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STOCHASTIC OPTIMIZATION APPLIED TO SCHEDULING MINING PROBLEMS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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Optimización Estocástica Aplicada a la Planificación Minera

En esta tesis, abordaremos el Problema de la Planificación de la Producción en una Mina Subterránea para una mina explotada mediante el método de hundimientos de bloques y/o paneles, a través de la formulación de un programa de optimización lineal estocástica multietapa. El objetivo es acrecentar la información considerada por los modelos existentes incorporando el riesgo sísmico, además de los tradicionalmente considerados como la incertidumbre financiera y geológica.

La formulación de un modelo que considere múltiples fuentes de incertidumbre implica la consideración de muchísimas variables que tornan imposible la resolución directa de este mediante métodos convecionales. Aún recurriendo a la clusterización de bloques, como se hace en este trabajo, el problema de optimización formulado continúa intratable. Para abordar la resolución del modelo propuesto utilizamos el bien conocido principio 'Divide y vencerás', más específicamente se echa mano a la Relajación Lagrangiana de restricciones que complican el problema. En este trabajo se relajan las restricciones de no anticipatividad, que relacionan los escenarios, lo que nos permite dividir el problema maestro en tantos subproblemas como escenarios, y se propone una metodología para obtener soluciones factibles para el problema maestro empleando las soluciones obtenidas para los subproblemas.

Se estudia la relación de la producción y la actividad sísmica, estableciendo un modelo empírico simple, que nos ayuda a entender y a estimar la intensidad de la micro sismicidad inducida por la minería. Este análisis nos posibilita estudiar y cuantificar los beneficios del fracturamiento de rocas, mediante explosivos y preacondicionamiento hidráulico. Se relaciona el preacondicionamiento y las tasas de extracción y el impacto en el Valor Presente Neto.

Para la realización de esta tesis, se consideraron datos reales proporcionados por la mina Codelco Chile División El Teniente, la mina de cobre más grande del mundo. Esta mina es explotada mediante hundimiento de bloques y paneles.

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STOCHASTIC OPTIMIZATION APPLIED TO SCHEDULING MINING PROBLEMS

In this PhD thesis we consider the Underground Mine Production Scheduling Problem for a mine exploited by Block-Panel caving by formulating it as a novel multistage stochastic linear program that includes more information than traditional models, which usually consider just economic and geologic uncertainty, by incorporating the seismic hazard.

The construction of a model considering many sources of uncertainty would lead us to a model with thousands of variables which is too big to solve by direct methods. Even using clusterization of blocks, as we do in this work, the model still remains intractable. To tackle this issue we combine the well known Lagrangian Relaxation over the Non Anticipative Constraints and a novel heuristic to construct a feasible solution from the solutions of the subproblems generated by the relaxation, which is based on a combination of the static and dynamic worst case approach.

In this work we study the relation between production and the micro-seismicity induced by mining, by proposing a simple empirical stochastic model, a Geometric Brownian Motion, for a seismic parameter which allows us to understand and predict the micro-seismicity. This model described satisfactorily even the case of preconditioning fracturing either by blasting or hydro-fracturing. The information given by the model for the seismic parameter is introduced in the stochastic optimization problem to quantify the economic and operational benefits from using preconditioning fracturing through the impact on the extraction rock rates and the Net Present Value.

In this thesis, we test our model on real data from El Teniente mine, the biggest underground copper mine in the world. El Teniente is mined by block and panel caving.

Contents

Chapter 1

Introduction

In this chapter, we introduce the problems that this work is interested to deal with. All of them are related to **mine planning** which will be explained in the first section of this chapter. In the next section we present the previous works in this area. After that the scope of our research will be exposed. Finally, the last section describes the main contributions and the structure of our work.

1.1 What is mine planning?

Mine planning is a research area of Operations Research that studies how the activities and strategic decisions in a mine have to be scheduled in order to optimize the benefits of the mine project along of a time horizon. Some activities and decisions in a mine are: to determine the initial investment and the capacity of the mine, to decide the number of rock tons to be extracted and the speed of the extraction in the time horizon, to select the rock that will be processed, to handle the waste and the residues produced, etc. The main challenge of mine planning is to provide mathematical and statistical models that capture the complexity and the risks inherent in the mining business to provide realistic estimates of the expected benefits to the investors. Examples of risks that the planner must consider are: uncertainty in the ore deposit and prices, the occurrence of accidents and strikes, tremors induced by mining, environmental problems, etc. The works developed up to now mostly consider the uncertainties and risks associated to metal prices and geological models and a few of them deal with production incidents such as accidents, strikes, tremors, etc. (Armstrong, 2012 [6]; Flores, 2015 [30] and this thesis).

Basically there are two types of mines: the open pit and the undergrounds mines. Up

to now, most works about mine planning have addressed the case of an open pit mine (see e.g., [4, 5, 7, 13, 14, 15, 24, 26, 27, 28, 41]); much less work has been done on underground mine planning (Lizotte and Elbrond, 1985 [48]; Chatterjee and Sridhar, 1986 [21]; Aguayo et al., 2012 [2]; Pourrahimian, 2013 [54]; Alonso-Aluyo et al., 2014 [3]). We devote Chapter 3 to the underground mining problem.

Currently, the planners use mainly three approaches to determine decisions and to estimate likely profits in a mining project (Tulcanaza, [60]). The first one is the determination of optimum production capacity; the second is the models that optimize the mining process in terms of the available ore reserves and the third is the stochastic mine planning.

The determination of the optimal production capacity needs mainly the following inputs: a geological model which defines the reserves, information about infrastructure and its limitations, market data, that is, investments, prices and costs, the operation capacity, etc. The results obtained through this approach are: the ultimate pit, the grade tonnage curve, the mine life, the cash flow of the project, etc.

Planning based on optimizing the process in terms of available ore reserves consists in extracting the rock with higher ore grade and lower processing cost, and deferring the extraction of rock with lower economic value until later. Lane formalized this approach by introducing the opportunity cost in [46], [49]. It focuses on extraction sequencing of rock by ore grade rather than on determination of optimal capacity. The results of this approach, in the most cases, is the increased Net Present Value.

In general these types of models could consider perfect information, i.e., these models suppose that all information is always available along the time horizon. Clearly, this is sub-optimal because it ignores the impact of the uncertainties in the mining business. The stochastic models take into account the uncertainties in geological models, prices, markets, etc. The uncertainty could be represented by random variables and stochastic processes whose distributions are determined from historical data and observations (Dimitrakopoulos 2010, [25]). Basically, the Net Present Value is maximized in a stochastic setting subject to constraints that represents the mining capacity, processing capacity, physical limitations, etc. The uncertain data could appear in the objective function and the constraints.

Stochastic mine planning provides more reliable and realistic results than deterministic models because it considers pessimistic, average and optimistic scenarios. The main difficulty in the stochastic models is their resolution because realistic stochastic models require too many variables in order to represent the uncertainty and the models obtained are

too big to solve them in a reasonable time. Indeed, typically mining scheduling problems can be reduced to the well-known Precedence-Constrained Knapsack Problem (Kellerer et al., 2004 [44]) and this problem is known to be NP-hard (You and Yamada, 2007 [62]), which implies that the exact resolution of large cases is very time consuming (see e.g., [2, 27]). For this reason heuristic methods are applied to solve the stochastic models, such as Lagrangean Relaxation [24], Progressive Hedging [26], Aggregation methods [41], Tabu Search [27], Genetic Algorithms [15] and many others.

1.2 Previous works

A vast literature exists on open pit optimization starting out from Lerchs & Grossman's seminal paper (1965) [47] on choosing the optimal pit contour. Having determined the pit contour, mine scheduling consists of specifying the sequence in which blocks should be removed in order to maximize the total discounted profit subject to a variety of physical and economic constraints. From this work up to now, many models were developed to optimize the processes in an open pit mine:

- Deterministic models. This class of models considers the all parameters known at the beginning of the time horizon of the economic valuation for the mine. Here, we find works focusing on cut-off grades estimation: the pioneer paper of Lane (1965) [46], Asad (2005) [7] which calculates the cut-off grades taking into account the stockpiling for a mine that exploits two minerals. Nieto et al. (2006) [52], Rendu (2014) [55], where the underground case is also treated, among others. Also, there are works based on block sequencing approach such as the cited paper of Lerchs & Grossman's (1965) [47] that presents the ulimate pit problem, Caccetta et al. (2003) [14] Schultz (2008) [58] which exposes an agregetation procedure to generate clusters of blocks, Cullenbine el al. (2011) [24], Jélvez et al. (2016) [41] which formulates the optimization in terms of macroblocks and presents a criterion to construct them, among many others.

- Robust and stochastic models. As was mentioned, these models take into account uncertainties: Dimitrakopoulos et al. (2007) [28], Abdel & Dimitrakopoulus (2010) [1], Boland et al. (2008) [13] which incorporate geological scenarios as exogeneuos uncertainty, Amaya et al. (2011) [4] in which a robust optimization model is developed considering ore grade uncertainty and compared to the Value-at-Risk and Conditional

Value-at Risk, Armstrong et al. (2012) [6] where a multistage model to manage major production incidents is presented, Franco (2017) [15] where a polymetalic stochastic model with uncertain geology, prices and costs, etc is considered.

Much less work has been done on underground mine scheduling. For the deterministic setting, the first applications were in the mid-1980s (Lizotte and Elbrond, 1985 [48]; Chatterjee and Sridhar, 1986 [21]). After describing the constraints involved in both open pit and underground mining, Aguayo et al (2012) [2] show how Codelco (the Chilean state copper mining company) has solved the long-term planning problem for combined open pit/underground mines since 1999 by relaxing a tight linear formulation. Pourrahimian et al. (2013) [54] developed a deterministic multi-step for an underground mine exploited by block caving model considering cluster of blocks and they solved it for hundreds of drawpoints.

Important constraints in the models for open pit mines and underground mines are the accessibility or precedence constraints because they condition the extraction sequences that have to be satisfied. Armstrong et al. (2011) [5] modelled them through an accesibility matrix for an open pit mine. This approach could be used for underground mines after obvious modifications. In the underground case Gangawat (2014) [31] labeled each block with spatial coordinates; Alonso-Aluyo et al. (2013) [3] used sets containing the predecessors of each unit (block or cluster). An interesting way to write these constraints for underground mines was presented in Aguayo at al (2012) [2] where the mine is divided in columns which have to satisfy a roof smoothness constraint: for each column, its neighbour columns must have similar heights to it. We use this to write the precedence constraints because it reduces the number of constraints and is accurate.

As in open pit mining, the next step was to develop models that take account of the uncertainty on key variables such as the commodity price and the ore grades. A set of geostatistical simulations of the deposit was integrated into a study on long-term scheduling of a nickel deposit (Carpentier et al., 2015 [17]). The two-stage stochastic integer program model developed considers a variable cut-off grade and takes account of maximal development, material handling flow conservation, mill and mine capacity, and activity precedencies.

Carvallo (2009) [18] and Alonso-Aluyo et al. (2014) [3] studied medium-term optimization of underground copper mines under uncertainty on future copper prices for finding an optimal extraction sequence for clusters of blocks. Multistage programming is used and the copper prices are modelled as a trinomial tree. One key constraint that has to be re-

spected is the non-anticipative principle (Garstka & Wets, 1974 [32]; Rockafellar & Wets, 1991 [56]; Heitsch & Romisch, 2003 [39]). This is done by setting up an equivalent deterministic model, which they solve by compact form. One point that is not clear in their paper is how the price scenarios were generated. Multi-stage programming has also been used to manage major production incidents in a mine, that is, incidents which prevent the mine from producing for several months or more, such as roof falls, strikes, etc. (Armstrong et al., 2012 [6]). Also Flores (2015) [30] addressed seismic risk but using a static model to calculate the optimal capacity of a underground mine exploited by block caving for long term.

Most models treat exogenous uncertainty and avoid endogenous uncertainty because the probability of occurrence of scenarios then depends on the decisions made earlier and this considerably increases the number of constraints and the size of the problem. For open pit mine the paper of Boland et al. (2008) [13] presents a model including the endogenous uncertainty just in ore grades: the decisions modify the initial information about ore grades and then it changes the weight probability of the scenarios. But, to the best of our knowledge, no papers treat endogenous uncertainty by seismic hazard for underground mines as we do in this thesis.

The problem of block sequencing in a mine, generally leads to optimization problems with as many binary variables as there are blocks, at least. In real applications, we have to model mines with thousands of blocks or macro blocks and the associated optimization problem becomes an intractable mixed integer linear problem , that is too big to find an exact solution. In fact most MIP formulation for mining schedule are NP-hard problems even for the deterministic formulations (Aguayo et al., 2012 [2]). Even after drastically diminishing the number of the variables by clustering (Pereira et al., 2008 [53]) most problems remains too big. Heuristic methods are required to tackle this problem in order to obtain reasonably feasible solutions. For example Caccetta & Hill (2003) [14] applied a well known branch & bound method to find solutions for the problem of determining an optimal production schedule over the life of the deposit. Cullenbine et al. (2011) [24] used a Sliding Time Window Heuristic to solve a deterministic Open Pit Mine Scheduling Problem. Dimitrakopoulos & Lamghari (2016) [26] developed an algorithm for a stochastic two stage problem that combines two techniques, Progressive Hedging (Rockafellar & Wets, 1991 [56]) and Sliding Time Windows, thus the scenario decomposition is done by the first one and feasible solutions are obtained by the second one. Jélvez et al. (2016) [41] solved the Ultimate Problem Pit by aggregating blocks with similar characteristics (e.g. ore grade, rock density, etc.) which

considerably reduces the number of variables, but the results are less accurate. Aguayo et al. (2012) [2] obtained feasible solutions starting with the solution of the linear relaxation and fixing the variables with integer values.

1.3 Objectives of the thesis

The objectives of this work are

- To formulate a model for the Net Present Value that includes price and seismicity uncertainty.
- To model the micro seismicity induced by mining in an underground mine exploited by block caving. Additionally we aim to construct scenarios representing the seismicity uncertainty.
- To construct scenarios for metal prices supposing that they are modeled by a Geometric Brownian Motion.
- To evaluate the economic impact of hydraulic fracturing in an underground mine exploited by block caving.
- To formulate a model an underground mine exploited by block caving with selective mining.
- To evaluate the economic impact of implementing selective mining on the NPV of an underground mine exploited by block caving.
- To develop an efficient algorithm that provides feasible solution for the models formulated.
- To apply the models presented to a real case.

1.4 Contributions and structure of the thesis

The key contributions made in this thesis are:

1. We present a multistage stochastic model applied to underground mines that considers uncertainty on both copper prices and induced seismicity. It is difficult to tackle the

uncertainty due to induced seismicity because it is endogenous. In other words, this uncertainty depends on the decision variables associated with extraction. To handle that, a piece-wise linear relation is assumed between the quantity of rock extracted and the induced seismicity, and then we find a probability distribution for local proportionality constants that links mining and induced seismicity. It allows us to construct scenarios without consider dependence on the decision variables. This model is also used to evaluate the economic impact of the preconditioning hydraulic fracturing used in underground mines.

2. We have used in a splitting representation in which the non-anticipative variables are explicit in order to write our multistage optimization models.

3. We consider macro blocks composed of clusters of blocks in the same column (if the grade is more or less uniform) rather than clusters of blocks in any column as in Alonso-Aluyo et al, (2014) [3]. This gives more accurate results.

4. We use Lagrangian relaxation applied to non anticipative constraints to solve both models, taking advantage of their specific structure. The relaxed problems are solved by the Volume Algorithm and we use a dynamic worst approach to generate feasible solutions.

5. We test our models by applying it to the El Teniente mine and compare the results with and without seismicity uncertainty.

Chapter 2

Mathematical Background

In this chapter we introduce the basic concepts and facts on stochastic optimization and its applications to scheduling mining and the approaches for solving big mixed linear programs that arises from stochastic optimization.

2.1 Stochastic optimization

Traditionally, to maximize the profit in a mining project, deterministic programming is used in which all information is always available [2, 14, 24, 41, 48]. But the uncertainties present along the time horizon, such as variation of prices, occurrences of accidents, etc., suggest that the results obtained by deterministic optimization are not reliable. For this reason, stochastic optimization which considers the presence of random quantities is better suited to the mining scheduling problem. Specifically, throughout this work we will use linear multistage stochastic programming (see [12, 59]).

Linear multistage stochastic programming

In the multistage stochastic setting the information is slowly revealed over time: firstly, a decision is made for today, namely period t, based on what we know today. After that, we observe the outcome from stochastic process at time t and make a decision for the next period $t+1$ and repeat the process. Additionally, the decisions made at time t can depend on information available up to this time, but not on the results of future observations. This requirement is called non anticipative principle.

Let $T \in \mathbb{Z}^+$ be a positive integer called time horizon. We denote the decision

variables by the vectors $x = (x_1, x_2, \ldots, x_T)$ where $x_t \in \mathbb{R}^{n_t}$ is the decision for period t and $n_t \in \mathbb{Z}^+$ for all $t = 1, \ldots, T$, and denote the uncertain information that is gradually revealed by the stochastic process $\xi = (\xi_1, \xi_2, \ldots, \xi_T)$ where $\xi_t \in \mathbb{R}^{d_t}$ is the uncertain data revealed at time t and $d_t \in \mathbb{Z}^+$ for all $t = 1, \ldots, T$. We use the notation $\xi_{[t]} = (\xi_1, \ldots, \xi_t)$ to denote the history of the stochastic process ξ up to time $t = 1, \ldots, T$.

