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Existence and Lyapunov Pairs for the Perturbed Sweeping Process Governed by a Fixed Set

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Abstract The aim of this paper is to prove existence results for a class of sweeping processes in Hilbert spaces by using the catching-up algorithm. These processes are governed by ball-compact non autonomous sets. Moreover, a full characterization of nonsmooth Lyapunov pairs is obtained under very general hypotheses. We also provide a criterion for weak invariance. Some applications to hysteresis and crowd motion are given.

Keywords Sweeping process · Lyapunov pair · Differential inclusions · Invariance · Normal cone

Mathematics Subject Classification (2010) 34A60 · 49J52 · 34G25 · 49J53 · 93D30

1 Introduction

In this paper we study several properties of the sweeping process governed by a fixed set, that is, the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N(S; x(t)) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in S, \end{cases} \tag{1}$$

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where $S \subseteq H$ is a merely ball-compact set, $N(S; x)$ denotes the Clarke normal cone to S at x and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty closed and convex values.

The study of differential inclusions involving normal cones goes back a long time. In the convex case, they are included in the so-called evolution equations governed by maximal monotone operators, which is a well known subject (see [11, 32] and the references therein). They also appear in the theory of projected dynamical systems which, as far as we know, began with the works of Henry [30, 31]. In these papers, to study some planning procedures in economy, Henry introduced the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in \operatorname{proj}_{T_S(x(t))}(F(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in S, \end{cases} \tag{2}$$

where $S \subseteq \mathbb{R}^n$ is a closed convex set, $T_S(\cdot)$ denotes tangent cone to S and $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued map. Henry showed existence and equivalence results for (1) and (2). Next, Cornet [23] relaxed the convexity assumption on S to tangential regularity. Since then, projected dynamical systems has been studied by several authors (see for instance [12, 20, 26, 48]) and the equivalence with differential variational inequalities and sweeping processes is well known.

Two years before the work of Henry, Moreau, in his seminal papers [40, 41], introduced the so-called sweeping process, which correspond to the differential inclusion (1) with a convex moving set $S(t)$ without perturbation. In these papers, to study some mechanical problems arising in elastoplasticity, Moreau introduced the so-called catching-up algorithm to deal with the existence of solutions. Since then, this algorithm has been used by several authors to show existence of solutions for the perturbed sweeping process. We can mention, e.g., [7, 10] for uniformly prox-regular sets, [43, 44] for uniformly subsmooth sets, among others. In the first part of the paper, we show existence through this algorithm for merely ball compact sets.

The work of Moreau was the starting point of several developments related to perturbed sweeping process with regular and nonregular moving sets. We refer to [6, 8, 10, 14, 15, 21, 24, 28, 34, 38, 39, 42, 47, 49] for more details. In this respect, it is worth pointing out that existence results for the sweeping process with merely closed moving sets already exist in the literature. We can mention the work of Benabdellah [6] and Colombo and Goncharov [21] for the unperturbed sweeping process and the work of Thibault [49] for the perturbed sweeping process in finite dimensions. Thus, our aim is to show existence of the sweeping process through the catching-up, which could be useful to deal with practical problems.

The second part of the paper is devoted to Lyapunov pairs for the sweeping process (1). Lyapunov pairs are the central idea behind the Lyapunov method. This indirect approach is relevant because it does not require an explicit expression for the solutions of the dynamical system. This is especially useful when dealing with complex real-world applications. Moreover, the Lyapunov method allows to address several stability properties of differential inclusions as asymptotic stability, existence of equilibria, stabilization among others (see, for example, [17–19]).

Characterizations of smooth and nonsmooth Lyapunov pairs has been considered for different dynamical systems by several authors (see [5, 17–19] and the references given there). In the present case, Adly, Hantoute and Théra [1, 2] give explicit criterion for Lyapunov pairs for maximal monotone evolution equations, which includes the sweeping process driven by a fixed convex set. Then, Hantoute and Mazade [29] give explicit criteria for Lyapunov functions for the sweeping process driven by a fixed uniformly prox-regular set.

Unfortunately, it is well known that some dynamical systems do not admit smooth Lyapunov pairs (see [18]). Thus, it is very important to deal with the nonsmooth Lyapunov pairs. Here is where the subdifferential theory has been very helpful. In this setting, the work of Clarke et al [19] has became a benchmark because they characterize Lyapunov pairs for differential inclusions by using the proximal subdifferential. The proximal subdifferential is the smallest reasonable subdifferential that allows a characterization of nonsmooth Lyapunov pairs. We follow this path and give an explicit criteria, involving the proximal subdifferential, of weak Lyapunov pairs for the sweeping process. It is worth pointing out that our result, in contrast with [1, 2, 29], does not involve the singular (horizon) subdifferential, which gives a simpler criterion.

The paper is organized as follows. After some preliminaries, in Section 3 we give an existence result through the catching-up algorithm. Then, in Section 4 we give a criteria for weak Lyapunov pairs for the sweeping process. As a result, we also give a criterion for weak invariance for the sweeping process. Finally, in Section 5, we give some applications to hysteresis and crowd motion.

