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Strategy-proof and group strategy-proof stable mechanisms: An equivalence

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We prove that group strategy-proofness and strategy-proofness are equivalent requirements on stable mechanisms in priority-based resource allocation problems with multi-unit demand. The result extends to the model with contracts.

Key words multi-unit demand, stability, strategy-proofness, group strategy-proofness

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1 Introduction

We study the compatibility of efficiency, stability, and group strategy-proofness in priority-based resource allocation problems with multi-unit demand, for example, situations when students with time constraints want to sign up for multiple courses with capacity constraints. However, in this problem, efficiency, stability, and group strategy-proofness are not compatible when we consider the general domain of priorities (see Roth and Sotomayor 1990).

Kojima (2013) shows that the existence of a stable and strategy-proof mechanism is equivalent to the existence of an efficient and stable mechanism. He characterizes those priority structures that allow stable and strategy-proof mechanisms as those that satisfy a condition called essential homogeneity. If essential homogeneity is satisfied, courses can have different priorities only for top-ranked students. If some courses have few seats available, essential homogeneity is required. For example, if one of the courses has only one seat available, essential homogeneity amounts to all courses having the same priorities. On the other hand, if each course has a large supply of seats, essential homogeneity has greater flexibility.

However, strategy-proofness does not prevent manipulation by coalitions of agents. This is a relevant concern when the coalitions that can be formed are small and easy to coordinate.[1] Group

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1 Consider, for example, the problem of assigning landing slots (see Schummer and Vohra 2013; Schummer and Abizada 2017).

strategy-proofness prevents this danger and guarantees efficiency, since it prevents profitable deviations from the grand coalition. Thus, we explore the possibility of designing mechanisms that are stable and group strategy-proof.

We exploit the characterization of essential homogeneity in terms of a serial dictatorship provided by Kojima (2013) to prove that group strategy-proofness and strategy-proofness are equivalent requirements when imposed on stable mechanisms (see also Barber *et al.* 2016). Thus, the essential homogeneity of a priority system characterizes both requirements and restricts our attention to serial dictatorships as implementing mechanisms. This equivalence is a surprising, albeit simple, result, since group strategy-proofness is more requiring than strategy-proofness. In particular, in the school assignment model (which is a many-to-one resource allocation model), the student-optimal stable mechanism always provides a stable and strategy-proof assignment (see Roth and Sotomayor 1990). However, efficiency and group strategy-proofness require priorities to satisfy an acyclicity condition (see Ergin 2002), which is similar but less restrictive than essential homogeneity. In the school assignment problem, Ergin (2002) proves that the existence of a stable and group strategy-proof mechanism is equivalent to the existence of a stable and efficient mechanism. Our characterization extends Ergin's results to the course assignment problem and contributes to explaining the restrictiveness of imposing strategy-proofness on stable mechanisms.[2] We also observe that our result extends to the multi-unit assignment model with contracts, thus completing the results by Pakzad-Hurson (2014).

Finally, we investigate whether restricting the demand of the students leads to results that are more permissive. We prove that it is not the case: even if a student can apply to at most two courses, essential homogeneity is still a necessary and sufficient condition for the existence of a stable and strategy-proof mechanism.

The paper is organized as follows. Section 2 presents the model. Section 3 presents the results. Section 4 concludes. The Proofs are in the Appendix.

2 The model

There are a finite set of courses C and a finite set of students S, with $S \cap C = \emptyset$. The model is characterized by a priority structure, which is the order by which students are given priority over courses. Formally, each course $c \in C$ has a priority, $\succ_c$, which is a strict, complete, and transitive binary relation over S. Each $c \in C$ has a supply of q_c, which is the maximum number of students who can enroll in c. A priority structure is a pair $(\succ, q_C)$, where $\succ = (\succ_c)_{c \in C}$ and $q_C = (q_c)_{c \in C}$.

