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Some universal solutions for incompressible elastic bodies that are not Green elastic

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ABSTRACT

Universal solutions are found for isotropic elastic bodies, for a class of constitutive equation that is not Green elastic, wherein the linearized strain is assumed to be a function of the Cauchy stress. The structure of the solutions is compared with one example from the classical theory of nonlinear elasticity, namely the case of the inflation and uniform extension/compression of a cylindrical annulus.

1. Introduction

In the classical theory of nonlinear elasticity[1], in the particular case of considering incompressible bodies, there exist some controllable deformations that are called universal solutions (see, for example, Saccomandi (2001) and Section 57 of Truesdell and Noll (2004)). Such solutions of the boundary value problem (in the quasi-static case) are valid for any constitutive equation within the class of equations for which they are obtained. A list of such solutions can be found, for example, in Section 57 of Truesdell and Noll (2004) (see also (Ericksen, 1954, 1955) and Section A of Volume I in Barenblatt & Joseph, 1997). An important element to find such solutions is the use of the scalar function p associated with the constraint of incompressibility, which is used to simplify the structure of the equations.

In the recent years some new classes of constitutive theories have been proposed for elastic bodies, which cannot be classified as Cauchy nor Green elastic bodies (Rajagopal, 2003, 2007, 2011a, 2011b; Rajagopal & Srinivasa, 2007, 2009). One of such relatively new class of constitutive equation corresponds to the case of having the linearized strain tensor $\boldsymbol{\varepsilon}$ as a function of the Cauchy stress tensor $\mathbf{T}$, i.e., $\boldsymbol{\varepsilon} = \mathfrak{g}(\mathbf{T})$ (see (Rajagopal, 2011b) and Section 4 of Bustamante and Rajagopal (2019)). That constitutive equation has potential uses in the modelling of concrete, some metal alloys, rock and in fracture mechanics, see (Buliček, Málek, Rajagopal, & Walton, 2015; Bustamante & Rajagopal, 2018; Devendiran, Sandeep, Kannan, & Rajagopal, 2017; Grasley et al., 2015; Kulvait, Málek, & Rajagopal, 2017) and Section 5 of Bustamante and Rajagopal (2019). The aim of the present paper is to analyse if the constraint of incompressibility can be used to simplify the boundary value problem, in order to obtain universal solutions similar to the case of the classical nonlinear elasticity theory. It was found that indeed the use of such constraint allows for the simplification of the problem and exact implicit universal solutions are found. The results presented in this communication correct some comments about this problem presented in Section 6.1.3 of Bustamante and Rajagopal (2019).

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[1] In this paper by the classical theory of nonlinear elasticity we mean the theory based on assuming that the stresses $\mathbf{T}$ are functions of the strains (or the left Cauchy-Green tensor $\mathbf{B}$), i.e., $\mathbf{T} = \mathfrak{f}(\mathbf{B})$, which in the case of incompressible bodies becomes $\mathbf{T} = -p\mathbf{I} + \mathfrak{f}(\mathbf{B})$, where p is a scalar field related with the constraint of incompressibility.

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This paper is divided in the following sections: In Section 2 some basic relations and equations concerning the theory of elasticity are shown. In Section 3 the constitutive equation $\boldsymbol{\varepsilon} = \mathfrak{g}(\mathbf{T})$ is studied for the case of incompressible bodies. In Section 4 a summary of the classical problem of the inflation and extension of a cylindrical annulus is presented within the context of the classical nonlinear elasticity theory, in order to compare the exact solution found in that theory with the solution (in implicit form) found in the case of $\boldsymbol{\varepsilon} = \mathfrak{g}(\mathbf{T})$. In Section 5 two more examples of universal solutions are presented. In Section 6 some final comments are given.

2. Basic equations

For a body $\mathscr{B}$ the reference and current configurations are denoted $\kappa_{\mathrm{r}}(\mathcal{B})$ and $\kappa_{\mathrm{t}}(\mathcal{B})$, respectively. The positions of a point $X \in \mathscr{B}$ in the reference and current configurations are denoted $\mathbf{X}$ and $\mathbf{x}$, respectively, and it is assumed that there exists a one-to-one function $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The deformation gradient, the left Cauchy-Green tensor, the displacement field and the linearized strain tensor are defined as

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}^{\mathrm{T}}\right), \tag{1}$$

respectively, where $J = \det \mathbf{F} > 0$ and the body it is said to be incompressible if $J = 1$ for any deformation.

The Cauchy stress tensor is denoted $\mathbf{T}$ and for the remaining of this paper we assume quasi-static deformations, therefore, the stress $\mathbf{T}$ must satisfy the equation of equilibrium

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{2}$$

where ρ is the density of the body and $\mathbf{b}$ represents the body forces, which for the boundary value problems studied in Sections 4 and 5 are assumed to be zero.

In Sections 4 and 5 some problems are studied considering cylindrical and spherical coordinates, therefore, (1_4) and (2) are listed for such systems of coordinates here. In the case of (1_4) in cylindrical coordinates (r, θ, z) we have

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \tag{3}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \tag{4}$$

and in spherical coordinates (r, θ, ϕ) we have

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{u_\phi}{r}\cot\phi, \quad \varepsilon_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, \tag{5}$$

$$\varepsilon_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{r}\cot\phi + \frac{1}{r}\frac{\partial u_\theta}{\partial \phi}\right), \quad \varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right), \tag{6}$$

$$\varepsilon_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right). \tag{7}$$

In the case of (2) in cylindrical coordinates that equation becomes

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \rho b_r = 0, \tag{8}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta} + \rho b_\theta = 0, \tag{9}$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + \rho b_z = 0, \tag{10}$$

while in spherical coordinates we have

$$\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\phi}}{\partial \theta} + \frac{3}{r}T_{r\phi} + \frac{\cos\phi}{r\sin\phi}(T_{\phi\phi} - T_{\theta\theta}) + \rho b_\phi = 0, \tag{11}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{3}{r}T_{r\theta} + \frac{2}{r}\frac{\cos\phi}{\sin\phi}T_{\phi\theta} + \rho b_\theta = 0, \tag{12}$$

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\cos\phi}{r\sin\phi}T_{r\phi} + \frac{1}{r}(2T_{rr} - T_{\phi\phi} - T_{\theta\theta}) + \rho b_r = 0. \tag{13}$$

More details about the kinematics of continuum media and the equations of equilibrium can be found, for example, in (Truesdell & Toupin, 1960).