2 Preliminaries

From now on H stands for a separable Hilbert space, whose norm is denoted by $\|\cdot\|$. The closed ball centered at x with radius r is denoted by $\mathbb{B}(x,r)$ and the closed unit ball is denoted by $\mathbb{B}$. The notation H_w stands for H equipped with the weak topology and $x_n \rightharpoonup x$ denotes the weak convergence of a sequence $(x_n)_n$ to x.

Recall that a vector $h \in H$ belongs to the Clarke tangent cone $T(S;x)$ (see [16]); when for every sequence $(x_n)_n$ in S converging to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex, and its negative polar $N(S;x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N(S;x) := \{v \in H\colon \langle v, h\rangle \le 0 \quad \forall h \in T(S;x)\}.$$

As usual, $N(S;x) = \emptyset$ if $x \notin S$. Through that normal cone, the *Clarke subdifferential* of a function $f\colon H \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\partial f(x) := \{v \in H\colon (v,-1) \in N(\operatorname{epi} f, (x, f(x)))\},$$

where $\operatorname{epi} f := \{(y,r) \in H \times \mathbb{R}\colon f(y) \le r\}$ is the epigraph of f. When the function f is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [19]) in the following simple and amenable way

$$\partial f(x) = \left\{v \in H\colon \langle v, h\rangle \le f^{\circ}(x;h) \text{ for all } h \in H\right\},$$

where

$$f^{\circ}(x;h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}\left[f(y+th) - f(y)\right],$$

is the *generalized directional derivative* of the locally Lipschitzian function f at x in the direction $h \in H$. The function $f^{\circ}(x;\cdot)$ is in fact the support of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function is a set-valued map with nonempty and convex values satisfying the important property of upper semicontinuity from H into H_w.

The weak tangent cone to a set S at $x \in S$ is defined as

$$T_S^w(x) := \{v \in H\colon \text{ there exists } t_n \searrow 0,\, v_n \rightharpoonup v \text{ such that } x + t_n v_n \in S\}.$$

Given $S \subseteq H$, we say that S is ball compact if, for all $r > 0$, the set $S \cap r\mathbb{B}$ is compact.

For a set $S \subseteq H$, the distance function of $S \subseteq H$, denoted by d_S is the function defined by $d_S(x) := \inf_{y \in S} \|x - y\|$. We denote $\operatorname{Proj}_S(x)$ the set (possibly empty)

$$\operatorname{Proj}_S(x) := \{y \in S \colon d_S(x) = \|x - y\|\}.$$

The equality (see [19])

$$N(S; x) = \operatorname{cl}^* \left(\mathbb{R}_+ \partial d_S(x)\right) \quad \text{for } x \in S,$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x, S)$ in place of $\partial d(\cdot, S)(x)$.

Let $f \colon H \to \mathbb{R} \cup \{+\infty\}$ be an lsc (lower semicontinuous) function and $x \in \operatorname{dom} f$. An element ζ belongs to the *proximal subdifferential* $\partial^P f(x)$ of f at x (see [19, Chapter 1]) if there exist two positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in B(x; \eta).$$

The proximal normal cone of a set $S \subseteq H$ at $x \in S$ is defined as

$$N^P(S; x) := \partial^P I_S(x),$$

where I_S is the indicator function of a set $S \subseteq H$ (recall that $I_S(x) = 0$ if $x \in S$ and $I_S(x) = +\infty$ if $x \notin S$).

We recall the following formula (see [19, Chapter 1]):

$$\zeta \in \partial^P f(x) \quad \Leftrightarrow \quad (\zeta, -1) \in N^P\left(\operatorname{epi} f; (x, f(x))\right). \tag{3}$$

The following well known result will be used in the proof of Theorem 1 below.

Lemma 1 *Let $S \subseteq H$ be a closed set. Then, for $x \notin S$ and $s \in \operatorname{Proj}_S(x)$ we have $x - s \in \|x - s\| \partial d_S(s)$.*

Recall that ζ is in the *Fréchet subdifferential* $\partial^F f(x)$ of a function f from H into $\mathbb{R} \cup \{+\infty\}$ with $f(x) < \infty$ provided for each $\varepsilon > 0$ there exists some neighborhood U of x such that for all $y \in U$ one has

$$\langle \zeta, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|.$$

When f is the indicator function I_S of a subset $S \subseteq H$ and $x \in S$, this amounts to saying that for some neighborhood U of x one has for all $y \in U \cap S$

$$\langle \zeta, y - x \rangle \leq \varepsilon \|y - x\|.$$

The obtained set is called the *Fréchet normal cone* to S at x and it is denoted by $N^F(S; x)$. The following formula holds (see [9] and the references therein)

$$\partial^F d_S(x) = N^F(S; x) \cap \mathbb{B}.$$

When $\partial^F g(x)$ coincides with the Clarke subdifferential $\partial f(x)$ of f at x, one says that f is *subdifferentially regular* at x. The regularity of the indicator function I_S is equivalent to the equality $N^F(S; x) = N(S; x)$, i.e., the Clarke and the Fréchet normal cones to S at $x \in S$ coincide. Generally, one says that the set S is *normally regular* at $x \in S$. It is well known (see [9]) that this is equivalent to the subdifferential regularity at $x \in S$ of the distance function d_S. When S normally regular at all points in S, we merely say that S is normally regular.

3 Existence Through the Catching-up Algorithm

In this section, we show existence for the sweeping process (1). More specifically, given a ball compact set S, we show that the catching-up algorithm converges uniformly (up to a subsequence) to a solution of (1).