With this notation we can schematically write the decision process described above in the form

$$\text{decision } (x_1) \rightsquigarrow \text{observation } (\xi_2) \rightsquigarrow \text{decision } (x_2) \rightsquigarrow$$

$$\ldots \rightsquigarrow \text{observation } (\xi_T) \rightsquigarrow \text{decision}(x_T),$$

and the non anticipative principle can be expressed by the equation $x_t = x_t(\xi_{[t]})$, that is, the decision variable x_t can depend just on information revealed up to time t. Since $\{\xi_t\}_{t=1}^T$ is a stochastic process, x_t is a stochastic process too.

Consider the case $T = 2$. This case is called a two-stage stochastic program. Suppose that we want to maximize the functions $f_1(x_1)$ for the first period and $f_2(x_2, \xi_2)$ for the second period (they could be revenue for example). If information ξ_2 is always available, then we have $f_2(x_2, \xi_2) \equiv f_2(x_2)$ and should solve

$$\max_{x_1 \in \mathcal{X}_1} \left[f_1(x_1) + \sup_{x_2 \in \mathcal{X}_2(x_1)} f_2(x_2) \right] \tag{2.1}$$

where $\mathcal{X}_1$ is a set and $\{\mathcal{X}_2(x_1)\}_{x_1 \in \mathcal{X}_1}$ is a family of sets.

But the information ξ_2 could be available just in the second period. In this case ξ_2 is a random variable and (2.1) does not make sense, instead it would be reasonable to consider the optimization problem

$$\max_{x_1 \in \mathcal{X}_1} \left[f_1(x_1) + \mathbb{E}_{\xi_2} \left[\sup_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) \right] \right] \tag{2.2}$$

where $\mathcal{X}_1$ is a set and $\{\mathcal{X}_2(x_1, \xi_2)\}_{x_1, \xi_2}$ is a family of sets.

Now, for the general case, that is, $T \in \mathbb{Z}^+$ arbitrary, consider $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ continuous functions for $t = 2, \ldots, T$ and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$ measurable and closed valued multifunctions for $t = 2, \ldots, T$. For the first stage consider the function $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$. Following the approach for two-stage case we formulate the problem

$$\max_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\sup_{x_2 \in \mathcal{X}(x_1, \xi_2)} f_2(x_2, \xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}\left[\sup_{x_T \in \mathcal{X}(x_{T-1}, \xi_T)} f_T(x_T, \xi_T) \right] \right] \right] \tag{2.3}$$

The problem above is called a nested formulation. Another way to formulate (2.3) is

$$\begin{array}{ll} \max_{x_1,\mathbf{x}_2,\ldots,\mathbf{x}_t} & \mathbb{E}\left[f_1(x_1) + f_2(\mathbf{x}_2(\xi_{[2]}),\xi_2) + \ldots + f_T(\mathbf{x}_T(\xi_{[T]}),\xi_T)\right] \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ \mathbf{x}_t(\xi_{[t]}) \in \mathcal{X}_t(\mathbf{x}_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\ldots,T, \end{array} \tag{2.4}$$

where the decision variables $\mathbf{x}_t : \mathbb{R}^{d_t} \times \ldots \times \mathbb{R}^{d_t} \to \mathbb{R}^{n_t}$, $\xi_t \mapsto \mathbf{x}_t(\xi_{[t]})$ $t = 1,\ldots,T$ are considered as functions of the data process $\xi_{[t]}$. Note that the optimization in (2.4) is performed over the deterministic vector $x_1 \in \mathbb{R}^{n_1}$ and over the functions $\mathbf{x}_t$ $(t > 1)$. So, unless the data process $\{\xi_t\}_{t=1}^T$ has a finite number of realizations, formulation (2.4) leads to an infinite dimensional optimization problem.

In particular we are interested in the so called multistage stochastic linear problem: the following is a multistage stochastic problem with linear objective functions and linear constraint functions:

$$\begin{array}{c} f_t(x_t,\xi_t) := c_t^{\mathsf{T}} x_t, \quad \mathcal{X}_1 := \{x_1 : A_{11}x_1 = b_1, x_1 \geq 0\}, \\ \mathcal{X}(x_{t-1},\xi_t) = \{x_t : \sum_{\tau=1}^t A_{t\tau}x_\tau = b_t, x_t \geq 0\} \quad t = 2,\ldots,T. \end{array} \tag{2.5}$$

Here $\xi_1 := (c_1, A_{11}, b_1)$ is known at the first-stage and is not random, and $\xi_t := (c_t, A_t, b_t) \in \mathbb{R}^{d_t}$, $t = 2,\ldots T$ are random data vectors.

Scenarios

In order to make (2.4) tractable we consider a finite number of possible realizations, N, for the data process $\{\xi_t\}_{t=1}^T$. For $t \in \{1,\ldots,T\}$ let the possible values for the random variable ξ_t be $\{\xi_t^1,\ldots\xi_t^{M_t}\}$, $t = 2,\ldots,T$ and let p_t^m be the probability of occurrence for the realization ξ_t^m with $m \in \{1,\ldots,M_t\}$. Then $\sum_{m=1}^{M_t} p_t^m$=1 for all t and the sets of possible realizations for the process $\{\xi_t\}_{t=1}^T$ is the set of T-tuples

$$\Xi = \{(\xi_1, \xi_2^{k_2},\ldots,\xi_T^{k_T}) : k_t \in \mathbb{Z}^+ \text{ and } k_t \leq M_t \text{ for } t = 2,\ldots,T\},$$

and the probability of occurrence of the scenario $(\xi_1, \xi_2^{k_2},\ldots,\xi_T^{k_T})$ is $p^{k_2,\ldots,k_T} = \cdot p_2^{k_2} \cdot \ldots \cdot p_T^{k_T}$. Clearly $\sum_{k_2,\ldots,k_T} p^{k_2,\ldots,k_T} = 1$.

We can represent the set Ξ, which is known as the scenario set or scenarios for short, by a scenario tree of T levels. Each level is also called stage. At level $t = 1$ we have only one root node, and we associate with it the deterministic value ξ_1. At level $t = 2$ we have as many nodes as different realizations of ξ_2 that can occur, that is, we have M_2 nodes. Each

of them is connected to the root by an arc. For each node g of level $t = 2$ (that corresponds to realization ξ_2^g of ξ_2) we have M_3 nodes and we connect them with the node g, etc.

Thus, the nodes at level $1 < t < T$ correspond to the possible values of ξ_t, $\{\xi_t^1, \ldots, \xi_t^{M_t}\}$. Each of them is connected to a unique node at level $t-1$, called the ancestor node and is also connected to M_{t+1} nodes at level $t+1$, named children nodes, which correspond to possible values at level $t+1$, that is $\{\xi_{t+1}^1, \ldots \xi_{t+1}^{M_{t+1}}\}$.

Deterministic Equivalent Model

Now, we proceed to formulate the optimization problem (2.4) with objective functions and constraints given by (2.5) when the uncertain data has finite possible realizations. Let $\xi = \{(c_t^\xi, A_t^\xi, b_t^\xi)\}_{t=1}^T \in \Xi$ be a scenario, p_ξ the probability of scenario ξ and $x^\xi = (x_1^\xi, \ldots, x_T^\xi)$ the corresponding sequence of decisions.

Firstly, look at the form of the objective function. The expectation in (2.4) calculated over the finite number of scenarios Ξ is

$$\mathbb{E}_\xi \left[\sum_{t=1}^{T} \left(c_t^\xi\right)^\mathsf{T} x_t^\xi \right] = \sum_{\xi=1}^{|\Xi|} p_\xi \left[c_1^\mathsf{T} x_1^\xi + \left(c_2^\xi\right)^\mathsf{T} x_2^\xi + \cdots + \left(c_T^\xi\right)^\mathsf{T} x_T^\xi \right]$$

where p_ξ is the probability of scenario $\xi \in \Xi$.

Now, we formulate the non anticipative constraints (NAC). To write the NAC, note that the condition that decisions cannot depend on future observations is equivalent to saying that two decision variables at time t, let us say x_t^ξ and $x_t^{\xi'}$, have to be equal if the information revealed up to this time is identical for the scenarios ξ and ξ', i.e., $\xi_{[t]} = \xi'_{[t]}$. Then, given $\xi, \xi' \in \Xi$ the NAC's have the form:

$$x_t^\xi = x_t^{\xi'} \quad \text{if} \quad \xi_{[t]} = \xi'_{[t]}. \tag{2.6}$$

Constraints (2.5) together with constraints (2.6) can be written as linear constraints for suitable matrices.

Thus, renaming variables and parameters the stochastic linear program (2.4)-(2.5) has the form:

$$\begin{array}{ll} \max_{x_p} & \sum_{p=1}^{P} c_p^\mathsf{T} x_p \\ \text{s.t.} & \sum_{\tau=1}^{p} A_\tau x_\tau = b_p \quad p = 1, \ldots, P \\ & A_0 \equiv 0 \\ & x_p \in \mathbb{R}^{n_p}, \end{array} \tag{2.7}$$

where the matrices $A_p \in \mathbb{R}^{m_p \times n_p}$ and the vectors $b_p \in \mathbb{R}^{m_p}$, $c_p \in \mathbb{R}^{n_p}$ are deterministic and $m_p, n_p, P \in \mathbb{Z}^+$ are suitable for all $p = 1, \dots P$. The linear (deterministic) programming problem (2.7) is called the deterministic equivalent model (DEM) for the stochastic linear programming problem (2.4)-(2.5).

The size of the problem (2.7) could grow dramatically depending on the number of scenarios considered and eventually this optimization problem cannot be solved exactly by conventional methods. In the next section we introduce special techniques to deal with big linear programming problems.

Scenario Generation

To model uncertain parameters in a stochastic optimization problem we used probability distributions (in a single period case) or a stochastic process (in a multi-period case) instead of single values. Except for some special cases, stochastic optimization problems cannot solved directly with continuous distributions, most solution methods need discrete distributions. In addition, the cardinality of the support of the discrete distribution is limited by computing power and the complexity of problem. Therefore, modelling the uncertain parameters requires constructing a finite number of scenarios that capture the uncertainty to provide robust solutions. The asymptotic properties of this discrete approximation are well-understood, see for instances, for instance, Mordukhovich [51]. Considering for $T \geq 3$, to the best of our knowledge there is not a implementable scenario generation method, which could approximate reasonably the solution of the original problem.

There are many techniques for scenario generation: Bundle-based sampling, moment-matching heuristics, the opinions of specialists, etc. (see, e.g., [43]). Here, we employ binomial trees which is a standard method for generating scenario that is widely used in finance to model Geometric Brownian Motion: (see Section 2.5).

Sensitivity & Stability Analysis

Sensitivity analysis is the study of how the uncertainty in the output of a mathematical model can be apportioned to different sources of uncertainty in its inputs. It helps us to identify the more relevant sources of uncertainty for a model and their influences on the objective function in a stochastic optimization problem.

To do a basic sensitivity analysis of (2.7) we perturb the uncertain data and observe

the changes on the objective function and compare them with a baseline case.

Stability analysis studies the impact on the solutions of variations in the scenario trees. Stochastic programs tend to have flat objective functions, so we can only require stability of the objective values, not of the solutions themselves.

Proper models and scenario generation should be stable, that is, if it generates several scenario trees, the solutions should not vary too much. Stability guarantees that the decisions do not depend on the choice of the scenario tree.

2.2 Lagrangian Relaxation and Volume Algorithm

Firstly, we present the Lagrangian Relaxation (see [29, 22, 36]) and after that a method to solve this relaxation called the Volume Algorithm (see [9]).

Lagrangean relaxation

Lagrangian Relaxation is a widely used technique to solve optimization problems. Basically, it consists in relaxing complicating constraints through the addition of them to the objective function and the penalization of them by factors called Lagrangean multipliers.

In order to make this description of the Lagrangian relaxation more precise, let us start with the following definition about relaxation of a maximization problem:

Definition 2.1 *Problem* $(RP_{max}) : \max\{g(x) : x \in W\}$ *is a relaxation of problem* $(P_{max}) : \max\{f(x) : x \in V\}$*, with the same variable* x*, if and only if*

1. *the feasible set of* (RP_{max}) *contains of* (P_{max})*, i.e.,* $W \supset V$*, and*
2. *over the feasible set of* (P_{max})*, the objective function of* (RP_{max}) *dominates that of* (P_{max})*, i.e.,* $\forall x \in V, g(x) \geq f(x)$*.*

Notation 2.1 *Given an optimization problem* (P) *let* $v(P)$ *be the optimal value of the problem* (P) *(if it exists).*

It follows that $v(RP_{max}) \geq v(P_{max})$ and (RP_{max}) is an optimistic version of (P_{max}).

From now on, let $A \in \mathbb{R}^{k_1 \times n}$ and $C \in \mathbb{R}^{k_2 \times n}$ be matrices and $b \in \mathbb{R}^{k_1}$ and $d \in \mathbb{R}^{k_2}$ be vectors and $X = \mathbb{R}_+^{n-p} \times \{0,1\}^p$. We are interested in dealing with linear problems, which have the form

$$\max\{f(x) : Ax = b, Cx \leq d, x \in X\}, \tag{2.8}$$

where the complicating constraints are $Ax = b$ (these will be the non anticipative constraints). Note that the equality constraints $Ax = b$ can be decomposed into the constraints $Ax \leq b$ and $-Ax \leq -b$. So, for our purposes is convenient to study the optimization problem:

$$\max\{f(x) : Ax \leq b, Cx \leq d, x \in X\}. \tag{P}$$

The constraints $Ax \leq b$ are assumed complicating, in the sense that without them problem (P) would be much simpler to solve. The constraints $Cx \leq d$ (possible empty) will be kept, together with X, to form de Lagrangean relaxation of (P) as follows.

Definition 2.2 *The Lagrangean relaxation of (P) relative to the complicating constraints $Ax \leq b$, with nonnegative Lagrangean multipliers λ, is the problem*

$$\max\{f(x) + \lambda(b - Ax) : Cx \leq d, x \in X\} \tag{LR_λ}$$

In (LR_λ), the slacks of the complicating constraints $Ax \leq b$ have been added to the objective function with weights λ and the complicating constraints have been dropped. This is called dualization of constraints $Ax \leq b$. Clearly, (LR_λ) is a relaxation of (P). Indeed, the feasible set of (LR_λ) contains the feasible set of (P) and

$$f(x) \leq f(x) + \lambda(b - Ax),$$

for any x feasible for (P) and any $\lambda \geq 0$. It follows that $v(P) \leq v(LR_\lambda)$, for all $\lambda \geq 0$.

Definition 2.3 *The problem of finding the lowest Lagrangean upper bound on $v(P)$ is:*

$$\min_{\lambda \geq 0} v(LR_\lambda) \tag{DP}$$

it is called the Lagrangean dual of (P) relative to the complicating constraints $Ax \leq b$.

The problem above is a problem in the dual space of the Lagrangian multipliers, whereas (LR_λ) is a problem in x.

Remark 2.1

1. *From now on, we will refer to $v(LR)$ as Lagrangean relaxation bound, or simply Lagrangean bound.*

2. *For (2.8) the Lagrangean relaxation is*

$$\max\{f(x) + \lambda(b - Ax) : Cx \leq d, x \in X\}, \quad \lambda \in \mathbb{R}^n, \tag{LR_λ}$$

and the dual problem become

$$\min_{\lambda \in \mathbb{R}^n} v(LR_\lambda) \tag{LR}$$

Now, we present a well known weak duality result that relates the Lagrangean bound $v(LRP)$ and $v(P)$.

Theorem 2.1 *1. If $x(\lambda)$ is an optimal solution of (LR_λ) for some $\lambda \geq 0$, then $v(P) \leq f(x(\lambda)) + \lambda(b - Ax(\lambda))$.*

2. If in addition $x(\lambda)$ is feasible for (P), then $f(x(\lambda)) \leq v(P) \leq f(x(\lambda)) + \lambda(b - Ax(\lambda))$.

3. If in addition $\lambda(b - Ax(\lambda)) = 0$, then $x(\lambda)$ is an optimal solution of (P), and $v(P) = f(x(\lambda))$.

Remark 2.2 *For (2.8) if $x(\lambda)$ is feasible for (P) automatically $\lambda(b - Ax(\lambda)) = 0$ and then $x(\lambda)$ is an optimal solution of (P).*

Volume Algorithm

We want to solve the maximization problem (LR) with $f(x) = cx$ for some fixed vector $c \in \mathbb{R}^n$. Thus, the problem (LR) becomes

$$\min_{\lambda \in \mathbb{R}^n} \max\{cx + \lambda(b - Ax) : Cx \leq d, x \in X\}, \qquad (LRP)$$

The maximization in (LRP) is over all $\lambda \in \mathbb{R}^n$ and to solve it numerically it is convenient to use a subgradient method. A subgradient method is an iterative procedure that can be used to solve the problem of minimizing a non-differentiable function $\phi(\lambda)$ on a closed convex set, Λ, i.e.,

$$\min\{\phi(\lambda) : \lambda \in \Lambda\},$$

using the following generic procedure:

1. Choose an initial point λ^0.

2. Construct a sequence of points $\{\lambda^n\} \subset X$ which eventually converges to an optimal solution using the rule:

$$\lambda^{n+1} = P_\Lambda(\lambda^n + \delta_n v^n) \qquad (2.9)$$

   where $P_\Lambda(.)$ is a projection on the set Λ, $\delta_n \geq 0$ is a positive scalar called step length and v_n is a vector, called step direction, which has to be determined at each iterate point.