Let us end this section showing the constitutive equation for an incompressible isotropic Green elastic body (see, for example, Section 49 of Truesdell & Noll (2004)):

$$\mathbf{T} = -p\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2, \tag{14}$$

where p is a scalar function and $-p\mathbf{I}$ is the part of the stress that does not do any work with any deformation compatible with the constraint, and α_1, α_2 are scalar functions that depend on the invariants of $\mathbf{B}$, and for Green elastic bodies are given in terms of derivatives of the energy function.

3. An incompressible body that is not Green elastic

In a series of relatively recent papers Rajagopal and co-workers have proposed some implicit constitutive theories, wherein in general the stresses cannot be expressed explicitly in terms of the strains (see, for example, Rajagopal, 2003; Rajagopal, 2007; Rajagopal & Srinivasa, 2007; Rajagopal & Srinivasa, 2009). One of such relations is $\mathfrak{F}(\mathbf{T},\mathbf{B}) = \mathbf{0}$. In the case it is assumed that the gradient of the displacement field is small $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right| \sim O(\delta)$, $\delta \ll 1$ we have that $\mathbf{B} \approx 2\boldsymbol{\varepsilon} + \mathbf{I}$, and from the above implicit relation the following constitutive equation has been found (see, for example, Rajagopal, 2011b) $\boldsymbol{\varepsilon} = \mathfrak{g}(\mathbf{T})$. In the particular case we assume there exists a scalar potential $\Pi = \Pi(\mathbf{T})$ such that $\mathfrak{g}(\mathbf{T}) = \frac{\partial \Pi}{\partial \mathbf{T}}$, if Π is an isotropic function then $\Pi = \Pi(I_1, I_2, I_3)$, where $I_1 = \text{tr}\ \mathbf{T}$, $I_2 = \frac{1}{2}\text{tr}\ (\mathbf{T}^2)$ and $I_3 = \frac{1}{3}\text{tr}\ (\mathbf{T}^2)$ and we obtain the representation (see Bustamante, 2009; Bustamante & Rajagopal, 2015a)

$$\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \mathbf{T}} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2, \tag{15}$$

where $\Pi_i = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3$.

In Bustamante and Rajagopal (2016) studied the case of considering (15) when modelling incompressible bodies. If the gradient of the displacement field is small the incompressibility constraint is

$$\text{tr}\boldsymbol{\varepsilon} = 0. \tag{16}$$

Using (15) in (16) we obtain the first order linear partial differential equation

$$3\Pi_1 + \Pi_2 I_1 + 2\Pi_3 I_2 = 0, \tag{17}$$

whose solution is (see Bustamante & Rajagopal, 2016, see also Section 6.1.3 of Bustamante & Rajagopal (2019))

$$\Pi = \bar{\Pi}(I_{D_2}, I_{D_3}), \tag{18}$$

where

$$I_{D_2} = \frac{1}{2}\text{tr}(\mathbf{T}_D^2), \quad I_{D_3} = \frac{1}{3}\text{tr}(\mathbf{T}_D^3), \tag{19}$$

where I_{D_k}, $k = 2, 3$ are invariants of the deviatoric stress $\mathbf{T}_D$ that is defined as

$$\mathbf{T}_D = \mathbf{T} - \frac{\text{tr}(\mathbf{T})}{3}\mathbf{I}. \tag{20}$$

The stress tensor can be written as

$$\mathbf{T} = -\sigma_S\mathbf{I} + \mathbf{T}_D \quad \text{where} \quad \sigma_S = -\frac{\text{tr}(\mathbf{T})}{3} \tag{21}$$

is the spherical stress. Using (18) in $\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \mathbf{T}}$ we obtain

$$\boldsymbol{\varepsilon} = -\frac{2I_{D_2}}{3}\frac{\partial \bar{\Pi}}{\partial I_{D_3}}\mathbf{I} + \frac{\partial \bar{\Pi}}{\partial I_{D_2}}\mathbf{T}_D + \frac{\partial \bar{\Pi}}{\partial I_{D_3}}\mathbf{T}_D^2 = \vartheta_0\mathbf{I} + \vartheta_1\mathbf{T}_D + \vartheta_2\mathbf{T}_D^2, \tag{22}$$

where $\vartheta_i = \vartheta_i(\mathbf{T}_D)$, and where $\vartheta_0 = -\frac{2}{3}I_{D_2}\frac{\partial \bar{\Pi}}{\partial I_{D_3}}$, $\vartheta_1 = \frac{\partial \bar{\Pi}}{\partial I_{D_2}}$ and $\vartheta_3 = \frac{\partial \bar{\Pi}}{\partial I_{D_3}}$. From (22) it is possible to see that $\boldsymbol{\varepsilon}(\mathbf{T}) = \boldsymbol{\varepsilon}(\mathbf{T}_D)$, i.e., the strain is only affected by the deviatoric stress. The case of obtaining linearized constitutive equations from (22) has been studied in Section 3.3 of Bustamante and Rajagopal (2016).

4. Universal solutions, two examples

In this section we review a well known exact solution obtained within the context of the classical theory of nonlinear elasticity, and then we compare with a similar solution found considering (22).

4.1. The inflation and expansion of a cylindrical annulus within the context of the classical theory of nonlinear elasticity

Let us review the problem of the inflation and expansion of a cylindrical annulus within the context of the classical theory of nonlinear elasticity, as it was done in Section 6.1.3 of Bustamante and Rajagopal (2019). We do this in order to have some concepts to compare what happens with the solutions obtained considering the constitutive Eq. (14), with what it is found using (22). In the reference configuration in cylindrical coordinates the annulus is defined by $R_{\rm i} \le R \le R_{\rm o}$, $0 \le \Theta \le 2\pi$, $0 \le Z \le L$.

