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NECESSARY CONDITIONS FOR TILING FINITELY GENERATED AMENABLE GROUPS

Benjamin Hellouin de Menibus*

Laboratoire de Recherche en Informatique
Université Paris-Sud - CNRS - CentraleSupélec, Université Paris-Saclay, France
https://orcid.org/0000-0001-5194-929X

Hugo Maturana Cornejo

Departamento de Ingeniería Matemática, DIM-CMM
Universidad de Chile, Chile

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Abstract. We consider a set of necessary conditions which are efficient heuristics for deciding when a set of Wang tiles cannot tile a group.

Piantadosi [19] gave a necessary and sufficient condition for the existence of a valid tiling of any free group. This condition is actually necessary for the existence of a valid tiling for an arbitrary finitely generated group.

We consider two other conditions: the first, also given by Piantadosi [19], is a necessary and sufficient condition to decide if a set of Wang tiles gives a strongly periodic tiling of the free group; the second, given by Chazottes et. al. [9], is a necessary condition to decide if a set of Wang tiles gives a tiling of $\mathbb{Z}^2$.

We show that these last two conditions are equivalent. Joining and generalising approaches from both sides, we prove that they are necessary for having a valid tiling of any finitely generated amenable group, confirming a remark of Jeandel [14].

1. Introduction.

$\mathbb{Z}^2$-subshifts of finite type (SFT) are a set of colourings of the 2-dimensional lattice $\mathbb{Z}^2$, or *tilings*, defined by a finite set of local restrictions. There are various equivalent ways to express the restrictions, such as the Wang tiles formalism introduced by Hao Wang [21]. This formalism was introduced to study the *domino problem*: given as input a set of restrictions (e.g. a set of Wang tiles), is there an algorithm that decides whether there is a tiling of $\mathbb{Z}^2$ that respects those restrictions?

R. Berger [7] showed that the domino problem is undecidable. The proof depends heavily on notions of periodicity and aperiodicity, more precisely on the existence of a set of Wang tiles that only tile $\mathbb{Z}^2$ in a strongly aperiodic manner. This is in stark

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* Corresponding author: Benjamin Hellouin de Menibus.

contrast with the situation on $\mathbb{Z}$ where the domino problem is decidable thanks to a graph representation [17].

There has been a recent interest in symbolic dynamics on more general contexts, such as where the lattice $\mathbb{Z}^2$ is replaced by the Cayley graph of an infinite, finitely generated group. Using again the existence of strongly aperiodic SFTs, the domino problem was shown to be undecidable, apart from $\mathbb{Z}^d$, on some semisimple Lie groups [18], the Baumslag-Solitar groups [2], the discrete Heisenberg group (announced, [20]), surface groups [10, 1], semidirect products on $\mathbb{Z}^2$ [6] or some direct products [4], polycyclic groups [13], some hyperbolic groups [11]... It is decidable on free groups [19] and on virtually free groups [3], and it is conjectured that these are the only groups where the domino problem is decidable (Conjecture 1 below).

As a consequence, outside of free and virtually free groups, one can not expect to find simple necessary and sufficient conditions for admitting a valid tiling. However, heuristics can be very useful when making an exhaustive search for SFTs with desired properties; necessary conditions in particular allow fast rejection of most empty SFTs. For example, a transducer-based heuristic was used in the search for the smallest set of Wang tiles that yield a strongly aperiodic $\mathbb{Z}^2$-SFT [15]. It is also of theoretical interest to understand how the group properties impact necessary conditions.

1.1. Statements of results.

We first consider a necessary and sufficient condition introduced by Piantadosi for an SFT on the free group to admit a valid tiling [19]. It is well-known that an SFT on a finitely generated group can only admit a tiling if the "corresponding" SFT on the free group does, so this becomes a necessary condition on an arbitrary f.g. group (Corollary 1).

The next two stronger conditions were introduced by Piantadosi (to decide if an SFT admits a strongly periodic tiling of the free group) and by Chazottes-Gambaudo-Gautero [9] in a more general context of tiling the euclidean plane by polygons, but which is necessary for an SFT to admit a tiling of $\mathbb{Z}^2$ [16]. We prove that the two conditions are equivalent (Theorem 3.7), and that they form a necessary condition for an SFT to admit a valid tiling on any finitely generated amenable group (Theorem 5.3), confirming a remark of Jeandel ([14], Section 3.1).

Finally, we provide for any non-free finitely generated group a counterexample that satisfies all conditions but does not provide a valid tiling.

2. Preliminaries.

2.1. Symbolic dynamics on groups.

In the whole article G is an infinite, finitely generated group with unit element 1_G. We write $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ where $\mathcal{S} = \{g_1, \dots, g_d\}$ is a finite set of generators and $\mathcal{R} = \{r_1, \dots, r_m, \dots\} \subset (\mathcal{S} \cup \mathcal{S}^{-1})^*$ is a (possibly infinite) set of relations. By convention $r \in \mathcal{R}$ means that $r = 1_G$.

For instance:

- the free group $\mathbb{F}_d$ is the group on d generators with no relations;
- $\mathbb{Z}^2 = \langle \{g_1, g_2\} \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$.

