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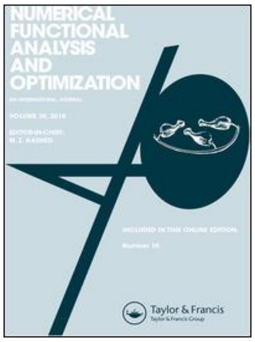

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Necessary and Sufficient Conditions for Qualitative Properties of Infinite Dimensional Linear Programming Problems

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ABSTRACT

Necessary and sufficient conditions for qualitative properties of infinite dimensional linear programing problems such as solvability, duality, and complementary slackness conditions are studied in this article. As illustrations for the results, we investigate the parametric version of Gale's example.

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1. Introduction

A linear program is an optimization problem with a linear objective function and linear constraints. It is an extremely powerful tool for addressing a wide range of applied optimization problems. Up to now, a complete theory of linear programing exists only for problems involving a finite number of decision variables subject to a finite number of constraints. Thus far, there exists a theory of infinite dimensional linear programing with many interesting results, yet the theory for strong duality has not been completed.

Infinite dimensional linear programing is both important and fascinating. Its importance arises from its position within the general theory of optimization, and from numerous real-world problems which can be modeled as infinite dimensional linear programing models [1, 2]. The fascination comes, in part at least, from the difficulty of extending the finite

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dimensional theory to the infinite dimensional case, when the infinite dimensional linear programing problem is superficially so similar to the finite dimensional linear programing problem. It is difficult to date the first attempts to extend the theory of linear programing to more general settings, but an early study of duality theory for infinite dimensional linear programs is due to Duffin. In his fundamental article [3], Duffin gave some results in locally convex topological vector spaces forming the base theory of infinite dimensional linear programing. Kretschmer [4] extended Duffin's results in the paired linear spaces. He posed the problem in a way which uses dual pairs of vector spaces, rather than a topological vector space and its (topological) dual; giving a formulation which is one of the most general possible.

Excellent reviews on duality theorems in infinite dimensional linear programing and some related topics were given by Anderson [5], Anderson and Nash [1]. In 2001, Shapiro discussed about the intimate relations between the duality theory and sensitivity analysis in [6]. This topic was also considered by Grestsky et al. [7], coming to conclusion that the necessary and sufficient condition of the zero duality gap and solvability is the existence of subdifferential of the value function at the primal constraint. Some information about recent achievements in infinite dimensional linear programing can be found in [8–12] and the references therein.

Most of the work that has been done on infinite dimensional linear programing has been aimed at producing an analogous theory, particularly of duality, to that of the finite case. Thus, many results which relate to the existence of primal or dual optimal values and solutions, duality, and complementary slackness conditions have been derived. The fundamental theorem of finite dimensional linear programing states that the finiteness of optimal value implies the existence of a solution. The second crucial property of finite linear programs is that the optimal values of the primal and dual problems are equal if either one of them is feasible. These are not always the cases for infinite linear programs (see, e.g., [1, Problem 3.9] and [1, Section 3.4.1]). Sufficient conditions for the fundamental theorem to hold were obtained in [1, Theorem 3.22] via the closedness of some convex cone constituted from the data of the primal problem. It is interesting that these conditions also ensure the absence of the duality gap between the primal and dual problems (see [1, Theorem 3.9] and [6, Proposition 2.6]). Examples can be found to point out that those conditions are not necessary (see Section 3 below). This leads to the following question: *What are the weakest conditions for the solvability of the primal problem and for the absence of the duality gap?*

The present article gives a complete solution for this question by establishing the necessary and sufficient conditions (the weakest conditions) for the solution existence and other qualitative properties of infinite linear programs such as duality and complementary slackness. This is done by using some well-known properties of the primal and dual problems and duality results in infinite dimensional linear programs. In addition, by using the parametric version of Gale's example, we are able to give a series of illustrative examples for our results.

The remaining of the article is organized as follows. Some preliminaries are given in the next section. Section 3 investigates the necessity of the closedness for qualitative properties in infinite dimensional linear programming. Necessary and sufficient conditions for the solvability, duality, and complementary slackness are given in Section 4.

2. Preliminaries

We initially need to define a certain amount of notation. The results that we need on topological vector spaces can be found in most books on the subject, for example those by Robertson and Robertson [13] and Schaefer [14]. Let (X, Y) and (Z, W) be two dual pairs of vector spaces. As there is no danger of confusion the bilinear forms on X and Y and on Z and W will both be represented by $\langle ., . \rangle$. Let $\sigma(X, Y)$ and $\sigma(Z, W)$ be weak topologies on X and Z, respectively, and A be a $\sigma(X, Y){-}\sigma(Z, W)$ continuous linear map from X to Z. The adjoint (transpose) of A is the linear map $A^* : W \to Y$ defined by the condition

$$\langle x, A^* w \rangle = \langle Ax, w \rangle \quad \forall x \in X, \forall w \in W.$$

It is well known that A^* is $\sigma(W, Z){-}\sigma(Y, X)$ continuous, where $\sigma(W, Z)$ and $\sigma(Y, X)$ are weak topologies on W and Y, respectively. Let P, Q be the convex cones in X and Z, respectively. The dual cones of P and Q are defined by setting

$$\begin{aligned} P^* &:= \{ y \in Y : \langle x, y \rangle \geq 0, \forall x \in P \}, \\ Q^* &:= \{ w \in W : \langle z, w \rangle \geq 0, \forall z \in Q \}. \end{aligned}$$

Consider the set constrained linear problem and its dual [1, Sect. 3.3] as follows

$$\begin{aligned} &(\text{ILP}) \qquad \min\{ \langle x, c \rangle : Ax - b \in Q, x \in P \}, \\ &(\text{ILP}^*) \qquad \max\{ \langle b, w \rangle : -A^* w + c \in P^*, w \in Q^* \}. \end{aligned}$$

Define the sets $H \subset Z \times \mathbb{R}$ and $K \subset Y \times \mathbb{R}$ as follows

$$H := \bigcup_{x \in P} \left((Ax - b - Q) \times [\langle x, c \rangle, +\infty[\right), \tag{1}$$

$$K := \bigcup_{w \in Q^*} \big((-A^* w + c - P^*) \times] - \infty, \langle b, w \rangle] \big). \tag{2}$$

Most of the work of this article will be done with respect to the weak topologies, $\sigma(Z \times \mathbb{R}, W \times \mathbb{R})$ on $Z \times \mathbb{R}$ and $\sigma(Y \times \mathbb{R}, X \times \mathbb{R})$ on $Y \times \mathbb{R}$. For the sets $A \subset Z \times \mathbb{R}$ and $B \subset Y \times \mathbb{R}$, we denote by $\overline{A}$ and $\overline{B}$ the closures of A and B, respectively.