3. Until: some stopping condition.

For a differentiable function $\phi(\lambda)$ this is the well known gradient method and for that case we have that the step direction in (2.9) is $v_n = \nabla\phi(\lambda)$. But for the problem (LRP) we have

$$\phi(\lambda) = \max\{cx + \lambda(b - Ax) : Cx \leq d, x \in X\}, \quad \lambda \in \mathbb{R}^n \tag{2.10}$$

which is a non differentiable convex function, so in this case the gradient cannot be defined. However, we can consider an element in the subdifferential set. Recall the definition of subdifferential of a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$:

$$\partial\psi(\lambda) = \{s \in \mathbb{R}^n : \psi(z) - \psi(\lambda) \geq s(z - \lambda)\},$$

and an element $s \in \partial\psi(\lambda)$ is called subgradient at $x \in \mathbb{R}^n$.

For the Volume Algorithm in (2.9) we take the step direction

$$v^n = \beta_n s^n + \eta_n v^{n-1} \tag{2.11}$$

where $s^n \in \partial\phi(\lambda^n)$ and $\beta_n, \eta_n \in \mathbb{R}$ are a suitable scalars. Note that here the current step direction depends on the previous one calculated in the prior iteration, this allows us to avoid a zigzagging behaviour and a slow convergence. These types of methods are known as the deflected subgradient method.

Next we have to determine how to construct the sequence $\{\lambda^n\}$, the step lengths $\{\delta_n\}$ and the step directions $\{v_n\}$ in (2.9), and the subgradient $s^n \in \partial\phi(\lambda^n)$ and scalars β_n, η_n in (2.11) for each n. To do that we start to rewrite the dual problem (DP) via the following result by Geoffrion (1974) [33]:

Theorem 2.2 *For $\phi(\lambda)$ defined by (2.10) we have*

$$\begin{aligned} \phi(\lambda^*) = \quad & \max \quad cx \\ & s.t. \quad Ax \leq b \\ & \qquad x \in conv(X) \end{aligned} \tag{2.12}$$

where λ^* is the optimal value of the problem (LR) and $conv(X)$ is the convex hull of X, that is, $conv(X) = \{\sum \alpha_n x^n : \sum_n \alpha_n = 1, \alpha_i \geq 0, x^i \in X, \forall i\}$

Notation 2.2 *Put $\phi^* = \phi(\lambda^*)$.*

Note that (2.12) can be written as

$$\begin{aligned} \phi^* = \quad & \max \quad \sum_i (cx^i)\alpha_i \\ & \text{s.t.} \quad (b - Ax^i)\alpha_i \leq 0 \\ & \qquad \sum_i \alpha_i = 1 \\ & \qquad \alpha_i \geq 0 \quad \forall i, \end{aligned}$$

Here the optimization is over α_i's, so, the solution of the problem (2.12) is $x = \sum_i \alpha_i^* x_i$ where α_i^* is the solution of the problem above and $x^i \in X$. To obtain the set of points $\{x^1, x^2, \ldots, x^i, \ldots\} \subset X$ which generates the solution of (2.12) consider the dual of the optimization problem above:

$$\begin{aligned} \phi^* = \quad & \min \quad z \\ & \text{s.t.} \quad z \geq (c + \lambda A)x^i - \lambda b, \quad \forall i \\ & \qquad \lambda \in \mathbb{R}^m_+, z \in \mathbb{R}. \end{aligned} \tag{2.13}$$

Setting $z = \phi(\lambda)$, then the problem above is equivalent to the Lagrangian dual:

$$\phi^* = \min\{\phi(\lambda) : \lambda \in \mathbb{R}^m_+\} \tag{LD}$$

where $\phi(\lambda)$ is the dual function determined pointwise by the subproblem:

$$\phi(\lambda) = \max\{(c + \lambda A)x^i : x^i \in X\} - \lambda b. \tag{2.14}$$

We can solve (LD) by a subgradient optimization procedure that we summarize below. From this, we obtain the wanted set of points $\{x^1, x^2, \ldots, x^i, \ldots\} \subset X$.

Algorithm 1 Subgradient algorithm

Step 0

Give an initial guess for the Lagrangian multiplier $\bar{\lambda}$ in (2.14).

Step 1

Solve (2.14) for $\bar{\lambda}$:

$$\begin{aligned} \max \quad & z = (c + \bar{\lambda}A)x - \bar{\lambda}b \\ s.t. \quad & x \in X \end{aligned}$$

Let $\bar{x}$ be the solution and $\bar{z}$ the optimal value of the problem above. Then $s = b - A\bar{x}$ is a subgradient at $\bar{\lambda}$.

Step 2

Updated $\bar{\lambda} = \bar{\lambda} + \delta d$. *Here* δ *is the step size, and the formula for it is*

$$\delta = \mu \frac{\bar{\phi} - LB}{||s||^2} \tag{2.15}$$

where $0 < \mu < 2$.

Step 3

If some stopping criteria is met then stop otherwise go to Step 1.

In the subgradient algorithm described above the value of the step size is justified empirically by Held, Wolfe and Crowder [40].

The basic idea to determine the corresponding weights α_i in the convex combination comes from applying the theorem on volume and duality due to Barahona and Anbil [9], where the α_i are estimating certain volumes associated to active faces at an optimal dual solution as shown in the following theorem.

Theorem 2.3 *Consider the problem*

$$\begin{aligned} \max \quad & \sum_i b^i \alpha_i \\ s.t. \quad & a_i \alpha_i \leq 0 \\ & \sum_i \alpha_i = 1 \\ & \alpha_i \geq 0 \quad i = 1, 2, \ldots, m, \end{aligned} \tag{2.16}$$

where $a_i \in \mathbb{R}^n$ *and* b_i *a vector, and its dual*

$$\begin{aligned} \min \quad & z \\ s.t. \quad & z + a_i \lambda \geq b_i, \quad \text{for } i = 1, 2, \ldots, m \\ & z \in \mathbb{R}, \end{aligned}$$

where $\lambda \in \mathbb{R}^n$. *Let* $(\hat{z}, \hat{\lambda}) \in \mathbb{R} \times \mathbb{R}^n$ *be an optimal solution, and suppose that constraints* $1, 2, \ldots, m'$ *with* $m' \leq m$ *are active at this point.*
Let $\bar{z} > \hat{z}$ *and assume that*

$$\begin{aligned} & z + a_i \lambda \geq b_i, \quad \text{for } i = 1, 2, \ldots, m' \\ & z \leq \bar{z}, \end{aligned}$$

defined a bounded polyhedron. For $1 \leq i \leq m'$, *let* γ_i *be the volume of this polyhedron. Then an optimal solution of (2.16) is given by*

$$\alpha_i = \frac{\gamma_i}{\sum_{j=1}^{m'} \gamma_j} \quad \text{for } i = 1, \ldots, m'.$$

Next we describe the Volume Algorithm, an extension of the subgradient, due to Barahona and Anbil [9] and based on the last theorem, that helps us to find an optimal solution for (LR).

Algorithm 2 Volume Algorithm

Initialization

Start with a vector $\bar{\lambda} := \lambda^0$ *and solve the corresponding subproblem (LD). Let* $\bar{x}$ *be an optimal solution of this subproblem. Set* $x^0 = \bar{x}$, $\bar{\phi} := \phi(\lambda^0)$, $k = 1$.

Step 1

Determine a direction of motion, $d^k = A\bar{x} - b$, *and*

$$\lambda^k = P_{(\mathbb{R}^m_+)}(\bar{\lambda} + \delta_k d^k),$$

for a step size δ_k *given by the Polyak's step length rule*

$$\delta_k = \mu_k \frac{\bar{\phi} - LB}{||d^k||^2}$$

where $0 < \mu_k < 2$.

Solve the problem (LD) with λ^k, *and let* x^k *and* $\phi(\lambda^k)$ *be the solutions obtained.*

Then $\bar{x}$ *is updated as*

$$\bar{x} := \alpha x + +(1 - \alpha)\bar{x} \tag{2.17}$$

for some $\alpha \in (0, 1)$, *to be discussed later.*

Step 2

If $\phi(\lambda^k) < \phi$, *update* $\bar{u}$ *and* $\bar{\phi}$ *as*

$$\bar{\phi} := \phi(\lambda^k), \quad \bar{\lambda} := \lambda^k$$

Let $k := k + 1$ *and go to Step 1.*

At the end of the Volume Algorithm, $\bar{\lambda}$ would be an output for the solution of the Lagrangian dual with the objective value $\bar{\phi}$. Note that it is a descent method because the update happens if $\phi(\lambda^k) < \phi$. Further, it provides an 'approximate' primal solution through (2.17) in the sense of the next definition:

Definition 2.4 *Suppose* $\bar{\lambda} \in \Omega^* = \{\bar{\lambda} \in \mathbb{R}^n_+ : \phi(\bar{\lambda}) \leq \phi(\lambda), \forall \lambda \in \mathbb{R}^n_+\}$ *and let*

$$X(\bar{\lambda}) = \{\bar{x} \in conv(X) : c\bar{x} + \bar{\lambda}(b - A\bar{x}) \geq cx + \bar{\lambda}(b - Ax), \forall x \in conv(X)\}.$$

An $\bar{x} \in X(\bar{\lambda})$ *is said to be a* ***near-optimal solution*** *of (P) if* $A\bar{x} \leq b$ *and* $\bar{\lambda}(b - A\bar{x}) = 0$.

Indeed, if $\{x^0, x^1, \ldots, x^k\}$ are produced by solving (2.14), then (2.17) implies

$$\bar{x} = (1-\alpha)^k x^0 + (1-\alpha)^{k-1}\alpha x^1 + \ldots + (1-\alpha)\alpha x^{k-1} + \alpha x^k,$$

i.e., $\bar{x}$ is a convex combination of $\{x^0, x^1, \ldots, x^k\} \subset X$, since for $\alpha \in (0,1)$, all coefficients are in $(0,1)$ and

$$\alpha \sum_{i=0}^{k-1} (1-\alpha)^i + (1-\alpha)^k = 1.$$

Thus, based on Theorem on Volume and Duality we conclude that $\bar{x}$ is near-optimal solution for (P).

2.3 Robust optimization: Worst Case & Dynamic Worst Case Approach

Decision Theory is the study of the reasoning underlying an agent's choices. In many cases, these choices have to be taken under uncertainty. To handle decisions under uncertainty some powerful tools as Scenario Analysis, Robust Optimization, Stochastic Optimization, Stochastic Games, etc have been developed. In this section we take tools from Scenario Analysis and Robust Optimization to provide feasible solutions for the Volume Algorithm scheme, which are needed to updated the Polyak step size (2.15).

Scenario analysis is a process of analyzing possible future events by considering alternative possible outcomes. For an extensive presentation of Scenario Analysis see e.g. Wright & Goodwin [61].

Robust Optimization aims to provide optimal responses that work in any scenario. In Robust Optimization a certain measure of robustness is sought against uncertainty that can be represented as deterministic variability in the value of the parameters of the problem itself and/or its solution. For more about Robust Optimization the reader can see Ben-Tal et al. [11].

Worst Case

The worst case approach is one of most popular methods in Robust Optimization. It consists of considering the worst scenario for the uncertain parameters to optimize. The worst scenario must be the lowest value if we are dealing with prices or the highest value if we are treating with costs.

Given a scenario tree the worst case scenario is the one such that each realization corresponds to the worst possibility at each period. The definition of worst realization depends on the decision maker.

To define the worst case scenario in a rigorous way consider a multi stage stochastic optimization problem:

$$\begin{array}{ll} \max_{x_t^\xi} & \sum_{\xi\in\Omega}\sum_{t=1}^{T}(c_t^\xi)^\intercal x_t^\xi \\ \text{s.t.} & \sum_{\tau=1}^{t} A_\tau^\xi x_\tau^\xi = b_t^\xi, \quad t=1,\dots,T, \quad \xi\in\Xi \\ & x_t^\xi = x_t^{\xi'} \quad \text{if} \quad \xi_{[t]}=\xi'_{[t]}, \quad t=1,\dots,T, \quad \xi\in\Xi \\ & A_0^\xi \equiv 0 \\ & x_t^\xi \in \mathbb{R}^{n_t}, \end{array} \tag{2.18}$$

where the uncertain parameters are $(A_t^\xi, b_t^\xi, c_t^\xi) \in \mathbb{R}^{m_p\times n_p}\times\mathbb{R}^{m_p}\times\mathbb{R}^{n_p}$. We are interested in finding a 'robust' solution $\{\hat{x}_t\}_t$ for the problem above in the sense of the next definition.

Definition 2.5 *We say that $\{\hat{x}_t\}_t$ is a robust solution for the problem (2.18) if*

$$\sum_{\tau=1}^{t} A_\tau^\xi \hat{x}_\tau = b_t^\xi$$

for all $t=1,\dots,T$ and $\xi\in\Xi$.

Note if $\{\hat{x}_t\}_t$ is a robust solution then $x_t^\xi := \hat{x}_t,\ \forall t=1,\dots,T,\ \xi\in\Xi$ is a feasible solution for the problem (2.18) (the non anticipative constraints are satisfied trivially).

Definition 2.6 *For the multistage problem (2.18) we define the **worst scenario** as the scenario ξ^{WC} such that the solution of the problem*

$$\begin{array}{ll} \max & \sum_{t=1}^{T}(c_t^{\xi^{WC}})^\intercal x_t \\ s.t. & \sum_{\tau=1}^{t} A_\tau^{\xi^{WC}} x_\tau = b_t^{\xi^{WC}}, \quad t=1,\dots,T \\ & A_0^\xi \equiv 0, \\ & x_t \in \mathbb{R}^{n_t}, \end{array}$$

denoted by $x_t^{\xi^{WC}}$ is a robust solution for (2.18) and $x_t^{\xi^{WC}} = \arg\min_{x_t^\xi\in\mathbb{R}^{n_t}}\{\sum_{t=1}^{T}(c_t^\xi)^\intercal x_t^\xi : \xi\in\Xi\}$ where x_t^ξ is the solution for the problem

$$\begin{array}{ll} \max & \sum_{t=1}^{T}(c_t^\xi)^\intercal x_t \\ \text{s.t.} & \sum_{\tau=1}^{t} A_\tau^\xi x_\tau = b_t^\xi, \quad t=1,\dots,T \\ & A_0^\xi \equiv 0, \\ & x_t \in \mathbb{R}^{n_t}, \end{array}$$

Thus, the worst case scenario is the worst scenario in terms of the objective function so that its realization at each period is the worst one.

If $x^{WC} = (x_1^{WC}, \ldots, x_T^{WC})$ corresponds to the worst scenario in the sense of the last definition, we have x^{WC} is a robust solution for (2.18) and then

$$x_t^{\xi} := x_t^{WC}, \qquad \forall \xi \in \Xi, t \in \mathcal{T},$$

is a feasible (sub)solution for (2.7).

Intuitively, the worst scenario approach gives us the decisions that should be taken if the worst scenario happens. It does not consider the information revealed about uncertain parameters during the time horizon. Therefore this is a very conservative and inflexible strategy to obtain a implementable policy.

A dynamic worst case approach that takes into account the information revealed along the time horizon to improve the decisions taken is presented next.

Dynamic Worst Case

The dynamic worst case approach is similar to the worst case, but instead of taking into account just one scenario, the worst one, the dynamic to solve (2.18) progressively updating the worst scenario scenario at stage t as one among the scenarios whose realizations are the same as the observed one up to stage t. Mathematically, it is expressed in the following way:

- For the first stage $t = 1$ we solve the optimization problem for the worst scenario and fix the variables for this stage.
- Consider $t > 1$. To determine the values of the decision variables for scenario ξ where $\xi_{[t]} = (\xi_1, \xi_2, \ldots, \xi_t)$ we solve the problem with variables fixed prior to stage $t - 1$ for the worst scenario among the scenarios with history $\xi_{[t]}$. We fix the variables up to stage t to the obtained values.
- We do $t \leftarrow t + 1$ and go back to the prior step.

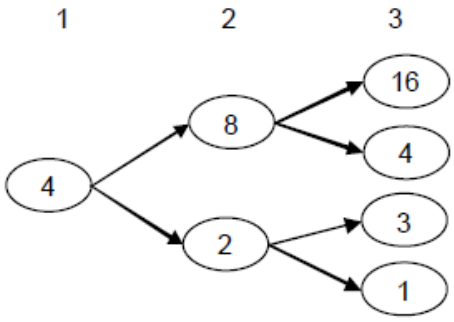

Figure 2.1: This scenario tree was constructed considering three periods and the tree is such that one scenario is worse than another one at the same period if it is located lower.

For example, in Figure 2.1 we solve the optimization problem considering the worst scenario $\{4, 2, 1\}$. After that at period $t = 2$, the decisions variables are fixed for the period $t = 1$ and the realization is observed: if the realization is up, that is, 8, the updated worst case scenario is $\{4, 8, 4\}$, otherwise the worst scenario is the same considered at stage $t = 1$, etc.

This dynamic approach is a wait & see type and it provides a feasible solution for the optimization problem that is less conservative than the one obtained by the (static) worst case. Anyway, the solution obtained by the dynamic approach is conservative since at each stage it considers the worst scenario given the realizations revealed at the current stage.

2.4 Geometric Brownian Motion

In this section we review the Geometric Brownian Motion or Log-Normal model which will be used to model the uncertainty in the metal prices (see e.g. [10, 35, 42, 45]).

Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ be a filtered probability space where Ω is a set space, $\mathcal{A}$ is $\sigma-$algebra of subsets of Ω, $\mathbb{P}$ is a probability measure over $(\Omega, \mathcal{A})$ and $\mathcal{F} = \cup_{i \in I} \mathcal{F}_i$ is a filtration.

Brownian Motion

A Brownian Motion is a stochastic process, that is, a sequence of random variables $\{B(t) : B(t)$ is a random variable for every $t \geq 0\}$, that satisfies the properties:

1. Starts at the origin with probability equal to one: $\mathbb{P}[B(0) = 0] = 1$.

2. The increments are independent random variables: for $0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n+1}$ we have $\{B(t_{i+1}) - B(t_i)\}_{i=0}^{n}$ are independent random variables.

