DOI: 10.1002/int.22223

RESEARCH ARTICLE

WILEY

Multiple attribute group decision-making based on order-α divergence and entropy measures under q-rung orthopair fuzzy environment

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Abstract
The q-rung orthopair fuzzy set (^{q}ROPFS), proposed by Yager, is a more effective and proficient tool to represent uncertain or vague information in real-life situations. Divergence and entropy are two important measures, which have been extensively studied in different information environments, including fuzzy, intuitionistic fuzzy, interval-valued fuzzy, and Pythagorean fuzzy. In the present communication, we study the divergence and entropy measures under the q-rung orthopair fuzzy environment. First, the work defines two new order-α divergence measures for ^{q}ROPFSs to quantify the information of discrimination between two ^{q}ROPFSs. We also examine several mathematical properties associated with order-α ^{q}ROPF divergence measures in detail. Second, the paper introduces two new parametric entropy functions called "order-α ^{q}ROPF entropy measures" to measure the degree of fuzziness associated with a ^{q}ROPFS. We show that the proposed order-α divergence and entropy measures include several existing divergence and entropy measures as their particular cases. Further, the paper develops a new decision-making approach to solve multiple attribute group decision-making problems under the ^{q}ROPF environment where the information about the attribute weights is completely unknown or partially known. Finally, an

example of selecting the best enterprise resource planning system is provided to illustrate the decision-making steps and effectiveness of the proposed approach.

KEYWORDS
divergence measure, entropy measure, ERP system selection, MAGDM, q-rung orthopair fuzzy set

1 | INTRODUCTION

The management of uncertainty is a very crucial and challenging issue in many decision support systems. Traditionally, the probability theory was used to handle the uncertainty that arises due to the random nature of the systems. However, in many real-world situations, uncertainty arises due to vagueness, lack of knowledge, imprecise data, and missing information. To cope with these situations appropriately, Zadeh[1] proposed the theory of fuzzy sets (FSs) in 1965. Afterward, several extensions/generalization of FSs have been introduced to solve many real-world decision problems in different areas. Intuitionistic fuzzy sets (IFSs), proposed by Atanassov,[2] has become one of the extensively studied and used generalizations of FSs in the past three decades.[3–12] In the intuitionistic fuzzy theory, the membership grade of each element is presented by a pair of values in between 0 and 1, in which the first component of each pair represents the membership value, and the second component denotes the nonmembership value of the corresponding element to IFS. The primary condition to use IFS theory is that the sum of the values of each pair should be less than or equal to 1. However, the sum of these two values may be higher than one in many real-life situations. For instance, an expert is invited to give his/her opinion about the feasibility of an investment plan in the share market. Assume that the expert provides the degree of feasibility as 0.7 and the degree of infeasibility as 0.6 for this investment plan. It is observed that $0.7+0.6>1$, so the IFS cannot be used to describe this information accurately.

The Pythagorean fuzzy set (PFS) theory was introduced by Yager[13] and Yager and Abbasov[14] as a new and remarkable generalization of IFS. A PFS is characterized by two functions, namely membership and nonmembership, and satisfying the condition that the square sum of the membership degree and the nonmembership degree is ≤ 1 for each element. We can observe that $0.7^2+0.6^2<1$, and hence the PFSs are more potent than IFSs to express uncertain and vague information. In the last 5 years, PFS theory has been gained much attention from researchers, and many valuable theoretical and practical results have been obtained to use this theory in different application areas.[15–24]

Recently, Yager[25] introduced the notion of q-rung orthopair fuzzy sets (^{q}ROPFSs) as a more general form of IFS and PFS in which the sum of the qth power of the membership degree and the qth power of the nonmembership degree is ≤ 1. It is worth mentioning that as the value of the parameter q increases, more and more orthopair satisfy the bounding constraint. It means ^{q}ROPFSs give more information space to describe uncertain or vague information. Let us revisit the above-discussed example if the expert provides the degree of feasibility as 0.8 and the degree of infeasibility as 0.9 for the investment plan. In this situation, IFS, as well as PFS, cannot be used to represent the expert preference information because of $0.8+0.9>1$ and $0.8^2+0.9^2>1$. Nevertheless, it is possible with ^{q}ROPFS as $0.8^5+0.9^5<1$. Thus, ^{q}ROPFSs are more proficient in handling the higher level of uncertain real-world information.

In a short span, the ^{q}ROPFS theory has been attracted considerable attention from researchers working in different application areas. Liu and Wang[26] proposed some arithmetic and geometric aggregation operators, and Liu and Liu[27] extended Bonferroni mean (BM) operator for decision-making with ^{q}ROPF information. Peng et al[28] defined exponential operational laws for ^{q}ROPFS and developed a decision-making approach by using a new score function. Peng and Dai[29] developed a classroom teaching quality assessment method with ^{q}ROPF information. Du[30] discussed correlation and correlation coefficient for ^{q}ROPFSs. Yang and Pang[31] explained partitioned BM operators; Wei et al[32] formulated Heronian mean (HM) operators, and Wei et al[33] defined Maclaurin symmetric mean (MSM) operators for aggregating ^{q}ROPF information. Peng and Liu[34] conducted a detailed study on the relationship between different information measures under the ^{q}ROPF environment. Liu and Wang[35] studied the BM operators for q-rung orthopair fuzzy numbers based on Archimedean norms. In addition, some new MADM approaches based on q-rung orthopair fuzzy point weighted aggregation operators were developed by Xing et al.[36]

In mathematics, while studying a set of objects, we like to associate various quantitative measures defined over the set. Two basic such measures are- quantitative measures with each object and the difference or divergence measures between any two objects. In uncertainty theory, entropy is an important tool for measuring uncertain information. In 1972, De Luca and Termini[37] defined a measure of fuzzy entropy analogous to Shannon entropy.[38] After that, several entropy functions have been derived by considering different points of view to measure the fuzziness associated with an FS.[39–45] In 2001, Burillo and Bustince[46] first introduced the notion of intuitionistic fuzzy entropy. Szmidt and Kacprzyk[47] extended De Luca and Termini's axioms on fuzzy entropy and developed an intuitionistic fuzzy entropy by employing a geometric interpretation of IFSs. Since then, the notion of intuitionistic fuzzy entropy has been extensively studied by researchers from all over the world, and several entropy functions have been proposed to measure the fuzziness associated with an IFS.[48–55] Recently, Peng and Liu[34] have defined some entropy measures for ^{q}ROPFS.

Inspired by the idea of divergence between two probability distributions, Bhandari and Pal[42] introduced the notion of fuzzy divergence, which gives the measure of information discrimination between two FSs. In 2007, Vlachos and Sergiadis[49] extended the idea of fuzzy divergence to IFSs and defined a measure of divergence between two IFSs. Later, a number of divergence measures between IFSs have been defined by various eminent researchers,[50,52,56–61] and the outcomes have been implemented in different application areas, including pattern recognition, decision-making, medical diagnosis, image segmentation problems. In the Pythagorean fuzzy environment, Xiao and Ding[62] studied the Jensen-Shannon divergence measure between PFSs and discussed its applications in medical diagnosis.

Although many studies have been done on divergence and entropy measures under fuzzy and intuitionistic fuzzy environments by several researchers, however, best of our knowledge, there is no investigation on divergence measures under the ^{q}ROPF environment. Therefore, the main objective of this work is to study the divergence and entropy measures with ^{q}ROPF information. For doing so, first, we introduce the notion of order-α divergence measures for ^{q}ROPFSs based on logarithmic and exponential functions to analyze the information of discrimination between two ^{q}ROPFSs. Then, we discuss several important mathematical properties of these measures in detail. It is noted that the proposed order-α ^{q}ROPF divergence measures include several well-known fuzzy, intuitionistic fuzzy and Pythagorean fuzzy divergence measures as their particular and limiting cases. Second, we propose two new order-α entropy functions to measure the degree of fuzziness associated with a ^{q}ROPF. The work proves their validity requirements and discusses several particular and limiting cases. Besides, the paper utilizes the proposed order-α divergence and entropy measures to develop a nonlinear optimization model for determining the attribute weights with completely unknown or partially known information about the attribute weights. We also discuss the application of the proposed divergence measures in multiple attribute group decision-making (MAGDM).

The remainder of the paper is organized as follows. Section 2 presents some basic concepts of ^{q}ROPFSs, which will be used for further developments. In Section 3 we introduce the standard definition of divergence measure for ^{q}ROPFSs and define order-α divergence measures between two ^{q}ROPFSs. We also prove their several important properties with particular and limiting cases. Further, two new entropy functions called "order-α ^{q}ROPF entropy measures" are introduced, which satisfy the axiomatic requirements.[34] Some mathematical properties of the proposed order-α ^{q}ROPF entropy measures are studied. We show that the various existing fuzzy and intuitionistic fuzzy entropy measures are the special cases of the proposed entropy measures. Section 4 develops a decision-making approach based on proposed order-α divergence and entropy measures to solve MAGDM problems with ^{q}ROPF information. Then, we consider a real-life best enterprise resource planning (ERP) system selection problem to demonstrate the effectiveness of the developed approach. A comparative study with existing methods is also provided to validate the obtained results. Section 5 concludes the works and highlights some future directions.

2 | PRELIMINARIES

This section introduces the definition of ^{q}ROPFS, basic operations, the concept of entropy, and the ^{q}ROPF weighted averaging operator.

Definition 1 (Yager[25]). ^{q}ROPFS P in a finite universe of discourse $Z = \{z_1, z_2, ..., z_n\}$ is given by

$$P = \{\langle z, \xi_P(z), \zeta_P(z)\rangle | z \in Z\}, \tag{1}$$

where $\xi_P : Z \to [0, 1]$ and $\zeta_P : Z \to [0, 1]$ denote the membership degree (MD) and the nonmembership degree (NMD) of the element z to the set P, respectively, satisfying the condition that $0 \leq \xi_P^q(z) + \xi_P^q(z) \leq 1 (q \geq 1)$. Moreover, the degree of the hesitancy of z to the set P is obtained by $\eta(z) = \sqrt[q]{1 - (\xi_P(z))^q + (\zeta_P(z))^q}$. Throughout this paper, we denote the family of all ^{q}ROPFS in Z by $^qROPFS(Z)$. For a given ^{q}ROPFS, the pair $(\xi_P(z), \zeta_P(z))$ is called the q-rung orthopair fuzzy number (^{q}ROPFN) and denoted by $\chi = (\xi_\chi, \zeta_\chi)$.

Note 1. If we put $q = 1$ and 2 in Definition 1, then ^{q}ROPFS is reduced to IFS[2] and PFS,[13] respectively.

Yager[25] and Liu and Wang[26] defined some basic operational laws on ^{q}ROPFSs as follows:

Definition 2. Let $P = \{\langle z, \xi_P(z), \zeta_P(z)\rangle \mid z \in Z\}$ and $Q = \{\langle z, \xi_Q(z), \zeta_Q(z)\rangle \mid z \in Z\}$ be two ^{q}ROPFSs in Z, then

i. $P \subseteq Q$ if and only if $\xi_P(z) \leq \xi_Q(z)$ and $\zeta_P(z) \geq \zeta_Q(z) \, \forall \, z \in Z$;
ii. $P = Q$ if and only if $P \subseteq Q$ and $P \supseteq Q$;
iii. $P^C = \{\langle z, \zeta_P(z), \xi_P(z)\rangle | z \in Z\}$;
iv. $P \overset{q}{\cup} Q = \{\langle z, \max(\xi_P(z), \xi_Q(z)), \min(\zeta_P(z), \zeta_Q(z))\rangle | z \in Z\}$;
v. $P \overset{q}{\cap} Q = \left\{\left\langle z, \min\left(\xi_P(z), \xi_Q(z)\right), \max\left(\zeta_P(z), \zeta_Q(z)\right)\right\rangle \middle| z \in Z\right\}$;

vi. $P \overset{q}{\oplus} Q = \left\{ \left\langle z, \sqrt[q]{\xi_P^q(z) + \xi_Q^q(z) - \xi_P^q(z)\xi_Q^q(z)}, \zeta_P(z)\zeta_Q(z) \right\rangle \middle| z \in Z \right\}$;

vii. $P \overset{q}{\oplus} Q = \left\{ \left\langle z, \xi_P(z)\xi_Q(z), \sqrt[q]{\zeta_P^q(z) + \zeta_Q^q(z) - \zeta_P^q(z)\zeta_Q^q(z)} \right\rangle \middle| z{\in}Z \right\}$;

viii. $P \overset{q}{\wedge} \lambda = \left\{ \left\langle z, \xi_P^\lambda(z), \sqrt[q]{1 - (1 - \zeta_P^q(z))^\lambda} \right\rangle \middle| z \in Z \right\}$;

ix. $\lambda \overset{q}{*} P = \left\{ \left\langle z, \sqrt[q]{1 - (1 - \xi_P^q(z))^\lambda}, \zeta_P^\lambda(z) \right\rangle \middle| z \in Z \right\}$.

Definition 3 (Peng and Liu[34]). An entropy on ${}^qROPFS(Z)$ is a real-valued mapping ${}^qE : {}^qROPFS(Z) \to [0, 1]$, which holds the following properties:

EP1: ${}^qE(P) = 0$ if and only if P is a crisp set.

EP2: ${}^qE(P) = 1$ if and only if $\xi_P(z) = \zeta_P(z)\ \forall\, z \in Z$.

EP3: ${}^qE(P) \leq {}^qE(Q)$ if P is less fuzzy than Q, that is, $\xi_P(z) \leq \xi_Q(z)$, $\zeta_P(z) \geq \zeta_Q(z)$ for $\xi_Q(z) \leq \zeta_Q(z)$ or $\xi_P(z) \geq \xi_Q(z)$, $\zeta_P(z) \leq \zeta_Q(z)$ for $\xi_Q(z) \geq \zeta_Q(z)$ for any $z \in Z$.

EP4: ${}^qE(P) = {}^qE(P^C)$.