Remark 2.1. Clearly, H and K are convex. Consider the set $H' \subset Z \times \mathbb{R}$ (see [5, p. 53] and [1, p. 387])

$$H' := \big\{ (Ax - z, \langle x, c \rangle + r) : x \in P, z \in Q, r \geq 0 \big\}. \tag{3}$$

The relations between H and H' are given by

$$H = H' - \big\{ (b, 0) \big\}, \quad \overline{H} = \overline{H'} - \big\{ (b, 0) \big\}. \tag{4}$$

We recall some basic concepts related to (ILP) and (ILP*).

Definition 2.1. For the problem (ILP), a vector $x \in X$ is said to be feasible if $x \in P$ and $Ax - b \in Q$. If (ILP) has a feasible vector then it is called consistent. And in this case, the infimum value of this problem, denoted by val(ILP), is defined as

$$\mathrm{val}(\mathrm{ILP}) := \inf\{ \langle x, c \rangle : Ax - b \in Q, x \in P \}.$$

If (ILP) has a feasible vector x satisfying $\langle x, c \rangle = \mathrm{val}(\mathrm{ILP})$ then (ILP) is said to be solvable and x is called a solution of (ILP).

Remark 2.2. (ILP) is consistent if and only if there exists $r \in \mathbb{R}$ such that $(0_Z, r) \in H$, or equivalently, $H \cap (\{0_Z\} \times \mathbb{R}) \neq \emptyset$. Moreover, (ILP) is solvable if and only if (ILP) is consistent and $(0_Z, \mathrm{val}(\mathrm{ILP}))$ belongs to $H \cap (\{0_Z\} \times \mathbb{R})$.

Definition 2.2. Problem (ILP) is called subconsistent if there exists $r \in \mathbb{R}$ such that $(0_Z, r) \in \overline{H}$. If (ILP) is subconsistent then its subvalue, denoted by subval(ILP), is defined as

$$\mathrm{subval}(\mathrm{ILP}) := \inf\big\{ r : (0_Z, r) \in \overline{H} \big\}.$$

Definition 2.3. For the problem (ILP*), a vector $w \in W$ is said to be feasible if $w \in Q^*$ and $-A^* w + c \in P^*$. If (ILP*) has a feasible vector then it is called consistent. And in this case, the infimum value of this problem, denoted by val(ILP*), is defined as

$$\mathrm{val}(\mathrm{ILP}^*) := \sup\{ \langle b, w \rangle : -A^* w + c^* \in P^*, w \in Q^* \}.$$

If (ILP*) has a feasible vector w satisfying $\langle b, w \rangle = \mathrm{val}(\mathrm{ILP}^*)$, then (ILP*) is said to be solvable and w is called a solution of (ILP*).

Remark 2.3. (ILP^*) is consistent if and only if there exists $r \in \mathbb{R}$ such that $(0_Y, r) \in K$, or equivalently $K \cap (\{0_Y\} \times \mathbb{R}) \neq \emptyset$. Moreover, (ILP^*) is solvable if and only if (ILP^*) is consistent and $(0_Y, \mathrm{val}(\mathrm{ILP}^*))$ belongs to $K \cap (\{0_Y\} \times \mathbb{R})$.

Definition 2.4. Problem (ILP^*) is called superconsistent if there exists $r \in \mathbb{R}$ such that $(0_Y, r) \in \overline{K}$. If (ILP^*) is superconsistent, then its supervalue, denoted by superval(ILP^*), is defined as

$$\mathrm{superval}(\mathrm{ILP}^*) := \sup\{r : (0_Y, r) \in \overline{K}\}.$$

A weak duality relation between (ILP) and (ILP^*) can be found in [1, Theorem 2.1].

Theorem 2.1. *If x is feasible for* (ILP) *and w is feasible for* (ILP^*), *then*

$$\langle x, c\rangle \geq \langle b, w\rangle.$$

The following complementary slackness results follow directly from Theorem 2.1.

Corollary 2.1. *If x is feasible for* (ILP), *w is feasible for* (ILP^*), *and the pair (x, w) satisfies the complementary slackness condition*

$$\begin{cases} \langle x, c - A^* w\rangle = 0, \\ \langle Ax - b, w\rangle = 0, \end{cases} \tag{5}$$

then x is optimal for (ILP) *and w is optimal for* (ILP^*).

Corollary 2.2. *Suppose that x is a solution of* (ILP) *and w is a solution of* (ILP^*). *Then* $\langle x, c\rangle = \langle b, w\rangle$ *if and only if the pair (x, w) satisfies the complementary slackness condition (5).*

The next theorem establishes the central result in the theory of duality that there is never a duality gap between the subvalue and dual value.

Theorem 2.2. (ILP) *is subconsistent with a finite subvalue M if and only if* (ILP^*) *is consistent with a finite value M.*

Proof. Using the relations (4), [5, Theorem 3], and the remarks after [5, Theorem 4], we deduce the conclusion of the theorem. □

If the positive cones P, Q are closed, then this theorem has an immediate corollary obtained by reversing the roles of the primal and dual programs.

Corollary 2.3. *Assume that the positive cones P, Q are closed. Then,* (ILP) *is consistent with a finite value M if and only if* (ILP^*) *is superconsistent with a finite supervalue M.*

The following theorem is an analog in a locally convex topological vector space setting of the separating hyperplane theorem. There are many roughly equivalent ways of stating such theorem. The following version [1, Proposition 6] is the most convenient one for our purposes.

Theorem 2.3. *Let (X, Y) be a dual pair of vector spaces, $B \subset X$ be a nonempty convex set, and $a \in X$. Then $a \notin \overline{B}$ if and only if there are $y \in Y$ and a real number α such that $\langle a, y\rangle < \alpha$ and $\langle x, y\rangle > \alpha$ for each $x \in B$. (In these circumstances we say that a is strictly separated from B by the hyperplane defined by y.)*

We will need a result on subduality which is in a sense parallel to weak duality theorem.

Theorem 2.4. *Consider the problems* (ILP) *and* (ILP*).

a. *If* (ILP) *is subconsistent and $(0_Z, r) \in \overline{H}$, then for any w feasible for* (ILP*), *we have $r \geq \langle b, w\rangle$;*
b. *If* (ILP*) *is superconsistent and $(0_Y, r) \in \overline{K}$, then for any x feasible for* (ILP), *we have $r \leq \langle x, c\rangle$.*

Proof. (a) Suppose, contrary to our claim, that there is $\overline{w} \in Q^*$ satisfying $-A^*\overline{w} + c \in P^*$ such that $r < \langle b, \overline{w}\rangle$. Then for any $x \in P$, any $q \in Q$ and any $\alpha \geq 0$, one has

$$\begin{aligned} r &< \langle b, \overline{w}\rangle + \langle q, \overline{w}\rangle + \langle x, -A^*\overline{w} + c\rangle \big(\text{as } \langle x, -A^*\overline{w} + c\rangle \geq 0 \text{ and } \langle q, \overline{w}\rangle \geq 0\big) \\ &\leq \langle Ax - b - q, -\overline{w}\rangle + \langle x, c\rangle + \alpha, \end{aligned}$$

which means that the hyperplane $(-\overline{w}, 1)$ strictly separates $(0_Z, r)$ from H. It follows from Theorem 2.3 that $(0_Z, r) \notin \overline{H}$, which is a contradiction.