3. The increments are stationary:

$$B(t + \Delta t) - B(t) = B(s + \Delta t) - B(s), \quad \forall s, t : 0 \leq s \leq t \in [0, +\infty]$$

where the equality is in the sense of the probability distributions.

4. The increments of the process are gaussians with mean 0 and variance $t - s$:

$$B(t) - B(s) \sim \mathcal{N}(0, \sqrt{t - s}).$$

Itô Calculus and Stochastic Differential Equations

Definition 2.7 *Let $b(t, x), \sigma(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$. A stochastic differential equation is given by:*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{2.19}$$

defined over $t \in [0, T]$ with initial condition the random variable X_0 which is $\mathcal{F}_0-$measurable and independent of the Brownian Motion. The equation (2.19) is in the sense of the integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

where the first integral is a Riemann integral and the second one is an Itô integral. The process $\{X_t\}_t$ is called an Itô process.

A Geometric Brownian Motion is described by the following Stochastic Equation:

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$
$$X_0 = x_0 \tag{2.20}$$

Through Itô Calculus can be shown that the solution of (2.20) is:

$$X_t = x_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right] \tag{2.21}$$

The equation (2.20) can be interpreted in the following way: when the stochastic term is dropped the equation is reduced to $dX_t = \mu X_t dt$ whose solution is $X_t = x_0 e^{\mu t}$ which corresponds to an initial capital x_0 that increases with $100\mu\%$ interest.

Theorem 2.4 *Let $\{X_t\}$ be an Itô stochastic process given by (2.19) and $f(t, x)$ a C^1 function on t and C^2 function on x, then the process $\{Y_t = f(X_t)\}$ is an Itô process and*

$$dY_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2. \tag{2.22}$$

Estimation of Geometric Brownian Motion parameters

To estimate the parameters μ and σ in the equation (2.20) from historical data we review two methods: The Maximum Likelihood and the non parametric method.

Maximum Likelihood Method

Consider the times $\{t_0 < t_1 < \ldots < t_N\}$ and a sample $\{x_0, x_1, \ldots, x_N\}$ of the stochastic process $\{X_t\}_t$ such that x_k is a realization of X_{t_k}.
We are searching an estimator that maximizes the joint probability $p(t_0, x_0, t_1, x_1, \ldots, t_N, x_N; \theta)$ dependent on a unknown vector parameter θ, let us say $\hat{\theta} = \text{argmax}_\theta \, L(\theta)$ where

$$L(\theta) := p(t_0, x_0, t_1, x_1, \ldots, t_N, x_N; \theta).$$

Thus, we have to solve the maximization problem:

$$p(t_0, x_0, t_1, x_1, \ldots, t_N, x_N) := \max_\theta L(\theta) \tag{2.23}$$

Now we are interested in determining the joint probability $p(t_0, x_0, t_1, x_1, \ldots, t_N, x_N)$ for the equation (2.20). As the solution (2.21) is a Markovian process and given the total probability theorem we have that the joint probability is:

$$\begin{aligned} p(t_0, x_0, t_1, x_1, \ldots, t_N, x_N; \theta) =& p(t_0, x_0; \theta) \times p(t_1, x_1 | t_0, x_0; \theta) \\ & \times p(t_2, x_2 | t_1, x_1, t_0, x_0; \theta) \ldots \\ & \times p(t_N, x_N | t_{N-1}, x_{N-1}, \ldots, t_1, x_1, t_0, x_0; \theta) \\ =& p(t_0, x_0; \theta) \times p(t_1, x_1 | t_0, x_0; \theta) \times p(t_2, x_2 | t_1, x_1; \theta) \\ & \times \ldots \times p(t_N, x_N | t_{N-1}, x_{N-1}; \theta) \end{aligned}$$

where $p(t_k, x_k | t_{k-1}, x_{k-1}; \theta)$ is the probability of transitioning from state x_k to x_{k-1} in a single step.

To determine the transition probability, discretize the equation (2.20) by the Euler-Maruyama scheme:

$$X_{t_k} - X_{t_{k-1}} = \mu X_{t_{k-1}} \Delta t + \sigma X_{t_{k-1}} (B_{t_k} - B_{t_{k-1}})$$

where $\Delta t = t_k - t_{k-1}$. Since $B_{t_k} - B_{t_{k-1}} \sim \mathcal{N}(0, \sqrt{\Delta t})$, the conditional distribution of the random variable given the event $\{X_{t_{k-1}} = x_{k-1}\}$ is

$$X_{t_k} | \{X_{t_{k-1}} = x_{k-1}\} \sim \mathcal{N}(\mu_k, \sigma_k),$$

where $\mu_k = x_{t_{k-1}} + \mu x_{t_{k-1}}$ and $\sigma_k = \sigma x_{t_{k-1}}$. So, the transition probability is

$$p(t_k, x_k | t_{k-1}, x_{k-1}; \theta) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{1}{2}\left(\frac{x_k - \mu_k}{\sigma_k}\right)^2\right], \quad k = 0, 1, \ldots, N$$

Note that the problem (2.23) is equivalent to $\min_\theta J(\theta)$ where $J(\theta) = -\ln L(\theta)$ and $\theta = (\mu, \sigma)$. The function $J(\theta)$ can be written as

$$\begin{aligned} J(\theta) &= -\ln p(t_0, x_0; \theta) + \frac{N}{2}\ln 2\pi + \sum_{k=0}^{N} \ln \sigma_k + \sum_{k=0}^{N} \frac{1}{2}\left(\frac{x_k - \mu_k}{\sigma_k}\right)^2 \\ &= \frac{N}{2}(\ln 2\pi + \ln \Delta t + 2\ln \sigma) + \sum_{k=1}^{N} \ln x_{k-1} + \frac{1}{2\sigma^t \Delta t}\sum_{k=1}^{N}\left(\frac{x_k}{x_{k-1}} - 1 - \mu\Delta t\right)^2. \end{aligned}$$

To find the best estimator $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ we search for the critical values of $J(\theta)$:

$$\begin{aligned} 0 = \frac{\partial J(\mu, \sigma)}{\partial \mu} &= -\frac{1}{\sigma^2}\sum_{k=1}^{N}\left(\frac{x_k}{x_{k-1}} - 1 - \mu\Delta t\right) \\ 0 = \frac{\partial J(\mu, \sigma)}{\partial \sigma} &= \frac{N}{\sigma} - \frac{1}{\sigma^3 \Delta t}\sum_{k=1}^{N}\left(\frac{x_k}{x_{k-1}} - 1 - \mu\Delta t\right)^2. \end{aligned}$$

To obtain the last equality we use the fact $p(t_0, x_0; \theta) = 0$.

Thus, the critical values are:

$$\begin{aligned} \hat{\mu} &= \frac{1}{N\Delta t}\sum_{k=1}^{N}\left(\frac{x_k}{x_{k-1}} - 1\right) \\ \hat{\sigma}^2 &= \frac{1}{N\Delta t}\sum_{k=1}^{N}\left(\frac{x_k}{x_{k-1}} - 1 - \hat{\mu}\Delta t\right)^2. \end{aligned}$$

Moreover, the Hessian matrix $H(J(\mu, \sigma))$ has positive eigenvalues at $(\hat{\mu}, \hat{\sigma})$, so it is the best estimator.

Non parametric method

Using the same discretization as in the last subsection, since the Geometric Brownian Motion is a diffusion process we obtain the following relations valid for $k = 0, 1, \ldots, N$:

$$\begin{aligned} \mathbb{E}\left[\frac{X_{t_{k+1}} - X_{t_k}}{\Delta t} - \mu X_{t_k}\right] &= o(\Delta t), \\ \mathbb{E}\left[\frac{(X_{t_{k+1}} - X_{t_k})^2}{\Delta t} - \sigma^2 X_{t_k}^2\right] &= o(\Delta t), \end{aligned}$$

where $o(\Delta)$ means that the order of the left sides in the equations above is Δt. Then,

$$\begin{aligned} \mu &= \frac{\frac{1}{\Delta t}\sum_{k=0}^{N-1}(x_{k+1} - x_k)}{\sum_{k=0}^{N-1} x_k}, \\ \sigma^2 &= \frac{\frac{1}{\Delta t}\sum_{k=0}^{N-1}(x_{k+1} - x_k)^2}{\sum_{k=0}^{N-1} x_k^2}. \end{aligned}$$

The expression for μ reduces to $\mu = \frac{x_N - x_0}{\Delta t \sum_{k=0}^{N-1} x_k}$ by the telescope property of the sum. Note that the last expression for μ reveals that the estimates of the parameters of the Geometric Brownian Motion depend strongly on the first and the last data of the period considered.

2.5 Option pricing

In general, the value of any asset is the present value of the expected cash flows on that asset. This section will consider a case where the assets have two specific characteristics:

- The assets derive their value from the values of other assets called the underlying security or commodity.
- The cash flows on the assets are contingent on the occurrence of specific events.

These assets are called options, and the present value of the expected cash flows on these assets will understate their true value. The metal prices holds the two conditions above, so they are options. Indeed, the metal price depends on the metal, the underlying commodity, and clearly it is depends on several kind of events: demand, strikes, economic crises, discoveries, etc.

Basically, there are two types of options: call and put options. A call option is a financial contract between two parties, the buyer and the seller of this type of option. The buyer of the call option has the right, but not the obligation, to buy an agreed quantity of the underlying security from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). The seller is obliged to sell the commodity or financial instrument to the buyer if the buyer so decides. The buyer pays a fee (called a premium) for this right. The term "call" comes from the fact that the owner has the right to "call the stock away" from the seller. Conversely, the put option gives the owner the right, but not the obligation, to sell the underlying, at the strike price, by the expiry date to a the seller of the put.

We will use the binomial price model which can be seen as an approximation of the widely used Black & Scholes model used to evaluate options. We start by introducing the Black & Scholes model.

Black & Scholes Model

The Black & Scholes Model is based in the following assumptions:

- The price of the option and the underlying asset related to the option have the same source of uncertainty.
- The rate of return on the riskless asset is constant and thus called the risk-free interest rate.
- The instantaneous log return of stock price is an infinitesimal random walk with drift; more precisely, it is a geometric Brownian motion, and we will assume its drift and volatility are constant.

Following these assumptions we can deduce the value for the call options, C, and for the put options, P:

$$C = S\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$
$$P = K\Phi(-d_2) - S\Phi(-d_1)$$

where Φ is the cumulative distribution of the normal standard distribution $\mathcal{N}(0,1)$ and

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

and

- S: current value of the underlying asset.
- K: strike price of the option.
- T: life to expiration of the option.
- r: riskless interest rate corresponding to the life of the option.
- σ^2: variance in the log return of the underlying asset.

Note that e^{-rT} is the discount factor, and reflects the fact that the exercise price on the option (call or put) does not have to be paid until expiration. This means that we are dealing with European options.

The portfolio that replicates the call option is created by buying $\Phi(d_1)$ units of the underlying asset, and borrowing $Ke^{-rt}\Phi(d_2)$. This portfolio will have the same cash flows as the call options. Analogously, for put options we can consider a portfolio whose cash flows are equal to the cash flows for this option.

Binomial trees

Binomial option pricing trees are based on a simple formulation for the asset price process in which the asset, in any time period, can move to one of two possible prices.

At each step, it is assumed that the underlying instrument will move up or down by a specific factor u and d per step of the tree respectively. Thus, by definition we have $u \geq 1$ and $0 < d \leq 1$ and additionally assume that $ud = 1$. So, if S is the current price, then in the next period the price will either be $S_{up} = S \cdot u$ or $S_{down} = S \cdot d$.

We suppose that the expected rate of return is constant (risk free), say r. Let p be the probability of the favourable scenario S_{up} then the probability for S_{down} is $1 - p$. Thus, if S_t is the scenario at period t and $S_{t+\Delta t}$ is the price for the next period we have:

$$pS_t + (1-p)S_t = \mathbb{E}[S_{t+\Delta t}] = e^{r\Delta t}S_t \Longrightarrow p = \frac{e^{r\Delta t} - d}{u - d},$$

and if σ denote the volatility it is obtained using approximations $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ and, consequently, $ud = 1$. Note these approximations are reasonable for Δt small. For p to be in the interval $(0, 1)$ the following condition on Δt has to be satisfied $\Delta t < \left(\frac{\sigma}{r}\right)^2$.

We can construct another binomial tree by considering the logarithmic stock price space and the price probability to up or down as $p = 1 - p = 1/2$. In that case, $\ln S_t$ could be increased to $\ln S_t + \ln u$ or it could be decreased to $\ln S_t + \ln d$. Then,

$$\ln S_{t+\Delta t} \sim \mathcal{N}(\ln S_t + (r^2 - \sigma/2)\Delta t, \sigma^2 \Delta t)$$

$$\Rightarrow \begin{cases} p(\ln S_t + \ln u) + (1-p)(\ln S_t + \ln d) = \ln S_t + (r^2 - \frac{\sigma}{2})\Delta t \\ p(\ln S_t + \ln u)^2 + (1-p)(\ln S_t + \ln d)^2 - (\ln S_t + (r^2 - \frac{\sigma}{2})\Delta t)^2 = \ln S_t + (r^2 - \frac{\sigma}{2})\Delta t \end{cases}$$

where $p = 1 - p = 1/2$. Solving these equations we obtain $u = e^{(r^2 - \sigma/2)\Delta t + \sigma\sqrt{r}}$ and $d = e^{(r^2 - \sigma/2)\Delta t - \sigma\sqrt{r}}$. This is not an approximation; the model is valid for every Δt. Note that in this case $ud \neq 1$.

Note that to construct a binomial tree in any case we need just the time step Δt, the initial price S_0, the riskfree interest rate r and the volatility σ.

We distinguish two types of binomial tree:

a) **Recombining tree**

If the underlying asset moves up and then down (u, d), the price will be the same as if it had moved down and then up (d, u) here the two paths merge or recombine. In general, recombining trees do not distinguish between two scenarios with the same

initial and final results. This property reduces the number of tree nodes, and thus accelerates the computation of the option price. Thus, a recombining scenario tree for T periods has $T + 1$ scenarios. This type of tree is used for non path-dependent options. In a multistage stochastic optimization setting it can be used if decisions are taken just at the initial and final stages.

This property also allows that the value of the underlying asset at each node can be calculated directly via formula, and does not require that the tree be built first. The node value will be:

$$S_n = S_0 \times u^{N_u - N_d},$$

where S_n is the spot price at node n, S_0 is the spot price at initial period, N_u is the number of up ticks and N_d is the number of down ticks, the probability to the price increase is p and the probability to the price decrease is $1 - p$.

When the time between periods are small, it can shown that the recombining tree with $ud = 1$ converges to the continuous Black and Scholes model (the Cox, Ross and Rubinstein tree [23]).

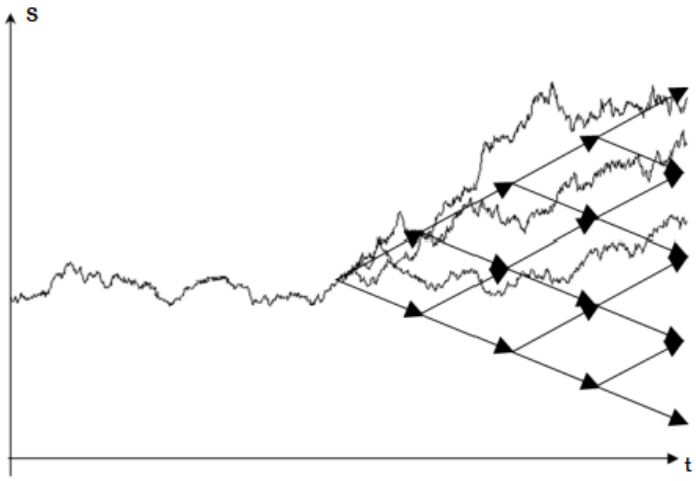

Figure 2.2: The Cox, Ross and Rubinstein tree converges to the Black & Scholes model if $\Delta t \to 0$

b) **Non-Recombining tree**

For path-dependent options we use a non-recombining tree. The distinctive characteristic of this type of tree is that two scenarios with the same initial and final results are not necessarily equal because the intermediate states are considered to distinguish

scenarios. Thus, a non-recombining scenario tree with T periods has 2^T scenarios. In the non-recombining tree, the u and d may change in each step, thus make the underlying asset more variable and has some application values. Even when u and d do not change, scenarios with initial and final prices are considered different.

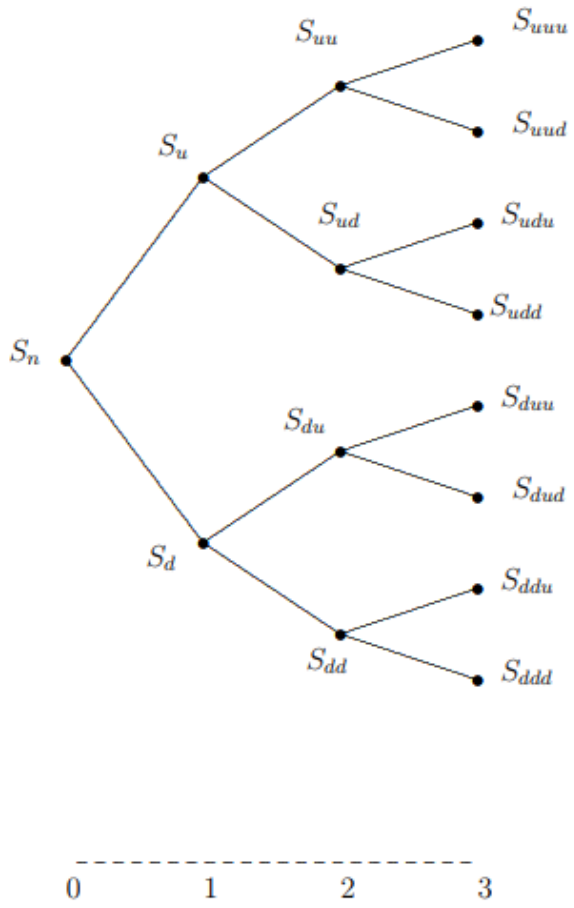

Figure 2.3: Non recombining binomial scenario tree for three periods.

