UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

EARLY DETECTION OF EXTREME WAVES BY ACOUSTIC-GRAVITY WAVES

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA MENCIÓN MATEMÁTICAS APLICADAS

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Este trabajo ha sido parcialmente financiado por los proyectos Centros de Excelencia Basal Conicyt PIA AFB 170001 CMM & UMI-CNRS 2807 y Fondecyt Regular 1171854

SANTIAGO DE CHILE
2019

ABSTRACT OF THE THESIS TO BE ELIGIBLE FOR
THE DEGREE OF MASTER OF SCIENCE IN ENGINEERING IN APPLIED MATHEMATICS
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DATE: 2019
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EARLY DETECTION OF EXTREME WAVES BY ACOUSTIC-GRAVITY WAVES

Extreme waves generated in the ocean are of high importance because various maritime structures in the world, including ships, are confronted to this type of wave events, both in deep waters and in coastal areas. Some extreme waves correspond to wave phenomena generated in an atypical way in the ocean, also called *monster waves, freak waves, rogue waves, extreme waves, solitons etc.*, since their generation differs from the common waves generated by wind. Assuming a slightly compressible ocean, the generation and analysis of acoustic-gravity waves (AGW or acoustic waves) in the ocean have been the subject of study for some time, because from them it is possible to obtain some information from the gravity wave, in this case a extreme wave that have generated them, and also to know other kind of phenomena induced by these AGW, as is the case of the bottom pressure.
In the present work, a mathematical model has been developed which represents the generation and propagation of an extreme wave represented by a pressure change in the surface of the ocean considering compressible fluid, from which the generation and propagation of acoustic waves is induced. Since sound travels at a speed of 1500 m/s in the ocean, these waves arrive first at any observation point, allowing early detection of the extreme wave from the pressure in the oceanic bottom due to propagation of the acoustic wave. The theoretical development and two-dimensional numerical simulations are presented in the document.
The implementation of this methodology and its results is relevant in the field of civil and maritime engineering in Chile since its high potential in coastal zones, due to the fact that for some years, the frequency of extreme wave events has been seen increased, and having an alternative detection system for extreme wave events can become a relevant factor in coastal management and natural disasters services.
It is important to mention that this type of work has not been developed previously in Chile.

DETECCIÓN TEMPRANA DE ONDAS EXTREMAS POR MEDIO DE ONDAS ACÚSTICAS

Olas extremas generadas en el océano son de alta importancia debido a que diversas estructuras marítimas en el mundo, inclyendo barcos, son enfrenatados a este tipo de eventos de oleaje, tanto en aguas profundas como en zonas costeras. Algunas olas extremas corresponden a fenómenos de oleaje generados de manera atípica en el océano, también llamadas *monster waves, freak waves, rogue waves, extreme waves, solitons etc.*, ya que su generación difiere de las olas comúnes generadas por viento.
Asumiendo un océano ligeramente compresible, la generación y análisis de ondas acústicas en el océano (acoustic waves o acoustic-gravity waves, en inglés) han sido tema de estudio desde hace algún tiempo, debido a que a partir de ellas es posible obtener alguna información de la onda de gravedad que las generó, y también conocer otros fenómenos inducidos por éstas, como es el caso de la presión en el fondo.
En el presente trabajo, se ha desarrollado un modelo matemático que representa la generación y propagación de una onda extrema representada por un cambio de presión en la superficie del océano considerando fluido compresible, a partir de la cual se induce la generación y propagación de ondas acústicas. Dado que el sonido viaja a una velocidad de 1500 m/s en el océano, éstas ondas llegan primero a cualquier punto de observación, permitiendo una detección anticipada de la onda extrema a partir de la presión en el fondo por efecto de la propagación de la onda acústica. El desarrollo teórico y simulaciones numéricas bidimensionales son presentadas en el documento.
La implementación de esta metodología y sus resultados es relevante en el ámbito de la ingeniería civil y marítima de Chile por su alto potencial en las zonas costeras, debido a que desde algunos años, la frecuencia de los eventos extremos de oleaje se han visto incrementada, y contar con un sistema de detección alternativo de eventos extremos de oleaje puede llegar a ser relevante en los servicios de gestión costera y desastres naturales.
Es importante mencionar que este tipo de trabajos no ha sido desarrollada anteriormente en Chile.

Acknowledgment

I want to thank my advisors Jaime Ortega and Usama Kadri for the high support in the process of this thesis work. To the Department of Mathematical Engineering and the Center for Mathematical Modeling (PIA AFB 170001 CMM) of the Universidad de Chile for letting me be part of the Applied Mathematics program and their economical support. I would like to thank to the people of the School of Mathematics in Cardiff University, they made my staying there very great and fun. Special thanks to my colleague of the Applied Mathematics program, Jorge Sepúlveda, for the long days of study to can overcome the difficulties of the Mathematics. To Juan Pablo Donoso, for his enormous patience explaining to me the specific concepts of the applied maths. To the kind people I met in the Department of Mathematical Engineering, Sebatián Tapia, Hugo Maturana, Javier Ramírez, Jorge Olivares and Guido Besomi.

Finally, I would like to express the highest gratitude to my wife Pamela, and to my children Kevin, Isabella and Emilia, for supporting me unconditionally. I always received a word of love from them to continue with this process.

To my wife Pamela, and to my children
Kevin, Isabella and Emilia...
I love you...

“Con la práctica diaria se llega a lo perfecto,
y la perfección única viene si usted trabaja para lograrlo”.

"Todo lo que se quiere se puede.
Siempre hay tiempo para mejorar.
El sacrificio, el trabajo serio y responsable conducen al éxito.
El éxito es una realidad, está en ti. Si tú quieres, puedes lograrlo".

Gran Maestro Marcos Beltrán Troncoso

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Coastal and offshore areas around the world are affected by extreme waves events commonly called *giant waves*, *rogue waves*, *monster waves*, *freak waves*, *solitons*, etc. These kinds of waves are generated in some point at the ocean and have a physical mechanism of generation (different to other water waves such as the generated by winds) that occurs in a sudden manner containing high energy associated to the wave height and period. Figure 1.1 shows an example of an extreme wave event impacting on an offshore platform.

Figure 1.1: Extreme wave event on Borgholm Dolphin platform, Scotland. Source: BBC News

By now, many of these extreme waves have been recorded by different devices, such as deep water buoys or ADCP's (*Acoustic Doppler Current Profiler*) in coastal areas; or have been detected by technologies, such as, satellite altimetry of the ocean surface. A typical example of an extreme wave is the famous "New Year Wave", measured on January 1^{st} 1995 in the North Sea with a very large height about 26m. Figure 1.2 shows a time series record of this event.

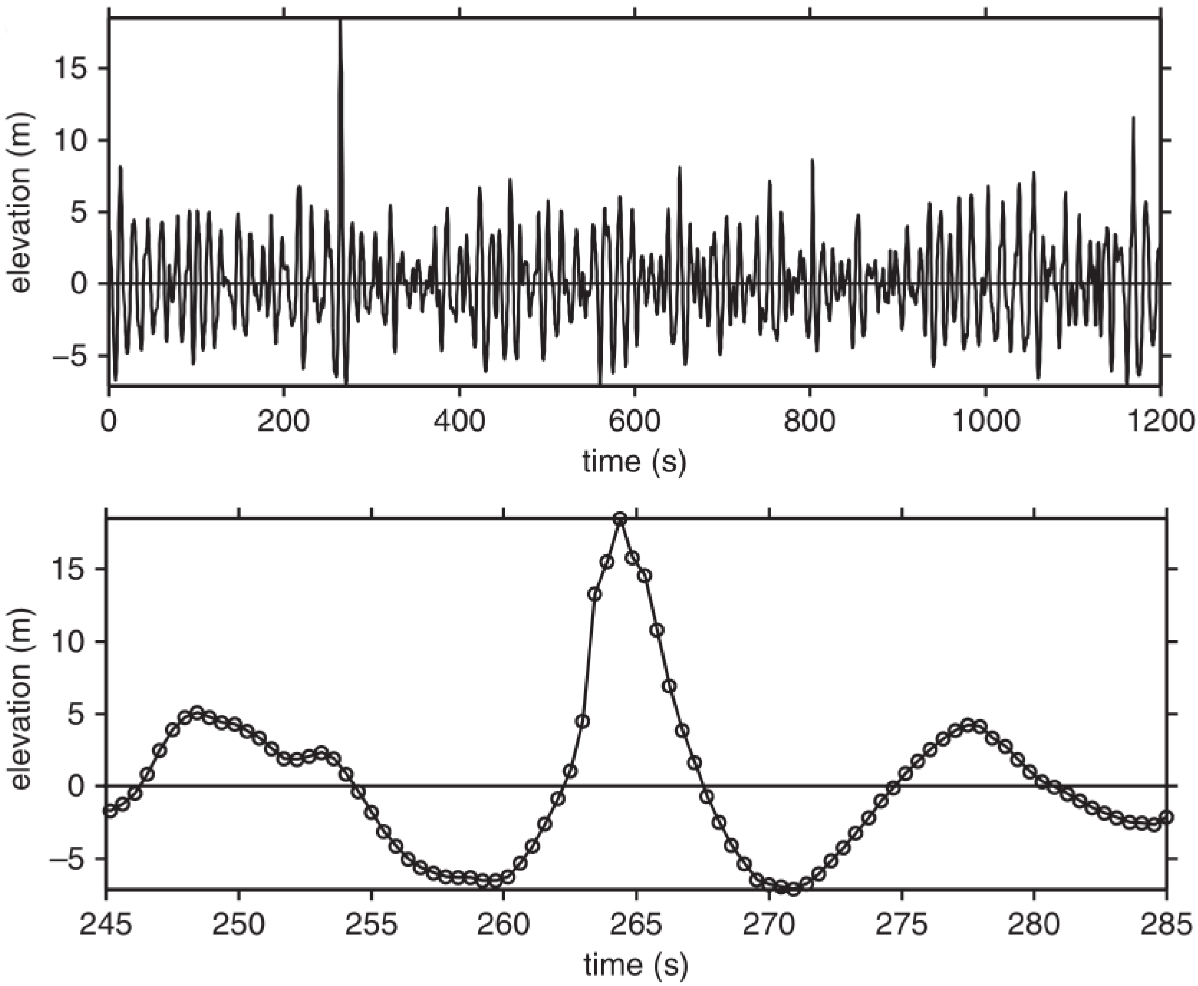

Figure 1.2: The famous “New Year Wave” measured on the 1st of January 1995 in the North Sea. (Wave height about 26m). Source: Rogue waves in the ocean. Kharif 2009.

Despite of the above, not too much efforts have been carried out on the prediction or early detection of these wave events, what can have a very high impact in saving people’s life, reducing the destruction of maritime facilities or to evacuate populated areas.

There are different theories around the generating mechanism of extreme waves in the ocean, such as the Benjamin-Feir instability, which is well understood and has been reproduced experimentally, such as in Chabchoub (2011). Recently, there have been some efforts predicting rogue waves arising in realistic ocean spectra by characterizing the trigger of rogue waves Cousins (2016). Nevertheless, to our knowledge, none of the previous works on extreme waves considers the slight compressibility of the ocean, which reveals a family of propagating acoustic-gravity waves (AGWs) that can be excellent precursors for an early warning system.

When the developments have been been treated as incompressible, that assumption is only valid so long as the time taken for a disturbance to be propagated to the bottom is small compared with the period of the waves, that is, $h/c \ll T$ or $h \ll cT$, where c is the velocity of sound in water, h is the depth and T is the wave period. For ocean waves h may be of the order of several kilometers, c is about 1500 m/s and T lies between about 5 and 20 s. The condition above is therefore no longer satisfied. It follows that in practice the compressibility of the water must be taken into account.

According the above, taking Cartesian axes (x, y, z) with the origin in the undisturbed free surface, the y-axis parallel to the wave crests, and the z-axis vertically downwards. It is assumed that the motion is periodic in the x-direction with wave-length λ. Let $z = h$ be the equation of the rigid bottom and $z = \eta$ the equation of the free surface. Also let $u =$ velocity, $p =$ pressure, $\rho =$ density, and let p_s, and ρ_s, denote the (constant) values of p and ρ at the free surface. We shall assume that viscosity is negligible and that the velocity is irrotational, so that

$$u = \nabla\phi \tag{1.1}$$

We assume also that ρ is a function of p only. Then the equations of motion may be integrated (Lamb 1932) to give

$$\phi_t - \frac{1}{2}u^2 + gz - P = 0 \tag{1.2}$$

where ϕ contains an arbitrary function of the time t and where

$$P = \int_{\rho_s}^{\rho} \frac{\mathrm{d}p}{\rho} \tag{1.3}$$

The relation between p and ρ is assumed as follow,

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2 = \text{constant} \tag{1.4}$$

that is, the velocity of sound c in the medium is constant. Then

$$P = c^2 \int_{\rho_s}^{\rho} \frac{\mathrm{d}p}{\rho} = c^2 \log(\rho/\rho_s) \tag{1.5}$$

Now, the equation of continuity can be written as

$$\frac{D\rho}{Dt} - \rho\nabla^2\phi = 0 \tag{1.6}$$

where $D\rho/Dt$ denotes the differentiation following the motion. Thus

$$\nabla^2\phi = \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{D}{Dt}\log(\rho) \tag{1.7}$$

and from (1.5) the next expression is obtained

$$\nabla^2\phi = \frac{1}{c^2}\frac{DP}{Dt} \tag{1.8}$$

To eliminate P between the above equations

$$c^2\nabla^2\phi = \frac{DP}{Dt}\left(\phi_t - \frac{1}{2}u^2 + gz\right)$$

$$= \phi_{tt} - \frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + u\cdot\nabla\phi_t - u\cdot\nabla\left(\frac{1}{2}u^2\right) + g\phi_z \tag{1.9}$$

but as

$$u\cdot\nabla\phi_t = u\cdot\frac{\partial}{\partial t}(\nabla\phi) = -\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) \tag{1.10}$$

then

$$\phi_{tt} - c^2\nabla^2\phi + g\phi_z - \frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) - u\cdot\nabla\left(\frac{1}{2}u^2\right) = 0 \tag{1.11}$$

which corresponds to the differential equation for ϕ, what can be expressed as follows once the non linear terms are neglected

$$\phi_{tt} - c^2\nabla^2\phi + g\phi_z = 0 \tag{1.12}$$

The above conducts to AGW theory considers wave disturbances in a compressible medium under the effects of gravity, accounting for two types of waves, gravity (surface) and acoustic (compression), that are virtually decoupled due to the disparity in time and length scales. It is essential to consider both gravity and compressibility effects, during one of the following scenarios: (1) when the frequency of the wave disturbance is extremely low, since the phase speed of the wave can then be altered (Kadri & Stiassnie, 2012; Kadri, 2019; Abdolali, 2019); (2) when studying energy exchange via a nonlinear triad mechanism (Kadri & Akylas, 2016; Kadri & Stiassnie, 2013; Kadri, 2017; Dias, 2018; Kadri, 2018); (3) when focusing on small scale drifting (Kadri, 2014); and (4) when analysing the propagation of AGWs from a disturbance at the bottom, such as a submarine earthquake, e.g. Kadri (2017); Mei & Kadri (2018), an impact on the sea-surface as in Kadri (2017), or a general disturbance on the free surface, e.g. Renzi & Dias (2014).

Based on the above and due to the pressure changes in the ocean surface because of extreme waves generation, they can induce the acoustic-gravity waves formation, which travel through the ocean at the speed of sound in the water (1500 m/s), velocity that is much higher than the phase velocity of a generated surface gravity wave, as the case of an extreme wave. As these AGW possess a high velocity, they can arrive the coast, or another point in the ocean earlier than an extreme gravity wave. Considering the aforementioned, these sound phenomena can be used for the early detection of these extreme waves, what have a great impact in to prevent this kind of extreme events in some point of the ocean, either deep water or coastal zones.

The present work is aimed to develop theoretically and numerically the generation and propagation of AGW by the generation of an extreme wave assuming compressible fluid. A multiple scale approach has been applied for the generation of the acoustic-gravity wave due to a pressure change on the ocean surface. This sudden change of pressure has been induced by the extreme wave once it has been generated. Bi-dimensional numerical simulations have been carried out to test the theory and to assess the developed calculations. The work has been developed with the aim to study the case of extreme waves and the potential to apply acoustic-gravity wave in live recordings as part of an early warning system.

The work presents the development of the mathematical model in Chapter 2, where the governing equations for the slightly compressible fluid are solved estimating the general form for the velocity potential of the gravity and acoustic-gravity waves using the Green's functions. In Chapter 3, the estimation of the potential for both, gravity and acoustic-gravity waves is presented. Chapter 4 shows the stationary phase approximation applied to obtain the gravity and acoustic–gravity wave potentials in the far field for rapidly varying solutions. In the same way, Chapter 5 presents the validation of the methodology compared with the theoretical results of other authors. Also, numerical simulations are presented in this chapter, where the bottom pressures due acoustic and acoustic–gravity waves are analyzed for different locations. Finally, conclusions and future work, and also the bibliography of the document are presented in Chapter 6 and Chapter 7, respectively.

Chapter 2

Mathematical Model

In the present chapter the considered mathematical model comes from Longuet-Higgins (1950), which corresponds to model of wave motion in a compressible fluid. The governing equations are developed as a basis to obtain the systems of equations for the gravity and acoustic-gravity waves. A scalement has been used to set and estimate the latter according to Kadri & Akylas (2015), where the scalement parameter $\mu = gh/c^2$ controls the effects of compressibility relative to gravity. At the end of the chapter, the Green's functions are calculated to obtain the potentials for each wave, gravity and acoustic-gravity, respectively.

2.1 Governing Equations

We consider the propagation of wave disturbances in a compressible ocean of a constant depth h under the effects of gravity. The sea bottom ($z = -h$) is assumed rigid, the water is treated as an inviscid barotropic fluid with constant sound speed c, and motion is irrotational. Following (Kadri & Akylas, 2016), we shall use dimensionless variables, employing h/c as time scale and μh as length scale, where $\mu = gh/c^2$ controls the effects of compressibility relative to gravity (typically $\mu \ll 1$).

The problem is formulated in terms of velocity potential $\phi(x, z, t)$, where $\mathbf{u} = \nabla\phi$ is the velocity field. Combining the unsteady Bernoulli equation with continuity, yields the field equation which governs the interior fluid (Longuet-Higgins (1950)),

$$\phi_{tt} - \frac{1}{\mu^2}\nabla^2\phi + \phi_z = 0 \qquad ; \text{on } -1/\mu < z < \eta \tag{2.1}$$

where $\mu = gh/c^2$. Here c corresponds to the speed of sound in water ($c = 1.5 \times 10^3$ m/s), h is the depth and g is the gravitational acceleration. The parameter μ controls the effects of compressibility relative to gravity (typically $\mu \ll 1$).