(b) Similarly, to obtain a contradiction, suppose that there exists $\overline{x} \in P$ satisfying $A\overline{x} - b \in Q$ such that $r > \langle \overline{x}, c\rangle$. Then for any $w \in Q^*$, any $p^* \in P^*$ and any $\beta \leq 0$, we get

$$\begin{aligned} r &> \langle \overline{x}, c\rangle - \langle A\overline{x} - b, w\rangle - \langle \overline{x}, p^*\rangle \big(\text{as } \langle A\overline{x} - b, w\rangle \geq 0 \text{ and } \langle \overline{x}, p^*\rangle \geq 0\big) \\ &\geq \langle \overline{x}, -A^*w + c - p^*\rangle + \langle b, w\rangle + \beta, \end{aligned}$$

which means that the hyperplane $(\overline{x}, 1)$ strictly separates $(0_Y, r)$ from K. We get from Theorem 2.3 that $(0_Y, r) \notin \overline{K}$. This contradicts our assumption. The proof is complete. □

3. On the necessity of the closedness for qualitative properties

Let us review some qualitative properties of the primal problem (ILP) and its dual (ILP*) under the closedness assumption of the set H given in (1) (see [1, Theorems 3.9, 3.22] and [5, Theorem 7]).

Theorem 3.1. *Assume that* (ILP) *is consistent and* val(ILP) *is finite. If H is closed with the weak topology $\sigma(Z \times \mathbb{R}, W \times \mathbb{R})$, then* (ILP) *is solvable and* val(ILP)= val(ILP*).

Proof. Using the relations (4), [1, Theorems 3.9, 3.22], and [5, Theorem 7] we deduce the conclusions of the theorem. □

Corollary 3.1. *Assume that H is closed with the weak topology $\sigma(Z \times \mathbb{R}, W \times \mathbb{R})$. If x is a solution of* (ILP) *and w is a solution of* (ILP*), *then the pair (x, w) satisfies the complementary slackness condition (5).*

Remark 3.1. In finite dimension, when P and Q are nonnegative orthants, H is closed automatically. Thus, Theorem 3.1 and Corolarry 3.1 are actually natural extensions of the classical results to infinite dimension spaces.

The following parametric version of Gale's example (see [1, Section 3.4.2]) will be used for two purposes. Firstly, we use it to show that the closedness assumption of the set H is sufficient but not necessary for the solvability of the problem (ILP) and the absence of a duality gap between the primal and dual problems. Secondly, it will also be used to illustrate the results obtained in Section 4. Note that Gale's example belongs to an important special class of infinite dimensional linear programing which is linear semi-infinite programing, where either the number of constraints or the number of variables is finite. This class of problems was developed by Gorbena and Lopéz [15]. The interested reader is referred to [2] for several results about linear and nonlinear semi-infinite programing.

Example 3.1. Let α, β be real parameters and consider the parametric linear program

$$(\mathrm{PG}_{\alpha,\beta}) \qquad \min\left\{x_0 : x_0 + \sum_{i=1}^{\infty} i x_i = \alpha, \sum_{i=1}^{\infty} x_i = \beta, x_i \geq 0, i = 0, 1, 2, \ldots\right\}.$$

The program $(\mathrm{PG}_{\alpha,\beta})$ is a model of (ILP) with the following data. The primal variable space $X = \mathbb{R}^{(\mathbb{N})}$, called the generalized finite sequence space, is formed by sequences with finitely many nonzero terms. The dual of X is $Y = \mathbb{R}^{\mathbb{N}}$, which is the space of all real sequences. In this article, we use the convention $0 \in \mathbb{N}$ and denote $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Note that (X, Y) is a dual pair with respect to the bilinear form

$$\langle x, y \rangle = \sum_{i=0}^{\infty} x_i y_i, \quad \forall x = (x_0, x_1, x_2, \ldots) \in X, \forall y = (y_0, y_1, y_2, \ldots) \in Y.$$

In what follows, X and Y are considered with the weak topologies. Let $Z = W = \mathbb{R}^2$. Clearly, with respect to the bilinear form

$$\langle z, w\rangle = z_1w_1 + z_2w_2, \quad \forall z = (z_1, z_2) \in Z, \forall w = (w_1, w_2) \in W,$$

$(Z,\ W)$ is a dual pair. Let $P = \mathbb{R}_+^{(\mathbb{N})} := \mathbb{R}^{(\mathbb{N})} \cap \mathbb{R}_+^{\mathbb{N}} \subset X, Q = \{(0,0)\} \subset Z$, where

$$\mathbb{R}_+^{\mathbb{N}} = \left\{x = (x_0, x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : x_i \geq 0, i = 0, 1, 2, \ldots\right\},$$

$b = (\alpha, \beta) \in Z$, and $c = (1, 0, 0, ...) \in Y$. Let $A : X \to Z$ and its adjoint $A^* : W \to Y$ be the linear maps represented by the infinite matrices

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \end{pmatrix} \in M(2 \times \infty),$$

$$A^* = A^T = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \end{pmatrix}^T \in M(\infty \times 2).$$

The set H given in (1) is represented by

$$\begin{aligned} H &= \bigcup_{x\in P} (Ax - b - Q) \times [\langle x, c\rangle, +\infty[\\ &= \left\{\left(x_0 + \sum_{i=1}^{\infty} ix_i - \alpha, \sum_{i=1}^{\infty} x_i - \beta, x_0 + r\right) : r \geq 0, (x_i) \in \mathbb{R}^{(\mathbb{N})} \cap \mathbb{R}_+^{\mathbb{N}}\right\}. \end{aligned}$$

We will show that $H = H_1 \bigcup H_2$, where

$$\begin{aligned} H_1 &= \left\{(t_1 - \alpha, -\beta, t_1 + t_2) : t_1, t_2 \geq 0\right\}, \\ H_2 &= \left\{(t_1 + t_2 - \alpha, t_2 - \beta, t_3) : t_1, t_3 \geq 0, t_2 > 0\right\}. \end{aligned}$$

Take any $(x_i) \in \mathbb{R}^{(\mathbb{N})} \cap \mathbb{R}_+^{\mathbb{N}}$ and $r \geq 0$. Then, $\sum_{i=1}^{\infty} x_i = 0$ if and only if $\sum_{i=1}^{\infty} ix_i = 0$. Therefore,

$$x_0 + \sum_{i=1}^{\infty} ix_i - \alpha = \begin{cases} x_0 - \alpha & \text{if } \displaystyle\sum_{i=1}^{\infty} x_i = 0, \\ \left(x_0 + \displaystyle\sum_{i=1}^{\infty}(i-1)x_i\right) + \displaystyle\sum_{i=1}^{\infty} x_i - \alpha & \text{if } \displaystyle\sum_{i=1}^{\infty} x_i > 0. \end{cases}$$