Each student $s \in S$ has a strict preference relation P_s over the set of subsets of C. We assume that the preference relation of each student is responsive (see Roth 1985), with demand q_s. Formally, we assume that for each $C' \subseteq C$ and for all $c, c' \in C \setminus C'$, the following hold: (i) if $\left|C'\right| < q_s$, then $\mathrm{C'} \cup \{\mathrm{c}\}\, \mathrm{P_s} \mathrm{C'} \cup \left\{\mathrm{c'}\right\}$ if and only if $\{\mathrm{c}\}\, \mathrm{P_s} \left\{\mathrm{c'}\right\}$; (ii) if $\left|C'\right| < q_s$, then $C' \cup \{c\}\, P_s C'$ if and only if $\{c\}\, P_s \emptyset$; (iii) if $\left|C'\right| > q_s$, then $\emptyset P_s C'$. The set of all responsive preferences is denoted by $\mathcal{P}$. For the preferences of students on individual courses we use the notation $P_s : c_1, c_2, ..., c_h$ meaning that $\{c_i\}\, P_s \left\{c_j\right\}$ for $i < j$ and $\{c_h\}\, P_s \emptyset$. For each $S' \subseteq S$, set $P_{S'} = (P_s)_{s \in S'} \in \mathcal{P}^{\left|S'\right|}$. For each $s \in S$ set $P_{-s} = P_{S \setminus \{s\}}$.

A matching is a function $\mu : S \cup C \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$, $|\mu(c)| \leq q_c$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all matchings is denoted

[2] Romero-Medina and Triossi (2018) study a similar problem in a two-sided market and prove that a stronger acyclicity condition is necessary and sufficient for the existence of a mechanism that is stable and strategy-proof for the agents on both sides of the market.

by $\mathcal{M}$. Matching μ is Pareto efficient if there is no matching μ' such that $\mu'(s)\, R_s \mu(s)$ for each $s \in S$ and $\mu'(s)\, P_s \mu(s)$ for at least one $s \in S$.

Matching μ is blocked by a pair $(s, c) \in S \times C$ if $s \notin \mu(c)$ and the following two conditions are satisfied: (1) $c P_s \emptyset$ and $|\mu(s)| < q_s$, or $c P_s c'$ for some $c' \in \mu(s)$; and (2) $|\mu(c)| < q_c$, or there exists $s' \in \mu(c)$ such that $s \succ_c s'$. Matching μ is individually rational if, for each $s \in S$ and each $c \in \mu(s)$, $c P_s \emptyset$. Finally, a matching μ is (pairwise) stable for $(S, C, P, \succ, q_C)$ if it is individually rational and there exists no pair blocking it.

A mechanism is a function $\varphi : \mathcal{P}^{|S|} \to \mathcal{M}$. It is efficient if $\varphi(P)$ is a Pareto efficient matching for each $P \in \mathcal{P}^{|S|}$. It is stable if $\varphi(P)$ is a stable matching for each $P \in \mathcal{P}^{|S|}$. It is strategy-proof if $\varphi(P)\, R_s \varphi\left(P'_s, P_{-s}\right)$ for each $P \in \mathcal{P}^{|S|}$, $s \in S$ and $P'_s \in \mathcal{P}$. It is group strategy-proof if there do not exist $S' \subseteq S$ and $P'_{S'} \in \mathcal{P}^{|S'|}$ such that $\varphi\left(P'_{S'}, P_{-S'}\right) R_s \varphi(P)$ for each $s \in S'$ and $\varphi\left(P'_{S'}, P_{-S'}\right) P_s \varphi(P)$ for at least one $s \in S'$. If each agent has responsive preferences there exists a stable mechanism that is Pareto superior to all other stable mechanisms, which is called student-optimal stable mechanism and is denoted by $\mu^S(P)$.

The priority structure $(\succ, q_C)$ satisfies essential homogeneity if there exist no $a, b \in C$ and $t, u \in S$ such that: (i) $t \succ_a u$ and $u \succ_b t$; (ii) there exist $S_a, S_b \subseteq S \setminus \{t, u\}$ such that $|S_a| = q_a - 1$, $|S_b| = q_b - 1$ and $s \succ_a u$ for each $s \in S_a$, $s \succ_b t$ for each $s \in S_b$. Theorem 1 in Kojima (2013) proves that the essential homogeneity of a priority structure is equivalent to the existence of a stable and efficient mechanism and to the existence of a stable and strategy-proof mechanism.

3 Results

First, we prove that the student-optimal stable mechanism is group strategy-proof if the priority structure satisfies essential homogeneity. Theorem 3 in Kojima (2013) shows that if a priority structure satisfies essential homogeneity, the student-optimal stable mechanism can be obtained as a serial dictatorship, where the students choose in the order determined by the priorities of any course of minimal capacity. Thus, the result follows from the fact that any serial dictatorship is group strategy-proof. For completeness, we include a proof of this result in the appendix.[3]

Lemma 1 *Assume that the priority structure $(\succ, q_C)$ satisfies essential homogeneity. Then the student-optimal stable mechanism is group strategy-proof.*

Integrating Lemma 1 and Theorem 1 in Kojima (2013) we obtain.