Definition 4. Let[26] $\chi_k = \left(\xi_{\chi_k}, \zeta_{\chi_k}\right)$ $(k = 1, 2, ..., t)$ be a collection of q-rung orthopair fuzzy number (^{q}ROPFN) and $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_t)^T$ be the weight vector of χ_k with $\upsilon_k \geq 0$ $(k = 1, 2, ..., t)$ and $\sum_{k=1}^t \upsilon_k = 1$. Then the function ${}^qROPFWA : V^t \to V$ defined as

$$
\begin{aligned}
{}^qROPFWA(\chi_1, \chi_2, ..., \chi_t) &= \left(\upsilon_1 \overset{q}{*} \chi_1\right) \overset{q}{\oplus} \left(\upsilon_2 \overset{q}{*} \chi_2\right) \overset{q}{\oplus} \cdots \overset{q}{\oplus} \left(\upsilon_t \overset{q}{*} \chi_t\right) \\
&= \left(\left(1 - \prod_{k=1}^{t}\left(1 - \xi_{\chi_k}^q\right)^{\upsilon_k}\right)^{1/q}, \prod_{k=1}^{t}\left(\zeta_{\chi_k}\right)^{\upsilon_k} \right),
\end{aligned} \tag{2}
$$

is called the q-Rung orthopair fuzzy weighted averaging (^{q}ROPFWA) operator.

Based on these concepts, in the next section, we introduce the order-α divergence and entropy measures for ^{q}ROPFSs and discuss their important properties with particular and limiting cases in detail.

3 | ORDER-α DIVERGENCE AND ENTROPY MEASURES FOR ^{q}ROPFSs

3.1 | Order-α divergence measures for ^{q}ROPFSs

Analogous to the Vlachos and Sergiadis,[49] first, we propose the standard definition of divergence measure for ^{q}ROPFSs as follows:

Definition 5. Let $P, Q \in {}^qROPFS(X)$, then the mapping $D : {}^qROPFS(Z) \times {}^qROPFS(Z) \to [0, 1]$ is called a divergence measure for ^{q}ROPFSs if it satisfies the following two properties:

DP1: $D(P \mid Q) \geq 0$ with equality if and only if $P = Q$;

DP2: $0 \leq D(P \mid Q) \leq 1$.

3.1.1 | Order-α divergence measures between ^{q}ROPFSs under single element universe

Definition 6. Let P and Q be two ^{q}ROPFSs defined in a single element universe $Z=\{z\}$, and from Definition 1, we have

$$\left.\begin{array}{ll}\xi_P^q(z)+\zeta_P^q(z)+\eta_P^q(z)=1, & 0\le\xi_P(z),\zeta_P(z),\eta_P(z)\le 1\\ \xi_Q^q(z)+\zeta_Q^q(z)+\eta_Q^q(z)=1, & 0\le\xi_Q(z),\zeta_Q(z),\eta_Q(z)\le 1\end{array}\right\}. \tag{3}$$

Equation (3) recommends that $\left(\xi_P^q(z),\zeta_P^q(z),\eta_P^q(z)\right)$ and $\left(\xi_Q^q(z),\zeta_Q^q(z),\eta_Q^q(z)\right)$ can be considered as two probability distributions associated with z. Then, based on the notion of order-α divergence between two probability distributions,[63] we define the following order-α divergence measures between two ^{q}ROPFSs P and Q given by

$$ {}_1^qD_\alpha^*(P\,|\,Q)=\frac{1}{(\alpha-1)}\log_2\left[\left(\xi_P^q(z)\right)^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha}+\left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha}\right], \tag{4}$$

$$ {}_2^qD_\alpha^*(P\,|\,Q)=\frac{1}{\left(e^{2^{\alpha-1}}-e\right)}\left[e^{\left(\left(\xi_P^q(z)\right)^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha}+\left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha}\right)}-e\right],$$

where $\alpha\in(0,1)$. (5)

Note 2. In all the formulas e denotes the exponential function.

In the next theorem, we prove that the measures ${}_\delta^qD_\alpha^*(P\,|\,Q)(\delta=1,2)$ defined in Equations (4) and (5) are valid divergence measures between ^{q}ROPFSs.

Theorem 1. *The divergence measures* ${}_\delta^qD_\alpha^*(P\,|\,Q)(\delta=1,2)$ *satisfy the properties DP1 and DP2, as listed in Definition 5.*

Proof.

(i) From Taneja,[64] we know that

$$\sum_{k=1}^m p_k^\alpha q_k^{1-\alpha}\begin{cases}\le 1, & 0<\alpha\le 1,\\ \ge 1, & \alpha\ge 1,\end{cases}\quad\text{where}\ \sum_{k=1}^m p_k=\sum_{k=1}^m q_k=1. \tag{6}$$

Now, by utilizing Equation (3) with the inequality given in Equation (6), we get

$$(\xi_P^q(z))^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha} + \left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha} \le 1 \ \forall\ \alpha\in(0,1). \tag{7}$$

Based on the above inequality, we obtain

$$\left.\begin{matrix}\frac{1}{(\alpha-1)}\log_2\left[\left(\xi_P^q(z)\right)^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha} + \left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha}\right] \\ \frac{1}{(e^{2^{\alpha-1}}-e)}\left[e^{\left(\left(\xi_P^q(z)\right)^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha}+\left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha}\right)}-e\right]\end{matrix}\right\}\ge 0. \tag{8}$$

Further, when $P=Q\Rightarrow\xi_P(z)=\xi_Q(z)$ and $\zeta_P(z)=\zeta_Q(z)$, then we get

$${}_{\delta}^{q}D_\alpha^*(P|Q)=0,(\delta=1,2). \tag{9}$$

Next, let us consider ${}_{\delta}^{q}D_\alpha^*(P|Q)=0$ $(\delta=1,2)$, which imply

$$\frac{1}{(\alpha-1)}\log_2\left[\left(\xi_P^q(z)\right)^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha} + \left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha}\right]=0, \tag{10}$$

$$\frac{1}{(e^{2^{\alpha-1}}-e)}\left[e^{\left(\left(\xi_P^q(z_j)\right)^\alpha\left(\frac{\xi_P^q(z_j)+\xi_Q^q(z_j)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z_j)\right)^\alpha\left(\frac{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}{2}\right)^{1-\alpha}+\left(\eta_P^q(z_j)\right)^\alpha\left(\frac{\eta_P^q(z_j)+\eta_Q^q(z_j)}{2}\right)^{1-\alpha}\right)}-e\right]=0. \tag{11}$$

From Equations (10) and (11), we get

$$(\xi_P^q(z))^\alpha\left(\frac{\xi_P^q(z)+\xi_Q^q(z)}{2}\right)^{1-\alpha}+\left(\zeta_P^q(z)\right)^\alpha\left(\frac{\zeta_P^q(z)+\zeta_Q^q(z)}{2}\right)^{1-\alpha} + \left(\eta_P^q(z)\right)^\alpha\left(\frac{\eta_P^q(z)+\eta_Q^q(z)}{2}\right)^{1-\alpha} = 1. \tag{12}$$

Since $\alpha\in(0,1)$, $\alpha\neq 1$, then Equation (12) holds only when $\xi_P(z)=\xi_Q(z)$ and $\zeta_P(z)=\zeta_Q(z)\Rightarrow P=Q$.

Hence based on results mentioned in Equations (8) to (12), we conclude

$$ {}^{q}_{\delta}D^{*}_{\alpha}(P|Q) \geq 0 \; (\delta = 1, 2) \text{ with equality if and only } P = Q. \tag{13}$$

(ii) It has already been proved that ${}^{q}_{\delta}D^{*}_{\alpha}(P|Q) \geq 0$ $(\delta = 1, 2)$, so we must show that the maximum value attains by the divergence measures ${}^{q}_{\delta}D^{*}_{\alpha}(P|Q)$ is 1.

Note that the divergence measures ${}^{q}_{\delta}D^{*}_{\alpha}(P|Q)$ $(\delta = 1, 2)$ attain their maximum for the following degenerate cases: (a) $P = (1, 0, 0)$ and $Q = (0, 1, 0)$; (b) $P = (0, 1, 0)$ and $Q = (1, 0, 0)$; (c) $P = (0, 0, 1)$ and $Q = (0, 1, 0)$; and (d) $P = (0, 0, 1)$ and $Q = (1, 0, 0)$. Therefore, in all these cases, we get

$$ {}^{q}_{\delta}D^{*}_{\alpha}(P|Q) = 1. \tag{14}$$

Hence

$$ 0 \leq {}^{q}_{\delta}D^{*}_{\alpha}(P|Q) \leq 1. \tag{15}$$

This proves the theorem. □

3.1.2 | Order-α divergence measures between ^{q}ROPFSs under finite universe of discourse

The idea of order-α divergence measures can be easily extended to any finite universe of discourse. We propose the following formal definition of the order-α divergence measures between ^{q}ROPFSs under the finite universe given as the following definition.

Definition 7. Let P and Q be two ^{q}ROPFSs defined in $Z = \{z_1, z_2, ..., z_n\}$. Then, we define the associated order-α divergence measures between two ^{q}ROPFSs P and Q as follows:

$$ {}^{q}_{1}D_{\alpha}(P \mid Q) = \sum_{j=1}^{n} {}^{q}_{1}D^{*}_{\alpha}(P(z_j)|Q(z_j)) = \frac{1}{n(\alpha - 1)} \sum_{j=1}^{n} \log_2 \left[\left(\xi_P^q(z_j)\right)^{\alpha} \left(\frac{\xi_P^q(z_j) + \xi_Q^q(z_j)}{2} \right)^{1-\alpha} + \left(\zeta_P^q(z_j)\right)^{\alpha} \left(\frac{\zeta_P^q(z_j) + \zeta_Q^q(z_j)}{2} \right)^{1-\alpha} + \left(\eta_P^q(z_j)\right)^{\alpha} \left(\frac{\eta_P^q(z_j) + \eta_Q^q(z_j)}{2} \right)^{1-\alpha} \right], \tag{16}$$

$$ {}^{q}_{2}D_{\alpha}(P \mid Q) = \sum_{j=1}^{n} {}^{q}_{2}D^{*}_{\alpha}\left(P(z_j)|Q(z_j)\right) = \frac{1}{n(e^{2^{\alpha-1}} - e)} \times \sum_{j=1}^{n} \left[e^{\left(\left(\xi_P^q(z_j)\right)^{\alpha} \left(\frac{\xi_P^q(z_j) + \xi_Q^q(z_j)}{2} \right)^{1-\alpha} + \left(\zeta_P^q(z_j)\right)^{\alpha} \left(\frac{\zeta_P^q(z_j) + \zeta_Q^q(z_j)}{2} \right)^{1-\alpha} + \left(\eta_P^q(z_j)\right)^{\alpha} \left(\frac{\eta_P^q(z_j) + \eta_Q^q(z_j)}{2} \right)^{1-\alpha} \right)} - e \right], \text{ where } \alpha \in (0, 1) \tag{17}$$

Note that the divergence measures ${}^{q}_{\delta}D_{\alpha}(P|Q)$ $(\delta = 1, 2)$ are not symmetric. To imbue the measures with symmetry, we propose the symmetric version of the divergence measures ${}^{q}_{\delta}D_{\alpha}(P|Q)(\delta = 1, 2)$ by the following expression:

$$ {}^{q}_{\delta}D_{\alpha}(P\|Q) = \frac{1}{2}\left({}^{q}_{\delta}D_{\alpha}(P|Q) + {}^{q}_{\delta}D_{\alpha}(Q|P)\right). \tag{18} $$

In the next theorem, we study some mathematical properties of the proposed symmetric divergence measures ${}^{q}_{\delta}D_{\alpha}(P\|Q)(\delta = 1, 2)$ in detail, which prepare their application ground in different areas.

Theorem 2. *For all $P, Q, R \in {}^{q}ROPFS(Z)$, the divergence measures ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$ hold the following properties:*

i. ${}^{q}_{\delta}D_{\alpha}\left(P\middle\|P \overset{q}{\cup} Q\right) = {}^{q}_{\delta}D_{\alpha}\left(Q\middle\|P \overset{q}{\cap} Q\right)$;

ii. ${}^{q}_{\delta}D_{\alpha}\left(P\middle\|P \overset{q}{\cap} Q\right) = {}^{q}_{\delta}D_{\alpha}\left(Q\middle\|P \overset{q}{\cup} Q\right)$;

iii. ${}^{q}_{\delta}D_{\alpha}\left(P \overset{q}{\cup} Q\middle\|P \overset{q}{\cap} Q\right) = {}^{q}_{\delta}D_{\alpha}(P\|Q)$;

iv. ${}^{q}_{\delta}D_{\alpha}\left(P\middle\|P \overset{q}{\cup} Q\right) + {}^{q}_{\delta}D_{\alpha}\left(P\middle\|P \overset{q}{\cap} Q\right) = {}^{q}_{\delta}D_{\alpha}(P\|Q)$;

v. ${}^{q}_{\delta}D_{\alpha}\left(P \overset{q}{\cup} Q\middle\|R\right) \leq {}^{q}_{\delta}D_{\alpha}(P\|R) + {}^{q}_{\delta}D_{\alpha}(Q\|R)$;

vi. ${}^{q}_{\delta}D_{\alpha}\left(P \overset{q}{\cap} Q\middle\|R\right) \leq {}^{q}_{\delta}D_{\alpha}(P\|R) + {}^{q}_{\delta}D_{\alpha}(Q\|R)$;

vii. ${}^{q}_{\delta}D_{\alpha}\left(P \overset{q}{\cup} Q\middle\|R\right) + {}^{q}_{\delta}D_{\alpha}\left(P \overset{q}{\cap} Q\middle\|R\right) = {}^{q}_{\delta}D_{\alpha}(P\|R) + {}^{q}_{\delta}D_{\alpha}(Q\|R|)$;

viii. ${}^{q}_{\delta}D_{\alpha}(P\|Q) = {}^{q}_{\delta}D_{\alpha}(P^{C}\|Q^{C})$;

ix. ${}^{q}_{\delta}D_{\alpha}(P\|Q^{C}) = {}^{q}_{\delta}D_{\alpha}(P^{C}\|Q)$;

x. ${}^{q}_{\delta}D_{\alpha}(P\|Q) + {}^{q}_{\delta}D_{\alpha}(P\|Q^{C}) = {}^{q}_{\delta}D_{\alpha}(P^{C}\|Q^{C}) + {}^{q}_{\delta}D_{\alpha}(P^{C}\|Q)$.