For more details about options read e.g. [8].

2.6 Block caving & micro-seismicity

One technique widely used to mine massive steeply dipping ore bodies (typically low grade) with high friability is block/panel caving: an undercut with haulage access is driven under the orebody, with "drawbells" are excavated between the top of the haulage level and the bottom of the undercut. The orebody is drilled and blasted above the undercut to create

a void at each draw-point so that the rock breaks and falls due to the friability of the rock and gravity. After that, the broken ore is removed via the haulage access.

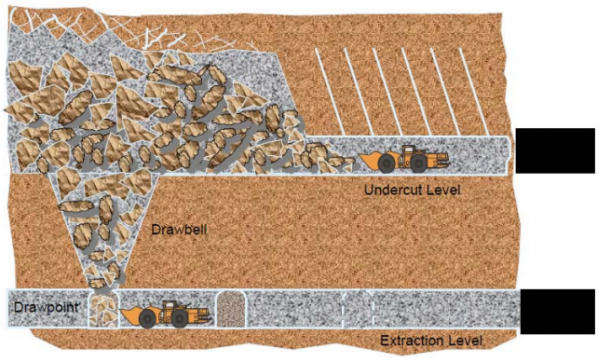

Figure 2.4: Block caving detail.

The rock removal generates instability in the ore body and new fracturing begins. This process produces seismic activity which fractures the ore body. This seismicity has to be controlled in order to prevent damage to the mine's structure and this can done mainly by controlling the extraction speed for the fracture rock from the draw-points.

If caving stops and removal of ore from the drawbells continues, a large void may form, resulting in the potential for a sudden and massive collapse and potentially catastrophic windblast throughout the mine. To avoid this situation sometimes the rock is preconditioned by hydraulic fracturing, blasting, or by a combination of both. Hydraulic fracturing has been applied to precondition strong roof rock over coal longwall panels, and to induce caving in both coal and hard rock mines. Where caving does continue, the ground surface may collapse into a surface depression. This phenomenon is called subsidence.

It is desirable to cave the whole ore body preserving the mine structure but many problems can occur in this process: tunnel collapse by rock burst due uncontrolled and strong seismicity, the caving stops and a rigid void is formed, the material extracted from a draw point is waste (dilution), etc. To avoid these problems it is fundamental to determine properly the mine design and the mine planning for short and middle terms which implies the control of the extraction rates and block sequencing.

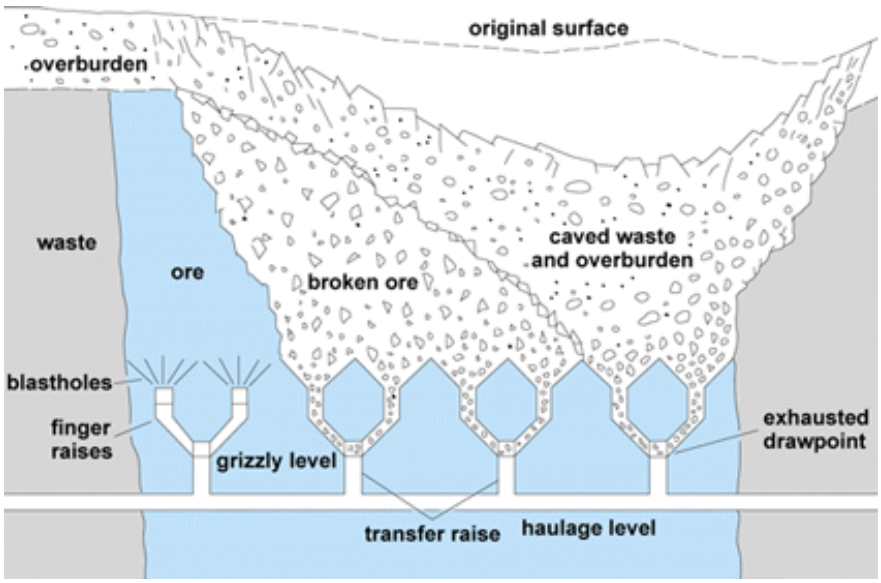

Figure 2.5: Block caving scheme.

Mining and micro-seismicity

Seismology is the study of earthquakes and the propagation of elastic waves through the earth or other bodies. Many studies on historical data of natural earthquake occurrences show that the magnitude of a earthquake can be modelled by bounded Pareto distribution (see e.g. [57]):

$$F_M(M) = \frac{M_-^{-\beta} - M^{-\beta}}{M_-^{-\beta} - M_+^{-\beta}},$$

where $M_- \leq M \leq M_+$ is the seismic moment and M_-, M_+ are the minimum and maximum seismic moment respectively, β is a parameter and F_M is a probability cumulative distribution for M.

Under proper assumptions, the frequency of the tremors can be modelled by Poisson Processes and it is independent of the event magnitudes [38]. Thus, the sum of the seismic moments of the tremors is a composite Poisson Process, i.e., if $M(t)$ is the cumulative sum of the magnitudes of the tremors up to period t and $(M_\tau)_{\tau=1}^{\infty}$ are independent and identically distributed variables representing the seismic moments of the tremors that happened at time τ with $M_\tau \sim Pareto(\beta, M^-, M^+)$ and the Poisson process $\{N(t)\}_{t\geq 0}$ is such that the number of seismic events that have happened up to and including time t is $N(t)$, then

$$M(t) = \sum_{\tau=1}^{N(t)} M_\tau.$$

In the case of micro-seismicity induced by mining the parameters β in the Pareto distribution and λ in the Poisson process depend on the mining (type of rock, mining technique, speed rate extraction, etc.).

An important relation between frequency and magnitude of the tremors is given by the Gutenberg- Richter law [37]:

$$\log n = a - bm, \tag{2.24}$$

where n is the number of earthquakes with magnitude m, and a and b are parameters. The parameter a is a measure of the level of seismicity, whereas the parameter b, which is typically close to 1, describes the relative number of small and large in a given interval of time.

The seismicity that results from ground deformation associated with a change in volume was described by McGarr [50] in a seminal paper. He observed the following relation between mining and seismicity:

$$\sum M_0 = K\mu|\Delta V|,$$

where ΔV is the volume change (ΔV is positive if the volume is added and ΔV is negative if the volume is removed), $\sum M_0$ is the sum of the seismic moments of the earthquake population, μ is the modulus of rigidity, and K is a factor close to 1. McGarr proved his claim by analyzing the East Rand Proprietary Gold Mine, South Africa, and the the Denver earthquakes in USA.

When mining occurs a rock mass is moved and by the McGarr work it triggers seismic activity. In 1994 Gibowicz & Kijko [34] proposed a similar relation:

$$M_t = K_t \times ton_t, \tag{2.25}$$

where M_t is the sum of seismic moments of the events that occurred in the period t; ton_t is the tonnage of the rock extracted in the period t and K_t is a proportionality constant for the period t, which depends on the type of rock and mining technique and the constant K_{t-1} for the prior period. So the intensity of the seismic activity in one period depends on what happened in the previous period. To the best of our knowledge there are no mechanical models to describe the relation between mining and micro-seismicity.

2.7 Modelling an Underground Mine

To model the mining process in an underground mine we divide the mine in sectors which are then divided in columns and then into blocks as shown in the figure below.

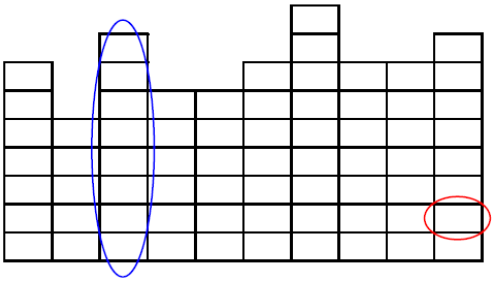

Figure 2.6: Scheme of a sector in an underground mine. In red we observe a block and in blue a column.

More precisely, let S be the set of sectors, $I(s)$ the set of columns in sector $s \in S$ and V_s the set of neighbours columns in sector $s \in S$. Each column $i_s \in I(s)$ is divided in $\{N_{i_s}\}$ blocks which are numbered from the bottom to the top. Let $I = \bigcup_{s \in S} I(s)$ be the set of columns in the whole mine, $N_s = \bigcup_{i_s \in I(s)} N_{i_s}$ the blocks in sector $s \in S$ and $N = \bigcup_{s \in S} N_s$ the set of blocks in the whole mine.

A generic block is represented by $n = (i, j, k, s)$ where i, j and k are the x-coordinate, y-coordinate and z-coordinate of n respectively, and $s \in S$ is the sector where the block is. So, (i, j) is the column to which the block n belongs to. We introduce the notation for a block $n_{i_s} \in \mathbb{Z}^+$ which means that the block is in column $i_s \in I(s)$ which is in sector s and the value of $n_{i_s} \in N_{i_s}$ corresponds to the height where the block is located (recall that the blocks in a column are numbered from the bottom to the top). When the context is clear we omit the subscripts indicating the sector and the column.

To model the extraction and processing in the mine we assign variables to each block. In this work we use three variables associated with blocks: if we denote by n any block, we have a binary variable that takes the value 1 if the block n is extracted and 0 if it is not, a continuous variable that takes real values between 0 and 1 and represents the fraction extracted of the block n and another continuous variable that takes real values between 0 and 1 and represents the fraction processed of the block n. Obviously, the fraction processed is lesser than the fraction extracted. We denote these variables as:

- $x^{\omega}_{n,t} = \begin{cases} 1 & \text{if excavation of block } n \in N \text{ begins at period } t \in \mathcal{T} \text{ and scenario } \omega \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$
- $y^{\omega}_{n,t} \in [0, 1]$: continuous variable representing the fraction of block $n \in N$ to be mined at period t and scenario $\omega \in \Omega$.

- $z^{\omega}_{n,t} \in [0,1]$: continuous variable representing the fraction of block $n \in N$ to be processed at period t and scenario $\omega \in \Omega$.

Here, Ω is the set of scenarios considered in the optimization of the Net Present Value. It can depend on many uncertainties such as prices, grades, seismic, demand, etc. Thus, we have as many variables as blocks, scenarios and periods in the time horizon. In order to reduce the size of the model some authors consider clusters of blocks (see [41],[58]). Instead of considering clusters as in those works, we divide each column in macro blocks with similar ore grade.

The accessibility conditions, also called precedence conditions, are important requirements that have to be fulfilled. These conditions are related to the coherent extraction and the physical stability of the mine and they establish the set of feasible extraction sequences. For underground mines any feasible extraction sequence has to hold the following condition: a block can only be extracted if the preceding blocks have been extracted, where precedents blocks are those close to and adjacent to this block (see Figure 2.7). The main stability criteria is the concavity of the mine cavity and the smoothness of the roof. The smoothness of the roof means that the height of any column cannot differ considerably from the height of its neighbouring columns (see [3]).

Some approaches to model the acessibility conditions are the accessibility matrix [5] which consists in defining a matrix for blocks $k_1, k_2 \in N_s$:

$$A_s(k_1,k_2) = \begin{cases} 1 & \text{if } k_2 \text{ can be extracted only after the extraction of } k_1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $s \in S$ and the precedence sets [3] that consists in defining the sets

$$P_n = \{\text{blocks that have to be extracted before the extraction of block } n\} \subset N_s,$$

for all $n \in N_s$ and $s \in S$.

In this work, we use the approach presented in [2] with clustering (each column is divided in macro blocks with similar ore grade). Our approach is summarized mathematically in the following relations:

$$h^{\omega}_{i,t} - h^{\omega}_{j,t} \leq \delta_s \quad \forall s \in S,\, i,j \in V_k,\, k \in I(s),\, t \in \mathcal{T}, \omega \in \Omega,$$
$$h^{\omega}_{i,t} = 0 \quad i \in B_s,\, \forall s \in S,\, t \in \mathcal{T}, \omega \in \Omega,$$

where $h^{\omega}_{i,t}$ is the height of the column $i \in I(s)$ at period $t \in \mathcal{T}$ and scenario $\omega \in \Omega$, V_k is the set of neighbouring columns to $k \in I(s)$, δ_s is the maximum differential of heights allowed

for sector $s \in S$ and B_s are the columns on the boundary of the sector $s \in S$. The first relation above enforces the smoothness of the roof and the second one, the concavity of the mine cavity.

In terms of the y variables the height is expressed as

$$h^{\omega}_{i,t} = h^{\omega}_{i,t-1} + \sum_{n \in N_i} \alpha_n y^{\omega}_{n,t} \quad \forall s \in S,\, i \in I(s),\, t \in \mathcal{T}, \omega \in \Omega.$$

The approach used in this thesis stands out, from others, because is simple, it can be expressed elegantly and is more precise than the clustering approach that allows clustering of blocks from different columns.

Figure 2.7: Precedence conditions. The point on the top represents the block that will be extracted and the points in the bottom represents the nine blocks that have to be extracted before the extraction of the upper block.

Chapter 3

Stochastic model with price and seismicity uncertainty

3.1 Modelling the uncertainty

3.1.1 Seismic risk scenarios

As was mentioned in Section 2.6, the occurrence of induced micro seismicity in an underground mine depends on the intensity of rock extraction during mining. We need to tackle the exogenous uncertainty due to the micro seismicity. To do this we construct the scenarios modelling the seismic activity by considering the random variable $k_t \equiv k_{s,t}$ in equation

$$M_t = \frac{K_t}{N_t} ton_t = k_t \times ton_t \quad \text{where} \quad k_t := \frac{K_t}{N_t} \tag{3.1}$$

where the time window considered is one month, $[t, t+1)$, M_t is the total seismic moment released in $[t, t+1)$, ton_t is the amount of rock extracted in $[t, t+1)$, K_t is a constant that depends on the value taken in the previous time window, K_{t-1}, N_t is the number of micro tremors that happened in $[t, t+1)$. As we suppose that K_t and N_t depend on the type of the rock and the mining technique, we avoid endogeneus uncertainty by modelling the parameter k_t which does not depend on the decision variables under these assumptions.

To model k_t from the historical data available, which gives the production and seismic events occurrence almost daily, we consider time windows of one month. k_t and suppose that it We have production data and data on seismicity for two sectors in the El Teniente mine: sector A (2000- 2015) and sector B (2006-2012). To estimate and validate the parameters we split the data into a training set and a testing period (for sector A: from 2000-2009 then

2010-2015; for sector B: from 2006-2012, then 2013-2016).

Figure 3.1.1 shows the $log(k_t)$ for the sector A of the El Teniente mine at monthly intervals over the period 2000 to 2016. Visually it is reasonable to use a GBM to model this variable. In the others sectors a similar pattern is observed. So let us suppose that $\{k_t\}$ is a Geometric Brownian motion (GBM).

To estimate its parameters we use Maximum Likelihood Method (Campbell et al., 1997):

$$\hat{\xi} = \frac{1}{nD}\sum_{t=1}^{t_n} \log\frac{k_t}{k_{t-1}},$$
$$\hat{\sigma}^2 = \frac{1}{nD}\sum_{t=1}^{t_n}\left(\log\frac{k_t}{k_{t-1}} - \hat{\alpha}D\right)^2, \tag{3.2}$$

where $\hat{\alpha}$ is the estimate for $\xi = \mu - \sigma^2/2$, $\hat{\sigma}$ is the estimate for σ, n is the cardinality of the train set and t_n is the time splitting training set and test set.

The estimates of the volatility are shown in Table 3.1.1 together with root mean square error (RMSE) and mean absolute percentage error (MAPE). To validate our estimates we plotted the 95% confidence interval around the observed values in the testing period.

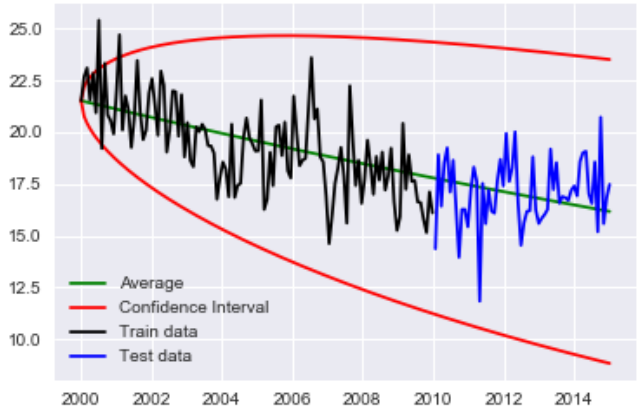

Figure 3.1: $\log(k(t))$ in the sector A.

ParameterSector	**A**	**B**
μ	-0.03539	-2.564742×10^{-5}
σ	0.68856	0.1377444
RMSE	2.9498599	1.92982669
MAPE	13.75909031	25.44987706

Table 3.1: μ, σ, RMSE, MAPE values for Sector A and Sector B sectors

In the sector B the train period considered was 2006-2012 and the test period was 2013-2016. The results are showing in Figure 3.1.1 and Table 3.1.1.

Figure 3.2: $\log(k(t))$ in the sector B.

We constructed the corresponding scenario trees for the period 2010-2014 with the parameters obtained from process k_t as a binomial tree with volatility obtained from (3.2). The drift from (3.2) is negative in both cases which means the seismicity is decreasing over time but we need a conservative approach and to do that it is considered null drift.