Following Rivlin (see Chapter A of Volume I in Barenblatt & Joseph (1997) and also Section 57 of Truesdell and Noll (2004)), let us study how p in (14) helps to find exact solutions for the boundary value problem. For the problem of inflation and extension of a cylindrical annulus, the deformation $\mathbf{x}$ is assumed to be of the form

$$r = f(R), \quad \theta = \Theta, \quad z = \lambda Z, \tag{23}$$

where λ is a constant. From (23) and $(1)_1$ we have

$$\mathbf{F} = f'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{f(R)}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda \mathbf{e}_z \otimes \mathbf{E}_Z, \tag{24}$$

where from the application of the constraint of incompressibility $\det \mathbf{F} = 1$ we obtain

$$r = f(R) = \sqrt{\frac{R^2 - R_{\rm i}^2}{\lambda} + r_{\rm i}^2}, \tag{25}$$

where $r_{\rm i}$ is the inner radius of the annulus in the current configuration.

If the notation $\tilde{\mathbf{T}} = \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2$ is used, from (24) it is possible to see that $\tilde{\mathbf{T}}$ only depends on the radial position, because α_1, α_2 depend on $\mathbf{B}$ that only depends on the radial position. The stress $\mathbf{T}$ only has normal components. From (8)–(10) and (14) we obtain that $p = p(r) = p(R)$ and

$$\frac{{\rm d}\tilde{T}_{rr}}{{\rm d}r} - \frac{{\rm d}p}{{\rm d}r} + \frac{1}{r}(\tilde{T}_{rr} - \tilde{T}_{\theta\theta}) = 0, \tag{26}$$

which can be solved easily for $p(r)$.

Now, if we assume that on the inner surface of the annulus there is a traction P, and that on the outer surface the annulus is traction free, from (26) we obtain

$$P = \int_{r_{\rm i}}^{r_{\rm o}} \frac{1}{\xi}\left[\tilde{T}_{\theta\theta}(\xi) - \tilde{T}_{rr}(\xi)\right]{\rm d}\xi, \tag{27}$$

where $r_{\rm o} = \sqrt{\frac{R_{\rm o}^2 - R_{\rm i}^2}{\lambda} + r_{\rm i}^2}$ is the outer radius in the current configuration.

Let us examine some characteristics of the above exact solution for that boundary value problem. From (25) we have an exact and explicit solution for $f(R)$ up to a constant $r_{\rm i}$. That constant must be found from (27) for a given P, and in general it is not possible to solve (27) exactly for $r_{\rm i}$. Regarding the components of $\mathbf{T}$, we can easily obtain p from (26) and from (14) we can have explicit solutions for T_{rr}, $T_{\theta\theta}$ and T_{zz}. Finally the solution (25) is valid for any α_1 and α_2, therefore, (25) is a universal solution.

4.2. The inflation and extension of a cylindrical annulus for an incompressible body that is not Green elastic

In the case of solving boundary value problems for the class of constitutive Eq. 22 using the semi-inverse method, it is convenient to assume simplified expressions for the stresses and the displacement field[2] (see, for example, Bustamante & Rajagopal, 2015b). Let us consider the problem of a cylindrical annulus[3] $r_{\rm i} \le r \le r_{\rm o}$, $0 \le \theta \le 2\pi$, $0 \le z \le L$, where the stress tensor is assumed to be

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{28}$$

We suppose that the above stress causes the displacement field

$$\mathbf{u} = f(r)\mathbf{e}_r + (\lambda - 1)z\mathbf{e}_z, \tag{29}$$

[2] In this paper in order to solve boundary value problems with the inverse method, we assume simplified expressions for the stresses 28 and the displacement field 29. This technique has been already used in some previous works by this author and his co-workers, in particular within the context of unconstrained elastic bodies (see, for example, Bustamante & Rajagopal (2015b,c)). On the other hand, if we assume some simplified expressions for the stresses, we could use the compatibility equations to find solutions of the problem without assuming from the beginning any especial form for the displacement field, as it is done in the classical theory of linear elasticity when using the stress potential. However, as discussed by Rajagopal and Srinivasa (2015), it is not necessary or mandatory to consider such compatibility equations if one uses from the beginning the displacement field as one of the main variables of the problem (along with the stresses). In Bustamante and Rajagopal (2010) (see Eq. (36) therein) the interested reader can see the rather complex structure of the final equation that appear when we use the compatibility equation and $\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T})$ for plane problems. In the case of assuming stresses that depend on one variable, the use of such compatibility equations has been explored in Bustamante and Rajagopal (2011).

[3] In this case because of the assumption of small deformations, the inner and outer radii in the current configuration are very similar to the radii in the reference configuration.

where λ is a constant. Using (29) in (3) and (4) we obtain

$$\boldsymbol{\varepsilon} = f'(r)\mathbf{e}_r \otimes \mathbf{e}_r + \frac{f(r)}{r}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (\lambda - 1)\mathbf{e}_z \otimes \mathbf{e}_z, \tag{30}$$

and using this in (16) we have the differential equation $f'(r) + \frac{f(r)}{R} + \lambda - 1 = 0$, whose solution is

$$f(r) = \frac{(1-\lambda)}{2}r + \frac{C}{r}, \tag{31}$$

where C is a constant.

Let us decompose the stress in a spherical and a deviatoric part as indicated in (20), (21), then from (28) we have

$$T_{rr} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{rr}}, \quad T_{\theta\theta} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{\theta\theta}}, \quad T_{zz} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{zz}}. \tag{32}$$

From now on we work with σ_{S} and $T_{\mathrm{D}_{rr}}$, $T_{\mathrm{D}_{\theta\theta}}$ and $T_{\mathrm{D}_{zz}}$ as the basic variables for the problem.