Proof. By using a similar methodology as adopted in the references,[8,57,59] we can obtain the proof of these properties easily. Therefore, we omit the proof from here. □

Special cases of the divergence measures ${}^{q}_{\delta}D_{\alpha}(P|Q)$ and ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$:

(1) When $\alpha \to 1$, then ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$ are reduced to the following measure:

$$
{}^{q}_{\delta}D_{\alpha}(P\|Q) = \frac{1}{2n}\sum_{j=1}^{n}\left[\begin{array}{l}
\xi_P^q(z_j)\log_2\left(\dfrac{2\xi_P^q(z_j)}{\xi_P^q(z_j)+\xi_Q^q(z_j)}\right) + \zeta_P^q(z_j)\log_2\left(\dfrac{2\zeta_P^q(z_j)}{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}\right)\\
\quad + \eta_P^q(z_j)\log_2\left(\dfrac{2\eta_P^q(z_j)}{\eta_P^q(z_j)+\eta_Q^q(z_j)}\right)\\
\xi_Q^q(z_j)\log_2\left(\dfrac{2\xi_Q^q(z_j)}{\xi_P^q(z_j)+\xi_Q^q(z_j)}\right) + \zeta_Q^q(z_j)\log_2\left(\dfrac{2\zeta_Q^q(z_j)}{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}\right)\\
\quad + \eta_Q^q(z_j)\log_2\left(\dfrac{2\eta_Q^q(z_j)}{\eta_P^q(z_j)+\eta_Q^q(z_j)}\right)
\end{array}\right], \tag{19}
$$

which gives the J-divergence measure between two ^{q}ROPFSs corresponding to Hung and Yang.[56]

(2) When $\alpha \to 1$ and $q = 2$, then divergence measures ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$ become the Pythagorean fuzzy Jenson-Shannon divergence proposed by Xiao and Ding.[62]
(3) When $\alpha \to 1$ and $q = 1$, then divergence measures ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$ become the J-divergence measure on IFSs proposed by Hung and Yang.[56]
(4) When $\alpha \to 1$ and $q = 1$, then divergence measures ${}^{q}_{\delta}D_{\alpha}(P|Q)$ $(\delta = 1, 2)$ reduce into the intuitionistic fuzzy divergence measure defined by Wei and Ye.[65]
(5) When $\alpha = 1/2$, then ${}^{q}_{\delta}D_{\alpha}(P\|Q)$ $(\delta = 1, 2)$ are reduced to the following measures:

$$
\begin{aligned}
&{}^{q}_{1}D_{1/2}(P\|Q)\\
&= \frac{-1}{n}\sum_{j=1}^{n}\left[\log_2\left\{\begin{array}{l}
\left(\sqrt{\left(\xi_P^q(z_j)\right)\left(\dfrac{\xi_P^q(z_j)+\xi_Q^q(z_j)}{2}\right)} + \sqrt{\left(\zeta_P^q(z_j)\right)\left(\dfrac{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}{2}\right)}\right.\\
\quad\left. + \sqrt{\left(\eta_P^q(z_j)\right)\left(\dfrac{\eta_P^q(z_j)+\eta_Q^q(z_j)}{2}\right)}\right)\\
\quad\times\left(\sqrt{\left(\xi_Q^q(z_j)\right)^{\alpha}\left(\dfrac{\xi_Q^q(z_j)+\xi_P^q(z_j)}{2}\right)} + \sqrt{\left(\zeta_Q^q(z_j)\right)\left(\dfrac{\zeta_Q^q(z_j)+\zeta_P^q(z_j)}{2}\right)}\right.\\
\quad\left. + \sqrt{\left(\eta_Q^q(z_j)\right)\left(\dfrac{\eta_Q^q(z_j)+\eta_P^q(z_j)}{2}\right)}\right)
\end{array}\right\}\right],
\end{aligned} \tag{20}
$$

$$
{}_{2}^{q}D_{1/2}(P\|Q) = \frac{1}{2n(e^{1/\sqrt{2}}-e)}\sum_{j=1}^{n} \times \left[\left\{ e^{\left(\sqrt{\left(\xi_P^q(z_j)\right)\left(\frac{\xi_P^q(z_j)+\xi_Q^q(z_j)}{2}\right)}+\sqrt{\left(\zeta_P^q(z_j)\right)\left(\frac{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}{2}\right)}+\sqrt{\left(\eta_P^q(z_j)\right)\left(\frac{\eta_P^q(z_j)+\eta_Q^q(z_j)}{2}\right)}\right)} \right\} + e^{\left(\sqrt{\left(\xi_Q^q(z_j)\right)^{\alpha}\left(\frac{\xi_Q^q(z_j)+\xi_P^q(z_j)}{2}\right)}+\sqrt{\left(\zeta_Q^q(z_j)\right)\left(\frac{\zeta_Q^q(z_j)+\zeta_P^q(z_j)}{2}\right)}+\sqrt{\left(\eta_Q^q(z_j)\right)\left(\frac{\eta_Q^q(z_j)+\eta_P^q(z_j)}{2}\right)}\right)} - 2e \right], \tag{21}
$$

which we called the Bhattacharyya distance measures between two ^{q}ROPFSs.

(6) When $q = 1$ and $\eta_P^q(z_j) = \eta_P^q(z_j) = 0 \ \forall\, j$, then ${}_{\delta}^{q}D_{\alpha}(P\,|\,Q)$ $(\delta = 1, 2)$ give modified version of fuzzy divergence of order-α defined by Hooda.[43]

(7) When $\alpha \to 1$, $q = 1$ and $\eta_P^q(z_j) = \eta_P^q(z_j) = 0 \ \forall\, j$, then ${}_{\delta}^{q}D_{\alpha}(P\,|\,Q)$ $(\delta = 1, 2)$ are reduced to fuzzy divergence proposed by Shang and Jiang.[66]

We know that the ^{q}ROPF divergence measures give information of discrimination between two ^{q}ROPFSs. In 2007, Vlachos and Sergiadis[49] defined a relationship between entropy and divergence measures for IFSs. It is expected that a similar relation will also valid for ^{q}ROPFSs. In the next theorem, based on the developed divergence measures between ^{q}ROPFSs, we will define two new entropy measures for ^{q}ROPFSs.

3.2 | Order-α ^{q}ROPF entropy measures

Theorem 3. *Let $P \in {}^{q}ROPFS(Z)$, then*

$$
{}_{1}^{q}E_{\alpha}(P) = -{}_{1}^{q}D_{\alpha}\left(P\,|\,P^{C}\right) + 1 = \frac{1}{n(1-\alpha)}\sum_{j=1}^{n}\log_2\left[\left(\left(\xi_P^q(z_j)\right)^{\alpha} + \left(\zeta_P^q(z_j)\right)^{\alpha}\right)\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j)\right], \tag{22}
$$

$$
{}_{2}^{q}E_{\alpha}(P) = -{}_{2}^{q}D_{\alpha}\left(P\,|\,P^{C}\right) + 1 = \frac{1}{n\left(e - e^{2^{\alpha-1}}\right)}\sum_{j=1}^{n}\left[e^{2^{\alpha-1}\left(\left(\left(\xi_P^q(z_j)\right)^{\alpha}+\left(\zeta_P^q(z_j)\right)^{\alpha}\right)\left(\xi_P^q(z_j)+\zeta_P^q(z_j)\right)^{1-\alpha}+2^{1-\alpha}\eta_P^q(z_j)\right)} - e^{2^{\alpha-1}}\right],
$$

where $\alpha \in (0, 1)$. (23)

are ^{q}ROPF entropy measures.

Proof. To prove the validity of the proposed entropy measures, it is enough to show that the ^{q}ROPF entropy measures given in Equations (22) and (23) satisfy the properties EP1 to EP4 mentioned in Definition 3.

EP1. Let P be a crisp set having membership values either 0 or 1 $\forall z_j \in Z$. Then entropy measures defined in Equations (22) and (23) become 0.

Next, if ${}_1^qE(P) = {}_2^qE_\alpha(P) = 0$, that is

$$\frac{1}{n(1-\alpha)}\sum_{j=1}^{n}\log_2\left[\left(\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha\right)\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j)\right] = 0, \quad (24)$$

$$\frac{1}{n(e - e^{2^{\alpha-1}})}\sum_{j=1}^{n}\left[e^{2^{\alpha-1}(((\xi_P^q(z_j))^\alpha + (\zeta_P^q(z_j))^\alpha)(\xi_P^q(z_j)+\zeta_P^q(z_j))^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j))} - e^{2^{\alpha-1}}\right] = 0. \quad (25)$$

From Equations (24) and (25), we get

$$\left(\xi_P^q(z_j) + \varsigma_Q^q(z_j)\right)\left(\frac{\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha}{\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^\alpha} - 2^{1-\alpha}\right) = 1 - 2^{1-\alpha} \ \forall \ z_j \in Z. \quad (26)$$

Since $\alpha \in (0,1)$, then Equation (26) will hold only when $\xi_P^q(z_j) = 0, \zeta_P^q(z_j) = 1$ or $\xi_P^q(z_j) = 1$, $\zeta_P^q(z_j) = 0 \ \forall \ z_j \in Z \Rightarrow \xi_P(z_j) = 0, \zeta_P(z_j) = 1$ or $\xi_P(z_j) = 1, \zeta_P(z_j) = 0 \ \forall \ z_j \in Z$, that is, P is a crisp set.

EP2. Let $\xi_P(z_j) = \zeta_P(z_j) \forall z_j \in Z$, then applying this condition on entropy measures ${}_\delta^qE_\alpha(P)(\delta = 1, 2)$ yield 1.

Conversely, let ${}_\delta^qE_\alpha(P) = 1$, then we have

$$\left(\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha\right)\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j) = 2^{1-\alpha}$$

or

$$\frac{\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha}{2} = \left(\frac{\xi_P^q(z_j) + \zeta_P^q(z_j)}{2}\right)^\alpha. \quad (27)$$

Now consider a function

$$f(y) = y^\alpha, \quad \text{where } y \in (0,1], \alpha \in (0,1). \quad (28)$$

Differentiating Equation (28) w.r.t. y, then we get

$$\frac{d(f(y))}{dy} = \alpha y^{\alpha-1} \text{ and } \frac{d^2(f(y))}{dy} = \alpha(\alpha-1)y^{\alpha-2}. \quad (29)$$

Since $\frac{d^2(f(y))}{dy} < 0$, when $0 < \alpha < 1$. So $f(y)$ is a concave function for all $\alpha \in (0,1)$. Consequently, for any $y_1, y_2 \in (0,1]$, we get the following inequality

$$\frac{f(y_1) + f(y_2)}{2} \le f\left(\frac{y_1 + y_2}{2}\right), \quad \text{when } \alpha \in (0,1), \quad (30)$$

with equality only for $y_1 = y_2$.

Therefore, by utilizing the inequalities given in Equation (30) with Equation (27), we conclude that

$$\xi_P(z_j) = \zeta_P(z_j) \quad \forall\, z_j \in Z. \tag{31}$$

EP3. To prove EP3, we construct the following functions as

$$\left.\begin{aligned} h_1(x,y) &= \frac{1}{(1-\alpha)}\log_2[(x^\alpha+y^\alpha)(x+y)^{1-\alpha}+2^{1-\alpha}(1-x-y)] \\ h_2(x,y) &= \frac{1}{(e-e^{2^{\alpha-1}})}[e^{2^{1-\alpha}((x^\alpha+y^\alpha)(x+y)^{1-\alpha}+2^{1-\alpha}(1-x-y))}-e^{2^{1-\alpha}}] \end{aligned}\right\}, \quad \text{where } x,y\in[0,1] \text{ and } \alpha\in(0,1). \tag{32}$$

Taking the partial derivatives of $h_1(x,y)$ and $h_2(x,y)$ with respect to x and y, respectively, yield:

$$\left.\begin{aligned} \frac{\partial h_1(x,y)}{\partial x} &= \left[\frac{(1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}x^{\alpha-1}-2^{1-\alpha}}{(1-\alpha)((x+y)^{1-\alpha}(x^\alpha+y^\alpha)+2^{1-\alpha}(1-x-y))}\right] \\ \frac{\partial h_1(x,y)}{\partial y} &= \left[\frac{(1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}y^{\alpha-1}-2^{1-\alpha}}{(1-\alpha)((x+y)^{1-\alpha}(x^\alpha+y^\alpha)+2^{1-\alpha}(1-x-y))}\right] \end{aligned}\right\}, \tag{33}$$

$$\left.\begin{aligned} \frac{\partial h_2(x,y)}{\partial x} &= \left[\frac{2^{1-\alpha}e^{2^{1-\alpha}((x^\alpha+y^\alpha)(x+y)^{1-\alpha}+2^{1-\alpha}(1-x-y))}((1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}x^{\alpha-1}-2^{1-\alpha})}{(e-e^{2^{\alpha-1}})}\right] \\ \frac{\partial h_2(x,y)}{\partial y} &= \left[\frac{2^{1-\alpha}e^{2^{1-\alpha}((x^\alpha+y^\alpha)(x+y)^{1-\alpha}+2^{1-\alpha}(1-x-y))}((1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}y^{\alpha-1}-2^{1-\alpha})}{(e-e^{2^{\alpha-1}})}\right] \end{aligned}\right\}. \tag{34}$$