Let $\mathcal{A}$ be a finite set endowed with the discrete topology; denote its cardinality $\#\mathcal{A}$. Let $\mathcal{A}^G = \{(x_g)_{g\in G} \mid \forall g \in G : x_g \in \mathcal{A}\}$ be the set of all functions from G to $\mathcal{A}$ endowed with the product topology. Given a finite subset $F \subset G$, an element $P \in \mathcal{A}^F$ is called a *pattern* and $F = \mathrm{supp}(P)$ its *support*; the set of all patterns is denoted $\mathcal{A}^*$.

$\mathcal{A}^G$ is a compact space called the *G-full shift.* It is a symbolic dynamical system under the following G-action, called the *G-shift*:

$$\forall x \in \mathcal{A}^G, \forall h \in G, (\sigma_h(x_g))_{g\in G} = (x_{h^{-1}g})_{g\in G}$$

We call *G-subshift* any closed shift-invariant subset $Y \subset \mathcal{A}^G$.

A pattern $P \in \mathcal{A}^F$ is said to *appear* in a configuration $x \in \mathcal{A}^G$ (and we write $P \sqsubset x$) if there exists $g \in G$ such that $\sigma_g(x)|_F = P$.

Given a set of forbidden patterns $\mathcal{F} \subset \mathcal{A}^*$, we can define the corresponding G-subshift:

$$Y = Y_{\mathcal{F}} = \{x \in \mathcal{A}^G \mid \forall P \sqsubset x : P \notin \mathcal{F}\}.$$

Every G-subshift can be defined in this way using a set of forbidden patterns. When a subshift can be defined by a finite set of forbidden patterns, we say it is a *G-subshift of finite type (G-SFT).* If furthermore the set of forbidden patterns can be chosen so that every pattern in $\mathcal{F}$ has support of the form $\{1_G, g_i\}$ where $g_i \in \mathcal{S}$ for some set of generators $\mathcal{S}$, we say it is a *G-nearest-neighbour subshift of finite type (G-NNSFT).* Notice that this definition depends on the choice of $\mathcal{S}$ which is usually clear in the context.

For example, If we consider $G = \mathbb{Z}$ with generator $+1$, $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \{11\}$ we obtain a $\mathbb{Z}$-NNSFT, the golden mean shift, a classical example in symbolic dynamics.

Definition 2.1 (Weakly & strongly aperiodic)**.** For a configuration $x \in \mathcal{A}^G$, we define the orbit of the element x under the shift action as $\text{orb}_\sigma(x) = \{\sigma_g(x)|g \in G\}$ and the set of elements on G that fix the configuration x by $\text{stab}_\sigma(x) = \{g \in G|\sigma_g(x) = x\}$. A configuration $x \in \mathcal{A}^G$ is

strongly periodic: if $\text{stab}_\sigma(x)$ has finite index or, equivalently, if $\text{orb}_\sigma(x)$ is finite;
strongly aperiodic: if $\text{stab}_\sigma(x) = \{1_G\}$.
weakly periodic: if it is not strongly aperiodic;
weakly aperiodic: if it is not strongly periodic.

More generally, a subshift $X \subset \mathcal{A}^G$ is weakly/strongly aperiodic if every configuration on X is weakly/strongly aperiodic.

Example 1. In $G = \mathbb{Z}^2$,

- the configuration x such that $x_g = 0$ for all g is strongly periodic;
- the configuration x such that $x_{g_1^n} = 0$ for all n, and $x_g = 1$ otherwise, is weakly periodic and weakly aperiodic;
- the configuration x such that $x_{(0,0)} = 0$, and $x_g = 1$ otherwise, is strongly aperiodic.

2.2. **Wang tiles, NNSFT and graphs.**

Definition 2.2 (Wang tiles, Wang subshifts)**.** Let $G = \langle \mathcal{S} \mid \mathcal{R}\rangle$ be a finitely generated group and $\mathcal{C}$ a finite set of colours. A *Wang tile* on $\mathcal{C}$ and $\mathcal{S}$ is a map $\mathcal{S} \cup \mathcal{S}^{-1} \to \mathcal{C}$.

Given a set T of Wang tiles, the corresponding *G-Wang subshift* is defined as:

$$X_T = \{(x_g) \in T^G \mid \forall g \in G, s \in \mathcal{S} \cup \mathcal{S}^{-1}, x_g(s) = x_{gs}(s^{-1})\}.$$

We call the elements in X_T *G-Wang tilings.*

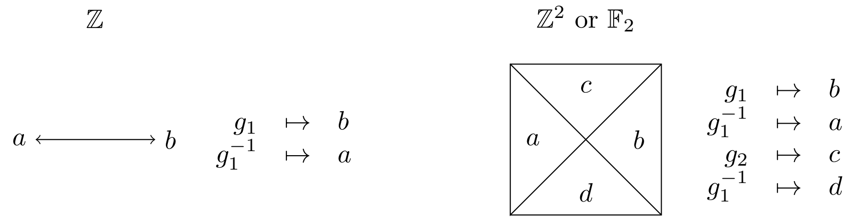


FIGURE 1. Examples of Wang tiles with colours $\mathcal{C} = \{a, b, c, d\}$ on one and two generators, respectively, with their corresponding maps.

