (0, T)) \times L^2(0, T)$ be a solution of

$$\begin{cases} \partial_t y_L(x,t) - \partial_x^2 y_L(x,t) = f(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ y_R(x,t) = y_L(0,t) = y_\Gamma(t), & \forall (x,t) \in \Omega_R \times (0,T), \\ (y(x,0), y_\Gamma(0)) = (y_0(x), y_{\Gamma,0}), & \forall x \in \Omega_L, \\ y_L(-L_1, t) = 0, & \forall t \in (0,T), \\ y_\Gamma'(t) + \partial_x y_L(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.1.2}$$

We emphasize that (3.1.2) is an extension of (3.1.1) where u_R is only a time-dependent function in Ω_R. This problem is well-posed in the sense of Hadamard. In addition, thanks to Theorem 3.1 problem (3.1.2) is null-controllable at time $T > 0$.

As we mentioned above, the problem (3.1.2) appears as the limit case of a heat equation with discontinuous diffusion coefficient of the form

$$\sigma^K(x) = \begin{cases} 1, & \text{if } x \in \Omega_L, \\ K^2, & \text{if } x \in \Omega_R, \end{cases} \tag{3.1.3}$$

with $K \geq 1$. More precisely, the second result of this paper is the following:

Theorem 3.2 *Let $u_0 \in L^2(\Omega)$, $\omega \subset \Omega$, $K > 0$ and $T > 0$. For $v^k \in L^2(\omega \times (0, T))$, let u^K be the solution of*

$$\begin{cases} \partial_t u^K - \partial_x(\sigma^K \partial_x u^K) = \chi_\omega v^K, & \forall (x,t) \in \Omega \times (0,T), \\ u^K(x,0) = u_0(x), & \forall x \in \Omega, \\ u^K(-L_1, t) = \partial_x u^K(L_2, t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.1.4}$$

Suppose that

$$v^K \rightharpoonup v \textit{ weakly in } L^2(\omega \times (0,T)). \tag{3.1.5}$$

Then, there exists a subsequence of the associated family of solutions $(u^K)_{K>0}$ defined *by* (3.1.4) *with initial data* $u_0 \in L^2(\Omega)$ *which converges as* $K \to +\infty$ *in the sense that*

$$u^K \rightharpoonup u \quad \textit{weakly in } L^2(0,T;H^1(\Omega)), \tag{3.1.6}$$
$$u^k \rightharpoonup u \quad \textit{weakly-star in } L^\infty(0,T;L^2(\Omega)), \tag{3.1.7}$$
$$u^K \to u \quad \textit{strongly in } L^2(\Omega \times (0,T)) \tag{3.1.8}$$

Moreover, u *is a weak solution of* (3.1.2) *with source term* $f = \chi_\omega v$ *and initial condition*

$$y_L(x,0) = u_{0,L}(x), \quad \forall x \in \Omega_L, \quad y_\Gamma(0) = y_R(x,0) = \int_{\Omega_R} u_{0,R}dx, \quad \forall x \in \Omega_R.$$

On the other hand, another interesting question concerning models (3.1.2) and (3.1.4) can be considered. In fact, for each $K \geq 1$, let u^K be the solution of

$$\begin{cases} \partial_t u_L^K(x,t) - \partial_x^2 u_L^K(x,t) = \chi_\omega(x)v(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ \partial_t u_R^K(x,t) - K^2\partial_x^2 u_R^K(x,t) = 0, & \forall (x,t) \in \Omega_R \times (0,T), \\ u_L^K(x,0) = u_{0,L}(x), & \forall x \in \Omega_L, \\ u_R^K(x,0) = u_{0,R}(x), & \forall x \in \Omega_R, \\ u_R^K(0^+,t) = u_L^K(0^-,t), & \forall t \in (0,T), \\ K^2\partial_x u_R(0^+,t) = \partial_x u_L(0^-,t), & \forall t \in (0,T), \\ u_L^K(-L_1,t) = \partial_x u_R^K(1,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.1.9}$$

In addition, let us consider (u, u_Γ) be the solution of

$$\begin{cases} \partial_t u_L(x,t) - \partial_x^2 u_L(x,t) = \chi_\omega v(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ u_R(x,t) = u_L(0,t) = u_\Gamma(t), & \forall (x,t) \in \Omega_R \times (0,T), \\ (u(x,0), u_\Gamma(0)) = (\tilde{u}_0(x), \tilde{u}_{0,\Gamma}), & \forall x \in \Omega, \\ u_L(-L_1,t) = 0, & \forall t \in (0,T), \\ u_\Gamma'(t) + \partial_x u_L(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.1.10}$$

Suppose that $v \in L^2(\omega \times (0,T))$ is a control which drives the initial state $u_0 \in L^2(\Omega)$ in (3.1.9) to zero at time $T > 0$. Then, we can formulate the following question: can we employ the limit control v of the problem (3.1.10) to control (in a suitable sense) system (3.1.9) too?

Concerning the above question, we have the following result

Theorem 3.3 *Let* $\varepsilon > 0$ *and choose* $u_0 \in L^2(\Omega)$ *such that*

$$\|u_{0,R} - \overline{u_{0,R}}\|^2_{L^2(\Omega)} \leq \frac{\varepsilon^2}{3}, \quad \textit{where} \quad \overline{u_{0,R}} = \int_{\Omega_R} u_{0,R}dx. \tag{3.1.11}$$

Let $\tilde{u}$ *be the solution of* (3.1.10) *with control* $v \in L^2(\omega \times (0,T))$ *and initial condition*

$$\tilde{u}_{0,L}(x) = u_{0,L}(x), \quad \forall x \in \Omega_L, \quad \tilde{u}_{\Gamma,0} = \tilde{u}_{0,R}(x) = \overline{u_{0,R}}, \quad \forall x \in \Omega_R.$$

Then, there exists $K_0 > 0$ such that for all $K \geq K_0$ the associated solution u^K of (3.1.9) *fulfills*

$$\|u^K(\cdot, T)\|_{L^2(\Omega)} \leq \varepsilon.$$

Roughly speaking, Theorem 3.3 asserts that for suitable initial conditions under consideration (in particular where $u_{0,R}$ is constant), problem (3.1.9) is approximately controllable at time $T > 0$.

To end this section, we give the outline of this paper. First, in Section 3.2 we fix the functional setting to study the problem (3.1.1). In particular, well-posedness in the sense of Hadamard is given. In Section 3.3 we prove the Theorem 3.1. This will be done by using a suitable Carleman estimate to the adjoint system. In Section 3.4 we prove Theorem 3.2 by using well-known arguments combined the definition of weak solution of such problems. In Section 3.5 we prove Theorem 3.3 by using the ideas developed in the proof of Theorem 3.2.

3.2 Well-posedness of the heat equation with dynamic boundary conditions

In this section, we study well-posedness results for parabolic equations with dynamic boundary conditions like (3.1.1). In particular, we restrict our attention to definitions and properties for this kind of problem.

Before going further, we shall point out that the used in this section are well-known in the literature, see for instance [25],[41] and [80] and for parabolic problems with dynamic boundary conditions we refer to [44], [31], [55] and [81]. However, for completeness we give these notions and results (some of them without proof).

3.2.1 Variational approach

In this section, well-posedness result in the appropriate functional spaces of problems in the form (3.1.1) is considered. First, for $T > 0$, let us consider (u, u_Γ) being a solution of the following problem:

$$\begin{cases} \partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ (u(x,0), u_\Gamma(0)) = (u_0(x), u_{\Gamma,0}), & \forall x \in \Omega_L, \\ u_\Gamma(t) = u(0,t), & \forall t \in (0,T), \\ u(-L_1, t) = 0, & \forall t \in (0,T), \\ \partial_t u_\Gamma(t) + \partial_x u(0,t) = g(t), & \forall t \in (0,T), \end{cases} \tag{3.2.1}$$

with $f \in L^2(\Omega_L \times (0,T))$, $(u_0, u_{\Gamma,0}) \in L^2(\Omega_L) \times \mathbb{R}$ and $g \in L^2(0,T)$. To state the result, we introduce the following space:

$$H_L^1(\Omega_L) = \{v \in H^1(\Omega_L)\,;\, v(-L_1) = 0\},$$

endowed by the usual norm of $H^1(\Omega_L)$. It is clear that $H^1_L(\Omega_L)$ is a Hilbert space. Now, we introduce the bilinear form $a : H^1_L(\Omega_L) \times H^1_L(\Omega_L) \to \mathbb{R}$ given by

$$a(u,v) = \int_{\Omega_L} \partial_x u \partial_x v dx, \quad \forall u, v \in V.$$

Additionally, let H be the completion of $H^1_L(\Omega_L)$ with respect to the norm induced by the inner product

$$(u,v)_H = \int_{\Omega_L} uv dx + u(0)v(0). \tag{3.2.2}$$

In this sense, it is clear that H is isomorphic to $L^2(\Omega_L, dx) \times \mathbb{R}$, where dx denotes the Lebesgue measure in Ω_L. In the same manner, $H^1_L(\Omega_L)$ is isomorphic to the space

$$V = \{(v, v_\Gamma) \in H^1(\Omega) \times \mathbb{R}\,,\, v\big|_\Gamma = v_\Gamma \text{ and } v(-L_1) = 0\},$$

endowed by the norm $\|(v, v_\Gamma)\|_V = \|\partial_x v\|_{L^2(\Omega_L)}$. For this reason, from now one we shall write

$$((u, u_\Gamma), (v, v_\Gamma))_H = \int_{\Omega_L} uv dx + u_\Gamma(0) v_\Gamma(0),$$

for each $(u, u_\Gamma), (v, v_\Gamma) \in H$ to represent the inner product (3.2.2). Similarly, the bilinear form $a : V \times V \to \mathbb{R}$ can be defined by

$$a((u, u_\Gamma), (v, v_\Gamma)) = \int_{\Omega_L} \partial_x u \partial_x v dx, \quad \forall (u, u_\Gamma), (v, v_\Gamma) \in V.$$

Now, we are interested in the following:

Problem: find $(u, u_\Gamma) \in C^0([0,T]; H) \cap L^2(0,T; V)$ such that for all $(v, v_\Gamma) \in V$ the following identity holds:

$$((\partial_t u(t), u'_\Gamma(t)), (v, v_\Gamma))_H + a((u, u_\Gamma), (v, v_\Gamma)) = ((f, g), (v, v_\Gamma))_H, \tag{3.2.3}$$

in the sense of distributions on $(0,T)$ with $(u(x,0), u_\Gamma(0)) = (u_0(x), u_{\Gamma,0})$ for each $x \in \Omega_L$.

It is not difficult to see that after integration by parts and well-known arguments (see [80], [85]), the above problem is equivalent to find a solution to (3.2.1). Then, concerning the above variational problem, we have the following result:

Proposition 3.4 *For each $(u_0, u_{\Gamma,0}) \in H$ and $(f, g) \in L^2(0,T; H)$, the problem (3.2.3) admits a unique solution*

$$(u, u_\Gamma) \in C^0([0,T]; H) \cap L^2(0,T; V).$$

Moreover, the following energy estimate holds true

$$\|(u, u_\Gamma)\|^2_{C^0([0,T];H)} + \|(u, u_\Gamma)\|^2_{L^2(0,T;V)} \leq C \left(\|(u_0, u_{\Gamma,0})\|^2_H + \|(f, g)\|^2_{L^2(0,T;H)} \right), \tag{3.2.4}$$

for some positive constant $C = C(\Omega_L, T)$.

It is clear that $a : V \times V \to \mathbb{R}$ is coercive and continuous on $V \times V$. Then, by standard arguments concerning parabolic problems (see e.g. [80]) the existence of (u, u_Γ) is guaranteed.

3.2.2 Semigroup approach

Using the notation previously introduced in the above section, let $A : D(A) \subset H \to H$ be the linear operator defined by

$$A(v, v_\Gamma) = \left(\partial_x^2 v, -\partial_x v(0)\right), \tag{3.2.5}$$

with domain

$$D(A) = \{(v, v_\Gamma) \in V \,;\, \partial_x^2 v \in L^2(\Omega_L)\}.$$

We have the following result

Proposition 3.5 *The operator A given by* (3.2.5) *is densely defined, self-adjoint and generates a contraction semigroup $(e^{tA})_{t\geq 0}$ on H.*

Proof. It is easy to check that $\{(y, y_\Gamma) \in V \,;\, y \in C^\infty(\overline{\Omega_L})\} \subset D(A)$ is dense in $L^2(\Omega_L)\times\mathbb{R}$. Hence, A is densely defined. In addition, for each $(v, v_\Gamma) \in D(A)$, we have

$$(A(v, v_\Gamma), (v, v_\Gamma))_H = \int_{\Omega_L} v\partial_x^2 v dx - v_\Gamma(0)\partial_x v(0),$$

and integration by parts shows that

$$(A(v, v_\Gamma), (v, v_\Gamma))_H = -\int_{\Omega_L} |\partial_x v|^2 dx \leq 0. \tag{3.2.6}$$

In the same manner, after integration by parts twice, we can assert that for each $(v, v_\Gamma), (w, w_\Gamma) \in D(A)$

$$(A(v, v_\Gamma), (w, w_\Gamma))_H = ((v, v_\Gamma), A(w, w_\Gamma))_H.$$

Thus, by Hille-Yosida's Theorem (see, for example, [30]), we conclude that A is the generator of a contraction semigroup on H. $\square$

Next we introduce different classes of solutions of (3.2.1).

Definition 3.6 *Let $f \in L^2(\Omega_L \times (0, T))$, $g \in L^2(0, T)$ and $Y_0 = (y_0, y_{0,\Gamma}) \in H$.*

(a) A strong solution of (3.2.1) *is a function $U = (u, u_\Gamma) \in H^1(0, T; H) \cap L^2(0, T; D(A))$ fulfilling* (3.2.1) *in $L^2(0, T; H)$.*

(b) A mild solution of (3.2.1) *is a function $U = (u, u_\Gamma) \in C^0([0, T]; H)$ satisfying*

$$U(t) = e^{tA}U_0 + \int_0^t e^{(t-\tau)A}(f(\tau), g(\tau))d\tau. \tag{3.2.7}$$

(c) A distributional solution of (3.2.1) *is a function* $U = (u, u_\Gamma) \in C^0([0,T]; H)$ *such that for all* $\tau \in [0,T]$ *we have*

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_L} u(\partial_t\phi + \partial_x^2\phi)dxdt + \int_{\Omega_L} (u(x,\tau)\phi(x,\tau) - u_0(x)\phi(x,0))dx \\
&+ u_\Gamma(\tau)\phi(0,\tau) - u_{\Gamma,0}\phi(0,0) + \int_0^\tau u_\Gamma(t)(\partial_x\phi(0,t) - \partial_t\phi(0,t))dt \\
=& \int_0^\tau \int_{\Omega_L} f\phi dxdt + \int_0^\tau g(t)\phi(0,t)dt,
\end{aligned}$$

for all $\phi \in C^\infty(\overline{\Omega_L} \times [0,T])$ *such that* $\phi(-L_1, t) = 0$ *for all* $t \in (0,T)$.

The next result asserts the existence of strong solutions for regular initial data:

Proposition 3.7 *Let* $f \in L^2(\Omega_L \times (0,T))$, $g \in L^2(0,T)$ *and* $(u_0, u_{0,\Gamma}) \in V$. *Then, there exists a unique strong solution*

$$U = (u, u_\Gamma) \in E_1(0,t) := H^1(0,T;H) \cap L^2(0,T;D(A))$$

of (3.2.1), *which is also a mild solution. In addition, there exists a constant* $C > 0$ *such that*

$$\|(u,u_\Gamma)\|_{E_1(0,T)} \leq C\left(\|(u,u_{0,\Gamma})\|_V + \|f\|_{L^2(\Omega_L)\times(0,T)} + \|g\|_{L^2(0,T)}\right).$$

However, for our purposes (specially for controllability results) we shall consider initial data $(u_0, u_{0,\Gamma}) \in H$. Then, the next result gives necessary conditions to get the uniqueness of a mild solution and describes the regularity of such solutions.

Proposition 3.8 *Let* $f \in L^2(\Omega_L \times (0,T))$, $g \in L^2(0,T)$ *and* $(u, u_{0,\Gamma}) \in H$. *Then,*

1. *there exists a unique mild solution* $U \in C([0,T]; H)$ *of* (3.2.1) *and the following energy estimate holds:*

   $$\|(u,u_\Gamma)\|_{C^0([0,T];H)} \leq C\left(\|(u_0,u_{0,\Gamma})\|_H + \|f\|_{L^2(\Omega_L\times(0,T))} + \|g\|_{L^2(0,T)}\right),$$

   for some positive constant $C = C(\Omega_L, T)$. *Moreover, for each* $\tau \in (0,T)$ *we get*

   $$(u,u_\Gamma) \in H^1(\tau,T;H) \cap L^2(\tau,T;D(A)).$$

2. *If* $(u, u_{0,\Gamma}) \in V$, *then there mild solution of* (3.2.1) *given by the first item is a strong one.*

3. *A function* (u, u_Γ) *is a distributional solution of* (3.2.1) *if and only if it is a mild solution.*

3.3 Controllability properties of the original problem

In this section, we devote to prove Theorem 3.1. First, using the well-known relation between null controllability and observability, we introduce the adjoint system

$$\begin{cases} \partial_t z(x,t) + \partial_x^2 z(x,t) = 0, & \forall (x,t) \in \Omega_L \times (0,T), \\ (z(x,T), z_\Gamma(T)) = (z_T(x), z_{\Gamma,T}), & \forall x \in \Omega_L, \\ z(-L_1,t) = 0, & \forall t \in (0,T), \\ z_\Gamma'(t) - \partial_x z(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.3.1}$$

Using the change of variables $t' = T - t$ and applying Proposition 3.4 we deduce that system (3.3.1) has a unique solution $z \in C^0([0,T]; H) \cap L^2(0,T;V)$, where H and V are the spaces defined in the previous section. Moreover, we have the following energy estimate:

$$\|(z, z_\Gamma)\|_{C^0([0,T];H)} + \|(z, z_\Gamma)\|_{L^2(0,T;V)} \le C\|(z_T, z_{\Gamma,T})\|_H, \tag{3.3.2}$$

for some constant $C = C(\Omega_L, T)$ Moreover, by Proposition 3.8 for all $\tau \in (0,T)$, we have

$$(z, z_\Gamma) \in H^1(0,\tau; H) \cap L^2(0,\tau; D(A)).$$

As we said before, the proof of (3.3.2) (or equivalently Theorem 3.1) is based on the observability inequality for the adjoint system (3.3.1):

$$\|z(\cdot,0)\|^2_{L^2(\Omega_L)} + |z_\Gamma(0)|^2 \le C \int_0^T \int_\omega |z|^2 dx dt, \tag{3.3.3}$$

for all $(z_T, z_{\Gamma,T}) \in L^2(\Omega_L) \times \mathbb{R}$, where z is the associated solution to (3.3.1), and for some positive constant $C = C(\Omega_L, T)$. This will be done by using a suitable Carleman estimate for (3.3.1).

In order to formulate next result, we shall introduce weight functions. For $\alpha \ge 1$, we define

$$\begin{aligned} \theta(t) =& (t(T-t))^{-\alpha}, \quad \forall t \in (0,T), \\ \psi(x) =& -\frac{1}{4L_1}x^2 + x + 2L_1, \quad \forall x \in \overline{\Omega_L}, \end{aligned}$$

with $\varphi(x,t) = \theta(t)\psi(x)$, for each $(x,t) \in \Omega_L \times (0,T)$. Notice that φ is a smooth positive function which blows up as $t \to 0^+$ and as $t \to T^-$.

Now we state the one-parameter Carleman estimate:

Lemma 3.9 *Let $T > 0$ and for $\alpha \ge 1$ define the function φ as above. Then, there exist positive constants $C = C(\alpha, \Omega_L, T)$ and $s_0 = s_0(\alpha, \Omega_L, T)$ such that for all $s \ge s_0$ the* *following inequality holds*

$$
\begin{aligned}
& s^3 \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{-3\alpha} |y|^2 dx dt + s \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{-\alpha} |\partial_x y|^2 dx dt \\
& + s^{-1} \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{\alpha} (|\partial_x^2 y|^2 + |\partial_t y|^2) dx dt \\
& + s^3 \int_0^T e^{-2s\varphi(0,t)} (t(T-t))^{-3\alpha} |y_\Gamma(t)|^2 dt \\
& + s \int_0^T e^{-2s\varphi(0,t)} (t(T-t))^{-\alpha} |\partial_x y(0,t)|^2 dt + \int_0^T e^{-2s\varphi(0,t)} |y'_\Gamma(t)|^2 dt \\
\leq & C \int_0^T \int_{\Omega_L} e^{-2s\varphi} |\partial_t y + \partial_x^2 y|^2 dx dt + C \int_0^T e^{-2s\varphi(0,t)} |y'_\Gamma(t) - \partial_x y(0,t)|^2 dt \\
& + Cs \int_0^T e^{-2s\varphi(-L_1,t)} (t(T-t))^{-\alpha} |\partial_x y(-L_1,t)|^2 dt,
\end{aligned} \tag{3.3.4}
$$

for all functions $(y, y_\Gamma) \in H^1(0,T;H) \cap L^2(0,T;D(A))$.

The proof is based on the works of A. Fursikov and O. Imanuvilov [50], [62], [61] in the case of Dirichlet and mixed boundary conditions. In our setting, we will see that some new boundary terms arise from the Wentzell dynamic condition. Then, the main difficulty is to prove that these new terms can be absorbed by choosing the parameter s large enough in the spirit of Carleman estimates.

Remark 3.10 *By using a cut-off function η localized close to $x = -L_1$, and standard observability properties for parabolic equations (see for example [48]), we can prove easily* (3.3.3) *when $\omega = (-L_1, -a)$ with $a > 0$.*

Remark 3.11 *One can consider Classical weight functions introduced by A. Fursikov and O. Imanuvilov [50], [62] in the unidimensional case. Indeed, these details are given in the appendix. However, in the context of multidimensional setting this strategy fails, see Remark 3.3 of [81]. In fact, there are some boundary terms depending on $\nabla_T y$ which we cannot absorb it.*

Proof of Lemma 3.9. Since all the terms in (3.3.4) are continuous with respect to the norm of $E_1(0,T)$, it suffices to consider smooth functions $y \in C^\infty(\overline{\Omega_L} \times [0,T])$. In fact, the general case follows by the classical approximation by convolution with mollifiers in space and time by the density of $C^\infty([0,T] \times \overline{\Omega_L})$ in $E_1(0,T)$. This allow us to deduce that $y'_\Gamma(t) = y(0,t)$ for all $t \in (0,T)$.

In what follows, C denotes a generic constant depending on α, Ω_L, ω and $T > 0$ that may change from line to line. For an easier comprehension, we divide the proof into four steps:

- **Step 1: Setting.** Let us introduce the conjugate variable

$$
z(x,t) = e^{-s\varphi(x,t)} y(x,t), \quad \forall (x,t) \in \Omega_L \times (0,T).
$$

Then, direct computations show that the space and time derivatives of z are given by

$$\begin{aligned}\partial_t z = -s\partial_t\varphi z + e^{-s\varphi}\partial_t y, \quad \partial_x z = -s\partial_x\varphi z + e^{-s\varphi}\partial_x y,\\ \partial_x^2 z = -s\partial_x^2\varphi z - s^2|\partial_x\varphi|^2 z - 2s\partial_x\varphi\partial_x\varphi z + e^{-s\varphi}\partial_x^2 y,\end{aligned}$$

for all $(x,t) \in \Omega_L \times (0,T)$. In order to simplify the computations, let us define the operators

$$\begin{aligned}M_1 = s\partial_t\varphi + \partial_x^2 + s^2|\partial_x\varphi|^2, \quad M_2 = \partial_t + 2s\partial_x\varphi + s\partial_x^2\varphi,\\ N_1 = s\partial_t\varphi - \partial_x, \quad N_2 = \partial_t - s\partial_x\varphi.\end{aligned}$$

In addition, due to the regularity of (y, y_Γ) we set

$$f(x,t) = \partial_t y(x,t) + \partial_x^2 y(x,t) \quad \forall (x,t) \in \Omega_L \times (0,T), \quad g(t) = \partial_t y(0,t) - \partial_x y(0,t) \quad \forall t \in (0,T).$$

Then, according to the above computations, it is clear that

$$\begin{aligned}e^{-s\varphi(x,t)} f(x,t) =& M_1 z(x,t) + M_2 z(x,t), \quad \forall (x,t) \in \Omega_L \times (0,T) \quad \text{and}\\ e^{-s\varphi(0,t)} g(t) =& N_1(z)(0,t) + N_2(z)(0,t), \quad \forall t \in (0,T).\end{aligned} \tag{3.3.5}$$

Taking $\|\cdot\|_{L^2(\Omega_L\times(0,T))}$ and $\|\cdot\|_{L^2(0,T)}$ to the equations in (3.3.5) we have

$$\begin{aligned}&\|e^{-s\varphi} f\|^2_{L^2(\Omega_L\times(0,T))} + \|e^{-s\varphi(0,t)} g\|^2_{L^2(0,T)}\\ =&\|M_1 z\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2 z\|^2_{L^2(\Omega_L\times(0,T))} + \|N_1 z(0,\cdot)\|^2_{L^2(0,T)} + \|N_2 z(0,\cdot)\|^2_{L^2(0,T)}\\ &+ 2\langle M_1 z, M_2 z\rangle_{L^2(\Omega_L\times(0,T))} + 2\langle N_1 z(0,\cdot), N_2 z(0,\cdot)\rangle_{L^2(0,T)}.\end{aligned} \tag{3.3.6}$$

Our next task is to compute the inner products in $L^2(\Omega_L \times (0,T))$ and $L^2(0,T)$. This will be done using integration by parts and applying boundary conditions at $x = -L_1$ and $x = 0$.