Since $x_0 \geq 0$ and $x_0 + \sum_{i=1}^{\infty}(i-1)x_i \geq 0$,

$$\left(x_0 + \sum_{i=1}^{\infty} ix_i - \alpha, \sum_{i=1}^{\infty} x_i - \beta, x_0 + r\right) \in H_1 \bigcup H_2.$$

Thus, $H \subset H_1 \bigcup H_2$. For any $z \in H_1$, there exist non-negative real numbers $t_1,\ t_2$ such that $z = (t_1 - \alpha, -\beta, t_1 + t_2)$. Put

$$\begin{cases} r = t_2, \\ x_0 = t_1, \\ x_i = 0 (i \in \mathbb{N}^*). \end{cases}$$

Since $t_1, t_2 \geq 0$, we have $z = (x_0 + \sum_{i=1}^{\infty} ix_i - \alpha, \sum_{i=1}^{\infty} x_i - \beta, x_0 + r) \in H$, and it follows that $H_1 \subset H$. For any $z \in H_2$, there exist non-negative real numbers t_1, t_2, t_3 such that $t_2 > 0$ and $z = (t_1 + t_2 - \alpha, t_2 - \beta, t_3)$. Since $t_2 > 0$, there exists $n \in \mathbb{N}^*$ such that

$$t_2 \geq \frac{t_1 - \min\{t_1, t_3\}}{n-1}. \tag{6}$$

Put $r = t_3 - \min\{t_1, t_3\}$ and

$$\begin{cases} x_0 = \min\{t_1, t_3\}, \\ x_1 = t_2 - \dfrac{t_1 - \min\{t_1, t_3\}}{n-1}, \\ x_n = \dfrac{t_1 - \min\{t_1, t_3\}}{n-1}, \\ x_i = 0 \big(i \in \mathbb{N}^* \setminus \{1, n\}\big). \end{cases}$$

Since $t_3 \geq \min\{t_1, t_3\} \geq 0, t_1 \geq \min\{t_1, t_3\} \geq 0$ and (6), we have $r \geq 0$, $x_i \geq 0$ for every $i \in \mathbb{N}$. Moreover,

$$\begin{cases} \displaystyle\sum_{i=1}^{\infty} x_i = t_2, \\ x_0 + \displaystyle\sum_{i=1}^{n} ix_i = t_1 + t_2, \\ x_0 + r = t_3. \end{cases}$$

Hence,

$$z = \left(x_0 + \sum_{i=1}^{\infty} ix_i - \alpha, \sum_{i=1}^{\infty} x_i - \beta, x_0 + r \right) \in H,$$

and so $H_2 \subset H$. Therefore, $H = H_1 \bigcup H_2$.

Claim 1. *The set H is not closed for every $\alpha, \beta \in \mathbb{R}$.*

The topology $\sigma(Z \times \mathbb{R}, W \times \mathbb{R})$ is indeed a usual topology on $\mathbb{R}^3$. Hence,

$$\overline{H_1} = H_1,$$
$$\overline{H_2} = \big\{(t_1 + t_2 - \alpha, t_2 - \beta, t_3) : t_1, t_2, t_3 \geq 0\big\}.$$

It follows from $\overline{H_1} \subset \overline{H_2}$ that

$$\overline{H} = \overline{H_2}.$$

Since $\overline{H} \neq H$ (for instance, $(1-\alpha, -\beta, 0) \in \overline{H} \setminus H$), the set H is not closed.

Claim 2. $(\mathrm{PG}_{\alpha,\beta})$ *is solvable if and only if* $\alpha \geq \beta \geq 0$. *Moreover,*

$$\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \begin{cases} \alpha & \text{if} \quad \alpha \geq 0, \beta = 0, \\ 0 & \text{if} \quad \alpha \geq \beta > 0. \end{cases}$$

Clearly, if $(\mathrm{PG}_{\alpha,\beta})$ is solvable then $\alpha \geq \beta \geq 0$. Now, we suppose that $\alpha \geq \beta \geq 0$. If $\beta = 0$ then $\overline{x} := (\alpha, 0, 0, \ldots)$ is the unique feasible solution of $(\mathrm{PG}_{\alpha,\beta})$, therefore, it is the only optimal solution of $(\mathrm{PG}_{\alpha,\beta})$. Thus, $(\mathrm{PG}_{\alpha,\beta})$ is solvable and $\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \alpha$. If $\beta > 0$ then there exists $n \in \mathbb{N}^*$ such that $n\beta > \alpha$. Put

$$\begin{cases} x_1 = \dfrac{n\beta - \alpha}{n-1}, \\ x_n = \dfrac{\alpha - \beta}{n-1}, \\ x_i = 0\big(i = \mathbb{N}^* \setminus \{1, n\}\big). \end{cases}$$

Then, (x_i) is a solution of $(\mathrm{PG}_{\alpha,\beta})$. Hence, $(\mathrm{PG}_{\alpha,\beta})$ is solvable and $\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = 0$.

Claim 3. *The dual problem of* $(\mathrm{PG}_{\alpha,\beta})$, *denoted by* $(\mathrm{PG}^*_{\alpha,\beta})$, *is solvable if and only if* $\alpha \geq \beta \geq 0$ *and in this case* $\mathrm{val}(\mathrm{PG}^*_{\alpha,\beta}) = 0$.

The dual problem $(\mathrm{PG}^*_{\alpha,\beta})$ is given by

$$\max\big\{\alpha w_1 + \beta w_2 : (w_1, w_2) \in W, w_1 \leq 1, i w_1 + w_2 \leq 0, i = 1, 2, 3, \ldots\big\}.$$

Suppose that $(\mathrm{PG}^*_{\alpha,\beta})$ is solvable. Let (w_1^*, w_2^*) be a solution of $(\mathrm{PG}^*_{\alpha,\beta})$. Since $(0, -n)$ and $(-n, n)$ are feasible for $(\mathrm{PG}^*_{\alpha,\beta})$ for every $n \in \mathbb{N}^*$, it follows that

$$\begin{cases} \alpha 0 + \beta(-n) \leq \alpha w_1^* + \beta w_2^*, \\ \alpha(-n) + \beta n \leq \alpha w_1^* + \beta w_2^*. \end{cases}$$

Hence,

$$\min\{\beta, \alpha - \beta\} \geq -\frac{\alpha w_1^* + \beta w_2^*}{n}.$$

Taking $n \to \infty$ in the above inequality, we obtain $\beta \geq 0$ and $\alpha \geq \beta$.