For a critical point of $h_1(x,y)$ and $h_2(x,y)$, we set $\frac{\partial h_1(x,y)}{\partial x} = \frac{\partial h_1(x,y)}{\partial y} = 0$ and $\frac{\partial h_2(x,y)}{\partial x} = \frac{\partial h_2(x,y)}{\partial y} = 0$. It gives

$$x = y. \tag{35}$$

Since $x,\ y \in [0,\ 1]$, we have

$$\left.\begin{aligned} \frac{\partial h_1(x,y)}{\partial x}, \frac{\partial h_2(x,y)}{\partial x} &\geq 0, \quad \text{when } x \leq y, \quad \alpha\in(0,1) \\ \frac{\partial h_1(x,y)}{\partial x}, \frac{\partial h_2(x,y)}{\partial x} &\leq 0, \quad \text{when } x \geq y, \quad \alpha\in(0,1) \end{aligned}\right\}, \tag{36}$$

Hence $h_1(x,y)$ and $h_2(x,y)$ are increasing functions of x and decreasing functions of y. Similarly, we obtain

$$\left.\begin{aligned} \frac{\partial h_1(x,y)}{\partial y}, \frac{\partial h_2(x,y)}{\partial y} &\leq 0, \quad \text{when } x \leq y, \quad \alpha\in(0,1) \\ \frac{\partial h_1(x,y)}{\partial y}, \frac{\partial h_2(x,y)}{\partial y} &\geq 0, \quad \text{when } x \geq y, \quad \alpha\in(0,1) \end{aligned}\right\}, \tag{37}$$

Let us consider two sets P, $Q \in {}^qROPFS(Z)$ with $P \subseteq Q$ and $Z = \{z_1, z_2, ..., z_n\}$ be partitioned into two disjoint sets Z_1 and Z_2 such that $Z_1 \cup Z_2 = Z$. Further, assume that $\forall\ z_j \in Z_1$ are dominated by the condition $\xi_P(z_j) \le \xi_Q(z_j) \le \zeta_Q(z_j) \le \zeta_P(z_j)$ while $\forall\ z_j \in Z_2$ satisfying $\xi_P(z_j) \ge \xi_Q(z_j) \ge \zeta_Q(z_j) \ge \zeta_P(z_j)$.

Hence considering the monotonicity of the functions $h_1(x, y)$ and $h_2(x, y)$, with Equations (22) and (23), we get

$$ {}^q_\delta E_\alpha(P) \le {}^q_\delta E_\alpha(Q)(\delta = 1, 2). \tag{38}$$

EP4. It is clear that $P^C = \{z_j,\ \zeta_P(z_j),\ \xi_P(z_j) \mid z_j \in Z\}$, then, from the definition of the entropy measures given in Equations (20) and (21), we get

$$ {}^q_\delta E_\alpha(P) = {}^q_\delta E_\alpha(P^C)(\delta = 1, 2). \tag{39}$$

This completes the proof. □

Theorem 4. *Let $P = \{z_j,\ \xi_P(z_j),\ \zeta_P(z_j) \mid z_j \in Z\}$ and $Q = \{z_j,\ \xi_Q(z_j),\ \zeta_Q(z_j) \mid z_j \in Z\}$ be two ^{q}ROPFSs defined in $Z = \{z_1, z_2, ..., z_n\}$ such that they satisfy for any $z_j \in Z$ either $P \subseteq Q$ or $P \supseteq Q$, then we have*

$$ {}^q_\delta E_\alpha\left(P \overset{q}{\cup} Q\right) + {}^q_\delta E_\alpha\left(P \overset{q}{\cap} Q\right) = {}^q_\delta E_\alpha(P) + {}^q_\delta E_\alpha(Q)(\delta = 1, 2). \tag{40}$$

Proof. Let $Z = \{z_1, z_2, ..., z_n\}$ be partitioned into two disjoint sets Z_1 and Z_2 such that $Z_1 = \{z_j \in Z\colon P \subseteq Q\}$ and $Z_2 = \{z_j \in Z : P \supseteq Q\}$. That is, for all $z_j \in Z_1$ hold $\xi_P(z_j) \le \xi_Q(z_j)$, $\zeta_Q(z_j) \ge \zeta_P(z_j)$ whereas $\forall\ z_j \in Z_2$ satisfy $\xi_P(z_j) \ge \xi_Q(z_j)$, $\zeta_Q(z_j) \le \zeta_P(z_j)$.

From Equations (22) and (23), we have

$$\begin{aligned} {}^q_1 E_\alpha\left(P \overset{q}{\cup} Q\right) &= \frac{1}{n(1-\alpha)} \\ &\times \Bigg[\sum_{z_j \in Z_1} \log_2\left\{\left(\left(\xi_Q^q(z_j)\right)^\alpha + \left(\zeta_Q^q(z_j)\right)^\alpha\right)\left(\xi_Q^q(z_j) + \zeta_Q^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \xi_Q^q(z_j) - \varsigma_Q^q(z_j)\right)\right\} \\ &+ \sum_{z_j \in Z_2} \log_2\left\{\left(\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha\right)\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \xi_P^q(z_j) - \varsigma_P^q(z_j)\right)\right\} \Bigg], \end{aligned} \tag{41}$$

$$\begin{aligned} &{}^q_2 E_\alpha\left(P \overset{q}{\cup} Q\right) \\ &= \frac{1}{n\left(e - e^{2^{\alpha-1}}\right)} \Bigg[\sum_{z_j \in Z_1} \left\{e^{2^{\alpha-1}\left(\left(\left(\xi_Q^q(z_j)\right)^\alpha + \left(\zeta_Q^q(z_j)\right)^\alpha\right)\left(\xi_Q^q(z_j) + \zeta_Q^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \xi_Q^q(z_j) - \zeta_Q^q(z_j)\right)\right)} - e^{2^{\alpha-1}}\right\} \\ &+ \sum_{z_j \in Z_2} \left\{e^{2^{\alpha-1}\left(\left(\left(\xi_P^q(z_j)\right)^\alpha + \left(\zeta_P^q(z_j)\right)^\alpha\right)\left(\xi_P^q(z_j) + \zeta_P^q(z_j)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \xi_P^q(z_j) - \zeta_P^q(z_j)\right)\right)} - e^{2^{\alpha-1}}\right\} \Bigg], \end{aligned} \tag{42}$$

$$
{}^q_1E_\alpha(P \overset{q}{\cap} Q) = \frac{1}{n(1-\alpha)}
$$
$$
\times\left[\sum_{z_j\in Z_1}\log_2\left\{\left(\left(\xi_P^q(z_j)\right)^\alpha+\left(\zeta_P^q(z_j)\right)^\alpha\right)\left(\xi_P^q(z_j)+\zeta_P^q(z_j)\right)^{1-\alpha}+2^{1-\alpha}\left(1-\xi_P^q(z_j)-\zeta_P^q(z_j)\right)\right\}\right.
$$
$$
\left.+\sum_{z_j\in Z_2}\log_2\left\{\left(\left(\xi_Q^q(z_j)\right)^\alpha+\left(\zeta_Q^q(z_j)\right)^\alpha\right)\left(\xi_Q^q(z_j)+\zeta_Q^q(z_j)\right)^{1-\alpha}+2^{1-\alpha}\left(1-\xi_Q^q(z_j)-\zeta_Q^q(z_j)\right)\right\}\right], \tag{43}
$$

and

$$
{}^q_2E_\alpha\left(P \overset{q}{\cap} Q\right) = \frac{1}{n(e-e^{2^{\alpha-1}})}
$$
$$
\times\left[\sum_{z_j\in Z_1}\left\{e^{2^{\alpha-1}(((\xi_P^q(z_j))^\alpha+(\zeta_P^q(z_j))^\alpha)(\xi_P^q(z_j)+\zeta_P^q(z_j))^{1-\alpha}+2^{1-\alpha}(1-\xi_P^q(z_j)-\zeta_P^q(z_j)))}-e^{2^{\alpha-1}}\right\}\right.
$$
$$
\left.+\sum_{z_j\in Z_2}\left\{e^{2^{\alpha-1}(((\xi_Q^q(z_j))^\alpha+(\zeta_Q^q(z_j))^\alpha)(\xi_Q^q(z_j)+\zeta_Q^q(z_j))^{1-\alpha}+2^{1-\alpha}(1-\xi_Q^q(z_j)-\zeta_Q^q(z_j)))}-e^{2^{\alpha-1}}\right\}\right]. \tag{44}
$$

Adding Equation (41) with Equation (43) and Equation (42) with Equation (44), we get ${}^q_1E_\alpha(P \overset{q}{\cup} Q) + {}^q_1E_\alpha(P \overset{q}{\cap} Q) = {}^q_1E_\alpha(P) + {}^q_1E_\alpha(Q)$ and ${}^q_2E_\alpha(P \overset{q}{\cup} Q) + {}^q_2E_\alpha(P \overset{q}{\cap} Q) = {}^q_2E_\alpha(P) + {}^q_2E_\alpha(Q)$.

This completes the proof. □

Theorem 5. *The entropy measures* ${}^q_\delta E_\alpha(P)$ $(\delta = 1, 2)$ *attain maximum value when* $\xi_P(z_j) = \zeta_P(z_j)\ \forall\ z_j \in Z$ *and minimum value when* $\xi_P(z_j) = 1$, $\zeta_P(z_j) = 0$ *or* $\xi_P(z_j) = 0$, $\zeta_P(z_j) = 1\ \forall\ z_j \in Z$. *Also, maximum and minimum values do not depend on the parameter* α.

Proof. It has already been proved in Theorem 3 that the entropy measures ${}^q_\delta E_\alpha(P)(\delta = 1, 2)$ attain maximum value if and only if $\xi_P(z_j) = \zeta_P(z_j)\ \forall\ z_j \in Z$ and minimum value when $\xi_P(z_j) = 1$, $\zeta_P(z_j) = 0$ or $\xi_P(z_j) = 0$, $\zeta_P(z_j) = 1\ \forall\ z_j \in Z$. Therefore, we must show that the maximum and minimum values of these entropy measures do not involve parameters.

First, let $\xi_P(z_j) = \zeta_P(z_j)\ \forall\ z_j \in Z$, then from Equations (22) and (23), we get ${}^q_1E_\alpha(P) = 1$, ${}^q_2E_\alpha(P) = 1$, which do not contain any parameter.

Next, if $\xi_P(z_j) = 1$, $\zeta_P(z_j) = 0$ or $\xi_P(z_j) = 0$, $\zeta_P(z_j) = 1\ \forall\ z_j \in Z$, then, utilizing Equations (22) and (23), we have ${}^q_1E_\alpha(P) = 0$, ${}^q_2E_\alpha(P) = 0$, which are also free from parameter.

This proves the theorem. □

Special cases of the entropy measures ${}^q_\delta E_\alpha(p)$ $(\delta = 1, 2)$:

(1) When $\alpha \to 1$, then ${}^q_\delta E_\alpha(P|Q)$ $(\delta = 1, 2)$ are reduced to the following measure:

$$ {}^q_\delta E_\alpha(P) = \frac{1}{n}\sum_{j=1}^{n}\Big[\xi_P^q(z_j)\log_2\Big(\xi_P^q(z_j)\Big) + \zeta_P^q(z_j)\log_2\Big(\zeta_P^q(z_j)\Big) + \Big(1 - \eta_P^q(z_j)\Big)\log_2\Big(1 - \eta_P^q(z_j)\Big) - \eta_P^q(z_j)\Big], \tag{45} $$

which is the entropy measure for ^{q}ROPFSs corresponding to Vlachos and Sergiadis.[49]

(2) When $q = 1$, then entropy measure ${}^q_1E_\alpha(P)$ reduces into the intuitionistic fuzzy entropy of order-α defined by Verma and Sharma.[67]
(3) When $\alpha \to 1$ and $q = 1$, then entropy measures ${}^q_\delta E_\alpha(P|Q)$ $(\delta = 1, 2)$ become the intuitionistic fuzzy entropy proposed by Vlachos and Sergiadis.[49]
(4) When $q = 1$ and $\eta_P^q(z_j) = \eta_P^q(z_j) = 0\ \forall\, j$, then ${}^q_1E_\alpha(P)$ is reduced to the fuzzy entropy of order-α proposed by Bhandari and Pal.[42]
(5) When $\alpha \to 1$, $q = 1$ and $\eta_P^q(z_j) = \eta_P^q(z_j) = 0\ \forall\, j$, then entropy measures mentioned in Equations (22) and (23) become De Luca and Termini's fuzzy entropy.[37]

Further, assume that the elements in the universe of discourse $Z = \{z_1, z_2, ..., z_n\}$ have the weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n}\omega_j = 1$. Then, corresponding to order-α divergence and entropy measures defined in Equations (16), (17), (22) and (23), we propose the following weighted order-α divergence and entropy measures for ^{q}ROPFS:

$$ \begin{aligned} &{}^q_1D^\omega_\alpha(P|Q) \\ &= \frac{1}{(\alpha - 1)}\sum_{j=1}^{n}\omega_j \log_2\Bigg[\Big(\xi_P^q(z_j)\Big)^\alpha\left(\frac{\xi_P^q(z_j) + \xi_Q^q(z_j)}{2}\right)^{1-\alpha} + \Big(\zeta_P^q(z_j)\Big)^\alpha\left(\frac{\zeta_P^q(z_j) + \zeta_Q^q(z_j)}{2}\right)^{1-\alpha} \\ &\quad + \Big(\eta_P^q(z_j)\Big)^\alpha\left(\frac{\eta_P^q(z_j) + \eta_Q^q(z_j)}{2}\right)^{1-\alpha}\Bigg], \end{aligned} \tag{46} $$

$$ \begin{aligned} &{}^q_2D^\omega_\alpha(P|Q) \\ &= \frac{1}{(e^{2^{\alpha-1}} - e)}\sum_{j=1}^{n}\omega_j\left[e^{\left(\left(\xi_P^q(z_j)\right)^\alpha\left(\frac{\xi_P^q(z_j)+\xi_Q^q(z_j)}{2}\right)^{1-\alpha} + \left(\zeta_P^q(z_j)\right)^\alpha\left(\frac{\zeta_P^q(z_j)+\zeta_Q^q(z_j)}{2}\right)^{1-\alpha} + \left(\eta_P^q(z_j)\right)^\alpha\left(\frac{\eta_P^q(z_j)+\eta_Q^q(z_j)}{2}\right)^{1-\alpha}\right)} - e\right], \end{aligned} \tag{47} $$

$$ {}^q_1E^\omega_\alpha(P) = \frac{1}{(1 - \alpha)}\sum_{j=1}^{n}\omega_j \log_2\Big[\Big(\Big(\xi_P^q(z_j)\Big)^\alpha + \Big(\zeta_P^q(z_j)\Big)^\alpha\Big)\Big(\xi_P^q(z_j) + \zeta_P^q(z_j)\Big)^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j)\Big], \tag{48} $$

$$ {}^q_2E^\omega_\alpha(P) = \frac{1}{(e - e^{2^{\alpha-1}})}\sum_{j=1}^{n}\omega_j\,[e^{2^{\alpha-1}(((\xi_P^q(z_j))^\alpha + (\zeta_P^q(z_j))^\alpha)(\xi_P^q(z_j) + \zeta_P^q(z_j))^{1-\alpha} + 2^{1-\alpha}\eta_P^q(z_j))} - e^{2^{\alpha-1}}], $$

where $\alpha \in (0, 1)$. (49)

In the next section, we develop a new decision-making approach to solve MAGDM problems under the ^{q}ROPF environment.