Using (30) and (32) in (22) (recalling (31)) we obtain

$$f'(r) = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{rr}} + \vartheta_2 T_{\mathrm{D}_{rr}}^2, \tag{33}$$

$$\frac{f(r)}{r} = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{\theta\theta}} + \vartheta_2 T_{\mathrm{D}_{\theta\theta}}^2, \tag{34}$$

$$\lambda - 1 = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{zz}} + \vartheta_2 T_{\mathrm{D}_{zz}}^2. \tag{35}$$

Regarding (28) from (8)–(10) we obtain $\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0$, which from (32) becomes

$$-\frac{\mathrm{d}\sigma_{\mathrm{S}}}{\mathrm{d}r} + \frac{\mathrm{d}T_{\mathrm{D}_{rr}}}{\mathrm{d}r} + \frac{1}{r}(T_{\mathrm{D}_{rr}} - T_{\mathrm{D}_{\theta\theta}}) = 0, \tag{36}$$

whose solution is

$$\sigma_{\mathrm{S}}(r) = T_{\mathrm{D}_{rr}}(r) - T_{rr}(r_{\mathrm{i}}) + \int_{r_{\mathrm{i}}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{rr}}(\xi) - T_{\mathrm{D}_{\theta\theta}}(\xi)]\,\mathrm{d}\xi, \tag{37}$$

where $T_{rr}(r_{\mathrm{i}})$ is the radial component of the stress on the inner surface of the annulus. Let us assume again that on the inner surface of the annulus we apply a traction P, so $T_{rr}(r_{\mathrm{i}}) = -P$, then from (32) and (37) we obtain $T_{rr}(r) = -P + \int_{r_{\mathrm{i}}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\,\mathrm{d}\xi$. If on the outer surface of the annulus there is no external traction then

$$P = \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\,\mathrm{d}\xi. \tag{38}$$

Finally, if we obtain $T_{zz}(r)$ from the above expressions, we can obtain the total axial load that is necessary to produce the displacement field (29) as

$$\mathcal{N} = 2\pi \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} rT_{zz}(r)\,\mathrm{d}r. \tag{39}$$

Let us study the above solution and compare with the classical case described in Section 4.1. From (29) and (31) we have an explicit solution for the displacement field up to a constant C. Let us assume that C and λ are known, then from (33)–(35) we can obtain $T_{\mathrm{D}_{rr}}$, $T_{\mathrm{D}_{\theta\theta}}$ and $T_{\mathrm{D}_{zz}}$ implicitly, i.e., they are exact solutions in implicit forms (which in general should be obtained numerically). Such solutions will depend in particular on the constant C, and such constant should be found numerically from (38). The above solution is valid for any $\bar{\Pi}$, therefore, it is also a universal solution. Regarding σ_{S}, if C and $T_{\mathrm{D}_{rr}}$, $T_{\mathrm{D}_{\theta\theta}}$ are known, then from (37) that function σ_{S} can also be obtained, and from (32) we can find T_{rr}, $T_{\theta\theta}$ and T_{zz}.

In the Appendix some results are presented for this boundary value problem, for a special expression for $\bar{\Pi}$ where we have a strain limiting behaviour.

5. Other examples

Two additional problems are studied in this section, where we use the constraint of incompressibility to obtain implicit solutions for the boundary value problem, following the method presented in the previous section.

5.1. The inflation, extension, torsion, circumferential shear, telescopic shear and closure of a cylindrical opened annulus

In this section the same problem presented in Section 6.1.2 of Bustamante and Rajagopal (2019) is examined, namely the inflation, axial extension, torsion, circumferential shear, telescopic shear and closure of the radially opened cylindrical annulus $r_{\mathrm{i}} \le r \le r_{\mathrm{o}}$, $0 \le \theta \le 2\pi - \alpha$. For such an annulus let us assume there is a stress distribution of the form

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z + T_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{rz}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + T_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \quad (40)$$

In such a case (8)–(10) become

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \quad \frac{\mathrm{d}T_{r\theta}}{\mathrm{d}r} + \frac{2}{r}T_{r\theta} = 0, \quad \frac{\mathrm{d}T_{rz}}{\mathrm{d}r} + \frac{1}{r}T_{rz} = 0. \quad (41)$$

Eqs. $(41)_{2,3}$ can be solved exactly and we find

$$T_{r\theta}(r) = T_{r\theta_\mathrm{i}}\left(\frac{r_\mathrm{i}}{r}\right)^2, \quad T_{rz}(r) = T_{rz_\mathrm{i}}\frac{r_\mathrm{i}}{r}, \quad (42)$$

where $T_{r\theta_\mathrm{i}}$ and T_{rz_i} are constants.

From $(21)_2$ and (40) we obtain

$$\sigma_\mathrm{S} = -\frac{[T_{rr}(r) + T_{\theta\theta}(r) + T_{zz}(r)]}{3}, \quad (43)$$

and the deviatoric stress is given by

$$\mathbf{T}_\mathrm{D} = T_{\mathrm{D}_{rr}}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{\theta\theta}}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz}}(r)\mathbf{e}_z \otimes \mathbf{e}_z + T_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{rz}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + T_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \quad (44)$$

It is assumed that the annulus deforms as

$$\mathbf{u} = f(r)\mathbf{e}_r + [kr\theta + g(r) + \tau_o rz]\mathbf{e}_\theta + [(\lambda - 1)z + h(r)]\mathbf{e}_z, \quad (45)$$

where $k > 0$, τ_o and λ are constants. In the especial case that $g(r) = 0$ and $\tau_o = 0$, it is possible to define k as $k = \frac{\alpha}{2\pi - \alpha}$, where α is the opening angle of the annulus, and in such a case $u_\theta = kr\theta$, which means that the annulus is closing completely. From (3) and (4) considering (45) we obtain

$$\boldsymbol{\varepsilon} = f'(r)\mathbf{e}_r \otimes \mathbf{e}_r + \left[k + \frac{f(r)}{r}\right]\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (\lambda - 1)\mathbf{e}_z \otimes \mathbf{e}_z + \frac{1}{2}\left[g'(r) + \frac{g(r)}{r}\right](\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \frac{h'(r)}{2}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \frac{\tau_o r}{2}(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \quad (46)$$

Using (46) in (16) we have $f'(r) + k + \frac{f(r)}{r} + \lambda - 1 = 0$, whose solution is

$$f(r) = (1 - \lambda - k)\frac{r}{2} + \frac{C}{r}, \quad (47)$$

where C is a constant.