Now we suppose that $\alpha \geq \beta \geq 0$. Then (0, 0) is a solution of $(\mathrm{PG}^*_{\alpha,\beta})$ and $\mathrm{val}(\mathrm{PG}^*_{\alpha,\beta}) = 0$. Indeed, let $(w_1,\ w_2)$ be a feasible vector of $(\mathrm{PG}^*_{\alpha,\beta})$, then $i w_1 + w_2 \leq 0$ for all $i \in \mathbb{N}$. It follows that $w_1 \leq 0$ and $w_1 + w_2 \leq 0$. Combine this with $\alpha \geq \beta \geq 0$ to obtain

$$\alpha w_1 + \beta w_2 \leq \beta w_1 + \beta w_2 = \beta(w_1 + w_2) \leq 0 = \alpha 0 + \beta 0.$$

It follows from Claims 1–3 that, although H is not closed, both of $(\mathrm{PG}_{\alpha,\beta})$ and $(\mathrm{PG}^*_{\alpha,\beta})$ are solvable and

$$\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \mathrm{val}\big(\mathrm{PG}^*_{\alpha,\beta}\big) = 0$$

for all pairs (α, β) such that $\alpha = \beta = 0$ or $\alpha \geq \beta > 0$.

4. Necessary and sufficient conditions for qualitative properties

We first recall some well-known properties of the sets H and K. For the sake of clear presentation, we provide a detailed proof.

Proposition 4.1.

a. *The set* $H \cap (\{0_Z\} \times \mathbb{R})$ *is* $\{0_Z\} \times [\mathrm{val}(\mathrm{ILP}), \infty[$ *or* $\{0_Z\} \times]\mathrm{val}(\mathrm{ILP}), \infty[$, *depending on whether* (ILP) *is solvable or not, respectively;*

b. *The set* $K \cap (\{0_Y\} \times \mathbb{R})$ *is* $\{0_Y\} \times]-\infty, \mathrm{val}(\mathrm{ILP}^*)]$ *or* $\{0_Y\} \times]-\infty, \mathrm{val}(\mathrm{ILP}^*)[$, *depending on whether* (ILP*) *is solvable or not, respectively;*

c. $\overline{H} \cap (\{0_Z\} \times \mathbb{R}) = \{0_Z\} \times [\mathrm{subval}(\mathrm{ILP}), \infty[$;

d. $\overline{K} \cap (\{0_Y\} \times \mathbb{R}) = \{0_Y\} \times]-\infty, \mathrm{superval}(\mathrm{ILP}^*)]$.

Proof. (a) We will show that

$$H \cap (\{0_Z\} \times \mathbb{R}) = \begin{cases} \{0_Z\} \times [\mathrm{val}(\mathrm{ILP}), \infty[& \text{if (ILP) is solvable,} \\ \{0_Z\} \times]\mathrm{val}(\mathrm{ILP}), \infty[& \text{if (ILP) is not solvable.} \end{cases} \tag{7}$$

Clearly, from the definitions of H and val(ILP), we have

$$\{0_Z\} \times]\mathrm{val}(\mathrm{ILP}), \infty[\subset H \cap (\{0_Z\} \times \mathbb{R}) \subset \{0_Z\} \times \big[\mathrm{val}(\mathrm{ILP}), \infty[.$$

Using the observation that (ILP) is solvable if and only if $(0_Z, \mathrm{val}(\mathrm{ILP})) \in H \cap (\{0_Z\} \times \mathbb{R})$, we immediately obtain (7).

(b) Similar to the proof of (a) given above, (b) follows from definitions of K, val(ILP*) and the observation that (ILP*) is solvable if and only if $(0_Y, \mathrm{val}(\mathrm{ILP}^*)) \in K \cap (\{0_Y\} \times \mathbb{R})$.

(c) Without loss of generality we can assume that (ILP) is subconsistent. From the definition of subval(ILP) and the closedness of $\overline{H} \cap (\{0_Z\} \times \mathbb{R})$ we have

$$\big(0_Z, \mathrm{subval}(\mathrm{ILP})\big) \in \overline{H} \cap (\{0_Z\} \times \mathbb{R}),$$

$$\{0_Z\} \times]\mathrm{subval}(\mathrm{ILP}), \infty[\subset \overline{H} \cap (\{0_Z\} \times \mathbb{R}) \subset \{0_Z\} \times [\mathrm{subval}(\mathrm{ILP}), \infty[.$$

It follows that (c) is satisfied.

(d) Similar to the proof of (c) given above, (d) follows from definition of superval(ILP*) and the closedness of $\overline{K} \cap (\{0_Y\} \times \mathbb{R})$. □

4.1. Solvability

The following theorem is an immediate consequence of part (a) of Proposition 4.1 above.

Theorem 4.1. *For the problem* (ILP) *the following statements are equivalent*

(a) $H \cap (\{0_Z\} \times \mathbb{R})$ *is nonempty, closed and* $\mathrm{val}(\mathrm{ILP})$ *is finite;*

(b) (ILP) *is solvable.*

We will use Theorem 4.1 to find the necessary and sufficient condition for the solvability of the parametric problem given in Example 3.1.

Example 4.1. Consider $(\mathrm{PG}_{\alpha,\beta})$ given in Example 3.1. The corresponding set H of $(\mathrm{PG}_{\alpha,\beta})$ is given by

$$H = \{(t_1 - \alpha, -\beta, t_1 + t_2) : t_1, t_2 \geq 0\} \bigcup \{(t_1 + t_2 - \alpha, t_2 - \beta, t_3) : t_1, t_3 \geq 0, t_2 > 0\}.$$

By some calculations, we obtain

$$H \cap (\{0_Z\} \times \mathbb{R}) = \begin{cases} \{(0,0)\} \times [\alpha, +\infty[& \text{if } \alpha \geq 0, \beta = 0, \\ \{(0,0)\} \times \mathbb{R}_+ & \text{if } \alpha \geq \beta > 0, \\ \emptyset & \text{otherwise.} \end{cases} \tag{8}$$

The set $H \cap (\{0_Z\} \times \mathbb{R})$ is closed for all $\alpha, \beta \in \mathbb{R}$. It follows from Theorem 4.1 that $(\mathrm{PG}_{\alpha,\beta})$ is solvable if and only if $H \cap (\{0_Z\} \times \mathbb{R})$ is nonempty and $\mathrm{val}(\mathrm{PG}_{\alpha,\beta})$ is finite. On the other hand, since the objective function of $(\mathrm{PG}_{\alpha,\beta})$ is bounded from below on its feasible set, $\mathrm{val}(\mathrm{PG}_{\alpha,\beta})$ is finite if and only if $(\mathrm{PG}_{\alpha,\beta})$ is consistent. Therefore, $(\mathrm{PG}_{\alpha,\beta})$ is solvable if and only if $H \cap (\{0_Z\} \times \mathbb{R})$ is nonempty. Combining this with (8) we deduce that $(\mathrm{PG}_{\alpha,\beta})$ is solvable if and only if $\alpha \geq \beta \geq 0$.

The next theorem is an immediate consequence of part (b) of Proposition 4.1 above.