4 | MAGDM APPROACH BASED ON ORDER-α DIVERGENCE AND ENTROPY MEASURES

Decision-making is an integral part of our day-to-day life activities. In MAGDM, a preferable alternative is selected by a group of decision-makers that satisfying a set of conflicting attributes. Due to the presence of uncertainty and vagueness in decision information, the traditional multiple attribute decision-making methods are incompetent to solve real-world decision problems. In the literature, a wide range of decision-making methods have been developed under uncertain environment based on fuzzy theory,[68–71] intuitionistic fuzzy theory,[3,4,6,8,9,52,58] and PFS theory.[15,21,72–74] As we know, the ^{q}ROPFS theory includes FS, IFS, and PFS as its particular cases. Therefore, it is essential and valuable to develop new decision-making methods under the ^{q}ROPF environment. For doing so, in this section, we utilize the developed entropy and divergence measures for ^{q}ROPFSs to formulate a new decision-making approach for solving MAGDM problems with ^{q}ROPF information.

4.1 | MAGDM problem formulation with ^{q}ROPF information

For a MAGDM problem with ^{q}ROPF information, let $Q = \{Q_1, Q_2, ..., Q_m\}$ be a group of m different alternatives characterized by another set of n attributes $A = \{A_1, A_2, ..., A_n\}$ with a weighting vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that $\omega_j \geq 0,\ j = 1, 2, ..., n$ and $\sum_{j=1}^{n} \omega_j = 1$. All the alternatives are evaluated by a group of l decision-makers $D = \{D^{(1)}, D^{(2)}, ..., D^{(l)}\}$. Due to insufficient expertise and limited knowledge about the problem domain, a decision-maker may only be capable of assessing the problem on one part rather than on all the aspects. Therefore, it is very significant to assign different weights to the various decision-makers according to their expertise, knowledge, and experiences. Suppose that the weight vector associated with the set of decision-makers is given as $\boldsymbol{\nu} = (\nu_1, \nu_2, ..., \nu_l)^T$ with $\nu_\tau \geq 0,\ \tau = 1, 2, ..., l$ and $\sum_{\tau=1}^{l} \nu_\tau = 1$. Further assume that the evaluation information related to all the alternatives $Q_i\ (i = 1, 2, ..., m)$ with respect to different attributes $A_j (j = 1, 2, ..., n)$, provided by the decision-makers $D^{(\tau)}\ (\tau = 1, 2, ..., l)$, may be summarized in the following ^{q}ROPF decision matrices given by

$$R^{(\tau)} = \begin{array}{c} \\ Q_1 \\ Q_2 \\ \vdots \\ Q_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} \langle \xi^{(\tau)}_{\chi_{11}}, \zeta^{(\tau)}_{\chi_{11}} \rangle & \langle \xi^{(\tau)}_{\chi_{12}}, \zeta^{(\tau)}_{\chi_{12}} \rangle & \cdots & \langle \xi^{(\tau)}_{\chi_{1n}}, \zeta^{(\tau)}_{\chi_{1n}} \rangle \\ \langle \xi^{(\tau)}_{\chi_{21}}, \zeta^{q(\tau)}_{\chi_{21}} \rangle & \langle \xi^{(\tau)}_{\chi_{22}}, \zeta^{(\tau)}_{\chi_{22}} \rangle & \cdots & \langle \xi^{(\tau)}_{\chi_{2n}}, \zeta^{(\tau)}_{\chi_{2n}} \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \xi^{(\tau)}_{\chi_{m1}}, \zeta^{(\tau)}_{\chi_{m1}} \rangle & \langle \xi^{(\tau)}_{\chi_{m2}}, \zeta^{(\tau)}_{\chi_{m2}} \rangle & \cdots & \langle \xi^{(\tau)}_{\chi_{mn}}, \zeta^{(\tau)}_{\chi_{mn}} \rangle \end{pmatrix} \end{array}, \tag{50}$$

where $\chi^{(\tau)}_{ij} = \langle \xi^{(\tau)}_{\chi_{ij}}, \zeta^{(\tau)}_{\chi_{ij}} \rangle$ represents the ^{q}ROPF evaluation information provided by the decision-maker $D^{(\tau)}$ of the alternative Q_i with respect to attribute A_j. The objective of the decision-makers is to select the most feasible alternative among the available alternatives.

4.2 | Decision-making steps

The following six steps are involved in the whole decision process:

Step 1: Normalize the decision matrices.

Generally, there are two types of attributes involve in any kind of MAGDM problem: (a) benefit attributes (b) cost attributes. After converting the cost attributes into benefit attributes, the ^{q}ROPF decision matrices $R^{(\tau)} = (\chi_{ij}^{(\tau)})_{m\times n}$ are transformed into normalized ^{q}ROPF decision matrices $\tilde{R}^{(\tau)} = (\tilde{\chi}_{ij}^{(\tau)})_{m\times n}$ where

$$\tilde{\chi}_{ij}^{(\tau)} = \begin{cases} \left\langle \xi_{\chi_{ij}}^{(\tau)}, \zeta_{\chi_{ij}}^{(\tau)} \right\rangle & \text{for benefit attributes;} \\ \left\langle \zeta_{\chi_{ij}}^{(\tau)}, \xi_{\chi_{ij}}^{(\tau)} \right\rangle & \text{for cost attributes.} \end{cases} \tag{51}$$

Step 2: Compute the attribute weights.

It is worth noting that the attribute weights play a very crucial role in solving MAGDM problems. In many situations, the attribute weights may be unknown or partially known due to imprecise data, time pressure, or limited knowledge of the decision-makers about the problem domain. We can determine the attribute weights based on decision-makers' subjective evaluation of each attribute, but this approach may be prejudiced by the decision-makers' personal judgments. So, it is not feasible to utilize in real-life decision problems. In the last few years, some methods, including the TOPSIS method,[9] maximizing deviation method,[75] entropy method,[76] entropy and divergence based method,[52] have been proposed to determine the attribute weights based on intuitionistic fuzzy information.

As we know, in the MAGDM problems, each DM evaluates all the alternatives based on all the attributes. By utilizing all the available information, we shall formulate a new optimization model to determine the attribute weights based on developed entropy and divergence measures with the dispersion measure of the attribute weights.

Yager[77] defined the dispersion measure of an attribute weighting vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ given by

$$Disp(\omega) = -\sum_{j=1}^{m} \omega_j \log_2 \omega_j. \tag{52}$$

Note that we should maximize the dispersion measure for determining the optimal attribute weights.

For the decision-maker $D^{(\tau)}$ and the attribute A_j, the divergence measure between the alternative Q_i and all other alternatives can be given as

$$Div_{ij}^{(\tau)} = \frac{1}{(m-1)} \sum_{\kappa=1}^{m} Div\left(\chi_{ij}^{(\tau)} \middle\| \chi_{kj}^{(\tau)}\right), \tag{53}$$

and the total measure of divergence among all the alternatives under the attribute A_j can be defined as

$$Div_{j}^{(\tau)} = \frac{1}{(m-1)} \sum_{i=1}^{m} \sum_{\kappa=1}^{m} Div\left(\chi_{ij}^{(\tau)} \middle\| \chi_{kj}^{(\tau)}\right). \tag{54}$$

Also, the total measure of divergence among all the alternatives with respect to all the attributes and DMs can be expressed by the following divergence matrix as

$$\widehat{Div} = \begin{pmatrix} Div_1^{(1)} & Div_2^{(1)} & \dots & Div_n^{(1)} \\ Div_1^{(2)} & Div_2^{(2)} & \dots & Div_n^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ Div_1^{(l)} & Div_2^{(l)} & \dots & Div_n^{(l)} \end{pmatrix}. \tag{55}$$

Taking the weights of all the DMs into account, the total divergence measure among all the alternatives for an attribute A_j can be represented by the following expression

$$Div_j = \sum_{\tau=1}^{l} \nu_\tau Div_j^{(\tau)} = \sum_{\tau=1}^{l} \nu_\tau \frac{1}{(m-1)} \sum_{i=1}^{m} \sum_{\kappa=1}^{m} Div\left(\chi_{ij}^{(\tau)} \middle\| \chi_{kj}^{(\tau)}\right). \tag{56}$$

Note that if the evaluation values of each alternative have very little difference under an attribute, then it shows that such an attribute gives a small contribution in the ranking process and should be assigned a small weight. On the other hand, if an attribute indicates the significant difference in evaluation values among all the alternatives, then such an attribute plays a vital role in the ranking process and should be assigned more considerable weight.

The entropy value of the ^{q}ROPF information under the attribute A_j given by the DM $D^{(\tau)}$ is defined as

$$E_j^{(\tau)} = \sum_{i=1}^{m} E\left(\chi_{ij}^{(\tau)}\right). \tag{57}$$

The entropy matrix constructed by the entropy values with respect to all the attributes and the DMs can be represented as

$$\widehat{E} = \begin{pmatrix} E_1^{(1)} & E_2^{(1)} & \dots & E_n^{(1)} \\ E_1^{(2)} & E_2^{(2)} & \dots & E_n^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ E_1^{(l)} & E_2^{(l)} & \dots & E_n^{(l)} \end{pmatrix}. \tag{58}$$

Utilizing entropy matrix given in Equation (58), the overall entropy value of the attribute A_j is obtained as

$$E_j = \sum_{\tau=1}^{l} \nu_\tau \sum_{i=1}^{m} E\left(\chi_{ij}^{(\tau)}\right). \tag{59}$$

Based on the above analysis, we conclude that the ideal attribute weights should maximize the dispersion and divergence but minimize the entropy of the total ^{q}ROPF decision matrices. Combining all these aspects with attribute weights, we have the following function

$$\begin{aligned}\widehat{F} &= -\sum_{j=1}^{n}\omega_j \log_2 \omega_j + \sum_{j=1}^{n}\left(\omega_j \sum_{\tau=1}^{l}\nu_\tau \frac{1}{(m-1)}\sum_{i=1}^{m}\sum_{\kappa=1}^{m} Div\left(\chi_{ij}^{(\tau)} \middle\| \chi_{\kappa j}^{(\tau)}\right)\right) - \sum_{j=1}^{n}\left(\omega_j \sum_{\tau=1}^{l}\nu_\tau \sum_{i=1}^{m} E\left(\chi_{ij}^{(\tau)}\right)\right) \\ &= \sum_{j=1}^{n}\omega_j\left(\sum_{\tau=1}^{l}\sum_{i=1}^{m}\nu_\tau\left(\frac{1}{(m-1)}\sum_{\kappa=1}^{m} Div\left(\chi_{ij}^{(\tau)} \middle\| \chi_{\kappa j}^{(\tau)}\right) - E\left(\chi_{ij}^{(\tau)}\right)\right) - \log_2 \omega_j\right) \\ &= \sum_{j=1}^{n}\omega_j(Div_j - E_j - \log_2 \omega_j).\end{aligned} \tag{60}$$

In terms of matrices given in Equations (55) and (58), Equation (60) can be written as

$$\widehat{F} = \omega(\nu^T(\widehat{Div} - \widehat{E}) - \log_2 \omega^T), \tag{61}$$

where

$$\log_2 \boldsymbol{\omega} = (\log_2 \omega_1, \log_2 \omega_2, \ldots, \log_2 \omega_n)^T. \tag{62}$$

We construct the following optimal model to determine the attribute weights:

(**MOD** 1)

$$\text{Max}\,\widehat{F} = \omega(\nu^T(\widehat{Div} - \widehat{E}) - \log_2 \omega^T)$$

$$\text{s.t.}\begin{cases}\omega \in J, \\ \sum_{j=1}^{n}\omega_j = 1, \\ \omega_j \geq 0, j = 1, 2, \ldots, n.\end{cases}$$

where J represents the set of all incomplete information about the attribute weights.
Based on our developed entropy and divergence measures, the following optimal models can be designed to determine the attribute weights:

(**MOD 2**)

$$\text{Max}\,\widehat{F} = \sum_{j=1}^{n}\omega_j\left(\sum_{\tau=1}^{l}\sum_{i=1}^{m}\nu_\tau\left(\frac{1}{(m-1)}\sum_{\kappa=1}^{m} {}^{q}_{\delta}D_\alpha\left(\chi_{ij}^{(\tau)} \middle\| \chi_{\kappa j}^{(\tau)}\right) - {}^{q}_{\delta}E_\alpha\left(\chi_{ij}^{(\tau)}\right)\right) - \log_2 \omega_j\right), \text{ where } \delta = 1, 2$$

$$\text{s.t.}\begin{cases}\omega \in J, \\ \sum_{j=1}^{n}\omega_j = 1, \\ \omega_j \geq 0, j = 1, 2, \ldots, n.\end{cases}$$

As per our choice and requirement, different pairs of entropy and divergence measures can be used in the optimal model presented in **MOD 2** to determine the weighting vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 3: Aggregation of all DMs Information.