The ∇ operator corresponds to

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right) \tag{2.2}$$

then ∇^2 becomes

$$\nabla^2 = \Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \tag{2.3}$$

where Δ is the Laplacian operator. The equation (1.1) can be rewritten as

$$\phi_{tt} - \frac{1}{\mu^2} \left(\phi_{xx} + \phi_{zz} \right) + \phi_z = 0 \qquad ; \text{on } -1/\mu < z < \eta \tag{2.4}$$

The bottom boundary condition in $z = -1/\mu$ is stated as

$$\phi_z = 0 \qquad ; \text{on } z = -1/\mu \tag{2.5}$$

By the other hand, on free surface $z = \eta(x, t)$, we consider the dynamic and kinematic conditions as follow

$$\phi_t + z \qquad ; \text{on } z = 0 \tag{2.6}$$

$$\nabla^2 \phi = 0 \qquad ; \text{on } z = 0 \tag{2.7}$$

what gives the combined boundary condition in the surface. After expanding the two free-surface conditions about $z = 0$, η can be expressed in terms of ϕ to this first order of approximation

$$\frac{\partial \phi_t}{\partial t} + \frac{\partial \eta}{\partial t} = \frac{\partial P(x,t)}{\partial t} \qquad ; \text{on } z = 0 \tag{2.8}$$

or

$$\phi_{tt} + \phi_z = P_t(x, t) \qquad ; \text{on } z = 0, \tag{2.9}$$

where $P(x, t)$ is the pressure acting as external force on the water surface. Now, we have to solve the system composed by the equations (1.4), (1.5) and (1.9)

$$\begin{cases} \phi_{tt} - \frac{1}{\mu^2} \left(\phi_{xx} + \phi_{zz} \right) + \phi_z = 0 & ; \text{on } -1/\mu < z < 0 \\ \phi_{tt} + \phi_z = P_t(x, t) & ; \text{on } z = 0 \\ \phi_z = 0 & ; \text{on } z = -1/\mu \end{cases} \tag{2.10}$$

Assuming the potential ϕ can be expressed by the next equation according to (Kadri & Akylas, 2016)

$$\phi(x, z, t) = f(z) \mathrm{e}^{\frac{1}{2}\mu^2 z} \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \tag{2.11}$$

we consider the Laplace transform and its inverse in the next forms

$$\bar{\phi}(x,z,\omega) = \int_0^\infty \phi(x,z,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t; \qquad \phi(x,z,t) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \bar{\phi}(x,z,\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \qquad (2.12\ a,b)$$

$$\hat{\bar{\phi}}(z,k,\omega) = \int_{-\infty}^\infty \bar{\phi}(x,z,\omega)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x; \qquad \bar{\phi}(x,z,\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{\bar{\phi}}(z,k,\omega)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \qquad (2.13\ a,b)$$

$$\hat{\bar{\phi}}(z,k,\omega) = \int_{-\infty}^\infty \int_0^\infty \phi(x,z,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x \qquad (2.14)$$

then, the potential can be obtained as follow

$$\hat{\bar{\phi}}(z,k,\omega) = \int_{-\infty}^\infty \left(\int_0^\infty f(z)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\right)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x \qquad (2.15)$$

$$\hat{\bar{\phi}}(z,k,\omega) = f(z)\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x. \qquad (2.16)$$

Using the potential of (2.16) and replacing it on the system (2.10) doing the corresponding derivatives we get

$$\hat{\bar{\phi}}_{tt}(z,k,\omega) = -\omega^2 f(z)\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x\right) \qquad (2.17)$$

$$\hat{\bar{\phi}}_{xx}(z,k,\omega) = -k^2 f(z)\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x\right) \qquad (2.18)$$

$$\hat{\bar{\phi}}_{z}(z,k,\omega) = (f_z + \frac{1}{2}\mu^2 f)\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x\right) \qquad (2.19)$$

$$\hat{\bar{\phi}}_{zz}(z,k,\omega) = (f_{zz} + \mu^2 f_z + \frac{1}{4}\mu^4 f)\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x\right). \qquad (2.20)$$

Replacing the derivatives in the field equation and boundary conditions of the system (2.10), we have that

$$\int_{-\infty}^\infty \int_0^\infty \left(-\omega^2 f + \frac{k^2}{\mu^2}f - \frac{1}{\mu^2}\left[f_{zz} + \mu^2 f_z + \frac{\mu^4}{4}f + f_z + \frac{\mu^2}{2}f\right]\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x = 0, \quad (2.21)$$

now, multiplying (2.21) by μ^2 and reducing the corresponding terms, we obtain,

$$\int_{-\infty}^\infty \int_0^\infty \left(-\omega^2\mu^2 f + k^2 f - f_{zz} + \frac{\mu^4}{4}f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x = 0 \qquad (2.22)$$

To the above integral be equal to 0, the next term has to be null

$$\left(-\omega^2\mu^2 f + k^2 f - f_{zz} + \frac{\mu^4}{4}f\right) = 0 \qquad (2.23)$$

and rearranging the terms, we get the next expression for the field equation

$$f_{zz} - \left(k^2 - \omega^2\mu^2 + \frac{\mu^4}{4}f\right) = 0 \quad ; \text{on} \quad -1/\mu < z < 0 \tag{2.24}$$

It is important to note that equation (2.24) holds also for $\hat{\hat{\phi}}$ as the integrals cancel out. Applying the same previous process to the surface boundary condition

$$\phi_{tt} + \phi_z = P_t(x,t) \quad ; \text{on } z = 0 \tag{2.25}$$

and using the corresponding expressions (2.17) to (2.20), we find

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(-\omega^2 f + f_z + \frac{\mu^2}{2}f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{0}^{\infty}P(x,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x \tag{2.26}$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(-\omega^2 f + f_z + \frac{\mu^2}{2}f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x = \int_{-\infty}^{\infty}\int_{0}^{\infty}P(x,t)\frac{\partial}{\partial t}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x \tag{2.27}$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(-\omega^2 f + f_z + \frac{\mu^2}{2}f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x = -\mathrm{i}\omega\int_{-\infty}^{\infty}\int_{0}^{\infty}P(x,t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x \tag{2.28}$$

By using the expression (2.14) on the right side of (2.28) we get the next form

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(-\omega^2 f + f_z + \frac{\mu^2}{2}f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x = -\mathrm{i}\omega\hat{\hat{P}}(k,\omega) \tag{2.29}$$

what gives the surface boundary condition

$$f_z - \left(\omega^2 - \frac{\mu^2}{2}\right)f = -\mathrm{i}\omega\hat{\hat{P}}(k,\omega) \quad ; \text{on } z = 0 \tag{2.30}$$

To get the bottom boundary condition, we apply the corresponding expressions (2.17) - (2.20) to the expression

$$\phi_z = 0 \quad ; \text{on } z = -1/\mu \tag{2.31}$$

obtaining the next form

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(f_z + \frac{1}{2}\mu^2 f\right)\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-ikx}\mathrm{d}t\mathrm{d}x = 0 \tag{2.32}$$

which implies that the term inside the parenthesis has also to be null

$$f_z + \frac{1}{2}\mu^2 f = 0 \quad ; \text{on } z = -1/\mu \tag{2.33}$$

With the use of the equations (2.24), (2.30) and (2.33), we get the new system of the equations as follows

$$\begin{cases} f_{zz} - \left(k^2 - \omega^2\mu^2 + \frac{1}{4}\mu^4\right) f = 0 & ; \text{on } \ -1/\mu < z < 0 \\ f_z - \left(\omega^2 - \frac{1}{2}\mu^2\right) f = -\mathrm{i}\omega\hat{\hat{P}}\mathrm{e}^{-\frac{1}{2}\mu^2 z} & ; \text{on } z = 0 \\ f_z + \frac{1}{2}\mu^2 f = 0 & ; \text{on } z = -1/\mu \end{cases} \tag{2.34}$$

where $\hat{\hat{P}} = \hat{\hat{P}}(k, z, \omega)$.

2.1.1 System of Equations for gravity and acoustic-gravity waves

As the problem involves two kind of waves, there are two limit cases to study, what corresponds to the relation of dispersion for gravity and acoustic-gravity waves.

1.- Gravity wave

First, the case when $\mu \to 0$ implies that only the gravity mode is present, and system can be stated as follows

$$\begin{cases} f_{zz} - k^2 f = 0 & ; \text{on } \ -1/\mu < z < 0 \\ f_z - \omega^2 f = -\mathrm{i}\omega\hat{\hat{P}} & ; \text{on } z = 0 \\ f_z = 0 & ; \text{on } z = -1/\mu \end{cases} \tag{2.35}$$

To solve the above system, the general solution corresponds to

$$f = A\mathrm{e}^{|k|z} + B\mathrm{e}^{-|k|z} \tag{2.36}$$

$$f_z = |k|\, A\mathrm{e}^{|k|z} - |k|\, B\mathrm{e}^{-|k|z} \quad \to f_z = |k|\left(A\mathrm{e}^{|k|z} - B\mathrm{e}^{-|k|z}\right) \tag{2.37}$$

from the equations(2.36) and (2.37), we can note that the terms $B\mathrm{e}^{-|k|z}$ and $kB\mathrm{e}^{-|k|z} \to 0$ because they decays exponentially due the negative sign. In this way, the general solution and its derivative can reformulated in the next form

$$f = A\mathrm{e}^{|k|z} \tag{2.38}$$

$$f_z = |k|\left(A\mathrm{e}^{|k|z}\right) \tag{2.39}$$

using (2.38) and (2.39) in the second expression of the system (2.35) is

$$|k|\, A\mathrm{e}^{|k|z} - \omega^2 A\mathrm{e}^{|k|z} = -\mathrm{i}\omega\hat{\hat{P}}, \tag{2.40}$$

then, evaluating the above equation on $z = 0$ and finding A we get

$$A = \frac{-\mathrm{i}\omega\hat{\hat{P}}}{|k| - \omega^2}. \tag{2.41}$$

By replacing (2.41) in the general solution (2.38) we can find the value of f, where

$$f = \frac{-\mathrm{i}\omega\hat{\hat{P}}}{|k| - \omega^2}\mathrm{e}^{|k|z} \tag{2.42}$$

which satisfies the complete system (2.35) and corresponds to the solution for the gravity wave of the problem. We note that the dispersion relation for this problem is given by the next form

$$\omega^2 = |k| \, . \tag{2.43}$$

2.- Acoustic-gravity wave

By the other hand, using the field equation of the system (2.35), the vertical profile $f(z)$ becomes oscillatory in the low-wavenumber limit, $k^2 < \mu^2\omega^2$. To analyze this possibility, we write

$$k = \mu\kappa \rightarrow k^2 = \mu^2\kappa^2 \quad ; Z = \mu z, \tag{2.44}$$

this rescaling amounts to using h instead μh as the characteristic length scale. Assuming $\gamma^2 = \omega^2 - \kappa^2 > 0$ and using the conditions (2.44) on the system (2.35), we have that the field equation and the boundary conditions are as follow

$$\begin{cases} \mu^2 f_{ZZ} - \left(\mu^2\kappa^2 - \omega^2\mu^2 + \frac{1}{4}\mu^4\right) f = 0 & ; \text{on } \; -1 < Z < 0 \\ \mu f_Z - \left(\omega^2 - \frac{1}{2}\mu^2\right) f = -\mathrm{i}\omega\hat{\hat{P}} & ; \text{on } Z = 0 \\ \mu f_Z + \frac{1}{2}\mu^2 f = 0 & ; \text{on } Z = -1 \end{cases} \tag{2.45}$$

neglecting the higher order terms (μ^2), the system reduces to

$$\begin{cases} f_{ZZ} + \gamma^2 f = 0 & ; \text{on } \; -1 < Z < 0 \\ \mu f_Z - \omega^2 f = -\mathrm{i}\omega\hat{\hat{P}} & ; \text{on } Z = 0 \\ f_Z + \frac{1}{2}\mu f = 0 & ; \text{on } Z = -1 \end{cases} \tag{2.46}$$

The general solution for the above system has the next form

$$f = A\cos(\gamma(Z+1)) + B(\sin\gamma(Z+1)) \tag{2.47}$$

then

$$f_Z = -A\gamma\sin(\gamma(Z+1)) + B\gamma\cos(\gamma(Z+1)) \tag{2.48}$$

$$f_{ZZ} = -A\gamma^2\cos(\gamma(Z+1)) - B\gamma^2\sin(\gamma(Z+1)) \tag{2.49}$$

$$f_{ZZ} = -\gamma^2[A\cos(\gamma(Z+1)) + B\sin(\gamma(Z+1))] = -\gamma^2 f \tag{2.50}$$

with the above the field equation is satisfied by the general solution. Let's apply now the obtained derivatives to the boundary conditions. Taking the bottom boundary condition from the system (2.46) and replacing the corresponding values, the next form is obtained

$$f_Z + \frac{1}{2}\mu f = 0 \tag{2.51}$$

$$-A\gamma\sin(\gamma(Z+1)) + B\gamma\cos(\gamma(Z+1)) + \frac{1}{2}\mu\left[A\cos(\gamma(Z+1)) + B(\sin\gamma(Z+1))\right] = 0 \tag{2.52}$$

on $Z=-1$, the above equation becomes to

$$-A\gamma\sin(\gamma(0))+B\gamma\cos(\gamma(0))+\frac{1}{2}\mu\left[A\cos(\gamma(0))+B(\sin(\gamma(0))\right]=0 \tag{2.53}$$

$$B\gamma+\frac{1}{2}\mu A=0 \tag{2.54}$$

$$\Rightarrow A=-\frac{2B\gamma}{\mu} \tag{2.55}$$

Replacing A in the surface boundary condition to obtain B

$$\mu f_Z-\omega^2 f=-\mathrm{i}\omega\hat{\hat{P}} \tag{2.56}$$

$$\mu\left[-\left(-\frac{2B\gamma}{\mu}\right)\gamma\sin(\gamma(Z+1))+B\gamma\cos(\gamma(Z+1))\right]- \\ -\omega^2\left[\left(-\frac{2B\gamma}{\mu}\right)\cos(\gamma(Z+1))+B\sin(\gamma(Z+1))\right]=-\mathrm{i}\omega\hat{\hat{P}} \tag{2.57}$$

on $Z=0$ and simplifying some terms, we get

$$(2B\gamma^2)\sin(\gamma)+\mu B\gamma\cos(\gamma)+\left(\frac{2B\omega^2\gamma}{\mu}\right)\cos(\gamma)-\omega^2 B\sin(\gamma)=-\mathrm{i}\omega\hat{\hat{P}} \tag{2.58}$$

$$B\left[(2\gamma^2)\sin(\gamma)+\mu\gamma\cos(\gamma)+\left(\frac{2\omega^2\gamma}{\mu}\right)\cos(\gamma)-\omega^2\sin(\gamma)\right]=-\mathrm{i}\omega\hat{\hat{P}} \tag{2.59}$$

$$B\left[(2\mu\gamma^2)\sin(\gamma)+\mu^2\gamma\cos(\gamma)+(2\omega^2\gamma)\cos(\gamma)-\mu\omega^2\sin(\gamma)\right]=-\mathrm{i}\mu\omega\hat{\hat{P}} \tag{2.60}$$

$$B\left[\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)\right]=-\mathrm{i}\mu\omega\hat{\hat{P}} \tag{2.61}$$

$$B=\frac{-\mathrm{i}\mu\omega\hat{\hat{P}}}{\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)} \tag{2.62}$$

Now, replacing B on (2.55) to get the expression for the parameter A

$$A=\frac{2\mathrm{i}\omega\gamma\hat{\hat{P}}}{\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)} \tag{2.63}$$

Taking A and B to the general equation (2.47)

$$f=A\cos(\gamma(Z+1))+B(\sin\gamma(Z+1)) \tag{2.64}$$

$$f=\frac{2\mathrm{i}\omega\gamma\hat{\hat{P}}}{\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)}\cos(\gamma(Z+1)) \\ -\frac{i\mu\omega\hat{\hat{P}}}{\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)}\sin(\gamma(Z+1)) \tag{2.65}$$

$$f=\mathrm{i}\omega\hat{\hat{P}}\left[\frac{2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))}{\sin(\gamma)(2\mu\gamma^2-\mu\omega^2)+\cos(\gamma)(\mu^2\gamma+2\omega^2\gamma)}\right] \tag{2.66}$$

Let's to analyze the denominator of the equation (2.66) in order to find the limiting cases for the acoustic-gravity wave.

$$\sin(\gamma)\frac{\mu}{\gamma}\left\{\frac{2\gamma^2-\omega^2}{\mu^2+2\omega^2}\right\}+\cos(\gamma)=O(\mu^2) \tag{2.67}$$

Since $\gamma^2 = \omega^2 - \kappa^2$, the dispersion relation can be rewritten as

$$\mu\left\{\frac{\gamma^2-\kappa^2}{2\gamma\omega^2}\right\}\sin(\gamma)+\cos(\gamma)=O(\mu^2) \tag{2.68}$$

When $\cos(\gamma) = 0 \Rightarrow \gamma = (n+1/2)\,\pi$, the term $(\gamma^2-\kappa^2)\sin(\gamma)$ survives because $\sin(\gamma) = 1$. In this case, the equation (2.68) can be rewritten as

$$\mu\left\{\frac{\gamma^2-\kappa^2}{\omega^2}\right\}\sin(\gamma)+2\gamma\cos(\gamma)=O(\mu^2) \tag{2.69}$$

as $\gamma = \left(n+\frac{1}{2}\right)\pi$ and $\omega^2 = \gamma^2+\kappa^2$, the expression (2.69) can be reformulated as

$$\mu\left\{\frac{\left(n+\frac{1}{2}\right)^2\pi^2-\kappa^2}{\left(n+\frac{1}{2}\right)^2\pi^2+\kappa^2}\right\}\sin(\gamma)+(2n+1)\pi\cos(\gamma)=O(\mu^2) \tag{2.70}$$

the expression (2.70) becomes

$$\mu\left\{\frac{\omega_n^2-\kappa^2}{\omega_n^2+\kappa^2}\right\}\sin(\gamma)+\cos(\gamma)=O(\mu^2) \tag{2.71}$$

thus, the dispersion relation has the following form

$$\omega^2=\omega_n^2+\kappa^2+\mu\left\{\frac{\omega_n^2-\kappa^2}{\omega_n^2+\kappa^2}\right\}+O(\mu^2)\quad ;\text{ with } n=0,1,2,..., \tag{2.72}$$

where $\omega_n = (n+1/2)\,\pi$.

Now, we can recover the potential ϕ by using the double inverse transform (2.12 b) and (2.13 b) on the expression (2.16) obtaining

$$\hat{\hat{\phi}}(z,k,\omega)=f(z)\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_{-\infty}^{\infty}\int_{0}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}t\mathrm{d}x \tag{2.73}$$

$$\phi(x,z,t)=\frac{1}{2\pi i}\int_{\Gamma}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\hat{\phi}}(z,k,\omega)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\right\}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{2.74}$$

2.1.2 Green's functions for the gravity and acoustic-gravity wave potentials

Considering the function f as in equations (2.42) and (2.66), we can solve them separately, first for the gravity wave and later for the acoustic-gravity wave as follows.