Theorem 4.2. *For the problem* (ILP^*) *the following statements are equivalent*

a. $K \cap (\{0_Y\} \times \mathbb{R})$ *is nonempty, closed and* $\mathrm{val}(\mathrm{ILP}^*)$ *is finite;*
b. (ILP^*) *is solvable.*

4.2. Duality

Theorem 4.3. *Assume that* (ILP) *and* (ILP^*) *are both consistent. The following statements are equivalent*

a. $\overline{H \cap (\{0_Z\} \times \mathbb{R})} = \overline{H} \cap (\{0_Z\} \times \mathbb{R})$;
b. $\mathrm{val}(\mathrm{ILP}) = \mathrm{val}(\mathrm{ILP}^*)$.

Proof. By parts (a) and (c) of Proposition 4.1 we have

$$\overline{H \cap (\{0_Z\} \times \mathbb{R})} = \{0_Z\} \times [\mathrm{val(ILP)}, \infty[,$$

$$\overline{H} \cap (\{0_Z\} \times \mathbb{R}) = \{0_Z\} \times [\mathrm{subval(ILP)}, \infty[.$$

By Theorem 2.2, subval(ILP)=val(ILP*). This implies the conclusion of the theorem. □

Combining Theorem 4.1 and Theorem 4.3, we obtain a generalization of Theorem 3.1.

Theorem 4.4. *Assume that* (ILP) *and* (ILP*) *are both consistent. The following statements are equivalent*

a. $H \cap (\{0_Z\} \times \mathbb{R}) = \overline{H} \cap (\{0_Z\} \times \mathbb{R})$;
b. (ILP) *has a solution and* $\mathrm{val(ILP)} = \mathrm{val(ILP^*)}$.

Proof. Observe that $H \cap (\{0_Z\} \times \mathbb{R}) \subset \overline{H \cap (\{0_Z\} \times \mathbb{R})} \subset \overline{H} \cap (\{0_Z\} \times \mathbb{R})$. Now the assertion of Theorem 4.4 follows from Theorem 4.1 and Theorem 4.3. □

Let us illustrate Theorem 4.4 by the following example.

Example 4.2. Consider the parametric problem (PG$_{\alpha,\beta}$) given in Example 3.1. The corresponding set H of (PG$_{\alpha,\beta}$) and its closure are given by

$$H = \{(t_1 - \alpha, -\beta, t_1 + t_2) : t_1, t_2 \geq 0\} \bigcup \{(t_1 + t_2 - \alpha, t_2 - \beta, t_3) : t_1, t_3 \geq 0, t_2 > 0\},$$
$$\overline{H} = \{(t_1 + t_2 - \alpha, t_2 - \beta, t_3) : t_1, t_2, t_3 \geq 0\}.$$

By some calculations, we obtain (8) and

$$\overline{H} \cap (\{0_Z\} \times \mathbb{R}) = \begin{cases} \{(0,0)\} \times \mathbb{R}_+ & \text{if } \alpha \geq \beta \geq 0, \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

Observe that (PG$_{\alpha,\beta}$) is consistent if and only if $H \cap (\{0_Z\} \times \mathbb{R}) \neq \emptyset$. This is equivalent to $\alpha \geq \beta \geq 0$. Clearly, the dual problem (PG$^*_{\alpha,\beta}$) is consistent. Hence, (PG$_{\alpha,\beta}$) and (PG$^*_{\alpha,\beta}$) are consistent if and only if $\alpha \geq \beta \geq 0$ and $H \cap (\{0_Z\} \times \mathbb{R}) = \overline{H} \cap (\{0_Z\} \times \mathbb{R})$ if and only if $\alpha \leq 0$ or $\beta \neq 0$. By Theorem 4.4, (PG$_{\alpha,\beta}$) is solvable and val(PG$_{\alpha,\beta}$)=val(PG$^*_{\alpha,\beta}$) if and only if

$$\begin{cases} \alpha \geq \beta \geq 0, \\ \alpha \leq 0 \quad \text{or} \quad \beta \neq 0. \end{cases}$$

This is equivalent to

$$\alpha = \beta = 0 \quad \text{or} \quad \alpha \geq \beta > 0.$$

Remark 4.1. It is interesting to know whether $\overline{H\cap(\{0_Z\}\times\mathbb{R})}=\overline{H}\cap(\{0_Z\}\times\mathbb{R})$ when $H\cap(\{0_Z\}\times\mathbb{R})$ is closed. If so, by Theorem 4.4, the solvability and strong duality can be checked by the closedness of $H\cap(\{0_Z\}\times\mathbb{R})$, which is usually easier. Unfortunately, the assertion is not true, it is demonstrated by Example 4.2 in the case $\alpha>\beta=0$.

If the positive cone P, Q are closed, then the above theorems have immediate corollaries obtained by reversing the roles of the primal and dual programs.

Corollary 4.1. *Assume that P and Q are closed,* (ILP) *and* (ILP*) *are both consistent. The following statements are equivalent*

a. $\overline{K\cap(\{0_Y\}\times\mathbb{R})}=\overline{K}\cap(\{0_Y\}\times\mathbb{R})$;
b. val(ILP)= val(ILP*).

Proof. Just as the proof of Theorem 4.3 given above, Corollary 4.1 follows from Corollary 2.3 and parts (b) and (d) of Proposition 4.1. □

Corollary 4.2. *Assume that P and Q are closed,* (ILP) *and* (ILP*) *are both consistent. The following statements are equivalent*

a. $K\cap(\{0_Y\}\times\mathbb{R})=\overline{K}\cap(\{0_Y\}\times\mathbb{R})$;
b. (ILP*) *has a solution and* val(ILP) = val(ILP*).

Proof. Similar to the proof of Theorem 4.4 given above, Corollary 4.2 follows from the observation $K\cap(\{0_Y\}\times\mathbb{R})\subset\overline{K\cap(\{0_Y\}\times\mathbb{R})}\subset\overline{K}\cap(\{0_Y\}\times\mathbb{R})$, Theorem 4.2 and Corollary 4.1. □

4.3. Complementary slackness

Combining Corollary 2.2 and Theorem 4.3, we obtain a generalization of Corollary 3.1.

Theorem 4.5. *Suppose that x is a solution of* (ILP) *and w is a solution of* (ILP*). *Then the following statements are equivalent*

(a) $\overline{H\cap(\{0_Z\}\times\mathbb{R})}=\overline{H}\cap(\{0_Z\}\times\mathbb{R})$;
(b) The pair (x, w) satisfies the complementary slackness condition (5).

We will use Theorem 4.5 and Corollary 2.1 to solve the parametric version of Gale's example.