• **Step 2.** Now let us compute the terms of the first inner product. In order to do that, let us use the following notation:

$$\langle M_1 z, M_2 z\rangle_{L^2(\Omega_L\times(0,T))} = \sum_{i,j=1}^{3} I_{j,k}$$

where J_{jk} stands for the scalar product in $L^2(\Omega_L \times (0,T))$ between the j^{th}-term of $M_1 z$ and the k^{th}-term of $M_2 z$. Then, we start with J_{11}. Using the fact that

$$z\partial_t z = \frac{1}{2}\partial_t |z|^2, \text{ in } \Omega_L \times (0,T)$$

and integrating by parts in time leads

$$I_{11} = s\int_0^T \int_{\Omega_L} \partial_t\varphi z \partial_t z dxdt = -\frac{1}{2}s\int_0^T \int_{\Omega_L} \partial_t^2\varphi |z|^2 dxdt, \tag{3.3.7}$$

where we have used the fact that $z(\cdot,t) \to 0$ as $t \to 0^+$ and $t \to T^-$. In the same manner, J_{12} can be estimated in the following way:

$$\begin{aligned} I_{12} =& 2s^2 \int_0^T \int_{\Omega_L} \partial_t\varphi \partial_x \varphi z \partial_x z dx dt \\ =& - s^2 \int_0^T \int_{\Omega_L} (\partial_t\partial_x\varphi\partial_x\varphi + \partial_t\varphi\partial_x^2\varphi)|z|^2 dxdt + s^2 \int_0^T \partial_t\varphi(0,t)\partial_x\varphi(0,t)|z(0,t)|^2 dt, \end{aligned} \tag{3.3.8}$$

where we have used the homogeneous Dirichlet boundary condition on $x = -L_1$. Furthermore, by definition J_{13} reads as follows

$$I_{13} = s^2 \int_0^T \int_{\Omega_L} \partial_t\varphi\partial_x^2\varphi|z|^2 dxdt. \tag{3.3.9}$$

Let us compute J_{21}. Integration by parts in space yields

$$\begin{aligned} I_{21} =& \int_0^T \int_{\Omega_L} \partial_t z \partial_x^2 z dxdt \\ =& - \int_0^T \int_{\Omega_L} \partial_t\partial_x z \partial_x z dxdt + \int_0^T \partial_t z(0,t)\partial_x z(0,t) dt. \end{aligned}$$

Let us compute the above terms. First, notice that

$$\partial_x z(\cdot,t) \to 0 \text{ as } t \to 0^+ \text{ and } t \to T^-.$$

Then, using the fact that $\partial_t\partial_x\partial_x z = \frac{1}{2}\partial_t|\partial_x z|^2$ in $\Omega_L \times (0,T)$ it follows that

$$\int_0^T \int_{\Omega_L} \partial_t(|\partial_x z|^2) dxdt = 0.$$

On the other hand, according to the definitions of N_1 and N_2, we have

$$\begin{aligned} &\int_0^T \partial_t z(0,t)\partial_x z(0,t)dt \\ =& \int_0^T |\partial_t z(0,t)|^2 dt - s\int_0^T \partial_x\varphi(0,t)z(0,t)\partial_t z(0,t)dt + s\int_0^T \partial_t\varphi(0,t)z(0,t)\partial_t z(0,t)dt \\ &- \int_0^T e^{-s\varphi(0,t)}g(t)\partial_t z(0,t)dt. \end{aligned}$$

Integration by parts yields

$$-s\int_0^T \partial_x\varphi(0,t)z(0,t)\partial_t z(0,t)dt = \frac{1}{2}s\int_0^T \partial_t\partial_x\varphi(0,t)|z(0,t)|^2 dt,$$

and

$$s \int_0^T \partial_t \varphi(0,t) z(0,t) \partial_t z(0,t) dt = -\frac{1}{2} \int_0^T \partial_t^2 \varphi(0,t) |z(0,t)|^2 dt.$$

Therefore, I_{21} is given by

$$\begin{aligned} I_{21} = & \int_0^T |\partial_t z(0,t)|^2 dt + \frac{1}{2} s \int_0^T \partial_t \partial_x \varphi(0,t) |z(0,t)|^2 dt \\ & - \frac{1}{2} s \int_0^T \partial_t^2 \varphi(0,t) |z(0,t)|^2 dt \end{aligned} \tag{3.3.10}$$

Let us emphasize that I_{21} contains the boundary term $|\partial_t z(0,t)|^2$, which plays an important role to eliminate the boundary terms from the next step. Once again, integration by parts in space we have

$$\begin{aligned} I_{22} = & - s \int_0^T \int_{\Omega_L} \partial_x^2 \varphi |\partial_x z|^2 dx dt + s \int_0^T \partial_x \varphi(0,t) |\partial_x z(0,t)|^2 dt \\ & - s \int_0^T \partial_x \varphi(-L_1,t) |\partial_x z(-L_1,t)|^2 dt. \end{aligned} \tag{3.3.11}$$

We point out that the third term of the right-hand side of I_{22} will be considered as an observation. In the same manner, I_{23} is given by

$$\begin{aligned} I_{23} = & s \int_0^T \int_{\Omega_L} \partial_x^2 \varphi \partial_x^2 z dx dt \\ = & - s \int_0^T \int_{\Omega_L} \partial_x^2 \varphi |\partial_x z|^2 dx dt + s \int_0^T \partial_x^2 \varphi(0,t) z(0,t) \partial_x z(0,t) dt, \end{aligned} \tag{3.3.12}$$

where we have used $z(-L_1,t) = 0$ for all $t \in (0,T)$ and the fact that $\partial_x^3 \varphi = 0$ in $\Omega_L \times (0,T)$. In addition, the term I_{31} reads as follows

$$I_{31} = -\frac{1}{2} s^2 \int_0^T \int_{\Omega_L} \partial_t (|\partial_x \varphi|^2 |z|^2) dx dt. \tag{3.3.13}$$

Moreover, integration by parts yields

$$I_{32} = 2s^3 \int_0^T \int_{\Omega_L} |\partial_x \varphi|^3 z \partial_x z dx dt \tag{3.3.14}$$

$$= - 3s^2 \int_0^T \int_{\Omega_L} |\partial_x \varphi|^2 \partial_x^2 \varphi |z|^2 dx dt + s^3 \int_0^T |\partial_x \varphi(0,t)|^3 |z(0,t)|^2 dt. \tag{3.3.15}$$

Finally, by definition I_{33} is given by

$$I_{33} = s^3 \int_0^T \int_{\Omega_L} |\partial_x \varphi|^2 \partial_x^2 \varphi |z|^2 dx dt. \tag{3.3.16}$$

In the next step, we compute the the second inner product in the equation (3.3.6). We emphasize that this step plays an important role on the proof of Lemma 3.9. In fact, we must ensure that all these new terms can be controlled or absorbed by taking the parameter s large enough.

• **Step 3: Boundary terms.** We introduce the notation

$$\langle N_1(z)(0,t), N_2(z)(0,t)\rangle_{L^2(0,T)} = \sum_{j,k=1}^{2} J_{j,k},$$

where J_{jk} stands for the scalar product in $L^2(0,T)$ between the j^{th}-term of $N_1(z)(0,\cdot)$ and the k^{th}-term of $N_2(z)(0,\cdot)$. Then, $J_1 1$ can be estimated in the following way:

$$J_{11} = s\int_0^T \partial_t\varphi(0,t)z(0,t)\partial_t z(0,t)dt = -\frac{1}{2}s\int_0^T \partial_t^2\varphi(0,t)|z(0,t)|^2dt. \tag{3.3.17}$$

Moreover, the other terms are given by

$$J_{12} = -\, s^2\int_0^T \partial_t\varphi(0,t)\partial_x\varphi(0,t)|z(0,t)|^2dt, \tag{3.3.18}$$

$$J_{21} = -\int_0^T \partial_x z(0,t)\partial_t z(0,t)dt, \tag{3.3.19}$$

$$J_{22} = -\, s\int_0^T \partial_x\varphi(0,t)z(0,t)\partial_x z(0,t)dt. \tag{3.3.20}$$

As we shall see in the next step, due to the definitions of Carleman weights ψ and θ with $\alpha \geq 1$, the terms given by the equations (3.3.17)-(3.3.20) can be bounded in a suitable way by taking s large enough. This enable us to control these new boundary terms.

• **Step 4:** Substituting (3.3.7)-(3.3.20) into (3.3.6) and gathering the terms we have

$$\begin{aligned}
&- 2s^3\int_0^T\int_{\Omega_L} |\partial_x\varphi|^2\partial_x^2\varphi|z|^2dxdt - 2s\int_0^T\int_{\Omega_L} \partial_x^2\varphi|\partial_x z|^2dxdt\\
&+ s^3\int_0^T |\partial_x\varphi(0,t)|^3|z(0,t)|^2dt + s\int_0^T \partial_x\varphi(0,t)|\partial_x z(0,t)|^2dt\\
&+ \int_0^T |\partial_t z(0,t)|^2dt + \|M_1 z\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2 z\|^2_{L^2(\Omega_L\times(0,T))}\\
&+ \|N_1(z)(0,t)\|^2_{L^2(0,T)} + \|N_2(z)(0,t)\|^2_{L^2(0,T)}\\
=&\|e^{-s\varphi}f\|^2_{L^2(\Omega_L\times(0,T))} + \|e^{-s\varphi(0,t)}g\|^2_{L^2(0,T)} + s\int_0^T \partial_x\varphi(-L_1,t)|\partial_x z(-L_1,t)|^2dt\\
&+ X + Y.
\end{aligned} \tag{3.3.21}$$

where X and Y are given by

$$X = \frac{1}{2}s\int_0^T\int_{\Omega_L} \partial_t^2\varphi|z|^2dxdt + s^2\int_0^T\int_{\Omega_L} \partial_t\partial_x\varphi\partial_x\varphi|z|^2dxdt,$$

and

$$
\begin{aligned}
Y = & -\frac{1}{2}s\int_0^T \partial_t\partial_x\varphi(0,t)|z(0,t)|^2dt - \frac{1}{2}s\int_0^T \partial_t^2\varphi(0,t)|z(0,t)|^2dt \\
& + \int_0^T e^{-s\varphi(0,t)}g\partial_t z(0,t)dt - s\int_0^T \partial_x^2\varphi(0,t)z(0,t)\partial_x z(0,t)dt \\
& - \frac{1}{2}s\int_0^T \partial_t^2\varphi(0,t)|z(0,t)|^2dt + s^2\int_0^T \partial_t\varphi(0,t)\partial_x\varphi(0,t)|z(0,t)|^2dt \\
& + \int_0^T \partial_x z(0,t)\partial_t z(0,t)dt + s\int_0^T \partial_x\varphi(0,t)z(0,t)\partial_x z(0,t)dt.
\end{aligned}
$$

Our next task is to eliminate the terms X and Y. In order to do that, let us point out that the derivatives of θ can be bounded as follows:

$$
|\theta'(t)| \le \alpha T(t(T-t))^{-(\alpha+1)}, \quad |\theta''(t)| \le C(t(T-t))^{-(\alpha+2)}, \quad \forall t \in (0,T), \tag{3.3.22}
$$

for some positive constant $C = C(\alpha, \Omega_L, T)$. On the other hand, ψ and their derivatives satisfy

$$
\frac{3}{2}L_1 \le \psi(x) \le 2L_1, \quad \frac{1}{2} \le \psi'(x) \le 1, \quad \psi''(x) = -\frac{1}{2L_1}, \quad \forall x \in \overline{\Omega_L}. \tag{3.3.23}
$$

Then, by using inequalities (3.3.22) and (3.3.23) in (3.3.21) we obtain

$$
\begin{aligned}
& s^3\int_0^T\int_{\Omega_L}(t(T-t))^{-3\alpha}|z|^2dxdt + s\int_0^T\int_{\Omega_L}(t(T-t))^{-\alpha}|\partial_x z|^2dxdt \\
& + s^3\int_0^T(t(T-t))^{-3\alpha}|z(0,t)|^2dt + s\int_0^T(t(T-t))^{-\alpha}|\partial_x z(0,t)|^2dt \\
& + \int_0^T|\partial_t z(0,t)|^2dt + \|M_1 z\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2 z\|^2_{L^2(\Omega_L\times(0,T))} \\
& \|N_1(z)(0,t)\|^2_{L^2(0,T)} + \|N_2(z)(0,t)\|^2_{L^2(0,T)} \le C_1\|e^{-s\varphi}f\|_{L^2(\Omega_L\times(0,T))} \\
& C_1\|e^{-s\varphi(0,t)}g\|^2_{L^2(0,T)} + C_1 s\int_0^T(t(T-t))^{-\alpha}|\partial_x z(-L_1,t)|^2dt + C_1|X| + C_1|Y|,
\end{aligned} \tag{3.3.24}
$$

for some constant $C_1 = C_1(\alpha, \Omega_L, T)$. Notice that

$$
C_1|X| \le C_2 s\int_0^T\int_{\Omega_L}(t(T-t))^{-\alpha+2}|z|^2dxdt + C_2 s^2\int_0^T\int_{\Omega_L}(t(T-t))^{-(2\alpha+1)}|z|^2dxdt.
$$

Moreover, since $\alpha \ge 1$ we can choose $s_1 > 0$ large enough to get

$$
C_1|X| \le \frac{1}{2}s^3\int_0^T\int_{\Omega_L}(t(T-t))^{-3\alpha}|z|^2dxdt, \quad \forall s \ge s_1.
$$

By using Young's inequality and the previous arguments we deduce the existence of a constant $s_2 = s_2(\alpha, C_1, \Omega, T) \ge s_1$ such that the following estimate for Y_1 holds:

$$
\begin{aligned}
C_1|Y| \le & \frac{1}{2}s^3\int_0^T(t(T-t))^{-3\alpha}|z(0,t)|^2 + \frac{1}{2}s\int_0^T(t(T-t))^{-\alpha}|\partial_x z(0,t)|^2dt \\
& + \frac{1}{2}\int_0^T|\partial_t z(0,t)|^2dt + \frac{1}{2}\int_0^T e^{-s\varphi(0,t)}|g|^2dt, \quad \forall s \ge s_2.
\end{aligned} \tag{3.3.25}
$$

Therefore, for each $s \geq s_3$ with $s_3 = \max\{s_1, s_2\}$ we get

$$
\begin{aligned}
& s^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt + s \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |\partial_x z|^2 dxdt \\
& + s^3 \int_0^T (t(T-t))^{-3\alpha} |z(0,t)|^2 dt + s \int_0^T (t(T-t))^{-\alpha} |\partial_x z(0,t)|^2 dt \\
& + \int_0^T |\partial_t z(0,t)|^2 dt + \|M_1 z\|^2_{L^2(\Omega_L \times (0,T))} + \|M_2 z\|^2_{L^2(\Omega_L \times (0,T))} \\
& + \|N_1(z)(0,t)\|^2_{L^2(0,T)} + \|N_2(z)(0,t)\|^2_{L^2(0,T)} \\
\leq & C_1 \|e^{-s\varphi} f\|^2_{L^2(\Omega_L \times (0,T))} + C_1 \|e^{-s\varphi(0,t)} g\|^2_{L^2(0,T)} + C_1 s \int_0^T (t(T-t))^{-\alpha} |\partial_x z(-L_1, t)|^2 dt.
\end{aligned}
\tag{3.3.26}
$$

It remains to deduce estimates for $\partial_t z$ and $\partial_x^2 z$. To do this, by definition of M_1 and M_2 we can assert that

$$
\begin{aligned}
& s^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |\partial_x^2 z|^2 dxdt \\
\leq & C s^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |M_1 z|^2 dxdt + C s \int_0^T \int_{\Omega_L} (t(T-t))^{-(\alpha+2)} |z|^2 dxdt \\
& + C s^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt, \quad \forall s > 0,
\end{aligned}
\tag{3.3.27}
$$

and

$$
\begin{aligned}
& s^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |\partial_t z|^2 dxdt \\
\leq & C s^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |M_2 z|^2 dxdt + C s \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |z|^2 dxdt \\
& + C s \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |\partial_x z|^2 dxdt, \forall s > 0.
\end{aligned}
\tag{3.3.28}
$$

Thus, the global terms of $\partial_t z$ and $\partial_x^2 z$ can be incorporated in the left-hand side of

(3.3.26). Thus, for each $s \geq s_3$ we have

$$
\begin{aligned}
& s^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt + s \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |\partial_x z|^2 dxdt \\
& + s^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} \left(|\partial_x^2 z|^2 + |\partial_t z|^2 \right) dxdt \\
& + s^3 \int_0^T (t(T-t))^{-3\alpha} |z(0,t)|^2 dt + s \int_0^T (t(T-t))^{-\alpha} |\partial_x z(0,t)|^2 dt \\
& + \int_0^T |\partial_t z(0,t)|^2 dt + \|M_1 z\|^2_{L^2(\Omega_L \times (0,T))} + \|M_2 z\|^2_{L^2(\Omega_L \times (0,T))} \\
& + \|N_1(z)(0,t)\|^2_{L^2(0,T)} + \|N_2(z)(0,t)\|^2_{L^2(0,T)} \\
\leq & C_2 \|e^{-s\varphi} f\|^2_{L^2(\Omega_L \times (0,T))} + C_2 \|e^{-s\varphi(0,t)} g\|^2_{L^2(0,T)} + C_2 s \int_0^T (t(T-t))^{-\alpha} |\partial_x z(-L_1,t)|^2 dt.
\end{aligned}
\tag{3.3.29}
$$

Finally, let us come back to the original variables. By definition of z, we know that

$$
s^3 \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{-3\alpha} |y|^2 dxdt = s^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt, \forall s > 0.
\tag{3.3.30}
$$

Moreover, since $\partial_x z = -se^{-s\varphi}\partial_x \varphi y + e^{-s\varphi}\partial_x y$, in $\Omega_L \times (0,T)$ we have

$$
\begin{aligned}
& s \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{-\alpha} |\partial_x y|^2 dxdt \\
\leq & Cs^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt + Cs \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |\partial_x z|^2 dxdt, \quad \forall s > 0.
\end{aligned}
\tag{3.3.31}
$$

In the same manner, the global terms of $\partial_x^2 y$ and $\partial_t y$ can be estimated in the following way:

$$
\begin{aligned}
& s^{-1} \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{\alpha} |\partial_x^2 y|^2 dxdt \\
\leq & Cs^3 \int_0^T \int_{\Omega_L} (t(T-t))^{-3\alpha} |z|^2 dxdt + Cs \int_0^T \int_{\Omega_L} (t(T-t))^{-\alpha} |\partial_x z|^2 dxdt \\
& + Cs^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |\partial_x^2 z|^2 dxdt, \quad \forall s > 0.
\end{aligned}
\tag{3.3.32}
$$

and

$$
\begin{aligned}
& s^{-1} \int_0^T \int_{\Omega_L} e^{-2s\varphi} (t(T-t))^{\alpha} |\partial_t y|^2 dxdt \\
\leq & Cs \int_0^T \int_{\Omega_L} (t(T-t))^{-2\alpha-1} |z|^2 dxdt + Cs^{-1} \int_0^T \int_{\Omega_L} (t(T-t))^{\alpha} |\partial_t z|^2 dxdt, \quad \forall s > 0.
\end{aligned}
\tag{3.3.33}
$$

Then, using (3.3.30)-(3.3.33) in (3.3.32) we obtain 3.3.4. This completes the proof of Lemma 3.9. □

Remark 3.12 *We point out that Lemma 3.9 can be used to prove a boundary controllability for problems in the form*

$$\begin{cases} \partial_t u_L(x,t) - \partial_x^2 u_L(x,t) = 0, & \forall (x,t) \in \Omega_L \times (0,T), \\ (u(x,0), u_\Gamma(0)) = (u_0(x), u_{\Gamma,0}), & \forall x \in \Omega_L, \\ u_\Gamma(t) = u(0,t), & \forall t \in (0,T), \\ u(-L_1,t) = v(t), & \forall t \in (0,T), \\ u_\Gamma'(t) + \partial_x u(0,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.3.34}$$

with $(u_0, u_{\Gamma,0}) \in L^2(\Omega_L) \times \mathbb{R}$ *the initial data and control* $v \in L^2(0,T)$ *acts only on the flux of solutions in the left-hand side of the domain* $x = -L_1$. *In fact, the adjoint of the control problem of* (3.3.34) *is the same as the one in* (3.3.1) *but in this case we have to prove the following observability inequality:*

$$\|z(\cdot,0)\|^2_{L^2(\Omega_L)} + |z_\Gamma(0)|^2 \leq C \int_0^T |\partial_x z(-L_1,t)|^2 dt,$$

for some constant $C = C(\Omega_L, T)$.

3.4 Convergence of the approximate system

In this section, the goal is to prove Theorem 2. To do this, we introduce the notions of weak solutions in the sense of distributions for the problems (3.1.2) and (3.1.4).

Let us start giving a remark on the approximate system. As we said before, sometimes parabolic equations with discontinuous diffusion coefficients can be viewed as a transmission problem. This means that (3.1.2) can be written in the following way:

$$\begin{cases} \partial_t u_L^K(x,t) - \partial_x^2 u_L^K(x,t) = \chi_\omega(x) v^K(x,t), & \forall (x,t) \in \Omega_L \times (0,T), \\ \partial_t u_R^K(x,t) - K^2 \partial_x^2 u_R^K(x,t) = 0, & \forall (x,t) \in \Omega_R \times (0,T), \\ u_L^K(x,0) = u_{0,L}(x), & \forall x \in \Omega_L, \\ u_R^K(x,0) = u_{0,R}(x), & \forall x \in \Omega_R, \\ u_R^K(0^+,t) = u_L^K(0^-,t), & \forall t \in (0,T), \\ K^2 \partial_x u_R(0^+,t) = \partial_x u_L(0^-,t), & \forall t \in (0,T), \\ u_L^K(-L_1,t) = \partial_x u_R^K(1,t) = 0, & \forall t \in (0,T). \end{cases} \tag{3.4.1}$$

We point out that equation $(3.4.1)_5$ and $(3.4.1)_6$ describes the continuity of the solution and the flux at $x = 0$.

Let $K > 0$ and $T > 0$. We say that $u \in C^0([0,T]; L^2(\Omega))$ is a weak solution of (3.4.1) if for all $\tau \in [0,T]$ the following inequality holds

$$
\begin{aligned}
&-\int_0^\tau\int_{\Omega_L} u_L^K(\partial_t\psi_L + \partial_x^2\psi_L)dxdt - \int_0^\tau\int_{\Omega_R} u_R^K(\partial_t\psi_R + K^2\partial_x^2\psi_R)dxdt\\
&+\int_{\Omega_R}(u_L^K(x,\tau)\psi_L(x,\tau) - u_{0,L}(x)\psi_L(x,0))dx + \int_{\Omega_R}(u_R^K(x,\tau)\psi_R(x,\tau) - u_{0,R}(x)\psi_R(x,0))dx\\
&+\int_0^\tau u_L^K(0,t)(\partial_x\psi_L(0,t) - K^2\partial_x\psi_R(0,t))dt = \int_0^\tau\int_{\Omega_L}\chi_\omega v^K\psi_L dxdt,
\end{aligned}
\tag{3.4.2}
$$

for all $\psi = \psi(x,t) \in \Psi(\tau)$, where

$$\Psi(\tau) = \{\psi \in C^0(\overline{\Omega}\times[0,\tau])\,;\, \psi_L, \psi_R \text{ are smooth and } \psi_L(-L_1,t) = \partial_x\psi_R(1,t) = 0\}.$$

On the other hand, we say that $y \in C^0([0,T];L^2(\Omega))$ is a weak solution of (3.1.2) if

$$
\begin{aligned}
&-\int_0^\tau\int_{\Omega_L} y_L(\partial_t\phi_L + \partial_x^2\phi_L)dxdt + \int_{\Omega_L}(y_L(x,\tau)\phi_L(x,\tau) - y_{0,L}(x)\phi_L(x,0))dx\\
&y_R(\tau)\phi_R(\tau) - y_{0,\Gamma}\phi(0) + \int_0^\tau y_\Gamma(t)(\partial_x\phi_L(0,t) - \phi_R'(t))dt\\
&= \int_0^\tau\int_{\Omega_L} f\phi_R dxdt + \int_0^\tau g(t)\phi_L(0,t)dt,
\end{aligned}
\tag{3.4.3}
$$

for all $\phi \in \Phi(\tau)$ where

$$\Phi(\tau) = \{\phi \in C^0(\overline{\Omega}\times[0,T])\,;\, \phi_L \text{ is smooth },\ \partial_x\phi_R = 0 \text{ in } \Omega_R \text{ and } \phi_L(-L_1,t) = \partial_x\phi_R(1,t) = 0\}$$

Now we have all the ingredients to start the proof of Theorem 3.2.