1 - Gravity wave

Let us consider the equation (2.42)

$$f = \frac{-\mathrm{i}\omega\hat{\hat{P}}(k,z,\omega)}{|k|-\omega^2}\mathrm{e}^{|k|z} \tag{2.75}$$

$$\phi_g(x,z,t) = \mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^t\int_{-\infty}^{\infty}\hat{\hat{P}}(\xi,\tau)G_g(x-\xi,t-\tau)\mathrm{d}\xi\mathrm{d}\tau \tag{2.76}$$

Where ϕ_g is the gravity wave potential and G is the Green's function associated. Let's take the Green's function of the gravity wave in the following form:

$$G_g(x,z,t) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\left\{\int_\Gamma\left[\frac{\omega\mathrm{e}^{|k|z}}{|k|-\omega^2}\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\right\}\mathrm{d}k \tag{2.77}$$

$$G_g(x,z,t) = -\frac{1}{2\pi^2}\int_0^{\infty}\mathrm{e}^{\mathrm{i}kx}\left\{\int_\Gamma\left[\frac{\omega\mathrm{e}^{|k|z}}{|k|-\omega^2}\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\right\}\mathrm{d}k. \tag{2.78}$$

We are going to analyze the two poles for the gravity wave when $\omega = \pm\sqrt{|k|}$ by using the Cauchy's Residue Theorem, where the integration domain $\Gamma \in [0,\infty)$. We need to find the integrals over the poles, points that are critical for the solution

$$\int_\Gamma = I_1 + I_2 \tag{2.79}$$

. Let's going to solve both critical cases for (2.78)

(a) Case: $\omega = -\sqrt{|k|}$

$$I_1 = 2\pi\mathrm{i}Res\left(f(\omega), -\sqrt{|k|}\right) \tag{2.80}$$

$$Res\left(f(\omega), -\sqrt{|k|}\right) = \lim_{\omega\to-\sqrt{|k|}}\left[\frac{\omega\mathrm{e}^{|k|z}}{(\sqrt{|k|}-\omega)(\sqrt{|k|}+\omega)}\mathrm{e}^{\mathrm{i}\omega t}\right](\sqrt{|k|}+\omega) \tag{2.81}$$

$$= -\frac{\sqrt{|k|}\mathrm{e}^{|k|z}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}}{2\sqrt{|k|}} \tag{2.82}$$

$$I_1 = -\pi\mathrm{i}\mathrm{e}^{|k|z}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \tag{2.83}$$

(b) Case: $\omega = +\sqrt{|k|}$

$$I_2 = 2\pi \mathrm{i} Res\left(f(\omega), \sqrt{|k|}\right) \tag{2.84}$$

$$Res\left(f(\omega), +\sqrt{|k|}\right) = \lim_{\omega \to +\sqrt{|k|}} \left[\frac{\omega \mathrm{e}^{|k|z}}{(\sqrt{|k|}+\omega)(\sqrt{|k|}-\omega)}\mathrm{e}^{\mathrm{i}\omega t}\right](\sqrt{|k|}-\omega) \tag{2.85}$$

$$= \frac{\sqrt{|k|}\mathrm{e}^{|k|z}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}}{2\sqrt{|k|}} \tag{2.86}$$

$$I_2 = \pi \mathrm{i} \mathrm{e}^{|k|z}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \tag{2.87}$$

Now, summing I_1 and I_2 and factorizing terms, we obtain the solution of the integral (2.79)

$$I_1 + I_2 = \pi \mathrm{i} \mathrm{e}^{|k|z}\left\{-\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} + \mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\right\} \tag{2.88}$$

$$I_1 + I_2 = -2\pi \mathrm{e}^{|k|z}\sin(\sqrt{|k|}t) \tag{2.89}$$

Now, replacing the equation (2.89) on (2.78) and simplifying terms we found the Green's Function for the gravity equation

$$G_g(x,z,t) = \frac{1}{\pi}\int_0^{\infty} \sin(\sqrt{|k|}t)\mathrm{e}^{|k|z}\mathrm{e}^{\mathrm{i}kx}dk \tag{2.90}$$

2 - Acoustic-gravity wave

In an analog way, let's take the equation (2.66)

$$f = \mathrm{i}\omega \hat{\hat{P}}(k,z,\omega)\left[\frac{2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))}{\sin(\gamma)(2\mu\gamma^2 - \mu\omega^2) + \cos(\gamma)(\mu^2\gamma + 2\omega^2\gamma)}\right] \tag{2.91}$$

$$\phi_s(x,z,t) = \mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^t \int_{-\infty}^{\infty} \hat{\hat{P}}(\xi,\tau)G_s(x-\xi,t-\tau)\mathrm{d}\xi\mathrm{d}\tau \tag{2.92}$$

Where ϕ_s is the potential associated to the acoustic-gravity wave and G is the Green's function considered in the next form

$$G_s(x,z,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\left\{\int_\Gamma \omega\left[\frac{2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))}{\sin(\gamma)(2\mu\gamma^2 - \mu\omega^2) + \cos(\gamma)(\mu^2\gamma + 2\omega^2\gamma)}\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\right\}\mathrm{d}k$$

$$G_s(x,z,t) = \frac{1}{2\pi^2}\int_0^{\infty} \mathrm{e}^{\mathrm{i}kx}\left\{\int_\Gamma \omega\left[\frac{2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))}{\sin(\gamma)(2\mu\gamma^2 - \mu\omega^2) + \cos(\gamma)(\mu^2\gamma + 2\omega^2\gamma)}\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\right\}\mathrm{d}k \tag{2.93}$$

Using now the Cauchy's Residue Theorem on the acoustic-gravity wave to get

$$\int_{\Gamma} = I_{n_+} + I_{n_-} \tag{2.94}$$

We are going to analyze the poles for the acoustic-gravity wave from (2.93) when $\omega = \pm\kappa_n$, where $\Gamma \in [0, \infty)$

(a) Case: $\omega = +\kappa_n$

$$I_{n_+} = 2\pi\mathrm{i} Res\left(f(\omega), +\kappa_n\right) \tag{2.95}$$

we have that

$$Res\left(f(\omega), +\kappa_n\right) = \lim_{\omega \to +\kappa_n} \omega\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}\omega t} \tag{2.96}$$

$$Res\left(f(\omega), +\kappa_n\right) = (+\kappa_n)\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}(+\kappa_n)t} \tag{2.97}$$

$$Res\left(f(\omega), +\kappa_n\right) = \kappa_n\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}\kappa_n t} \tag{2.98}$$

replacing (2.98) on (2.95), we get

$$I_{n_+} = 2\pi\mathrm{i}\kappa_n\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}\kappa_n t} \tag{2.99}$$

(b) Case: $\omega = -\kappa_n$

$$I_{n_-} = 2\pi\mathrm{i} Res\left(f(\omega), -\kappa_n\right) \tag{2.100}$$

we have that

$$Res\left(f(\omega), -\kappa_n\right) = \lim_{\omega \to -\kappa_n} \omega\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}\omega t} \tag{2.101}$$

$$Res\left(f(\omega), -\kappa_n\right) = (-\kappa_n)\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{\mathrm{i}(-\kappa_n)t} \tag{2.102}$$

$$Res\left(f(\omega), -\kappa_n\right) = -\kappa_n\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{-\mathrm{i}\kappa_n t} \tag{2.103}$$

replacing (2.103) on (2.100) yields

$$I_{n_-} = -2\pi\mathrm{i}\kappa_n\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\mathrm{e}^{-\mathrm{i}\kappa_n t} \tag{2.104}$$

$$\int_{\Gamma} = I_{n_+} + I_{n_-} \tag{2.105}$$

$$2\pi\mathrm{i}\kappa_n[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))]\mathrm{e}^{\mathrm{i}\kappa_n t} + \left(-2\pi\mathrm{i}\kappa_n[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))]\mathrm{e}^{-\mathrm{i}\kappa_n t}\right) \tag{2.106}$$

$$2\pi \mathrm{i}\kappa_n \left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right](\mathrm{e}^{\mathrm{i}\kappa_n t} - \mathrm{e}^{-\mathrm{i}\kappa_n t}) \tag{2.107}$$

$$2\pi \mathrm{i}\kappa_n \left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right](2\mathrm{i}\sin(\kappa_n t)) \tag{2.108}$$

$$\int_\Gamma = -4\pi\kappa_n \left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right](\sin(\kappa_n t)). \tag{2.109}$$

Replacing the equation (2.109) on (2.93) we have the expression of the Green's Function for the acoustic-gravity wave equation

$$G_s(x,z,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty} \kappa_n \left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right](\sin(\kappa_n t))\mathrm{e}^{\mathrm{i}kx}dk. \tag{2.110}$$

Chapter 3

Estimation of the potential

To estimate the potential $\phi(x,z,t)$, let us analyze the pressures on the ocean assuming that they can be exponentially gaussian distributed in order to simplify the physics of the phenomena, by using a double gaussian in time and space with the next form

$$P(x,t) = \frac{2I_0}{\pi\epsilon\sigma} \mathrm{e}^{-\left(\frac{x}{\sigma}\right)^2} \mathrm{e}^{-\left(\frac{t}{\epsilon}\right)^2} \quad ; x \in \mathbb{R}, t \geq 0, \tag{3.1}$$

where I_0 is the total impulse per unit width (considering y as the transversal coordinate equal to 1). Replacing the expression (3.1) for the pressure and the found Green's equations (2.90) on (2.76) and (2.110) on (2.92) for the gravity wave and for the acoustic-gravity wave respectively, we can find the related potentials ϕ_g and ϕ_s.

1 - Calculation of gravity wave potential ϕ_g

Once the mentioned replacements have been made, the following expression is obtained for the gravity wave potential

$$\phi_g(x,z,t) = \mathrm{e}^{\frac{1}{2}\mu^2 z} \int_0^t \int_{-\infty}^{\infty} \frac{2I_0}{\pi\epsilon\sigma} \mathrm{e}^{-\left(\frac{x}{\sigma}\right)^2} \mathrm{e}^{-\left(\frac{t}{\epsilon}\right)^2} \left(\frac{1}{\pi} \int_0^{\infty} \sin(\sqrt{|k|}(t-\tau)) \mathrm{e}^{|k|z} \mathrm{e}^{\mathrm{i}k(x-\xi)} \mathrm{d}k \right) \mathrm{d}\xi \mathrm{d}\tau \tag{3.2}$$

rearranging terms

$$\phi_g(x,z,t) = \frac{2I_0}{\epsilon\sigma\pi^2} \mathrm{e}^{\frac{1}{2}\mu^2 z} \int_0^t \int_{-\infty}^{\infty} \mathrm{e}^{-\left(\frac{x}{\sigma}\right)^2} \mathrm{e}^{-\left(\frac{t}{\epsilon}\right)^2} \left(\int_0^{\infty} \sin(\sqrt{|k|}(t-\tau)) \mathrm{e}^{|k|z} \mathrm{e}^{\mathrm{i}k(x-\xi)} \mathrm{d}k \right) \mathrm{d}\xi \mathrm{d}\tau \tag{3.3}$$

In order to solve the above equation, a grouping of the corresponding terms is made, resulting in the next expression, which consists of two integrals I_{g_1} and I_{g_2}

$$\phi_g(x,z,t) = \frac{2I_0}{\epsilon\sigma\pi^2} \mathrm{e}^{\frac{1}{2}\mu^2 z} \int_0^{\infty} \mathrm{e}^{|k|z} \left(\underbrace{\int_0^t \sin(\sqrt{|k|}(t-\tau)) \mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2} \mathrm{d}\tau}_{I_{g_1}} \underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2} \mathrm{e}^{\mathrm{i}k(x-\xi)} \mathrm{d}\xi}_{I_{g_2}} \right) \mathrm{d}k \tag{3.4}$$

rewriting the equation (3.4) in exponential form, yields

$$\phi_g(x,z,t) = \frac{2I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^\infty \mathrm{e}^{|k|z}\left(\underbrace{\frac{1}{2\mathrm{i}}\int_0^t\left\{\mathrm{e}^{\mathrm{i}\sqrt{|k|}(t-\tau)}-\mathrm{e}^{-\mathrm{i}\sqrt{|k|}(t-\tau)}\right\}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau}_{I_{g_1}}\underbrace{\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{d}\xi}_{I_{g_2}}\right)\mathrm{d}k \tag{3.5}$$

Now, to try this formulation, let us solve first the integral I_{g_1}

$$I_{g_1} = \frac{1}{2\mathrm{i}}\left\{\int_0^t \mathrm{e}^{\mathrm{i}\sqrt{|k|}(t-\tau)}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau - \int_0^t \mathrm{e}^{-\mathrm{i}\sqrt{|k|}(t-\tau)}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau\right\}, \tag{3.6}$$

going through the algebra, the solving is made as follows

$$I_{g_1} = \frac{1}{2\mathrm{i}}\left\{\int_0^t \mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau - \int_0^t \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\mathrm{e}^{\mathrm{i}\sqrt{|k|}\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau\right\} \tag{3.7}$$

$$I_{g_1} = \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{-\mathrm{i}\sqrt{|k|}\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{\mathrm{i}\sqrt{|k|}\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau\right\} \tag{3.8}$$

$$I_{g_1} = \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{-\left[\left(\frac{\tau}{\epsilon}\right)^2+\mathrm{i}\sqrt{|k|}\tau\right]}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{-\left[\left(\frac{\tau}{\epsilon}\right)^2-\mathrm{i}\sqrt{|k|}\tau\right]}\mathrm{d}\tau\right\} \tag{3.9}$$

$$I_{g_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left\{\mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)^2}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)^2}\mathrm{d}\tau\right\}. \tag{3.10}$$

Setting the variables $u_{g_1} = \tau/\epsilon + \mathrm{i}\sqrt{|k|}\epsilon/2$ and $u_{g_2} = \tau/\epsilon - \mathrm{i}\sqrt{|k|}\epsilon/2$, we have that equation (3.10) becomes

$$I_{g_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left\{\mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\int_{\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}}^{\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}} \mathrm{e}^{-u_{g1}^2}\mathrm{d}u_{g_1} - \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\int_{\frac{-\mathrm{i}\sqrt{|k|}\epsilon}{2}}^{\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}} \mathrm{e}^{-u_{g2}^2}\mathrm{d}u_{g_2}\right\}, \tag{3.11}$$

the above expression (3.11) has two integrals that have the form of the error function. According this, the next equation is obtained

$$\begin{aligned}I_{g_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}&\left\{\epsilon\frac{\sqrt{\pi}}{2}\mathrm{e}^{\mathrm{i}\sqrt{|k|}t}\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\right]\right.\\ &\left.-\epsilon\frac{\sqrt{\pi}}{2}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\left[\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-\mathrm{erf}\left(-\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\right]\right\}\end{aligned} \tag{3.12}$$

$$I_{g_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2} \left\{ \epsilon\frac{\sqrt{\pi}}{2}\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} \left[\mathrm{erf}\left(\frac{t}{\epsilon} + \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right) - \mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right) \right] \right.$$
$$\left. - \epsilon\frac{\sqrt{\pi}}{2}\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \left[\mathrm{erf}\left(\frac{t}{\epsilon} - \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right) + \mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right) \right] \right\} \tag{3.13}$$

$$I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2} \left\{ \mathrm{erf}\left(\frac{t}{\epsilon} + \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} - \mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} \right.$$
$$\left. - \mathrm{erf}\left(\frac{t}{\epsilon} - \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} - \mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \right\} \tag{3.14}$$

$$I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2} \left\{ \mathrm{erf}\left(\frac{t}{\epsilon} + \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} - \mathrm{erf}\left(\frac{t}{\epsilon} - \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \right.$$
$$\left. - \mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\left(\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} + \mathrm{e}^{-\mathrm{i}\sqrt{|k|}t}\right) \right\} \tag{3.15}$$

converting the exponential terms to functions of sin and cos, (3.15) can be rewritten as

$$I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2} \left\{ \mathrm{erf}\left(\frac{t}{\epsilon} + \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\sqrt{|k|}t} - \mathrm{erf}\left(\frac{t}{\epsilon} - \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\sqrt{|k|}t} \right.$$
$$\left. - 2\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\cos(\sqrt{|k|}t) \right\} \tag{3.16}$$

$$I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2} \left\{ \mathrm{erf}\left(\frac{t}{\epsilon} + \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\left[\cos(\sqrt{|k|}t) + \mathrm{i}\sin(\sqrt{|k|}t)\right] \right. \tag{3.17}$$
$$\left. - \mathrm{erf}\left(\frac{t}{\epsilon} - \frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\left[\cos(\sqrt{|k|}t) - \mathrm{i}\sin(\sqrt{|k|}t)\right] - 2\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\cos(\sqrt{|k|}t) \right\}$$

Equation (3.17) can be rewritten as

$$
\begin{aligned}
I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\Bigg\{&\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\cos(\sqrt{|k|}t)+\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\sin(\sqrt{|k|}t)\\
&-\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\cos(\sqrt{|k|}t)+\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\sin(\sqrt{|k|}t)\\
&-2\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\cos(\sqrt{|k|}t)\Bigg\}
\end{aligned}
\tag{3.18}
$$

grouping terms we get

$$
\begin{aligned}
I_{g_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\Bigg\{&\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-2\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\right]\cos(\sqrt{|k|}t)\\
&+\left[\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\right]\sin(\sqrt{|k|}t)\Bigg\}.
\end{aligned}
\tag{3.19}
$$

Multiplying (3.19) by i/i = 1 and considering only the terms that have positive imaginary part or the expression (3.19) and naming the rest as complex conjugated (*c.c.*), we obtain the next form for the integral I_{g_1}

$$
\begin{aligned}
I_{g_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\Bigg\{&\Bigg[\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\\
&-2\mathrm{i}\mathrm{erf}\left(\frac{\mathrm{i}\epsilon\sqrt{|k|}}{2}\right)\Bigg]\cos(\sqrt{|k|}t)\\
&+\left[\mathrm{i}^2\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+\mathrm{i}^2\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\right]\sin(\sqrt{|k|}t)\Bigg\}
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
I_{g_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\Bigg\{&\left[\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)-\mathrm{i}\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\epsilon\sqrt{|k|}}{2}\right)\right]\cos(\sqrt{|k|}t)\\
&-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)\right]\sin(\sqrt{|k|}t)\Bigg\}
\end{aligned}
\tag{3.21}
$$

thus,

$$I_{g_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\epsilon\sqrt{|k|}}{2}\right)+c.c.\right]\cos(\sqrt{|k|}t)\right.$$
$$\left.-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+c.c.\right]\sin(\sqrt{|k|}t)\right\} \quad (3.22)$$

Now, we are going to solve the integral I_{g_2}.

$$I_{g_2} = \int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{d}\xi \quad (3.23)$$

$$I_{g_2} = \int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}k\xi}\mathrm{d}\xi \quad (3.24)$$

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{-\mathrm{i}k\xi}\mathrm{d}\xi \quad (3.25)$$

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{-\left[\left(\frac{\xi}{\sigma}\right)^2+\mathrm{i}k\xi\right]}\mathrm{d}\xi \quad (3.26)$$

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}+\mathrm{i}\frac{k\sigma}{2}\right)^2}\mathrm{d}\xi \quad (3.27)$$

Setting the variable $v_{g_1} = \xi/\sigma + \mathrm{i}k\sigma/2$, we get the following expression

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-v_{g_1}^2}\mathrm{d}v_{g_1} \quad (3.28)$$