Throughout this section, $F\colon [T_0, T] \times H \rightrightarrows H$ will be a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_1^F$) F is upper semicontinuous from $[T_0, T] \times H$ into H_w.
($\mathcal{H}_2^F$) There exists $h\colon H \to \mathbb{R}^+$ Lipschitz continuous such that

$$d\,(0, F(t,x)) := \inf\{\|w\|\colon w \in F(t,x)\} \leq h(x),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

The following theorem, which is the main result of this section, asserts the existence of solutions for the sweeping process (1) for a merely ball-compact set S. This result is in line with [6, 21, 49] and extends the result given in [4, Theorem 10.1.1] for sleek sets. Moreover, its proof is strongly based on ideas from [43, Chapter 1], where the author uses the catching-up algorithm to deal with perturbed state-dependent sweeping processes governed by uniformly subsmooth moving sets.

Theorem 1 *Assume that S is a ball compact subset of H and that $F\colon [T_0, T] \times H \rightrightarrows H$ satisfies ($\mathcal{H}_1^F$) and ($\mathcal{H}_2^F$). Then, for any $x_0 \in S$, there exists at least one Lipschitz solution x of the sweeping process* (1). *Moreover,*

$$\|\dot{x}(t)\| \leq 2h(x(t)) \quad \text{a.e. } t \in [T_0, T].$$

Proof Let $n \in \mathbb{N} \setminus \{0\}$ and define $\mu_n := (T - T_0)/n$. Consider the partition of $[T_0, T]$ defined by $t_k^n := T_0 + k \cdot \mu_n$ for $k = 0, \ldots, n$. For each $(t,x) \in [T_0, T] \times H$ denote by $f(t,x)$ the element of minimal norm of the closed convex set $F(t,x)$, that is,

$$f(t,x) := \operatorname{proj}_{F(t,x)}(0).$$

Then, due to ($\mathcal{H}_2^F$), $\|f(t,x)\| \leq h(x)$ for all $(t,x) \in [T_0, T] \times H$.

We will construct a sequence of Lipschitz functions $(x_n)_n$ which converges (up to a subsequence) to a solution of the sweeping process (1).

Define the functions δ_n and θ_n as

$$\delta_n(t) = \begin{cases} t_k^n & \text{if } t \in [t_k^n, t_{k+1}^n[\\ t_{n-1}^n & \text{if } t = T, \end{cases}$$

and

$$\theta_n(t) = \begin{cases} t_{k+1}^n & \text{if } t \in [t_k^n, t_{k+1}^n[\\ T & \text{if } t = T. \end{cases}$$

It is clear that $\theta_n(t) \to t$ and $\delta_n(t) \to t$ uniformly as $n \to \infty$.

Put $x_0^n := x_0 \in S$ and for $k = 0, \ldots, n-1$ we define

$$x_{k+1}^n \in \operatorname{Proj}_S\left(x_k^n + \mu_n \cdot f(t_k^n, x_k^n)\right),$$

where the right-hand side is non empty because S is ball compact. Moreover, due to ($\mathcal{H}_2^F$), we observe that for $k = 0, \ldots, n-1$

$$\|x_{k+1}^n - x_k^n\| \leq d_S\left(x_k^n + \mu_n f(t_k^n, x_k^n)\right) + \mu_n \|f(t_k^n, x_k^n)\| \leq 2\mu_n h(x_k^n).$$

Thus, if L_h is the Lipschitz constant of h,

$$\|x_{k+1}^n\| \leq (1+2\mu_n L_h)\|x_k^n\| + 2\mu_n h(0) \quad \text{for all } k=0,\ldots,n-1,$$

which, due to [19, p. 183], entails

$$\begin{aligned}\|x_{k+1}^n\| &\leq (\|x_0\| + 2(k+1)\mu_n h(0)) \exp\left(2L_h(k+1)\mu_n\right)\\ &\leq M := (\|x_0\| + 2(T-T_0)h(0)) \exp\left(2L_h(T-T_0)\right).\end{aligned}$$

For any $t \in [t_k^n, t_{k+1}^n]$ with $k=0,\ldots,n-1$, we put

$$x_n(t) := \frac{t_{k+1}^n - t}{\mu_n} x_k^n + \frac{t - t_k^n}{\mu_n} x_{k+1}^n.$$

Then, for a.e. $t \in [t_k^n, t_k^{n+1}]$, $\|x_n(t)\| \leq M$ and

$$\|\dot{x}_n(t)\| = \|x_{k+1}^n - x_k^n\|/\mu_n \leq 2h(x_n(t_k^n)) \leq 2(L_h M + h(0)). \tag{4}$$

Since $x - y \in \|x-y\|\partial d_S(y)$ for any $y \in \operatorname{Proj}_S(x)$ (see Lemma 1), we obtain

$$\dot{x}_n(t) \in -2h(x_k^n)\partial d_S\left(x_n(t_{k+1}^n)\right) + f(t_k^n, x_n(t_k^n)) \quad \text{a.e. } t \in [t_k^n, t_{k+1}^n]. \tag{5}$$