Notice that the definition of a Wang tile depends only on the chosen set of generators, so that the same Wang tile can be used for $\mathbb{F}_2$ and $\mathbb{Z}^2$, for example.

Take any G-NNSFT X on the alphabet $\mathcal{A}$, where $G = \langle \{g_1, \ldots, g_d\} \mid \mathcal{R} \rangle$ is an arbitrary finitely generated group. Let $\mathcal{F}$ be a set of forbidden patterns with each support of the form $\{1_G, g_i\}$.

We associate to X a set of d graphs $\Gamma_1, \ldots, \Gamma_d$, where the set of vertices is $\mathcal{A}$ for all Γ_i, and

$$\forall a, b \in \mathcal{A}, \qquad a \to b \text{ in } \Gamma_i \iff \left\{ \begin{array}{ccc} 1_G & \to & a \\ g_i & \to & b \end{array} \right. \notin \mathcal{F}.$$

By definition of a G-NNSFT, it follows that a configuration x belongs to X if, and only if, $x_h \to x_{hg_i}$ is an edge in Γ_i for all $h \in G$ and all $1 \le i \le d$.

Definition 2.3 (Cycles)**.** A *cycle* on a graph Γ is a path - with possible edge and vertex repetitions - that starts and ends on the same vertex. A cycle through the vertices $a_1 \ldots a_n a_1$, with $a_i \in \mathcal{A}$, is denoted $\overline{a_1 \ldots a_n}$.

A cycle is *simple* if it does not contain any vertex repetition. Denote $\mathcal{SC}(\Gamma)$ the set of simple cycles on Γ, which is a finite set.

Remark 1. In graph theory, cycles are sometimes called *closed walks*, in which case cycle means simple cycle. We decided to follow Piantadosi's conventions [19] for convenience.

Let w be a cycle and $a \in \mathcal{A}$. We define:

$$|w|_a = \#\{i \mid w_i = a, 1 \le i \le |w|\}.$$

In any cycle, the path between the closest repetitions is a simple cycle. By removing this simple cycle and iterating the argument, we can see that any cycle w can be decomposed into simple cycles, in the sense that there are integers λ_ω for $\omega \in \mathcal{SC}(\Gamma)$ such that:

$$\forall a \in \mathcal{A}, |w|_a = \sum_{\omega \in \mathcal{SC}(\Gamma)} \lambda_\omega |\omega|_a.$$

We say that two G-subshifts $X, Y \subset \mathcal{A}^G$ are (topologically) *conjugate* if there is a shift-commuting homeomorphism Φ (that is, $\Phi \circ \sigma_g = \sigma_g \circ \Phi$ for all $g \in G$) such that $\Phi(X) = Y$. A shift-commuting homeomorphism (or *conjugacy*) corresponds to a reversible cellular automaton: there is a finite subset $H \subset G$ and a local rule $\varphi : \mathcal{A}^H \to \mathcal{A}$ such that

$$\forall x \in X, \forall g \in G, \ \Phi(x)_g = \varphi(\sigma_{g^{-1}}(x)|_H),$$

and Φ^{-1} is itself a cellular automaton.

Proposition 1. *For any set of generators, each G-SFT is conjugate to a G-NNSFT and each G-NNSFT is conjugate to a G-Wang subshift.*

This is folklore. A detailed proof for the SFT - NNSFT part can be found in [5] (Propositions 1.6 and 1.7), and a proof of the NNSFT - Wang subshift part in [12].

Since the conjugacy from a G-Wang subshift to a G-NNSFT can be chosen letter-to-letter (i.e. $H = \{1_G\}$ in the definition), it is easy to see that the conjugacy does not depend on G, so we could say that a set of graphs and a set of Wang tiles are conjugate.

Proposition 2. *Let X and Y be two conjugate G-subshifts. X admits a valid tiling if and only if Y admits a valid tiling. The same is true for weakly/strongly (a)periodic tilings.*

3. Piantadosi's and Chazottes-Gambaudo-Gautero's conditions.

3.1. State of the art on the free group and $\mathbb{Z}^2$.

The first two conditions were introduced by Piantadosi in the context of symbolic dynamics on the free group $\mathbb{F}_d$.

Definition 3.1 (Condition $(\star)$ [19])**.** A family of graphs $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ whose vertices are an alphabet $\mathcal{A}$ satisfies *condition* $(\star)$ if and only if there is some nonempty $\mathcal{A}' \subset \mathcal{A}$ with a *colouring function* $\Psi : \mathcal{A}' \times \mathcal{S} \to \mathcal{A}'$ such that, for any colour $a \in \mathcal{A}'$ and any generator $g_i \in \mathcal{S}$, $a \to \Psi(a, g_i)$ is an edge in Γ_i.

Theorem 3.2 ([19])**.** *Let X be a $\mathbb{F}_d$-NNSFT on the alphabet $\mathcal{A}$. X is nonempty if and only if the corresponding set of graphs satisfies condition $(\star)$.*

This theorem provides a decision procedure for the domino problem in free groups of any rank: find a subalphabet such that every letter admits a valid neighbour in the subalphabet for every generator.