Example 4.3. Consider (PG$_{\alpha,\beta}$) given in Example 3.1. Let F, F^* be the feasible sets of (PG$_{\alpha,\beta}$) and (PG$^*_{\alpha,\beta}$), respectively. Clearly F^* is nonempty and rewritten as

$$F^* = \{(w_1, w_2) \in W : w_1 \leq 0, w_1 + w_2 \leq 0\}.$$

Since $H \cap (\{0_Z\} \times \mathbb{R})$ is closed, as in Example 4.2, $\overline{H \cap (\{0_Z\} \times \mathbb{R})} = \overline{H} \cap (\{0_Z\} \times \mathbb{R})$ if and only if $(\alpha, \beta) \in P_0$, where

$$P_0 := \left\{(p, q) \in \mathbb{R}^2 : p \leq 0 \text{ or } q \neq 0\right\}.$$

Suppose now that $(\alpha, \beta) \in P_0$. It follows from Theorem 4.5 and Corollary 2.1 that $x^* = (x_0, x_1, \ldots)$ solves $(\mathrm{PG}_{\alpha,\beta})$ and $w^* = (w_1, w_2)$ solves $(\mathrm{PG}^*_{\alpha,\beta})$ if and only if

$$\begin{cases} x^* \in F, \\ w^* \in F^*, \\ \langle x^*, c - A^* w^* \rangle = 0, \\ \langle Ax^* - b, w^* \rangle = 0. \end{cases}$$

The latter system of equations is rewritten as

$$\begin{cases} x_0 + \sum_{i=1}^{\infty} i x_i = \alpha, \quad \sum_{i=1}^{\infty} x_i = \beta, \quad x_i \geq 0 (i \in \mathbb{N}), \\ w_1 \leq 0, \quad w_1 + w_2 \leq 0, \\ x_0(1 - w_1) + \sum_{i=1}^{\infty} x_i(-i w_1 - w_2) = 0, \\ \left(x_0 + \sum_{i=1}^{\infty} i x_i - \alpha\right) w_1 + \left(\sum_{i=1}^{\infty} x_i - \beta\right) w_2 = 0. \end{cases} \tag{10}$$

By some calculations, (10) is equivalent to

$$\begin{cases} \sum_{i=1}^{\infty} i x_i = \alpha, \quad \sum_{i=1}^{\infty} x_i = \beta, \quad x_i \geq 0 (i \in \mathbb{N}), \\ w_1 \leq 0, \quad w_1 + w_2 \leq 0, \\ x_0 = 0, \\ x_i(i w_1 + w_2) = 0 (i \in \mathbb{N}^*). \end{cases} \tag{11}$$

Let $P_1, P_2, P_3, P_4 \subset \mathbb{R}^2$ be defined by

$$\begin{cases} P_1 := \{(0, 0)\}, \\ P_2 = \left\{(p, p) \in \mathbb{R}^2 : p > 0\right\}, \\ P_3 = \left\{(p, q) \in \mathbb{R}^2 : p > q > 0\right\}, \\ P_4 = P_0 \setminus (P_1 \bigcup P_2 \bigcup P_3). \end{cases}$$

We consider four cases of the pair (α, β).

Case 1. $(\alpha, \beta) \in P_1$

Solving (11), we get

$$\begin{cases} x_i = 0 (i \in \mathbb{N}), \\ w_1 \le 0, \quad w_1 + w_2 \le 0. \end{cases}$$

Therefore, the solution sets and the values of $(\mathrm{PG}_{\alpha,\beta})$ and $(\mathrm{PG}^*_{\alpha,\beta})$ are given by

$$\mathrm{Sol}(\mathrm{PG}_{\alpha,\beta}) = \{(0, 0, \ldots)\}, \quad \mathrm{Sol}(\mathrm{PG}^*_{\alpha,\beta}) = F^*,$$
$$\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \mathrm{val}(\mathrm{PG}^*_{\alpha,\beta}) = 0.$$

Case 2. $(\alpha, \beta) \in P_2$

Solving (11), we get

$$\begin{cases} x_i = 0 (i \in \mathbb{N} \setminus \{1\}), \\ x_1 = \alpha, \\ w_1 \le 0 \\ w_1 + w_2 = 0. \end{cases}$$

Therefore, the solution sets and the values of $(\mathrm{PG}_{\alpha,\beta})$ and $(\mathrm{PG}^*_{\alpha,\beta})$ are given by

$$\mathrm{Sol}(\mathrm{PG}_{\alpha,\beta}) = \{(0, \alpha, 0, \ldots)\}, \quad \mathrm{Sol}(\mathrm{PG}^*_{\alpha,\beta}) = \{(w_1, w_2) \in F^* : w_1 + w_2 = 0\},$$
$$\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \mathrm{val}(\mathrm{PG}^*_{\alpha,\beta}) = 0.$$

Case 3. $(\alpha, \beta) \in P_3$

In this case, (11) implies (is in fact equivalent, as we show later)

$$\begin{cases} \sum_{i=1}^{\infty} i x_i = \alpha, \quad \sum_{i=1}^{\infty} x_i = \beta, \quad x_i \ge 0 (i \in \mathbb{N}), \\ w_1 \le 0, \quad w_1 + w_2 \le 0, \\ x_0 = 0, \\ \alpha w_1 + \beta w_2 = 0. \end{cases}$$

Observe that $\alpha w_1 + \beta w_2 = (\alpha - \beta) w_1 + \beta (w_1 + w_2)$. Invoking the assumption $\alpha > \beta > 0$, the above system is equivalent to

$$\begin{cases} \sum_{i=1}^{\infty} i x_i = \alpha, \quad \sum_{i=1}^{\infty} x_i = \beta, \quad x_i \ge 0 (i \in \mathbb{N}), \\ x_0 = 0, \\ w_1 = w_2 = 0. \end{cases}$$

This is obviously implies (11). Hence, the solution sets and the values of $(\mathrm{PG}_{\alpha,\beta})$ and $(\mathrm{PG}^*_{\alpha,\beta})$ are given by

$$\mathrm{Sol}(\mathrm{PG}_{\alpha,\beta}) = \left\{(0, x_1, x_2, \ldots) : \sum_{i=1}^{\infty} i x_i = \alpha, \sum_{i=1}^{\infty} x_i = \beta, x_i \geq 0 (i \in \mathbb{N})\right\},$$

$$\mathrm{Sol}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = \{(0,0)\},$$
$$\mathrm{val}(\mathrm{PG}_{\alpha,\beta}) = \mathrm{val}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = 0.$$

Case 4. $(\alpha, \beta) \in P_4$

In this case, we will show that either $(\mathrm{PG}_{\alpha,\beta})$ or $(\mathrm{PG}^*_{\alpha,\beta})$ has no solution. Suppose on the contrary that there exist $x^* = (x_0, x_1, \ldots) \in \mathrm{Sol}(\mathrm{PG}_{\alpha,\beta})$ and $w^* = (w_1, w_2) \in \mathrm{Sol}(\mathrm{PG}^*_{\alpha,\beta})$. Then (x^*, w^*) satisfies (11). Then

$$\alpha = x_0 + \sum_{i=1}^{\infty} i x_i \geq \sum_{i=1}^{\infty} x_i = \beta \geq 0.$$

On the other hand, it follows from $(\alpha, \beta) \in P_0$ that $\alpha = \beta = 0$ or $\alpha \geq \beta > 0$. Then $(\alpha, \beta) \in P_1 \bigcup P_2 \bigcup P_3$. This is a contradiction.