Replacing (46) and (44) in (22) we have

$$f'(r) = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{rr}} + \vartheta_1(T_{\mathrm{D}_{rr}}^2 + T_{r\theta}^2 + T_{rz}^2), \quad (48)$$

$$k + \frac{f(r)}{r} = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{\theta\theta}} + \vartheta_1(T_{r\theta}^2 + T_{\mathrm{D}_{\theta\theta}}^2 + T_{\theta z}^2), \quad (49)$$

$$\lambda - 1 = \vartheta_0 + \vartheta_1 T_{\mathrm{D}_{zz}} + \vartheta_2(T_{rz}^2 + T_{\theta z}^2 + T_{\mathrm{D}_{zz}}^2), \quad (50)$$

$$\frac{1}{2}\left[g'(r) - \frac{g(r)}{r}\right] = \vartheta_1 T_{r\theta} + \vartheta_2(T_{\mathrm{D}_{rr}}T_{r\theta} + T_{r\theta}T_{\mathrm{D}_{\theta\theta}} + T_{rz}T_{\theta z}), \quad (51)$$

$$\frac{h'(r)}{2} = \vartheta_1 T_{rz} + \vartheta_1(T_{\mathrm{D}_{rr}}T_{rz} + T_{r\theta}T_{\theta z} + T_{rz}T_{\mathrm{D}_{zz}}), \quad (52)$$

$$\frac{\tau_o r}{2} = \vartheta_1 T_{\theta z} + \vartheta_2(T_{r\theta}T_{rz} + T_{\mathrm{D}_{\theta\theta}}T_{\theta z} + T_{\theta z}T_{\mathrm{D}_{zz}}). \quad (53)$$

Eqs. (51) and (52) can be solved exactly obtaining

$$g(r) = 2r\int_{r_\mathrm{i}}^{r}\left\{\frac{1}{\xi}[\vartheta_1(\mathbf{T}_\mathrm{D}(\xi))T_{r\theta}(\xi) + \vartheta_2(\mathbf{T}_\mathrm{D}(\xi))\{T_{\mathrm{D}_{rr}}(\xi)T_{r\theta}(\xi) + T_{r\theta}(\xi)T_{\theta\theta}(\xi) + T_{rz}(\xi)T_{\theta z}(\xi)\}]\right\}\mathrm{d}\xi + \frac{g_\mathrm{i} r}{r_\mathrm{i}}, \quad (54)$$

$$h(r) = 2\int_{r_\mathrm{i}}^{r}\{\vartheta_1(\mathbf{T}_\mathrm{D}(\xi))T_{rz}(\xi) + \vartheta_2(\mathbf{T}_\mathrm{D}(\xi))[T_{\mathrm{D}_{rr}}(\xi)T_{rz}(\xi) + T_{r\theta}(\xi)T_{\theta z}(\xi) + T_{rz}(\xi)T_{\mathrm{D}_{zz}}(\xi)]\}\,\mathrm{d}\xi + h_\mathrm{i}, \quad (55)$$

where g_i and h_i are constants.

Therefore, considering (48)–(50) and (53) (recalling (47)), we have four equations to find $T_{D_{rr}}$, $T_{D_{\theta\theta}}$, $T_{D_{zz}}$ and $T_{\theta z}$ in terms of $f(r)$, k, λ and τ_o. The system of Eqs. (48)–(50) are implicit solutions for such components of the deviatoric stress. From $(41)_1$ we obtain the same expression for σ_S as in (37).

Regarding the boundary conditions, the surfaces of the annulus where we need to analyze the boundary conditions are $r = r_i$, r_o, $\theta = 0$, $\theta = 2\pi - \alpha$, $z = 0$ and $z = L$:

Surfaces $r = r_i$, $r = r_o$: Let us use the notation $\hat{\mathbf{t}}_i$ for the external traction on the surface $r = r_i$, we have

$$T_{rr}(r_i) = -\hat{t}_{i_r}, \quad T_{r\theta}(r_i) = T_{r\theta_i} = -\hat{t}_{i_\theta}, \quad T_{zr}(r_i) = T_{rz_i} = -\hat{t}_{i_z}. \tag{56}$$

If the notation $\hat{\mathbf{t}}_o$ is used for the external traction on $r = r_o$, then

$$T_{rr}(r_o) = \hat{t}_{o_r}, \quad T_{r\theta}(r_o) = T_{r\theta_o}\left(\frac{r_i}{r_o}\right)^2 = \hat{t}_{o_\theta}, \quad T_{zr}(r_o) = T_{rz_i}\frac{r_i}{r_o} = \hat{t}_{o_z}. \tag{57}$$

From $(41)_1$ considering $(56)_1$ and $(57)_1$ we have

$$\hat{t}_{o_r} + \hat{t}_{i_r} = \int_{r_i}^{r_o} \frac{1}{\xi}[T_{D_{\theta\theta}}(\xi) - T_{D_{rr}}(\xi)]\, d\xi. \tag{58}$$

The above equation can be used to obtain, for example, the constant C (see (47)–(51), (53)) as a function of $\hat{t}_{r_o} + \hat{t}_{r_i}$. If $\hat{t}_{r_o} = 0$ and $\hat{t}_{r_i} = P$ we obtain the same Eq. (38).

Surfaces $\theta = 0$, $\theta = 2\pi - \alpha$: For these surfaces it is possible to calculate the total moment in the axial direction z that is denoted $\mathcal{M}$, and the total shear forces $\mathcal{S}_1$ (in the plane (r, θ)) and $\mathcal{S}_2$ (in the plane (θ, z)) that are necessary to produce the displacement field as

$$\mathcal{M} = L\int_{r_i}^{r_o} rT_{\theta\theta}(r)\, dr, \quad \mathcal{S}_1 = L\int_{r_i}^{r_o} T_{r\theta}(r)\, dr = T_{r\theta_i}\frac{r_i}{r_o}L(r_o - r_i), \tag{59}$$

$$\mathcal{S}_2 = L\int_{r_i}^{r_o} T_{\theta z}(r)\, dr. \tag{60}$$

Surfaces $z = 0$, $z = L$: For these surfaces it is possible to calculate the total axial force $\mathcal{N}$ (in the direction z), and the total shear forces $\mathcal{S}_3$ (in the plane (θ, z)) and $\mathcal{S}_4$ (in the plane (r, z)) as

$$\mathcal{N} = (2\pi - \alpha)\int_{r_i}^{r_o} rT_{zz}(r)\, dr, \quad \mathcal{S}_3 = (2\pi - \alpha)\int_{r_i}^{r_o} rT_{\theta z}(r)\, dr, \tag{61}$$

$$\mathcal{S}_4 = (2\pi - \alpha)\int_{r_i}^{r_o} rT_{rz}(r)\, dr = (2\pi - \alpha)T_{rz_i} r_i(r_o - r_i). \tag{62}$$