To aggregate all the individual ^{q}ROPF decision matrices $\tilde{R}^{(\tau)} = (\tilde{\chi}_{ij}^{(\tau)})_{m\times n}(\tau = 1, 2, ..., l)$ into a collective one $\tilde{R} = (\tilde{\chi}_{ij})_{m\times n}$, we utilize the ^{q}ROPF weighted averaging (^{q}ROPFWA) operator given by

$$\tilde{\chi}_{ij} = {}^{q}ROPFWA\left(\tilde{\chi}_{ij}^{(1)}, \tilde{\chi}_{ij}^{(2)}, ..., \tilde{\chi}_{ij}^{(l)}\right) = \left\langle \left(1 - \prod_{\tau=1}^{l}\left(1 - \left(\xi_{\tilde{\chi}_{ij}}^{(\tau)}\right)^{q}\right)^{\nu_\tau}\right)^{1/q}, \prod_{\tau=1}^{l}\left(\zeta_{\tilde{\chi}_{ij}}^{(\tau)}\right)^{\nu_\tau} \right\rangle,$$

$$i = 1, 2, ..., m; \quad j = 1, 2, ..., n. \tag{63}$$

Step 4: Determine the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS).

Use the following expressions to obtain the positive ideal solution (PIS) and the negative ideal solution (NIS) given as

$$PIS = \{\langle A_j, \xi_{PIS}(A_j), \zeta_{PIS}(A_j)\rangle | A_j \in A\} \text{ and } NIS = \{\langle A_j, \xi_{NIS}(A_j), \zeta_{NIS}(A_j)\rangle | A_j \in A\}, \tag{64}$$

where for each $j = 1, 2, ..., n$

$$\left.\begin{array}{l} \xi_{PIS}(A_j) = \max\limits_{i}\left\{\xi_{Q_i}(A_j)\right\}, \quad \zeta_{PIS}(A_j) = \min\limits_{i}\left\{\zeta_{Q_i}(A_j)\right\} \\ \xi_{NIS}(A_j) = \min\limits_{i}\left\{\xi_{Q_i}(A_j)\right\}, \quad \zeta_{NIS}(A_j) = \max\limits_{i}\left\{\zeta_{Q_i}(A_j)\right\} \end{array}\right\}. \tag{65}$$

Step 5: Calculate the Measure of Divergence of all Alternatives with PIS and NIS, respectively.

We can utilize the divergence measures defined in Equations (46) and (47) to calculate the measure of divergence of all the alternatives Q_i with PIS and NIS, respectively.

Step 6: Calculate the Relative Divergence Coefficients $\mathfrak{I}_i$'s.

To determine the relative divergence coefficients $\mathfrak{I}_i$'s corresponding to each alternative Q_i, we use the following formula defined as

$$\mathfrak{I}_i = \frac{{}_{\delta}^{q}D_{\alpha}^{\omega}(Q_i \| PIS)}{{}_{\delta}^{q}D_{\alpha}^{\omega}(Q_i \| PIS) + {}_{\delta}^{q}D_{\alpha}^{\omega}(Q_i \| NIS)}, \quad i = 1, 2, ..., m; \quad \delta = 1, 2. \tag{66}$$

Step 7: Finally, rank all the alternatives according to the values of relative divergence coefficients $\mathfrak{I}_i$'s in ascending order. The alternative corresponding to the lowest relative divergence coefficient value will be the best alternative.

Next, we present the application of the developed MAGDM approach in the ERP selection problem.

4.3 | Numerical example

Example 1. In today's dynamic and competitive environment, companies face many challenges to expand market share and fulfill customers' expectations. This requires reducing the total costs in the entire supply chain, shorten lead-time, reduce inventories, provide more choices for product selection, timely delivery, better customer services, improve the quality of the products to sustain in the global market, and efficiently coordinate globe demand, supply, and production. To achieve these objectives, more and more companies are implementing ERP systems. An ERP system is a packaged enterprise information system that mechanizes and integrates whole business tasks such as product planning, purchasing, inventory control, sales, human resource management, and finance. Implementation of ERP systems is one of the most significant investment projects due to the difficulty, high cost, and adaptation risks. It is worth mentioning that any ERP software available in the market cannot adequately meet the requirements and expectations of companies because every company runs its business with different strategies and goals. Therefore, ERP software selection is a significant and challenging decision problem for managers because it provides high-quality services for end-users.

Let us suppose a company plans to implement ERP systems. There are five possible alternative ERP systems, say, Q_i $(i = 1, 2, ..., 5)$ available for selection. To assess the ERP systems, the company decides to form a committee of four experts $D^{(\tau)}$; $(\tau = 1, 2, 3, 4)$ from different professional organizations, whose weight vector is $\boldsymbol{\nu} = (0.20, 0.30, 0.15, 0.35)^T$. The selection committee recommends six attributes to evaluate the available alternatives: (a) the function and technology A_1, (b) cost A_2, (c) strategic fitness A_3, (d) vendor's reputation and references A_4, (e) support and training A_5, and (f) ease of use A_6. The experts $D^{(\tau)}$ $(\tau = 1, 2, 3, 4)$ evaluate the five potential ERP systems concerning the attributes $A_j (j = 1, 2, ..., 6)$ and form the following ^{q}ROPF decision matrices $R^{(\tau)}$; $\tau = 1, 2, 3, 4$, as given in Tables 1 to 4.

Now, we apply the developed MAGDM approach to select the best alternative. The computational process as follows:

Step 1: Since A_2 is a cost attribute, therefore, we convert A_2 into benefit attribute by using Equation (51). The normalized decision matrices so obtained are given in Tables 5 to 8.

We consider the following two cases:

Case (i): When the information about the attribute weights is completely unknown.

Step 2: We shall utilize **MOD 2** to determine the attribute weights $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)^T$. Here, we are using the divergence measure ${}_1^qD_\alpha(P|Q)$ and entropy measure ${}_1^qE_\alpha(P)$ to calculate the total divergence and entropy values for each decision-maker concerning all the attributes. The following divergence and entropy matrices are obtained as (here; we have taken $q = 3$ and $\alpha = 0.5$)

$$\widehat{Div} = \begin{pmatrix} 0.3251 & 0.7734 & 0.4527 & 0.5886 & 0.2379 & 0.1461 \\ 0.0993 & 0.3685 & 0.5731 & 0.7289 & 0.5953 & 0.3048 \\ 0.0392 & 0.2109 & 0.3793 & 0.3881 & 0.3202 & 0.6880 \\ 0.6180 & 0.4663 & 0.4631 & 0.1135 & 0.3564 & 0.2068 \end{pmatrix}, \tag{67}$$

and

TABLE 1 ^{q}ROPF decision matrix $R^{(1)}$ provided by the expert $D^{(1)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.6, 0.7⟩	⟨0.8, 0.6⟩	⟨0.5, 0.4⟩	⟨0.8, 0.7⟩	⟨0.4, 0.3⟩	⟨0.6, 0.4⟩
Q_2	⟨0.7, 0.4⟩	⟨0.5, 0.5⟩	⟨0.4, 0.8⟩	⟨0.6, 0.3⟩	⟨0.5, 0.5⟩	⟨0.7, 0.3⟩
Q_3	⟨0.5, 0.7⟩	⟨0.9, 0.6⟩	⟨0.6, 0.5⟩	⟨0.4, 0.8⟩	⟨0.8, 0.3⟩	⟨0.5, 0.2⟩
Q_4	⟨0.8, 0.4⟩	⟨0.3, 0.7⟩	⟨0.6, 0.6⟩	⟨0.3, 0.5⟩	⟨0.4, 0.5⟩	⟨0.7, 0.5⟩
Q_5	⟨0.3, 0.5⟩	⟨0.7, 0.1⟩	⟨0.9, 0.4⟩	⟨0.7, 0.4⟩	⟨0.6, 0.2⟩	⟨0.4, 0.4⟩

TABLE 2 ^{q}ROPF decision matrix $R^{(2)}$ provided by the expert $D^{(2)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.7, 0.5⟩	⟨0.4, 0.7⟩	⟨0.6, 0.4⟩	⟨0.7, 0.5⟩	⟨0.3, 0.4⟩	⟨0.5, 0.7⟩
Q_2	⟨0.6, 0.5⟩	⟨0.3, 0.4⟩	⟨0.3, 0.5⟩	⟨0.5, 0.3⟩	⟨0.7, 0.7⟩	⟨0.8, 0.4⟩
Q_3	⟨0.7, 0.3⟩	⟨0.7, 0.6⟩	⟨0.8, 0.6⟩	⟨0.9, 0.6⟩	⟨0.5, 0.8⟩	⟨0.6, 0.3⟩
Q_4	⟨0.6, 0.6⟩	⟨0.2, 0.5⟩	⟨0.5, 0.3⟩	⟨0.4, 0.6⟩	⟨0.9, 0.3⟩	⟨0.6, 0.6⟩
Q_5	⟨0.5, 0.4⟩	⟨0.6, 0.2⟩	⟨0.8, 0.7⟩	⟨0.9, 0.2⟩	⟨0.6, 0.6⟩	⟨0.3, 0.5⟩

TABLE 3 ^{q}ROPF decision matrix $R^{(3)}$ provided by the expert $D^{(3)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.5, 0.8⟩	⟨0.4, 0.7⟩	⟨0.9, 0.3⟩	⟨0.6, 0.5⟩	⟨0.5, 0.5⟩	⟨0.8, 0.2⟩
Q_2	⟨0.6, 0.4⟩	⟨0.4, 0.8⟩	⟨0.5, 0.6⟩	⟨0.4, 0.6⟩	⟨0.6, 0.7⟩	⟨0.7, 0.4⟩
Q_3	⟨0.7, 0.2⟩	⟨0.7, 0.6⟩	⟨0.7, 0.4⟩	⟨0.8, 0.3⟩	⟨0.4, 0.5⟩	⟨0.2, 0.5⟩
Q_4	⟨0.5, 0.3⟩	⟨0.5, 0.5⟩	⟨0.6, 0.2⟩	⟨0.7, 0.4⟩	⟨0.8, 0.2⟩	⟨0.4, 0.3⟩
Q_5	⟨0.8, 0.4⟩	⟨0.3, 0.6⟩	⟨0.8, 0.6⟩	⟨0.9, 0.5⟩	⟨0.5, 0.3⟩	⟨0.8, 0.7⟩

TABLE 4 ^{q}ROPF decision matrix $R^{(4)}$ provided by the expert $D^{(4)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.4, 0.5⟩	⟨0.7, 0.6⟩	⟨0.3, 0.8⟩	⟨0.6, 0.5⟩	⟨0.3, 0.4⟩	⟨0.7, 0.5⟩
Q_2	⟨0.8, 0.5⟩	⟨0.2, 0.7⟩	⟨0.6, 0.6⟩	⟨0.8, 0.3⟩	⟨0.6, 0.7⟩	⟨0.5, 0.6⟩
Q_3	⟨0.9, 0.2⟩	⟨0.4, 0.6⟩	⟨0.8, 0.3⟩	⟨0.7, 0.4⟩	⟨0.5, 0.4⟩	⟨0.7, 0.2⟩
Q_4	⟨0.5, 0.6⟩	⟨0.3, 0.4⟩	⟨0.7, 0.5⟩	⟨0.6, 0.2⟩	⟨0.8, 0.5⟩	⟨0.5, 0.5⟩
Q_5	⟨0.7, 0.8⟩	⟨0.2, 0.9⟩	⟨0.8, 0.2⟩	⟨0.7, 0.5⟩	⟨0.4, 0.4⟩	⟨0.8, 0.3⟩

TABLE 5 Normalized ^{q}ROPF decision matrix $\tilde{R}^{(1)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.6, 0.7⟩	⟨0.6, 0.8⟩	⟨0.5, 0.4⟩	⟨0.8, 0.7⟩	⟨0.4, 0.3⟩	⟨0.6, 0.4⟩
Q_2	⟨0.7, 0.4⟩	⟨0.5, 0.5⟩	⟨0.4, 0.8⟩	⟨0.6, 0.3⟩	⟨0.5, 0.5⟩	⟨0.7, 0.3⟩
Q_3	⟨0.5, 0.7⟩	⟨0.6, 0.9⟩	⟨0.6, 0.5⟩	⟨0.4, 0.8⟩	⟨0.8, 0.3⟩	⟨0.5, 0.2⟩
Q_4	⟨0.8, 0.4⟩	⟨0.7, 0.3⟩	⟨0.6, 0.6⟩	⟨0.3, 0.5⟩	⟨0.4, 0.5⟩	⟨0.7, 0.5⟩
Q_5	⟨0.3, 0.5⟩	⟨0.1, 0.7⟩	⟨0.9, 0.4⟩	⟨0.7, 0.4⟩	⟨0.6, 0.2⟩	⟨0.4, 0.4⟩