Proof of Theorem 3.2. First, we recall that for each $K > 0$, (3.4.1) admits a unique weak solution

$$u^K \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_L^1(\Omega)),$$

with $u_0 \in L^2(\Omega)$ and $v \in L^2(\omega\times(0,T))$. Moreover, the following energy estimate holds:

$$\|u^K\|_{C^0([0,T];L^2(\Omega))} + \|\sigma^K\partial_x u^K\|^2_{L^2(\Omega\times(0,T))} \le C\left(\|u_0\|^2_{L^2(\Omega)} + \|v^K\|^2_{L^2(\omega\times(0,T))}\right), \tag{3.4.4}$$

for some positive constant $C = (\Omega,\omega,T)$ independent of K. Since v^K converges weakly to v, we deduce that

$$(u^K)_{K>1} \text{ is uniformly bounded } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_L^1(\Omega)).$$

Then, there exists a subsequence $(u^K)_{K>1}$ (which denotes by the same index for simplicity) such that (3.1.6) and (3.1.7) holds. Moreover, we can use classical compactness results (see for instance [25],[41] and [89]) to deduce that

$$u^K \to u \text{ strongly in } L^2(\Omega\times(0,T)).$$

It remains to identify the limit problem for u. First, from (3.4.4) it is easy to see that

$$\|\partial_x u_R^K\|_{L^2(\Omega_R\times(0,T))} \leq CK^{-2},$$

and as $K \to +\infty$ we see that $\partial_x u_R = 0$ a.e. in Ω_R, i.e., $u_R = u_R(t)$ is a function of t only. Moreover, Trace Theorem implies that $u_L(-L_1,t) = 0$ and $u_R(t) = u_L(0,t)$, $\forall t \in (0,T)$.

Now we focus on the weak solution (3.4.2). In order to avoid the explicit dependence of K, we choose $\psi = \phi$ with $\phi \in \Phi(\tau)$. Then, we have

$$\begin{aligned}
&-\int_0^\tau\int_{\Omega_L} u_L^K(\partial_t\phi_L + \partial_x^2\phi_L)dxdt - \int_0^\tau\int_{\Omega_R} u_R^K\phi'(t)dxdt \\
&+\int_{\Omega_L}(u_L^K(x,\tau)\phi_L(x,\tau) - u_{0,L}\phi_L(x,0))dx + \phi(\tau)\int_{\Omega_R} u_R^K dx - \phi(0)\int_{\Omega_R} u_{0,R}dx \\
&+\int_0^\tau u_L^K(0,t)\partial_x\phi_L(0,t)dt = \int_0^\tau\int_{\Omega_L}\chi_\omega v^K\phi_L dxdt.
\end{aligned} \tag{3.4.5}$$

Letting $K \to +\infty$ in (3.4.5), using the fact that $u_R(t) = u_L(0,t)$ and $|\Omega_R| = 1$ we get

$$\begin{aligned}
&-\int_0^\tau\int_{\Omega_L} u_L(\partial_t\phi_L + \partial_x^2\phi_L)dxdt + \int_{\Omega_L}(u_L(x,\tau)\phi_L(x,\tau) - u_{0,L}(x)\phi_L(x,0))dx \\
&+ u_R(\tau)\phi_R(\tau) - \phi(0)\int_{\Omega_R} u_{0,R}dx + \int_0^\tau u_L(0,t)(\partial_x\phi_L(0,t) - \phi_R'(t))dt \\
&= \int_0^\tau\int_{\Omega_L}\chi_\omega v\phi_L dxdt,
\end{aligned}$$

which is the definition of weak solution (3.4.3) for (3.1.2) with $f = \chi_\omega v$, $g = 0$, $y_{0,L} = u_{0,L}$ and $y_{0,\Gamma} = \int_{\Omega_R} u_{0,R}dx$. This completes the proof of Theorem 3.2. $\square$

3.5 Plugging the limit control in the approximate system

In this section we focus on the proof of Theorem 3.3. In general terms, the proof is based on the convergence result given in Theorem 3.2 together with regularity results of (3.1.9) and (3.1.10).

Proof of Theorem 3.3. Let us define $y = u - u^K$ in $\overline{\Omega} \times [0,T]$. It is clear that y depends on K and therefore we shall write y^K instead of y, however in this case we avoid this dependence for simplicity. Then, according to the equations (3.1.9) and (3.1.10) y is a solution of

$$\begin{cases}
\partial_t y_L(x,t) - \partial_x^2 y_L(x,t) = 0, & \forall (x,t) \in \Omega_L \times (0,T), \\
\partial_t y_R(x,t) - K^2\partial_x^2 y_R(x,t) = \partial_x u_L(0,t), & \forall (x,t) \in \Omega_R \times (0,T), \\
y_L(x,0) = 0, & \forall x \in \Omega_L, \\
y_R(x,0) = u_{0,R}(x) - \int_{\Omega_R} u_{0,R}dx, & \forall x \in \Omega_R, \\
y_R(0,t) = y_L(0,t), & \forall t \in (0,T), \\
K^2\partial_x y_R(0,t) = \partial_x y_L(0,t) - \partial_x u_L(0,t), & \forall t \in (0,T), \\
y(-L_1,t) = \partial_x y_R(1,t) = 0, & \forall t \in (0,T).
\end{cases} \tag{3.5.1}$$

Due to the regularity of u and u^K (see for instance Propositions 3.7 and 3.8 in Section 2 and [80] or [41]), and classical arguments we can write for all $t \in (0,T)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_L}|y_L(t)|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega_R}|y_R(t)|^2dx+\int_{\Omega}\sigma^K|\partial_x y(t)|^2dx\\ =&-\partial_x u_L(0,t)\int_{\Omega_R}y_R dx-y_L(0,t)\partial_x u_L(0,t).\end{aligned}$$

Thus, for all $\tau \in [0,T]$ we have

$$\begin{aligned}\int_{\Omega_L}|y(\tau)|^2dx \leq &2\int_0^T\int_{\Omega_L}y_R\partial_x u_L(0,t)dxdt+2\int_0^T y_L(0,t)\partial_x u_L(0,t)dt\\ &+\int_{\Omega_R}|u_{0,R}-\overline{u_{0,R}}|^2dx.\end{aligned}\tag{3.5.2}$$

Now, by (3.1.6) we choose $K_1 > 1$ such that for all $K \geq K_1$ we have

$$\left|\int_0^T\int_{\Omega}ywdxdt\right| \leq \frac{\varepsilon^2}{3}, \quad \forall w \in L^2(\Omega\times(0,T)).\tag{3.5.3}$$

Moreover, by Trace Theorem and the strong convergence of $(y)_{K>1}$ of we choose $K_2 > 1$ such that for all $K \geq K_2$ we get

$$\left|\int_0^T y(0,t)zdt\right| \leq \frac{\varepsilon^2}{3}, \quad \forall z \in L^2(0,T).\tag{3.5.4}$$

Then, applying the inequalities (3.1.11), (3.5.2) and (3.5.3), choosing $\tau = T$ and by using the fact that $u(T) = 0$ in $\overline{\Omega}$ we get

$$\|u^K(\cdot,T)\|_{L^2(\Omega\times(0,T))} \leq \varepsilon, \quad \forall K \geq K_0$$

with $K_0 = \max\{K_1, K_2\}$. This completes the proof of Theorem 3.3. $\square$

Chapter 4

Controllability of 1-D heat equation with discontinuous diffusion coefficients

In this chapter, we will study the null-controllability of the heat equation with Lipschitz diffusion coefficient and mixed boundary conditions. In this case, we consider a boundary control acting in the left-hand side of the domain. This will be done by using a suitable Carleman estimate, which implies the observability inequality for the adjoint system asociated to the original problem.

4.1 Introduction, setting and main result

Let $\Omega = (-L_1, L_2)$ be an open interval of $\mathbb{R}$. Throughout this chapter, we consider the following heat equation with mixed boundary conditions:

$$\begin{cases} \partial_t u(x,t) - \partial_x\left(\sigma(x)\partial_x u(x,t)\right) = 0, & \forall (x,t) \in \Omega \times (0,T), \\ u(x,0) = u^0(x), & \forall x \in \Omega, \\ u(-L_1,t) = v(t), & \forall t \in (0,T), \\ \partial_x u(L_2,t) = 0, & \forall t \in (0,T). \end{cases} \tag{4.1.1}$$

Here, the piecewise diffusion coefficient $\sigma = \sigma(x)$ is defined by

$$\sigma(x) = \begin{cases} \sigma_1^2, & \forall x \in (-L_1, 0), \\ \sigma_2^2, & \forall x \in (0, L_2). \end{cases}$$

where $\sigma_1, \sigma_2 > 0$. Moreover, the initial datum u^0 belongs in $L^2(\Omega)$ is given and $v \in L^2(0,T)$ is the boundary control. It is well-known that, under these assumptions, the problem (4.1.1) is well-posed in the Hadamard's sense. As we said before, we are interested in the problem of null-controllability of (4.1.1). In other words, we focus in the following question: given $u^0 \in L^2(\Omega)$ in (4.1.1) and $T > 0$, can we find a boundary control $v \in L^2(0,T)$ such that the associated solution u satisfy

$$u(x,T) = 0, \quad \forall x \in \Omega?$$

It is well-known that the question above is equivalent to prove the observability inequality for the adjoint system of (4.1.1). To be more precise, let w be the solution of

$$\begin{cases} \partial_t w(x,t) + \partial_x(\sigma(x)\partial_x w(x,t)) = 0, & \forall (x,t) \in \Omega \times (0,T), \\ w(T) = w^T, & \forall x \in \Omega, \\ w(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x w(L_2,t) = 0, & \forall t \in (0,T). \end{cases} \tag{4.1.2}$$

Then, the observability property of (4.1.2) is the following: can we find a constant $C_{obs} > 0$ such that every solution w of (4.1.2) satisfies

$$\|w(x,0)\|^2_{L^2(\Omega)} \leq C_{obs} \int_0^T |\partial_x w(-L_1,t)|^2 dt?$$

The main ingredient in the proof of the observability inequality is a suitable Carleman estimate where the observation is on the flux of the left-hand side of the domain. More precisely, we will get an estimate of the following form

$$\begin{aligned} &s^3 \int_0^T \int_\Omega \rho |z|^2 dxdt + s \int_0^T \int_\Omega \mu |\partial_x z|^2 dxdt \\ \leq &C \int_0^T \int_\Omega \nu |\partial_t \pm \partial_x(\sigma \partial_x z)|^2 dxdt + Cs \int_0^T \mu(t,-L_1)|\partial_x z(t,-L_1)|^2 dt, \end{aligned}$$

for some functions ρ, μ and ν depending on space and time.

Controllability issues of parabolic equations have been intensely studied by several authors since the 70's. For this reason, we will mention some of the most important results concerning scalar parabolic equations with smooths coefficients. In [42] and [43] H.O. Fattorini and D.L. Russel obtained for first time results about null boundary controllability for the one dimensional heat equation using the so-called method of moments. In contrast, in [88] the author proved a null controllability result for N-dimensional heat equation with a boundary control supported on the whole boundary of the domain. To be more specific, he proved that the null controllability of wave equation at a positive time implies the null controllability of the heat equation at any positive time.

In 1995, the null-controllability of the heat equation for high-dimensional case was solved by G. Lebeau and L. Robbiano [78] using a spectral inequality which was proved using local Carleman estimates. On the other hand, in 1996 the same problem was solved by A. Fursikov and O. Imanuvilov in [50]. Besides, the authors consider a general parabolic operator. This result was obtained by proving Carleman estimate for a general parabolic equation and for an arbitrary internal observation region.

The literature is also rich about controllability of other types of parabolic equations like Stokes or Navier-Stokes. For a deeper discussion, see for instance the survey of E. Fernandez-Cara and S. Guerrero [48] (see also [8] and the references given there).

Now we reduce our scope to controllability issues for parabolic problems in the case of non-smooth coefficients. In 2002, E. Fernandez-Cara and E. Zuazua in [49] proved a controllability result for $1-D$ linear parabolic equations for coefficients with bounded variations using the Russel's method (see [88]).

In [39] the authors proved a Carleman estimate and consequently a null controllability result for a semilinear heat equation in the case where the control is supported in the region where the diffusion coefficient is the 'lowest'. The key is the construction of a non-smooth weight function satisfying the same transmission condition as the solution.

In [20], the authors achieve a Carleman estimate for the operators of the form $\partial_t \pm \partial_x(\sigma\partial_x)$ without any restriction on the observation region, but this strategy does not extend to higher-dimensional cases. In the one-dimensional case, this monotonicity assumption of [39] on the diffusion coefficient was relaxed in [20] and [19] introducing more requirements on the non-smooth weight function. Also, they achieve a Carleman estimate with boundary observation in the Dirichlet case and finite jumps on the diffusion coefficient on the domain.

In [74], J. Le Rousseau derived a Carleman estimate for the problem above where the diffusion coefficient σ is a bounded variation function. The proofs relies in the idea of approximate the diffusion coefficient σ of $\partial_t \pm \partial_x(\sigma\partial_x)$ by a sequence of piecewise functions σ_ε and study the controllability properties of each problem with $\partial_t \pm \partial_x(\sigma_\varepsilon\partial_x)$ and later pass to the limit. The main issue in this limiting process is to keep both the weight functions and constants in the Carleman estimate under control. They also obtain Carleman estimates in the case of boundary observation considering Dirichlet boundary conditions for bounded variation diffusion coefficients.

On the other hand, in [76] the authors obtain a Carleman estimate for an operator of the type $\nabla\cdot(c(x)\nabla z)$ without any isotropy assumption. Specifically, in this article c is a symmetric positive-definite matrix with a jump discontinuity across a smooth hypersurface. Also, they give conditions on the Carleman weight functions that are rather simple to handle, and they prove that these functions are sharp.

Recently, in [82] the authors achieve a null controllability result for one-dimensional parabolic equation with generalized Robin-Neumann conditions at both extremities, with one boundary control. They following the flatness approach. Also, they obtain some numerical results on the reconstruction of the control for the problem above.

Our next task is formulate the main result of this work. To do this, we will introduce some notation. Let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator formally defined by

$$A = \partial_x\left(\sigma\partial_x\right),$$

and its domain of A is given by

$$D(A) = \left\{u \in H^1(\Omega)\,;\, \sigma\partial_x u \in H^1(\Omega)\,,\, u(-L_1) = 0\right\}.$$

Throught this section, we consider the following system:

$$\begin{cases} \partial_t z(x,t) - \partial_x\left(\sigma(x)\partial_x z(x,t)\right) = f(x,t), & \forall (x,t) \in \Omega\times(0,T), \\ z(x,0) = z^0(x), & \forall x \in \Omega, \\ z(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x z(L_2,t) = 0, & \forall t \in (0,T). \end{cases} \tag{4.1.3}$$

Suppose that $f \in L^2(\Omega \times (0,T))$ and $z^0 \in L^2(\Omega)$. It is clear that, under these assumptions, the problem (4.1.3) is well-posed. Moreover, due to the classical semigroup approach, the solution z of (4.1.3) satisfies

$$z(\cdot,t) \in D(A), \text{ for all } t \in (0,T). \tag{4.1.4}$$

Before going further, it is convenient to write the system (4.1.3) in terms of each parts of the domain separated by the interface located in $\{0\}$. To be more specific, let us $\Omega_1 = (-L_1, 0)$ and $\Omega_2 = (0, L_2)$. Here and consequently, for a spatial function h defined on the domain Ω, h_j stands its restriction to the subdomains Ω_j, for each $j = 1, 2$.

Thus, with this notation, system (4.1.3) can be written as follows:

$$\begin{cases} \partial_t z_1(x,t) - \sigma_1(x)^2 \partial_x^2 z_1(x,t) = f_1(x,t), & \forall (x,t) \in \Omega_1 \times (0,T), \\ \partial_t z_2(x,t) - \sigma_2(x)^2 \partial_x^2 z_2(x,t) = f_2(x,t), & \forall (x,t) \in \Omega_2 \times (0,T), \\ z_1(x,0) = z_1^0(x), & \forall x \in \Omega_1, \\ z_2(x,0) = z_2^0(x), & \forall x \in \Omega_2, \\ z_2(0,t) = z_1(0,t), & \forall t \in (0,T), \\ \sigma_2^2 \partial_x z_2(0,t) = \sigma_1^2 \partial_x z_1(0,t), & \forall t \in (0,T), \\ z_1(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x z_2(L_2,t) = 0, & \forall t \in (0,T), \\ \partial_x z_1(-L_1,t) = g_N(t), & \forall t \in (0,T). \end{cases} \tag{4.1.5}$$

The boundary conditions $(4.1.5)_3$ and $(4.1.5)_4$ are the transmission conditions of the system. Let us emphasize that these conditions arise due to the regularity of the solution z in (4.1.4).

Let us introduce the weight functions that we will use for state the Carleman inequality for systems like (4.1.3). For each $j = 1, 2$, we consider the positive functions $\phi_j = \phi_j(x) \in C^2(\overline{\Omega_j}, \mathbb{R})$ such that $\phi_1(0) = \phi_2(0)$ and

$$\begin{cases} m_j \leq \sigma_j \phi_j'(x) \leq M_j, & \text{in } \Omega_j, \\ r_j \leq -\sigma_j \phi_j''(x) \leq R_j, & \text{in } \Omega_j, \end{cases} \tag{4.1.6}$$

for some positive constants m_j, M_j, r_j and R_j with $M_2 < m_1$. Note that the assumptions in (4.1.6) imply:

$$m_1 = \sigma_1 \phi_1'(0), \; M_1 = \sigma_1 \phi_1(-L_1)', \; m_2 = \sigma_2 \phi_2(L_2)', \; M_2 = \sigma_2 \phi_2(0)'.$$

The Figure 2.1 sketch a prototype of functions ϕ_1 and ϕ_2 that we will use:

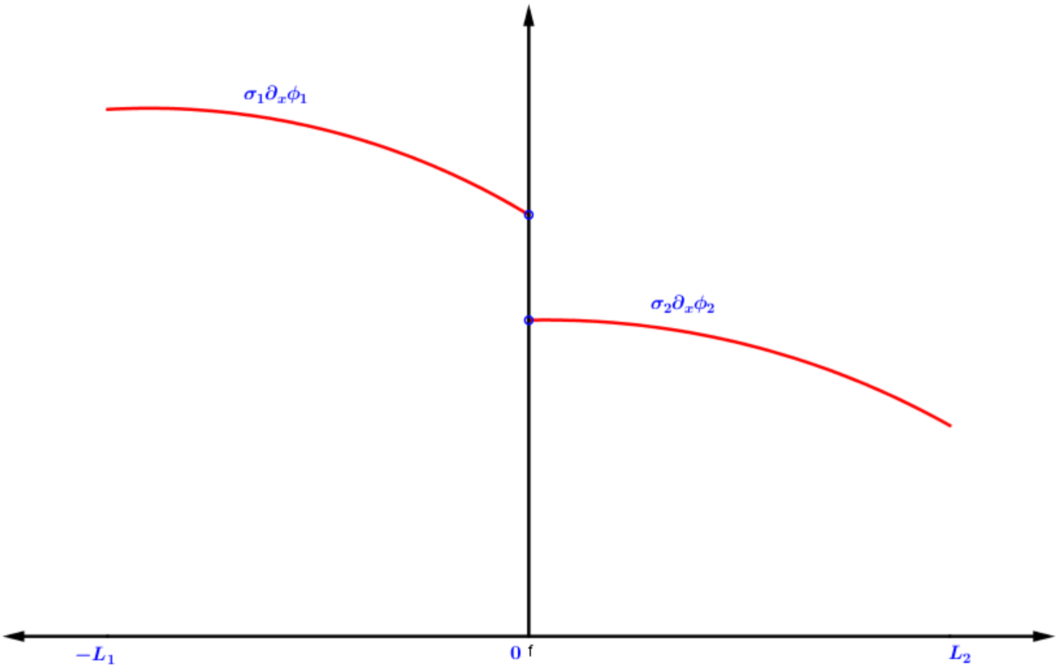

Figure 4.1: Sketch of the jump of the derivative of functions ϕ_1 and ϕ_2

For instance, note that the following class of functions:

$$\phi_1(x) = -\frac{(M_1 - m_1)}{2L_1\sigma_1}x^2 + \frac{m_1}{\sigma_1}x + a, \quad \phi_2(x) = -\frac{(M_2 - m_2)}{2L_2\sigma_2}x^2 + \frac{M_2}{\sigma_2}x + a, \quad a \in \mathbb{R},$$

satisfy the above assumptions, for $a \in \mathbb{R}$ sufficiently large such that ϕ_1 and ϕ_2 are positive.

Additionally, for a parameter $\alpha > 0$ and $j = 1, 2$, let us denote by φ_j the following functions:

$$\varphi_j(x,t) = \theta^\alpha(t)\phi_j(x), \quad \forall (x,t) \in \Omega_j \times (0,T), \tag{4.1.7}$$

where $\theta(t) = t^{-1}(T-t)^{-1}$. For simplicity of notation, we ignore the dependence of α and σ_j on φ_j.

We can now formulate the Carleman estimate for 1-D heat equation:

Theorem 4.1 *Consider the functions φ_j in (4.1.7) for $\alpha \geq 2$. Moreover, assume that $3\alpha \geq 2\beta$, with $\beta \geq 1$. Then, there exist two positive constants C and s_* such that for all*

$s \geq s_*$, the following inequality holds:

$$\begin{aligned}
&s^3 \int_0^T \int_{\Omega_1} e^{-2s\varphi_1}\theta^{3\alpha-\beta}|z_1|^2 dxdt + s^3 \int_0^T \int_{\Omega_2} e^{-2s\varphi_2}\theta^{3\alpha-\beta}|z_2|^2 dxdt \\
&+ s \int_0^T \int_{\Omega_1} e^{-2s\varphi_1}\theta^{\alpha-\beta}|\partial_x z_1|^2 dxdt + s \int_0^T \int_{\Omega_2} e^{-2s\varphi_2}\theta^{\alpha-\beta}|\partial_x z_2|^2 dxdt \\
&+ s^3 \int_0^T e^{-2s\varphi_1(t,0)}\theta^{3\alpha-\beta}|z_1(t,0)|^2 dt + s^3 \int_0^T e^{-2s\varphi_2(t,L_2)}\theta^{3\alpha-\beta}|z_2(t,L_2)|^2 dt \\
\leq& C \int_0^T \int_{\Omega_1} e^{-2s\varphi_1}\theta^{-\beta}|\partial_t z_1 \pm \sigma_1^2\partial_x^2 z_1|^2 dxdt + C \int_0^T \int_{\Omega_2} e^{-2s\varphi_2}\theta^{-\beta}|\partial_t z_2 \pm \sigma_2^2\partial_x^2 z_2|^2 dxdt \\
&+ Cs \int_0^T e^{-2s\varphi_1(t,-L_1)}\theta^{\alpha-\beta}|\partial_x z_1(-L_1)|^2 dt,
\end{aligned}$$

for each function $z_j \in L^2(\Omega_j \times (0,T))$ such that $\partial_t z_j \pm \sigma_j^2 \partial_x^2 z_j \in L^2(\Omega_j \times (0,T))$, for each $j = 1,2$, and $z_1(-L_1,t) = \partial_x z_2(L_2,t) = 0$ for each $t \in (0,T)$.

As a consequence of Theorem 4.1, we have the following:

Corollary 4.2 *Let (u_1, u_2) be the solution of the following system*

$$\begin{cases}
\partial_t u_1(x,t) - \sigma_1^2\partial_x^2 u_1(x,t) = 0, & \forall (x,t) \in \Omega_1 \times (0,T), \\
\partial_t u_2(x,t) - \sigma_2^2\partial_x^2 u_2(x,t) = 0, & \forall (x,t) \in \Omega_2 \times (0,T), \\
u_1(x,0) = u_1^0(x), & \forall x \in \Omega_1, \\
u_2(x,0) = u_2^0(x), & \forall x \in \Omega_2, \\
u_1(-L_1,t) = v(t), & \forall t \in (0,T), \\
\partial_x u_2(L_2,t) = 0, & \forall t \in (0,T),
\end{cases} \tag{4.1.8}$$

where $u_j^0 \in L^2(\Omega_j)$, for each $j = 1,2$ and $v \in L^2(0,T)$. Then, system (4.1.8) is null-controllable at time $T > 0$.

We will prove the Corollary 4.2 at the end of this chapter.