Furthermore, the definitions of δ_n and θ_n together with (5) give, for a.e. $t \in [T_0, T]$

$$\dot{x}_n(t) \in -2h(x_n(\delta_n(t)))\partial d_S\left(x_n(\theta_n(t))\right) + f\left(\delta_n(t), x_n(\delta_n(t))\right). \tag{6}$$

Moreover, due to the definition of x_n, for all $t \in [T_0, T]$

$$\begin{aligned} d_S(x_n(t)) &\leq 2\mu_n h(x_n(\delta_n(t)))\\ &\leq 2\mu_n L_h\|x_n(\delta_n(t))\| + 2\mu_n h(0)\\ &\leq 2\mu_n\left(L_h M + h(0)\right).\end{aligned} \tag{7}$$

Fix $t \in [T_0, T]$ and define $K(t) := \{x_n(t) \colon n \in \mathbb{N}\}$. We claim that $K(t)$ is relatively compact. Indeed, let $(x_m(t))_m \subseteq K(t)$ and take $y_m(t) \in \operatorname{Proj}_S(x_m(t))$ (the projection exists due to the ball compactness of S and the boundedness of $K(t)$). Moreover, according to (7),

$$\begin{aligned}\|y_n(t)\| &\leq d_S(x_n(t)) + \|x_n(t)\|\\ &\leq 2\mu_n\left(L_h M + h(0)\right) + M\\ &\leq R := 2(T-T_0)\left(L_h M + h(0)\right) + M.\end{aligned}$$

This entails that $y_n(t) \in S \cap R\mathbb{B}$. Thus, by the ball compactness of S, there exists a subsequence $(y_{m_k}(t))_{m_k}$ of $(y_m(t))_m$ converging to some $y(t)$ as $k \to +\infty$. Then,

$$\begin{aligned}\|x_{m_k}(t) - y\| &\leq d_S(x_{m_k}(t)) + \|y_{m_k}(t) - y(t)\|\\ &\leq 2\mu_{m_k}\left(L_h M + h(0)\right) + \|y_{m_k}(t) - y(t)\|,\end{aligned}$$

which implies that $K(t)$ is relatively compact. Moreover, it is not difficult to see by (4) that $K := (x_n)$ is equicontinuous. Therefore, by virtue of (4), Arzela-Ascoli's and Dunford-Pettis's Theorems, we obtain the existence of a Lipschitz function x and a subsequence $(x_k)_k$ of $(x_n)_n$ such that

(i) (x_k) converges uniformly to x on $[T_0, T]$.
(ii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$.
(iii) $x_k(\theta_k(t)) \to x(t)$ for all $t \in [T_0, T]$.
(iv) $x_k(\delta_k(t)) \to x(t)$ for all $t \in [T_0, T]$.

These conditions, the convergence theorem (see [3, Proposition 5] for more details) and (6) imply that x satisfies

$$\begin{cases} \dot{x}(t) \in -2h(x(t))\partial d_S(x(t)) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0. \end{cases}$$

Furthermore, according to (7), and since S is closed we obtain that $x(t) \in S$ for all $t \in [T_0, T]$. Finally, x is a solution of (1) because $\partial d_S(x) \subseteq N(S; x)$ for all $x \in S$. □

Remark 1 If the set S is r-uniformly prox-regular (see [45]) and F is single-valued, then there exists a unique solution of (1) which satisfies

$$\|\dot{x}(t)\|^2 = \langle \dot{x}(t), F(t, x(t))\rangle \quad \text{a.e. } t \in [T_0, T].$$

Thus, in particular,

$$\|\dot{x}(t) - F(t, x(t))\| \leq \|F(t, x(t))\| \quad \text{a.e. } t \in [T_0, T].$$

These facts are well known and the compactness of S is not needed here.

4 Lyapunov Pairs and Invariance

In this section we give an explicit criterion for weak Lyapunov pairs and weak Lyapunov functions for the sweeping process (1). Throughout this section we assume that $F\colon [T_0, T] \times H \rightrightarrows H$ is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_3^F$) $F(\cdot, \cdot)$ is scalarly $\mathcal{L} \otimes \mathcal{B}$ measurable on $[T_0, T] \times H$.
($\mathcal{H}_4^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.
($\mathcal{H}_5^F$) There exist $h\colon H \to \mathbb{R}^+$ Lipschitz such that

$$\|F(t, x)\| := \sup\{\|w\|\colon w \in F(t, x)\} \leq h(x),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

Let $V\colon H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function and $W\colon H \to \mathbb{R}$ be continuous. We say that (V, W) forms a *weak Lyapunov* pair for the sweeping process (1) if for every $x_0 \in S$ there exists x solution of (1) such that

$$V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0) \quad \text{for all } t \in [T_0, T].$$

We will consider the following Hypotheses on V and W

($\mathcal{H}^V$) $V\colon H \to \mathbb{R} \cup \{+\infty\}$ is a proper lsc function with $\operatorname{dom} V \subseteq S$.
($\mathcal{H}^W$) $W\colon H \to \mathbb{R}$ is an lsc function with

$$0 \leq W(x) \leq \beta(1 + \|x\|) \quad \text{for all } x \in H,$$

for some $\beta \geq 0$.