TABLE 6 Normalized ^{q}ROPF decision matrix $\tilde{R}^{(2)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.7, 0.5⟩	⟨0.7, 0.4⟩	⟨0.6, 0.4⟩	⟨0.7, 0.5⟩	⟨0.3, 0.4⟩	⟨0.5, 0.7⟩
Q_2	⟨0.6, 0.5⟩	⟨0.4, 0.3⟩	⟨0.3, 0.5⟩	⟨0.5, 0.3⟩	⟨0.7, 0.7⟩	⟨0.8, 0.4⟩
Q_3	⟨0.7, 0.3⟩	⟨0.6, 0.7⟩	⟨0.8, 0.6⟩	⟨0.9, 0.6⟩	⟨0.5, 0.8⟩	⟨0.6, 0.3⟩
Q_4	⟨0.6, 0.6⟩	⟨0.5, 0.2⟩	⟨0.5, 0.3⟩	⟨0.4, 0.6⟩	⟨0.9, 0.3⟩	⟨0.6, 0.6⟩
Q_5	⟨0.5, 0.4⟩	⟨0.2, 0.6⟩	⟨0.8, 0.7⟩	⟨0.9, 0.2⟩	⟨0.6, 0.6⟩	⟨0.3, 0.5⟩

TABLE 7 Normalized ^{q}ROPF decision matrix $R^{(3)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.5, 0.8⟩	⟨0.7, 0.4⟩	⟨0.9, 0.3⟩	⟨0.6, 0.5⟩	⟨0.5, 0.5⟩	⟨0.8, 0.2⟩
Q_2	⟨0.6, 0.4⟩	⟨0.8, 0.4⟩	⟨0.5, 0.6⟩	⟨0.4, 0.6⟩	⟨0.6, 0.7⟩	⟨0.7, 0.4⟩
Q_3	⟨0.7, 0.2⟩	⟨0.6, 0.7⟩	⟨0.7, 0.4⟩	⟨0.8, 0.3⟩	⟨0.4, 0.5⟩	⟨0.2, 0.5⟩
Q_4	⟨0.5, 0.3⟩	⟨0.5, 0.5⟩	⟨0.6, 0.2⟩	⟨0.7, 0.4⟩	⟨0.8, 0.2⟩	⟨0.4, 0.3⟩
Q_5	⟨0.8, 0.4⟩	⟨0.6, 0.3⟩	⟨0.8, 0.6⟩	⟨0.9, 0.5⟩	⟨0.5, 0.3⟩	⟨0.8, 0.7⟩

$$\widehat{E} = \begin{pmatrix} 4.6720 & 4.4341 & 4.5252 & 4.6408 & 4.6259 & 4.7322 \\ 4.8038 & 4.7317 & 4.8548 & 4.2684 & 4.4936 & 4.6972 \\ 4.4705 & 4.6692 & 4.3455 & 4.4322 & 4.6205 & 4.5087 \\ 4.3544 & 4.2284 & 4.1303 & 4.5064 & 4.8775 & 4.5000 \end{pmatrix}. \tag{68}$$

Then, utilizing available information, we construct the following optimal model to determine the weight vector corresponding to the attributes:

$$\begin{aligned} \text{Max.}\,\widehat{F} = (&-4.1992\omega_1 - \omega_1 \log \omega_1 - 4.0266\omega_2 - \omega_2 \log \omega_2 - 3.9774\omega_3 - \omega_3 \log \omega_3 \\ &- 4.0164\omega_4 - \omega_4 \log \omega_4 - 4.2746\omega_5 - \omega_5 \log \omega_5 - 4.3107\omega_6 - \omega_6 \log \omega_6), \end{aligned} \tag{69}$$

$$\text{s.t.} \begin{cases} \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 = 1, \\ \omega_j \geq 0, j = 1, 2, 3, 4, 5, 6. \end{cases} \tag{70}$$

Solving the above nonlinear programming model with the help of MATLAB software, the following weight vector is obtained

$$\omega_1 = (0.1587, 0.1788, 0.1850, 0.1801, 0.1506, 0.1469)^T. \tag{71}$$

Step 3: We utilize the ^{q}ROPF weighted averaging (^{q}ROPFWA) operator given in Equation (63) to aggregate all the individual ^{q}ROPF decision matrices $\tilde{R}^{(\tau)} = (\tilde{\chi}_{ij}^{(\tau)})_{m\times n} (\tau = 1, 2, 3, 4)$ into the collective one $\tilde{R} = (\tilde{\chi}_{ij})_{m\times n}$. The collective ^{q}ROPF decision matrix $\tilde{R}$ is represented in Table 9.

Step 4: We obtain the positive ideal solution *PIS* and the negative ideal solution *NIS* given as

$$PIS = \begin{Bmatrix} \langle A_1, 0.7938, 0.2902\rangle, \langle A_2, 0.7313, 0.2868\rangle, \langle A_3, 0.8273, 0.3878\rangle, \\ \langle A_4, 0.8273, 0.3329\rangle, \langle A_5, 0.7538, 0.3739\rangle, \langle A_6, 0.6950, 0.2591\rangle \end{Bmatrix}, \tag{72}$$

TABLE 8 Normalized ^{q}ROPF decision matrix $R^{(4)}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.4, 0.5⟩	⟨0.6, 0.7⟩	⟨0.3, 0.8⟩	⟨0.6, 0.5⟩	⟨0.3, 0.4⟩	⟨0.7, 0.5⟩
Q_2	⟨0.8, 0.5⟩	⟨0.7, 0.2⟩	⟨0.6, 0.6⟩	⟨0.8, 0.3⟩	⟨0.6, 0.7⟩	⟨0.5, 0.6⟩
Q_3	⟨0.9, 0.2⟩	⟨0.6, 0.4⟩	⟨0.8, 0.3⟩	⟨0.7, 0.4⟩	⟨0.5, 0.4⟩	⟨0.7, 0.2⟩
Q_4	⟨0.5, 0.6⟩	⟨0.4, 0.3⟩	⟨0.7, 0.5⟩	⟨0.6, 0.2⟩	⟨0.8, 0.5⟩	⟨0.5, 0.5⟩
Q_5	⟨0.7, 0.8⟩	⟨0.9, 0.2⟩	⟨0.8, 0.2⟩	⟨0.7, 0.5⟩	⟨0.4, 0.4⟩	⟨0.8, 0.3⟩

TABLE 9 Collective ^{q}ROPF decision matrix $\tilde{R}$

	A_1	A_2	A_3	A_4	A_5	A_6
Q_1	⟨0.5807, 0.5739⟩	⟨0.6510, 0.5589⟩	⟨0.6407, 0.4883⟩	⟨0.6866, 0.5348⟩	⟨0.3678, 0.3905⟩	⟨0.6624, 0.4610⟩
Q_2	⟨0.7107, 0.4624⟩	⟨0.6383, 0.3010⟩	⟨0.4918, 0.6017⟩	⟨0.6658, 0.3329⟩	⟨0.6214, 0.6544⟩	⟨0.6950, 0.4352⟩
Q_3	⟨0.7882, 0.2902⟩	⟨0.6000, 0.6052⟩	⟨0.7600, 0.4271⟩	⟨0.7846, 0.4970⟩	⟨0.5977, 0.4807⟩	⟨0.6033, 0.2591⟩
Q_4	⟨0.7938, 0.4986⟩	⟨0.5895, 0.2868⟩	⟨0.6794, 0.3878⟩	⟨0.5941, 0.3706⟩	⟨0.7538, 0.3739⟩	⟨0.6459, 0.4892⟩
Q_5	⟨0.6377, 0.5331⟩	⟨0.7313, 0.3797⟩	⟨0.8273, 0.3945⟩	⟨0.8237, 0.3633⟩	⟨0.5337, 0.3767⟩	⟨0.6813, 0.4206⟩

and

$$NIS = \left\{\begin{matrix} \langle A_1, 0.5807, 0.5739\rangle, \langle A_2, 0.5895, 0.6052\rangle, \langle A_3, 0.4918, 0.6017\rangle, \\ \langle A_4, 0.5941, 0.5348\rangle, \langle A_5, 0.3678, 0.6544\rangle, \langle A_6, 0.6033, 0.4892\rangle \end{matrix}\right\}. \quad (73)$$

Step 5: Using the divergence measure given in Equation (46) with ω_1 to calculate the measure of divergence of the alternatives Q_i with PIS and NIS, respectively. The results are presented in Table 10.

Step 6: Based on Equation (66), we get the relative divergence coefficients $\mathfrak{I}_i's$ corresponding to each alternative as

$$\mathfrak{I}_1 = 0.7540, \quad \mathfrak{I}_2 = 0.6126, \quad \mathfrak{I}_3 = 0.3278, \quad \mathfrak{I}_4 = 0.3929, \quad \mathfrak{I}_5 = 0.3273.$$

Step 7: The ranking of the alternatives according to the relative divergence coefficients $\mathfrak{I}_i's$ in descending order is obtained as

$$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1.$$

Hence Q_5 is the best ERP system.

Case (ii) When the information about the attribute weights is partially known.

Suppose that the known information about the attribute weights is expressed as

$$J = \{\omega_1 \geq 0.10, 0.15 \leq \omega_2 \leq 0.20, 0.25 \leq \omega_3 \leq 0.35, \omega_4 \geq 0.15, 0.20 \leq \omega_5 \leq 0.30, 0.10 \leq \omega_6 \leq 0.15\}.$$

Then we construct the following optimization model to derive the attributes' weighting vector:

$$\text{Max. } \widehat{F} = \begin{pmatrix} -4.1992\omega_1 - \omega_1 \log \omega_1 - 4.0266\omega_2 - \omega_2 \log \omega_2 - 3.9774\omega_3 - \omega_3 \log \omega_3 \\ - 4.0164\omega_4 - \omega_4 \log \omega_4 - 4.2746\omega_5 - \omega_5 \log \omega_5 - 4.3107\omega_6 - \omega_6 \log \omega_6 \end{pmatrix}, \quad (74)$$

TABLE 10 The divergence measures ${}^{q}_{1}D^{\omega_1}_{0.5}(Q_i \| PIS)$ and ${}^{q}_{1}D^{\omega_1}_{0.5}(Q_i \| NIS)$

${}^{q}_{1}D^{\omega_1}_{0.5}(Q_1 \| PIS)$	0.0472	${}^{q}_{1}D^{\omega_1}_{0.5}(Q_1 \| NIS)$	0.0154
${}^{q}_{1}D^{\omega_1}_{0.5}(Q_2 \| PIS)$	0.0438	${}^{q}_{1}D^{\omega_1}_{0.5}(Q_2 \| NIS)$	0.0277
${}^{q}_{1}D^{\omega_1}_{0.5}(Q_3 \| PIS)$	0.0157	${}^{q}_{1}D^{\omega_1}_{0.5}(Q_3 \| NIS)$	0.0322
${}^{q}_{1}D^{\omega_1}_{0.5}(Q_4 \| PIS)$	0.0319	${}^{q}_{1}D^{\omega_1}_{0.5}(Q_4 \| NIS)$	0.0493
${}^{q}_{1}D^{\omega_1}_{0.5}(Q_5 \| PIS)$	0.0271	${}^{q}_{1}D^{\omega_1}_{0.5}(Q_5 \| NIS)$	0.0557

$$\text{s.t.}\begin{cases}\omega_1 \geq 0.10,\\ 0.15 \leq \omega_2 \leq 0.20,\\ 0.25 \leq \omega_3 \leq 0.35,\\ \omega_4 \geq 0.15,\\ 0.20 \leq \omega_5 \leq 0.30,\\ 0.10 \leq \omega_6 \leq 0.15,\\ \omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6=1,\\ \omega_j \geq 0, j=1,2,3,4,5,6.\end{cases} \tag{75}$$

After solving the above optimization model with the help of MATLAB software, we get the following optimal weight vector of the attributes given by:

$$\omega_2 = (0.1298, 0.1500, 0.2500, 0.1501, 0.2000, 0.1201)^T. \tag{76}$$

By repeating the above steps with ω_2, the obtained measures of divergence of the alternatives Q_i with *PIS* and *NIS* are summarized in Table 11, respectively.and the relative divergence coefficients $\mathfrak{I}_i's$ corresponding to each alternative are calculated as

$$\mathfrak{I}_1 = 0.7436,\ \mathfrak{I}_2 = 0.6591,\ \mathfrak{I}_3 = 0.3012,\ \mathfrak{I}_4 = 0.3542,\ \mathfrak{I}_5 = 0.2989.$$

Therefore, the ranking of the alternatives according to the relative divergence coefficients $\mathfrak{I}_i$'s in descending order is obtained as

$$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1.$$

Hence Q_5 is still the best ERP system.

Besides, we have been considered different values of α to analyze the influence of the parameter on the ranking order of the alternatives. The obtained attributes' weighting vectors, relative divergence coefficients $\mathfrak{I}_i$'s and the ranking order of the alternatives are summarized in Table 12.

Furthermore, if we utilize exponential function based order-α divergence and entropy measures in the proposed method to solve the above-discussed numerical example, and then Table 13 presents the obtained results.

The results presented in Tables 13 and 14 indicate that the ranking order may be different depending on the considered value of α, which shows the flexibility of the developed method.