Modelling of micro seismicity induced by mining

Hydraulic Fracturing

Hydraulic fracturing (also fracking or hydrofracturing) is a stimulation technique in which rock is fractured by a pressurized liquid. In the oil and gas industry is widely used to create cracks in the deep-rock formations through which natural gas, petroleum, and brine will flow

more freely. In mining by block caving it is used for preconditioning the rock mass fracturing and relaxing it before to be mined. It is a complement of preconditioning by blasting which is used for the same purpose. even though the use of hydraulic fracturing in mining is not new there are few works on the benefits of its use ([19],[20]).

Hydraulic fracturing is designed to establish an efficient cave initiation, to carry out a rapid cave propagation and to obtain smaller magnitude micro tremors. Additionally, finer fragmentation, better management of the stresses are hoped for. From the point of view of mine planning, higher draw rates, faster undercutting rates, etc.

To quantify the achievement of the objectives we present the seismic behaviour with and without hydraulic fracturing in El Teniente mine and we propose a methodology to quantify the benefits of its use.

3.1.2 Study case at El Teniente mine

Hydraulic fracturing (HF) was used to precondition a sector (called Sector A) in the El Teniente underground copper mine. In order to compare the seismicity with and without HF, Codelco provided production data and data on seismicity for Sector A from 1994 to 2017, with no HF until 2007.

3.1.3 Case without hydraulic fracturing

Figure 3.3 shows the magnitude of each seismic event that occurred and Figure 3.4 shows the average magnitude calculated over 30 day periods, that is, the sum of the magnitudes of all that occurred in 30 days divided by the number of events in that time window. The number of seismic events normalized by the production is shown in Figure 3.5. Note the increase after 2015.

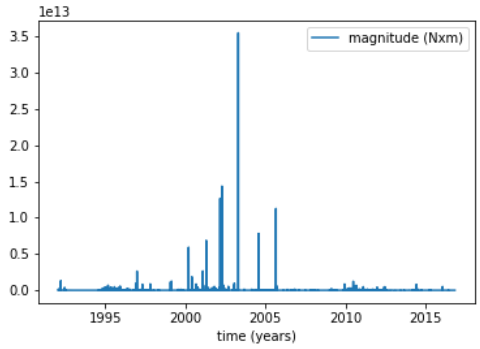

Figure 3.3: M_t in sector A without HF.

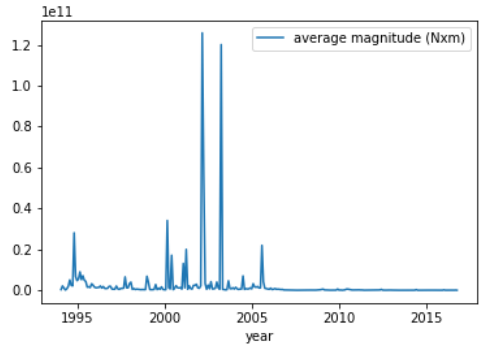

Figure 3.4: m_t in sector A without HF.

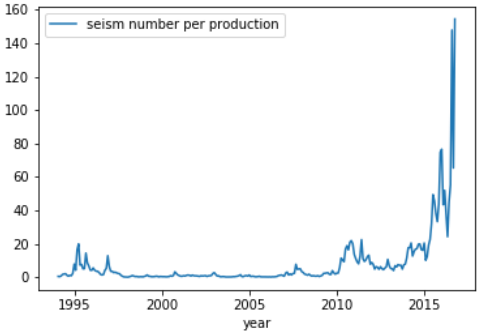

Figure 3.5: Number of earth tremors per meter in sector A without HF.

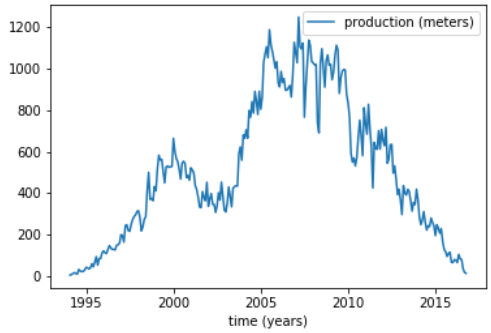

Figure 3.6: Evolution over time of the production in sector A without HF.

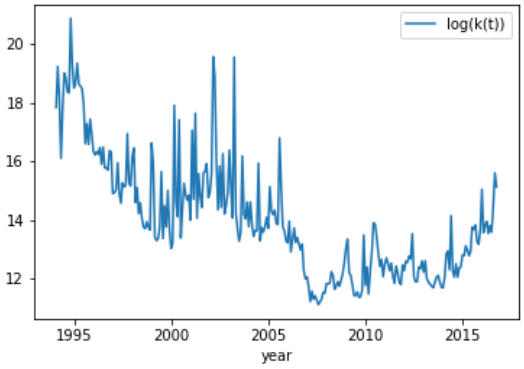

Figure 3.7: $\log(k(t))$ in sector A with HF.

The production in the sector A without preconditioning was increasing up to 2007 but decreased afterwards (3.6).

Before sector A in this mine was preconditioned by HF, there were irregular and highly unpredictable events. After 2007 a slight increase is observed.

The relation between the logarithm of frequency and the magnitude seisms are show-

ing in the figures below. As expected, at sectors A and B is observed the behaviour predicted by Richter-Gutenberg law.

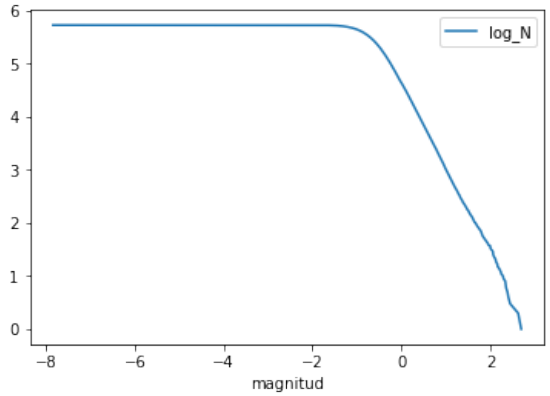

Figure 3.8: Logarithm of frequency vs magnitude seisms in sector A without HF.

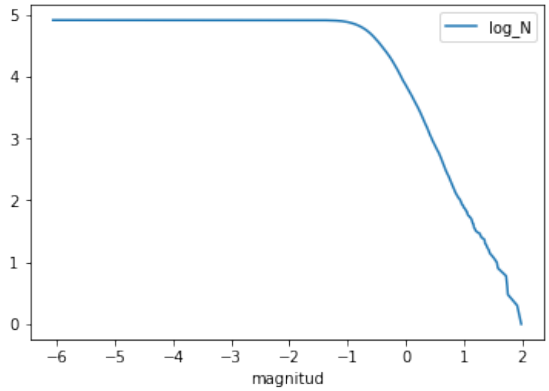

Figure 3.9: Logarithm of frequency vs magnitude seisms in sector A with HF.

3.1.4 Case with hydraulic fracturing

The magnitude of each event occurred is depicted in 3.10.

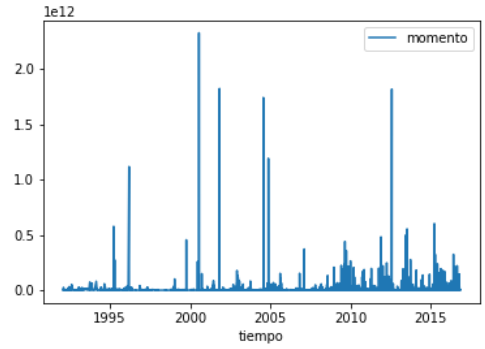

Figure 3.10: M_t in sector A with HF.

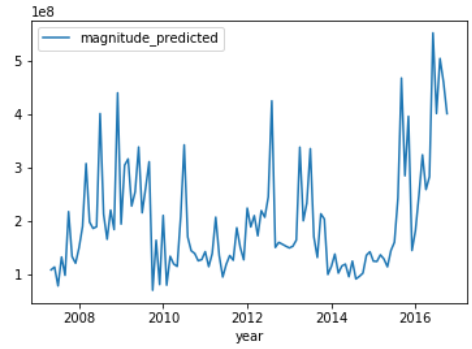

Figure 3.11: m_t in sector A with HF.

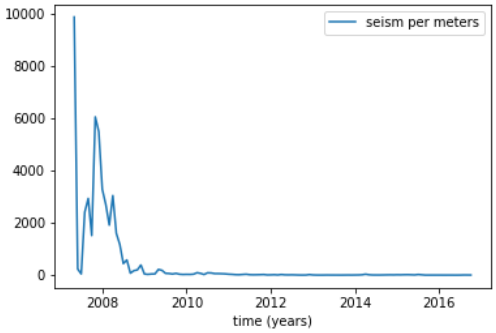

Figure 3.12: Seismic events per meter in sector A with HF.

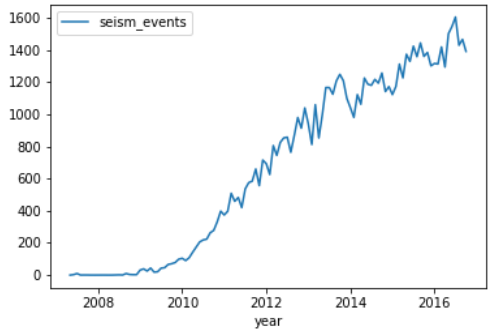

Figure 3.13: Evolution overtime of the production in sector A with HF.

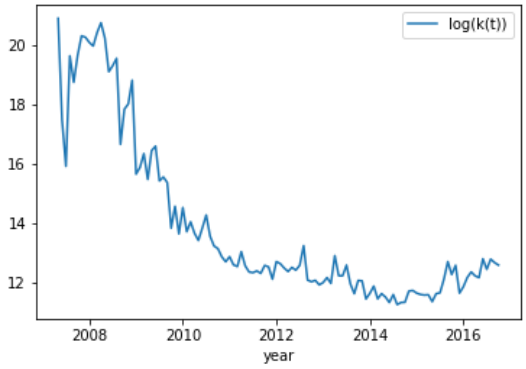

Figure 3.14: $\log(k(t))$ in sector A with HF.

Figure 3.11 shows the average expected magnitude given by (3.1) . The number of seismic events normalized by the production is shown in Figure 3.12. There was a considerably activity when mining started (2007-2008). After that the frequency normalized by production stabilized. The production in sector A with preconditioning increased until today (3.13).

When sector A is preconditioned with HF we observed a abrupt fall in the induced seismicity. Figure 3.14 shows a similar behaviour to the non preconditioning case when mining is beginning. After 2015 a slight increase in the seismicity is observed.

Modelling of metal prices

One of the most traditional approaches for modelling asset prices is the Geometric Brownian Motion. As we can observe in the Figure 3.15 copper prices present an erratic behaviour with a trend to increase which can be modeled as a Geometric Brownian Motion with positive drift.

3.1.5 Prices scenarios

In finance, the Black & Scholes model is widely used to model the continuous option pricing and it is appropriate for metal prices. But for our stochastic model we need to provide a finite number of scenarios for each period and to do that we construct a binomial scenario tree with a time step of 1 year. In contrast to works [3, 18, 35] which use historical data to construct scenarios for prices we prefer to use binomial trees to construct the price scenarios as these are widely used in finance (see Section 2.5). Despite of the tractability of the recombining scenario we have to use a non-recombining binomial tree because decisions are taken at one stage can influence subsequent ones. To construct one for copper prices we have taken initial price equal to $S_0 = 2.5$, annual riskless interest rate $r = 5\%$ and annual volatility $\sigma = 20\%$ \$US/lb. We used the parameters $u = e^{(r^2-\sigma/2)\Delta t+\sigma\sqrt{r}}$ and $d = e^{(r^2-\sigma/2)\Delta t-\sigma\sqrt{r}}$.

Figure 3.15: Historical copper prices.

3.2 Stochastic model with price and seismicity uncertainty

We use the following notation for the sets in our multi-stage stochastic model:

- $\mathcal{T}$: set of periods in the time horizon.
- S: set of sectors.
- $I(s)$: set of columns in sector $s \in S$.
- $I = \cup_{s \in S} I(s)$: set of all columns in the mine.
- V_i: set of columns adjacent to column $i \in I$. Note that if $i \in I(s)$ then $V_i \subset I(s)$, for $s \in S$.
- N_i: set of blocks belonging to column $i \in I(s)$ with $s \in S$. The blocks of a column are numbered in increasing order from the bottom to the top.
- $N(s) = \cup_{i \in I(s)} N_i$: set of blocks belonging to sector $s \in S$.
- $N = \cup_{s \in S} N(s)$: set of all blocks in the mine.
- $Pred(s)$: the set that defines the extraction precedence relationship among columns, that is $(i, j) \in Pred(s)$ if the extraction of column i must begin before it takes place at column j.

We also define the following deterministic parameters:

- T: number of periods in the time horizon.
- D_t: duration of period $t \in \mathcal{T}$.
- ton_s^{ini}: number of tons of rock that have been processed in sector $s \in S$ before the time horizon considered.
- ton_n: tonnage of block $n \in N$.
- α_n: height of block $n \in N$.
- λ_n: ore grade of block $n \in N$.

- $area_i$: basal area of column $i \in I$.
- Δ: maximum number of periods a drawpoint can remain open.
- δ_s: maximum height differential allowed between two neighboring columns in sector $s \in S$.
- $h_{i,min}$: minimum height that can be mined before abandoning a column $i \in I$.
- PC_t: processing capacity of mine at time $t \in \mathcal{T}$.
- $ton^t_{s,max}$ and $ton^t_{s,min}$: maximum and minimum number of tons that must be processed at period $t \in \mathcal{T}$ in sector $s \in S$ respectively.
- $ton^{inc}_{s,t}$ and $ton^{dec}_{s,t}$: maximum increase and decrease of tons of rocks processed in sector s from period $t-1$ to period t, respectively, for $s \in S$, $t \in \mathcal{T} - \{1\}$.
- $area^t_{s,max}$ and $area^t_{s,min}$: maximum and minimum areas that must be incorporated at period $t \in \mathcal{T}$ in sector $s \in S$ respectively.
- ϵ_t: metallurgical recovery factor at time $t \in \mathcal{T}$.
- C^t_s: cost per ton unit of mining and processing in sector $s \in S$ at period $t \in \mathcal{T}$.
- a^t_i: cost of column $i \in I$ that starts production at period $t \in \mathcal{T}$.
- $c^+_{s,t}$: cost per ton unit of production increase in sector s from period $t-1$ to period t, for $s \in S$, $t \in \mathcal{T} - \{1\}$. Note that in a mine there is no cost associated with decreasing production.
- M_{max}: maximum seismic moment that can be resisted by the mine.
- $R_s(M_{max})$: cost to construct a support structure in sector $s \in S$ to withstand induced earthquakes whose seismic moment is equal to or less than M_{max}.
- γ: security factor.

To model uncertainty we introduce the following parameters and notation:

- Ω: set of scenarios.
- $[\omega]_t$: history of $\omega \in \Omega$ up to period $t \in \mathcal{T}$, i.e., if $\omega = (\omega_1, \cdots, \omega_T)$ then $[\omega]_t = (\omega_1, \cdots, \omega_t)$.

- $\sim_t$: equivalence relation over Ω defined by $\omega \sim_t \omega'$ if only if $[\omega]_t = [\omega']_t$, for $t \in \mathcal{T}$.
- w^ω: probability of occurrence of scenario $\omega \in \Omega$.
- p_t^ω: price in scenario $\omega \in \Omega$ at time $t \in \mathcal{T}$.
- $r_{n,t}^\omega = (p_t^\omega \lambda_n \epsilon_n - C_s^t) ton_n$: revenue factor for block $n \in N_s$ in sector $s \in S$ at period $t \in \mathcal{T}$ and scenario $\omega \in \Omega$.
- $k_{s,t}^\omega$: constant to calculate seismic activity by relation (3.1) for sector $s \in S$, scenario $\omega \in \Omega$ at time $t \in \mathcal{T}$.

The variables to model rock extraction are:

- $x_{n,t}^\omega = \begin{cases} 1 & \text{if excavation of block } n \in N \text{ begins in period } t \in \mathcal{T} \text{ and scenario } \omega \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$
- $y_{n,t}^\omega \in [0,1]$: continuous variable representing the fraction of block $n \in N$ to be mined in period t and scenario $\omega \in \Omega$.
- $ton_{\omega,s,t}^-, ton_{\omega,s,t}^+ \in [0,\infty)$: decrease and increase in the number of tons extracted in sector $s \in S$ at period $t \in \mathcal{T}$ and scenario $\omega \in \Omega$, respectively.

Maximizing the NPV of whole mine

We want to maximize the NPV of whole mine, i.e.:

$$\max \sum_{\omega\in\Omega}\sum_{t\in\mathcal{T}} w^\omega \sum_{s\in S}\left\{\sum_{n\in N_s} r_{n,t}^\omega y_{n,t}^\omega - \sum_{i\in I(s)} a_i x_{1_i,t}^\omega - \sum_{n\in N_s} c_{s,t}^+ ton_{\omega,s,t}^+\right\} - \sum_{s\in S} R_s(M_{max}). \quad (3.3)$$

The first term represents the profit due to selling the metal, the second one is the cost associated with starting production in the columns, the third one is the cost to increase the production and the last term is the cost to construct a resistant structure in the mine.