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\sigma\frac{\sqrt{\pi}}{2}\left\{\mathrm{erf}(\infty)-\mathrm{erf}(-\infty)\right\} \quad (3.29)$$

$$I_{g_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\sigma\frac{\sqrt{\pi}}{2}\left\{2\right\} \quad (3.30)$$

$$I_{g_2} = \sigma\sqrt{\pi}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2} \quad (3.31)$$

Rewriting the equation (3.5) with the found expressions for I_{g_1} and I_{g_2}

$$\phi_g(x,z,t) \tag{3.32}$$
$$= \frac{2I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^{\infty}\mathrm{e}^{|k|z}\left(-\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\epsilon\sqrt{|k|}}{2}\right)+c.c.\right]\cos(\sqrt{|k|}t)\right.\right.$$
$$\left.\left.-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+c.c.\right]\sin(\sqrt{|k|}t)\right\}\sigma\sqrt{\pi}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)\mathrm{d}k$$

where

$$a_g = \mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\epsilon\sqrt{|k|}}{2}\right)+c.c.\right]$$

and

$$b_g = \mathrm{e}^{-\left(\frac{\epsilon\sqrt{|k|}}{2}\right)^2}\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\sqrt{|k|}\epsilon}{2}\right)+c.c.\right]$$

Simplifying terms of the above equation, we get the solution for the gravity wave potential,which has the next form

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^{\infty}\mathrm{e}^{|k|z}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\left\{a_g\cos(\sqrt{|k|}t)-b_g\sin(\sqrt{|k|}t)\right\}\mathrm{d}k. \tag{3.33}$$

2 - Calculation of acoustic-gravity wave potential ϕ_s

Now, we are going to calculate the potential for the acoustic-gravity wave using the replacements mentioned in the beginning of the section. The calculations are as follows

$$\phi_s(x,z,t) = \mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^t\int_{-\infty}^{\infty}\frac{2I_0}{\pi\epsilon\sigma}\mathrm{e}^{-\left(\frac{x}{\sigma}\right)^2}\mathrm{e}^{-\left(\frac{t}{\epsilon}\right)^2}$$
$$\left(-\frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\sin(\kappa_n(t-\tau)\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{d}k\right)d\xi d\tau \tag{3.34}$$

rearranging terms

$$\phi_s(x,z,t) = -\frac{4I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}$$
$$\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right](\sin(\kappa_n(t-\tau))\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{d}k\right)d\xi d\tau \tag{3.35}$$

In order to solve the above equation, a grouping of the corresponding terms is made, resulting in the next expression, which is expressed in exponential form and consists of two integrals

I_{s_1} and I_{s_2} as follows

$$\phi_s(x,z,t) = -\frac{4I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))\right.\right.$$
$$\left.\left.-\mu\sin(\gamma(Z+1))\right]\left\{\underbrace{\int_0^t(\sin(\kappa_n(t-\tau))\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau}_{I_{s_1}}\underbrace{\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{d}\xi}_{I_{s_2}}\right\}\right)\mathrm{d}k \tag{3.36}$$

rearranging the above equation, it yields

$$\phi_s(x,z,t)$$
$$= -\frac{4I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))\right.\right.$$
$$\left.\left.-\mu\sin(\gamma(Z+1))\right]\left\{\underbrace{\frac{1}{2\mathrm{i}}\int_0^t\left\{\mathrm{e}^{\mathrm{i}\kappa_n(t-\tau)}-\mathrm{e}^{-\mathrm{i}\kappa_n(t-\tau)}\right\}\mathrm{e}^{-(\frac{\tau}{\epsilon})^2}\mathrm{d}\tau}_{I_{s_1}}\underbrace{\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{e}^{-(\frac{\xi}{\sigma})^2}\mathrm{d}\xi}_{I_{s_2}}\right\}\right)\mathrm{d}k \tag{3.37}$$

To try the (3.35) formulation, let's solve first the integral I_{s_1}.

$$I_{s_1} = \frac{1}{2\mathrm{i}}\left\{\int_0^t\mathrm{e}^{\mathrm{i}\kappa_n(t-\tau)}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau - \int_0^t\mathrm{e}^{-\mathrm{i}\kappa_n(t-\tau)}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau\right\} \tag{3.38}$$

going though the algebra, the following expressions are obtained

$$I_{s_1} = \frac{1}{2\mathrm{i}}\left\{\int_0^t\mathrm{e}^{\mathrm{i}\kappa_n t}\mathrm{e}^{-\mathrm{i}\kappa_n\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}d\tau - \int_0^t\mathrm{e}^{-\mathrm{i}\kappa_n t}\mathrm{e}^{\mathrm{i}\kappa_n\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}d\tau\right\} \tag{3.39}$$

$$I_{s_1} = \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}\kappa_n t}\int_0^t\mathrm{e}^{-\mathrm{i}\kappa_n\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\kappa_n t}\int_0^t\mathrm{e}^{\mathrm{i}\kappa_n\tau}\mathrm{e}^{-\left(\frac{\tau}{\epsilon}\right)^2}\mathrm{d}\tau\right\} \tag{3.40}$$

$$I_{s_1} = \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}\kappa_n t}\int_0^t\mathrm{e}^{-\left[\left(\frac{\tau}{\epsilon}\right)^2+\mathrm{i}\kappa_n\tau\right]}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\kappa_n t}\int_0^t\mathrm{e}^{-\left[\left(\frac{\tau}{\epsilon}\right)^2-\mathrm{i}\kappa_n\tau\right]}\mathrm{d}\tau\right\} \tag{3.41}$$

$$I_{s_1} = \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}\kappa_n t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)^2-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\kappa_n t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)^2-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\mathrm{d}\tau\right\} \tag{3.42}$$

$$I_{s_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\mathrm{e}^{\mathrm{i}\kappa_n t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)^2}\mathrm{d}\tau - \mathrm{e}^{-\mathrm{i}\kappa_n t}\int_0^t \mathrm{e}^{-\left(\frac{\tau}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)^2}\mathrm{d}\tau\right\} \tag{3.43}$$

Setting now the variable $u_{s_1} = \tau/\epsilon + \mathrm{i}\kappa_n\epsilon/2$ and $u_{s_2} = \tau/\epsilon - \mathrm{i}\kappa_n\epsilon/2$, we have that

$$I_{s_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\mathrm{e}^{\mathrm{i}\kappa_n t}\int_{\frac{\mathrm{i}\kappa_n\epsilon}{2}}^{\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}} \mathrm{e}^{-u_{s_1}^2}\mathrm{d}u_{s_1} - \mathrm{e}^{-\mathrm{i}\kappa_n t}\int_{-\frac{\mathrm{i}\kappa_n\epsilon}{2}}^{\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}} \mathrm{e}^{-u_{s_2}^2}\mathrm{d}u_{s_2}\right\} \tag{3.44}$$

the above expression (3.44) has two integrals that have the form of the error function. According this, the next equation is obtained

$$\begin{aligned} I_{s_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\Bigg\{&\mathrm{e}^{\mathrm{i}\kappa_n t}\frac{\sqrt{\pi}}{2}\epsilon\left[\operatorname{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right] \\ &- \mathrm{e}^{-\mathrm{i}\kappa_n t}\frac{\sqrt{\pi}}{2}\epsilon\left[\operatorname{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\operatorname{erf}\left(-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\Bigg\} \end{aligned} \tag{3.45}$$

re-accommodating terms, the step by step calculations are performed as follow

$$\begin{aligned} I_{s_1} = \frac{1}{2\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\Bigg\{&\mathrm{e}^{\mathrm{i}\kappa_n t}\frac{\sqrt{\pi}}{2}\epsilon\left[\operatorname{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right] \\ &- \mathrm{e}^{-\mathrm{i}\kappa_n t}\frac{\sqrt{\pi}}{2}\epsilon\left[\operatorname{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\Bigg\} \end{aligned} \tag{3.46}$$

$$\begin{aligned} I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\Bigg\{&\mathrm{e}^{\mathrm{i}\kappa_n t}\left[\operatorname{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right] \\ &- \mathrm{e}^{-\mathrm{i}\kappa_n t}\left[\operatorname{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\Bigg\} \end{aligned} \tag{3.47}$$

$$\begin{aligned} I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\Bigg\{&\operatorname{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\kappa_n t}-\operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\kappa_n t}-\operatorname{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\kappa_n t} \\ &- \operatorname{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\kappa_n t}\Bigg\} \end{aligned} \tag{3.48}$$

grouping terms and expressing the formulation in terms of sin and cos

$$I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\kappa_n t}-\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\kappa_n t}-\mathrm{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\left(\mathrm{e}^{\mathrm{i}\kappa_n t}+\mathrm{e}^{-\mathrm{i}\kappa_n t}\right)\right\} \tag{3.49}$$

$$I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{\mathrm{i}\kappa_n t}-\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\mathrm{e}^{-\mathrm{i}\kappa_n t}-2\mathrm{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\cos(\kappa_n t)\right\} \tag{3.50}$$

$$\begin{aligned}I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}&\left\{\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\left[\cos(\kappa_n t)+\mathrm{i}\sin(\kappa_n t)\right]\right.\\ &\left.-\,\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\left[\cos(\kappa_n t)-\mathrm{i}\sin(\kappa_n t)\right]-2\mathrm{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\cos(\kappa_n t)\right\}\end{aligned} \tag{3.51}$$

the equation (3.51) can be rewritten as

$$\begin{aligned}I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}&\left\{\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\cos(\kappa_n t)+\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\sin(\kappa_n t)\right.\\ &\left.-\,\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\cos(\kappa_n t)+\mathrm{ierf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\sin(\kappa_n t)-2\mathrm{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\cosh(\kappa_n t)\right\}\end{aligned} \tag{3.52}$$

grouping terms, we get

$$\begin{aligned}I_{s_1} = \epsilon\frac{\sqrt{\pi}}{4\mathrm{i}}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}&\left\{\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-2\mathrm{erf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\cos(\kappa_n t)\right.\\ &\left.+\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+\mathrm{ierf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\sin(\kappa_n t)\right\}\end{aligned} \tag{3.53}$$

Multiplying (3.53) by $\mathrm{i}/\mathrm{i} = 1$ and considering only the terms that have positive imaginary part or the expression (3.53) and naming the rest as complex conjugated (cc), we obtain the next form for the integral I_{s_1}

$$\begin{aligned}I_{s_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}&\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\mathrm{ierf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-2\mathrm{ierf}\left(\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\cos(\kappa_n t)\right.\\ &\left.+\left[\mathrm{i}^2\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+\mathrm{i}^2\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\sin(\kappa_n t)\right\}\end{aligned} \tag{3.54}$$

$$\begin{aligned}I_{s_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}&\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)-\mathrm{ierf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\kappa_n\epsilon}{2}\right)\right]\cos(\kappa_n t)\right.\\ &\left.-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+\mathrm{erf}\left(\frac{t}{\epsilon}-\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)\right]\sin(\kappa_n t)\right\}\end{aligned} \tag{3.55}$$

Thus,

$$I_{s_1} = -\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\kappa_n\epsilon}{2}\right)+c.c\right]\cos(\kappa_n t)\right.$$
$$\left.-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+c.c\right]\sin(\kappa_n t)\right\} \tag{3.56}$$

Now, we are going to solve the integral I_{s_2}.

$$I_{s_2} = \int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{\mathrm{i}k(x-\xi)}\mathrm{d}\xi \tag{3.57}$$

$$I_{s_2} = \int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}k\xi}\mathrm{d}\xi \tag{3.58}$$

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}\right)^2}\mathrm{e}^{-\mathrm{i}k\xi}\mathrm{d}\xi \tag{3.59}$$

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{-\left[\left(\frac{\xi}{\sigma}\right)^2+\mathrm{i}k\xi\right]}\mathrm{d}\xi \tag{3.60}$$

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{\xi}{\sigma}+\frac{\mathrm{i}k\sigma}{2}\right)^2}\mathrm{d}\xi \tag{3.61}$$

Setting the variable $v_{s_1} = \xi/\sigma + \mathrm{i}k\sigma/2$, we get the following expression

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-v_{s_1}^2}\mathrm{d}v_{s_1} \tag{3.62}$$

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\sigma\frac{\sqrt{\pi}}{2}\left\{\mathrm{erf}(\infty)-\mathrm{erf}(-\infty)\right\} \tag{3.63}$$

$$I_{s_2} = \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\sigma\frac{\sqrt{\pi}}{2}\left\{2\right\} \tag{3.64}$$

$$I_{s_2} = \sigma\sqrt{\pi}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2} \tag{3.65}$$

Rewriting the equation (3.37) with the found expressions for I_{s_1} and I_{s_2}

$$\begin{aligned}
&\phi_s(x,z,t) \\
&= -\frac{4I_0}{\epsilon\sigma\pi^2}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))\right.\right. \\
&\qquad \left.-\mu\sin(\gamma(Z+1))\right]\left[-\epsilon\frac{\sqrt{\pi}}{4}\mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left\{\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\kappa_n\epsilon}{2}\right)+c.c.\right]\cos(\kappa_n t)\right.\right. \\
&\left.\left.\left.-\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+c.c.\right]\sin(\kappa_n t)\right\}\right]\sigma\sqrt{\pi}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)\mathrm{d}k
\end{aligned} \tag{3.66}$$

where

$$a_s = \mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left[\mathrm{ierf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+2\mathrm{erfi}\left(\frac{\kappa_n\epsilon}{2}\right)+c.c.\right]$$

and

$$b_s = \mathrm{e}^{-\left(\frac{\kappa_n\epsilon}{2}\right)^2}\left[\mathrm{erf}\left(\frac{t}{\epsilon}+\frac{\mathrm{i}\kappa_n\epsilon}{2}\right)+cc\right]$$

Simplifying terms of the above equation (3.66), we get the solution for the acoustic-gravity wave potential, which has the next form

$$\begin{aligned}
\phi_s(x,z,t) = \frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\left\{a_s\cos(\kappa_n t)\right.\right. \\
\left.\left.-b_s\sin(\kappa_n t)\right\}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)\mathrm{d}k.
\end{aligned} \tag{3.67}$$

Chapter 4

Stationary phase approximation

For rapidly varying solutions, as the case for the obtained potentials for gravity and acoustic-gravity waves, they present stationary points along the integration domain. Due to this, a Stationary Phase Approximation can be applied in a direct manner to the solutions. This rises when the integration domain becomes larger (for a large distance, x) and an integration by parts can be applied to develop an asymptotic expansion in inverse powers of the domain so long as the boundary terms are finite and the resulting integrals exist.

The method of stationary phase gives the leading asymptotic behavior of generalized Fourier integrals having stationary points.

1 - Stationary phase approximation for the gravity wave potential ϕ_g

Let us consider the equation (3.33) from the previous section

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\int_0^\infty \mathrm{e}^{|k|z}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\left\{a_g\cos(\sqrt{|k|}t) - b_g\sin(\sqrt{|k|}t)\right\}\mathrm{d}k \tag{4.1}$$

which by replacing $\mu \to 0$ and $\omega(k) = \sqrt{|k|}$ results in the next expression

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi}\int_0^\infty \mathrm{e}^{\omega^2(k)z}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\left\{a_g(k)\cos(\omega(k)t) - b_g(k)\sin(\omega(k)t)\right\}\mathrm{d}k \tag{4.2}$$

Now, the equation (4.2) can be reformulated as

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi}\int_0^\infty \mathrm{e}^{\omega^2(k)z}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\left\{a_g(k)\left(\frac{\mathrm{e}^{\mathrm{i}\omega(k)t}+\mathrm{e}^{-\mathrm{i}\omega(k)t}}{2}\right) - b_g(k)\left(\frac{\mathrm{e}^{\mathrm{i}\omega(k)t}-\mathrm{e}^{-\mathrm{i}\omega(k)t}}{2\mathrm{i}}\right)\right\}\mathrm{d}k \tag{4.3}$$

factorizing terms and separating (4.3) in two integrals I_{g_a} and I_{g_b} showed next

$$\phi_g(x,z,t) = -\frac{I_0}{4\pi}\left\{\underbrace{\int_0^\infty e^{\omega^2(k)z}\left(a_g(k)-\frac{b_g(k)}{\mathrm{i}}\right)e^{\mathrm{i}(kx+\omega(k)t)}e^{-\left(\frac{k\sigma}{2}\right)^2}\mathrm{d}k}_{I_{g_a}} + \underbrace{\int_0^\infty e^{\omega^2(k)z}\left(a_g(k)+\frac{b_g(k)}{\mathrm{i}}\right)e^{\mathrm{i}(kx-\omega(k)t)}e^{-\left(\frac{k\sigma}{2}\right)^2}\mathrm{d}k}_{I_{g_b}}\right\} \tag{4.4}$$

To solve the equation (4.4) we are going to solve the integrals I_{g_a} and I_{g_b} independently by using the Stationary Phase Approximation formulation according Bender (1999), that can be written as

$$I = \int_a^b F(k)e^{\mathrm{i}(kx\mp\omega(k)t)}\mathrm{d}k \tag{4.5}$$

and which has a direct solution in the next form,

$$I = \frac{|F(k_0)|}{\pi}\sqrt{\frac{2\pi}{t\,|\omega''(k_0)|}}\cos(k_0x-\omega(k_0)t\pm\frac{\pi}{4}) \tag{4.6}$$

due to integrals I_{g_a} and I_{g_b} have the form exposed in equation (4.4), the equation (4.5) can be applied directly on them. Regarding the above, the integrals of the equation (4.4) can be expressed in the next form

$$I_{g_a} = \int_0^\infty \underbrace{e^{\omega^2(k)z}\left(a_g(k)-\frac{b_g(k)}{\mathrm{i}}\right)e^{-\left(\frac{k\sigma}{2}\right)^2}}_{F_{g_a}(k)} e^{\mathrm{i}(kx+\omega(k)t)}\mathrm{d}k; \tag{4.7a}$$

(4.7b)

$$I_{g_b} = \int_0^\infty \underbrace{e^{\omega^2(k)z}\left(a_g(k)+\frac{b_g(k)}{\mathrm{i}}\right)e^{-\left(\frac{k\sigma}{2}\right)^2}}_{F_{g_b}(k)} e^{\mathrm{i}(kx-\omega(k)t)}\mathrm{d}k \tag{4.7b}$$

for which the phase term is stated as as $g_g^+(k) = (kx+\omega(k)t)$ and $g_g^-(k) = (kx-\omega(k)t)$. The point of stationary phase is $k = k_0$ where its derivative corresponds to

$$\frac{\mathrm{d}g_g^\pm(k)}{\mathrm{d}k} = 0 \tag{4.8}$$

$$\frac{\mathrm{d}g_g^+(k)}{dk} = x\frac{\mathrm{d}k}{\mathrm{d}k} + t\frac{\mathrm{d}\omega(k)}{\mathrm{d}k} \tag{4.9}$$

$$x + t\frac{\mathrm{d}\omega(k)}{\mathrm{d}k} = 0 \tag{4.10}$$

$$\frac{\mathrm{d}\omega(k)}{\mathrm{d}k} = -\frac{x}{t} \tag{4.11}$$

In an analog way, when the phase term g_g^- is considered, its first derivative will be

$$\frac{\mathrm{d}\omega(k)}{\mathrm{d}k} = \frac{x}{t} \tag{4.12}$$

For this case $\omega(k_0) = (k_0)^{1/2}$, then the equations (4.11) and (4.12) can be rewritten as

$$\frac{1}{2}(k_0)^{-\frac{1}{2}} = -\frac{x}{t}; \qquad \frac{1}{2}(k_0)^{-\frac{1}{2}} = \frac{x}{t} \tag{4.13 a, b}$$

