PAPER

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Paper

Phys. Educ. **54** (2019) 055031 (4pp) iopscience.org/ped

Computing areas and integrals with random numbers

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Abstract

We introduce the basics of the Monte Carlo method that allows computing areas and definite integrals, by means of the generation of long sequences of random numbers. The areas of some nontrivial shapes are computed, showing the convergence towards their exact values. The application of the method to the computation of definite integrals is also given, showing an important example of the kinematics of a particle under the action of an arbitrary time-dependent force.

1. The method

The computation of the area of a given two-dimensional shape, or the evaluation of a definite integral, are common tasks in physics. We will show a very simple procedure based on the Monte Carlo method [1–6] for estimating the areas of arbitrary shapes and computing integrals using a random number generator.

Consider the problem of computing the area enclosed by a simple curve in two dimensions. Let S be the region enclosed by the curve and S_0 a region that includes S, but whose perimeter is simple enough, e.g. a rectangle, so that its area is easily computed (figure 1). Let us now generate random points (x, y) inside S_0. Some of these points will fall inside S. As the number of points increases, the proportion of points inside S will be directly proportional to the ratio of the areas of S and S_0. The area of S can then be estimated as:

$$\text{Area}(S) \approx \frac{N(S)}{N(S_0)}\text{Area}(S_0) \tag{1}$$

where $N(S)$, $N(S_0)$ are the number of points inside S and S_0, respectively.

The algorithm for this type of computation is

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For $n = 1$ up to $n = N$:
(a) Generate a random point $(x, y)$ inside $S_0$
(b) If points falls inside $S$, $N(S) = N(S) + 1$;
    otherwise do nothing
(c) Repeat
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At the end of the loop, we compute the ratio $N(S)/N$ and multiply it by the (known in advance) area of S_0. This will give an approximate value for the area of S.

The precision of the result will improve with the number of random points used. This is the theoretical equivalent of throwing darts against a board containing a flat shape. After many (unbiased) throws, we take the ratio of the number of hits inside the shape to the total number of hits and multiply it by the area of the board, to get an estimate for the area of the surface. This is the essence of the Monte Carlo method [1–5]. The method is conceptually simple, but it has a defect: its slow convergence. Typically the error is of order $O(1/\sqrt{N})$ which makes necessary the use of a large set of random numbers to obtain a good accuracy. This is not a huge obstacle nowadays since random numbers can be generated with great speed and are usually included in most pocket calculators. This Monte Carlo method can be simply generalized to higher dimensions and is quite useful when dealing with shapes with complex boundaries.

2. Applications

Equation (1) looks simple enough, but the main difficulty in practice is to decide whether a given point (x, y) belongs to S or not. To do that we need, for instance, an equation for the perimeter that encloses S. This equation has the general form $f(x, y) = 0$, and a given point (x, y) belongs to S if $f(x, y) < 0$ or $f(x, y) > 0$, depending on the shape of the surface $z = f(x, y)$. As an example, let us consider the case of the four-leaved shape, known as a *quadrifolium*, and shown in figure 2(d):

$$f(x, y) = (x^2 + y^2)^3 - 4x^2y^2 = 0. \tag{2}$$

A 3D plot (not shown here) reveals that all points (x, y) inside the leaves satisfy $f(x, y) < 0$. Next, we use algorithm (1) for a number N of random points and obtain the estimate S_N. As N is increased the value of S_N approaches the true value of the enclosed area S. In the *quadrifolium* case, the values converge to 2.83 at large N.

We have also applied the method to several other different shapes (figure 2):

(a) A unit circle: $x^2 + y^2 - 1 = 0$.
(b) An *astroid*: $(x^2 + y^2 - 1)^2 + 27x^2y^2 = 0$.
(c) A bicuspid: $(x^2 - 1)(x - 1)^2 + (y^2 - 1)^2 = 0$.

In all cases we see the convergence of S_N for sufficiently large N. In the case of the unit circle, the value should converge to π, which implies that our method can be used to estimate π. From figure 2(d) we obtain $\pi \approx 3.14$ for $N = 5 \times 10^5$. Should these results change if we compute them with another random sequence? The answer is yes, there will be fluctuations, but these fluctuations will become smaller and smaller as N increase.