5.2. The inflation of a sphere

In this section we study briefly the problem of the inflation of an incompressible sphere (see Bustamante & Rajagopal, 2015). Let us consider the sphere $r_i \le r \le r_o$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, where the stress tensor is assumed to be

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\phi\phi}(r)\mathbf{e}_\phi \otimes \mathbf{e}_\phi. \tag{63}$$

From (11) it is easy to see that $T_{\phi\phi} = T_{\theta\theta}$.

Let us assume that the displacement field is

$$\mathbf{u} = u_r(r)\mathbf{e}_r, \tag{64}$$

then from (5)–(7) we obtain

$$\boldsymbol{\varepsilon} = f'(r)\mathbf{e}_r \otimes \mathbf{e}_r + \frac{f(r)}{r}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi). \tag{65}$$

From (16) we obtain $f'(r) + 2\frac{f(r)}{r} = 0$, from where we have

$$f(r) = \frac{C}{r^2}, \tag{66}$$

where C is a constant.

From $(21)_1$ we have

$$T_{rr}(r) = -\sigma_S(r) + T_{D_{rr}}(r), \quad T_{\theta\theta}(r) = -\sigma_S(r) + T_{D_{\theta\theta}}(r), \quad T_{\phi\phi}(r) = -\sigma_S(r) + T_{D_{\phi\phi}}(r), \tag{67}$$

and the equilibrium Eqs. (11)–(13) become

$$-\frac{d\sigma_S}{dr}+\frac{dT_{D_{rr}}}{dr}+\frac{2}{r}(T_{D_{rr}}-T_{D_{\theta\theta}})=0. \tag{68}$$

Using (65) and (67) in (22) we have

$$-2\frac{C}{r^3}=\vartheta_0(\mathbf{T}_D)+\vartheta_1(\mathbf{T}_D)T_{D_{rr}}+\vartheta_2(\mathbf{T}_D)T_{D_{rr}}^2, \tag{69}$$

$$\frac{C}{r^3}=\vartheta_0(\mathbf{T}_D)+\vartheta_1(\mathbf{T}_D)T_{D_{\theta\theta}}+\vartheta_2(\mathbf{T}_D)T_{D_{\theta\theta}}^2, \tag{70}$$

from where we can obtain $T_{D_{rr}}(r)$ and $T_{D_{\theta\theta}}(r)$ that depend implicitly in C.

As in Section 4 we can assume that for the surfaces $r=r_i$ and $r=r_o$ we have the boundary conditions $T_{rr}(r_i)=-P$ and $T_{rr}(r_o)=0$, and from (68) that implies (compare with (27))

$$P=\int_{r_i}^{r_o}\frac{2}{\xi}\left[T_{D_{\theta\theta}}(\xi)-T_{D_{rr}}(\xi)\right]d\xi. \tag{71}$$

Using $T_{D_{rr}}(r)$ and $T_{D_{\theta\theta}}(r)$ obtained from (69), (70), from (71) we can find C as a function of P, as in the case of the boundary value problems presented in Sections 4 and 5.1.

6. Final comments

In the present communication some exact universal solutions have been obtained for a class of constitutive equation, where the linearized strain is a nonlinear function of the stresses (22). Unlike the universal solutions that have been obtained in the classical theory of nonlinear elasticity, where we have explicit expressions for the deformation and the stresses (14), in the present case we obtained solutions, where we have explicit expressions for the displacement field, but where the stresses (the deviatoric part of the stress) must be found (implicitly) by solving a system of nonlinear algebraic equations (see, for example, (33)–(35)).

In order to solve boundary value problems considering 15 or 22 with the semi-inverse method, we assumed expressions for the stresses and the deformation field (see, for example, (28) and (29)). It is possible that for some expressions for Π or $\bar{\Pi}$, non-unique solutions can be found. This is particularly clear when one sees the systems of algebraic Eqs. (33)–(35), (48)–(50), (53) and (69), (70). Moreover, we assumed that real solutions existed for the components of the deviatoric stress tensor from (33)–(35), (48)–(50), (53) and (69), (70), but in general it would be necessary to impose some restrictions on $\bar{\Pi}$ to assure existence of real solutions from such equations. In the present work we do not study those problems.

The methodology to obtain such exact universal solutions presented in this paper, can be easily extended to the case of large deformations, where some measure of the strains (large strains) is assumed to be a function of the stresses. That analysis will be part of a future work.

In the case of the universal solutions that have been obtained in the classical theory of nonlinear elasticity (see Section 57 of Truesdell & Noll (2004)), in general some constants must be found using numerical methods (such as is the case of r_i that is the inner radius of the annulus in (25)), for which we need specific expressions for the functions α_0 and α_1 in (14). For the class of constitutive equation studied here 22, some constants must also be found solving numerically some algebraic equations (see, for example, (38)), but the stresses (the deviatoric components of the stress) must also be found numerically solving some algebraic equations (see, for example, (33)–(35), (48)–(50), (53) and (69), (70)). So the structure of the universal solutions in the case of (22) is more complex than in the case of (14). Nevertheless, even if we need to solve a system of algebraic equations such as (33)–(35), that is in general much better and simpler than to solve the original system of ordinary differential and algebraic Eqs. $(1)_4$, (2) and (15) (see, for example, Bustamante & Rajagopal, 2015a, 2015b).

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Appendix. A numerical example

In this Appendix some numerical results are presented concerning in particular with the solutions of (33)–(35). The results depend on an specific expression for $\bar{\Pi}$. First let us consider some simple problems with homogeneous deformations and stresses for a cylinder and a slab.