4.2 Proof of the Theorem 4.1

4.2.1 Setting

For simplicity, we prove the case where the operator involve is $\partial_t z_j - \sigma_j^2 \partial_x^2 z_j$ for each $j = 1,2$. The other case is completely analogous. First, we start reformulating our parabolic problem (4.1.5). This will be done into three steps:

• Step 1: Localization in time

Let us define η as a smooth function compactly supported in $]-1,1[$ with $\eta(0) = 1$. For $\beta \geq 0$, $\lambda \geq 1$ and $t_0 \in (0,T)$, we define the function η_{t_0} as follows:

$$\eta_{t_0}(t) = \eta\left(\frac{\lambda(t-t_0)}{t_0^\beta (T-t_0)^\beta}\right), \quad \forall t \in (0,T), \tag{4.2.1}$$

and shorten notation, we write η_{t_0} instead of η_{β,λ,t_0}. Moreover, for each $j = 1, 2$, we define

$$z_{j,t_0}(x,t) = \eta_{t_0}(t) z_j(x,t) \quad \forall (x,t) \in \Omega_j \times (0,T),$$

where (z_1, z_2) is the solution of system (4.1.5). Then, the new variables (z_{1,t_0}, z_{2,t_0}) solve the following problem:

$$\begin{cases} \partial_t z_{1,t_0} - \sigma_1^2 \partial_x^2 z_{1,t_0} = \eta_{t_0} f_1 + \partial_t \eta_{t_0} z_1, & \forall (x,t) \in \Omega_1 \times (0,T), \\ \partial_t z_{2,t_0} - \sigma_2^2 \partial_x^2 z_{2,t_0} = \eta_{t_0} f_2 + \partial_t \eta_{t_0} z_2, & \forall (x,t) \in \Omega_2 \times (0,T), \\ z_{2,t_0}(0,t) = z_{1,t_0}(0,t), & \forall t \in (0,T), \\ \sigma_2^2 \partial_x z_{2,t_0}(0,t) = \sigma_1^2 \partial_x z_{1,t_0}(0,t), & \forall t \in (0,T), \\ z_{1,t_0}(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x z_{2,t_0}(L_2,t) = 0, & \forall t \in (0,T), \\ \partial_x z_{1,t_0}(-L_1,t) = g_{N,t_0}(t), & \forall t \in (0,T), \end{cases}$$

where $g_{N,t_0} = \eta_{t_0} z(t, -L_1)$, for each $t \in (0,T)$ is the observation of our new system.

Notice that the form of η_{t_0} in (4.2.1) does not play an important role here. Besides, we have defined in this section for indicate the dependence of t_0 of the support of z_{1,t_0} and z_{2,t_0}.

• Step 2: Conjugation

For $\alpha > 0$, we consider the weight functions defined in (4.1.7). For each $j = 1, 2$, let us denote by Z_j the following function:

$$Z_j = e^{-s\varphi_j} z_{j,t_0} = e^{-s\varphi_j} \eta_{t_0} z_j, \quad \forall (x,t) \in \Omega_j \times (0,T),$$

where s is the Carleman parameter associated to the weight functions (4.1.7). Once again, for abbreviation, we write Z_j instead of Z_{j,t_0}. Then the unknown variables (Z_1, Z_2) solve the following problem:

$$\begin{cases} \partial_t Z_1 - \sigma_1^2 (\partial_x + s\theta^\alpha(t) \partial_x \phi_1)^2 Z_1 = g_1, & \forall (x,t) \in \Omega_1 \times (0,T), \\ \partial_t Z_2 - \sigma_2^2 (\partial_x + s\theta^\alpha(t) \partial_x \phi_2)^2 Z_2 = g_2, & \forall (x,t) \in \Omega_2 \times (0,T), \\ Z_2(0,t) = Z_1(0,t), & \forall t \in (0,T), \\ \sigma_2^2 \partial_x Z_2(0,t) = \sigma_1^2 \partial_x Z_1(0,t) + (m_1 \sigma_1 - M_2 \sigma_2) s\theta^\alpha(t) Z_2(0,t), & \forall t \in (0,T), \\ Z_1(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x Z_2(L_2,t) = -m_2 \sigma_2^{-1} \theta^\alpha(t) s Z_2(L_2,t), & \forall t \in (0,T), \\ \partial_x Z_1(-L_1,t) = G_N(t), & \forall t \in (0,T). \end{cases} \tag{4.2.2}$$

Here, the function G_N is defined by $G_N = e^{-s\varphi_1(-L_1,t)} g_{N,t_0}(t)$, in $(0,T)$. Moreover, for each $j = 1, 2$, the function g_j is given by

$$g_j = e^{-s\varphi_j} \eta_{t_0} f_j + e^{-s\varphi_j} \partial_t \eta_{t_0} z_j + \alpha \theta^{\alpha-1}(t) \partial_t \theta(t) \phi_j Z_j, \quad \forall (x,t) \in \Omega_j \times (0,T). \tag{4.2.3}$$

• **Step 3: Freeze in time**

Now, our next task is to avoid the dependence of t on the coefficients of the right-hand side of $(4.2.2)_1$ and $(4.2.2)_2$ and also on the boundary conditions $(4.2.2)_4$ and $(4.2.2)_6$.

To simplify our notation, for t_0 fixed, we denote by s_0 the following expression:

$$s_0 = s\theta^\alpha(t_0). \tag{4.2.4}$$

Then, we rewrite the system (4.2.2) as follows:

$$\begin{cases} \partial_t Z_1 - \sigma_1^2(\partial_x + \partial_x\phi_1 s_0)^2 Z_1 = F_1, & \forall (x,t) \in \Omega_1 \times (0,T), \\ \partial_t Z_2 - \sigma_2^2(\partial_x + \partial_x\phi_2 s_0)^2 Z_2 = F_2, & \forall (x,t) \in \Omega_2 \times (0,T), \\ Z_2(0,t) = Z_1(0,t), & \forall t \in (0,T), \\ \sigma_2^2\partial_x Z_2(0,t) = \sigma_1^2\partial_x Z_1(0,t) + (m_1\sigma_1 - M_2\sigma_2)s_0 Z_2(0,t) + H(t), & \forall t \in (0,T), \\ Z_1(-L_1,t) = 0, & \forall t \in (0,T), \\ \partial_x Z_2(L_2,t) = -m_2\sigma_2^{-1}s_0 Z_2(L_2,t) + J(t), & \forall t \in (0,T), \\ \partial_x Z_1(-L_1,t) = G_N(t), & \forall t \in (0,T), \end{cases} \tag{4.2.5}$$

where the residual functions H and J are defined by

$$H(t) = (m_1\sigma_1 - M_2\sigma_2)(\theta^\alpha(t) - \theta^\alpha(t_0))sZ_2(0,t), \quad t \in (0,T), \tag{4.2.6}$$
$$J(t) = (\theta^\alpha(t_0) - \theta^\alpha(t))m_2\sigma_2^{-1}sZ_2(L_2,t), \quad t \in (0,T). \tag{4.2.7}$$

Furthermore, the source term F_j in (4.2.5) is given by

$$\begin{aligned} F_j =& e^{-s\varphi_j}\eta_{t_0} f_j + e^{-s\varphi_j}\partial_t\eta_{t_0} z_j + \alpha\theta^{\alpha-1}(t)\partial_t\theta(t)\phi_j Z_j + 2\sigma_j^2(s\theta^\alpha(t) - s_0)\partial_x\phi_j\partial_x Z_j \\ &+\sigma_j^2(s\theta^\alpha(t) - s_0)\partial_x^2\phi_j Z_j + \sigma_j^2(s^2\theta^{2\alpha}(t) - s_0^2)|\partial_x\phi_j|^2 Z_j, \end{aligned} \tag{4.2.8}$$

for each $(x,t) \in \Omega_j \times (0,T)$ and $j = 1,2$.

Let us emphasize that in this step, several residual functions of different nature appears, see the definition of H, J and F_j for instance. At the moment, we will treat these ones as a source terms and later we will choose the parameters α, β, λ and s in order to eliminate them in the classical spirit of Carleman estimates.

As we said before, the proof of the main result is is deduced by the following:

Lemma 4.3 *There exists a positive constant $C_1 = C_1(\alpha,\beta,m_1,M_1,m_2,M_2,r_1,r_2,\sigma_1,\sigma_2)$ independent of s such that each solution (Z_1,Z_2) of (4.2.5) with $F_j \in L^2(\Omega_j \times (0,T))$,*

$j=1,2$, $G_N \in L^2(0,T)$ *and* $H, J \in L^2(0,T)$ *satisfies*

$$\begin{aligned}
&s_0^3\int_0^T\int_{\Omega_1}|Z_1|^2dxdt + s_0^3\int_0^T\int_{\Omega_2}|Z_2|^2dxdt + s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1|^2dxdt\\
&+s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2|^2dxdt + s_0^3\int_0^T|Z_2(0,t)|^2dt + s_0^3\int_0^T|Z_2(L_2,t)|^2dt\\
\leq& C_1\int_0^T\int_{\Omega_1}|F_1|^2dxdt + C_1\int_0^T\int_{\Omega_2}|F_2|^2dxdt + C_1 s_0\int_0^T|G_N|^2dt\\
&+C_1 s_0\int_0^T|H|^2dt + C_1 s_0\int_0^T|J|^2dt,
\end{aligned} \tag{4.2.9}$$

for all $s \geq s_*$.

We will give the proof of the Lemma 4.3 later. Now, with this Lemma at hand, we can conclude the proof of Theorem 4.1.

4.2.2 End of the proof of Theorem 4.1

Our main goal is eliminate the residual terms appeared in Lemma 1 for each solution of the system (4.2.5). For an easier comprehension, the rest of the proof of Theorem 4.1 falls naturally into three parts:

- **Step 1: First residual terms:** H **and** J

Proposition 4.4 *We assume the same hypotheses as Lemma 4.3. Additionally, suppose that* $2\beta+\alpha \geq 2$. *Then, there exist a positive constant* λ_* *such that for every* $\lambda \geq \lambda_*$, *the following inequality holds:*

$$C_1 s_0\int_0^T|H|^2dt + C_1 s_0\int_0^T|J|^2dt \leq \frac{1}{2}s_0^3\int_0^T|Z_2(0,t)|^2dt + \frac{1}{2}s_0^3\int_0^T|Z_2(L_2,t)|^2dt, \tag{4.2.10}$$

where C_1 *is the constant associated to Lemma 4.3.*

Proof. Our proof starts with the observation that for each $t \in \operatorname{supp}(\eta_{t_0})$ satisfies the following:

$$|t-t_0| \leq \lambda^{-1}(t_0(T-t_0))^{\beta}. \tag{4.2.11}$$

Then, for each $t \in \operatorname{supp}(\eta_{t_0})$ we have

$$|\theta^\alpha(t)-\theta^\alpha(t_0)| \leq C(\theta^\alpha(t_0))'|t-t_0| \leq C\lambda^{-1}|t_0(T-t_0)|^{\beta-\alpha-1}, \tag{4.2.12}$$

for some constant $C>0$. Then, the $L^2(0,T)$-norm of H can be bounded as follows:

$$C_1 s_0\int_0^T|H|^2dt \leq C|t_0(T-t_0)|^{2\beta-\alpha-2}\lambda^{-2}s^3\int_0^T|Z_2(0,t)|^2dt. \tag{4.2.13}$$

Furthermore, notice that if we choose λ_* such that

$$2C\left|\frac{T^2}{4}\right|^{2\beta+2\alpha-2} \leq \lambda_*^2, \tag{4.2.14}$$

where C is the constant appear in (4.2.13), it is evident that

$$C|t_0(T-t_0)|^{2\beta-\alpha-2}\lambda^{-2} \leq \frac{1}{2}|t_0(T-t_0)|^{-3\alpha}, \quad \forall\lambda \geq \lambda_*. \tag{4.2.15}$$

Thus, combining (4.2.13) with (4.2.15), we obtain

$$C_1 s_0 \int_0^T |H|^2 dt \leq \frac{1}{2} s_0^3 \int_0^T |Z_2(0,t)|^2 dt. \tag{4.2.16}$$

Analogously, straightforward computations show that

$$C_1 s_0 \int_0^T |J|^2 dt \leq C\left|t_0(T-t_0)\right|^{2\beta-\alpha-2}\lambda^{-2} s^3 \int_0^T |Z_2(L_2,t)|^2 dt. \tag{4.2.17}$$

Then, if we choose λ_* such that

$$2C\left|\frac{T^2}{4}\right|^{2\beta+2\alpha-2} \leq \lambda_*^2, \tag{4.2.18}$$

where the constant C is as the right-hand side of (4.2.17), we deduce that

$$C\left|t_0(T-t_0)\right|^{2\beta-\alpha-2}\lambda^{-2} \leq \frac{1}{2}|t_0(T-t_0)|^{-3\alpha}, \tag{4.2.19}$$

for each $\lambda \geq \lambda_*$. Hence, we have the following upper-bound for $L^2(0,T)-$norm of J:

$$C_1 s_0 \int_0^T |J|^2 dt \leq \frac{1}{2} s_0^3 \int_0^T |Z_2(0,t)|^2 dt. \tag{4.2.20}$$

Consequently, we add the inequalities (4.2.16) and (4.2.20), with λ_* the maximum value which satisfies (4.2.14) and (4.2.18) and the proof of Proposition 4.4 is complete. $\square$

Applying the Proposition 1 into (4.2.9), we see that

$$\begin{aligned}
&s_0^3 \int_0^T \int_{\Omega_1} |Z_1|^2 dxdt + s_0^3 \int_0^T \int_{\Omega_2} |Z_2|^2 dxdt + s_0 \int_0^T \int_{\Omega_1} |\partial_x Z_1|^2 dxdt \\
&+ s_0 \int_0^T \int_{\Omega_2} |\partial_x Z_2|^2 dxdt + s_0^3 \int_0^T |Z_2(0,t)|^2 dt + s_0^3 \int_0^T |Z_2(L_2,t)|^2 dt \\
\leq& C_2 \int_0^T \int_{\Omega_1} |F_1|^2 dxdt + C_2 \int_0^T \int_{\Omega_2} |F_2|^2 dxdt + C_2 s_0 \int_0^T |G_N|^2 dt,
\end{aligned} \tag{4.2.21}$$

for some constant $C_2 > 0$ independent of s and for every $\lambda \geq \lambda_*$.

• **Step 2: Residual terms of F_j**

Now, we are interested in eliminate the residual terms of F_1 (and F_2 respectively) in (4.2.8) depending on Z_1 and $\partial_x Z_1$ (and Z_2 and $\partial_x Z_2$ respectively). First, we rewrite the terms of F_j as follows:

$$F_j = \tilde{f}_j + K_j,$$

where $\tilde{f}_j$ and K_j is defined by

$$\tilde{f}_j = e^{-s\varphi_j}\eta_{t_0} f_j + e^{-s\varphi_j}\partial_t \eta_{t_0} z_j,$$

and

$$\begin{aligned} K_j =& \alpha s \theta^{\alpha-1}\partial_t\theta \phi_j Z_j + \sigma_j^2\left(\theta^{2\alpha}(t) - \theta^{2\alpha}(t_0)\right)|\partial_x\phi_j|^2 s^2 Z_j + \sigma_j^2(\theta^\alpha(t) - \theta^\alpha(t_0))\partial_x^2\phi_j s Z_j \\ &- \sigma_j^2(\theta^\alpha(t) - \theta^\alpha(t_0)) s\partial_x Z_j \end{aligned}$$

in $(0,T)\times\Omega_j$. We recall that η_{t_0} is defined as follows:

$$\eta_{t_0}(t) = \eta\left(\frac{\lambda(t-t_0)}{(t_0(T-t_0))^\beta}\right),$$

where the support of the function η belongs in $]-1,1[$ and $\eta(0)=1$.

Proposition 4.5 *Let $t_0 \in (0,T)$ fixed and suppose that $\beta \geq 1$ in the definition of η_{t_0} in (4.2.1). Then, there exist three positive constants c_1, c_2 and λ_* independent of t_0 such that*

$$c_1 \leq \frac{t(T-t)}{t_0(T-t_0)} \leq c_2, \quad \forall \lambda \geq \lambda_*.$$

Proof. By definition of η_{t_0} we can write each term of its support as follows:

$$t = t_0 + a\lambda^{-1}t_0^\beta(T-t_0)^\beta, \quad \text{with } -1 < a < 1.$$

Then, we have

$$\begin{aligned} \psi_{t_0}(t) :=& \frac{t(T-t)}{t_0(T-t_0)} \\ =& 1 + a\lambda^{-1}t_0^{\beta-1}(T-t_0)^{\beta-1}\left(T - 2t_0 - a\lambda^{-1}t_0^\beta(T-t_0)^\beta\right), \end{aligned} \tag{4.2.22}$$

for each t lying in the support of ψ_{t_0}.We divide the proof into four cases:

• **Case 1:** Suppose that $0 < t_0 \leq T/2$ and $a \geq 0$. In this case, notice that

$$\begin{aligned} \psi_{t_0}(t) \geq& 1 - s^2\lambda^{-2}(t_0(T-t_0))^{2\beta-1} \\ \geq& 1 - \lambda^{-2}\frac{T^{2(2\beta-1)}}{4^{2\beta-1}}. \end{aligned}$$

It is clear that if we choose λ_* such that

$$\sqrt{2}\frac{T^{2\beta-1}}{2^{\beta-1}} \leq \lambda_*,$$

it is clear that

$$1-\lambda^{-2}\frac{T^{2(2\beta-1)}}{4^{2\beta-1}}\geq\frac{1}{2},\quad \lambda\geq\lambda_*,$$

Therefore, from the estimates above, we can assert that

$$\psi_{t_0}(t)\geq\frac{1}{2},\quad \forall\lambda\geq\lambda_*.$$

On the other hand,

$$\begin{aligned}\psi_{t_0}(t)\leq&1+a\lambda^{-1}t_0^{\beta-1}(T-t_0)^{\beta-1}\\ \leq&1+\lambda_*^{-1}\left(\frac{T^2}{4}\right)^{\beta-1}.\end{aligned}$$

Therefore, we obtain the following bounds for ψ_{t_0}:

$$\frac{1}{2}\leq\psi_{t_0}(t)\leq 1+a\lambda_*\left(\frac{T}{4}\right)^{\beta-1}, \tag{4.2.23}$$

where λ_* does not depend on t_0.

• **Case 2:** Suppose that $0<t_0\leq T/2$ and $a<0$. For simplicity, we will use the temporary notation $b=-a$ that is, $b>0$. Then, (4.2.22) can be rewritten as follows:

$$\psi_{t_0}(t)=1-b\lambda^{-1}t_0^{\beta-1}(T-t_0)^{\beta-1}\left(T-2t_0+b\lambda^{-1}t_0^{\beta}(T-t_0)^{\beta}\right).$$

It is easy to see that

$$\psi_{t_0}(t)\leq 1.$$

On the other hand,

$$\psi_{t_0}(t)\geq 1-b(T-2t_0)\frac{T^{2(\beta-1)}}{2^{2(\beta-1)}}\lambda^{-1}-b^2\frac{T^{2(2\beta-1)}}{2^{2(2\beta-1)}}\lambda^{-2}.$$

Straightforward computations show that if we choose λ_* such that

$$\lambda_*\geq\left(2+\sqrt{\frac{3}{2}}\right)\frac{T^{2\beta-1}}{2^{\beta-2}},$$

we can assert that

$$1-b(T-2t_0)\frac{T^{2(\beta-1)}}{2^{2(\beta-1)}}\lambda^{-1}-b^2\frac{T^{2(2\beta-1)}}{2^{2(2\beta-1)}}\lambda^{-2}\geq\frac{1}{2},\quad \forall\lambda\geq\lambda_*.$$

Hence, we have

$$\frac{1}{2}\leq\psi_{t_0}(t)\leq 1,\quad \forall\lambda\geq\lambda_*. \tag{4.2.24}$$

• **Case 3:** Suppose that $T/2<t_0<T$ and $a\geq 0$. Then, it is easy to see that

$$\psi_{t_0}(t)\leq 1.$$

On the other hand, a simple computation shows that if we choose λ_* fullfilling

$$\lambda_* \geq \frac{5}{2}\frac{T^{2\beta-1}}{2^{2\beta-1}},$$

we obtain

$$\psi_{t_0}(t) \geq 1-(2t_0-T)a\lambda^{-1}\left|\frac{T^2}{4}\right|^{\beta-1} - a^2\lambda^{-2}\left|\frac{T^2}{4}\right|^{2\beta-1} \geq \frac{1}{2}, \quad \lambda \geq \lambda_*.$$

Thus, we have

$$\frac{1}{2} \leq \psi_{t_0}(t) \leq 1, \quad \lambda \geq \lambda_*. \tag{4.2.25}$$

• **Case 4:** Suppose that $T/2 < t_0 < T$ and $a < 0$. Once again, we set $b = -a$. In this case, we note that

$$\begin{aligned}\psi_{t_0}(t) =& 1 - b\lambda^{-1}t_0^{\beta-1}(T-t_0)^{\beta-1}\left(T-2t_0+b\lambda^{-1}t_0^{\beta}(T-t_0)^{\beta}\right)\\ \leq& 1+b(2t_0-T)\lambda^{-1}t_0^{\beta-1}(T-t_0)^{\beta-1}\\ \leq& 1+2^{2-2\beta}b\lambda_*^{-1}T^{2\beta-1}.\end{aligned}$$

Furthermore, it is clear that if we choose λ_* such that

$$\lambda \geq 2^{2\beta-1}\sqrt{2}bT^{2\beta-1},$$

we obtain

$$1-b^2\lambda^{-2}2^{2-4\beta}T^{4\beta-2} \geq \frac{1}{2}, \quad \lambda \geq \lambda_*.$$

Therefore,

$$\frac{1}{2} \leq \psi_{t_0}(t) \leq 1+b2^{2-2\beta}\lambda_*^{-1}T^{2\beta-1}, \quad \forall \lambda \geq \lambda_*. \tag{4.2.26}$$

From (4.2.23), (4.2.24), (4.2.25) and (4.2.26), we conclude the proof of the Proposition 4.5. □

Proposition 4.6 *We Suppose the same hypotheses of Lemma 1. Additionally, let α and β such that $\alpha \geq 2$ and $2\beta - \alpha \geq 2$. Then, for each $j = 1,2$, there exist two positive constants λ_* and s_* such that for every $\lambda \geq \lambda_*$ and $s \geq s_*$ we have*

$$C_2\int_0^T\int_{\Omega_j}|K_j|^2dxdt \leq \frac{1}{2}s_0^3\int_0^T\int_{\Omega_j}|Z_j|^2dxdt + \frac{1}{2}s_0\int_0^T\int_{\Omega_j}|\partial_x Z_j|^2dxdt, \quad j=1,2,$$

where C_2 is the constant previously defined in (4.2.21).

Proof. By definition of K_j, we have

$$\begin{aligned}&C_2\int_0^T\int_{\Omega_j}|K_j|^2dxdt\\ \leq& C(t_0(T-t_0))^{-2(\alpha+1)}s^2\int_0^T\int_{\Omega_j}|Z_j|^2dxdt + Cs^4\int_0^T\int_{\Omega_j}|\theta^{2\alpha}(t)-\theta^{2\alpha}(t_0)|^2|Z_j|^2dxdt\\ &+Cs^2\int_0^T\int_{\Omega_j}|\theta^{\alpha}(t)-\theta^{\alpha}(t_0)|^2|Z_j|^2dxdt + Cs^2\int_0^T\int_{\Omega_j}|\theta^{\alpha}(t)-\theta^{\alpha}(t_0)|^2|\partial_x Z_j|^2dxdt.\end{aligned} \tag{4.2.27}$$

Let us estimate each term of the right-hand side of the inequality above. First, notice that if we choose s_* such that

$$4C_2\left|\frac{T^2}{4}\right|^{\alpha-2} \leq s_*, \tag{4.2.28}$$

the following inequality holds:

$$C_2|t_0(T-t_0)|^{-2(\alpha+1)} \leq \frac{1}{4}|t_0(T-t_0)|^{-3\alpha}s, \quad s \geq s_*. \tag{4.2.29}$$

This implies,

$$C_2|t_0(T-t_0)|^{-2(\alpha+1)}s^2\int_0^T\int_{\Omega_j}|Z_j|^2dxdt \leq \frac{1}{4}s_0^3\int_0^T\int_{\Omega_j}|Z_j|^2dxdt, \quad s \geq s_*. \tag{4.2.30}$$

On the other hand, we note that for each $t \in \operatorname{supp}(\eta_{t_0})$, we have

$$|\theta^{2\alpha}(t)-\theta^{2\alpha}(t_0)| \leq C\theta^{2\alpha-1}(t_0)\partial_t\theta(t_0)|t-t_0| \leq C|t_0(T-t_0)|^{\beta-2\alpha-1}\lambda^{-1}, \tag{4.2.31}$$

for some constant $C>0$. Then, notice that for each $t \in \operatorname{supp}(\eta_{t_0})$, we can assert that

$$\begin{aligned}
&C_2s^4\int_0^T\int_{\Omega_j}|\theta^{2\alpha}(t)-\theta^{2\alpha}(t_0)|^2|Z_j|^2dxdt \\
\leq &C|t_0(T-t_0)|^{2\beta-4\alpha-2}\lambda^{-2}s^4\int_0^T\int_{\Omega_j}|Z_j|^2dxdt.
\end{aligned} \tag{4.2.32}$$

If we choose the parameters s_* and λ_* such that

$$4C\left|\frac{T^2}{4}\right|^{2\beta-\alpha-2} \leq \lambda_*^2 s_*^{-1}, \tag{4.2.33}$$

where the constant $C>0$ is given in (4.2.32), it is easy to check that

$$C|t_0(T-t_0)|^{2\beta-4\alpha-2}\lambda^{-2}s^4 \leq \frac{1}{4}s^3(t_0)(T-t_0)|^{-3\alpha}, \quad \lambda \geq \lambda_*,\ s \geq s_*. \tag{4.2.34}$$

Thus, combining (4.2.32) and (4.2.34), the first term of the right-hand side of (4.2.27) can be bounded as follows:

$$C_2s^4\int_0^T\int_{\Omega_j}|\theta^{2\alpha}(t)-\theta^{2\alpha}(t_0)|^2|Z_j|^2dxdt \leq \frac{1}{4}s_0^3\int_0^T\int_{\Omega_j}|Z_j|^2dxdt. \tag{4.2.35}$$

Now we deal with the second term of the right-hand side of (4.2.27). Using the estimate of θ^α in (4.2.12), we see that

$$C_2s^2\int_0^T\int_{\Omega_j}|\theta^{\alpha}(t)-\theta^{\alpha}(t_0)|^2|Z_j|^2dxdt \leq C|t_0(T-t_0)|^{2\beta-2\alpha-2}\lambda^{-2}s^2\int_0^T\int_{\Omega_j}|Z_j|^2dxdt. \tag{4.2.36}$$

If we choose the parameters λ_* and s_* such that

$$4C\left|\frac{T^2}{4}\right|^{2\beta+\alpha-2} \leq \lambda_*^2 s_*, \tag{4.2.37}$$

where C is the constant appeared in (4.2.36), it is easy to seen that

$$C|t_0(T-t_0)|^{2\beta-\alpha-2} \leq \frac{1}{4}|t_0(T-t_0)|^{-3\alpha}s^3\int_0^T\int_{\Omega_j}|Z_j|^2dxdt, \quad \lambda\geq\lambda_*,\ s\geq s_*. \tag{4.2.38}$$

Substituting (4.2.38) into (4.2.36) yields

$$Cs^2\int_0^T\int_{\Omega_j}|\theta^\alpha(t)-\theta^\alpha(t_0)|^2|Z_j|^2dxdt \leq \frac{1}{4}s_0^3\int_0^T\int_{\Omega_j}|Z_j|^2dxdt. \tag{4.2.39}$$

Once again, applying the estimate (4.2.12), the third term of the right-hand side of (4.2.27) can be bounded as follows:

$$Cs^2\int_0^T\int_{\Omega_j}|\theta^\alpha(t)-\theta^\alpha(t_0)|^2|\partial_x Z_j|^2dxdt \leq C|t_0(T-t_0)|^{2\beta-2\alpha-2}\lambda^{-2}s^2\int_0^T\int_{\Omega_j}|\partial_x Z_j|^2dxdt. \tag{4.2.40}$$

It is inmediate that if we choose λ_* and s_* satisfying

$$2C\left|\frac{T^2}{4}\right|^{2\beta-\alpha-2} \leq \lambda_*^2 s_*^{-1}, \tag{4.2.41}$$

it follows that

$$C|t_0(T-t_0)|^{2\beta-2\alpha-2} \leq \frac{1}{2}|t_0(T-t_0)|^{-\alpha}, \quad \forall\lambda\geq\lambda_*,\ \forall s\geq s_*. \tag{4.2.42}$$

Substituting (4.2.42) into (4.2.40), we get

$$Cs^2\int_0^T\int_{\Omega_j}|\theta^\alpha(t)-\theta^\alpha(t_0)|^2|\partial_x Z_j|^2dxdt \leq \frac{1}{2}s_0\int_0^T\int_{\Omega_j}|\partial_x Z_j|^2dxdt. \tag{4.2.43}$$

Finally, we add the inequalities (4.2.35), (4.2.39) and (4.2.43), and the proof of the Proposition 4.6 is complete. □

Thus, applying the Proposition 4.4 and 4.6 into (4.2.9), we conclude that

$$\begin{aligned} &s_0^3\int_0^T\int_{\Omega_1}|Z_1|^2dxdt + s_0^3\int_0^T\int_{\Omega_2}|Z_2|^2dxdt + s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1|^2dxdt \\ &+ s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2|^2dxdt \\ \leq& C\int_0^T\int_{\Omega_1}|\tilde{f}_1|^2dxdt + C\int_0^T\int_{\Omega_2}|\tilde{f}_2|^2dxdt + Cs_0\int_0^T|G_N|^2dt, \end{aligned} \tag{4.2.44}$$

for some constant $C > 0$ independent of λ and s.