What implies for $g_g^{\pm}(k_0)$ that

$$k_0 = \frac{1}{4}\left(\frac{t}{x}\right)^2 \tag{4.14}$$

Considering the above results and formulations, we can express the integrals I_{g_a} and I_{g_b} as

$$I_{g_a} = \frac{\left|\mathrm{e}^{\omega^2(k_0)z}\left(a_g(k_0) - \frac{b_g(k_0)}{\mathrm{i}}\right)\mathrm{e}^{-\left(\frac{k_0\sigma}{2}\right)^2}\right|}{\pi}\sqrt{\frac{2\pi}{t\left|\omega''(k_0)\right|}}\cos(k_0x - \omega(k_0)t + \frac{\pi}{4}) \tag{4.15}$$

$$I_{g_b} = \frac{\left|\mathrm{e}^{\omega^2(k_0)z}\left(a_g(k_0) + \frac{b_g(k_0)}{\mathrm{i}}\right)\mathrm{e}^{-\left(\frac{k_0\sigma}{2}\right)^2}\right|}{\pi}\sqrt{\frac{2\pi}{t\left|\omega''(k_0)\right|}}\cos(k_0x - \omega(k_0)t - \frac{\pi}{4}) \tag{4.16}$$

Let us consider the second derivative of $\omega(k)$ as

$$\omega''(k) = -\frac{1}{4}k^{-3/2}$$

Replacing the obtained value k_0 on the above derivative, we get

$$\begin{aligned}\omega''(k_0) &= -\frac{1}{4}\left(\frac{1}{4}\frac{t^2}{x^2}\right)^{-3/2}\\ &= -\frac{1}{4}(2)^3\left(\frac{x}{t}\right)^3\\ &= -2\left(\frac{x^3}{t^3}\right)\end{aligned}$$

replacing the terms accordingly, the next expressions are obtained

$$I_{g_a} = \left| \frac{1}{\pi} \mathrm{e}^{\omega^2(k_0)z} \left(a_g(k_0) - \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{\frac{1}{4}\left(\frac{t}{x}\right)^2 \sigma}{2}\right)^2} \right| \sqrt{\frac{2\pi}{t\left|-2\left(\frac{x^3}{t^3}\right)\right|}} \cos\left(\frac{1}{4}\left(\frac{t}{x}\right)^2 x - \frac{1}{2}\frac{t}{x}t + \frac{\pi}{4} \right) \tag{4.17}$$

$$I_{g_a} = \left| \frac{1}{\pi} \mathrm{e}^{\omega^2(k_0)z} \left(a_g(k_0) - \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)} \right| \sqrt{\frac{t^2\pi}{x^3}} \cos\left(-\frac{t^2}{4x} + \frac{\pi}{4} \right) \tag{4.18}$$

$$I_{g_b} = \left| \frac{1}{\pi} \mathrm{e}^{\omega^2(k_0)z} \left(a_g(k_0) + \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{\frac{1}{4}\left(\frac{t}{x}\right)^2 \sigma}{2}\right)^2} \right| \sqrt{\frac{2\pi}{t\left|-2\left(\frac{x^3}{t^3}\right)\right|}} \cos\left(\frac{1}{4}\left(\frac{t}{x}\right)^2 x - \frac{1}{2}\frac{t}{x}t - \frac{\pi}{4} \right) \tag{4.19}$$

$$I_{g_b} = \left| \frac{1}{\pi} \mathrm{e}^{\omega^2(k_0)z} \left(a_g(k_0) + \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)} \right| \sqrt{\frac{t^2\pi}{x^3}} \cos\left(-\frac{t^2}{4x} - \frac{\pi}{4} \right) \tag{4.20}$$

thus, the gravity wave potential can be rewritten as

$$\phi_g(x,z,t) = \frac{I_0}{4\pi} \left\{ I_{g_a} + I_{g_b} \right\} \tag{4.21}$$

$$\begin{aligned} \phi_g(x,z,t) = -\frac{I_0}{4\pi} & \\ & \left\{ \frac{1}{\pi} \mathrm{e}^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z} \left(a_g(k_0) - \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)} \sqrt{\frac{t^2\pi}{x^3}} \cos\left(-\frac{t^2}{4x} + \frac{\pi}{4} \right) \right. \\ & \left. + \frac{1}{\pi} \mathrm{e}^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z} \left(a_g(k_0) + \frac{b_g(k_0)}{\mathrm{i}} \right) \mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)} \sqrt{\frac{t^2\pi}{x^3}} \cos\left(-\frac{t^2}{4x} - \frac{\pi}{4} \right) \right\} \end{aligned} \tag{4.22}$$

$$\begin{aligned} \phi_g(x,z,t) = & -\frac{I_0}{4\pi^2} \mathrm{e}^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z} \mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)} \frac{t}{x} \sqrt{\frac{\pi}{x}} \\ & \left\{ \left(a_g(k_0) - \frac{b_g(k_0)}{\mathrm{i}} \right) \cos\left(-\frac{t^2}{4x} + \frac{\pi}{4} \right) + \left(a_g(k_0) + \frac{b_g(k_0)}{\mathrm{i}} \right) \cos\left(-\frac{t^2}{4x} - \frac{\pi}{4} \right) \right\} \end{aligned} \tag{4.23}$$

Thus, from the equation (4.4) the derivative of the potential $\phi_g(x,z,t)$ with respect to the time is

$$\phi_g'(x,z,t) = -\frac{I_0}{4\pi}\Bigg\{\underbrace{\int_0^\infty \mathrm{i}\omega(k)\mathrm{e}^{\omega^2(k)z}\left(a_g(k)-\frac{b_g(k)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(kx+\omega(k)t)}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}dk}_{I_{g_a}} - \underbrace{\int_0^\infty \mathrm{i}\omega(k)\mathrm{e}^{\omega^2(k)z}\left(a_g(k)+\frac{b_g(k)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}dk}_{I_{g_b}}\Bigg\} \tag{4.24}$$

factorizing the corresponding terms, we get

$$\phi_g'(x,z,t) = -\frac{\mathrm{i}\omega(k)I_0}{4\pi}\Bigg\{\underbrace{\int_0^\infty \mathrm{e}^{\omega^2(k)z}\left(a_g(k)-\frac{b_g(k)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(kx+\omega(k)t)}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}dk}_{I_{g_a}} - \underbrace{\int_0^\infty \mathrm{e}^{\omega^2(k)z}\left(a_g(k)+\frac{b_g(k)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}dk}_{I_{g_b}}\Bigg\} \tag{4.25}$$

Once k_0 is replaced, the next expression is found

$$\phi_g'(x,z,t) = -\mathrm{i}\frac{t}{2x}\frac{I_0}{4\pi^2}\mathrm{e}^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z}\mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)}\frac{t}{x}\sqrt{\frac{\pi}{x}} \left\{\left(a_g(k_0)-\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}+\frac{\pi}{4}\right)-\left(a_g(k_0)+\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}-\frac{\pi}{4}\right)\right\} \tag{4.26}$$

finally,

$$\phi_g'(x,z,t) = -\mathrm{i}\frac{tI_0}{8x\pi^2}e^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z}\mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)}\frac{t}{x}\sqrt{\frac{\pi}{x}} \left\{\left(a_g(k_0)-\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}+\frac{\pi}{4}\right)-\left(a_g(k_0)+\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}-\frac{\pi}{4}\right)\right\}. \tag{4.27}$$

Now, to obtain the surface elevation η_g due to gravity modes, we can employ the following form

$$\partial\phi_g/\partial t + \eta_g(x,t) = 0, \qquad (z=0). \tag{4.28}$$

from which we can state η_g as

$$\eta_g(x,t) = -\partial\phi_g/\partial t, \qquad (z=0). \tag{4.29}$$

Thus, considering expression (4.27), we get

$$\eta_g(x,t) = \mathrm{i}\frac{tI_0}{8x\pi^2}e^{\frac{1}{4}\left(\frac{t}{x}\right)^2 z}\mathrm{e}^{-\left(\frac{t^4\sigma^2}{64x^2}\right)}\frac{t}{x}\sqrt{\frac{\pi}{x}}$$
$$\left\{\left(a_g(k_0)-\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}+\frac{\pi}{4}\right)-\left(a_g(k_0)+\frac{b_g(k_0)}{\mathrm{i}}\right)\cos\left(-\frac{t^2}{4x}-\frac{\pi}{4}\right)\right\}, \tag{4.30}$$

for the gravity wave.

2 - Stationary phase approximation for the acoustic-gravity wave potential ϕ_s

In the same way than the in previous calculations, lets apply the Stationary Phase Method to the expression (3.67) corresponding to the AGW potential. The calculations are as follows

$$\phi_s(x,z,t) = \frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\kappa_n\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\{a_s(\kappa_n)\cos(\kappa_n t)\right.$$
$$\left. - b_s(\kappa_n)\sin(\kappa_n t)\}\,\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)\mathrm{d}k \tag{4.31}$$

or in an equivalent manner

$$\phi_s(x,z,t) = \frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\omega_n\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\{a_s(\omega_n)\cos(\omega_n t)\right.$$
$$\left. - b_s(\omega_n)\sin(\omega_n t)\}\,\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)\mathrm{d}k$$

Now, doing the replacement of $k=\mu\kappa$ and considering $n=0$, we have that

$$\phi_s(x,z,t) = \frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^{\infty}\omega_0\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\{a_s(\omega_0)\cos(\omega_0 t)\right.$$
$$\left. - b_s(\omega_0)\sin(\omega_0 t)\}\,\mathrm{e}^{\mathrm{i}\mu\kappa_0 x}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)\mathrm{d}\mu\kappa$$

rearranging terms,

$$\phi_s(x,z,t) = \mu\frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^{\infty}\omega_0\left[2\gamma\cos(\gamma(Z+1))-\mu\sin(\gamma(Z+1))\right]\{a_s(\omega_0)\cos(\omega_0 t)\right.$$
$$\left. - b_s(\omega_0)\sin(\omega_0 t)\}\,\mathrm{e}^{\mathrm{i}\mu\kappa_0 x}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)\mathrm{d}\kappa \tag{4.32}$$

rewriting (4.32) with exponential forms for cos and sin

$$\phi_s(x,z,t) = \mu\frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^\infty \omega_0\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\left\{a_s(\omega_0)\left(\frac{\mathrm{e}^{i\omega_0 t}+\mathrm{e}^{-i\omega_0 t}}{2}\right)\right.\right.$$
$$\left.\left. - b_s(\omega_0)\left(\frac{\mathrm{e}^{i\omega_0 t}-\mathrm{e}^{-i\omega_0 t}}{2\mathrm{i}}\right)\right\}\mathrm{e}^{\mathrm{i}\mu\kappa_0 x}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)\mathrm{d}\kappa \tag{4.33}$$

the equation becomes to

$$\phi_s(x,z,t) = \mu\frac{I_0}{2\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^\infty \omega_0\left[2\gamma\cos(\gamma(Z+1)) - \mu\sin(\gamma(Z+1))\right]\left\{\left(a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{i\omega_0 t}\right.\right.$$
$$\left.\left. + \left(a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{-i\omega_0 t}\right\}\mathrm{e}^{\mathrm{i}\mu\kappa_0 x}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)\mathrm{d}\kappa \tag{4.34}$$

making the corresponding calculations in order to reduce terms, we get

$$\begin{aligned}\phi_s(x,z,t) = \mu\frac{I_0}{2\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^\infty \omega_0\left[2\gamma\cos(\gamma(Z+1))\right.\right. \\ \left. - \mu\sin(\gamma(Z+1))\right]\left\{\left(a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x+\omega_0 t)}\right. \\ \left.\left. + \left(a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x-\omega_0 t)}\right\}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)\mathrm{d}\kappa\end{aligned} \tag{4.35}$$

Similar to the calculations made for the gravity wave, (4.35) is composed of two integrals, I_{s_a} and I_{s_b}, respectively

$$\phi_s(x,z,t) = \mu\frac{I_0}{2\pi}$$

$$\Bigg(\underbrace{\int_0^{\infty}\omega_0[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1))-\mu\sin(\gamma(\kappa_0)(Z+1))]\,\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\Bigg\{\Bigg(a_s(\omega_0)-\frac{b_s(\omega_0)}{\mathrm{i}}\Bigg)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x+\omega_0 t)}\Bigg\}d\kappa}_{I_{s_a}}$$

$$+\underbrace{\int_0^{\infty}\omega_0[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1))-\mu\sin(\gamma(\kappa_0)(Z+1))]\,\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\Bigg\{\Bigg(a_s(\omega_0)+\frac{b_s(\omega_0)}{\mathrm{i}}\Bigg)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x-f\omega_0 t)}\Bigg\}\mathrm{d}\kappa}_{I_{s_b}}\Bigg) \tag{4.36}$$

To solve the equation (4.36), the Stationary Phase Approximation is considered to reduce the integrals I_{s_a} and I_{s_b}. The formulation can be expressed as follows

$$I = \int_a^b F(\kappa_n)\mathrm{e}^{\mathrm{i}(\mu\kappa_n x\mp\omega(\kappa_n)t)}\mathrm{d}\mu\kappa \tag{4.37}$$

with its direct solution

$$I = \mu\frac{|F(\kappa_n)|}{\pi}\sqrt{\frac{2\pi}{t\,|\omega''(\kappa_n)|}}\cos(\mu\kappa_n x-\omega(\kappa_n)t\pm\frac{\pi}{4}) \tag{4.38}$$

Now, the integrals of the equation (4.36) can be expressed in the next form

$$I_{s_a}=\int_0^{\infty}\underbrace{\omega_0[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1))-\mu\sin(\gamma(\kappa_0)(Z+1))]\,\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\Bigg(a_s(\omega_0)-\frac{b_s(\omega_0)}{\mathrm{i}}\Bigg)}_{F_{s_a}}\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x+\omega_0 t)}\mathrm{d}\kappa \tag{4.39a}$$

$$I_{s_b}=\int_0^{\infty}\underbrace{\omega_0[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1))-\mu\sin(\gamma(\kappa_0)(Z+1))]\,\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\Bigg(a_s(\omega_0)+\frac{b_s(\omega_0)}{\mathrm{i}}\Bigg)}_{F_{s_b}}\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x-\omega_0 t)}\mathrm{d}\kappa \tag{4.39b}$$

The phase term of the above integrals is stated as $g_s^+(\kappa_0) = (\mu\kappa_0 x + \omega(\kappa_0)t)$ and $g_s^-(\kappa_0) = (\mu\kappa_0 x - \omega(\kappa)t)$. The point of stationary phase is $\kappa = \kappa_0$ where the derivative is

$$\frac{\mathrm{d}g_s^{\pm}(\kappa_0)}{\mathrm{d}\omega_0} = 0 \tag{4.40}$$

$$\frac{\mathrm{d}g_s^{-}(\kappa_0)}{\mathrm{d}\omega_0} = \mu x \frac{\mathrm{d}\kappa_0}{\mathrm{d}\omega_0} - t\frac{\mathrm{d}\omega(\kappa_0)}{\mathrm{d}\omega_0} \tag{4.41}$$

$$\mu x \frac{\mathrm{d}\kappa_0}{\mathrm{d}\omega} - t = 0 \tag{4.42}$$

$$\frac{\mathrm{d}\kappa_0}{\mathrm{d}\omega} = \frac{t}{\mu x} \tag{4.43}$$

In an analog way, when the phase term g_g^+ is considered, its first derivative will be

$$\frac{\mathrm{d}\kappa_0}{\mathrm{d}\omega} = -\frac{t}{\mu x} \tag{4.44}$$

Recalling that the dispersion relation for the AGW has the following form

$$\omega^2 = \omega_n^2 + \kappa^2 + \mu \left\{ \frac{\omega_n^2 - \kappa^2}{\omega_n^2 + \kappa^2} \right\} + O(\mu^2) \quad \text{; with } n = 0, 1, 2, ..., \tag{4.45}$$

Neglecting the term associated to μ, we can rewrite ω as

$$\omega^2 = \omega_n^2 + \kappa^2 \rightarrow \kappa_n(\omega) = \sqrt{\omega^2 - \omega_n^2} \tag{4.46}$$

Doing the derivative of κ_n with respect to ω we will have

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\omega} = \frac{\omega}{\sqrt{\omega^2 - \omega_n^2}} \tag{4.47}$$

Replacing the equation (4.47) on (4.43) and (4.44)

$$\frac{\omega}{\sqrt{\omega^2 - \omega_n^2}} = \frac{t}{\mu x}; \qquad \frac{\omega}{\sqrt{\omega^2 - \omega_n^2}} = -\frac{t}{\mu x} \tag{4.49 a, b}$$

respectively. The point of stationary phase is at $\omega = \psi_n$ where $\mathrm{d}g_s^{\pm}/\mathrm{d}\omega = 0$. According this, we can state the next expression

$$\frac{\psi_n}{\sqrt{\psi_n^2 - \omega_n^2}} = \pm\frac{t}{\mu x} \tag{4.49}$$

which follows that

$$\psi_n = \frac{\omega_n}{\sqrt{1-\left(\pm\frac{\mu x}{t}\right)^2}} = \frac{\omega_n}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.50}$$

$$\psi_n^2 - \omega_n^2 = \omega_n^2\left(\frac{\left(\frac{\mu x}{t}\right)^2}{1-\left(\frac{\mu x}{t}\right)^2}\right) \tag{4.51}$$

$$\sqrt{\psi_n^2-\omega_n^2} = \omega_n\frac{\left(\frac{\mu x}{t}\right)}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.52}$$

In this way, we will have for κ_n

$$\kappa_n(\psi_n) \equiv K_n = \sqrt{\psi_n^2-\omega_n^2} = \omega_n\frac{\left(\frac{\mu x}{t}\right)}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.53}$$

As we are considering only the zero mode, where $n = 0$, and recalling from (2.72) that $\omega_n = (n + 1/2)\pi \to \omega_0 = (0 + 1/2)\pi = \pi/2$. Thus, the equation (4.53) becomes

$$\kappa_0(\psi_0) \equiv K_0 = \sqrt{\psi_0^2-\omega_0^2} = \omega_0\frac{\left(\frac{\mu x}{t}\right)}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.54}$$

$$K_0 = \left(\frac{\pi}{2}\right)\frac{\left(\frac{\mu x}{t}\right)}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.55}$$

and (4.50) yields

$$\psi_0 = \frac{\frac{\pi}{2}}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}} \tag{4.56}$$