Theorem 1 *Let $(\succ, q_C)$ be a priority structure. A stable mechanism is strategy-proof if and only if it is group strategy-proof.*

Obviously, Theorem 1 also implies that the existence of a group strategy-proof stable mechanism is equivalent to the priority structure being essentially homogeneous. Kojima (2013) also provides a characterization of essential homogeneity in terms of a generalized version of the consistency property

[3] The literature usually cites Bird (1984), Elhers and Klaus (2003), or Svensson (1999) as a reference for the result. However, none of the papers provide a proof. Bird (1984) only studies the Top Trading Cycles mechanism. Ehlers and Klaus (2003, theorem 1, p. 271) refers to the result as "straightforward". Svensson (1999, p. 557), who studies a house allocation problem, refers to Bird (1984) for coalitional results and proves that "A strategy-proof, non-bossy and neutral mechanism f is serially dictatorial (theorem 1, p. 562)" but not the reverse.

(see Thomson 2018). Ergin (2002), employing the same argument as Pápai (2000, lemma 1) proves the equivalence between acyclicity and the consistency of a priority structure. An alternative but more complex proof of Theorem 1 would consist of proving directly that the generalized consistency of a priority structure implies that the student-optimal stable rule is group strategy-proof in course assignment problems.

Pakzad-Hurson (2014) extends the results by Ergin (2002) and Kojima (2013) to priority-based resource allocation problems with contracts. He adapts the definition of essential homogeneity to this setup and introduces a student-lexicographic condition on the priorities of the courses. He proves that a stable and strategy-proof mechanism exists if and only if the priority structure satisfies essential homogeneity and the student-lexicographic condition. The same argument used in the proof of theorem 3 in Kojima (2013) implies that if the priority system satisfies essential homogeneity and the student-lexicographic condition, the student optimal stable matching can be obtained through a serial dictatorship. Thus, the result we obtained for the model without contracts extends directly to the model with contracts.

Allowing for multi-unit demand makes strategy-proofness a required condition in assignment models with priorities. The reader might wonder whether this is a consequence of the fact that the designer must consider any possible demand of the students. This assumption may appear too restrictive since in real world applications students can enroll in a limited number of courses. We prove that this is not the case: if students can enroll in at most two courses, essential homogeneity is still a necessary requirement for the existence of a strategy-proof mechanism.

Proposition 1 *Assume that $(\succ, q_C)$ is not essentially homogeneous. Then there exists $P \in \mathcal{P}^{|S|}$ such that $q_s \leq 2$ for each $s \in S$ and $P'_t \in \mathcal{P}$ with $q'_t \leq 2$, for some $t \in S$ such that, for any stable mechanism, φ, $\varphi\left(P'_t, P_{-t}\right)(t)\; P_t \varphi(P)(t)$.*

The intuition behind Proposition 1 can be explained through a simple example. Let us assume that there are only two students, t and u and two courses, a and b, each with one vacant seat. Suppose $(\succ, q_C)$ is not essentially homogeneous. Let $a, b \in C$, and $t, u \in S$ as in the definition of essential homogeneity and assume that $P_t : b, a, q_t = 2$, $P_u : a, b, q_u = 1$. There is a unique stable matching μ, where $\mu(t) = a$ and $\mu(u) = b$. In this situation, student t competes with student u for both courses, losing her favorite course b. However, if she reveals preferences $P'_t : b$, she no longer competes for course b since student u's favorite course is a. Indeed, this deviation is profitable for t when any stable mechanism is employed because she obtains course b.

4 Conclusions

In this paper, we show the equivalence of imposing group strategy-proofness and strategy-proofness on stable mechanisms when studying the allocation of indivisible goods to a set of agents with multi-unit demand. Essential homogeneity in the priority structure is necessary and sufficient for the existence of such mechanisms. We also find that it is not possible to relax the characterization by imposing caps on agents' demands. In addition, our results extend to the model with contracts. Future research looking for positive results on a larger set of priority structures should move in a different direction. For instance, it could explore non-revelation mechanisms, relaxing the equilibrium concept. An alternative to this approach is to reduce the stability requirement, when looking for strategy-proof mechanisms.