- **Step 3: Last residual terms depending on z_j, $j = 1, 2$**

Our next task is to deal with the residual terms depending on z_j from $\tilde{f}_j$, $j = 1, 2$. In order to do that, we introduce the following functions:

$$\omega_0(t) = \int_0^T |\eta_{t_0}(t)|^2 dt_0, \quad \omega_1(t) = \int_0^T |\partial_t \eta_{t_0}(t)| dt_0.$$

In the reminder of this step, we need the following technical result:

Proposition 4.7 *There exist positive constants c_3, c_4, c_5, c_6 and λ_* such that, for all $t \in$ supp(η_{t_0}),*

$$c_3 \lambda^{-1} \theta^{-\beta}(t) \leq \omega_0(t) \leq c_4 \lambda^{-1} \theta^{-\beta}(t), \quad \forall \lambda \geq \lambda_*, \tag{4.2.45}$$

and

$$c_5 \lambda \theta^{\beta}(t) \leq \omega_1(t) \leq c_6 \lambda \theta^{\beta}(t), \quad \forall \lambda \geq \lambda_*. \tag{4.2.46}$$

Proof. By definition,

$$\begin{aligned} \omega_0(t) &= \int_0^T \left| \eta \left(\frac{\lambda(t - t_0)}{(t_0(T - t_0))^\beta} \right) \right|^2 dt_0 \\ &= \int_0^T |\eta(\tilde{t})|^2 \left| \frac{d\tilde{t}}{dt_0} \right|^{-1} dt_0, \end{aligned} \tag{4.2.47}$$

where $\tilde{t}$ is defined by

$$\tilde{t} = \frac{\lambda(t - t_0)}{t_0^\beta (T - t_0)^\beta}.$$

A direct compute show that

$$\frac{d\tilde{t}}{dt_0} = -\lambda \left(\frac{1}{(t_0(T - t_0))^\beta} + \frac{\beta(T - 2t_0)(t - t_0)}{(t_0(T - t_0))^{\beta+1}} \right).$$

Since every $t \in \text{supp}(\eta_{t_0})$ satisfy

$$|t - t_0| \leq \lambda^{-1}(t_0(T - t_0))^\beta,$$

we can assert that

$$\frac{\lambda}{(t_0(T - t_0))^\beta} - \beta T \leq \left| \frac{d\tilde{t}}{dt_0} \right| \leq \frac{\lambda}{(t_0(T - t_0))^\beta} + \beta T.$$

Taking λ_* large enough and using the Proposition 4.5, we have for each $t \in \text{supp}(\eta_{t_0})$

$$c_1 \lambda (t(T - t_0))^{-\beta} \leq \left| \frac{d\tilde{t}}{dt_0} \right| \leq c_2 \lambda (t(T - t))^{-\beta}, \quad \lambda \geq \lambda_*.$$

Consequently, we have for all $t \in \operatorname{supp}(\eta_{t_0})$

$$c_3\lambda^{-1}(t(T-t))^{\beta} \le \omega_0(t) \le c_4\lambda^{-1}(t(T-t))^{\beta}, \quad \lambda \ge \lambda_*,\ t \in \operatorname{supp}(\eta_{t_0}).$$

This completes the proof of the first inequality. To proof the other one, notice that

$$\partial_t \eta_{t_0}(t) = \frac{\lambda}{(t_0(T-t_0))^{\beta}} \partial_{\tilde{t}} \eta(\tilde{t}),$$

where $\tilde{t}$ is defined as before. Then,

$$\omega_1(t) = \lambda^2 \int_0^T t_0^{-2\beta}(T-t)^{-2\beta} |\partial_{\tilde{t}}\eta(\tilde{t})|^2 \left|\frac{d\tilde{t}}{dt_0}\right|^{-1} dt_0.$$

The rest of the proof runs as before. Thus, the proof of the Proposition 4.7 is complete. $\square$

Now, we integrate in t_0 on $(0,T)$ in (4.2.44) and by definition of Z_1, Z_2 and G_N, we obtain

$$\begin{aligned}
&\int_0^T\int_0^T\int_{\Omega_1} s_0^3 e^{-2s\varphi_1}|\eta_{t_0}z_1|^2 dxdtdt_0 + \int_0^T\int_0^T\int_{\Omega_2} s_0^3 e^{-2s\varphi_2}|\eta_{t_0}z_2|^2 dxdtdt_0 \\
&+\int_0^T\int_0^T\int_{\Omega_1} s_0 e^{-2s\varphi_1}|\eta_{t_0}\partial_x z_1|^2 dxdtdt_0 + \int_0^T\int_0^T\int_{\Omega_2} s_0 e^{-2s\varphi_2}|\eta_{t_0}\partial_x z_2|^2 dxdtdt_0 \\
&+\int_0^T\int_0^T s_0^3 e^{-2s\varphi(0,t)}|\eta_{t_0}z_1(0,t)|^2 dtdt_0 + \int_0^T\int_0^T s_0^3 e^{-2s\varphi(L_2,t)}|\eta_{t_0}z_2(L_2,t)|^2 dtdt_0 \\
\le& C\int_0^T\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}|\tilde{f}_1|^2 dxdtdt_0 + C\int_0^T\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}|\tilde{f}_2|^2 dxdtdt_0 \\
&+C\int_0^T\int_0^T s_0 e^{-2s\varphi_1(-L_1,t)}|\eta_{t_0}z_1(-L_1,t)|^2 dtdt_0,
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^T\int_0^T\int_{\Omega_j} |\tilde{f}_j|^2 dxdt \\
\le& C\int_0^T\int_0^T\int_{\Omega_j} e^{-2s\varphi_j}|\eta_{t_0}|^2|f_j|^2 dxdtdt_0 + C\int_0^T\int_0^T\int_{\Omega_2} e^{-2s\varphi_j}|\partial_t\eta_{t_0}|^2|z_j|^2 dxdtdt_0
\end{aligned}$$

Applying Fubini's Theorem and the results of Proposition 4.7, we deduce that

$$\begin{aligned}
&\lambda^{-1}s^3\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{3\alpha-\beta}|z_1|^2 dxdt + \lambda^{-1}s^3\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{3\alpha-\beta}|z_2|^2 dxdt \\
&+\lambda^{-1}s\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{\alpha-\beta}|\partial_x z_1|^2 dxdt + \lambda^{-1}s\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{\alpha-\beta}|\partial_x z_2|^2 dxdt \\
&+\lambda^{-1}s^3\int_0^T e^{-2s\varphi_1(t,0)}\theta^{3\alpha-\beta}|z_1(t,0)|^2 dt + \lambda^{-1}s^3\int_0^T e^{-2s\varphi_2(t,L_2)}\theta^{3\alpha-\beta}|z_2(t,L_2)|^2 dt \\
\le& C\lambda^{-1}\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{-\beta}|f_1|^2 dxdt + \lambda^{-1}\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{-\beta}|f_2|^2 dxdt \\
&+C\lambda\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{\beta}|z_1|^2 dxdt + C\lambda\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{\beta}|z_2|^2 dxdt \\
&+C\lambda^{-1}s\int_0^T e^{-2s\varphi_1(-L_1)}\theta^{\alpha-\beta}|g_N|^2 dt,
\end{aligned}$$

or equivalently

$$
\begin{aligned}
&s^3\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{3\alpha-\beta}|z_1|^2dxdt + s^3\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{3\alpha-\beta}|z_2|^2dxdt\\
&+ s\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{\alpha-\beta}|\partial_x z_1|^2dxdt + s\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{\alpha-\beta}|\partial_x z_2|^2dxdt\\
&+ s^3\int_0^T e^{-2s\varphi_1(0,t)}\theta^{3\alpha-\beta}|z_1(0,t)|^2dt + s^3\int_0^T e^{-2s\varphi_2(L_2,t)}\theta^{3\alpha-\beta}|z_2(L_2,t)|^2dt\\
\le & C_3\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{-\beta}|f_1|^2dxdt + C_3\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{-\beta}|f_2|^2dxdt\\
&+ C_3\lambda^2\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{\beta}|z_1|^2dxdt + C_3\lambda^2\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{\beta}|z_2|^2dxdt\\
&+ C_3 s\int_0^T e^{-2s\varphi_1(-L_1)}\theta^{\alpha-\beta}|g_N|^2dt,
\end{aligned}
$$

To eliminate the terms of z_1 and z_2 on the right-hand side of the inequality above, we will use the following result:

Proposition 4.8 *Suppose that $\alpha \ge 2$ and $3\alpha \ge 2\beta$. Then, there exists two positive constants s_* and λ_* such that for all $s \ge s_*$ and $\lambda \ge \lambda_*$, we have*

$$
C_3\lambda^2\int_0^T\int_{\Omega_j} e^{-2s\varphi_j}\theta^{\beta}|z_j|^2dxdt \le \frac{1}{2}s^3\int_0^T\int_{\Omega_j} e^{-2s\varphi_j}\theta^{3\alpha-\beta}|z_j|^2dxdt, \qquad (4.2.48)
$$

for each $j = 1, 2$.

Finally, using the estimates (4.2.48) we obtain

$$
\begin{aligned}
&s^3\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{3\alpha-\beta}|z_1|^2dxdt + s^3\int_0^T\int_{\Omega_2} e^{-2s\varphi_1}\theta^{3\alpha-\beta}|z_2|^2dxdt\\
&+ s\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{\alpha-\beta}|\partial_x z_1|^2dxdt + s\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{\alpha-\beta}|\partial_x z_2|^2dxdt\\
&+ s^3\int_0^T e^{-2s\varphi_1(0,t)}\theta^{3\alpha-\beta}|z_1(0,t)|^2dt + s^3\int_0^T e^{-2s\varphi_2(L_2,t)}\theta^{3\alpha-\beta}|z_2(L_2,t)|^2dt\\
\le & C\int_0^T\int_{\Omega_1} e^{-2s\varphi_1}\theta^{-\beta}|f_1|^2dxdt + C\int_0^T\int_{\Omega_2} e^{-2s\varphi_2}\theta^{-\beta}|f_2|^2dxdt\\
&+ Cs\int_0^T e^{-2s\varphi_1(L_1,t)}\theta^{\alpha-\beta}|z_1(-L_1,t)|^2dt.
\end{aligned}
$$

This completes the proof of Theorem 4.1.

4.3 Proof of Lemma 4.3

In this section, we devote to prove the Lemma 4.3. The proof of this one falls naturally into two parts:

• The first one concerns in the L^2 estimates for the global terms of Z_1 and Z_2, the local term of Z_1 at the interface and the local term of Z_2 at the right-hand side of the domain:

$$\begin{aligned}
&s_0^3\int_0^T\int_{\Omega_1}|Z_1|^2dxdt+s_0^3\int_0^T\int_{\Omega_2}|Z_2|^2dxdt+s_0^3\int_0^T|Z_1(0,t)|^2dt+s_0^3\int_0^T|Z_2(L_2,t)|^2dt\\
\leq &C\int_0^T\int_{\Omega_1}|F_1|^2dxdt+C\int_0^T\int_{\Omega_2}|F_2|^2dxdt+Cs_0\int_0^T|G_N|^2dt\\
&+Cs_0\int_0^T|H|^2dt+Cs_0\int_0^T|J|^2dt,
\end{aligned}\tag{4.3.1}$$

where C is a positive constant independent of t_0 and $s\geq s^*$.

• The second part consists in prove a similar estimate for the $L^2((0,T)\times\Omega_j)$-norm of the spatial derivatives of Z_j for $j=1,2$:

$$\begin{aligned}
&s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1|^2dxdt+s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2|^2dxdt\\
\leq &C\int_0^T\int_{\Omega_1}|F_1|^2dxdt+C\int_0^T\int_{\Omega_2}|F_2|^2dxdt+Cs_0\int_0^T|G_N|^2dt+Cs_0\int_0^T|H|^2dt\\
&+Cs_0\int_0^T|J|^2dt.
\end{aligned}\tag{4.3.2}$$

Clearly, if we add the estimates (4.3.1) and (4.3.2) the proof of the Lemma 4.3 is complete.

Then, we start proving the inequality (4.3.1). Before going further, let us bring a brief orientation of the proof. First, we will use the Fourier transform in time in order to get good estimates in frequency domain. However, it is not evident that these estimates are uniform on the frequency parameter in the Fourier domain. In order to prove that, we divide in some frequency ranges to analyze our estimates saying, Low, intermediate and high frequencies. That is, this phenomenon depends where the information comes from it.

For a function $h\in L^2(\mathbb{R})$, we introduce the partial Fourier transform in time defined by

$$\hat{h}(\tau)=\mathcal{F}(h)(\tau)=\int_{\mathbb{R}}h(t)e^{-it\tau}dt.\tag{4.3.3}$$

where i is the imaginary unit. Now, we extend the variables Z_1 and Z_2 of system (4.2.5) by zero, and we do the same for the functions H, J and F_1 and F_2.

Thus, applying the Fourier transform (4.3.3) to the system (4.2.5), we obtain

$$\begin{cases}(\sigma_1\partial_x+\sigma_1\partial_x\phi_1 s_0+\gamma)(\sigma_1\partial_x+\sigma_1\partial_x\phi_1 s_0-\gamma)\hat{Z}_1=-\hat{F}_1, & \forall(x,\tau)\in\Omega_1\times\mathbb{R},\\ (\sigma_2\partial_x+\sigma_2\partial_x\phi_2 s_0+\gamma)(\sigma_2\partial_x+\sigma_2\partial_x\phi_2 s_0-\gamma)\hat{Z}_2=-\hat{F}_2, & \forall(x,\tau)\in\Omega_2\times\mathbb{R},\\ \hat{Z}_2(0,\tau)=\hat{Z}_1(0,\tau), & \forall t\in\mathbb{R},\\ \sigma_2^2\partial_x\hat{Z}_2(0,\tau)=\sigma_1^2\partial_x\hat{Z}_1(0,\tau)+(m_1-M_2\sigma_2)s_0\hat{Z}_1(0,\tau)+\hat{H}(\tau), & \forall\tau\in\mathbb{R},\\ \hat{Z}_1(-L_1,\tau)=0, & \forall\tau\in\mathbb{R},\\ \partial_x\hat{Z}_2(L_2,\tau)=-m_2\sigma_2^{-1}s_0\hat{Z}_2(L_2,\tau)+\hat{J}(\tau), & \forall\tau\in\mathbb{R},\\ \partial_x\hat{Z}_1(-L_1,\tau)=\hat{G}_N(\tau), & \forall\tau\in\mathbb{R}.\end{cases} \tag{4.3.4}$$

Here, $\gamma=\gamma(\tau)$ is defined by

$$\gamma=\begin{cases}\sqrt{|\tau|}e^{i\frac{\pi}{4}}, & \text{if } \tau\geq 0,\\ \sqrt{|\tau|}e^{-i\frac{\pi}{4}}, & \text{if } \tau<0.\end{cases} \tag{4.3.5}$$

It is clear that $\Re(\gamma(\tau))=\sqrt{|\tau|/2}\geq 0$ and $\gamma^2=i\tau$, for all $\tau\in\mathbb{R}$. Now, inspired in the structure of the operator in $(4.3.4)_1$ and $(4.3.4)_2$, we define the auxiliar variables:

$$\hat{W}_j=(\sigma_j\partial_x+\sigma_j\partial_x\phi_j s_0-\gamma)\hat{Z}_j,\quad j=1,2.$$

4.3.1 First estimates

Proposition 4.9 *Let $\hat{W}_1$ be the solution of the following system:*

$$\begin{cases}(\sigma_1\partial_x+\sigma_1\partial_x\phi_1 s_0+\gamma)\hat{W}_1(x,\tau)=-\hat{F}_1(x,\tau), & \forall(x,\tau)\in\Omega_1\times\mathbb{R},\\ \hat{W}_1(-L_1,\tau)=\sigma_1\hat{G}_N(\tau), & \forall\tau\in\mathbb{R}.\end{cases} \tag{4.3.6}$$

Then, each solution $\hat{W}_1$ of (4.3.6) satisfies

$$(s_0^2+\sqrt{|\tau|}s_0)\int_{\Omega_1}|\hat{W}_1(x,\tau)|^2dx+s_0|\hat{W}_1(0,\tau)|^2\leq C\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dx+Cs_0|\hat{G}_N(\tau)|^2, \tag{4.3.7}$$

for all $\tau\in\mathbb{R}$ for some constant $C=C(\sigma_1,m_1)$ independent of s_0.

Proof. Let $\tau\in\mathbb{R}$ fixed. We multiply the equation $(4.3.6)_1$ by $s_0\overline{\hat{W}_1}(\tau)$ and we integrate on Ω_1:

$$\begin{aligned}&\sigma_1 s_0\int_{\Omega_1}\partial_x\hat{W}_1(\tau)\overline{\hat{W}_1}(\tau)dx+\sigma_1 s_0^2\int_{\Omega_1}\partial_x\phi_1|\hat{W}_1(\tau)|^2dx+\gamma(\tau)s_0\int_{\Omega_1}|\hat{W}_1(\tau)|^2dx\\ =&-s_0\int_{\Omega_1}\hat{F}_1(\tau)\overline{\hat{W}_1}(\tau)dx.\end{aligned} \tag{4.3.8}$$

Integration by parts yields

$$\sigma_1 s_0 \Re \int_{\Omega_1} \partial_x \hat{W}_1(\tau)\overline{\hat{W}_1}(\tau)dx = -\frac{\sigma_1 s_0}{2}|\hat{W}_1(0,\tau)|^2 - \frac{\sigma_1^2 s_0}{2}|\hat{G}_N(\tau)|^2. \tag{4.3.9}$$

Taking the real part in (4.3.8) and using (4.3.9), we see that

$$\begin{aligned}
&\frac{\sigma_1 s_0}{2}|\hat{W}_1(0,\tau)|^2 + \sigma_1 s_0^2 \int_{\Omega_1} \partial_x\phi_1 |\hat{W}_1(x,\tau)|^2 dx + \frac{\sqrt{|\tau|}}{2} s_0 \int_{\Omega_1} |\hat{W}_1(x,\tau)|^2 dx \\
=&-s_0 \Re \int_{\Omega_1} \hat{F}_1(x,\tau)\overline{\hat{W}_1}(x,\tau)dx + \frac{\sigma_1^2 s_0}{2}|\hat{G}_N(\tau)|^2.
\end{aligned}$$

Applying Young's inequality, we have

$$\begin{aligned}
&\sigma_1 s_0^2 \int_{\Omega_1} \partial_x\phi_1 |\hat{W}_1(x,\tau)|^2 dx + \Re(\gamma(\tau)) s_0 \int_{\Omega_1} |\hat{W}_1(x,\tau)|^2 dx + \frac{\sigma_1 s_0}{2}|\hat{W}_1(0,\tau)|^2 \\
\leq& \frac{m_1}{2} s_0^2 \int_{\Omega_1} |\hat{W}_1(x,\tau)|^2 dx + \frac{1}{2m_1}\int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + \frac{\sigma_1^2 s_0}{2}|\hat{G}_N(\tau)|^2.
\end{aligned}$$

The proof of Proposition 4.9 is complete since $m_1\sigma^{-1}$ is a lower bound of $\partial_x\phi_1$. □

Proposition 4.10 *Let $\hat{W}_2$ be the solution of the system*

$$\begin{cases}
(\sigma_2\partial_x + \sigma_2\partial_x\phi_2 s_0 + \gamma)\hat{W}_2(x,\tau) = -\hat{F}_2(x,\tau), & \forall (x,\tau) \in \Omega_2 \times \mathbb{R}, \\
\sigma_2 \hat{W}_2(0,\tau) = \sigma_1 \hat{W}_1(0,\tau) + (\sigma_1 - \sigma_2)\gamma \hat{Z}_2(0,\tau) + \hat{H}(\tau), & \forall \tau \in \mathbb{R}, \\
\hat{W}_2(L_2,\tau) = \gamma(\tau)\hat{Z}_2(L_2,\tau) + \hat{J}(\tau), & \forall \tau \in \mathbb{R}.
\end{cases} \tag{4.3.10}$$

Then, there exists a constant $C = C(m_1, m_2, \sigma_1, \sigma_2)$ such that each solution of (4.3.10) *satisfies*

$$\begin{aligned}
&(s_0^2 + \sqrt{|\tau|}s_0)\int_{\Omega_2} |\hat{W}_2(\tau)|^2 dx + s_0|\hat{W}_2(\tau, L_2)|^2 \\
\leq& C\int_{\Omega_1} |\hat{F}_1(\tau)|^2 dx + C\int_{\Omega_2} |\hat{F}_2(\tau)|^2 dx + Cs_0|\hat{G}_N(\tau)|^2 + C|\tau|s_0|\hat{Z}_2(\tau,0)|^2 \\
&+ Cs_0|\hat{H}(\tau)|^2,
\end{aligned} \tag{4.3.11}$$

for each $\tau \in \mathbb{R}$.