Now, we are going to obtain the second derivative $\partial^2 g_s^{\pm}/\partial\omega^2$ in the following way

$$\frac{\partial^2 g_s^{\pm}}{\partial\omega^2} = \frac{\mu x}{t}\left(\frac{\frac{\partial\psi_n}{\partial\omega}}{\sqrt{\psi_n^2-\omega_n^2}} - \frac{\omega\psi_n}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right) \tag{4.57}$$

Taking $\omega = \psi_n$, we can rewrite

$$\frac{\partial^2 g_s^{\pm}}{\partial\omega^2} = \frac{\mu x}{t}\left(\frac{\frac{\partial\omega}{\partial\omega}}{\sqrt{\psi_n^2-\omega_n^2}} - \frac{\omega^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right) = \frac{\mu x}{t}\left(\frac{1}{\sqrt{\psi_n^2-\omega_n^2}} - \frac{\omega^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right) \tag{4.58}$$

$$\frac{\partial^2 g_s^{\pm}}{\partial\omega^2} = \frac{\mu x}{t}\left(\frac{\psi_n^2-\omega_n^2-\omega^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right) = \frac{\mu x}{t}\left(\frac{\omega^2-\omega_n^2-\omega^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right) \tag{4.59}$$

$$\frac{\partial^2 g_s^{\pm}}{\partial \omega^2} = \frac{\mu x}{t} \frac{\omega_n^2}{(\psi_n^2 - \omega_n^2)^{\frac{3}{2}}} \tag{4.60}$$

for $\omega_0 = \pi/2$ the equation (4.60) is

$$\frac{\partial^2 g_s^{\pm}}{\partial \omega^2} = \frac{\left(\frac{\pi}{2}\right)^2 \left(\frac{\mu x}{t}\right)}{\left(\left(\frac{\omega_n}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}}\right)^2 - \left(\frac{\pi}{2}\right)^2\right)^{\frac{3}{2}}} \tag{4.61}$$

$$\frac{\partial^2 g_s^{\pm}}{\partial \omega^2} = \frac{\left(\frac{\pi}{2}\right)^2 \left(\frac{\mu x}{t}\right)}{\left(\left(\frac{\frac{\pi}{2}}{\sqrt{1-\left(\frac{\mu x}{t}\right)^2}}\right)^2 - \left(\frac{\pi}{2}\right)^2\right)^{\frac{3}{2}}} \tag{4.62}$$

Taking the above results and formulations, we can express the integrals I_{s_a} and I_{s_b} (eqs. 4.39a and 4.39b) using the solution for stationary phase approximation (4.38) as

$$I_{s_a} = \left| \frac{1}{\pi} \left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu \sin(\gamma(K_0)(Z+1))\right] \mathrm{e}^{-\left(\frac{\mu K_0 \sigma}{2}\right)^2} \left(a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right)\right| \sqrt{\frac{2\pi}{t\left|\frac{\mu x}{t}\frac{\omega_n^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right|}} \cos(\mu K_0 x - \psi_0 t + \frac{\pi}{4}) \tag{4.63}$$

$$I_{s_a} = \left| \frac{1}{\pi} \left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu \sin(\gamma(K_0)(Z+1))\right] \mathrm{e}^{-\left(\frac{\mu K_0 \sigma}{2}\right)^2} \left(a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right)\right| \sqrt{\frac{2\pi}{\mu x\frac{\omega_n^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}}} \cos(\mu K_0 x - \psi_0 t + \frac{\pi}{4}) \tag{4.64}$$

$$I_{s_b} = \left| \frac{1}{\pi} \left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu \sin(\gamma(K_0)(Z+1))\right] \mathrm{e}^{-\left(\frac{\mu K_0 \sigma}{2}\right)^2} \left(a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right)\right| \sqrt{\frac{2\pi}{t\left|\frac{\mu x}{t}\frac{\omega_n^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}\right|}} \cos(\mu K_0 x - \psi_0 t - \frac{\pi}{4}) \tag{4.65}$$

$$I_{s_b} = \left| \frac{1}{\pi} \left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu \sin(\gamma(K_0)(Z+1))\right] \mathrm{e}^{-\left(\frac{\mu K_0 \sigma}{2}\right)^2} \left(a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right)\right| \sqrt{\frac{2\pi}{\mu x\frac{\omega_n^2}{(\psi_n^2-\omega_n^2)^{\frac{3}{2}}}}} \cos(\mu K_0 x - \psi_0 t - \frac{\pi}{4}) \tag{4.66}$$

Once obtained the formulations for I_{s_a} and I_{s_b}, we can replace the to get the general form for the acoustic-gravity wave potential

$$\phi_s(x,z,t) = \mu\frac{I_0}{2\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left\{I_{s_a} + I_{s_b}\right\} \tag{4.67}$$

what yields

$$\begin{aligned}
&\phi_s(x,z,t) = \\
&\mu\frac{I_0}{2\pi^2}\omega_0\left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu\sin(\gamma(K_0)(Z+1))\right]\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\left(\frac{K_0\Sigma}{2}\right)^2}\sqrt{\frac{2\pi}{\mu x\frac{\omega_0^2}{\left(\psi_0^2-\omega_0^2\right)^{\frac{3}{2}}}}} \\
&\left\{\left[a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t + \frac{\pi}{4}\right) + \left[a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t - \frac{\pi}{4}\right)\right\}
\end{aligned} \tag{4.68}$$

for $n = 0$, where $X = \mu x$ and $\Sigma = \mu\sigma$. To obtain the derivative of the potential $\phi_s(x,z,t)$, lets derivate the equation (4.36) with respect to the time

$$\begin{aligned}
&\phi_s'(x,z,t) = \mathrm{i}\mu\omega_0^2\frac{I_0}{2\pi^2} \\
&\left(\underbrace{\int_0^\infty \left[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1)) - \mu\sin(\gamma(\kappa_0)(Z+1))\right]\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\left\{\left(a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x + \omega_0 t)}\right\}d\kappa}_{I_{s_a}}\right. \\
&\left.-\underbrace{\int_0^\infty \left[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1)) - \mu\sin(\gamma(\kappa_0)(Z+1))\right]\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\left\{\left(a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}(\mu\kappa_0 x - \omega_0 t)}\right\}d\kappa}_{I_{s_b}}\right)
\end{aligned} \tag{4.69}$$

after applying the stationary phase approximation on I_{s_a} and I_{s_b} and doing the corresponding replacements, we obtain the expression for the derivative of the AGW potential with respect to the variable t

$$\begin{aligned}
&\phi_s'(x,z,t) = \\
&\mathrm{i}\mu\omega_0^2\frac{I_0}{2\pi^2}\left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu\sin(\gamma(K_0)(Z+1))\right]\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\left(\frac{K_0\Sigma}{2}\right)^2}\sqrt{\frac{2\pi}{\mu x\frac{\omega_0^2}{\left(\psi_0^2-\omega_0^2\right)^{\frac{3}{2}}}}} \\
&\left\{\left[a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t + \frac{\pi}{4}\right) - \left[a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t - \frac{\pi}{4}\right)\right\}.
\end{aligned} \tag{4.70}$$

In an analogue manner, to obtain the surface elevation η_s due to acoustic-gravity modes, we can employ the same form (4.29) to get the wave elevation

$$\eta_g(x,t) = -\partial\phi_g/\partial t, \qquad (z=0) \tag{4.71}$$

resulting in

$$\eta_s(x,z,t) = \\ -\mathrm{i}\mu\omega_0^2\frac{I_0}{2\pi^2}\left[2\gamma(K_0)\cos(\gamma(K_0)(Z+1)) - \mu\sin(\gamma(K_0)(Z+1))\right]\mathrm{e}^{\frac{1}{2}\mu^2 z}\mathrm{e}^{-\left(\frac{K_0\Sigma}{2}\right)^2}\sqrt{\frac{2\pi}{\mu x\frac{\omega_0^2}{\left(\psi_0^2-\omega_0^2\right)^{\frac{3}{2}}}}} \\ \left\{\left[a_s(\omega_0) - \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t + \frac{\pi}{4}\right) - \left[a_s(\omega_0) + \frac{b_s(\omega_0)}{\mathrm{i}}\right]\cos\left(K_0X - \psi_0 t - \frac{\pi}{4}\right)\right\} \tag{4.72}$$

for the acoustic-gravity wave.

Chapter 5

Numerical Simulations

5.1 Validation

To validate the calculations and results obtained until here, we are going to compare the two found potentials ϕ_g and ϕ_s; with the potentials obtained by Renzi & Dias (2014). In this case, the validations will be carried out in a qualitative and not quatitative manner, due to the scarce information of measured in-situ data. One future goal is to perform physical experiments in order to can validate the solutions of this kind of theories for AGW's.

1 - Validation for the gravity wave potential ϕ_g

Let us take the expression (2.31) of the section 2

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi} e^{\frac{1}{2}\mu^2 z} \int_0^\infty e^{|k|z} e^{ikx} e^{-\left(\frac{k\sigma}{2}\right)^2} \left\{ a_g(k) \sin(\sqrt{|k|}t) - b_g(k) \cos(\sqrt{|k|}t) \right\} dk \quad (5.1)$$

Considering that $\mu \to 0$, then the term $e^{\frac{1}{2}\mu^2 z} \to 1$. Now, the equation (5.1) can be written as this latter equation can be also expressed as

$$\phi_g(x,z,t) = -\frac{I_0}{2\pi} \int_0^\infty \{\cosh(|k|\, z - kx) + \sinh(|k|\, z - kx)\} \{a_g(k) \sin(\omega(k)t) - b_g(k) \cos(\omega(k)t)\} e^{-\left(\frac{k\sigma}{2}\right)^2} dk \quad (5.2)$$

which is in agreement equation (3.11) of the gravity wave potential according to Renzi & Dias (2014).

2 - Validation for the acoustic-gravity wave potential ϕ_s

Let us take now the expression (3.64) of the section 3

$$\phi_s(x,z,t) = \frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}[2\gamma(\kappa_n)\cos(\gamma(\kappa_n)(Z+1)) - \mu\sin(\gamma(\kappa_n)(Z+1))]\left\{a_s(\kappa_n)\cosh(\kappa_n t)+b_s(\kappa_n)\sinh(\kappa_n t)\right\}\mathrm{e}^{ikx}\mathrm{e}^{-\left(\frac{k\sigma}{2}\right)^2}\right)dk \tag{5.3}$$

Doing the replacement of $k=\mu\kappa$, the next expression is obtained

$$\phi_s(x,z,t) = \mu\frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\sum_{n=1}^{\infty}\int_0^{\infty}[2\gamma(\kappa_n)\cos(\gamma(\kappa_n)(Z+1)) - \mu\sin(\gamma(\kappa_n)(Z+1))]\left\{a_s(\kappa_n)\cosh(\kappa_n t)-b_s(\kappa_n)\sinh(\kappa_n t)\right\}\mathrm{e}^{i\mu\kappa_n x}\mathrm{e}^{-\left(\frac{\mu\kappa_n\sigma}{2}\right)^2}\right)d\kappa \tag{5.4}$$

For the zero mode ($n=0$), the expression (5.4) becomes to

$$\phi_s(x,z,t) = \mu\frac{I_0}{\pi}\mathrm{e}^{\frac{1}{2}\mu^2 z}\left(\int_0^{\infty}[2\gamma(\kappa_0)\cos(\gamma(\kappa_0)(Z+1)) - \mu\sin(\gamma(\kappa_0)(Z+1))]\left\{a_s(\kappa_0)\cosh(\kappa_0 t) - b_s(\kappa_0)\sinh(\kappa_0 t)\right\}\mathrm{e}^{i\mu\kappa_0 x}\mathrm{e}^{-\left(\frac{\mu\kappa_0\sigma}{2}\right)^2}\right)d\kappa \tag{5.5}$$

In the same way than than the gravity wave, the equation (5.5) is comparable to the equation (3.10) corresponding to the Acoustic-Gravity wave potential according to Renzi & Dias (2014). Doing again a qualitative comparison between them, it can be noted that both have similar form respect to their components, nevertheless, the found expression (4.5) is more simplified due to the implemented escalations.

5.2 Double-Gaussian pressure

The present section is aimed to expose the numerical results from the developed method. The simulations where carried out applying the equations obtained in section 4, where the general free surface elevation (η) was calculated. We are going to consider all the results non-dimensionally in space and time.

In order to have a physical insight on the results we have carried out the following examples. Consider a sudden double Gaussian pressure given in (3.1), with $I_0 = 1$, $\epsilon = 2$ and $h = 500$ m, so that $\mu = 2.18 \times 10^{-3}$. Also, the simulated cases consider values of $\Sigma = 1$ and $\sigma = 1$ in order to test the variability of the gravity and acoustic-gravity signals according to these parameters. The length scale is defined by $\mu h = 1.09$ and the timescale by $h/c = 1/3$. The simulations where carried out applying equations (4.30) and (4.72) obtained in the previous section, where a closed form relation for the bottom pressure (P_b) was derived.

Figure 5.1 shows the arrival of the bottom pressure due acoustic-gravity wave considering $\Sigma = 1 = 459$ for different modes in an observation point located at 550 m away from the origin, which corresponds to $x = 500$. Different modes arrival are exposed in a time interval $0 <$ t < 20, where the critical time corresponds to $t = 1.09$. In the same manner, Figure 5.2 presents the arrival of different modes at an observation point at $x = 15000$ or 16350 m in a time interval $0 <$ t < 500. The critical time for this case corresponds to $t = 32.7$.

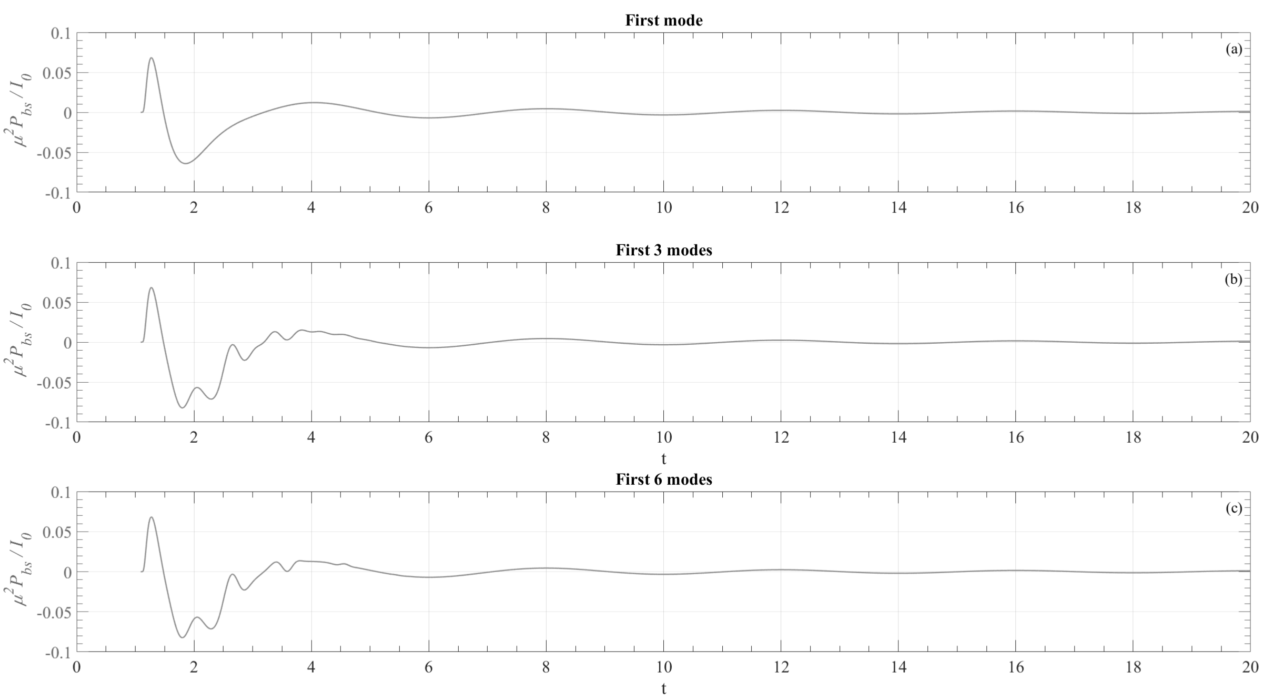

Figure 5.1: Comparison of the bottom pressure due different acoustic modes between $0 <$ t < 20 at a location x $= 500$ with $\Sigma = 1$ and $\epsilon = 2$ at $x = 500$. (a) First mode, (b) First 3 modes, and (c) First 6 modes.

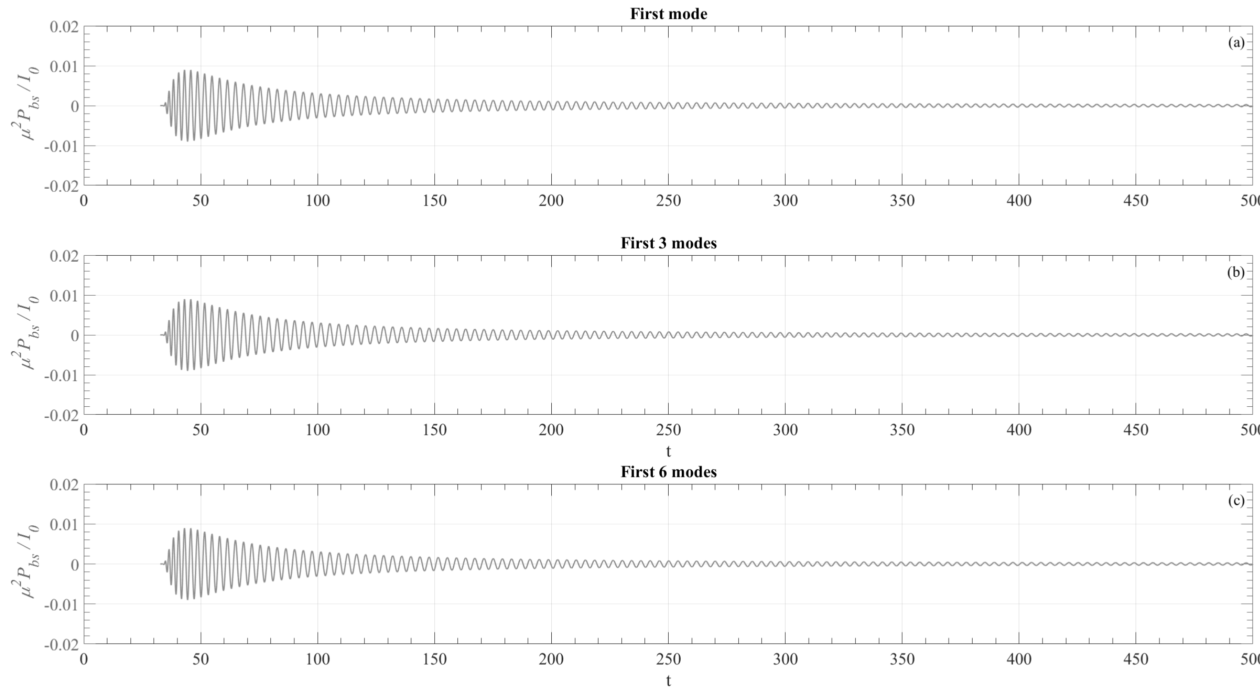

Figure 5.2: Comparison of the bottom pressure due different acoustic modes between $0 < \text{t} < 500$ at a location x = 500 with $\Sigma = 1$ and $\epsilon = 2$ at $x = 15000$. (a) First mode, (b) First 3 modes, and (c) First 6 modes.