Homogeneous tension/compression of a cylinder: For the cylinder $0\le r\le r_o$, $0\le\theta\le 2\pi$, $0\le z\le L$, let us assume that the it deforms due to the homogeneous stress distribution $\mathbf{T}=\sigma\mathbf{e}_z\otimes\mathbf{e}_z$ (where σ is constant). The spherical stress $(21)_2$ is given as $\sigma_S=-\frac{\sigma}{3}$, and as a results the non-zero components of the deviatoric stress (20) are $T_{D_1}=T_{D_2}=-\frac{\sigma}{3}$ and $T_{D_3}=\frac{2\sigma}{3}$. From (19) we have $I_{D_2}=\frac{\sigma^2}{3}$, $I_{D_3}=\frac{2\sigma^3}{27}$. Let us assume the following simplified expression for $\bar{\Pi}$:

$$\bar{\Pi}(I_{D_2},I_{D_3})=\Gamma(I_{D_2})+\Psi(I_{D_3}), \tag{72}$$

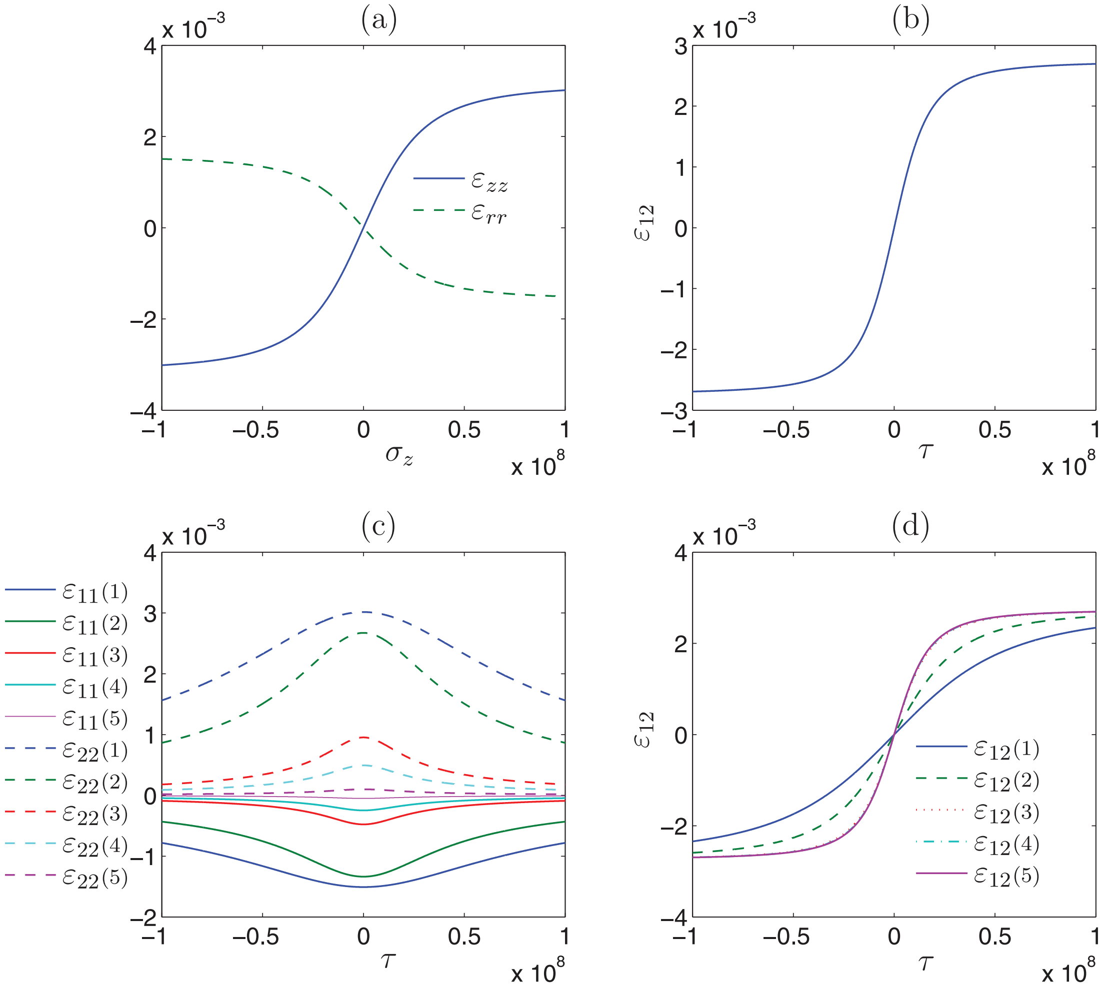

Fig. 1. Results for some homogeneous distributions of stresses and strains. (a) Uniaxial tension and compression of a cylinder (see 73). (b) Simple shear of a slab (see (74)). (c) Shear and compression of a slab (see (75) and (76)), components ε_{11} and ε_{22} of the strain tensor for the following cases: (1) $\sigma = 10^8$, (2) $\sigma = 5 \times 10^7$, (3) $\sigma = 10^7$, (4) $\sigma = 5 \times 10^6$, (5) $\sigma = 10^6$. (d) Shear and compression of a slab component ε_{12} of the strain. All the stresses are in Pa.

then from (22) we obtain

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = -\frac{\sigma^2}{9}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}} - \frac{\sigma}{3}\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_2}}, \quad \varepsilon_{zz} = \frac{2\sigma^2}{9}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}} + \frac{2\sigma}{3}\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_2}}. \tag{73}$$

Simple shear: For the slab $L_i \leq x_i \leq L_i$, $i = 1, 2, 3$ let us assume that this slab deforms due to the presence of the simple homogeneous shear stress $\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$. In this case from (19) we have $I_{\mathrm{D}_2} = \tau^2$, $I_{\mathrm{D}_3} = 0$, and considering (72) from (22) we obtain

$$\varepsilon_{11} = \varepsilon_{22} = \frac{\tau^2}{3}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}}(0), \quad \varepsilon_{33} = -\frac{2\tau^2}{3}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}}(0), \quad \varepsilon_{12} = \tau\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_3}}(I_{\mathrm{D}_2}). \tag{74}$$