Definition 3.3 (Condition $(\star\star)$ [19])**.** Consider a family of graphs $\Gamma = \{\Gamma_i\}_{1\leq i\leq d}$ and $\mathcal{SC}(\Gamma_i) = \{\omega_i^j\}_{1\leq j\leq \#\mathcal{SC}(\Gamma_i)}$ the set of simple cycles for each graph Γ_i.

We denote by $(\star\star)$ the following equation on real numbers $x_{i,j}$:

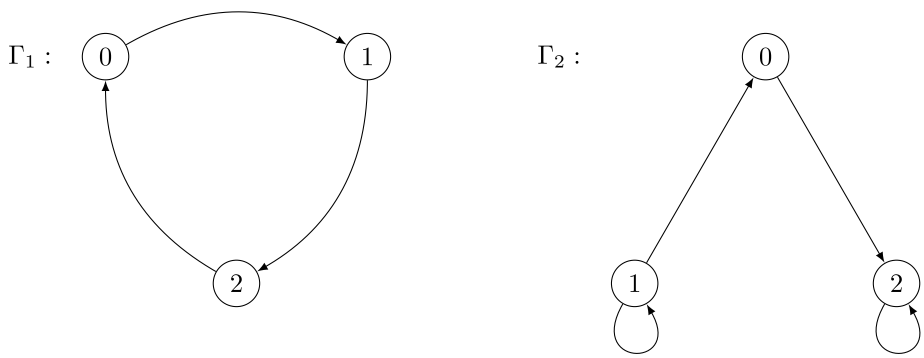
$$\forall a \in \mathcal{A}, \quad \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j}|\omega_1^j|_a = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_2)} x_{2,j}|\omega_2^j|_a = \cdots = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_d)} x_{d,j}|\omega_d^j|_a.$$

We say that the graph family satisfies *condition* $(\star\star)$ if equation $(\star\star)$ is not empty (e.g. all graphs contain at least a cycle) and admits a nontrivial positive solution.

Remark 2. We formulated the previous condition in terms of simple cycles (using the formalism from Theorem 3.6 instead of Theorem 3.4 in [19]) because they form a finite set, making it easier to prove formally when the condition is not satisfied.

Theorem 3.4 ([19], Theorem 3.6)**.** *A $\mathbb{F}_d$-NNSFT contains a strongly periodic configuration if and only the associated family of graphs satisfies condition $(\star\star)$.*

Example 2. We illustrate Piantadosi's conditions on the following example:



The corresponding $\mathbb{F}_2$-NNSFT admits a tiling, because it satisfies condition ($\star$) on the alphabet $\mathcal{A}' = \mathcal{A}$. However, it does not admit a periodic tiling: the simple cycles of Γ_1 are (up to shifting) $\{\overline{012}\}$ and the simple cycles of Γ_2 are $\{\overline{1}, \overline{2}\}$, so Equation ($\star\star$) is:

$$x_{1,1} = 0 \qquad (a = 0)$$
$$x_{1,1} = x_{2,1} \qquad (a = 1)$$
$$x_{1,1} = x_{2,2} \qquad (a = 2)$$

which obviously doesn't admit a nontrivial solution. As we will see later, the corresponding $\mathbb{Z}^2$-NNSFT doesn't admit any tiling.

Remark 3. For example, if all graphs Γ_i share a common cycle w (say $\omega_i^1 = w$ for all graphs Γ_i), then condition ($\star\star$) admits a solution: for all i, $x_{i,1} = 1$ and $x_{i,j} = 0$ when $j \neq 1$. Therefore the corresponding $\mathbb{F}_d$-NNSFT contains a periodic configuration.

Definition 3.5 (Condition $(\star\star)'$ [9])**.** Let T be a set of Wang tiles on colours $\mathcal{C}$ and set of generators $\mathcal{S}$. For each $g \in \mathcal{S} \cup \mathcal{S}^{-1}$ and each colour $c \in \mathcal{C}$, define c_g the subset of Wang tiles $\tau_i \in T$ such that $\tau_i(g) = c$. We call $(\star\star)'$ the following equation:

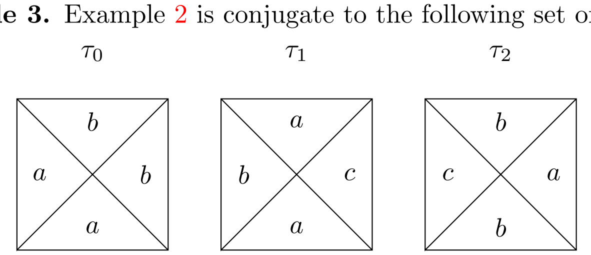
$$\forall g \in \mathcal{S}, \forall c \in \mathcal{C}, \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j.$$

We say that T satisfies *condition* $(\star\star)'$ if Equation $(\star\star)'$ admits a positive nontrivial solution.

Theorem 3.6 ([9])**.** *If a set T of Wang tiles admits a valid tiling of $\mathbb{Z}^2$, then it satisfies condition $(\star\star)'$.*

This condition and result were introduced in [9], but a much easier presentation in our context is given in [16].

Example 3. Example 2 is conjugate to the following set of Wang tiles.



$0 \mapsto \tau_0$
$1 \mapsto \tau_1$
$2 \mapsto \tau_2$

Equation $(\star\star)'$ becomes the following, where next to each equation is the corresponding generator and colour.