Figures 5.3 and 5.4 shows the gravity (left) and acoustic-gravity (right) bottom pressures at different locations considering the first 6 modes for the acoustic-gravity signal. Results show that the arrival time of the acoustic bottom pressures are shorter than the gravity signal for locations $x = 200, 500$ and 1000. By the other hand, when the distances are further, as the case of $x = 10000, 15000$ and 20000, arrival times are shorter for the gravity signal. Despite the above, in both figures it can be noted that the values of the bottom pressure for the acoustic-gravity wave are higher than the gravity wave.

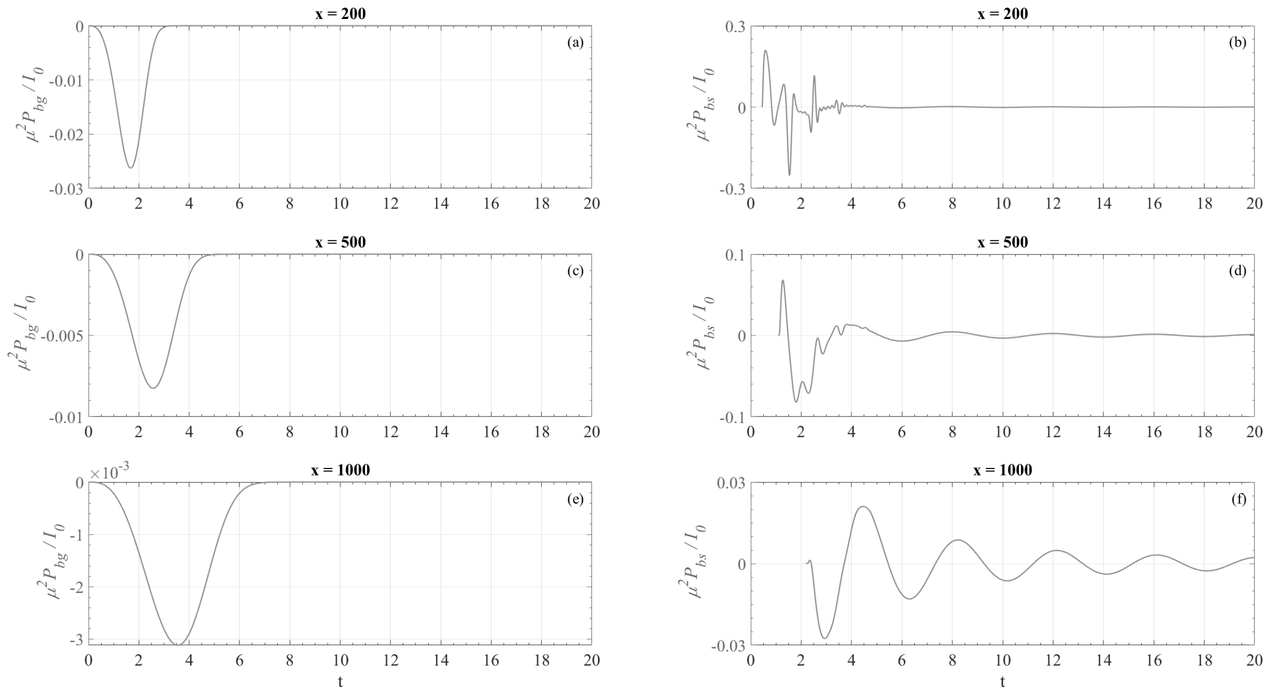

Figure 5.3: Gravity and acoustic-gravity bottom pressures at different locations with $\Sigma = 1$ and $\epsilon = 2$.

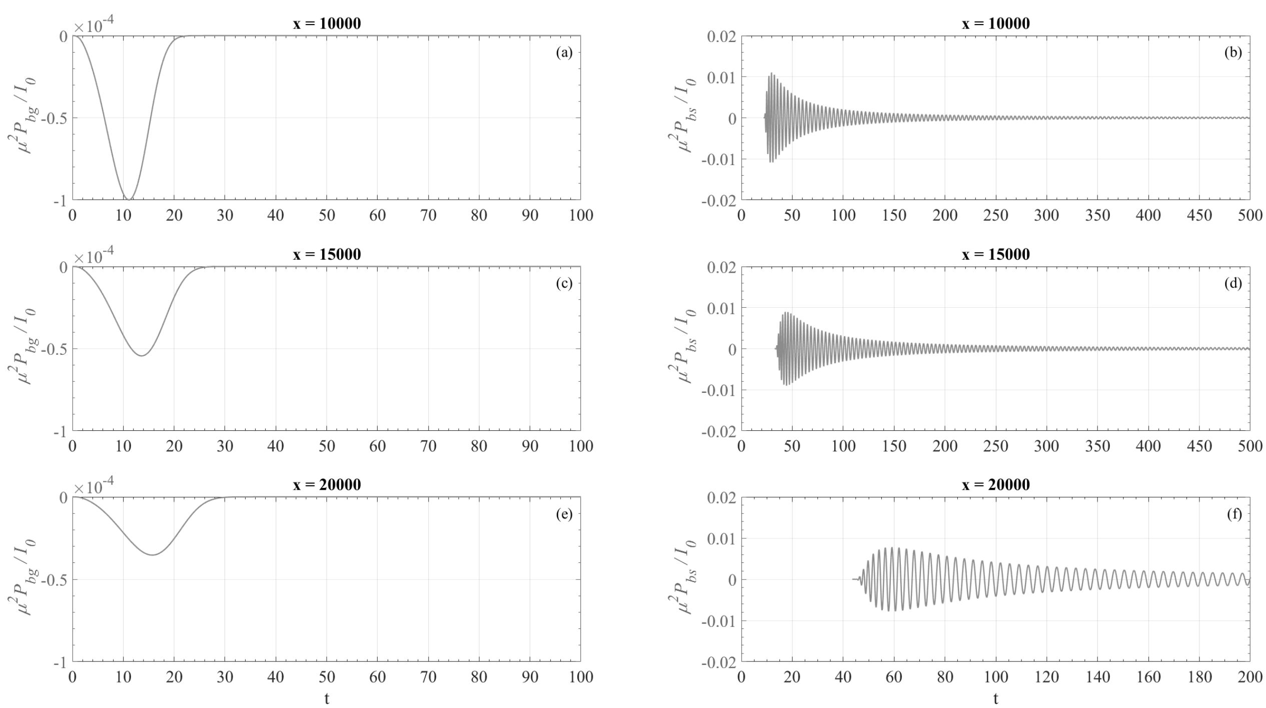

Figure 5.4: Gravity and acoustic-gravity bottom pressures at different locations with $\Sigma = 1$ and $\epsilon = 2$.

Similarly to the previous simulations, the next figures expose the arrival of different modes for the bottom pressure signal due acoustic-gravity wave considering $\sigma = 1$. Figure 5.5 shows the results an observation point located $x = 500$ and Figure 5.6 at $x = 15000$ with arrival times of $t = 1.09$ and $t = 32.7$, respectively.

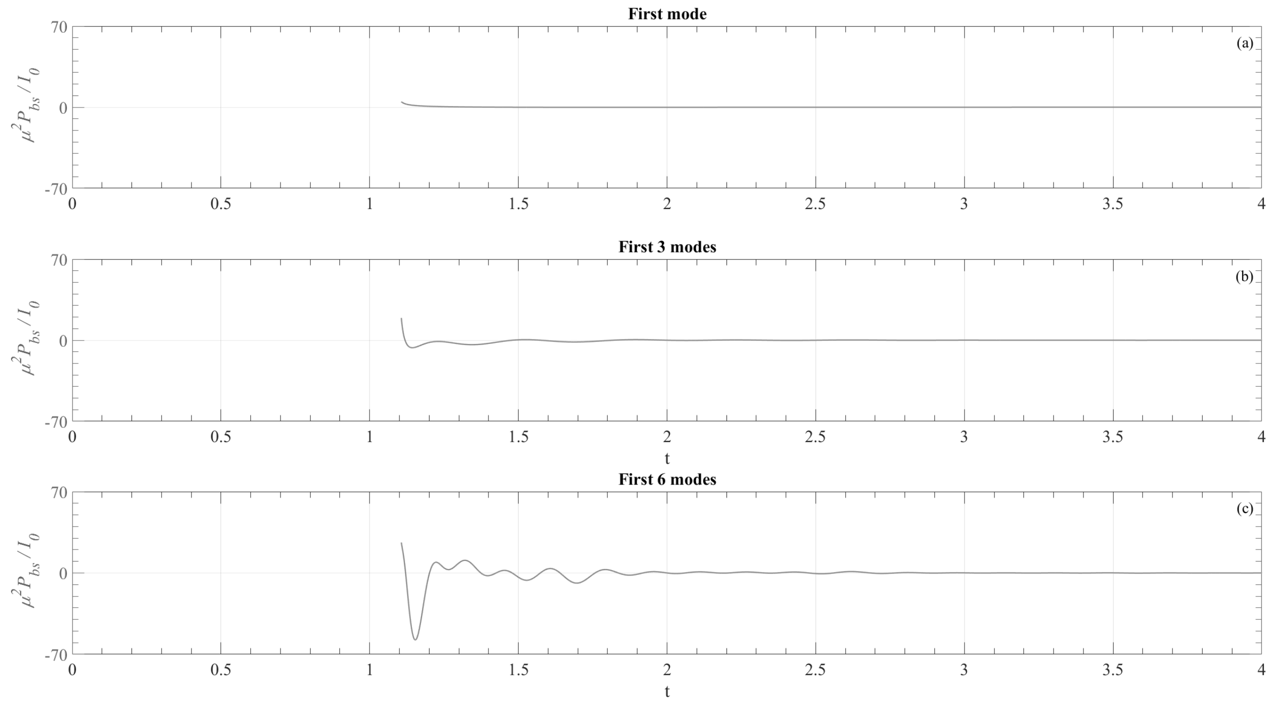

Figure 5.5: Comparison of the bottom pressure due different acoustic modes between 0 < t < 4 at a location x = 500 with $\sigma = 1$ and $\epsilon = 2$ at $x = 500$. (a) First mode, (b) First 3 modes, and (c) First 6 modes.

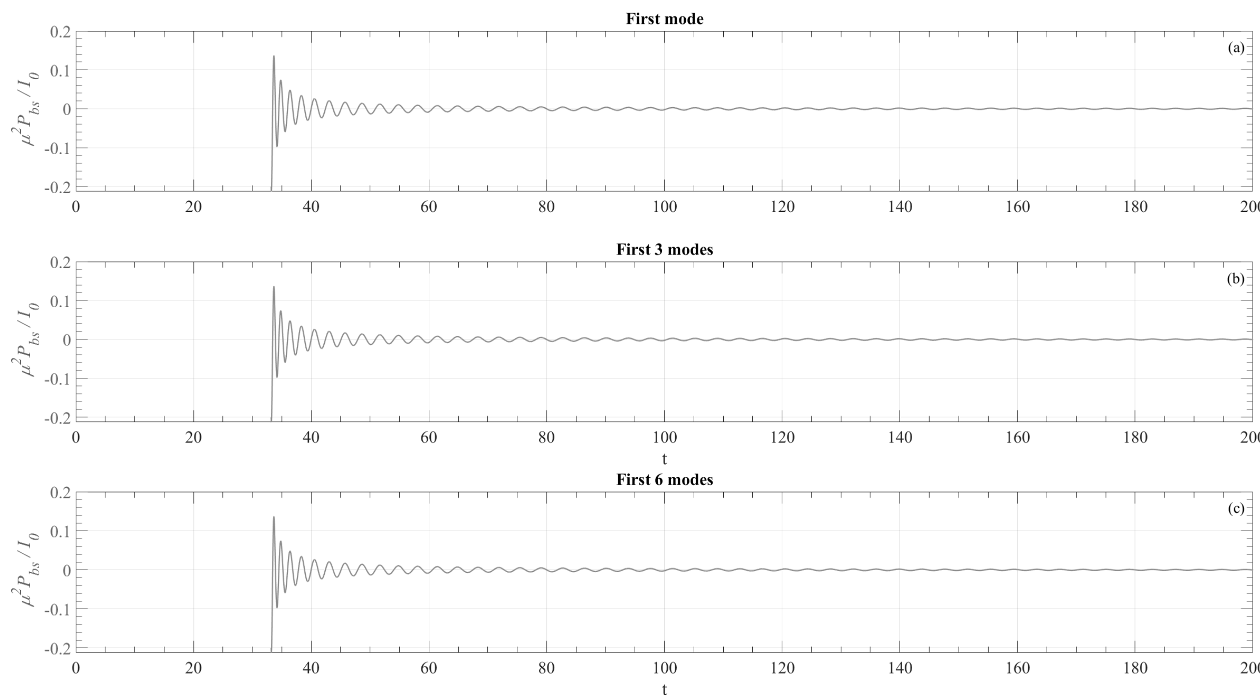

Figure 5.6: Comparison of the bottom pressure due different acoustic modes between 0 < t < 200 at a location x = 500 with $\sigma = 1$ and $\epsilon = 2$ at $x = 15000$. (a) First mode, (b) First 3 modes, and (c) First 6 modes.

Results show that the arrival time of the acoustic bottom pressures are shorter than the gravity signal for locations when a value of $\sigma = 1$. Figure 5.7 and Figure 5.8 expose the results for locations at $x = 200, 500$ and 1000 and also at $x = 10000, 15000$ and 20000, respectively. In the same way that the previous simulations the values of the bottom pressure for the acoustic-gravity wave are higher than the gravity wave.

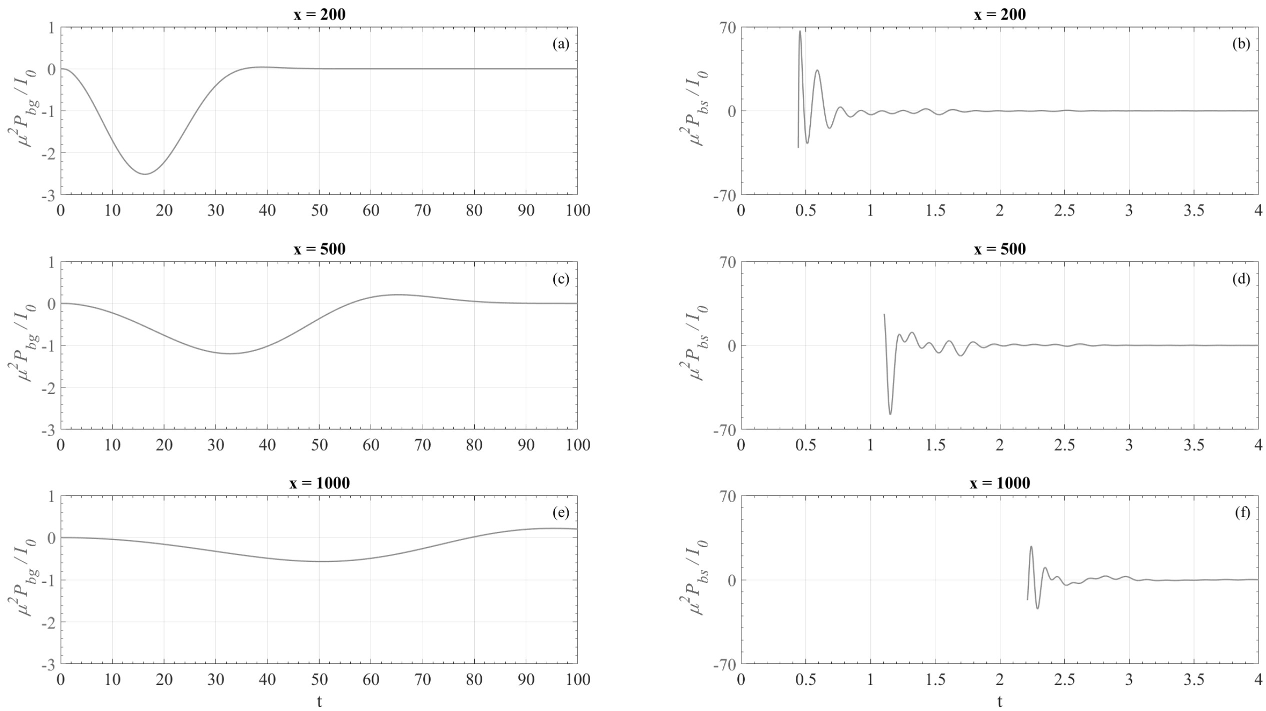

Figure 5.7: Gravity and acoustic-gravity bottom pressures at different locations with $\sigma = 1$ and $\epsilon = 2$.

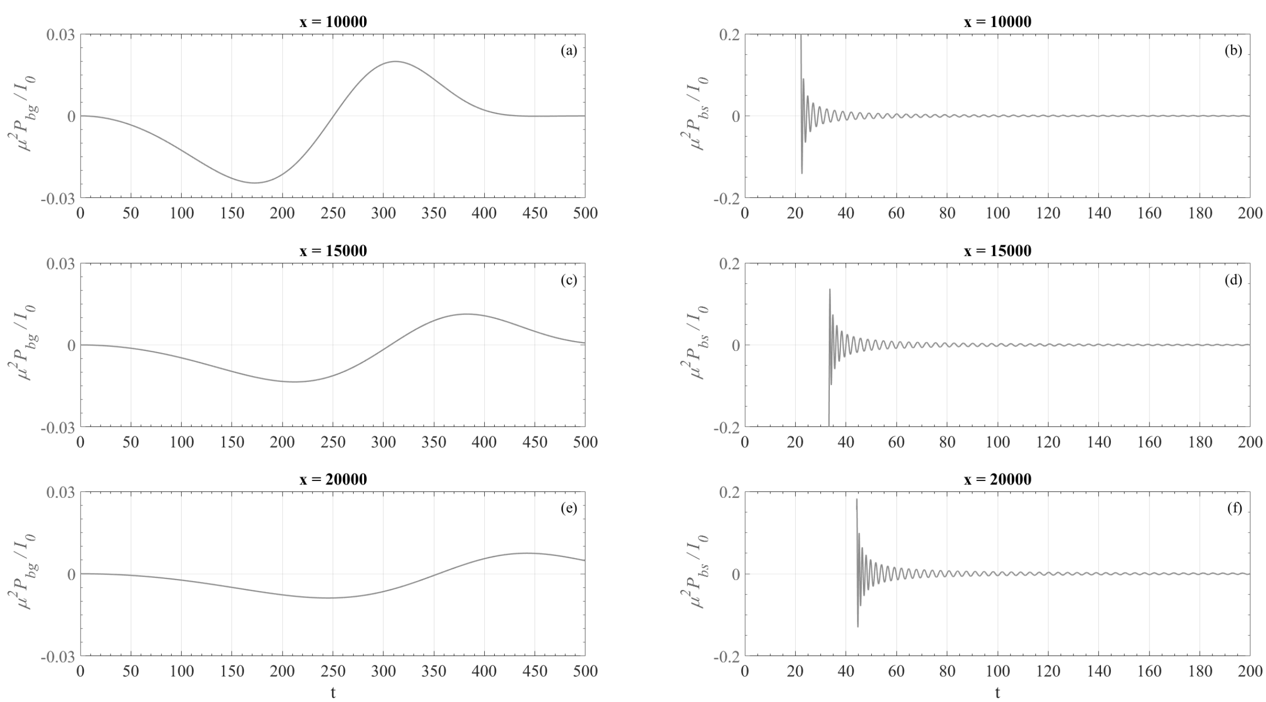

Figure 5.8: Gravity and acoustic-gravity bottom pressures at different locations with $\sigma = 1$ and $\epsilon = 2$.