Another straightforward application of this Monte Carlo method is the computation of definite integrals. In this case, we want to evaluate the definite integral of a bounded function $f(x)$ from $x = a$ to $x = b$ (figure 3(a)). This is nothing more than the total area between the x-axis and $f(x)$. This kind of situation arises when computing the

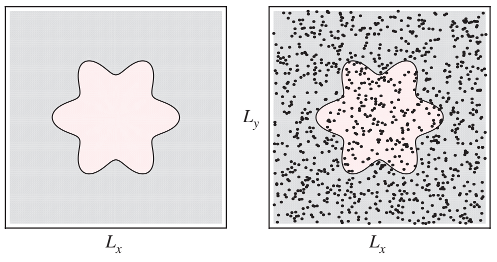

Figure 1. Left: rectangle $S_0 = L_x \times L_y$ containing an arbitrary flat shape of area S. Right: random points covering of S_0. A fraction of these fall inside S.

anti-derivative of some function $f(x)$, with $f(x)$ positive or negative depending on the domain. If $f(x)$ is negative for some segments, it is convenient to shift $f(x)$ by a judiciously chosen height A, so that $G(x) = f(x) + A$ is always positive (figure 3(b)). The integral can be cast as $I = \int_a^b f(x)\mathrm{d}x = \int_a^b G(x)\mathrm{d}x - A(b - a)$. The integral over $G(x)$ can now be computed using the random number method outlined in equation (1) (figure 3(c)). In this manner we obtain an estimate for $\int_a^b f(x)\mathrm{d}x$. Figure 3(d) shows the results obtained for $f(x) = \sin(x)/x$ for $a = -2\pi$, $b = 2\pi$ and $A = 0.3$, as a function of the total number of random numbers used.

A very important example from physics is provided by the motion of a particle in one dimension subjected to an arbitrary time-dependent force $F(t)$. Newton's equation reads:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \frac{1}{m}F(t), \tag{3}$$

where m is the particle's mass and $x(t)$ is the particle's position. Integrating this equation one time, we obtain

$$v(t) - v(0) = (1/m)\int_0^t F(s)\mathrm{d}s \tag{4}$$

where $v(t)$ is the velocity of the particle. We can see that the value of the particle's speed at a given time calls for the computation of the area under $F(s)$ from $s = 0$ up to $s = t$ which is carried out by using the Monte Carlo algorithm outlined before. Once we know $v(t)$, the position of the particle at time t is obtained by integrating one more time:

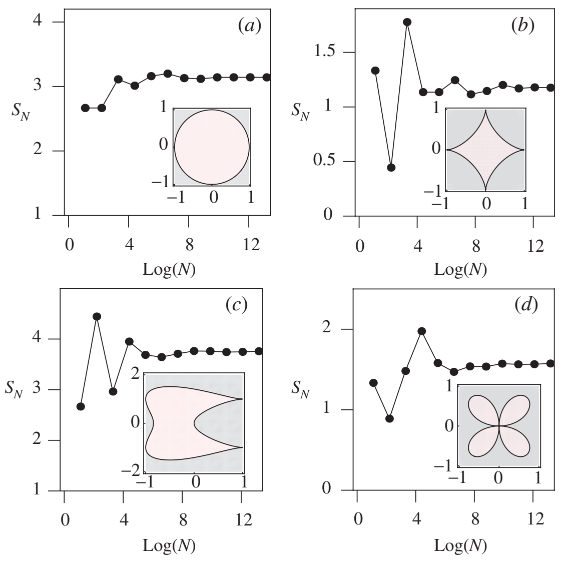

Figure 2. Areas S_N for different shapes computed according to the Monte Carlo method versus the total number N of random points (x, y) used. The insets show the shapes whose area we want to compute: (a) unit circle. (b) Astroid. (c) Bicuspid. (d) Quadrifolium.