(g_1, a)	$x_2 = x_0$	(g_2, a)	$x_1 = x_0 + x_1$
(g_1, b)	$x_0 = x_1$	(g_2, b)	$x_0 + x_2 = x_2$
(g_1, c)	$x_1 = x_2$	(g_2, c)	$0 = 0$

This equation does not admit a positive nontrivial solution, so the corresponding $\mathbb{Z}^2$-Wang subshift is empty.

3.2. **Conditions $(\star\star)$ and $(\star\star)'$ are equivalent.** Although conditions $(\star\star)$ and $(\star\star)'$ were introduced in very different contexts (periodic tilings of the free group and tilings of the Euclidean plane, respectively), it turns out that they are equivalent. The fact that $(\star\star)$ is a condition on graphs (NNSFTs) and $(\star\star)'$ is a condition on sets of Wang tiles (Wang subshifts) is only cosmetic since Proposition 1 lets us go from one model to the other.

Theorem 3.7. *Let T be a set of Wang tiles over the set of colours $\mathcal{C}$ and the set of generators $\mathcal{S}$.*

T satisfies condition $(\star\star)'$ if, and only if, the associated graphs satisfy condition $(\star\star)$.

Proof. ($\Leftarrow$) Let $(x_{i,j})$ be a nonnegative solution to equation $(\star\star)$. For every tile τ_i, put $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j} |\omega_1^j|_{\tau_i}$.

Because each simple cycle of Γ_1 is a cycle, it contains as many tiles in c_{g_1} as in $c_{g_1^{-1}}$; that is, $\sum_{\tau_i \in c_{g_1}} |\omega_1^j|_{\tau_i} = \sum_{\tau_j \in c_{g_1^{-1}}} |\omega_1^j|_{\tau_i}$. Summing over all simple cycles ω_1^j, we get $\sum_{\tau_i \in c_{g_1}} x_i = \sum_{\tau_i \in c_{g_1^{-1}}} x_j$.

Since $(x_{i,j})$ is a solution to Equation $(\star\star)$, we also have $x_i = \sum_{j=1}^{\#\mathcal{SC}(\Gamma_n)} x_{n,j} |\omega_n^j|_{\tau_i}$ for every n, so the same argument shows that (x_i) is a nonnegative solution of equation $(\star\star)'$.

($\Rightarrow$) Because equation $(\star\star)'$ admits a solution, it admits a rational solution, and therefore an integer solution. Let (x_i) be an integer, nonnegative solution of equation $(\star\star)'$.

For the generator g_1, consider the graph Γ_1 obtained by the letter-to-letter conjugacy of Proposition 1: concretely, it is the graph on vertices $\{\tau_i\}_{1 \leq i \leq n}$ with $\tau_i \to \tau_j \Leftrightarrow \exists c \in \mathcal{C}, \tau_i \in c_{g_1}$ and $\tau_j \in c_{g_1^{-1}}$.

We define an auxiliary graph G_1 on vertices $\{\tau_i^k\}_{1 \leq i \leq n, 1 \leq k \leq x_i}$ (that is, x_i copies for each tile τ_i) as follows.

Because

$$\forall c \in \mathcal{C}, \sum_{\tau_i \in c_{g_1}} x_i = \sum_{\tau_j \in c_{g_1^{-1}}} x_j,$$

we can fix an arbitrary bijection

$$\Psi_1^c : \{\tau_i^k \,:\, \tau_i \in c_{g_1}, 1 \leq k \leq x_i\} \to \{\tau_{i'}^{k'} \,:\, \tau_{i'} \in c_{g_1^{-1}}, 1 \leq k' \leq x_{i'}\},$$

and put an edge $\tau_i^k \to \tau_{i'}^{k'}$ if and only if $\Psi_1^c(\tau_i^k) = \tau_{i'}^{k'}$ for some $c \in \mathcal{C}$. Because each vertex has indegree and outdegree 1, it is a (not necessarily connected) Eulerian graph and admits a finite set of cycles covering every vertex exactly once.

Notice that by construction, if G_1 has an edge $\tau_i^k \to \tau_{i'}^{k'}$, then Γ_1 has an edge $\tau_i \to \tau_{i'}$. Therefore each cycle of G_1 can be sent on a cycle in Γ_1 through the projection $\tau_i^k \mapsto \tau_i$. In this way, project the finite set of cycles obtained above and decompose them into simple cycles of Γ_1. Denote $x_{1,j}$ the total number of each simple cycle ω_1^j obtained in this way.

Because each tile τ_i was present in G_1 as a vertex in x_i copies, we have for every i: $\sum_{j=1}^{\#\mathcal{SC}(\Gamma_1)} x_{1,j}|\omega_1^j|_{\tau_i} = x_i$.