Proposition 1 *Assume, in addition to* ($\mathcal{H}_3^F$), ($\mathcal{H}_4^F$) *and* ($\mathcal{H}_5^F$), *that S is closed and ball compact and* ($\mathcal{H}^V$) *and* ($\mathcal{H}^W$) *hold. Then (V, W) forms a weak Lyapunov pair for* (1) *if and only if for all $n \in \mathbb{N}$, (V, W_n) forms a weak Lyapunov pair for* (1), *where*

$$W_n(x) := \inf\{W(y) + n\|x - y\|\colon y \in H\}.$$

Proof If (V, W) forms a weak Lyapunov pair for (1), then for $x_0 \in S$ there exists a solution x of (1) such that

$$V(x(t)) + \int_{T_0}^{t} W_n(x(s))ds \leq V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0),$$

that is, (V, W_n) forms a weak Lyapunov pair for (1). Reciprocally, if for all $n \in \mathbb{N}$, (V, W_n) forms a weak Lyapunov pair for (1), for all $x_0 \in S$, there exists x_n solution of (1) such that

$$V(x_n(t)) + \int_{T_0}^{t} W_n(x_n(s))ds \leq V(x_0). \tag{8}$$

Since S is compact, the set of solutions of the sweeping process is compact in $\mathrm{Lip}([T_0, T]; H)$. Therefore, there exists a subsequence $(x_k)_k$ of $(x_n)_n$ converging uniformly to a solution x of the sweeping process. Then, by passing to the inferior limit in (8), we obtain that (V, W) forms a Lyapunov pair for (1). □

The following result, which is the main result of this section, gives a fully characterization of the weak Lyapunov pairs for the sweeping process (1).

Theorem 2 *Assume, in addition to* $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ *and* $(\mathcal{H}_5^F)$, *that* S *is normally regular and ball compact and* $(\mathcal{H}^V)$ *and* $(\mathcal{H}^W)$ *hold. Then the following conditions are equivalent:*

(i) *For a.e.* $t \in [T_0, T]$, $x \in \mathrm{dom}\, V$ *and* $\zeta \in \partial^P V(x)$

$$\inf\{\langle v, \zeta\rangle : v \in -h(x)\partial d_S(x) + F(t, x)\} \leq -W(x).$$

(ii) (V, W) *forms a weak Lyapunov pair for the sweeping process* (1).

Proof According to Proposition 1, without loss of generality, we can assume that W is continuous.

Let $G\colon [T_0, T] \times H \times \mathbb{R} \to H \times \mathbb{R}$ defined by

$$G(t, x, y) = \begin{pmatrix} -h(x)\partial d_S(x) + F(t, x) \\ -W(x). \end{pmatrix}$$

Then G has closed and convex values. Moreover, for a.e. $t \in [T_0, T]$ $G(t, \cdot, :)$ is upper semicontinuous from $H \times \mathbb{R}$ into $H_w \times \mathbb{R}$ and for a.e. $t \in [T_0, T]$ and all $(x, y) \in [T_0, T] \times H \times \mathbb{R}$

$$\begin{aligned} \|G(t, x, y)\| &:= \sup\{\|v\| : v \in G(t, x, y)\} \\ &\leq h(x) + \|F(t, x)\| + |W(x)| \\ &\leq 2h(x) + \beta(1 + \|x\|) \\ &\leq (2L_h + \beta)\|x\| + (2h(0) + \beta) \\ &\leq (2(L_h + h(0)) + \beta)(1 + \|x\|), \end{aligned} \tag{9}$$

where L_h is the Lipschitz constant of h. Moreover, since $\mathrm{epi}\, V \subseteq S \times \mathbb{R}$ and S is ball compact, $\mathrm{epi}\, V$ is also ball compact. Therefore, due to [13, Theorem 3.3], the following conditions are equivalent:

(a) For a.e. $t \in [T_0, T]$ and $(x, r) \in \mathrm{epi}\, V$

$$G(t, x, r) \cap T^w_{\mathrm{epi}\, V}(x, r) \neq \emptyset.$$

(b) For a.e. $t \in [T_0, T]$ and $(x, r) \in \mathrm{epi}\, V$

$$G(t, x, r) \cap \overline{\mathrm{co}}\, T^w_{\mathrm{epi}\, V}(x, r) \neq \emptyset.$$

(c) For a.e. $t \in [T_0, T]$, $(x, r) \in \operatorname{epi} V$ and $(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, r))$

$$\inf\{\langle v, \zeta\rangle + s\theta \colon (v, s) \in G(t, x, r)\} \leq 0.$$

(d) $(\operatorname{epi} V, G)$ is weakly invariant, that is, for any $(x_0, r_0) \in \operatorname{epi} V$ there exists a solution (x, r) of the differential inclusion $(\dot{x}(t), \dot{r}(t)) \in G(t, x(t), r(t))$ on $[T_0, T]$ with $(x(T_0), r(T_0)) = (x_0, r_0)$ such that $(x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$.

To finish the proof, it suffices to show that (c) is equivalent to (i) and (d) is equivalent to (ii).

(c) $\Rightarrow$ (i): Let N the set of null Lebesgue measure on which (c) is not satisfied, let $t \in [T_0, T] \setminus N$ and $\zeta \in \partial^P V(x)$. Then, by virtue of (3),

$$(\zeta, -1) \in N^P(\operatorname{epi} V; (x, V(x))).$$

Therefore, by using (c),

$$\inf\{\langle v, \zeta\rangle - s \colon (v, s) \in G(t, x, V(x))\} \leq 0,$$

which implies (i).