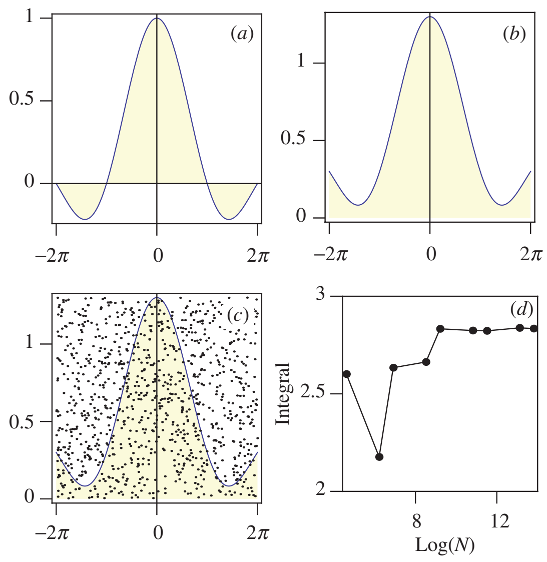

Figure 3. Computing the definite integral of function $f(x) = \sin(x)/x$. (a) The given function $f(x)$. (b) The shifted function $f(x) + 0.3$ which is positive inside the range of interest. (c) Computation of the area under the shifted function using the method of random numbers. (d) The values of the original integral as a function of the total number of random points used. For $N = 10^6$ the estimate gives 2.83 which is close to the exact value 2.8363.

$$x(t) - x(0) = \int_0^t v(s)\mathrm{d}s, \qquad (5)$$

which now calls for the area under the function $v(t)$. Since we do not have an explicit expression for $v(t)$ but only a set $\{t_i, v_i\}$, we must resort to an approximation for $v(t)$ for arbitrary t. The simplest way is to join each pair $\{t_i, v_i\}$ and $\{t_{i+1}, v_{i+1}\}$ with a straight line. In this way $v(t)$ is approximated by a sequence of continuous straight lines. Another option is to use a data interpolation function available in some

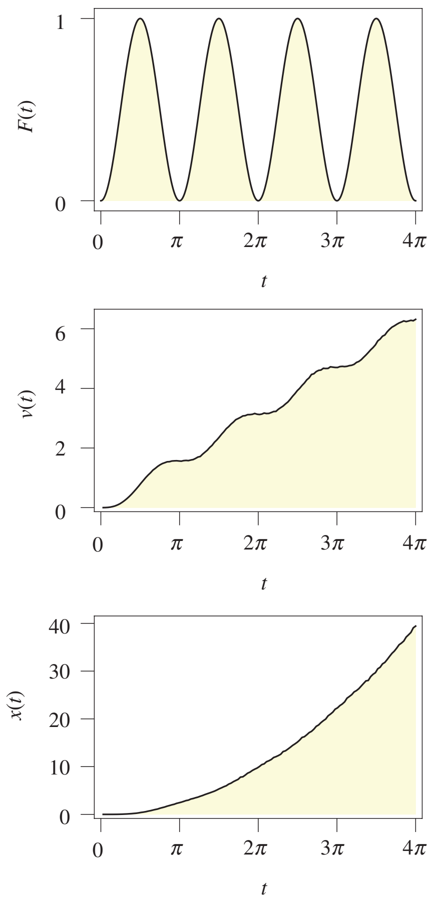

Figure 4. Kinematics in 1D for a particle of unit mass subjected to a force $F = \sin(t)^2$. Top, middle and bottom panels show the time evolution of the force, speed and particle's position, respectively (number of Monte Carlo points used: 10^5).

symbolic manipulators (like Mathematica) or in software packages. After this is done, we can compute $x(t)$ from equation (5), using the Monte Carlo approach. As a simple example, let us take $F(t) = \sin(t)^2, m = 1, x(0) = 0,$ and $v(0) = 0$. Results for $N = 10^5$ are shown in figure 4. The curve for $x(t)$ looks nearly 'parabolic' which can be understood from the fact that the external force is constant, on average.

The obtained curves for $v(t)$ and $x(t)$ will look smoother the larger the number of random points used. The number of points needed will depend on how 'wiggly' $F(t)$ is in the interval of interest. Also, it should be noted that for the computation of the area under $F(t)$ both, a single long random sequence or an average over a large number of shorter random sequences, can be used.

3. Conclusions

We have introduced the very basics of the Monte Carlo method that allows for the computation of integrals that appear in the calculations of areas and of definite integrals. We exemplified the basic algorithm with the calculation of areas for some non-trivially shaped objects. As a physically relevant example, we have computed the kinematics of a particle subjected to a time-dependent force.

Acknowledgments

This work was supported by Fondo Nacional de Desarrollo Cientifico y Tecnológico (Fondecyt) Grant 1160177.

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Received 23 June 2019
Accepted for publication 12 July 2019
https://doi.org/10.1088/1361-6552/ab31ba

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