Now apply the same argument for each generator $g_2, \ldots, g_n$ and the variables $(x_{i,j})$ thus obtained are a solution to equation ($\star\star$). □

4. Necessary conditions for tiling arbitrary groups.

Since the above conditions apply on sets of Wang tiles or set of graphs, they actually are conditions on a family of G-SFT where G ranges over all groups with a fixed number of generators. The following proposition relates the properties of these SFT. It can be found (under a different form) in [8] (Proposition 10 and remark below)

Proposition 3. *Let $G_1 = \langle\{g_1, \ldots, g_d\}|\mathcal{R}\rangle$, $G_2 = \langle\{g_1, \ldots, g_d\}|\mathcal{R}'\rangle$ be finitely generated groups, with $\mathcal{R}' \subset \mathcal{R}$. Consider the canonical surjective morphism $\pi : G_2 \to G_1$ defined by $\pi(g_i) = g_i$, $\forall 1 \le i \le d$. Let $\Phi : \mathcal{A}^{G_1} \to \mathcal{A}^{G_2}$ be defined by $\Phi(x)_g = x_{\pi(g)}$. Let X_1 and X_2 be the corresponding G_1-NNSFT and G_2-NNSFT respectively, such that X_2 has the same local rules as X_1.*

We have:

1. *If x is a valid tiling for X_1 then $\Phi(x)$ is a valid tiling for X_2.*
2. *If x is weakly periodic then $\Phi(x)$ is weakly periodic. In particular, if X_1 admits a weakly periodic tiling, then X_2 admits a weakly periodic tiling.*
3. *If x is weakly aperiodic then $\Phi(x)$ is weakly aperiodic. In particular, if X_1 admits a weakly aperiodic tiling, then X_2 admits a weakly aperiodic tiling.*

The strong properties are not preserved by Φ, but of course the image of a strongly (a)periodic tiling remains weakly (a)periodic. Stronger versions with different hypotheses can be found in [8, 14].

Proof.

1. Since X_2 is an NNSFT, it is enough to check that, for all $h \in G_2$ and all $1 \le i \le d$, $\Phi(x)_h \to \Phi(x)_{hg_i}$ is an edge in Γ_i, that is to say, that it is not a forbidden pattern for X_2. By definition of Φ, $\Phi(x)_h = x_{\pi(h)}$ and $\Phi(x)_{hg_i} = x_{\pi(h)\pi(g_i)} = x_{\pi(h)g_i}$. Because x is a valid tiling for X_1, we have that $x_{\pi(h)} \to x_{\pi(h)g_i}$ is an edge in Γ_i, which proves the result.
2. If x is a weakly periodic tiling in X_1, then $\mathrm{stab}_\sigma(x)$ is nontrivial by definition. We have:
$$\begin{aligned}\mathrm{stab}_\sigma(\Phi(x)) &= \{g \in G_2 \,:\, \forall h \in G_2, \Phi(x)_{hg} = \Phi(x)_h\} \\ &= \{g \in G_2 \,:\, \forall h \in G_2, x_{\pi(h)\pi(g)} = x_{\pi(h)}\}.\end{aligned}$$
Since π is surjective, this means that $\pi(\mathrm{stab}_\sigma(\Phi(x))) = \mathrm{stab}_\sigma(x)$. $\mathrm{stab}_\sigma(x)$ is nontrivial so $\mathrm{stab}_\sigma(\Phi(x)) = \pi^{-1}(\mathrm{stab}_\sigma(x))$ is nontrivial as well.
3. If x is a weakly aperiodic tiling in X_1, then $\mathrm{stab}_\sigma(x)$ does not have finite index. The canonical morphism $\pi : G_2 \to G_1$ yields a morphism on the quotient:
$$\tilde{\pi} : G_2/\pi^{-1}(\mathrm{stab}_\sigma(x)) \to G_1/\,\mathrm{stab}_\sigma(x),$$
and $\tilde{\pi}$ is surjective since π is surjective. Remember that $\mathrm{stab}_\sigma(\Phi(x)) = \pi^{-1}(\mathrm{stab}_\sigma(x))$ by the previous point. Since $\mathrm{stab}_\sigma(x)$ does not have finite

index, $G_1/\operatorname{stab}_\sigma(x)$ is infinite, so $G_2/\pi^{-1}(\operatorname{stab}_\sigma(x))$ is infinite as well, and $\operatorname{stab}_\sigma(\Phi(x)) = \pi^{-1}(\operatorname{stab}_\sigma(x))$ does not have finite index.

□

Remark 4. In the last proposition, the converse of the point (1) does not hold. For instance, consider $G = \mathbb{Z}^2 = \langle g_1, g_2 \mid g_1g_2g_1^{-1}g_2^{-1}\rangle$. Example 2 provided an example of a set of graphs that satisfies condition $(\star)$ (so the corresponding $\mathbb{F}_2$-NNSFT admits a valid tiling) but does not satisfy condition $(\star\star)$ (so the corresponding $\mathbb{Z}^2$-NNSFT does not admit any valid tiling).

To understand why, notice that $ker(\pi)$ contains $g_1g_2g_1^{-1}g_2^{-1}$, so if a tiling $x \in \mathcal{A}^{\mathbb{F}_2}$ is such that $x_{1_{\mathbb{F}_2}} \neq x_{g_1g_2g_1^{-1}g_2^{-1}}$, then $\Phi^{-1}(x) = \emptyset$. If this happens for all $x \in X_2$ then X_1 is empty.