Proof. Using the same ideas of Proposition 4.9, it is easy to check that

$$(s_0^2 + \sqrt{|\tau|}s_0)\int_{\Omega_2} |\hat{W}_2(\tau)|^2 dx + s_0|\hat{W}_2(\tau, L_2)|^2 \leq C\int_{\Omega_2} |\hat{F}_2(\tau)|^2 dx + Cs_0|\hat{W}_2(\tau,0)|^2, \tag{4.3.12}$$

for all $\tau \in \mathbb{R}$, where the constant C in the inequality above depends only on m_2 and σ_2. Our next task is to estimate the local term of $\hat{W}_2$ at the interface. In order to do that, we use the boundary condition $(4.3.10)_2$ to get the following estimate:

$$s_0|\hat{W}_2(\tau,0)|^2 \leq Cs_0|\hat{W}_1(\tau,0)|^2 + C|\tau|s_0|\hat{Z}_2(\tau,0)|^2 + Cs_0|\hat{H}(\tau)|^2. \tag{4.3.13}$$

Substituting (4.3.13) into (4.3.12) and applying the Proposition 4.9, we can assert that

$$
\begin{aligned}
&(s_0^2+\sqrt{|\tau|}s_0)\int_{\Omega_2}|\hat{W}_2(\tau)|^2dx+s_0|\hat{W}_2(L_2,\tau)|^2\\
\leq &C\int_{\Omega_1}|\hat{F}_1(\tau)|^2dx+C\int_{\Omega_2}|\hat{F}_2(\tau)|^2dx+Cs_0|\hat{G}_N(\tau)|^2+C|\tau|s_0|\hat{Z}_2(0,\tau)|^2\\
&+Cs_0|\hat{H}(\tau)|^2,
\end{aligned}
\tag{4.3.14}
$$

which is the desired conclusion. □

Proposition 4.11 *Let Z_1 be the solution of*

$$
\begin{cases}
(\sigma_1\partial_x+\sigma_1\partial_x\phi_1 s_0-\gamma)\hat{Z}_1(x,\tau)=\hat{W}_1(x,\tau), & \forall(x,\tau)\in\Omega_1\times\mathbb{R},\\
\hat{Z}_1(-L_1,t)=0, & \forall\tau\in\mathbb{R}.
\end{cases}
\tag{4.3.15}
$$

Then, there exists a constant $C=C(m_1,\sigma_1)$ such that each solution $\hat{Z}_1$ of (4.3.15) with source term $\hat{W}_2(\tau)\in L^2(\Omega)$ for all $\tau\in\mathbb{R}$ satisfies

$$
\begin{aligned}
&r_1\sigma_1 s_0^3\int_{\Omega_1}|\hat{Z}_1(x,\tau)|^2dx+s_0^2\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma(\tau)))^2|\hat{Z}_1(x,\tau)|^2dx\\
&+\sigma_1(m_1s_0-\Re(\gamma(\tau)))s_0^2|\hat{Z}_1(0,\tau)|^2\leq C\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dx+Cs_0|\hat{G}_N(\tau)|^2,\quad\forall\tau\in\mathbb{R}.
\end{aligned}
\tag{4.3.16}
$$

Proof. Let $\tau\in\mathbb{R}$. We multiply the equation (4.3.15) by $(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))\overline{\hat{Z}}_1$ and we integrate on Ω_1:

$$
\begin{aligned}
&\sigma_1\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))\partial_x\hat{Z}_1(x,\tau)\overline{\hat{Z}_1}(\tau)dx\\
&+\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\gamma)(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))|\hat{Z}_1(x,\tau)|^2dx\\
=&\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))\hat{W}_1(x,\tau)\overline{\hat{Z}_1}(x,\tau)dx.
\end{aligned}
\tag{4.3.17}
$$

Integration by parts yields,

$$
\begin{aligned}
&\sigma_1\Re\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))\partial_x\hat{Z}_1(x,\tau)\overline{\hat{Z}_1}(x,\tau)dx\\
=&-\frac{\sigma_1^2 s_0}{2}\int_{\Omega_1}\partial_x^2\phi_1|\hat{Z}_1(x,\tau)|^2dx-\frac{\sigma_1}{2}(m_1s_0-\Re(\gamma))|\hat{Z}_1(0,\tau)|^2.
\end{aligned}
\tag{4.3.18}
$$

Taking the real part in (4.3.17) and using (4.3.18), we have

$$
\begin{aligned}
&-\frac{\sigma_1^2}{2}s_0\int_{\Omega_1}\partial_x\phi_1|\hat{Z}_1(x,\tau)|^2dx+\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))^2|\hat{Z}_1(x,\tau)|^2dx\\
&-\sigma_1(m_1s_0-\Re(\gamma))|\hat{Z}_1(0,\tau)|^2\\
=&\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))\hat{W}_1(x,\tau)\overline{\hat{Z}}_1(x,\tau)dx.
\end{aligned}
$$

By the Young inequality and the assumptions on ϕ_1, we see that

$$\begin{aligned}&r_1\sigma_1 s_0\int_{\Omega_1}|\hat{Z}_1(x,\tau)|^2dx+\int_{\Omega_1}(\sigma_1\partial_x\phi_1 s_0-\Re(\gamma))^2|\hat{Z}_1(x,\tau)|^2dx\\&+\sigma_1(m_1s_0-\Re(\gamma))|\hat{Z}_1(0,\tau)|^2\leq\int_{\Omega_1}|\hat{W}_1(x,\tau)|^2dx.\end{aligned}\tag{4.3.19}$$

Finally, we multiply (4.3.19) by s_0^2 and apply the Proposition 4.9, and the proof is complete. □

Proposition 4.12 *Let $\hat{Z}_2$ be the solution of*

$$\begin{cases}(\sigma_2\partial_x+\sigma_2\partial_x\phi_2 s_0-\gamma)\hat{Z}_2(x,\tau)=\hat{W}_2(x,\tau), & \forall(x,\tau)\in\Omega_2\times\mathbb{R},\\ \hat{Z}_2(0,\tau)=\hat{Z}_1(0,\tau), & \forall\tau\in\mathbb{R}.\end{cases}\tag{4.3.20}$$

Then, there exists a constant $C>0$ independent of s_0 such that for every solution $\hat{Z}_2$ of (4.12)*, we have*

$$\begin{aligned}&r_2\sigma_2 s_0^3\int_{\Omega_2}|\hat{Z}_2(x,\tau)|^2dx+s_0\int_{\Omega_2}(\sigma_2\partial_x\phi_2 s_0-\Re(\gamma(\tau)))|\hat{Z}_2(x,\tau)|^2dx\\&+\sigma_2(m_2s_0-\Re(\gamma(\tau)))s_0^2|\hat{Z}_2(L_2,\tau)|^2\\ \leq& C\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dx+C\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dx\\&+Cs_0|\hat{G}_N(\tau)|^2+C|\tau|s_0|\hat{Z}_2(0,\tau)|^2+Cs_0|\hat{H}(\tau)|^2,\end{aligned}\tag{4.3.21}$$

for each $\tau\in\mathbb{R}$.

Proof. Firstly, as in the proof of Proposition 4.12 each solution Z_2 of (4.3.20) satisfies

$$\begin{aligned}&r_2\sigma_2 s_0^3\int_{\Omega_2}|\hat{Z}_2(x,\tau)|^2dx+s_0^2\int_{\Omega_2}(\sigma_2\partial_x\phi_2-\Re(\gamma(\tau)))|\hat{Z}_2(x,\tau)|^2dx\\&+\sigma_2\left(m_2s_0-\Re(\gamma(\tau))\right)s_0^2|\hat{Z}_2(L_2,\tau)|^2\\ \leq& s_0^2\int_{\Omega_2}|\hat{W}_2(x,\tau)|^2dx+\sigma_2\left(M_2s_0-\Re(\gamma(\tau))\right)s_0|\hat{Z}_2(0,\tau)|^2,\quad\forall\tau\in\mathbb{R}.\end{aligned}\tag{4.3.22}$$

Then, it remains to estimate the global term of $\hat{W}_2$ and the local term of $\hat{Z}_2$ at the interface. To do this, notice that from the Proposition 4.12, the global term of $\hat{W}_2$ can be bounded as follows:

$$\begin{aligned}s_0^2\int_{\Omega_2}|\hat{W}_2(x,\tau)|^2dx\leq& C\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dx+C\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dx+Cs_0|\hat{G}_N(\tau)|^2\\&+C|\tau|s_0|\hat{Z}_2(0,\tau)|^2+Cs_0|\hat{H}(\tau)|^2.\end{aligned}\tag{4.3.23}$$

On the other hand, since $M_2<m_1$ and $Z_1=Z_2$ at the interface for all $\tau\in\mathbb{R}$, we deduce that

$$\begin{aligned}(M_2s_0-\Re(\gamma(\tau)))s_0^2|\hat{Z}_2(0,\tau)|^2\leq&(m_1s_0-\Re(\gamma(\tau)))s_0^2|\hat{Z}_1(0,\tau)|^2\\ \leq& C\int_{\Omega_2}|\hat{F}_1(x,\tau)|^2dx+Cs_0|\hat{G}_N(\tau)|^2,\quad\forall\tau\in\mathbb{R}.\end{aligned}\tag{4.3.24}$$

Finally, we substitute (4.3.23) and (4.3.24) into (4.3.24) and the proof is complete. □

4.3.2 Global estimates in the Fourier domain

Our next task is prove the following inequality

$$
\begin{aligned}
& s_0^3 \int_{\Omega_1} |\hat{Z}_1(x,\tau)|^2 dx + s_0^3 \int_{\Omega_2} |\hat{Z}_2(x,\tau)|^2 dx + s_0^3 |Z_1(0,\tau)|^2 + s_0^3 |\hat{Z}_2(L_2,\tau)|^2 \\
\leq & C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C \int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2 + C s_0 |\hat{H}(\tau)|^2 + C s_0 |\hat{J}(\tau)|^2,
\end{aligned} \tag{4.3.25}
$$

for all $\tau \in \mathbb{R}$ by using the estimates of Propositions 4.9, 4.10, 4.11 and 4.12. In order to do that, let δ and δ' be two positive numbers such that

$$\delta < \sqrt{2} m_2 < \sqrt{2} M_2 < \delta' < \sqrt{2} m_1.$$

Then, we divide the real line into three subsets, namely Low frequencies, Intermediate frequencies and High frequencies. These intervals are ilustrated in the following Figure:

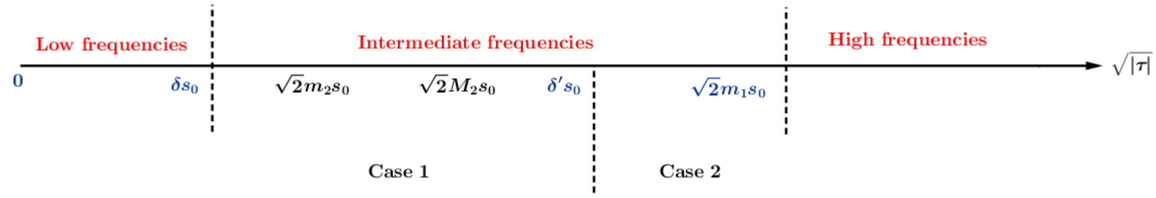

Figure 4.2: Sketch of different ranges in frequency domain

- **Case 1: Low frequencies** Suppose that $\tau \in \mathbb{R}$ is such that

$$0 \leq \sqrt{|\tau|} \leq \delta s_0. \tag{4.3.26}$$

We note that the condition above implies

$$\frac{1}{\sqrt{2}} \left(\sqrt{2} m_1 - \delta' \right) s_0 < m_1 s_0 - \sqrt{\frac{|\tau|}{2}}.$$

By Proposition 4.11, we see that

$$\frac{\sigma_1}{\sqrt{2}} \left(\sqrt{2} m_1 - \delta' \right) s_0^3 |\hat{Z}_1(0,\tau)|^2 \leq C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2. \tag{4.3.27}$$

Then, we can estimate the global term of $\hat{Z}_1$ and its local term at the interface as follows:

$$s_0^3 \int_{\Omega_1} |\hat{Z}_1(x,\tau)|^2 dx + s_0^3 |\hat{Z}_2(0,\tau)|^2 \leq C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2. \tag{4.3.28}$$

On the other hand, the condition (4.3.26) also implies

$$\frac{1}{\sqrt{2}} (\sqrt{2} m_2 - \delta) s_0 < m_2 s_0 - \sqrt{\frac{|\tau|}{2}}.$$

Thus, similarly to the estimate (4.3.27), we can assert that

$$s_0^3 \int_{\Omega_2} |\hat{Z}_2(x,\tau)|^2 dx + s_0^3 |\hat{Z}_2(L_2,\tau)|^2 \leq C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C \int_{\Omega_2} |\hat{F}_2(\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2 + C s_0 |\tilde{H}(\tau)|^2, \tag{4.3.29}$$

where we applied the Proposition 4.12. Finally, we add the inequalities (4.3.27) and (4.3.29), and the proof of (4.3.25) in the case of Low frequencies is complete.

- **Case 2: Intermediate frequencies** Suppose that $\tau \in \mathbb{R}$ is chosen such that

$$\delta s_0 < \sqrt{|\tau|} \leq \sqrt{2} m_1 s_0.$$

To do this, we consider two cases (see figure 1.2)

- We assume that $\tau \in \mathbb{R}$ is such that

$$\delta s_0 < \sqrt{|\tau|} \leq \delta' s_0.$$

In this case, we already have the estimate (4.3.27). Besides, the estimate (4.3.29) does not hold. To deal with this issue, by the boundary condition $(4.3.10)_3$ we can assert that

$$|\tau| s_0 |\hat{Z}_2(L_2,\tau)|^2 \leq 2 s_0 |\hat{W}_2(L_2,\tau)|^2 + 2 s_0 |\tilde{J}(\tau)|^2. \tag{4.3.30}$$

From Proposition 4.10, we have the following upper bound of $W_2(\tau, L_2)$:

$$s_0 |\hat{W}_2(L_2,\tau)|^2 \leq C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C \int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + C|\tau| s_0 |\hat{Z}_2(0,\tau)|^2 + C s_0 |\hat{H}(\tau)|^2. \tag{4.3.31}$$

Combining (4.3.30) and (4.3.31) with (4.3.27) we see that

$$s_0^3 |\hat{Z}_2(L_2,\tau)|^2 \leq C \int_{\Omega_1} |F_1(x,\tau)|^2 dx + C \int_{\Omega_2} |F_2(x,\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2 + C s_0 |\hat{H}(\tau)|^2 + C s_0 |\hat{J}(\tau)|^2. \tag{4.3.32}$$

Hence, by Proposition 4.11 and 4.12, we get

$$s_0^3 \int_{\Omega_2} |\hat{Z}_2(x,\tau)|^2 dx \leq C \int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C \int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + C s_0 |\hat{G}_N(\tau)|^2 + C s_0 |\hat{H}(\tau)|^2 + C s_0 |\hat{J}(\tau)|^2. \tag{4.3.33}$$

Thus, we add (4.3.28), (4.3.32) and (4.3.33), this is precisely the claim in the case 1 of Intermediate frequencies.

- We consider $\tau \in \mathbb{R}$ such that

$$\delta' s_0 < \sqrt{|\tau|} < \sqrt{2} m_1 s_0.$$

In this case, we adopt a different strategy in order to estimate the local terms of $\hat{Z}_1$ and $\hat{Z}_2$ at the interface. Roughly speaking, we will descompose the solution $\hat{Z}_2$ into two components: the first one is unknown in the sense that this one depends of the local term $\hat{Z}_2(\tau,0)$, and the second one can be estimated using the same machinery introduced in the Propositions above.

Before to start with the proof in this case, let us state some definitions. We consider the functions $\hat{Z}_u$ and $\hat{Z}_k$, defined as the solution of the following problems:

$$\begin{cases} (\sigma_2\partial_x + \sigma_2\partial_x\phi_2 s_0 - \gamma)\hat{Z}_u(x,\tau) = u(x,\tau), & \forall (x,\tau)\Omega_2\times\mathbb{R}, \\ \hat{Z}_u(L_2,\tau) = \frac{1}{\gamma}u(L_2,\tau), & \forall \tau\in\mathbb{R}, \end{cases} \tag{4.3.34}$$

and

$$\begin{cases} (\sigma_2\partial_x + \sigma_2\partial_x\phi_2 s_0 - \gamma)\hat{Z}_k(x,\tau) = k(x,\tau), & \forall (x,\tau)\in\Omega_2\times\mathbb{R}, \\ \hat{Z}_k(L_2,\tau) = \frac{1}{\gamma}k(L_2,\tau) - \frac{1}{\gamma}\hat{J}(\tau), & \forall \tau\in\mathbb{R}, \end{cases} \tag{4.3.35}$$

respectively. Here, the functions u and k solve

$$\begin{cases} (\sigma_2\partial_x + \sigma_2\partial_x\phi_2 s_0 + \gamma)u(x,\tau) = 0, & \forall (x,\tau)\in\Omega_2\times\mathbb{R}, \\ u(0,\tau) = \frac{(\sigma_1-\sigma_2)}{\sigma_2}\gamma\hat{Z}_2(0,\tau), & \forall \tau\in\mathbb{R}, \end{cases} \tag{4.3.36}$$

and

$$\begin{cases} (\sigma_2\partial_x + \sigma_2\partial_x\phi_2 s_0 + \gamma)k(x,\tau) = -F_2(x,\tau), & \forall (x,\tau)\in\Omega_2\times\mathbb{R}, \\ k(0,\tau) = \frac{\sigma_1}{\sigma_2}\hat{W}_1(0,\tau) + \frac{1}{\sigma_2}\hat{H}(\tau), & \forall \tau\in\mathbb{R}. \end{cases} \tag{4.3.37}$$

respectively. It is clear that $\hat{Z}_2 = \hat{Z}_u + \hat{Z}_k$ and $\hat{W}_2 = u + k$ in $\mathbb{R}\times\Omega_2$. Let us compute the explicit solution of $\hat{Z}_u$. Firstly, by Duhamel's formula, the expression of u in (4.3.36) is given by

$$u(x,\tau) = \left(\frac{\sigma_1-\sigma_2}{\sigma_2}\right)\gamma(\tau)\hat{Z}_2(0,\tau)\exp\left((\phi_2(0)-\phi_2(x))s_0 - \frac{\gamma(\tau)}{\sigma_2}x\right). \tag{4.3.38}$$

Then, using Duhamel's formula once again, the explicit solution of $\hat{Z}_u$ in terms of u is given by

$$\begin{aligned} \hat{Z}_u(x,\tau) = & \frac{1}{\gamma(\tau)}u(\tau,L_2)\exp\left((\phi_2(L_2)-\phi_2(x))s_0 - \frac{\gamma(\tau)}{\sigma_2}(L_2-x)\right) \\ & - \frac{1}{\sigma_2}\exp\left(\frac{\gamma(\tau)}{\sigma_2}x - \phi_2(x)s_0\right)\int_x^{L_2}\exp\left(\phi_2(\tilde{x})s_0 - \frac{\gamma(\tau)}{\sigma_2}\tilde{x}\right)u(\tilde{x},\tau)d\tilde{x}. \end{aligned} \tag{4.3.39}$$

Substituting (4.3.38) by (4.3.39) and evaluating at $x=0$, we have

$$\hat{Z}_u(0,\tau) = \frac{\sigma_1-\sigma_2}{2\sigma_2}\hat{Z}_2(0,\tau)\left(3\exp\left(-\frac{2\gamma(\tau)L_2}{\sigma_2}\right) - 1\right).$$

Therefore, since $\hat{Z}_2(0,\tau) = \hat{Z}_u(0,\tau) + \hat{Z}_k(0,\tau)$, we obtain

$$\Lambda(\tau)\hat{Z}_2(0,\tau) = \hat{Z}_k(0,\tau), \tag{4.3.40}$$

where $\Lambda = \Lambda(\tau)$ is defined by

$$\Lambda(\tau) := \frac{\sigma_1+\sigma_2}{2\sigma_2} + 3\frac{(\sigma_2-\sigma_1)}{2\sigma_2}\exp\left(-\frac{2\gamma(\tau)L_2}{\sigma_2}\right).$$

Now we will show that there exists a constant $C>0$ independent of s_0 such that

$$|\hat{Z}_2(\tau,0)|^2 \leq C|\hat{Z}_k(\tau,0)|^2.$$

Indeed, suppose that $0<\sigma_1<\sigma_2$. Then, it is easy to check that

$$\frac{1}{2} \leq \frac{\sigma_1+\sigma_2}{2\sigma_2} \leq \Re(\Lambda(\tau)),$$

and the assertion follows directly. Now, we want to show the same inequality when $0<\sigma_2<\sigma_1$. In that case, we consider the following assumptions: we suppose that

$$s \geq \frac{T^{2\alpha}}{2^{2\alpha}}$$

and

$$M_2 \geq \frac{\log(3)\sigma_2}{2L_2}.$$

In particular these conditions implies that $s_0 \geq 1$ and

$$\sqrt{|\tau|} \geq \frac{\log(3)\sigma_2}{\sqrt{2}L_2},$$

or equivalently

$$\exp\left(-\frac{\sqrt{2|\tau|}L_2}{\sigma_2}\right) \leq \frac{1}{3}.$$

Hence, notice that under these conditions, we can assert that

$$\Re(\Lambda(\tau)) \geq 1. \tag{4.3.41}$$

This implies the desired claim. It remains to prove a estimate for $\hat{Z}_k$ at the interface. In order to do that, using the same ideas of Proposition 4.10, we can assert that

$$\begin{aligned}&(s_0^2+\sqrt{|\tau|})s_0\int_{\Omega_2}|k(x,\tau)|^2dx + s_0|k(L_2,\tau)|^2\\ \leq& C\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dx + Cs_0|\hat{W}_1(0,\tau)|^2 + Cs_0|\hat{H}(\tau)|^2.\end{aligned} \tag{4.3.42}$$

Furthermore, we can prove the following estimate for $\hat{Z}_2$:

$$\begin{aligned}&r_2\sigma_2 s_0\int_{\Omega_2}|\hat{Z}_k(0,\tau)|^2dx + \int_{\Omega_2}(\sigma_2\partial_x\phi_2 s_0 - \Re(\gamma(\tau)))^2|\hat{Z}_k(x,\tau)|^2dx\\ &+\sigma_2(\Re(\gamma(\tau)) - M_2 s_0)|\hat{Z}_k(0,\tau)|^2\\ \leq& C\int_{\Omega_2}|k(x,\tau)|^2dx + C\frac{1}{\sqrt{|\tau|}}(|k(L_2,\tau)|^2 + |\hat{J}(\tau)|^2).\end{aligned} \tag{4.3.43}$$

Now, since we are in the second case of Intermediate frequencies, we have

$$\Re(\gamma(\tau)) - M_2 s_0 > (\delta' - M_2)\, s_0.$$

Then, from (4.3.43), we deduce that

$$\sqrt{|\tau|} s_0 |\hat{Z}_k(0,\tau)|^2 \leq C\sqrt{|\tau|}\int_{\Omega_2} |k(x,\tau)|^2 dx + Cs_0|k(L_2,\tau)|^2 + Cs_0|\hat{J}(\tau)|^2. \quad (4.3.44)$$

Multiplying by s_0 the inequality above and using the lower and upper bound of $\sqrt{|\tau|}$, we see that

$$s_0^3|\hat{Z}_k(0,\tau)|^2 \leq C\int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + Cs_0|\hat{W}_1(0,\tau)|^2 + Cs_0|H(\tau)|^2 + Cs_0|\hat{J}(\tau)|^2.$$

Moreover, applying the Proposition 4.9, we obtain

$$\begin{aligned} s_0^3|\hat{Z}_k(0,\tau)|^2 \leq & C\int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C\int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + Cs_0|\hat{G}_N(\tau)|^2 \\ & + Cs_0|\hat{H}(\tau)|^2 + Cs_0|\hat{J}(\tau)|^2. \end{aligned}$$

Therefore, using the relation (4.3.40), we obtain

$$\begin{aligned} s_0^3|\hat{Z}_2(0,\tau)|^2 \leq & C\int_{\Omega_1} |\hat{F}_1(x,\tau)|^2 dx + C\int_{\Omega_2} |\hat{F}_2(x,\tau)|^2 dx + Cs_0|\hat{G}(\tau)|^2 \\ & + Cs_0|H(\tau)|^2 + Cs_0|J(\tau)|^2. \end{aligned} \quad (4.3.45)$$

The rest of the proof runs as before. Thus, we proved the desired inequality in the second case of Intermediate frequencies.

- **High frequencies:**

Now, we consider the case of high frequencies, that is, we take $\tau \in \mathbb{R}$ such that

$$\sqrt{2} m_1 s_0 \leq \sqrt{|\tau|}. \quad (4.3.46)$$

In this case, we will apply the same strategy as the case before. However, we can not estimate directly the term $|\tau| s_0 |\hat{Z}_2(\tau,0)|^2$ in (4.3.44). To avoid this difficulty, we note that the condition (4.3.46) implies

$$\frac{\sqrt{|\tau|}}{2\sqrt{2}} \leq \Re(\gamma(\tau)) - M_2 s_0.$$

Therefore, from the estimate of $\hat{Z}_k$ in (4.3.43), we have

$$\sqrt{|\tau|}|\hat{Z}_k(\tau,0)|^2 \leq C\int_{\Omega_2} |k(\tau)|^2 dx + C\frac{\sigma_2}{\sqrt{|\tau|}}(|k(\tau,L_2)|^2 + |\hat{J}(\tau)|^2). \quad (4.3.47)$$

Equivalently,

$$|\tau|s_0|\hat{Z}_k(\tau,0)|^2 \le C\sqrt{|\tau|}\int_{\Omega_2}|k(\tau)|^2dx + C\sigma_2 s_0|k(\tau,L_2)|^2 + C\sigma_2 s_0|\hat{J}(\tau,L_2)|^2.$$

Using the estimate of k in (4.3.42), we obtain

$$\begin{aligned}
&|\tau|s_0|\hat{Z}_2(0,\tau)|^2\\
\le& C|\tau|s_0|\hat{Z}_k(0,\tau)|^2\\
\le& C\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dx + C\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dx + Cs_0|\hat{G}_N(\tau)|^2 + Cs_0|\hat{H}(\tau)|^2 + Cs_0|\hat{J}(\tau)|^2.
\end{aligned}$$

We proceed for estimate the term of $\hat{Z}_2(\tau,L_2)$ as before. This conclude the proof of (4.3.25) in the case of high frequencies. Hence, we have proved the estimate (4.3.25) for all $\tau\in\mathbb{R}$. Now, integrating on $\mathbb{R}$ in τ, we deduce that

$$\begin{aligned}
&s_0^3\int_{\mathbb{R}}\int_{\Omega_1}|\hat{Z}_1(x,\tau)|^2dxdt + s_0^3\int_{\mathbb{R}}\int_{\Omega_2}|\hat{Z}_2(x,\tau)|^2dxdt\\
\le& C\int_{\mathbb{R}}\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dxd\tau + \int_{\mathbb{R}}\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dxd\tau + Cs_0\int_{\mathbb{R}}|\hat{G}_N|^2d\tau\\
&+ Cs_0\int_{\mathbb{R}}|\hat{H}(\tau)|^2d\tau + Cs_0\int_{\mathbb{R}}|\hat{J}(\tau)|^2d\tau.
\end{aligned}\tag{4.3.48}$$

Finally, we use the Parseval's identity and the fact that all functions are supported in an open subset of $(0,T)$. This concludes the proof of the inequality 4.3.1.