(i) $\Rightarrow$ (c): Let N' the set of null Lebesgue measure on which (i) is not satisfied and $\alpha < \infty$, let $t \in [T_0, T] \setminus N'$ and $(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, r))$. Then, due to [19, Exercise 2.1], $\theta \leq 0$ and

$$(\zeta, \theta) \in N^P(\operatorname{epi} V; (x, V(x))).$$

<u>First case</u>: $\theta < 0$:
It is not difficult to prove that $r = V(x)$. Then, due to (3) and (i), we obtain

$$\begin{aligned}
&\inf\{\langle v, \zeta\rangle + s\theta \colon (v, s) \in G(t, x, V(x))\}\\
&= \inf\{\langle v, \frac{\zeta}{|\theta|}\rangle \colon v \in -h(x)\partial d_S(x) + F(t, x)\}|\theta| - \theta W(x)\\
&\leq -W(x)|\theta| - \theta W(x)\\
&= 0,
\end{aligned}$$

which proves (c).
<u>Second case</u>: $\theta = 0$:
According to [50, Proposition 2.6], for all $n \in \mathbb{N}$ there exist

$$(\zeta_n, \theta_n) \in N^P(\operatorname{epi} V; (x_n, V(x_n))),$$

with $x_n \to x$, $V(x_n) \to V(x)$, $\zeta_n \to \zeta$, $\theta_n \to 0$ and $\theta_n < 0$. Thus, by the argument given in the first case, for all $n \in \mathbb{N}$

$$\inf\{\langle v, \zeta_n\rangle + s\theta_n \colon (v, s) \in G(t, x_n, V(x_n))\} \leq 0. \tag{10}$$

Moreover, since $G(t, x_n, V(x_n))$ is closed, convex and bounded (because inequality (9)), the infimum in (10) is attained at some points (v_n, s_n) with $s_n = -W(x_n)$ and $v_n \in -h(x_n)\partial d_S(x_n) + F(t, x_n)$. This implies that

$$v_n \in 2h(x_n)\mathbb{B} \subseteq 2\left(L_h\|x_n\| + h(0)\right)\mathbb{B},$$

where L_h is the Lipschitz constant of h. Hence, since $(x_n)_n$ is bounded, $(v_n)_n$ is bounded and we can assume that $v_n \rightharpoonup \bar{v}$. The upper semicontinuity from H into H_w of F and $\partial d_S(\cdot)$, shows that $\bar{v} \in -h(x)\partial d_S(x) + F(t, x)$. Therefore, by using (10), we get

$$\begin{aligned}
&\inf\{\langle v, \zeta\rangle : (v, s) \in G(t, x, r)\}\\
&\qquad = \inf\{\langle v, \zeta\rangle : (v, s) \in G(t, x, V(x))\}\\
&\qquad \leq \langle \bar{v}, \zeta\rangle\\
&\qquad = \lim_{n\to\infty} (\langle v_n, \zeta_n\rangle + s_n\theta_n)\\
&\qquad = \lim_{n\to\infty} \inf\{\langle v, \zeta_n\rangle + s\theta_n : (v, s) \in G(t, x_n, V(x_n))\}\\
&\qquad \leq 0,
\end{aligned}$$

which proves (c).
(d) $\Rightarrow$ (ii): Let $x_0 \in S$. We have to prove the existence of a solution x of (1) with $x(T_0) = x_0$ and such that

$$V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0) \quad \text{for all } t \in [T_0, T]. \tag{11}$$

Recall that $\operatorname{dom} V \subseteq S$.
First case: $x_0 \in S \setminus \operatorname{dom} V$: by virtue of Theorem 1, there exists x solution of (1) which obviously satisfies (11) because $V(x_0) = +\infty$.
Second case: $x_0 \in \operatorname{dom} V$: we have that $(x_0, V(x_0)) \in \operatorname{epi} V$. Thus, by virtue of assertion (d), there exists a solution (x, r) of the differential inclusion $(\dot{x}(t), \dot{r}(t)) \in G(t, x(t), r(t))$ on $[T_0, T]$ with $(x(T_0), r(T_0)) = (x_0, V(x_0))$ such that $(x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$. Moreover, since $\operatorname{dom} V \subseteq S$, $x(t) \in S$ and, hence, $\partial d_S(x(t)) \subseteq N(S; x(t))$ for all $t \in [T_0, T]$. Therefore, x is a solution of (1). Finally, since $(x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$,

$$V(x(t)) \leq r(t) = \int_{T_0}^{t} \dot{r}(s)ds = V(x_0) - \int_{T_0}^{t} W(x(s))ds,$$

which proves (ii).