The problem is subject to the following constraints:
Each block can be extracted at most once during the time horizon:

$$\sum_{t\in\mathcal{T}} x_{n,t}^\omega \leq 1 \quad \forall s \in S,\, n \in N(s),\, \omega \in \Omega. \quad (3.4)$$

Logical relations between binary and continuous variables have to be satisfied:

$$\sum_{t'=1}^{t} y_{n,t'}^\omega \leq \sum_{t'=1}^{t} x_{n,t'}^\omega \quad \forall s \in S,\, n \in N(s),\, t \in \mathcal{T},\, \omega \in \Omega. \quad (3.5)$$

The extraction of block n must finish before the extraction of block $n+1$ can start at each drawpoint:

$$\sum_{t'=1}^{t} x^{\omega}_{n+1,t'} \leq \sum_{t'=1}^{t} y^{\omega}_{n,t'} \quad \forall s \in S,\, n \in I(s),\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.6}$$

The order in which the drawpoints can enter production is determined:

$$\sum_{t'=1}^{t} x^{\omega}_{1_i,t'} \leq \sum_{t'=1}^{t} y^{\omega}_{1_j,t'} \quad \forall s \in S,\, i,j \in Pred(s),\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.7}$$

We define the height of the column in each period:

$$h^{\omega}_{i,t} = h^{\omega}_{i,t-1} + \sum_{n \in N_i} \alpha_n y^{\omega}_{n,t} \quad \forall s \in S,\, i \in I(s),\, t \in \mathcal{T}, \omega \in \Omega. \tag{3.8}$$

Neighboring columns must have similar height so that the rock breaks smoothly:

$$h^{\omega}_{i,t} - h^{\omega}_{j,t} \leq \delta_s \quad \forall s \in S,\, i,j \in V_k,\, k \in I(s),\, t \in \mathcal{T}, \omega \in \Omega \tag{3.9}$$

The variation in the production is defined:

$$ton^{+}_{\omega,s,t} - ton^{-}_{\omega,s,t} = \begin{cases} \sum_{n \in N_s} y^{\omega}_{n,1} ton_n - ton^{ini}_s & \text{if } t = 1, \\ \sum_{n \in N_s} (y^{\omega}_{n,t} - y^{\omega}_{n,t-1}) ton_n & \text{if } t > 1, \end{cases} \quad \forall s \in S,\, \omega \in \Omega, t \in \mathcal{T}. \tag{3.10}$$

The minimum height must be removed if a column enters production:

$$\sum_{n \in N_i} \sum_{t \in \mathcal{T}} y^{\omega}_{n,t} \geq h_{i,min} \cdot \sum_{t \in \mathcal{T}} x^{\omega}_{1_i,t} \quad \forall s \in S,\, i \in I(s),\, \omega \in \Omega. \tag{3.11}$$

The processing capacity cannot be exceed:

$$\sum_{s \in S} \sum_{n \in N(s)} y^{\omega}_{n,t} ton_n \leq PC_t \quad \forall t \in \mathcal{T}, \omega \in \Omega. \tag{3.12}$$

Each drawpoint has a maximum duration:

$$x^{\omega}_{1_i,t} + \sum_{t'=t+\Delta}^{T} y^{\omega}_{n,t} \leq 1 \quad \forall s \in S,\, t \in \mathcal{T},\, i \in I(s),\, n \in N_i,\, \omega \in \Omega. \tag{3.13}$$

The area incorporated per period is bounded:

$$area^{t}_{s,min} \leq \sum_{i \in I(s)} x^{\omega}_{1_i,t} area_{1_i} \leq area^{t}_{s,max} \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.14}$$

The rock extracted per period is bounded:

$$ton^{t}_{s,min} \leq \sum_{n \in N(s)} y^{\omega}_{n,t} ton_n \leq ton^{t}_{s,max} \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.15}$$

The increase and decrease of production are bounded:

$$0 \leq ton^{+}_{\omega,s,t} \leq ton^{inc}_{s,t} \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega, \tag{3.16}$$

$$0 \leq ton^{-}_{\omega,s,t} \leq ton^{dec}_{s,t} \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.17}$$

Strong earth tremors must be prevented:

$$k^{\omega}_{s,t} \sum_{n \in N(s)} y^{\omega}_{s,t} ton_n \leq \gamma \cdot M_{max} \quad \forall s \in S,\, \forall t \in \mathcal{T},\, \omega \in \Omega. \tag{3.18}$$

Non anticipative constraints must be satisfied:

$$x^{\omega}_{n,t} = x^{\omega'}_{n,t}, \tag{3.19}$$

$$y^{\omega}_{n,t} = y^{\omega'}_{n,t}, \tag{3.20}$$

$$ton^{+}_{\omega,s,t} = ton^{+}_{\omega',s,t}, \tag{3.21}$$

$$ton^{-}_{\omega,s,t} = ton^{-}_{\omega',s,t}, \;\; \omega \sim_t \omega', \tag{3.22}$$

The decision variables $x^{\omega}_{n,t}$ are binary:

$$x^{\omega}_{n,t} \in \{0,1\} \quad \forall s \in S,\, n \in N(s),\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.23}$$

The decision variables $y^{\omega}_{n,t}$ are continuous and bounded:

$$y^{\omega}_{n,t} \in [0,1] \quad \forall s \in S,\, n \in N(s),\, t \in \mathcal{T},\, \omega \in \Omega. \tag{3.24}$$

The decision variables $ton^{+}_{s,t}$ and $ton^{-}_{s,t}$ are non negative:

$$ton^{+}_{\omega,s,t} \geq 0 \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega, \tag{3.25}$$

$$ton^{-}_{\omega,s,t} \geq 0 \quad \forall s \in S,\, t \in \mathcal{T},\, \omega \in \Omega \tag{3.26}$$

Remark 3.1 • *As [3] the constraints related to increments and decrements on the production are given by (3.10):*

$$ton^{+}_{\omega,s,t} - ton^{-}_{\omega,s,t} = \begin{cases} \sum_{n \in N_s} y^{\omega}_{n,1} ton_n - ton^{ini}_{s} & \text{if } t = 1, \\ \sum_{n \in N_s} (y^{\omega}_{n,t} - y^{\omega}_{n,t-1}) ton_n & \text{if } t > 1, \end{cases} \quad \forall s \in S,\, \omega \in \Omega, t \in \mathcal{T}.$$

Clearly, the production cannot increase and decrease in the same period, therefore for any $s \in S, \omega \in \Omega, t \in \mathcal{T}$ we should have $ton^{+}_{\omega,s,t} = 0$ or $ton^{-}_{\omega,s,t} = 0$, i.e., $ton^{+}_{\omega,s,t} \cdot ton^{-}_{\omega,s,t} = 0$. In general, this condition is not true, and it is necessary to linearize the nonlinear condition $ton^{+}_{\omega,s,t} \cdot ton^{-}_{\omega,s,t} = 0$ by introducing binary auxiliary variables which

would add more numerical complications to our problem. Fortunately, for our problem it is not necessary, because the condition $ton^{+}_{\omega,s,t} \cdot ton^{-}_{\omega,s,t} = 0$ *is implied from (3.10). Indeed, if* $(x^{\omega}_{n,t}, y^{\omega}_{n,t}, ton^{+}_{\omega,t}, ton^{-}_{\omega,t})$ *denote a solution for problem (3.3)-(3.26), suppose by contradiction that* $ton^{+}_{\omega,s,t} > 0$ *and* $ton^{-}_{\omega,s,t} > 0$. *Then, we can write* $ton^{-}_{\omega,s,t} = A + B$ *where* $A, B > 0$. *Thus,* $\overline{ton}^{+}_{\omega,s,t} = ton^{+}_{\omega,s,t} - A$ *and* $\overline{ton}^{+}_{\omega,s,t} = B$ *satisfied (3.10) and the objective function (3.3) would increase for these values. Therefore,* $(x^{\omega}_{n,t}, y^{\omega}_{n,t}, ton^{+}_{\omega,t}, ton^{-}_{\omega,t})$ *would not be optimal.*

- *The constraint (3.12) connects the production in all sectors of the mine which requires calculating the NPV using all blocks but it will present computational problems due the problem size. To avoid the curse of dimensionality we present a strategy in the next Section.*

- *Note that the constraints (3.10) and (3.12)-(3.18) limit the extraction speed. In other works such as [2] and [3] these constraints are considered except the constraint (3.18) which limits the extraction speed for safety reasons.*

3.3 Solution method based on Lagrangian Relaxation

3.3.1 Lagrangian Relaxation (LR)

In order to simplify the notation we consider only one sector which allows us to drop the index s referring to the sectors. The approach presented in this section for a mine with more than one sector follows trivially.

As in Carvallo (2009) we apply LR to relax the non anticipative constraints (Carøe & Schultz, 1999 [16]). This will allow us to divide the problem (3.3)-(3.26) into $|\Omega|$ sub-problems.

Now we introduce some notation to rewrite the non anticipative constraints NAC. Let Ω_t be the equivalence class determined by $\sim_t$ and let $[\omega]_t = \{\omega^t_1, \cdots, \omega^t_{m_t}\}$ be an element of Ω_t where $\omega^t_1 < \cdots < \omega^t_{m_t}$, $t \in \mathcal{T}$. This can be represented through a scenario tree where each node at level t represents an equivalence class $[\omega]_t$ (called scenario group) and a child of node $[\omega]_t \in \Omega_t$ is a element $[\omega]_{t+1} \in \Omega_{t+1}$ such that $[\omega]_{t+1} \subset [\omega]_t$. Note that if any node has the same number of children as the cardinality of any equivalence class determined by

$\sim_t$ is the same for $t = 1 \dots \mathcal{T}$. Then, the constraints (3.19) can be rewritten as:

$$(x^{\omega}_{n,t}, y^{\omega}_{n,t}, ton^{+}_{\omega,t}, ton^{-}_{\omega,t}) - (x^{\omega+1}_{n,t}, y^{\omega+1}_{n,t}, ton^{+}_{\omega+1,t}, ton^{-}_{\omega+1,t}) = 0,$$

$\forall n \in N$, $\omega \in [\omega]_t - \{\omega^t_{m_t}\}, [\omega]_t \in \Omega_t, t \in \mathcal{T}$.

We proceed to define the Lagrangian associated with the problem (3.3)-(3.26) where the non anticipative constraints (3.19) are relaxed. Define the Lagrangian

$$L := L(x^{\omega}_{n,t}, y^{\omega}_{n,t}, ton^{+}_{\omega,t}, ton^{-}_{\omega,t}, \lambda^{\omega}_{n,t}, \mu^{\omega}_{n,t}, \rho^{\omega}_{t}, \theta^{\omega}_{t})$$

by

$$\begin{aligned} L = \sum_{t\in\mathcal{T}} \Bigg\{ \sum_{\omega\in\Omega} w^{\omega} \Bigg[\sum_{n\in N} p^{\omega}_{t} \lambda_n ton_n y^{\omega}_{n,t} - \sum_{i\in I} a_i x^{\omega}_{1_i,t} - \sum_{n\in N} c^{+}_{t} ton^{+}_{\omega,t} \Bigg] \\ + \sum_{\omega\in\Omega-\{|\Omega|\}} \Bigg[\sum_{n\in N} \lambda^{\omega}_{n,t}(x^{\omega}_{n,t} - x^{\omega+1}_{n,t}) + \sum_{n\in N} \mu^{\omega}_{n,t}(y^{\omega}_{n,t} - y^{\omega+1}_{n,t}) + \rho^{\omega}_{t}(ton^{+}_{\omega,t} - ton^{+}_{\omega+1,t}) \\ + \theta^{\omega}_{t}(ton^{-}_{\omega,t} - ton^{-}_{\omega+1,t}) \Bigg] \Bigg\}, \end{aligned}$$

where $\Lambda_{\omega,n,t} := (\lambda^{\omega}_{n,t}, \mu^{\omega}_{n,t}, \rho^{\omega}_{t}, \theta^{\omega}_{t})$ are the Lagrange multipliers and $\lambda^{\omega}_{n,t} = \mu^{\omega}_{n,t} = \rho^{\omega}_{t} = \theta^{\omega}_{t} = 0$ if $\omega = \omega^t_{m_t}$. If we are using a scenario tree where each node has d children, the equality $\omega = \omega^t_{m_t}$ is equivalent to saying that ω is a multiple of $d^{T-(t-1)}$, i.e., $\omega = kd^{T-(t-1)}$ for $k = 1, \dots, d^{t-1}$.

Reordering terms the Lagrangian can be written in the form $L = \sum_{\omega\in\Omega} L_\omega$ where L_ω depends on $x^{\omega}_{n,t}, y^{\omega}_{n,t}, ton^{+}_{\omega,t}, ton^{-}_{\omega,t}, \lambda^{\omega}_{n,t}, \mu^{\omega}_{n,t}, \rho^{\omega}_{t}, \theta^{\omega}_{t}$ and it is defined by:

$$\begin{aligned} L_\omega = \sum_{t\in\mathcal{T}} \Bigg\{ w^{\omega} \Bigg[\sum_{n\in N} r^{\omega}_{n,t} y^{\omega}_{n,t} - \sum_{i\in I(s)} a_i x^{\omega}_{1_i,t} - \sum_{n\in N} c^{+}_{t} ton^{+}_{\omega,t} \Bigg] + \sum_{n\in N} (\lambda^{\omega}_{n,t} - \lambda^{\omega-1}_{n,t}) x^{\omega}_{n,t} \\ + \sum_{n\in N} (\mu^{\omega}_{n,t} - \mu^{\omega-1}_{n,t}) y^{\omega}_{n,t} + (\rho^{\omega}_{t} - \rho^{\omega-1}_{t}) ton^{+}_{\omega,t} + (\theta^{\omega}_{t} - \theta^{\omega-1}_{t}) ton^{-}_{\omega,t} \Bigg\}, \end{aligned}$$

for $\omega \neq 1$ and $\omega \neq |\Omega|$. For $\omega = 1$ we have

$$\begin{aligned} L_1 = \sum_{t\in\mathcal{T}} \Bigg\{ w^{1} \Bigg[\sum_{n\in N} r^{1}_{n,t} y^{1}_{n,t} - \sum_{i\in I(s)} a_i x^{1}_{1_i,t} - \sum_{n\in N} c^{+}_{t} ton^{+}_{1,t} \Bigg] + \sum_{n\in N} \lambda^{1}_{n,t} x^{1}_{n,t} \\ + \sum_{n\in N} \mu^{1}_{n,t} y^{1}_{n,t} + \rho^{1}_{t} ton^{+}_{1,t} + \theta^{1}_{t} ton^{-}_{1,t} \Bigg\}, \end{aligned}$$

and for $\omega = |\Omega|$ we have:

$$L_{|\Omega|} = \sum_{t\in\mathcal{T}}\Bigg\{ w^{|\Omega|}\left[\sum_{n\in N} r_{n,t}^{|\Omega|}y_{n,t}^{|\Omega|} - \sum_{i\in I(s)} a_i x_{1_i,t}^{|\Omega|} - \sum_{n\in N} c_t^{+} ton_{|\Omega|,t}^{+}\right] - \sum_{n\in N}\left[\lambda_{n,t}^{|\Omega|-1}x_{n,t}^{|\Omega|} + \sum_{n\in N}\mu_{n,t}^{|\Omega|-1}y_{n,t}^{|\Omega|} + \rho_t^{|\Omega|-1}ton_{|\Omega|,t}^{+} + \theta_t^{|\Omega|-1}ton_{|\Omega|,t}^{-}\right]\Bigg\},$$

We are interested in solving

$$z = \min_{\Lambda_{\omega,n,t}} \Theta(\Lambda_{\omega,n,t}) \tag{3.27}$$

where Θ is the dual function defined by

$$\Theta(\Lambda_{\omega,n,t}) := \max\{L : (x_{n,t}^{\omega}, y_{n,t}^{\omega}, ton_{\omega,t}^{+}, ton_{\omega,t}^{-}) \text{ satifies } (3.4)-(3.18),\ (3.23)-(3.26)\}. \tag{3.28}$$

The following weak duality result is well known and can be found in Nemhauser & Wolsey (1988).

Proposition 3.1 *The optimal value of the Lagrangian dual (3.27) is an upper bound of the optimal value of (3.3)-(3.26). If for some choice $(\lambda_{n,t}^{\omega}, \mu_{n,t}^{\omega}, \rho_t^{\omega}, \theta_t^{\omega})$ of Lagrangian multipliers the corresponding solution $(x_{n,t}^{\omega}, y_{n,t}^{\omega}, ton_{\omega,t}^{+}, ton_{\omega,t}^{-})$ with $n \in N$, $t \in \mathcal{T}$, $\omega \in \Omega$, of the Lagrangian relaxation (3.28), then $(x_{n,t}^{\omega}, y_{n,t}^{\omega}, ton_{\omega,t}^{+}, ton_{\omega,t}^{-})$ is an optimal solution of the problem (3.3)-(3.26) and $(\lambda_{n,t}^{\omega}, \mu_{n,t}^{\omega}, \rho_t^{\omega}, \theta_t^{\omega})$ is an optimal solution of (3.27).*

The Lagrangian dual (3.27) is a convex non-smooth program which can be solved by subgradient methods. A major advantage is that it splits into separate subproblems for each scenario,

$$\Theta(\Lambda_{\omega,n,t}) = \sum_{\omega\in\Omega}\Theta_{\omega}(\Lambda_{\omega,n,t})$$

where

$$\Theta_{\omega}(\Lambda_{\omega,n,t}) = \max\{L_{\omega} : (x_{n,t}^{\omega}, y_{n,t}^{\omega}, ton_{\omega,t}^{+}, ton_{\omega,t}^{-}) \text{ satifies } (3.4)-(3.18),\ (3.23)-(3.26)\}. \tag{3.29}$$

3.3.2 The volume algorithm and obtaining feasible solutions

To solve the minimization problem (3.27) we need to update the Lagrangian multipliers $\Lambda_{\omega,t}$ in an efficient way (the index for blocks is omitted in this section). For this, we use the volume algorithm, an extension of the subgradient algorithm, which can be seen as a fast way to approximate the Dantzig-Wolfe decomposition (Barahona & Anbil, 2000).