4.3.3 Estimates of the spatial derivatives

The goal of this section is show that the spatial derivatives Z_1 and Z_2 of (4.2.5) with source terms $F_j\in L^2(\Omega_j\times(0,T))$ $j=1,2$, Neumann data $G_N\in L^2(0,T)$ and residual terms $H,J\in L^2(0,T)$ satisfies the following:

$$\begin{aligned}
&s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1(x,\tau)|^2dxdt + s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2(x,\tau)|^2dxdt\\
\le& C\int_0^T\int_{\Omega_1}|\hat{F}_1(x,\tau)|^2dxdt + C\int_0^T\int_{\Omega_2}|\hat{F}_2(x,\tau)|^2dxdt + Cs_0\int_0^T|\hat{G}_N|^2dt + Cs_0\int_0^T|\hat{H}|^2dt\\
&+ Cs_0\int_0^T|\hat{J}|^2dt,
\end{aligned}\tag{4.3.49}$$

where the constant $C=C(m_1,m_2,r_1,r_2,\sigma_1,\sigma_2)$ is independent of s_0. We start multiplying the first equation by Z_1 and we integrate on Ω_1:

$$\frac{d}{dt}\int_{\Omega_1}|Z_1(x,t)|^2dx - \sigma_1\int_{\Omega_1}(\partial_x + s_0\partial_x\phi_1)^2Z_1(x,t)\cdot Z_1(x,t)dx = \int_{\Omega_1}F_1(x,t)Z_1(x,t)dx,$$

for all $t \in (0,T)$ where by definition

$$(\partial_x + s\partial_x\phi_1)^2 Z_1 = \partial_x^2 Z_1 + s_0\partial_x\phi_1\partial_x Z_1 + s_0\partial_x^2\phi_1 Z_1 + |s_0\partial_x\phi_1|^2 Z_1, \quad \forall (x,t) \in \Omega_1 \times (0,T).$$

Then,

$$\begin{aligned} &-\sigma_1^2 \int_{\Omega_1} (\partial_x + s_0\partial_x\phi_1)^2 Z_1(x,t) \cdot Z_1(x,t)dx \\ =& -\sigma_1^2 \int_{\Omega_1} \partial_x Z_1(x,t) Z_1(x,t)dx - 2\sigma_1^2 s_0 \int_{\Omega_1} \partial_x\phi_1\partial_x Z_1(x,t)Z_1(x,t)dx \\ &-\sigma_1^2 s_0 \int_{\Omega_1} \partial_x^2\phi_1 |Z_1(x,t)|^2 dx - s_0^2 \int_{\Omega_1} |\partial_x\phi_1|^2 |Z_1(x,t)|^2 dx. \end{aligned} \tag{4.3.50}$$

Integration by parts yields

$$-\int_{\Omega_1} \partial_x^2 Z_1(x,t)Z_1(x,t)dx = \int_{\Omega_1} |\partial_x Z_1(x,t)|^2 dx - \partial_x Z_1(0,t)Z_1(0,t),$$

where we used the condition $Z_1(L_2,t) = 0$ for each $t \in (0,T)$ and

$$-2s_0 \int_{\Omega_1} \partial_x\phi_1 Z_1(x,t)Z_1(x,t)dx = s_0 \int_{\Omega_1} \partial_x^2\phi_1 |Z_1(x,t)|^2 dx - s_0\partial_x\phi_1(0)|Z_1(0,t)|^2.$$

Therefore, we can rewrite the equation (4.3.50) as follows:

$$\begin{aligned} &-\sigma_1^2 \int_{\Omega_1} (\partial_x + s_0\partial_x\phi_1)^2 Z_1(x,t)Z_1(x,t)dx \\ =&\sigma_1^2 \int_{\Omega_1} |\partial_x Z_1(x,t)|^2 dx - \sigma_1^2 s_0^2 \int_{\Omega_1} |\partial_x\phi_1|^2 |Z_1(x,t)|^2 dx - m_1\sigma_1 s_0 |Z_1(0,t)|^2 \\ &-\sigma_1^2 Z_1(0,t)\partial_x Z_1(0,t). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_1} |Z_1(x,t)|^2 dx + \sigma_1 \int_{\Omega_1} |\partial_x Z_1(x,t)|^2 dx \\ =&\sigma_1 s_0^2 \int_{\Omega_1} |\partial_x\phi_1|^2 |Z_1(x,t)|^2 dx + \int_{\Omega_1} F_1(x,t)Z_1(x,t)dx + m_1\sigma_1 s_0 |Z_1(0,t)|^2 \\ &+\sigma_1^2 Z_1(0,t)\partial_x Z_1(0,t). \end{aligned} \tag{4.3.51}$$

On the other hand, multiplying by $\hat{Z}_2$ the second equation of (4.2.5) and integrating on Ω_2, we have:

$$\frac{d}{dt}\int_{\Omega_2} |Z_2(t)|dx - \sigma_2^2 \int_{\Omega_2} (\partial_x + s_0\partial_x\phi_2)^2 Z_2(t)Z_2(t)dx = \int_{\Omega_2} F_2(t)Z_2(t)dx. \tag{4.3.52}$$

Integration by parts yields

$$\begin{aligned}
&-\int_{\Omega_2}(\partial_x+s_0\partial_x\phi_2)^2 Z_2(x,t)\cdot Z_2(x,t)dx\\
=&\int_{\Omega_2}|\partial_x Z_2(x,t)|^2dx - s_0^2\int_{\Omega_2}(\partial_x\phi_2)^2|Z_2(x,t)|^2dx+\partial_x Z_2(0,t)Z_2(0,t)\\
&-s(t_0)\partial_x\phi_2(L_2)|Z_2(L_2,t)|^2+\partial_x\phi_2|Z_2(0,t)|^2-\partial_x Z_2(L_2,t).
\end{aligned}\tag{4.3.53}$$

Moreover, from the boundary condition $(4.2.5)_6$, we can assert that

$$-\partial_x Z_2(L_2,t)Z_2(L_2,t)=s_0\partial_x\phi_2(L_2)|Z_2(L_2,t)|^2-J(L_2,t)Z_2(L_2,t),\quad \forall t\in(0,T).\tag{4.3.54}$$

Furthermore, applying the boundary conditions $(4.2.5)_3$ and $(4.2.5)_4$

$$\begin{aligned}
&-\sigma_2^2\partial_x Z_2(0,t)Z_2(0,t)\\
=&-\sigma_1^2\partial_x Z_1(0,t)Z_1(0,t)-\sigma_2^2(\sigma_1^2\partial_x\phi_1(0)-\sigma_2\partial_x\phi_2(0))s_0|Z_1(0,t)|^2\\
&-\sigma_2^2H(0,t)Z_2(0,t).
\end{aligned}\tag{4.3.55}$$

Combining (4.3.52) with (4.3.53), (4.3.54) and (4.3.55), gives:

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_2}|Z_2(x,t)|^2dx+\sigma_2^2\int_{\Omega_2}|\partial_x Z_2(x,t)|^2dx\\
=&\sigma_2^2s_0^2\int_{\Omega_2}|\partial_x\phi_2|^2|Z_2(x,t)|^2dx+\int_{\Omega_2}F(x,t)Z_2(x,t)dx-\sigma_1^2\partial_x Z_1(0,t)Z_1(0,t)\\
&-(m_1\sigma_1-M_2\sigma_2)s_0|Z_2(0,t)|^2-\sigma_2^2J(t)Z_2(L_2,t).
\end{aligned}\tag{4.3.56}$$

Combining the inequality (4.3.51) with (4.3.56) and integrating on $(0,T)$, we obtain:

$$\begin{aligned}
&\sigma_1^2s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1(x,t)|^2dxdt+\sigma_2^2s_0\int_0^T\int_{\Omega_1}|\partial_x Z_2(x,t)|^2dxdt\\
\leq&Cs_0^3\int_0^T\int_{\Omega_1}|Z_1(x,t)|^2dx+Cs_0^3\int_0^T\int_{\Omega_2}|Z_2(x,t)|^2dxdt+C\int_0^T\int_{\Omega_1}|F_1(x,t)|^2dxdt\\
&Cs_0\int_0^T|Z_1(0,t)|^2dx+Cs_0\int_0^T|Z_2(L_2,t)|^2dt+Cs_0\int_0^T|H|^2dt\\
&+C\int_0^T|J|^2dt.
\end{aligned}$$

Consequently, combining the inequality above with (4.3.48), we have

$$\begin{aligned}
&s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1(x,t)|^2dxdt+s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2(x,t)|^2dxdt\\
\leq&C\int_0^T\int_{\Omega_1}|F_1(x,t)|^2dxdt+C\int_0^T\int_{\Omega_2}|F_2(x,t)|^2dxdt+Cs_0\int_0^T|G_N|^2dt+Cs_0\int_0^T|H|^2dt\\
&+Cs_0\int_0^T|J|^2dt,
\end{aligned}\tag{4.3.57}$$

which is the desired conclusion.

Finally, adding the inequalities (4.3.48) and (4.3.57) yields

$$
\begin{aligned}
&s_0^3\int_0^T\int_{\Omega_1}|Z_1(x,t)|^2dxdt+s_0^3\int_0^T\int_{\Omega_2}|Z_2(x,t)|^2dxdt+s_0\int_0^T|Z_1(-L_1,t)|^2dt\\
&+s_0\int_0^T|Z_2(L_2,t)|^2dt+s_0\int_0^T\int_{\Omega_1}|\partial_x Z_1(x,t)|^2dxdt+s_0\int_0^T\int_{\Omega_2}|\partial_x Z_2(x,t)|^2dxdt\\
\leq &C\int_0^T\int_{\Omega_1}|F_1(x,t)|^2dxdt+C\int_0^T\int_{\Omega_2}|F_2(x,t)|^2dxdt+Cs_0\int_0^T|G_N|^2dt+Cs_0\int_0^T|H|^2dt\\
&+Cs_0\int_0^T|J|^2dt,
\end{aligned}
$$

and the proof of the Lemma 4.3 is complete. □

4.4 Proof of the Corollary 4.2

This section is devoted to proof the Corollary 4.2. In order to do that, let (y_1, y_2) be the solution of

$$
\begin{cases}
\partial_t y_1-\sigma_1^2\partial_x^2 y_1=0, & \forall(x,t)\in\Omega_1\times(0,T),\\
\partial_t y_2-\sigma_1^2\partial_x^2 y_2=0, & \forall(x,t)\in\Omega_2\times(0,T),\\
y_1(x,0)=y_1^0(x), & \forall x\in\Omega_1,\\
y_2(x,0)=y_2^0(x), & \forall x\in\Omega_2,\\
y_1(-L_1,t)=v(t), & \forall t\in(0,T),\\
\partial_x y_2(L_2,t)=0, & \forall t\in(0,T),
\end{cases}
\tag{4.4.1}
$$

where $\sigma_1,\sigma_2>0$, $y_j^0\in L^2(\Omega_j)$, $j=1,2$ and $v\in L^2(0,T)$.

For an easier comprehesion, we divide the proof into three steps:

• **Step 1: Duality**

Let us consider the following adjoint system:

$$
\begin{cases}
-\partial_t w_1-\sigma_1^2\partial_x^2 w_1=0, & \forall(x,t)\in\Omega_1\times(0,T),\\
-\partial_t w_2-\partial_x^2\partial_x^2 w_2=0, & \forall(x,t)\in\Omega_2\times(0,T),\\
w_1(x,T)=w_1^T(x), & \forall x\in\Omega_1,\\
w_1(x,T)=w_2^T(x), & \forall x\in\Omega_2,\\
w_1(-L_1,t)=0, & \forall t\in(0,T),\\
\partial_x w_2(L_2,t)=0, & \forall t\in(0,T).
\end{cases}
\tag{4.4.2}
$$

It is clear that the null controllability of the system (4.4.1) is equivalent to proof the so-called observability inequality of the adjoint system (4.4.2): There exists a constant $C>0$ such that each solution of (4.4.2) satisfy

$$
\int_{\Omega_1}|w_1(x,0)|^2dx+\int_{\Omega_2}|w_2(x,0)|^2dx\leq C\int_0^T|\partial_x w_1(-L_1,t)|^2dt. \tag{4.4.3}
$$

Thus, we restrict to attention to prove the inequality (4.4.3). In order to do that, we will use the Carleman estimate of the Theorem 4.1.

• Step 2: Applying the Carleman estimate

We apply the Theorem 4.1 to the system 4.4.2, i.e., there exist constants $C > 0$ and $s_* > 0$ such that for each $s \geq s_*$, each solution of 4.4.2 satisfy

$$\begin{aligned} &s^3 \int_0^T \int_{\Omega_1} e^{-2s\varphi_1} \theta^{3\alpha-2\beta} |w_1(x,t)|^2 dxdt + s^3 \int_0^T \int_{\Omega_2} e^{-2s\varphi_2} \theta^{3\alpha-2\beta} |w_2(x,t)|^2 dxdt \\ \leq &Cs \int_0^T e^{-2s\varphi_1(-L_1,t)} \theta^{\alpha-\beta} |\partial_x w_1(-L_1,t,)|^2 dt, \end{aligned} \tag{4.4.4}$$

where we use $\partial_t w_j + \sigma_j^2 \partial_x^2 w_j = 0$, for each $j = 1, 2$. Since the Carleman weights φ_1 and φ_2 are bounded, it is easy to check that

$$\begin{aligned} &C \int_{T/4}^{3T/4} \int_{\Omega_1} |w_1(x,t)|^2 dxdt + C \int_{T/4}^{3T/4} \int_{\Omega_2} |w_2(x,t)|^2 dxdt \\ \leq &{s_*}^3 \int_{\Omega_1} e^{-2s_*\varphi_1} \theta^{3\alpha-2\beta} |w_1(x,t)|^2 dx + {s_*}^3 \int_{\Omega_1} e^{-2s_*\varphi_1} |w_2(x,t)|^2 dxdx, \end{aligned} \tag{4.4.5}$$

and

$$s_* \int_0^T e^{-2s\varphi_1(-L_1)} \theta^{\alpha-\beta} |\partial_x w_1(-L_1), t|^2 \leq C \int_0^T |\partial_x w_1(-L_1,t)|^2 dt. \tag{4.4.6}$$

Combining (4.4.5) and (4.4.6) with (4.4.6) we obtain

$$C \int_{T/4}^{3T/4} \int_{\Omega_1} |w_1(x,t)|^2 dxdt + C \int_{T/4}^{3T/4} \int_{\Omega_2} |w_2(x,t)| dxdt \leq C \int_0^T |\partial_x w_1(-L_1,t)|^2 dt. \tag{4.4.7}$$

• Step 3: Observability inequality

Multiplying the equation $(4.4.2)_j$ by w_j for $j = 1, 2$, we obtain

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt} \int_{\Omega_1} |w_1(x,t)|^2 dx - \frac{1}{2}\frac{d}{dt} \int_{\Omega_2} |w_2(x,t)|^2 dx + \sigma_1^2 \int_{\Omega_1} |\partial_x w_1(x,t)|^2 dx \\ &+ \sigma_2^2 \int_{\Omega_2} |\partial_x w_2(x,t)|^2 dx = 0, \end{aligned} \tag{4.4.8}$$

where we used integration by parts. Now, integrating on $(0, \tilde{t})$, $\tilde{t} \in (0,T)$, we have

$$\begin{aligned} \int_{\Omega_1} |w_1(x,0)|^2 dx + \int_{\Omega_2} |w_2(x,0)|^2 dx = &\int_{\Omega_1} |w_1(\tilde{t})|^2 dx + \int_{\Omega_2} |w_2(\tilde{t})|^2 dx \\ &- \sigma_1^2 \int_0^{\tilde{t}} \int_{\Omega_1} |\partial_x w_1|^2 dxdt - \sigma_2^2 \int_0^{\tilde{t}} \int_{\Omega_2} |\partial_x w_2|^2 dxdt. \end{aligned}$$

Integrating on $(T/4, 3T/4)$ on $\tilde{t}$, we see that

$$\int_{\Omega_1} |w_1(x,0)|^2 dx + \int_{\Omega_2} |w_2(x,0)|^2 dx \tag{4.4.9}$$

$$\leq C \int_{T/4}^{3T/4} \int_{\Omega_1} |w_1(x,t)|^2 dx dt + C \int_{T/4}^{3T/4} \int_{\Omega_1} |w_2(x,t)|^2 dx dt. \tag{4.4.10}$$

Thus, combining (4.4.7) with (4.4.9), we obtain

$$\int_{\Omega_1} |w_1(x,0)|^2 dx + \int_{\Omega_2} |w_2(x,0)|^2 dx \leq C \int_0^T |\partial_x w_1(-L_1,t)|^2 dt, \tag{4.4.11}$$

and the proof of observability inequality is complete. Hence, system (4.4.1) is null-controllable.

Conclusions

In this thesis we have obtained theoretical results about inverse and control problems on some hyperbolic and parabolic problems. In particular, we have focused in wave systems with potential in cascade and heat equation with dynamic boundary conditions. We conclude this thesis with some final remarks and perspectives related to these subjects.

In Chapter 2 we studied the simultaneous potential reconstruction for a hyperbolic system in cascade when some components of the system are not accesible. Specifically, we analyzed this inverse problem where we cannot get any measurements on the last component. Our results are based on a suitable Carleman estimate on a hyperbolic system with measurements of all components except the last one. Then, we have adapted the Bukhgeim-Klibanov method to get a Lipschitz stability result for this inverse problem.

First of all, concerning the special structure of the cascade system we considered in this study, notice that in (2.3.7), the source terms $f_j, \dots, f_n$ arise in the estimate of F_j, for each $j = 1, \dots, n$, because of the cascade structure of system (2.1.1) and the Carleman estimate of Proposition 2.6, see also Remark 2.7. This is the main difficulty to recover the potentials $(q_1, \dots, q_n)$ with less components of (2.1.1). Then, the stability of the inverse problem treated in this thesis with two or more inaccessible components is open.

Regarding relationships of the present work with controllability, let us notice that in the particular case of $h_j = 0$ for each $j = 1, \dots, n$ in (2.2.15) and under strong assumptions on the regularity of the solutions of (2.2.14), one can obtain a Carleman inequality of (2.2.14) with internal measurements of the first component of the system. To be more precise, for each $j = 1, \dots, n$, we define α_j such that

$$\begin{cases} \alpha_{j+1} + 1 < \alpha_j < \alpha_{j+1} + 2, \quad j = 1, \dots, n-1, \\ \alpha_n = 0. \end{cases}$$

Then, there exist two constants $C = C(\Omega, \omega, T, x_0) > 0$ and $s_0 \geq 1$ such that for all $s \geq s_0$, the following inequality holds:

$$\sum_{j=1}^{n} I(\alpha_j, v_j, \Omega) \leq C \sum_{j=0}^{2^{n-1}} s^{\beta_j} \int_{-T}^{T} \int_{\omega_\omega} e^{2s\varphi} |\partial_t^j v_1|^2 dx dt,$$

for each solution of system (2.2.14) and for some positive constants β_j, $j = 0, ..., 2^{n-1}$. In principle, this would allow to construct a control that would require stronger regularity assumptions.

Finally, let us remark that a slight change in the proof of Proposition 2.6 shows that

$$I(0,v,\Omega) \leq C\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\Box v + pv|^2 dxdt + Cs\int_{-T}^{T}\int_{\omega_2} e^{2s\varphi}|\nabla v|^2 dxdt, \tag{4.4.12}$$

for all $v \in L^2(-T,T;H_0^1(\Omega))$ such that $\Box v + pv \in L^2(\Omega\times(-T,T))$, $\partial_\nu v \in L^2(\partial\Omega\times(-T,T))$ and $v(\pm T) = 0$ in Ω. The main ingredient of the proof are the part b) of Lemma 2.4 and the weighted Poincaré inequality (see [11]). Under that form, estimate (4.4.12) can be used in the study of wave systems with first order coupling terms.

In Chapter 3, we studied the null controllability for a suitable class of parabolic equations with dynamic boundary conditions. The main result is based on the proof of the observability inequality for the associated adjoint system. In order to get it, we used a suitable Carleman estimate for a heat equation with dynamic boundary conditions.

Moreover, we present other results based on the fact that parabolic equations with this kind of boundary conditions can be viewed as a limit of heat equations with discontinuous diffusion coefficients.

The results presented in this chapter can be extended naturally to higher dimensions. Indeed, let $d \geq 1$ and set $\Omega \subset \mathbb{R}^d$ be an open set with smooth boundary. In addition, let us consider $\Gamma \subset \partial\Omega$ be a nonempty open subset. Let $(u, u_\Gamma) \in L^2(\Omega\times(0,T)) \times L^2(\Gamma\times(0,T))$ be a solution of

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega v & \text{in } \Omega\times(0,T), \\ (u(\cdot,0), u_\Gamma(\cdot,0)) = (u_0, u_{\Gamma,0}), & \text{in } \Omega\times\Gamma, \\ u_\Gamma = u, & \text{on } \Gamma\times(0,T), \\ u = 0, & \text{on } (\partial\Omega\setminus\overline{\Gamma})\times(0,T), \\ \partial_t u + \partial_\nu u = 0, & \text{on } \Gamma\times(0,T), \end{cases} \tag{4.4.13}$$

where ∂_ν denotes the outward normal derivative and $\omega \subset \Omega$. Then, one can formulate the problem of null controllability for system (4.4.13) for any time $T > 0$, i.e., given $T > 0$ and $(u_0, u_{0,\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$, there exists a control $v \in L^2(\omega\times(0,T))$ such that the associated solution of (4.4.13) fulfills

$$u(T) = 0, \text{ in } \overline{\Omega}.$$

Then, following the approach given in Section 3.3 we have to prove the observability inequality associated to the adjoint system of (4.4.13) by using a suitable Carleman estimate. In this context, one can use weight functions which satisfy similar estimates as in (3.3.22) and (3.3.23). In particular, when Ω has radial symmetry we shall consider explicit weight functions based on ψ and θ used in Section 3.

As we mentioned in Section 3.1, parabolic equations with discontinuous diffusion coefficients can be used to approximate parabolic equations with dynamic boundary conditions. Additionally, it is well known that for each $K > 0$, $u_0 \in L^2(\Omega)$ and $T > 0$, (3.1.4) is null-controllable at time $T > 0$, see for example [38],[19],[20] and [21]. However, the constant $C > 0$ appeared in the observability inequality depends (for instance) on the diffusivity parameter, and therefore if we adapt these settings to problem (3.1.4), the observability constant may depends on $K \geq 1$. This means that the sequence of controls $(v^K)_{K>0}$ in (3.1.4) may not be uniformly bounded in $L^2(\omega \times (0,T))$.

In order to avoid this difficulty, one can build a Carleman estimate where the weight functions depends on $K > 0$. In the following, we present a result in this direction. From now on, for $\alpha \geq 1$ and $K \geq 1$, $\psi : \overline{\Omega} \subset \mathbb{R} \to \mathbb{R}$ and $\theta : (0,T) \subset \mathbb{R} \to \mathbb{R}$ denotes the functions given by

$$
\begin{aligned}
\psi_L(x) =& -\frac{1}{4L_1}x^2 + x + 2L_1, \quad \forall x \in \overline{\Omega_L}, \\
\psi_R(x) =& -\frac{1}{4K^2}x^2 + \frac{1}{K^2}x + 2L_1, \quad \forall x \in \overline{\Omega_R}, \\
\theta(t) =& (t(T-t))^{-\alpha}, \quad \forall t \in (0,T).
\end{aligned}
$$

We point out that $\psi \geq 2L_1$ in Ω and

$$
\psi_R(0) = \psi_L(0), \quad K^2\psi_R'(0) = \psi_L'(0),
$$

i.e., ψ satisfies the same transmission conditions of u^K in $(3.4.1)_5$ and $(3.4.1)_6$ across the interface $x = 0$. Moreover, notice that ψ_L stands for the same weight function used in Section 3.3.