Homogeneous shear plus compression/tension: In this final case let us consider the same slab described previously, but now deforming due to the application of the homogeneous distribution of stresses $\mathbf{T} = \sigma\mathbf{e}_1 \otimes \mathbf{e}_2 + \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$. In this case from (19) we obtain $I_{\mathrm{D}_2} = \frac{\sigma^2}{3} + \tau^2$, $I_{\mathrm{D}_3} = \frac{\sigma}{3}(\frac{2\sigma^2}{9} + \tau^2)$, and on the other hand (72) from (22) we have

$$\varepsilon_{11} = \frac{1}{3}(\tau^2 - \sigma^2)\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}} - \frac{\sigma}{3}\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_3}}, \quad \varepsilon_{22} = \frac{\tau^2}{3}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}} + \frac{2\sigma}{3}\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_3}}, \tag{75}$$

$$\varepsilon_{33} = -\frac{1}{3}(\sigma^2 + 2\tau^2)\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}} - \frac{\sigma}{3}\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_3}}, \quad \varepsilon_{12} = \tau\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}I_{\mathrm{D}_3}} + \frac{\sigma}{3}\frac{\mathrm{d}\Psi}{\mathrm{d}I_{\mathrm{D}_3}}\right). \tag{76}$$

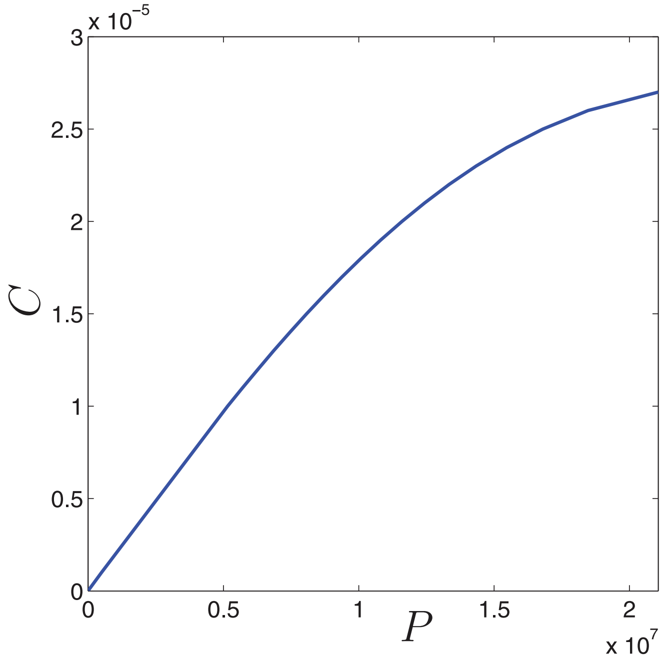

Fig. 2. Behaviour of the constant C [m^2] (see (31) and (38)) as a function of P [Pa].

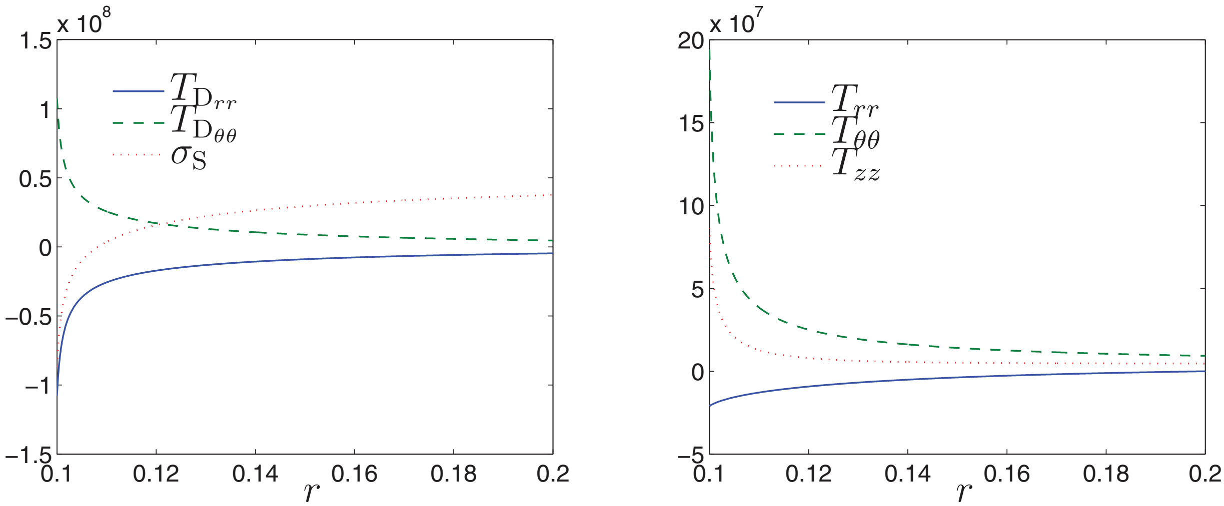

Fig. 3. Behaviour of σ_{S} and the nonzero components of $\mathbf{T}_{\mathrm{D}}$ and $\mathbf{T}$ as functions of r for a specific value for P. The stresses are given in Pa, the radial position is in metres.

Numerical example: Let us consider the following expressions for the functions Γ and Ψ presented in (72)

$$\Gamma(I_{\mathrm{D}_2}) = \frac{\gamma}{\iota}\sqrt{1+3\iota I_{\mathrm{D}_2}}, \quad \Psi = 0, \tag{77}$$

where $\gamma = 10^{-10}\,1/\mathrm{Pa}$, $\iota = 10^{-15}\,1/\mathrm{Pa}^2$. If the above specific expressions for Γ and Ψ are used in (73)–(76) we obtain a strain limiting behaviour for the simple problems presented previously as depicted in Fig. 1.

If (77) is used in (33)–(35), from (38) we can obtain, for example, the behaviour of C as a function of P as presented in Fig. 2 (under the assumption that $\lambda = 1$).

For the maximum magnitude for $P = 2.106 \times 10^7\mathrm{Pa}$ we have $C = 2.7 \times 10^{-5}\mathrm{m}^2$, and in such a case the behaviour of the non-zero components of $\mathbf{T}_{\mathrm{D}}$ and $\mathbf{T}$ and σ_{S} are depicted in Fig. 3.

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