Corollary 1. *Let $\Gamma_1, \ldots, \Gamma_d$ be a set of graphs that does not satisfy condition $(\star)$. Then the corresponding G-NNSFT is empty for an arbitrary group G with d generators.*

Proof. If there was a valid tiling in $G = \langle g_1, \ldots, g_d \mid \mathcal{R}\rangle$ then, applying Proposition 3, we would obtain a tiling on $\mathbb{F}_d = \langle g_1, \ldots, g_d \mid \emptyset\rangle$, which is in contradiction with Theorem 3.2. □

5. Necessary conditions for tiling amenable groups.

Definition 5.1 (Følner sequence)**.** Let G be a finitely generated group. A *Følner sequence* for G is a sequence of finite subsets $S_n \subset G$ such that:

$$G = \bigcup_n S_n \quad \text{and} \quad \forall g \in G, \frac{\#(S_n g \triangle S_n)}{\# S_n} \xrightarrow[n\to\infty]{} 0,$$

where $S_n g = \{hg \ : \ h \in S_n\}$ and $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference.

In the previous definition, it is easy to see that the second condition only has to be checked for g in a finite generating set. The set $S_n g \triangle S_n$ can be understood as the border of S_n, so an element of a Følner sequence must have a small border relative to its interior.

Definition 5.2 (Amenable group)**.** A finitely generated group G is *amenable* if it admits a Følner sequence.

This definition applies more generally for all countable groups. A few examples:

- $\mathbb{Z}^d$ is amenable and a Følner sequence is given by $S_n = [-n, n]^d$. Indeed, if $(g_i)_{1\le i\le d}$ is the canonical set of generators, then $\#S_n = (2n+1)^d$ and $\#((S_n + g_i)\triangle S_n) = 2\cdot(2n+1)^{d-1}$.
- $\mathbb{F}_d$ for $d \geq 2$ is not amenable. In particular, the balls S_n of radius n - that is, reduced[1] words of length $\leq n$ on the set of generators $(g_i)_{1\le i\le d}$ - are not a Følner sequence. Indeed, one can easily check that $\#S_n = \Omega(d^n)$ and $\#(S_n g_i \triangle S_n) = \Omega(d^n)$.

Theorem 5.3 (Heuristic for tiling an amenable group)**.** *Let G be a finitely generated amenable group, $\mathcal{S}$ a finite set of generators, and T a set of Wang tiles.*

If there is a tiling of G with the tiles T, then condition $(\star\star)$ (or equivalently $(\star\star)'$) is satisfied.

[1] with no $g_i^{-1}g_i$ or $g_ig_i^{-1}$ factors

This results confirms a remark by Jeandel in [14], Section 3.1.

Proof. Let $x \in T^G$ be a tiling of G and S_n be a Følner sequence for G. Using notations from Definition 3.3, for a colour $c \in \mathcal{C}$ and a generator $g \in \mathcal{S}$, c_g is the set of tiles τ such that $\tau(g) = c$.

For any $h \in S_n \cap S_n g^{-1}$, we have $x_h \in c_g \Leftrightarrow x_{hg} \in c_{g^{-1}}$ (and in this case, $hg \in S_n \cap S_n g$). This means that, for all $c \in \mathcal{C}, g \in \mathcal{S}$ and $n \in \mathbb{N}$:

$$\#\{h \in S_n \cap S_n g^{-1} \ : \ x_h \in c_g\} = \#\{h \in S_n \cap S_n g \ : \ x_h \in c_{g^{-1}}\},$$

so in particular

$$|\#\{h \in S_n \ : \ x_h \in c_g\} - \#\{h \in S_n \ : \ x_h \in c_{g^{-1}}\}| \leq \#(S_n g \triangle S_n) + \#(S_n g^{-1} \triangle S_n).$$

For each tile τ_i, let $x_i^n = \frac{\#\{h \in S_n \ : \ x_h = \tau_i\}}{\#S_n}$. The previous computation implies that:

$$\forall g \in \mathcal{S},\ \forall c \in \mathcal{C},\ \left|\sum_{\tau_i \in c_g} x_i^n - \sum_{\tau_j \in c_{g^{-1}}} x_j^n\right| \leq \frac{\#(S_n g \triangle S_n)}{\#S_n} + \frac{\#(S_n g^{-1} \triangle S_n)}{\#S_n}.$$

Notice that the right-hand side tends to 0 as n tends to infinity by definition of a Følner sequence. Consider the sequence of vectors $((x_i^n)_i)_{n \in \mathbb{N}}$ and, by compactness, let (x_i) be any limit point of this sequence. Since $\sum_i x_i^n = 1$ for all n by definition, $\sum_i x_i = 1$ as well, and we have

$$\forall g \in \mathcal{S},\ \forall c \in \mathcal{C},\ \sum_{\tau_i \in c_g} x_i = \sum_{\tau_j \in c_{g^{-1}}} x_j,$$

so (x_i) is a nontrivial solution to Equation $(\star\star)$. Condition $(\star\star)'$ follows by Theorem 3.7. ∎

6. Counterexamples.

It is clear that none of the $(\star)$, $(\star\star)$ or $(\star\star)'$ conditions can be a sufficient condition to admit a $\mathbb{Z}^d$-tiling, since it would be a decision procedure for the Domino problem; this argument applies to any group where the Domino problem is undecidable. For completeness, we provide explicit counterexamples for any non-free finitely generated group.