In the following, let F, F^* be the feasible sets of (ILP) and (ILP*), respectively, i.e.

$$F = \{x \in X : Ax - b \in Q, x \in P\},$$
$$F^* = \{w \in W : -A^* w + c \in P^*, w \in Q^*\}.$$

Theorem 4.6. *The following statements are equivalent*

a. $H \cap (\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}) = \overline{H} \cap (\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\})$;
b. If (ILP*) *has a solution* w^* *then* (ILP) *has a solution* $x^* \in F$ *and the pair* (x^*, w^*) *satisfies the complementary slackness condition* (5).

Proof. (a) $\Rightarrow$ (b) Assume that w^* is a solution of (ILP*). Then $\mathrm{val}(\mathrm{ILP}^*) = \langle b, w^*\rangle \in \mathbb{R}$. From Theorem 2.2, (ILP) is subconsistent and $\mathrm{subval}(\mathrm{ILP}) = \langle b, w^*\rangle$. By (a) and the definition of subval(ILP), we have

$$(0_Z, \langle b, w^*\rangle) \in \overline{H} \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) = H \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right).$$

Thus, there exists $x^* \in P$ such that $Ax^* - b \in Q$ and $\langle x^*, c\rangle \leq \langle b, w^*\rangle$. It means that $x^* \in F$ and $\langle b, w^*\rangle \geq \langle x^*, c\rangle$. By Theorem 2.1, $\langle b, w^*\rangle = \langle x^*, c\rangle$, and so x^* is optimal for (ILP). It follows from Corollary 2.2 that the pair (x^*, w^*) satisfies the complementary slackness (5).

(b) $\Rightarrow$ (a) Without loss of generality we can assume that $\overline{H} \cap (\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\})$ is nonempty. Let $w^* \in F^*$ such that $(0_Z, \langle b, w^*\rangle) \in \overline{H}$. By Theorem 2.4(a), we have $\langle b, w^*\rangle \geq \langle b, w\rangle$ for every

$w \in F^*$. Hence, w^* is optimal for (ILP*). From Theorem 2.2, (ILP) is subconsistent and $\mathrm{subval(ILP)} = \mathrm{val(ILP^*)} = \langle b, w^* \rangle$. Since (b) holds, (ILP) has a solution x^* and the pair (x^*, w^*) satisfies the complementary slackness (5). Hence, by Corollary 2.2,

$$\mathrm{val(ILP)} = \langle x^*, c \rangle = \langle b, w^* \rangle = \mathrm{subval(ILP)}.$$

By parts (a) and (c) of Proposition 4.1, we have

$$\begin{aligned} &H \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) \\ &= \left(\{0_Z\} \times [\mathrm{val(ILP)}, \infty[\right) \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) \\ &= \left(\{0_Z\} \times [\mathrm{subval(ILP)}, \infty[\right) \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) \\ &= \overline{H} \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right). \end{aligned}$$

Therefore, (a) is satisfied. □

We will illustrate Theorem 4.6 by finding the solution set of (PG$^*_{\alpha,\beta}$) in parametric version of Gale's example.

Example 4.4. Consider (PG$_{\alpha,\beta}$) and (PG$^*_{\alpha,\beta}$) given in Example 3.1. Let F, F^* be the feasible sets of (PG$_{\alpha,\beta}$) and (PG$^*_{\alpha,\beta}$), respectively. Clearly F^* is nonempty and can be rewritten as

$$F^* = \left\{(w_1, w_2) \in W : w_1 \le 0, w_1 + w_2 \le 0\right\}.$$

By some calculations, we have

$$H \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) = \begin{cases} \{(0,0,0)\} & \text{if} \quad \alpha = \beta = 0, \\ \{(0,0,0)\} & \text{if} \quad \alpha \ge \beta > 0, \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\overline{H} \cap \left(\{0_Z\} \times \{\langle b, w\rangle : w \in F^*\}\right) = \begin{cases} \{(0,0,0)\} & \text{if} \quad \alpha \ge \beta \ge 0, \\ \emptyset & \text{otherwise}. \end{cases}$$

Therefore, $H \cap (\{0_Z\} \times \{\langle b,w\rangle : w \in F^*\}) = \overline{H} \cap (\{0_Z\} \times \{\langle b,w\rangle : w \in F^*\})$ if and only if $(\alpha, \beta) \in P_0$ where P_0 is defined as in Example 4.3.

Suppose now that $(\alpha, \beta) \in P_0$. It follows from Theorem 4.6 that $w^* = (w_1, w_2) \in F^*$ solves (PG$^*_{\alpha,\beta}$) if and only if there exists $x^* = (x_0, x_1, \ldots) \in F$ such that

$$\begin{cases} \langle x^*, c - A^* w^* \rangle = 0, \\ \langle A x^* - b, w^* \rangle = 0. \end{cases} \tag{12}$$

Let $P_1, P_2, P_3, P_4 \subset \mathbb{R}^2$ be defined as in Example 4.3. We also consider four cases of the pair (α, β).

Case 1. $(\alpha, \beta) \in P_1$

As calculated in Case 1 of Example 4.3, (12) holds for $x^* = (0, 0, \ldots) \in F$ and every $w^* \in F^*$. Hence,

$$\mathrm{Sol}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = F^*.$$

Case 2. $(\alpha, \beta) \in P_2$

As calculated in Case 2 of Example 4.3, $w^* = (w_1, w_2) \in F^*$ is such that there exists $x^* \in F$ and (x^*, w^*) satisfies (12) if and only if $w_1 + w_2 = 0$. Hence,

$$\mathrm{Sol}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = \left\{(w_1, w_2) \in F^* : w_1 + w_2 = 0\right\}.$$

Case 3. $(\alpha, \beta) \in P_3$

As calculated in Case 3 of Example 4.3, there is only $w^* = (0, 0) \in F^*$ such that there exists $x^* \in F$ and (x^*, w^*) satisfies (12). Hence,

$$\mathrm{Sol}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = \left\{(0, 0)\right\}.$$

Case 4. $(\alpha, \beta) \in P_4$

As calculated in Case 4 of Example 4.3, we cannot find $w^* \in F^*$ and $x^* \in F$ such that (12) holds. Therefore,

$$\mathrm{Sol}\left(\mathrm{PG}^*_{\alpha,\beta}\right) = \emptyset.$$

Theorem 4.7. *If P, Q are closed, then the following statements are equivalent*

a. $K \cap (\{0_Y\} \times \{\langle x, c\rangle : x \in F\}) = \overline{K} \cap (\{0_Y\} \times \{\langle x, c\rangle : x \in F\})$;
b. *If* (ILP) *has a solution x^* then* (ILP*) *has a solution w^* and the pair (x^*, w^*) satisfies the complementary slackness* (5).

Proof. Just as in the proof of Theorem 4.6, (a)⇒(b) follows from Corollary 2.3 and Theorem 2.1 and (b)⇒(a) follow from Theorem 2.4(b), Corollaries 2.2–2.3, and parts (b) and (d) of Proposition 4.1. □

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