4.3.1 | The validity of the proposed method

It is worth mentioning that the above considered ERP selection problem cannot be solved by using the existing multiple attribute decision-making approaches developed under IF and PF environments

TABLE 11 The divergence measures ${}_1^qD_{0.5}^{\omega_2}(Q_i\|PIS)$ and ${}_1^qD_{0.5}^{\omega_2}(Q_i\|NIS)$

${}_1^qD_{0.5}^{\omega_2}(Q_1\|PIS)$	0.0493	${}_1^qD_{0.5}^{\omega_2}(Q_1\|NIS)$	0.0170
${}_1^qD_{0.5}^{\omega_2}(Q_2\|PIS)$	0.0491	${}_1^qD_{0.5}^{\omega_2}(Q_2\|NIS)$	0.0254
${}_1^qD_{0.5}^{\omega_2}(Q_3\|PIS)$	0.0153	${}_1^qD_{0.5}^{\omega_2}(Q_3\|NIS)$	0.0355
${}_1^qD_{0.5}^{\omega_2}(Q_4\|PIS)$	0.0289	${}_1^qD_{0.5}^{\omega_2}(Q_4\|NIS)$	0.0527
${}_1^qD_{0.5}^{\omega_2}(Q_5\|PIS)$	0.0252	${}_1^qD_{0.5}^{\omega_2}(Q_5\|NIS)$	0.0591

TABLE 12 The ranking order of the alternatives by taking different values of α in ${}^{q}_{1}D^{\omega}_{\alpha}(P|Q)$ and ${}^{q}_{1}E^{\omega}_{\alpha}(P)$

α	Attribute weights, ω_1	Relative divergence coefficients, $\Im_i$'s	Ranking order
Case (i): When the information about the attribute weights is completely unknown			
0.2	$(0.1631\ 0.1746\ 0.1746\ 0.1723\ 0.1584\ 0.1570)^T$	$\Im_1 = 0.7184$, $\Im_2 = 0.5783$, $\Im_3 = 0.3235$, $\Im_4 = 0.4090$, $\Im_5 = 0.3849$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.5	$(0.1587\ 0.1788\ 0.1850\ 0.1801\ 0.1506\ 0.1469)^T$	$\Im_1 = 0.7540$, $\Im_2 = 0.6126$, $\Im_3 = 0.3278$, $\Im_4 = 0.3929$, $\Im_5 = 0.3273$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
0.8	$(0.1553\ 0.1793,\ 0.1931\ 0.1866\ 0.1457\ 0.1401)^T$	$\Im_1 = 0.7633$, $\Im_2 = 0.6304$, $\Im_3 = 0.3174$, $\Im_4 = 0.3948$, $\Im_5 = 0.3004$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
→1	$(0.1535\ 0.1791\ 0.1971\ 0.1901\ 0.1431\ 0.1372)^T$	$\Im_1 = 0.7712$, $\Im_2 = 0.6532$, $\Im_3 = 0.3343$, $\Im_4 = 0.3907$, $\Im_5 = 0.2511$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
Case (ii): When the information about the attribute weights is partially known			
α	**Attribute weights, ω_2**	**Relative divergence coefficients, $\Im_i$'s**	**Ranking order**
0.2	$(0.1274\ 0.1500\ 0.2500\ 0.1500\ 0.2000\ 0.1226)^T$	$\Im_1 = 0.7130$, $\Im_2 = 0.6216$, $\Im_3 = 0.3033$, $\Im_4 = 0.3799$, $\Im_5 = 0.3533$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.5	$(0.1298\ 0.1500\ 0.2500\ 0.1501\ 0.2000\ 0.1201)^T$	$\Im_1 = 0.7436$, $\Im_2 = 0.6591$, $\Im_3 = 0.3012$, $\Im_4 = 0.3542$, $\Im_5 = 0.2989$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
0.8	$(0.1288\ 0.1501\ 0.2500\ 0.1548\ 0.2000\ 0.1162)^T$	$\Im_1 = 0.7487$, $\Im_2 = 0.6717$, $\Im_3 = 0.2996$, $\Im_4 = 0.3525$, $\Im_5 = 0.2798$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
→1	$(0.1277\ 0.1501\ 0.2500\ 0.1581\ 0.2000\ 0.1141)^T$	$\Im_1 = 0.7471$, $\Im_2 = 0.6861$, $\Im_3 = 0.3242$, $\Im_4 = 0.3508$, $\Im_5 = 0.2488$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$

TABLE 13 The ranking order of the alternatives taking different values of α in ${}^{q}_{2}D^{\omega}_{\alpha}(P|Q)$ and ${}^{q}_{2}E^{\omega}_{\alpha}(P)$

α	Attribute weights, ω_1	Relative divergence coefficients, $\Im_i$'s	Ranking order
Case (i): When the information about the attribute weights is completely unknown			
0.2	$(0.1579\ 0.1800\ 0.1856\ 0.1779\ 0.1547\ 0.1438)^T$	$\Im_1 = 0.7128$, $\Im_2 = 0.5814$, $\Im_3 = 0.3264$, $\Im_4 = 0.4091$, $\Im_5 = 0.3829$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.5	$(0.1575\ 0.1833\ 0.1917\ 0.1850\ 0.1480\ 0.1345)^T$	$\Im_1 = 0.7515$, $\Im_2 = 0.6127$, $\Im_3 = 0.3231$, $\Im_4 = 0.3943$, $\Im_5 = 0.3271$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.8	$(0.1540\ 0.1804\ 0.1953\ 0.1882\ 0.1437\ 0.1384)^T$	$\Im_1 = 0.7622$, $\Im_2 = 0.6287$, $\Im_3 = 0.3182$, $\Im_4 = 0.3968$, $\Im_5 = 0.3008$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
→1	$(0.1535\ 0.1791\ 0.1971\ 0.1901\ 0.1431\ 0.1372)^T$	$\Im_1 = 0.7712$, $\Im_2 = 0.6532$, $\Im_3 = 0.3343$, $\Im_4 = 0.3907$, $\Im_5 = 0.2511$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
Case (ii): When the information about the attribute weights is partially known			
α	**Attribute weights, ω_2**	**Relative divergence coefficients, $\Im_i$'s**	**Ranking order**
0.2	$(0.1307\ 0.1501\ 0.2500\ 0.1501\ 0.2000\ 0.1190)^T$	$\Im_1 = 0.7109$, $\Im_2 = 0.6205$, $\Im_3 = 0.3038$, $\Im_4 = 0.3781$, $\Im_5 = 0.3552$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.5	$(0.1291\ 0.1506\ 0.2500\ 0.1518\ 0.2000\ 0.1185)^T$	$\Im_1 = 0.7400$, $\Im_2 = 0.6553$, $\Im_3 = 0.3020$, $\Im_4 = 0.3557$, $\Im_5 = 0.3021$	$Q_3 \succ Q_5 \succ Q_4 \succ Q_2 \succ Q_1$
0.8	$(0.1280\ 0.1504\ 0.2500\ 0.1565\ 0.2000\ 0.1151)^T$	$\Im_1 = 0.7465$, $\Im_2 = 0.6701$, $\Im_3 = 0.3012$, $\Im_4 = 0.3542$, $\Im_5 = 0.2812$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$
→1	$(0.1277\ 0.1501\ 0.2500\ 0.1581\ 0.2000\ 0.1141)^T$	$\Im_1 = 0.7471$, $\Im_2 = 0.6861$, $\Im_3 = 0.3242$, $\Im_4 = 0.3508$, $\Im_5 = 0.2488$	$Q_5 \succ Q_3 \succ Q_4 \succ Q_2 \succ Q_1$

TABLE 14 The ranking order of the alternatives by utilizing Liu and Wang's method[26] based on ^{q}ROFWA operator

	Score values					
Attribute weights	$S(Q1)$	$S(Q2)$	$S(Q3)$	$S(Q4)$	$S(Q5)$	**Ranking order**
$(0.1631\ 0.1746\ 0.1746\ 0.1723\ 0.1584\ 0.1570)^T$	0.1160	0.1827	0.2884	0.2675	0.3331	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1587\ 0.1788\ 0.1850\ 0.1801\ 0.1506\ 0.1469)^T$	0.1171	0.1851	0.2897	0.2658	0.3380	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1553\ 0.1793\ 0.1931\ 0.1866\ 0.1457\ 0.1401)^T$	0.1179	0.1803	0.2911	0.2646	0.3430	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1535\ 0.1791\ 0.1971\ 0.1901\ 0.1431\ 0.1372)^T$	0.1184	0.1798	0.2919	0.2639	0.3456	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1274\ 0.1500\ 0.2500\ 0.1500\ 0.2000\ 0.1226)^T$	0.1142	0.1493	0.2834	0.2728	0.3428	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1298\ 0.1500\ 0.2500\ 0.1501\ 0.2000\ 0.1201)^T$	0.1149	0.1505	0.2849	0.2741	0.3434	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1288\ 0.1501\ 0.2500\ 0.1548\ 0.2000\ 0.1162)^T$	0.1398	0.1766	0.3061	0.2894	0.3630	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1579\ 0.1800\ 0.1856\ 0.1779\ 0.1547\ 0.1438)^T$	0.1164	0.1804	0.2884	0.2665	0.3372	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1575\ 0.1833\ 0.1917\ 0.1850\ 0.1480\ 0.1345)^T$	0.1168	0.1803	0.2902	0.2656	0.3420	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1540\ 0.1804\ 0.1953\ 0.1882\ 0.1437\ 0.1384)^T$	0.1181	0.1801	0.2912	0.2640	0.3445	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1307\ 0.1501\ 0.2500\ 0.1501\ 0.2000\ 0.1190)^T$	0.1136	0.1493	0.2842	0.2736	0.3425	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1291\ 0.1506\ 0.2500\ 0.1518\ 0.2000\ 0.1185)^T$	0.1138	0.1494	0.2839	0.2732	0.3432	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$
$(0.1280\ 0.1504\ 0.2500\ 0.1565\ 0.2000\ 0.1151)^T$	0.1140	0.1496	0.2842	0.2729	0.3446	$Q_5 > Q_3 > Q_4 > Q_2 > Q_1$

TABLE 15 The ranking order of the alternatives by utilizing Liu et al[78] TOPSIS method

	Closeness index values					
Attribute weights	$\rho(Q1)$	$\rho(Q2)$	$\rho(Q3)$	$\rho(Q4)$	$\rho(Q5)$	**Ranking order**
$(0.1631\ 0.1746\ 0.1746\ 0.1723\ 0.1584\ 0.1570)^T$	0.7430	0.5538	0.2584	0.1844	0.0565	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1587\ 0.1788\ 0.1850\ 0.1801\ 0.1506\ 0.1469)^T$	0.7445	0.5666	0.2724	0.2036	0.0606	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1553\ 0.1793\ 0.1931\ 0.1866\ 0.1457\ 0.1401)^T$	0.7459	0.5779	0.2808	0.2182	0.0651	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1535\ 0.1791\ 0.1971\ 0.1901\ 0.1431\ 0.1372)^T$	0.7463	0.5832	0.2844	0.2256	0.0669	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1274\ 0.1500\ 0.2500\ 0.1500\ 0.2000\ 0.1226)^T$	0.7727	0.6941	0.2974	0.2085	0.0918	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1298\ 0.1500\ 0.2500\ 0.1501\ 0.2000\ 0.1201)^T$	0.7747	0.6947	0.2986	0.2098	0.0959	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1288\ 0.1501\ 0.2500\ 0.1548\ 0.2000\ 0.1162)^T$	0.7757	0.6963	0.3027	0.2150	0.0991	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1579\ 0.1800\ 0.1856\ 0.1779\ 0.1547\ 0.1438)^T$	0.7467	0.5704	0.2762	0.2014	0.0633	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1575\ 0.1833\ 0.1917\ 0.1850\ 0.1480\ 0.1345)^T$	0.7488	0.5779	0.2873	0.2175	0.0714	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1540\ 0.1804\ 0.1953\ 0.1882\ 0.1437\ 0.1384)^T$	0.7456	0.5799	0.2836	0.2222	0.0653	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1307\ 0.1501\ 0.2500\ 0.1501\ 0.2000\ 0.1190)^T$	0.7753	0.6950	0.2993	0.2104	0.0976	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1291\ 0.1506\ 0.2500\ 0.1518\ 0.2000\ 0.1185)^T$	0.7750	0.6951	0.3006	0.2115	0.0966	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$
$(0.1280\ 0.1504\ 0.2500\ 0.1565\ 0.2000\ 0.1151)^T$	0.7760	0.6966	0.3041	0.2165	0.0944	$Q_5 \succ Q_4 \succ Q_3 \succ Q_2 \succ Q_1$

because the preference information provided by the experts does not satisfy the condition $0 \leq MD + NMD \leq 1$ or $0 \leq (MD)^2 + (NMD)^2 \leq 1$. Therefore, to validate the obtained results, we have been used Liu and Wang's method[26] based on ^{q}ROFWA operator and Liu et al.[78] TOPSIS method to solve the above considered ERP selection problem. The obtained score values/closeness index values and the ranking order of the alternatives are shown in Tables 14 and 15.

From Tables 14 and 15, we can see that the best alternative is Q_5, which has an agreement with our obtained results. This validates that our developed method is reasonable and flexible in solving real-life MAGDM problems under the ^{q}ROPF environment.

5 | CONCLUSIONS

This study has presented a valuable study on divergence and entropy measures for ^{q}ROPFSs. We have defined two new order-α divergence measures between ^{q}ROsPFSs based on logarithmic and exponential functions. Several basic and important mathematical properties of these divergence measures have been proved. Further, the paper has defined two new parametric entropy functions called "order-α ^{q}ROPF entropy measures" to quantify the degree of fuzziness associated with a ^{q}ROPFS. The limiting and particular cases of the developed order-α entropy and divergence measures have been discussed in detail. It is interesting to note that several known information measures under fuzzy and intuitionistic fuzzy environments are the special cases of the developed order-α entropy and divergence measures. Besides, the paper has formulated a decision-making approach for solving MAGDM problems in which the attribute weights are completely unknown or partially known. To determine the attribute weights, we have constructed a nonlinear optimization model based on our developed divergence and entropy measures. Finally, a numerical example has been considered for demonstrating the decision-making process and the effectiveness of the developed approach. Note that our developed approach can also be applied to solve the MAGDM problems with intuitionistic fuzzy and Pythagorean fuzzy information by selecting the appropriate value of the parameter q. In addition, if there is only one decision-maker, then the developed approach can be utilized to solve the MADM problems mentioned in.[75,76,79]

In future work, we shall explore the applications of the developed decision-making approach in different application areas, including green supplier selection, facility location selection, and faculty recruitment problems.

ACKNOWLEDGMENT

Support from the Chilean Government (Conicyt) through the Fondecyt Postdoctoral Program (Project Number 3170556) is gratefully acknowledged.

CONFLICT OF INTERESTS

The authors declare that there are no conflict of interests.

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How to cite this article: Verma R. Multiple attribute group decision-making based on order-α divergence and entropy measures under q-rung orthopair fuzzy environment. *Int J Intell Syst*. 2020;35:718–750. https://doi.org/10.1002/int.22223