The update of $\Lambda_{\omega,t}$ is given by:

$$Z^1_{\omega,t} = \eta((X^\omega_t, Y^\omega_t) - (X^{\omega+1}_t, Y^{\omega+1}_t)), \tag{3.30}$$

$$Z^i_{\omega,t} = \eta((X^\omega_t, Y^\omega_t) - (X^{\omega+1}_t, Y^{\omega+1}_t)) + (1-\eta)Z^{i-1}_{\omega,t}, \quad \forall i > 1 \tag{3.31}$$

$$\overline{\Lambda}^i_{\omega,t} = \text{argmin}\{L(\Lambda^k_{\omega,t}) : k = 1, \ldots, i-1\}, \quad \forall i \geq 1 \tag{3.32}$$

$$\Phi^1_{\omega,t} = \Lambda^1_{\omega,t}, \tag{3.33}$$

$$\Phi^i_{\omega,t} = \begin{cases} \Lambda^i_{\omega,t} & \text{if } L(\Lambda^i_{\omega,t}) < L(\overline{\Lambda}^i_{\omega,t}), \\ \Phi^{i-1}_{\omega,t} & \text{in other case,} \end{cases} \quad \forall i > 1 \tag{3.34}$$

$$FS^i = \max_{0 \leq k \leq i} FS_k \tag{3.35}$$

$$\beta^i_{\omega,t} = \rho \frac{L(\overline{\Lambda}^i_{\omega,t}) - FS^i}{||(X^\omega_t, Y^\omega_t) - (X^{\omega+1}_t, Y^{\omega+1}_t)||^2}, \tag{3.36}$$

$$\Lambda^{i+1}_{\omega,t} = \Phi^i_{\omega,t} + \beta^i_{\omega,t} Z^i_{\omega,t}, \tag{3.37}$$

where X^ω_t and Y^ω_t represent the binary and continuous solution for maximization problem (3.28) respectively, $\eta, \rho \in \mathbb{R}^+$, FS_k is the feasible solution of (3.3)-(3.26) obtained at iteration k. Note that $Z^i_{\omega,t}$ is the value of violation of the NAC constraints at iteration i, $L(\overline{\Lambda}^i_{\omega,t})$ is the best Lagrangian solution at the iteration i and FS^i is the best feasible solution of problem (3.3)-(3.26) at the iteration i.

Next we use the wait and see with worst scenario approach (Carvallo, 2009) to obtain feasible solutions FS_k from solutions of the relaxed subproblems (3.29). The worst scenario corresponds to the highest seismic activity and the lowest metal price. Let $\xi_{WC}(t, \omega)$ be the worst case scenario passing on node (t, ω).

The main difficulty to solve (3.3)-(3.26) arises from the binary variables x's. So we proceed to fix them using a wait and see with worst scenario approach: Firstly, we fix the variables $x^\omega_{n,t'} = x^{\xi_{WC}(\omega,T)}_{n,t'} = x^{\xi_{WC}}_{n,t'}$ for $t' = 1, \cdots T-1$ and for period $t = T$ solve the lagrangian subproblems with the variables just fixed. Let $\bar{x}^{\omega,T-1}_{n,t'}$ be the feasible solution obtained.

For the period $t \in \mathcal{T} - \{T\}$ and scenario $\omega \in \Omega$ we fix the variables $x^\omega_{n,t'} = \bar{x}^\omega_{n,t'}$ for $t' \neq t$ and solve the lagrangian subproblems with these fixed variables to obtained a solution $x^{\xi_{WC}(\omega,t'),t}_{n,t'}$. If the problem is infeasible, the feasible solution is $\{\bar{x}^{\omega,t-1}_{n,t'}\}_{t',\omega,n}$, otherwise, let $\bar{x}^{\omega,t}_{n,t'}$ be the current feasible solution where $\bar{x}^{\omega,t}_{n,t'} = \bar{x}^{\omega,t-1}_{n,t'}$ for every $t' \neq t$ and $\bar{x}^{\omega,t}_{n,t} = x^{\xi_{WC}(\omega,t),t}_{n,t}$.

Algorithm 3 to obtain a feasible solution at iteration i

Initialization

$t = T$. Solve the subproblem for scenario ξ_{WC} and put $\bar{X}_t^{\omega} = X_t^{\xi_{WC}}$ for every $\omega \in \Omega$ and $t \in \mathcal{T} - \{t\}$.

Step 1 *Solve the Lagrangian Relaxation Subproblems (3.29) for scenarios* $\{\xi_{WC}(\omega, t) : \omega \in \Omega\}$ *with the additional constraints* $X_{t'}^{\omega} = \bar{X}_{t'}^{\omega}$ *for all* $t' \in \mathcal{T} - \{t\}$.

if $t > 1$ *and the subproblems are feasible and the solution's are* $\bar{X}_t^{\xi_{WC}(\omega,t)}$ *for every* $\xi_{WC}(\omega, t)$ *for all* $\omega \in \Omega$ ***then***

Put $\bar{X}_t^{\omega} = \bar{X}_t^{\xi(\omega,t)}$ *and* $\tilde{X}_{t'}^{\omega} := \bar{X}_{t'}^{\omega}$ *for every* $\omega \in \Omega$, $t' \in \mathcal{T}$.

if $OBJ(\bar{X}_t^{\xi(\omega,t)}) > BF$ ***then***

do $BF = OBJ(\bar{X}_t^{\xi(\omega,t)})$ *and* $t = t - 1$ *and repeat the Step 1.*

end if

else

Stop. The feasible solution is $\tilde{X}_t^{\omega}$.

end if

The feasible solution is $\bar{X}_t^{\omega}$.

Algorithm 4 based on Volume Algorithm

Initialization

$Z_{feasible} := 0$ *initial value of the feasible solution.*

$\bar{Z} = Z_{LP}$ *the value of linear relaxation is the initial value for the upper bound.*

$\Lambda := 0$ *initial values of the Lagrangian multipliers.*

Step 1

Split the relaxed problem for the multipliers Λ *in* $|\Omega|$ *sub problems and solve each one separately. Obtain the solution of the Lagrangian Relaxation problem,* Z_{LR}*, summing the values of the subproblems.*

Step 2

Obtain a feasible solution from Z_{LR} *from the Algorithm 1.*

Step 3

if $Z_{Heuristic} < Z_{feasible}$ **then**

$Z_{feasible} := Z_{Heuristic}$

end if

Step 4

if $Z_{LR} < \bar{Z}$ **then**

$\bar{Z} := Z_{LR}$ *update the best upper bound.*

end if

Step 5

if $\frac{\bar{Z} - Z_{feasible}}{\bar{Z}} \times 100 < tolerance$ **then**

stop.

else *Update the Lagrangian multipliers* Λ *using (3.30) and go to step 1.*

end if

3.3.3 Solving the subproblems

The resolution of the problem (3.3)-(3.26) using the heuristic presented before implies the resolution of many subproblems as many scenarios we consider at each iteration. We consider two sources of uncertainty (prices and seismicity) which in numerical experiments means 256 scenarios. Therefore, we need an heuristic to solve quickly the subproblems which consider binary variables.

To solve the subproblems associated to scenarios quickly we used a simple but effec-

tive heuristic presented on [2]. It consists in solving the linear relaxation of each subproblem.

3.4 Numerical results

In the numerical experiments we used Lagrangian Decomposition on the NAC constraints and Volume Algorithm to update the Lagrangian multipliers. To find feasible solutions we used two approaches: one based one the static worst case approach and the other one based on the dynamic worst case approach.

The strategy based on the static worst case approach consists in obtaining a feasible solution fixing the variables to the solutions under the worst case up to stage t (in our experiments $t = 4$) and in solving a new problem with the fixed variables up to stage t. It presents marginal improvements on the feasible solutions obtained, which implies that the best feasible solution found is similar to the initial one. It suggests that this methodology is inflexible and inefficient because the worst case is so conservative.

On other hand, the strategy on the dynamic worst case scenario implies firstly, that we fix all variables to the solution of the worst case and regressively we re-fix the variables from the last stage by the dynamic worst case scenario approach. This strategy gives better results than the static case.

We test our model with real data from El Teniente the largest underground copper mine located in O'Higgins Region, Chile. The time horizon for our experiment is 5 years and two of the nine sectors of El Teniente are considered active.

Note that the size of our optimization problem grows dramatically if we consider several sectors because parameters $k^{\omega}_{s,t}$ in equation (3.18) depend on the sectors. Thus, if two sectors and two scenarios are considered at each stage for parameter $k^{\omega}_{s,t}$ we have $2^5 \times 2^5 \times 2^5 = 32768$ scenarios. It makes our problem too big and too hard to solve. To avoid this difficulty we solve the problem just for price uncertainty, (P_{price}), and replace the production constraints (3.12) by the condition,

$$\sum_{s \in S} ton^{t}_{s,max} \leq PC_{s,t} \quad \forall s \in S, t \in \mathcal{T}, \tag{3.38}$$

where $PC_{s,t}$ is the rock extracted in sector s at time t for the problem (P_{price}). Since constraints (3.12) are the only constraints that link the sectors, their substitution by (3.38) allows us to split our problem involving all sectors into smaller subproblems considering just one sector. These problems can be solved independently of each other, and this considerably

reduces the size of the problems that we have to solve. It is reasonable because firstly we fix the amount of rock to be extracted in each sector and at each period considering just price uncertainty and limits on capacity process and after that we applied a 'correction' considering seismic uncertainty.

Thus each subproblem involves $2^4 \times 2^4 = 512$ scenarios representing price and seismic risk uncertainty. To determine seismic risk scenarios we use historical data corresponding to the period 1992 to 2011. Our time horizon corresponds to the period 2012-2017.

As was mentioned, the binomial non-recombining price tree is constructed with volatility equal to 20% and initial price equal to 2.5 $US/lb.

In sector A, for our time horizon we consider 900 columns. The numerical results for NPV corresponding to sector A taking into account seismic risk are summarized in the Table (3.4).

NPV (millions of dollars)	1759.01
Binary Variables	144,000
Continuous Variables	144,160
Constraints	3,025,312
GAP(%)	1.10
Time(seconds)	7,690.56

Table 3.2: NPV of the sector A with finance & seismic risk.

Table (3.4) gives the expected, the maximum (best case) and the minimum (worst case) NPV for the proposed stochastic model and its standard deviation.

Expected value	1759.01
Maximum	2601.97
Minimum	1202.76
Deviation	842.296

Table 3.3: NPV values of the stochastic model with finance in millions of dollars & seismic risk.

We note that the stochastic optimization provides considerably better results than the worst case, which justifies the benefits of using stochastic models in mine planning instead

of using very conservative approaches such as the worst case scenario methodology. But the results are still conservative and reliable.

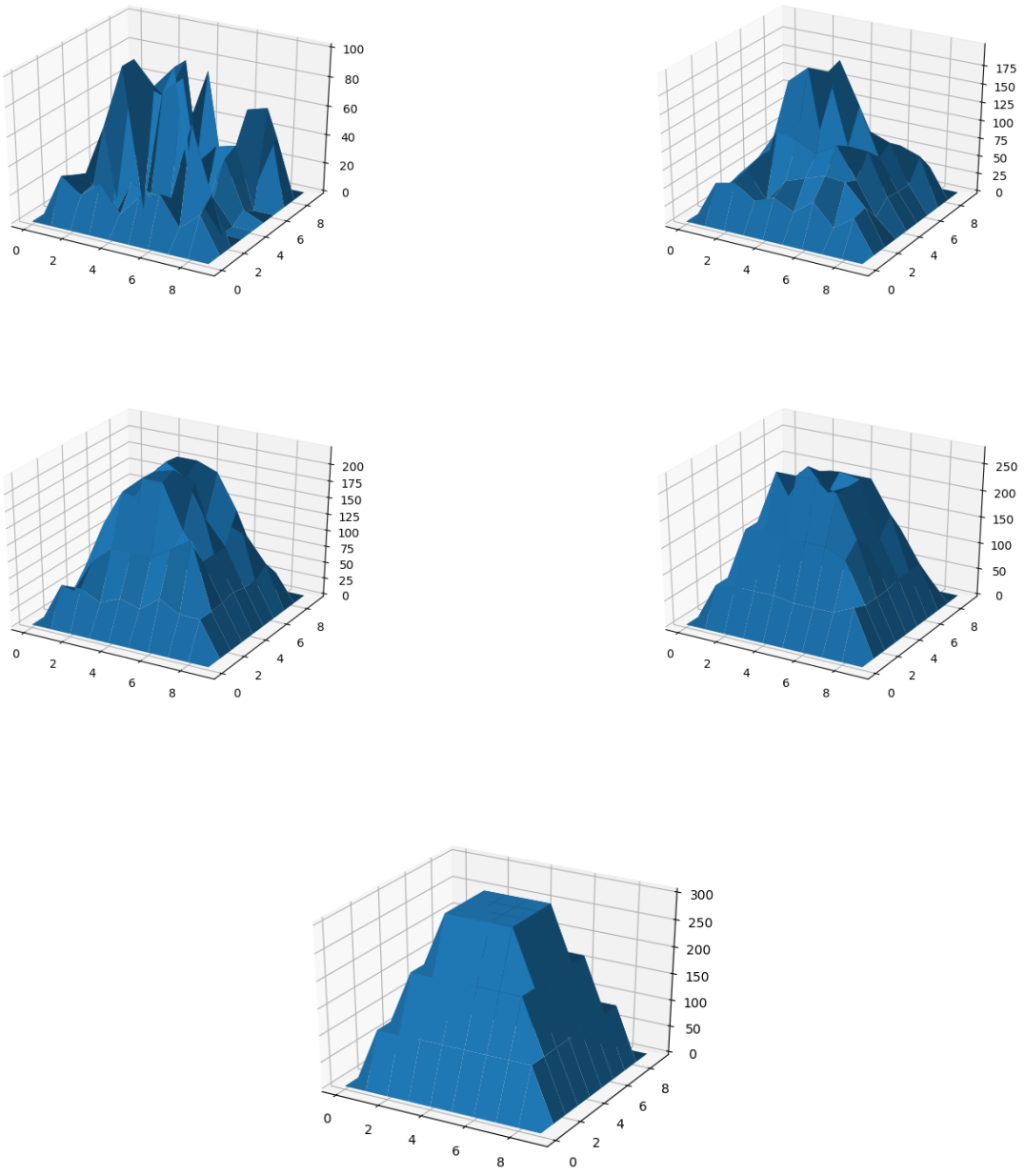

Figure 3.16: Evolution over time of the production in sector A with HF along the time planning horizon of five years.

3.4.1 Sensitivity Analysis

Sensitivity analysis was performed for prices and seismicity scenarios. As expected, the model is highly sensitive to the copper prices because the operational income depends directly on

that. Additionally, a high sensitivity is observed on seismic activity because the rates of extraction depend strongly on that.

Supposing that the NPV depends just on the price and the seismic activity and both are independent, using multiple linear regression we obtain:

$$NPV_h = aP_0 + \sum_{j=1}^{S} b_j K_0^j$$
$$NPV_l = aP_0 + \sum_{j=1}^{S} b_j K_0^j$$

where P_0 is the price at time $t = 0$ and K_0^j is the value for the parameter K of sector j in (3.18) at period $t = 0$.

Price ($ US per pound)	NPV (millions of dollars)	Rate extraction (tons per day)
2.4	1459.11	92959
2.5	1759.01	92959
2.6	2058.91	92959

Table 3.4: Sensitivity on prices

3.4.2 Model with price uncertainty vs model with price and seismic uncertainty

As expected, the model with uncertainty on price and seismicity gives a lower NPV than the model with just price uncertainty. At the same time, the model with more uncertainty is more reliable than model with less because of the value of standard deviation.

3.4.3 Comparison between models with and without HF

Clearly, the use of HF improves the block caving performance by diminishing the occurrence of induced micro tremors during the mining. The mining operations become safer, allowing more efficient rates of rock extraction. However, the disadvantage of HF is the high cost that it implies.

	with HF	without HF
Expected value (millions of dollars)	1759.01	
Average extraction rate (tons per day)	92958.90	

Table 3.5: Comparison between case with HF and case without HF

Chapter 4

Conclusions and future work

4.1 Conclusions and future work

As more and more open-pit mines are moving underground, it is becoming increasingly important to develop methods for optimizing mining schedules subject to the specific access and security constraints involved. The removal of large volumes of ore during underground mining causes tension in the remaining rock mass. The faster material is removed, the more support is required to avoid rock falls and injury and loss of life and this support is costly.

This thesis addresses the question of finding the optimal mining sequence subject to two sources of uncertainty: firstly the uncertainty due to the seismicity induced by the mining itself and secondly that caused by the commodity price. We used multistage stochastic programming as a function of the copper price taking account of the impact of the speed of the mining on the micro-seismicity. One of the key difficulties was that the uncertainty is endogenous. The level of stress induced in the rock (a state variable) depends on the rate at which mining advances, which is a decision variable. To get around this difficulty we modelled the average moment k_t per tremor per time period (where the number of tremors is a Poisson process).

Nowadays the rock mass is "preconditioned" by blasting or by hydraulic fracturing to reduce the in-situ stress. Intensive preconditioning consists of either drilling from above the zone to be mined and blasting, or drilling from below and fracturing the rock hydraulically. It has been used successfully in Australia and Chile to reduce the stress in the rock mass, thereby preventing rock-bursts. we have also incorporated this into our model.

As this research project is being sponsored by the state mining company Codelco, the

method was tested using data from two of the nine sectors at the El Teniente underground copper mine in the O'Higgins Region, Chile. we succeeded in optimizing the production for up to 5 years. In the future this should be extended to longer time horizons.

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