(ii) $\Rightarrow$ (d): Fix $(x_0, r_0) \in \operatorname{epi} V$. Then, $V(x_0) \leq r_0$ and, since $\operatorname{dom} V \subseteq S$, $x_0 \in S$. Moreover, due to (ii), there exists x solution of (1) such that

$$V(x(t)) + \int_{T_0}^{t} W(x(s))ds \leq V(x_0) \quad \text{for all } t \in [T_0, T]. \tag{12}$$

Let $f \in L^1([T_0, T]; H)$ such that for a.e. $t \in [T_0, T]$

$$-\dot{x}(t) + f(t) \in N(S; x(t)) \text{ and} f(t) \in F(t, x(t)).$$

Then, since S is Fréchet normally regular (see [49, Proposition 2.1]), for a.e. $t \in [T_0, T]$

$$\langle \dot{x}(t) - f(t), \dot{x}(t)\rangle \leq 0.$$

Hence, for a.e. $t \in [T_0, T]$

$$\begin{aligned}
\|\dot{x}(t) - f(t)\|^2 &= \langle \dot{x}(t) - f(t), \dot{x}(t)\rangle + \langle \dot{x}(t) - f(t), -f(t)\rangle\\
&\leq \langle \dot{x}(t) - f(t), -f(t)\rangle\\
&\leq \|\dot{x}(t) - f(t)\| \times \|f(t)\|\\
&\leq \|\dot{x}(t) - f(t)\| \times h(x(t)),
\end{aligned}$$

where we have used $(\mathcal{H}_5^F)$. Therefore, $\|\dot{x}(t) - f(t)\| \leq h(x(t))$ for a.e. $t \in [T_0, T]$. Moreover, since S is normally regular and $\partial^F d_S(\cdot) = N^F(S;\cdot) \cap \mathbb{B}$. Then, for a.e. $t \in [T_0, T]$

$$\begin{aligned}\dot{x}(t) &\in -N(S;x(t)) \cap h(x(t))\mathbb{B} + F(t,x(t))\\ &\subseteq -h(x(t))\partial d_S(x(t)) + F(t,x(t)).\end{aligned}$$

Hence, x is a solution of (1). Now, define $r(t) := r_0 - \int_{T_0}^t W(x(s))ds$. Hence, by (12), the pair (x, r) satisfies for all $t \in [T_0, T]$

$$V(x(t)) \leq V(x_0) - \int_{T_0}^t W(x(s))ds \leq r_0 - \int_{T_0}^t W(x(s))ds = r(t),$$

that is, $(x(t), r(t)) \in \operatorname{epi} V$ for all $t \in [T_0, T]$. Finally, it is clear that (x, r) is a solution of the differential inclusion $(\dot{x}(t), \dot{r}(t)) \in G(t, x(t), r(t))$ on $[T_0, T]$ with $(x(T_0), r(T_0)) = (x_0, r_0)$, which proves (d). □

As an immediate consequence of Theorem 2, by taking V as the indicator function of S and W equals to 0, we obtain the existence for the sweeping process (1). The following result is consistent with Theorem 1.

Theorem 3 *Assume that S is a normally regular and ball compact subset of H and that $F\colon [T_0, T] \times H \rightrightarrows H$ satisfies $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ and $(\mathcal{H}_5^F)$. Then, for any $x_0 \in S$, there exists at least one Lipschitz solution x of the sweeping process* (1). *Moreover,*

$$\|\dot{x}(t)\| \leq 2h(x(t)) \quad a.e.\ t \in [T_0, T].$$

Proof Let $V\colon H \to \mathbb{R}\cup\{+\infty\}$ be defined by $V = I_S$ and $W \equiv 0$. Then, for a.e. $t \in [T_0, T]$ and $\zeta \in N^P(S;x) \setminus \{0\}$

$$\begin{aligned}\inf\{\langle v, \zeta\rangle : v \in -h(x)\partial d_S(x) + F(t,x)\} &\leq -\frac{h(x)}{\|\zeta\|}\langle \zeta, \zeta\rangle + h(x)\|\zeta\|\\ &\leq 0.\end{aligned}$$

Therefore, all the conditions of Theorem 2 hold. Thus, for all $x_0 \in S$, there exists at least one solution x of the sweeping process (1). □

Example 1 Let $V\colon H \to \mathbb{R}$ be differentiable function whose gradient ∇V is Lipschitz continuous and S be a normally regular and ball-compact set. Consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -\nabla V(x(t)) - N(S;x(t)) & \text{a.e. } t \in [T_0, T],\\ x(T_0) = x_0 \in S. \end{cases} \tag{13}$$

Consider the function $\tilde{V}(x) := V(x) + I_S(x)$ and fix $\zeta \in \partial^P \tilde{V}(x)$. Then, by the classical sum rule, $\zeta \in \nabla V(x) + N(S;x)$ and

$$\inf\{\langle v, \zeta\rangle\colon v \in -\|\nabla V(x)\|\partial d_S(x) - \nabla V(x)\} \leq 0,$$

which, by virtue of Theorem 2, shows that V is a Lyapunov function for (13). This result had been already obtained in [36, Proposition 3.1] for r-uniformly prox-regular sets.

4.1 Weak Invariance

In this subsection, as a consequence of Theorem 2, we give a characterization of weak invariance for the sweeping process.

Definition 1 (weak invariance) We say that K is weakly invariant with respect to the sweeping process (1) if for all $x_0 \in K$ there exists a solution of the sweeping process (1) with $x(T_0) = x_0$ and $x(t) \in K$ for all $t \in [T_0, T]$.

The following result is an improvement of [22, Theorem 4.3] for a fixed set.