Theorem 6.1. *Let G be an arbitrary finitely generated group. If G is not free, then there exists a Wang tile set that satisfies the three conditions $(\star)$, $(\star\star)$ and $(\star\star)'$ and such that the corresponding G-Wang subshift is empty.*

Proof. Write $G = \langle g_1, \dots, g_d \mid \mathcal{R}\rangle$, and take $r_1 : w_1 \dots w_n \in \mathcal{R}$, with $w_1 \dots w_n$ a reduced word on generators $g_1 \dots g_d$ (no generator is next to its inverse).

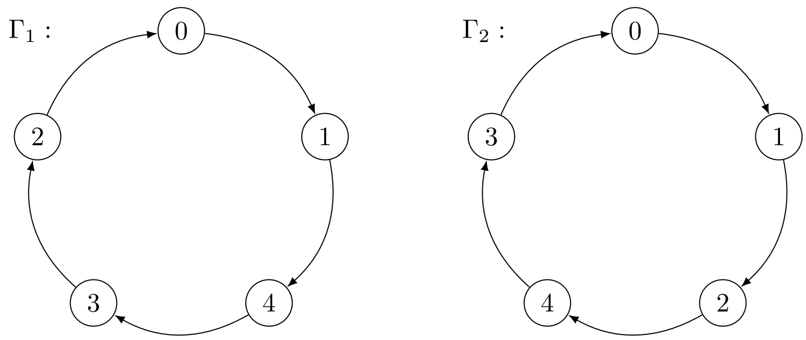
We build a family of graphs Γ_d on vertices $\{0, \dots, n\}$ with the following edges:

$$\forall i \leq n, \begin{cases} \text{if } w_i = g_j, \text{ then } \Gamma_j \text{ has an edge } i-1 \to i; \\ \text{if } w_i = g_j^{-1}, \text{ then } \Gamma_j \text{ has an edge } i \to i-1. \end{cases}$$

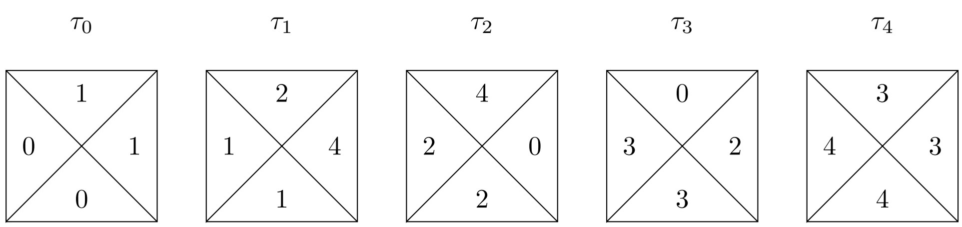
Notice that every vertex has indegree and outdegree at most 1 and we did not create any cycle in the process, so we can complete every Γ_j to be isomorphic to a n-cycle graph C_n.

Now we define a set of $n+1$ Wang tiles on $n+1$ colours $\{0, \dots, n\}$ as follows. Tile τ_i has the following colours: for all j, $g_j^{-1} \to i$ and $g_j \to k$ if there is an edge $\tau_i \to \tau_k$ in Γ_j.

Example 4. For $\mathbb{Z}^2$, we have $r_1 : g_1 g_2 g_1^{-1} g_2^{-1} = 1$. Therefore Γ_1 contains $0 \to 1$ and $3 \to 2$, and Γ_2 contains $1 \to 2$ and $4 \to 3$. One possible completion for Γ_1 and Γ_2 is the following:



The corresponding G-NNSFT is conjugate to the G-Wang subshift defined by the following tiles through the rewriting $i \leftrightarrow \tau_i$:



This tiling satisfies condition $(\star\star)'$ since we can assign the same weight $\frac{1}{n}$ to each tile.

It is clear that a tiling x of G using tiles $\tau_0, \ldots, \tau_n$ must contain every tile. Assume w.l.o.g that $x_1 = \tau_0$. By construction we must have $x_{w_1} = \tau_1$, $x_{w_1 w_2} = \tau_2$, and by an easy induction $x_w = \tau_n$. But since $w = 1$ in G, we have $\tau_0 = x_1 = x_w = \tau_n$, a contradiction. Therefore there is no tiling of G using tiles $\tau_0, \ldots, \tau_n$. □

7. Conclusion.

We would like to mention the two following conjectures that relate the fact of admitting a valid (periodic) tiling and the underlying group structure:

Conjecture 1 ([3]). *A finitely generated group has a decidable domino problem if and only if it is virtually free.*

Conjecture 2 ([8]). *A finitely generated group has an SFT with no strongly periodic point if and only if it is not virtually cyclic.*

In both cases, the "if" direction is proven and the "only if" direction is open.

If Conjecture 1 holds, every infinite amenable groupe has an undecidable domino problem. We ask whether the domino problem could be decidable when considering all amenable groups "at the same time", with a decision procedure given by Conditions $(\star\star)$ and $(\star\star)'$.

Problem. *Is there a set of Wang tiles that satisfies condition $(\star\star)'$ but that does not tile any (infinite) amenable group?*

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E-mail address: hellouin@lri.fr
E-mail address: hmaturana@dim.uchile.cl