Then, we have the following result:

Lemma 4.13 *Let $\alpha \geq 1$, $0 < K_0 \leq K$, $T > 0$, define $\varphi = \psi\theta$, with ψ and θ defined as above and σ^K given by* (3.1.3). *Then, there exists two positive constants $C = C(\alpha, \Omega, T)$ and $s^* = s^*(\alpha, \Omega, T)$ independent of K such that for all $s \geq s^*$ we have*

$$
\begin{aligned}
& s^3 \int_0^T \int_\Omega e^{-2s\varphi}(\sigma^K)^{-1}(t(T-t))^{-3\alpha}|y|^2 dx dt + s \int_0^T \int_\Omega e^{-2s\varphi}\sigma^K (t(T-t))^{-\alpha}|\partial_x y|^2 dx dt \\
& + s^3 \int_0^T e^{-2s\varphi(0,t)}(t(T-t))^{-3\alpha}|y(0,t)|^2 dt + s \int_0^T e^{-2s\varphi(0,t)}(t(T-t))^{-\alpha}|\partial_x y(0,t)|^2 dt \\
& + K^{-2}s^3 \int_0^T e^{-2s\varphi(L_1,t)}(t(T-t))^{-3\alpha}|y(-L_1,t)|^2 dt \leq C \int_0^T \int_\Omega e^{-2s\varphi}|\partial_t y + A^K y|^2 dx dt \\
& + Cs \int_0^T e^{-2s\varphi(-L_1,t)}|\partial_x y(-L_1,t)|^2 dt + Cs^2 \int_0^T e^{-2s\varphi(L_2,t)}(t(T-t))^{-2\alpha-1}|y(1,t)|^2 dt,
\end{aligned} \tag{4.4.14}
$$

for all $y \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; D(A^K))$, where A^K is defined by

$$
A^K y = \partial_x(\sigma^K \partial_x y),
$$

with domain

$$
D(A^K) = \{y \in H_L^1(\Omega)\,;\, \sigma^K y' \in H^1(\Omega)\}.
$$

The proof is based on the classical approach of Carleman estimates introduced in the context of parabolic equations by O. Imanuvilov et al. We emphasize that, for our purposes, the main difficulty here is to track the dependence on $K \geq 1$ of the constant

C in (4.4.14). In fact, in this context we cannot absorb the last term of the right-hand side of (4.4.14). Thus, the question of uniform controllability for parabolic problems in the form (3.1.4) remains open.

Inspired in the ideas of [9] (see also [67] and the references therein) problems in the form (3.1.1) can be viewed as limit of another class of parabolic problems. In order to get an idea, for simplicity we set $I = (0,1)$ and denote $x^* = 1 - K^{-1}$ with $K > 0$. Then, we define the subsets

$$I_L = (0, x^*) \text{ and } I_R = (x^*, 1).$$

For $K > 0$, let us consider the following problem

$$\begin{cases} (1 + (K-1)\chi_{I_L})\,\partial_t u^K - \partial_x^2 u^K = f^K, & \forall (x,t) \in I \times (0,T), \\ u^K(x,0) = u_0, & \forall x \in I, \\ u^K(0,t) = \partial_x u^K(1,t) = 0, & \forall t \in (0,T). \end{cases} \tag{4.4.15}$$

On the other hand, we introduce the problem

$$\begin{cases} \partial_t y - \partial_x^2 y = g, & \forall (x,t) \in I \times (0,T), \\ y(x,0) = y_0(x), & \forall x \in I, \\ y(0,t) = 0, & \forall t \in (0,T), \\ \partial_t y(1,t) + \partial_x y(1,t) = 0, & \forall t \in (0,T), \end{cases} \tag{4.4.16}$$

with $g \in L^2(I \times (0,T))$ and $y_0 \in H^1(I)$. Then, we have the following result

Lemma 4.14 *Let $0 < K_0 \le K$ and $u_0 \in H^1(\Omega)$. Suppose that*

$$f^K \rightharpoonup f \text{ weakly in } L^2(I \times (0,T)).$$

Then, there exists a subsequence $(u^K)_{K>0}$ of solutions for the problem (4.4.15) which converges to u in the following way

$$u^K \rightharpoonup u \text{ weakly in } L^2(0,T;H^2(I)) \cap H^1(0,T;L^2(I)).$$

Moreover, u is a strong solution of (4.4.16) with $g = f$ and $y_0 = u_0$.

Of course, all the above questions in the context of controllability can be considered for (4.4.15).

In Chapter 4, the null controllability of heat equation with discontinuous diffusion coefficients was studied. Following the arguments presented above, the idea is to prove the observability inequality for the associated adjoint system. This was done for a suitable Carleman estimate for this kind of problems. The novelty is based on the combination of microlocal analysis ideas with localization in time functions.

In this sense, we believe that the proof of this Carleman estimate allow us to deduce some insights about these systems. The original idea was to use this kind of estimates to prove uniform observability results for systems like (3.2). However, some difficulties on the proof suggest that we might use localization in time functions which also depends on space. This is a work in progress.

Appendix A

Carleman estimate for heat equation with dynamic boundary conditions by using Classical weights

A.1 Introduction and main result

The goal is to prove a Carleman estimate which allow us to prove the following Observability inequality

$$\|z(\cdot,0)\|^2_{L^2(\Omega_L)} + |z_\Gamma(0)|^2 \leq C\int_0^T\int_\omega |z|^2dxdt, \tag{A.1.1}$$

for each $(z_T, z_{T,\Gamma}) \in L^2(\Omega_L)\times\mathbb{R}$, where $(z, z_\Gamma) \in L^2(\Omega_L\times(0,T))\times L^2(0,T)$ is a solution of the adjoint system

$$\begin{cases} \partial_t z(x,t) + \partial_x^2 z(x,t) = 0, & \forall (x,t)\in\Omega_L\times(0,T),\\ (z(x,T), z_\Gamma(T)) = (z_T(x, z_{T,\Gamma})), & x\in\Omega_L,\\ z(-L_1,t) = 0, & \forall t\in(0,T),\\ z'_\Gamma(t) - \partial_x z(0,t) = 0, & \forall t\in(0,T). \end{cases} \tag{A.1.2}$$

In order to do that, we will consider the Classical weight functions introduced by A. Fursikov and O. Imanuvilov. We recall that these ones are based on an auxiliary function whose existence is given by the following result:

Lemma A.1 *Given nonempty open set $\omega \subset\subset \Omega_L$, there is a function $\eta_0 \in C^2(\overline{\Omega_L})$ such that*

$$\eta_0 > 0 \text{ in } \Omega_L, \quad \eta_0(-L_1) = \eta_0(0) = 0, \quad |\eta_0'| > 0 \text{ in } \overline{\Omega_L\setminus\omega}.$$

We notice that the functions given by the above lemma fulfills

$$\eta_0'(-L_1) > 0, \quad \eta_0'(0) < 0.$$

From now on, we fix $\omega' \subset\subset \Omega_L$, $\lambda, m > 1$ and η_0 as in the previous lemma. We define the weight functions α and η by

$$\alpha(x,t) =(t(T-t))^{-1}\left(e^{2\lambda m\|\eta_0\|_\infty} - e^{\lambda(m\|\eta_0\|_\infty+\eta_0(x))}\right), \tag{A.1.3}$$

$$\eta(x,t) =(t(T-t))^{-1}e^{\lambda(m\|\eta_0\|_\infty+\eta_0(x))}, \tag{A.1.4}$$

for each $(x,t) \in \overline{\Omega_L} \times (0,T)$. Now we have all the ingredients to state the Carleman estimate for heat equation with dynamic boundary conditions:

Theorem A.2 *Let $T > 0$, $\omega \subset\subset \Omega_L$ be a nonempty and open interval. In addition, we choose $\omega' \subset\subset \omega$. Define α, η_0, ξ as above with respect to ω'. Then, there exists constants $C > 0$, $\lambda_1 \geq 1$ and $s_1 \geq 1$ such that the following inequality holds*

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\Omega_L} e^{2s\alpha}\xi^3|\varphi|^2dxdt + s\lambda\int_0^T\int_{\Omega_L} e^{2s\varphi}\xi|\partial_x\varphi|^2dxdt\\
&+ s^{-1}\int_0^T\int_{\Omega_L} e^{2s\alpha}\xi^{-1}|\partial_t\varphi|^2dxdt + s^{-1}\int_0^T\int_{\Omega_L} e^{2s\alpha}\xi^{-1}|\partial_x^2\varphi|^2dxdt\\
&+ s^3\lambda^3\int_0^T e^{2s\alpha(0,t)}\xi^3(0,t)|\varphi(0,t)|^2dt + s\lambda\int_0^T e^{2s\alpha(0,t)}|\partial_x\varphi(0,t)|^2dt\\
&+ \int_0^T e^{2s\alpha(0,t)}|\partial_t\varphi(0,t)|^2dt + s\lambda\int_0^T e^{2s\alpha(-L_1,t)}\xi(-L_1,t)|\partial_x\varphi(-L_1,t)|^2dt\\
&\leq C\int_0^T\int_{\Omega_L} e^{2s\alpha}|\partial_t\varphi + \partial_x^2\varphi|^2dxdt + C\int_0^T e^{2s\alpha(0,t)}|\partial_t\varphi(0,t) - \partial_x\varphi(0,t)|^2dt\\
&+ Cs^3\lambda^4\int_0^T\int_\omega e^{2s\alpha}\xi^3|\varphi|^2dxdt,
\end{aligned} \tag{A.1.5}$$

for all $\lambda \geq \lambda_1$ and $s \geq s_1$ and for all $\varphi \in C^2(\overline{\Omega} \times [0,T])$.

The rest of this appendix is devoted to prove the above Theorem.

A.2 Proof of the Carleman estimate

Let $\varphi \in C^\infty(\overline{\Omega_L} \times [0,T])$, $\lambda \geq 1$ and $s \geq s \geq 1$ be given. Define

$$\psi = e^{-s\varphi}\varphi, \quad f = e^{-s\alpha}(\partial_t\varphi + \partial_x^2\varphi), \quad g = e^{-s\alpha}(\partial_t\varphi - \partial_x\varphi), \quad \forall (x,t) \in \overline{\Omega_L} \times (0,T).$$

Direct computations show that

$$e^{-s\alpha}\partial_t\varphi = \partial_t\varphi + s\partial_t\alpha\psi, \quad e^{-s\alpha}\partial_x\varphi = \partial_x\psi + s\partial_x\alpha\psi,$$
$$e^{-s\alpha}\partial_x^2\varphi = \partial_x^2\psi + 2s\partial_x\alpha\partial_x\psi + s^2|\partial_x\alpha|^2\psi + s\partial_x^2\alpha\psi.$$

In the following, we shall use the abbreviations

$$M_1\psi = s^2|\partial_x\alpha|^2\psi + \partial_x^2\psi + s\partial_t\alpha\psi, \quad M_2\psi = s\partial_x^2\alpha + \partial_t\psi + 2s\partial_x\alpha\partial_x\psi,$$
$$N_1\psi = s\partial_t\alpha\psi - \partial_x\psi, \quad N_2\psi = \partial_t\psi - s\partial_x\alpha\psi.$$

Then, according to this notation, it is clear that ψ satisfies the following equations:

$$M_1\psi + M_2\psi = f, \quad \text{and } N_1\psi + N_2\psi = g, \text{ in } \overline{\Omega_L} \times (0,T). \tag{A.2.1}$$

Applying $\|\cdot\|_{L^2(\Omega_L\times(0,T))}$ and $\|\cdot\|_{L^2(0,T)}$ to the equations (A.2.1) we get

$$\begin{aligned} &\|M_1\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|N_1\psi(0,\cdot)\|^2_{L^2(0,T)} + \|N_2\psi(0,\cdot)\|^2_{L^2(0,T)} \\ &\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega_L\times(0,T))} + \langle N_1\psi(0,\cdot), N_2\psi(0,\cdot)\rangle_{L^2(0,T)}. = \|f\|^2_{L^2(\Omega_L\times(0,T))} + \|g(0,\cdot)\|^2_{L^2(0,T)} \end{aligned} \tag{A.2.2}$$

Our next task is to compute the inner products in the right-hand side of (A.2.2). In order to do that, we shall use the notation

$$\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega_L\times(0,T))} = \sum_{j,k=1}^{3} I_{j,k},$$

where I_{jk} stands for the scalar product in $L^2(\Omega_L \times (0,T))$ between the j^{th} term of $M_1\psi$ and the k^{th} term of $M_2\psi$. Then, I_{11} reads as follows

$$I_{11} = s^3 \int_0^T \int_{\Omega_L} |\partial_x\alpha|^2 \partial_x^2\alpha |\psi|^2 dxdt,$$

On the other hand, using the identity $\frac{1}{2}\psi\partial_t\psi = \partial_t(|\psi|^2)$ in $\Omega_L \times (0,T)$, we have

$$I_{12} = s^2 \int_0^T \int_{\Omega_L} |\partial_x\alpha|^2 \psi\partial_t\psi dxdt = -\int_0^T \int_{\Omega_L} \partial_x\alpha\partial_t\partial_x\alpha|\psi|^2 dxdt,$$

where we are used the fact that α blows up as $t \to 0^+$ and $t \to T-$. Moreover, integration by parts shows that

$$\begin{aligned} I_{13} =& 2s^3 \int_0^T \int_{\Omega_L} |\partial_x\alpha|^3 \psi\partial_x\psi dxdt \\ =& -3s^3 \int_0^T \int_{\Omega_L} |\partial_x\alpha|^2\partial_x^2\alpha|\psi|^2 dxdt + s^3 \int_0^T |\partial_x\alpha(0,t)|^3|\psi(0,t)|^2 dt, \end{aligned}$$

where we used $\psi(-L_1,t) = 0$ for all $t \in (0,T)$. In the same way, the term I_{21} can be estimated as follows

$$\begin{aligned} I_{21} =& -s \int_0^T \int_{\Omega_L} \partial_x^2\alpha\psi\partial_x^2\psi dxdt \\ =& -s \int_0^T \int_{\Omega_L} \partial_x^3\alpha\psi\partial_x\psi dxdt - s\int_0^T \int_{\Omega_L} \partial_x^2\alpha|\partial_x\psi|^2 dxdt + s\int_0^T \partial_x^2\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt. \end{aligned}$$

In addition, the term I_{22} reads as follows:

$$\begin{aligned} I_{22} =& \int_0^T \int_{\Omega_L} \partial_x^2\psi\partial_t\psi dxdt \\ =& -\int_0^T \int_{\Omega_L} \partial_t(|\partial_x\psi|^2) dxdt + \int_0^T \partial_x\psi(0,t)\partial_t\psi(0,t)dt. \end{aligned}$$

Notice that the first term in the above equality is given by

$$-\frac{1}{2}\int_0^T\int_{\Omega_L}\partial_t(|\partial_x\psi|^2)dxdt = 0.$$

On the other hand, using the equations (A.2.1) for g we have

$$\begin{aligned}
I_{22} &= \int_0^T \partial_x\psi(0,t)\partial_t\psi(0,t)dt \\
&= \int_0^T |\partial_t\psi(0,t)|^2dt + s\int_0^T \partial_t\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt \\
&\quad - s\int_0^T \partial_x\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt - \int_0^T \partial_x\psi(0,t)g(0,t)dt.
\end{aligned}$$

Moreover, I_{23} can be estimated in the following way

$$\begin{aligned}
I_{23} =&2s\int_0^T\int_{\Omega_L}\partial_x\alpha\partial_x\psi\partial_x^2\psi dxdt \\
=&-s\int_0^T\int_{\Omega_L}\partial_x^2\alpha|\partial_x\psi|^2dxdt + s\int_0^T\partial_x\alpha(0,t)|\partial_x\psi(0,t)|^2dt \\
&- s\int_0^T\partial_x\alpha(-L_1,t)|\partial_x\psi(-L_1,t)|^2dt.
\end{aligned}$$

By definition, I_{31} reads as follows

$$I_{31} = s^2\int_0^T\int_{\Omega_L}\partial_t\alpha\partial_x^2\alpha|\psi|^2dxdt.$$

Once again, since α blows up as $t\to 0^+$ and $t\to T^-$ we get

$$I_{32} = s\int_0^T\int_{\Omega_L}\partial_t\alpha\psi\partial_t\psi dxdt = -\frac{1}{2}s\int_0^T\int_{\Omega_L}\partial_t^2\alpha|\psi|^2dxdt.$$

Finally, I_{33} is given by

$$\begin{aligned}
I_{33} =&2s^2\int_0^T\int_{\Omega_L}\partial_x\alpha\partial_t\alpha\psi\partial_x\psi dxdt \\
=&-s^2\int_0^T\int_{\Omega_L}\partial_x(\partial_x\alpha\partial_t\alpha)|\psi|^2dxdt + s^2\int_0^T\partial_x\alpha(0,t)\partial_t\alpha(0,t)|\psi(0,t)|^2dt.
\end{aligned}$$

Gathering all the terms we have

$$
\begin{aligned}
&\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega_L\times(0,T))}\\
=&-2s^3\int_0^T\int_{\Omega_L}|\partial_x\alpha|^2\partial_x^2\alpha|\psi|^2dxdt-2s\int_0^T\int_{\Omega_L}\partial_x^2\alpha|\partial_x\psi|^2dxdt\\
&+s^3\int_0^T|\partial_x\alpha(0,t)|^3|\psi(0,t)|^2dt+s\int_0^T\partial_x\alpha(0,t)|\partial_x\psi(0,t)|^2dt\\
&+\int_0^T|\partial_t\psi(0,t)|^2dt-s\int_0^T\partial_x\alpha(-L_1,t)|\partial_x\psi(-L_1,t)|^2dt\\
&-s\int_0^T\int_{\Omega_L}\partial_x^3\alpha\psi\partial_x\psi dxdt+s\int_0^T\partial_x^2\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt\\
&+s\int_0^T\partial_t\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt-s\int_0^T\partial_x\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt\\
&-\int_0^T g(0,t)\partial_x\psi(0,t)dt+s^2\int_0^T\int_{\Omega_L}\partial_t\alpha\partial_x^2\alpha|\psi|^2dxdt\\
&-\frac{1}{2}s\int_0^T\int_{\Omega_L}\partial_t^2\alpha|\psi|^2dxdt+s^2\int_0^T\partial_x\alpha(0,t)\partial_t\alpha(0,t)|\psi(0,t)|^2dt.
\end{aligned}
\tag{A.2.3}
$$

Similar computations shows that the second inner product of (A.2.2) is given by

$$
\begin{aligned}
&\langle N_1\psi(0,t), N_2\psi(0,t)\rangle_{L^2(0,T)}\\
=&s\int_0^T\partial_t\alpha(0,t)\psi(0,t)\partial_t\psi(0,t)dt-s^2\int_0^T\partial_t\alpha(0,t)\partial_x\alpha(0,t)|\psi(0,t)|^2dt\\
&-\int_0^T\partial_x\psi(0,t)\partial_t\psi(0,t)dt+s\int_0^T\partial_x\alpha(0,t)\psi(0,t)\partial_x\psi(0,t)dt.
\end{aligned}
\tag{A.2.4}
$$

Now we focus on some estimates on weight functions. According to the definitions of α and η, we get

$$
|\partial_t\alpha(x,t)|\le C(t(T-t))^{-1}\xi(x,t),\quad |\partial_t^2\alpha(x,t)|\le C(t(T-t))^{-2}\xi(x,t),\tag{A.2.5}
$$

for each $(x,t)\in\overline{\Omega_L}\times(0,T)$ and for some constant C dependent of T but independent of λ, m and s. On the other hand, a direct computations on spatial derivatives of α gives

$$
\partial_x\alpha(x,t)=-\lambda\eta_0'(x)\xi(x,t),\quad \partial_x^2\alpha(x,t)=-\lambda(\eta_0''(x)+\lambda|\eta_0'(x)|^2)\xi(x,t),\tag{A.2.6}
$$

for each $(x,t)\in\overline{\Omega_L}\times(0,T)$. We point out that the second derivative of α can be bounded by below in the following way

$$
\partial_x^2\alpha(x,t)\ge-\lambda^2|\eta_0'(x)|^2\xi(x,t),\quad \forall(x,t)\in\overline{\Omega_L}\times(0,T).\tag{A.2.7}
$$

Then, substituting (A.2.3) and (A.2.4) into (A.2.2) and by using the estimates (A.2.5), (A.2.6) and (A.2.7) we obtain

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|N_1\psi(0,t)\|^2_{L^2(0,T)} + \|N_2\psi(0,T)\|^2_{L^2(0,T)} \\
&+ s^3\lambda^4\int_0^T\int_{\Omega_L}\xi^3|\psi|^2dxdt + s\lambda\int_0^T\int_{\Omega_L}\xi|\partial_x\psi|^2dxdt \\
&+ s^3\lambda^3\int_0^T\int_{\Omega_L}\xi^3(0,t)|\psi(0,t)|^2dxdt + s\lambda\int_0^T\xi(0,t)|\partial_x\psi(0,t)|^2dt \\
&+ \int_0^T|\partial_t\psi(0,t)|^2dt + s\lambda\int_0^T\xi(-L_1,t)|\partial_x\psi(-L_1,t)|^2dt \\
\le& C_1\int_0^T\int_{\Omega_L}|f|^2dxdt + C_1\int_0^T|g(0,t)|^2dt + C_1s^3\lambda^4\int_0^T\int_{\omega'}\xi|\psi|^2dxdt \\
&+ C_1s\lambda\int_0^T\int_{\omega'}\xi|\partial_x\psi|^2dxdt + X + Y
\end{aligned}
\tag{A.2.8}
$$

where X and Y are defined by

$$
\begin{aligned}
X =& C_1s^2\lambda^2\int_0^T\int_{\Omega_L}(t(T-t))^{-1}\xi^2|\psi|^2dxdt + C_1s^2\lambda^2\int_0^T\int_{\Omega_L}(t(T-t))^{-2}\xi^2|\psi|^2dxdt \\
&+ C_1s\lambda^3\int_0^T\int_{\Omega_L}\xi|\psi||\partial_x\psi|dxdt,
\end{aligned}
$$

and

$$
\begin{aligned}
Y =& C_1s\lambda^2\int_0^T\xi(0,t)|\psi(0,t)||\partial_x\psi(0,t)|dt + C_1s\int_0^T(t(T-t))^{-1}\xi(0,t)|\psi(0,t)||\partial_x\psi(0,t)|dt \\
&+ C_1s\lambda\int_0^T\xi(0,t)|\psi(0,t)||\partial_x\psi(0,t)|dt + C_1\int_0^T|g(0,t)||\partial_x\psi(0,t)|dt \\
&+ C_1s^2\lambda\int_0^T(t(T-t))^{-1}\xi^2(0,t)|\psi(0,t)|^2dt + C_1\int_0^T|\partial_x\psi(0,t)||\partial_t\psi(0,t)|dt \\
&+ C_1s\int_0^T(t(T-t))^{-1}\xi(0,t)|\psi(0,t)||\partial_t\psi(0,t)|dt + C_1s\lambda\int_0^T\xi(0,t)|\psi(0,t)||\partial_x\psi(0,t)|dt.
\end{aligned}
$$

Now, it is clear that there exists $\lambda_1\ge 1$, $s_1\ge 1$ such that for all $\lambda\ge\lambda_1$ and $s\ge s_1$ we have the following estimates:

$$
X \le \frac{1}{2}s^3\lambda^4\int_0^T\int_{\Omega_L}\xi^3|\psi|^2dxdt + \frac{1}{2}s\lambda\int_0^T\int_{\Omega_L}\xi|\partial_x\psi|^2dxdt, \tag{A.2.9}
$$

and

$$
\begin{aligned}
Y \le& \frac{1}{2}s^3\lambda^3\int_0^T\int_{\Omega_L}\xi^3(0,t)|\psi(0,t)|^2dxdt + \frac{1}{2}s\lambda\int_0^T\xi(0,t)|\partial_x\psi(0,t)|^2dt \\
&+ \frac{1}{2}\int_0^T|\partial_t\psi(0,t)|^2dt + \frac{1}{2}C_1^2\int_0^T|g(0,t)|^2dt.
\end{aligned}
\tag{A.2.10}
$$

Then, using (A.2.9) and (A.2.10) in (A.2.8) we get

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|M_2\psi\|^2_{L^2(\Omega_L\times(0,T))} + \|N_1\psi(0,\cdot)\|^2_{L^2(0,T)} + \|N_2\psi(0,\cdot)\|^2_{L^2(0,T)} \\
&+ s^3\lambda^4 \int_0^T \int_{\Omega_L} \xi^3|\psi|^2 dxdt + s\lambda \int_0^T \int_{\Omega_L} \xi|\partial_x\psi|^2 dxdt \\
&+ s^3\lambda^3 \int_0^T \int_{\Omega_L} \xi^3(0,t)|\psi(0,t)|^2 dxdt + s\lambda \int_0^T \xi(0,t)|\partial_x\psi(0,t)|^2 dt \\
&+ \int_0^T |\partial_t\psi(0,t)|^2 dt + s\lambda \int_0^T \xi(-L_1,t)|\partial_x\psi(-L_1,t)|^2 dt \\
\leq& C_2 \int_0^T \int_{\Omega_L} |f|^2 dxdt + C_2 \int_0^T |g(0,t)|^2 dt + C_2 s^3\lambda^4 \int_0^T \int_{\omega'} \xi^3|\psi|^2 dxdt,
\end{aligned}
$$

where the local term of $\partial_x\psi$ in ω' can be absorbed as in [48]. Moreover, global terms of $\partial_t\psi$ and $\partial_x^2\psi$ can be obtained by using the equations (A.2.1) and Young's inequality. Finally, we come back on the original variables and the proof is complete. □

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