Theorem 4 *Assume, in addition to* $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ *and* $(\mathcal{H}_5^F)$, *that* S *is normally regular and ball compact. Let* $K \subseteq S$ *be a closed set. Then the following conditions are equivalent:*

(i) *For a.e.* $t \in [T_0, T]$, *for all* $x \in K$ *and* $\zeta \in N^P(K; x)$

$$\inf\{\langle v, \zeta\rangle : v \in -h(x)\partial d_S(x) + F(t, x)\} \leq 0.$$

(ii) *For all* $x_0 \in K$ *there exists a solution of the sweeping process* (1) *with* $x(T_0) = x_0$ *and* $x(t) \in K$ *for all* $t \in [T_0, T]$.

5 Applications

In this section we give some applications of our existence result (Theorem 1) to hysteresis and to the modeling of crowd motion in emergency evacuation.

5.1 Hysteresis

In this subsection, we study the so-called Play operator, which arises in hysteresis (see, for instance [27, 46]). Several properties in hysteresis can be described in terms of some hysteresis operators. One of these hysteresis operators is the so-called Play operator [46], which to a given Lipschitz function y associates the set of solutions of the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N(S; x(t)) + \dot{y}(t) & \text{a.e. } t \in [T_0, T]; \\ x(T_0) = x_0 \in S. \end{cases} \tag{14}$$

The case where S is convex, uniformly prox-regular and α-far has been studied, respectively, in [27, 35, 46]. For a normally regular and ball compact set S and given y Lipschitz with $x_0 \in S$, due to Theorem 3 and since $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ and $(\mathcal{H}_5^F)$ trivially hold, there exists at least one solution of (14) with $\|\dot{x}(t)\| \leq 2\|\dot{y}(t)\|$ for a.e. $t \in [T_0, T]$. Therefore, the Play operator is well defined in $\mathrm{Lip}([T_0, T]; H)$.

5.2 Crowd Motion

In this subsection, we consider a model of crowd motion in emergency evacuation. We refer to [7, 37, 51] for a detailed description. Our discussion is based on [7].

The model handles contacts in order to deal with local interactions between people and to describe the whole dynamics of the pedestrian traffic. This model for crowd motion (where people are identified to rigid disks) rests on two principles. On the one hand, each individual has a spontaneous velocity that he would like to have in the absence of other people. On the

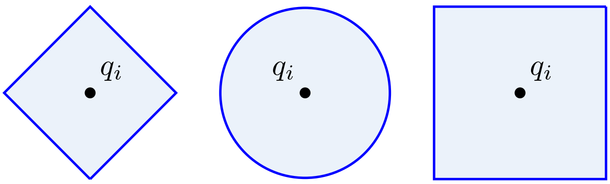

Fig. 1 Disks, respectively, for $d = \|\cdot\|_1$. $d = \|\cdot\|_2$ and $d = \|\cdot\|_\infty$

other hand, the actual velocity must take into account congestion. Those two principles lead to defining the actual velocity as the Euclidean projection of the spontaneous velocity over the set of admissible velocities (regarding the non overlapping constrains between sets).

More precisely, we consider N persons identified to rigid disks (for some distance d in $\mathbb{R}^2$). For convenience, the disks are supposed to have the same radius r. The center of the ith disk is denoted by $q_i \in \mathbb{R}^2$. Since overlapping is forbidden, the vector of positions $q = (q_1, \ldots, q_n) \in \mathbb{R}^{2N}$ has to belong to the "set of feasible configurations", defined by

$$Q := \{q \in \mathbb{R}^{2N} : D_{ij}(q) \geq 0 \quad \forall i \neq j\},$$

where $D_{ij}(q) = d(q_i, q_j) - 2r$ is the distance between the disk i and j and d is some distance in $\mathbb{R}^2$ (see Fig. 1).

It is worth emphasizing that Q is not uniformly prox-regular if, for instance, $d(x, y) = \|(x, y)\|_1$ or $d(x, y) = \|(x, y)\|_\infty$.

If the global spontaneous velocity of the crowd is denoted by

$$V(t, q) = (V_1(t, q_1), \ldots, V_N(t, q_N)) \in \mathbb{R}^{2N},$$

the previous crowd motion model can be described by the following differential inclusion:

$$\frac{dq}{dt} \in -N(Q; q) + V(t, q),$$

which fits in our context. Therefore, Theorems 1 and 3 give the existence for the crowd motion model. Moreover, one solution for this model can be obtained through the catching-up algorithm, described in the proof of Theorem 1.

Remark 2 In the last example, the underlying space H is of finite dimension ($H = \mathbb{R}^{2N}$), therefore, to get the existence of solutions we can apply the results from [49]. These existence results are based on a viability result from Frankowska and Plaskacz [25]. Thus, our contribution in finite dimensional spaces is to show the existence of solutions through the catching-up algorithm.

6 Concluding Remarks

In this paper we have seen, under ball-compactness of the fixed set S, how we can use the catching-up algorithm to show existence of solutions of the perturbed sweeping process. Moreover, we give a characterization of weak-Lyapunov pairs and invariance for the sweeping process. The ball-compactness of the fixed set S may appear to be a strong hypothesis but it is totally needed. We refer to [33], where it is shown an example of a perturbed sweeping process driven by a noncompact and convex set S without existence.

Even though, the study of sweeping processes reached certain maturity, there still remain several issues to be addressed. We can mention, asymptotic behavior, existence of periodic solutions, regularity of solutions, etc. We will pursue these in future works.

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