In order to assess the behavior of the bottom pressure signal due acoustic and acoustic-gravity wave for different values of ϵ, σ and x a sensitivity analysis has been carried out. The following figures expose maximum bottom pressures due to gravity and acoustic-gravity waves for different values of Σ, ϵ and different locations x. The values of the parameters considered in the analysis are shown in each subplot. In Figures 5.9 and 5.10 it can be noted that for different increasing values of Σ (subplots a and b) and ϵ (subplots c and d), the maximum bottom pressure tends to decay at a specific location of $x = 1000$ and $x = 20000$. When $\Sigma = 1 = 459$ and $\epsilon = 2$, the maximum bottom pressure due gravity wave and acoustic-gravity decays with an increasing distance as it is shown in subplots (e) and (f). As a generalized result for the simulated conditions, bottom pressure due acoustic-gravity waves is always higher than gravity's.

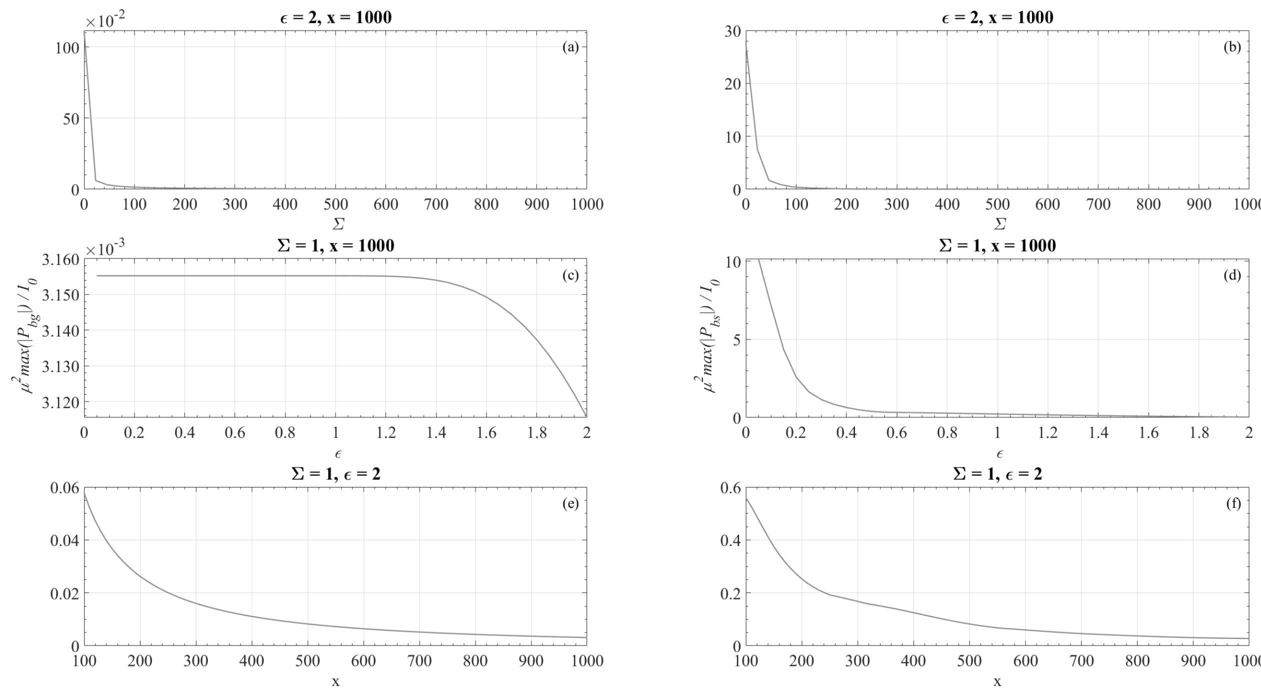

Figure 5.9: Gravity and acoustic-gravity maximum bottom pressures due different parameters Σ, ϵ and locations x. (Top) Results due different values of Σ (x = 1000, $\epsilon = 2$). (Middle) Results due different values of ϵ (x = 1000, $\Sigma = 1$). (Bottom) Results due different values of x ($\Sigma = 1$, $\epsilon = 2$)

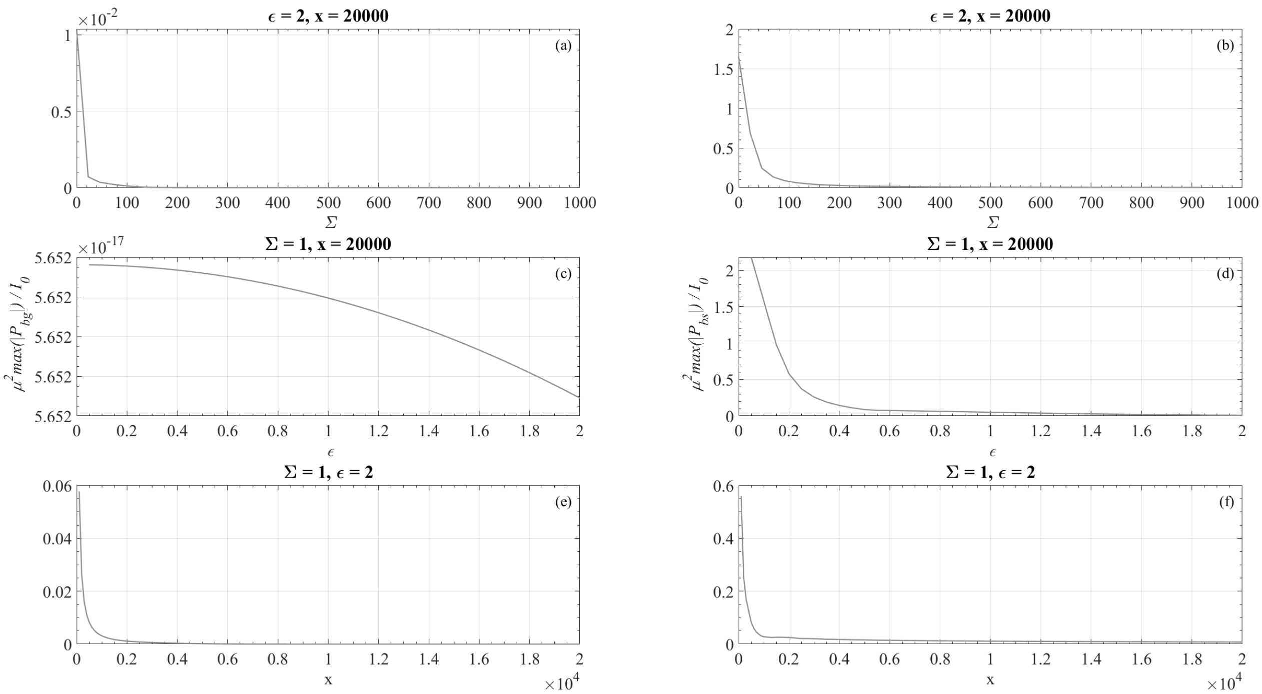

Figure 5.10: Gravity and acoustic-gravity maximum bottom pressures due different parameters Σ, ϵ and locations x. (Top) Results due different values of Σ (x = 20000, $\epsilon = 2$). (Middle) Results due different values of ϵ (x = 20000, $\Sigma = 1$). (Bottom) Results due different values of x ($\Sigma = 1$, $\epsilon = 2$)

In the same way, Figure 5.11 and Figure 5.12 show signals considering varying values of σ and ϵ in the same locations at $x = 1000$ and $x = 20000$. It can be noted that similarly to the previous simulated cases the bottom pressure decays. Subplots (a), (b), (c) and (d) show this behavior for the signals. Subplots (e) and (f) show the gravity and acoustic-gravity signals when parameters $\sigma = 1$ and $\epsilon = 2$. Here, the acoustic-gravity signal shows variability in lengths between 100 and 1000 and a specific decaying can not appreciated. However, for further distances, the signal due acoustic-gravity wave shows a decay, even when a variation in the signal is present. It is due to mention that in these cases the magnitude of the bottom pressure due to the acoustic wave is higher than gravity's similarly to the other simulated cases.

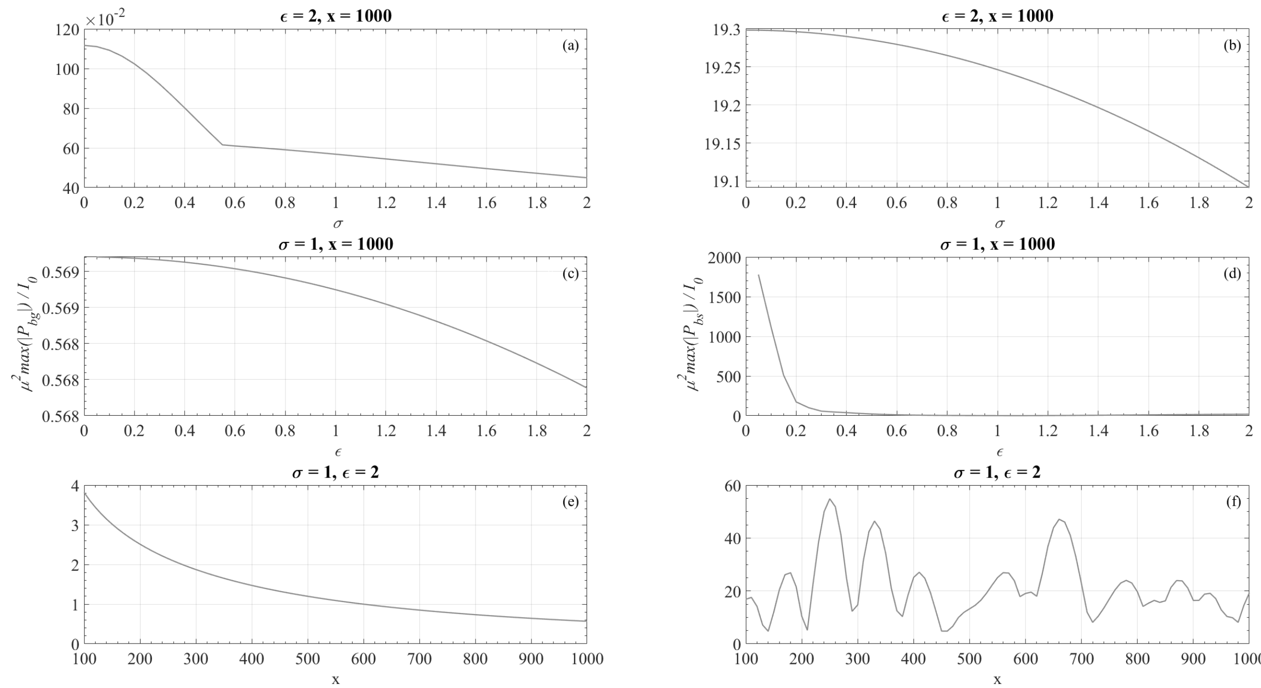

Figure 5.11: Gravity and acoustic-gravity maximum bottom pressures due different parameters σ, ϵ and locations x. (Top) Results due different values of σ (x = 1000, $\epsilon = 2$). (Middle) Results due different values of ϵ (x = 1000, $\sigma = 1$). (Bottom) Results due different values of x ($\sigma = 1$, $\epsilon = 2$)

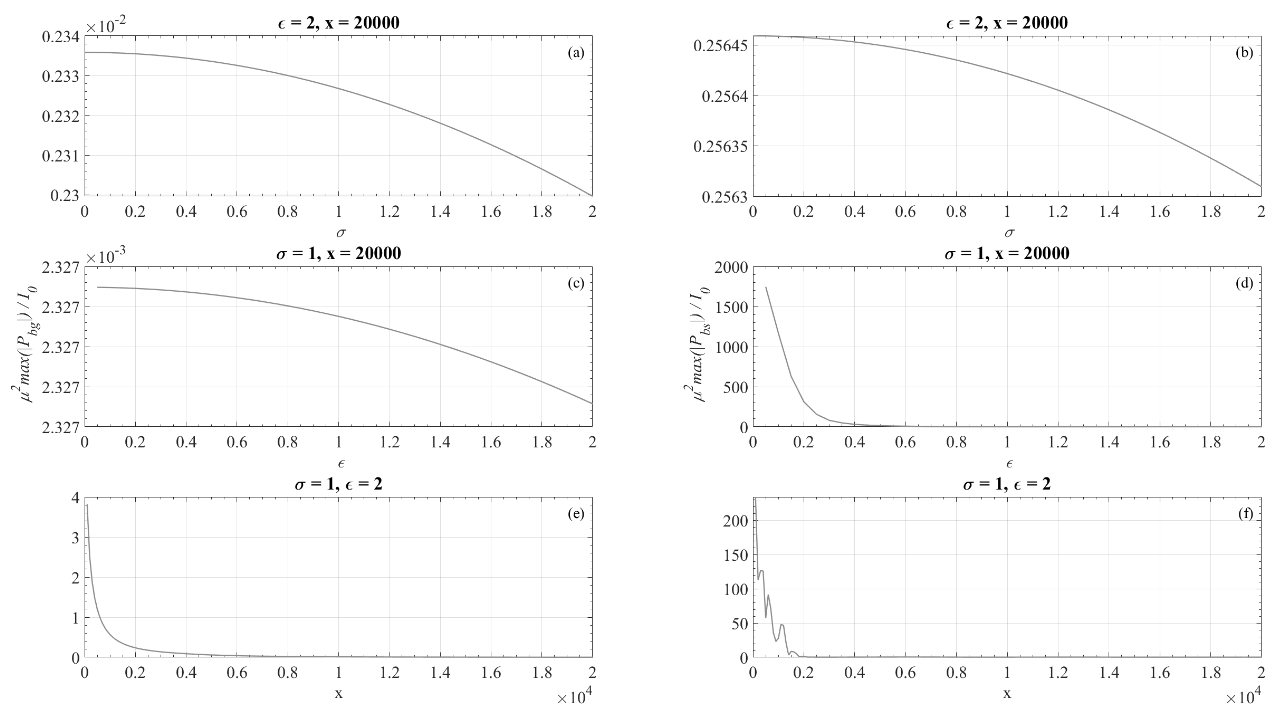

Figure 5.12: Gravity and acoustic-gravity maximum bottom pressures due different parameters σ, ϵ and locations x. (Top) Results due different values of σ (x = 20000, $\epsilon = 2$). (Middle) Results due different values of ϵ (x = 20000, $\sigma = 1$). (Bottom) Results due different values of x ($\sigma = 1$, $\epsilon = 2$)

5.3 Generation time and location

In this section, an inverse calculus has been applied using the arrival of the acoustic-gravity wave signal at different specific locations in order to calculate the distance in what the gravity wave was generated. To try this and for simplicity, we consider only the first mode $n = 0$, similar to the considered in Mei & Kadri (2018), and denote quantities obtained from the recordings, assuming a recording device at some specific locations (x) listed in Table 5.1. The frequencies at the specific locations are estimated using the following equation,

$$\hat{\psi}_{\hat{t}_j} = \frac{\pi}{2\sqrt{1 - \left(\frac{X}{\hat{t}_j - t_i}\right)^2}} \qquad j, i = 1, 2, ... \tag{5.6}$$

what represents the measured frequency at the selected instant $\hat{t}_j$ in the acoustic signal, where $j, i = 1, 2, \ldots$, with $j \neq i$. Thus, the equation 4.50 can be rewritten in terms of the measured quantities as

$$X_0 = \frac{\hat{t}_2 - \hat{t}_1}{\left\{1 - \left[\pi/(2\hat{\psi}_{\hat{t}_2})\right]^2\right\}^{-1/2} - \left\{1 - \left[\pi/(2\hat{\psi}_{\hat{t}_1})\right]^2\right\}^{-1/2}} \tag{5.7}$$

which allows to estimate the inverse location X_0. Table 5.1 shows the input locations where the signal frequencies were measured. Here can be noted that after the inverse calculation there is no difference between the modified location X and X_0, which represents a high accuracy of the implemented model and method.

Input location, x	**Modified location, $X = \mu x$**	**Calculated location (inverse) - X_0**
100	0.218	0.218
200	0.436	0.436
500	1.090	1.090
1000	2.180	2.180
2000	4.360	4.360
5000	10.900	10.900
10000	21.800	21.800
15000	32.700	32.700
20000	43.600	43.600
50000	109.000	109.000

Table 5.1: Inverse calculation of the distance X_0

The results allows to infer that if only the measurement of the acoustic signal exists, it would be possible to know the location of the gravity wave source or the existing distance between the record device and the gravity wave.

Chapter 6

Conclusions and future work

The aim of the present work has been to develop theoretically and numerically the generation and propagation of acoustic-gravity waves (AGWs) because of the generation of an extreme wave assuming compressible fluid, with the aim of implement the methodology as an early detection of extreme wave events due to sound waves in the water travel faster than gravity waves. A pressure change in the ocean surface has been considered as an acoustic-gravity wave percusor. Furthermore, as part of the analysis, a multi scaling approach was applied to the generation of the acoustic-gravity wave due to a pressure change on the ocean surface. Also, the stationary phase approximation was implemented to analyze and solve the rapidly varying solutions for the gravity and acoustic-gravity waves in long distances (Kadri, 2017). The solutions of this work were validated against results from another authors such as Renzi & Dias (2014), theoretically and numerically. A numerical example was performed in order to assess the method, it corresponds to simplified 2D simulations what was developed considering an arbitrary Double-Gaussian pressure on the ocean surface as an AGW precursor. Different parameter values for the scale and time parameters were tested in order to asses the behavior of both, gravity and acoustic gravity waves. In general, the magnitude of the bottom pressures due acoustic-gravity waves results higher than the associated to the gravity wave regardless the distance. Also, results for the simulations present that the acoustic signals arrive first to the pressure associated to the gravity wave in a specific fixed point in the far field when the scale parameter σ is 1 or lower. Nevertheless, the results show that when a high value of $sigma$ is considered, the arrival time of the acoustic signal is higher only until a distance of $x = 1000$. From the arrival time of the acoustic-gravity wave at different locations, an inverse estimation of gravity wave generation distance was implemented. The results show good results for the inverse calculation, due there is no difference between the input location and the inverse calculated location. This implies a high precision of the proposed mathematical model and methodology. One future work is to extended this methodology to 3D simulations numerically. Also, to perform physical measurements in order to compare simulations with laboratory experiments would be very helpful to asses how robust and precise could the method faced to physical measurements.

It is concluded that the development of the theory and also the numerical experiments in this work have been carried our successfully, due acoustic-gravity and gravity waves shows the strong relationship between them when a compressible fluid is considered.

Chapter 7

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