Appendix A

Proof of Lemma 1

Let c be a course of minimal capacity, which is let c such that $q_c = \min \left\{ q_{c'} \mid c' \in C \right\}$. For each $l = 1, 2, ..., |S|$, let $s_l \in S$ be the l-th ranked student according to $\succ_c$, formally $s = s_l$ if and only if $\left| \left\{ s' \in S \mid s' \succ_c s \right\} \right| = l - 1$. For each $P \in \mathcal{P}^{|S|}$, for each $s \in S$ and each $A \in 2^C$, let $Ch_s(A)$ be student s favorite subset of A, formally $Ch_s(A) = \max_{P_s} \{B \mid B \subseteq A\}$. For each l, $1 \leq l \leq |S| - 1$ define $A_l(P)$ recursively as follows: $A_1(P) = C$ and $A_{l+1}(P) = \left\{ c' \in C \mid \bigcup_{l' \leq l, c \in Ch_{s_{l'}}(A_{l'}(P))} \{s_{l'}\} < q_{c'} \right\}$. Finally, define a serial dictatorship $\mu(P)$ as $\mu(P)(s_l) = Ch_{s_l}(A_l(P))$ for all l, $1 \leq l \leq |S|$. For each $c \in C$, set $\mu(P)(c) = \bigcup_{c \in \mu(P)(s_l)} \{s_l\}$. At her turn, in the order $s_1, s_2, ..., s_l$, each student chooses her favorite set of courses that still has vacant seats.

Since $(\succ, q_C)$ satisfies essential homogeneity, theorem 3 in Kojima (2013) implies $\mu(P) = \mu^S(P)$ for each $P \in \mathcal{P}^{|S|}$. Thus, in order to complete the proof of the claim, it suffices to show that mechanism $\mu(P)$ is group strategy-proof. The proof is by contradiction. Assume that there exists a nonempty set of agents $S' \subset S$, P and $P'_{S'} = \left(P'_s\right)_{s \in S'}$ such that $\mu(P'_{S'}, P_{S \setminus S'})(s)\, R_s \mu(P)(s)$ for each $s \in S'$ and $\mu(P'_{S'}, P_{S \setminus S'})\left(s'\right) P_{s'} \mu(P)\left(s'\right)$ for some $s' \in S'$.

Let $l = \min \left\{ i \mid \mu(P'_{S'}, P_{S \setminus S'})(s_i) \neq \mu(P)(s_i) \right\}$. For each $i < l$, $\mu(P'_{S'}, P_{S \setminus S'})(s_i) = \mu(P)(s_i)$, then $A_l\left(P'_{S'}, P_{S \setminus S'}\right) = A_l(P)$. First assume $s_l \notin S'$. In this case $\mu\left(P'_{S'}, P_{S \setminus S'}\right)(s_l) = \mu(P)(s_l)$, which yields a contradiction. Next assume $s_l \in S'$, then $\mu(P'_{S'}, P_{S \setminus S'})(s_l)\, P_{s_l} \mu(P)(s_l)$. Since $A_l\left(P'_{S'}, P_{S \setminus S'}\right) = A_l(P)$, $\mu(P)(s_l)\, R_{s_l} \mu(P'_{S'}, P_{S \setminus S'})(s_l)$, which yields a contradiction.

Proof of Theorem 1

By definition, a group strategy-proof mechanism is always strategy-proof. In order to complete the proof of the claim we next prove that a stable and strategy-proof mechanism is also group strategy-proof. By theorem 1 in Kojima 2013 if a stable and strategy-proof mechanism exists, the priority structure $(\succ, q_C)$ is essentially homogeneous and the student-optimal stable mechanism is the unique strategy-proof mechanism. From Lemma 1 the student-optimal stable mechanism is group strategy-proof, which completes the proof of the claim.

Proof of Proposition 1

We use the same argument as in example 1 in Kojima (2013). Assume that $(\succ, q_C)$ is not essentially homogeneous. Let $a, b \in C$ and $t, u \in S$ as in the definition of essential homogeneity. Let P_t and P_u such that $P_t : b, a$ and $P_u : a, b$. Let $q_t = 2$ and $q_u = 1$. Let P'_t be such that $P'_t : b$. For each $s \in S_a \setminus S_b$, let $P_s : a, b$ and $q_s = 1$. For each $s \in S_b \setminus S_a$, let $P_s : b, a$ and $q_s = 1$. For each $s \in S_b \cap S_a$, set $P_s : a, b$ and $q_s = 2$. For each $s \in S \setminus (S_a \cup S_c \cup \{t, u\})$, let P_s be such that $\emptyset P_s a$, $\emptyset P_s b$. Then, in any stable mechanism φ, $\varphi(P)(t) = \{a\}$ and $\varphi\left(P'_t, P_{-t}\right)(t) = \{b\}$, which completes the proof of the claim.

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