ON DWELL PROBABILITIES, RESONANCES AND LIFETIMES

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"Me alejo con espanto y horror de la triste maldad de las funciones que no tienen derivada".

Charles Hermite

Biografía

Camilo nació en la ciudad de Concepción, Chile, el año 1987. Realizó sus estudios básicos y medios en el Colegio San Agustín e ingresó el año 2005 a la carrera de Licenciatura en Matemáticas en la Universidad de Concepción. El año 2013 obtiene el grado de Magíster en Matemáticas en la misma institución con la tesis titulada *Medidas e Integración No-Arquimedeana* bajo la dirección del Dr. José Aguayo. Posteriormente ingresa al programa de Doctorado en Ciencias, mención en Matemáticas de la Universidad de Chile. Luego de aprobar los exámenes de calificación comenzó su especialización en el área de la Física Matemática bajo la dirección del Dr. Claudio Fernández y el Dr. Olivier Bourget, específicamente en modelamientos matemáticos para fenómenos de la Mecánica Cuántica. La presente tesis es el fruto de dicha investigación.

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Contents

Resumen

En esta tesis presentamos dos modelos de fenómenos cuánticos donde estudiamos comportamientos resonantes. El primero corresponde a una partícula atrapada por una barrera definida por una Delta de Dirac en el semieje positivo. El segundo modelo corresponde a una perturbación regular de un valor propio sumergido en un espectro absolutamente continuo. Es este modelo además mostramos estimaciones para el Dwell Time y su relación con el Sojourn Time.

Abstract

In this thesis we present two quantum models where we study resonant behaviours. The first corresponds to a particle trapped by a delta barrier in the positive real line and the second to a regular perturbation of an eigenvalue embedded in some continuous spectrum. In the later model, we also show estimates for the Dwell Time and its relation to the Sojourn Time.

Chapter 1

Introduction

In Quantum Mechanics, the *observables* are the measurable quantities. Mathematically they are represented by self-adjoint operators on a Hilbert Space $\mathcal{H}$. In this work, the *pure state*, or just states, are the unitary vector of $\mathcal{H}$ and when the eigenvalues of the operator exist, they correspond to the possible outcomes of measuring the observable. The expected value of an observable A in a certain state φ is the real number $\langle \varphi, A\varphi \rangle$.

Another ingredient is the dynamics of the system. This is defined by its time evolution, which is written as

$$\varphi(t) = U(t)\varphi(0)$$

where $(U(t))_{t\in\mathbb{R}}$ is a family of unitary operators. Moreover, it is required that this family is a group in t ($U(0) = Id$, $U(t+s) = U(t)U(s)$) and that it is strongly continuous. By the Stone Theorem, there exist a unique possibly unbounded operator H that is self-adjoint on its domain, such that $U(t) = e^{-iHt}$. The operator H is known as *Hamiltonian* or the generator of the group, and it is given by

$$H\varphi = \lim_{t\to 0} \frac{i}{t}(U(t)\varphi - \varphi).$$

In other words, if $\varphi(0)$ belongs to the domain of H, then $\varphi(t)$ solves the *Schrödinger Equation*

$$i\frac{\partial}{\partial t}\varphi = H\varphi$$

and Stone Theorem assures that

$$\varphi(t) = e^{-iHt}\varphi(0).$$

Generally speaking, the operator H represents the total amount of energy of the system. Historically, Quantum Mechanics was formulated on the Hilbert space $L^2(\mathbb{R}^n)$ and the usual Hamiltonian had the form $H = -\Delta + V$ where $-\Delta$ is the self-adjoint realization of the negative Laplacian Operator (kinetic energy) and $V(x)$ a real valued suitable function defined on $\mathbb{R}^n$ (potential energy).

All this entails that the quantity

$$\langle e^{-iHt}\varphi, Ae^{-iHt}\varphi\rangle$$

represents the expected value of A at time t given the initial state φ. In particular, if P is a orthogonal projection, then

$$\langle e^{-iHt}\varphi, Pe^{-iHt}\varphi\rangle = ||Pe^{-iHt}\varphi||^2 \tag{1.1}$$

corresponds to the probability of finding the system at time t in the range of P, with initial state $\varphi \in \mathcal{H}$. In the present thesis, two cases are considered:

i) $P = |\varphi\rangle\langle\varphi|$, orthogonal projection onto the one dimensional subspace generated by φ. Thus,

$$\langle e^{-iHt}\varphi, Pe^{-iHt}\varphi\rangle = |\langle\varphi, e^{-iHt}\varphi\rangle|^2, \tag{1.2}$$

which measures the probability that the particle, initially in state φ, remains in such a state at time t. This number is known as the *survival probability.* Integrating over t we obtain the total amount of time that the system stays in its initial state. That value is known as *Sojourn Time.*

$$\mathfrak{S}_H(\varphi) = \int_{\mathbb{R}} |\langle\varphi, e^{-iHt}\varphi\rangle|^2 dt \tag{1.3}$$

ii) For $\Omega \subset \mathbb{R}^n$, $\mathcal{H} = L^2(\mathbb{R}^n)$, and $P = \chi_\Omega$ (characteristic function of Ω) it holds

$$\langle e^{-iHt}\varphi, \chi_\Omega e^{-iHt}\varphi\rangle = ||\chi_\Omega e^{-iHt}\varphi||^2 = \int_B |e^{-iHt}\varphi(x)|^2 dx.$$

This quantity can be thought as the probability that the particle lives in the region Ω at time t given the initial state φ. The *Dwell Time* is then defined as the total amount of time that the particle lives in region B given the initial state φ. This is

$$\mathfrak{D}_H(\varphi) := \int_{\mathbb{R}} ||\chi_\Omega e^{-iHt}\varphi||^2 dt. \tag{1.4}$$

In case (i) when a Hamiltonian H_0 has an eigenvalue λ_0 with associated eigenvector ϕ_0, then the survival probability is identically 1:

$$\langle\varphi_0, e^{-iH_0t}\varphi_0\rangle = e^{-i\lambda_0 t}$$

which implies

$$|\langle\varphi_0, e^{-iH_0t}\varphi_0\rangle|^2 = |\langle\varphi_0, e^{-i\lambda_0 t}\varphi_0\rangle|^2 = 1$$

In other words, a quantum system which starts in an eigenstate remains in the same state forever, and the associated Sojourn Time is infinity.

In case (ii), if $supp\{\varphi\} \subset \Omega$ then

$$||\chi_\Omega e^{-iHt}\varphi||^2 = \sup\{|\langle\phi, e^{-iHt}\varphi\rangle|^2 : ||\phi|| = 1, supp\{\phi\} \subset \Omega\} \tag{1.5}$$

then the survival probability (1.2) plays an important role in the value of this supremum. In this work, we will show that in many models this diagonal term is the main contribution to the quantity $||\chi_\Omega e^{-iHt}\varphi||^2$.

The concept of *quantum resonance* is a very important notion with a lot of relevance in Physics. Several mathematical approaches have been formulated in order to define this phenomenon but none encloses all its physical meaning. Important attempts can be seen in [8] and [18], each approach captures different properties that make it more suitable according to the settings. This concept tries to capture the behavior of a particle spending a long time in a suitable state. Here we present two of the most prolific approaches, both contained in [18]:

- The quantum resonances of a Schrödinger operator H associated with a dense set of vector $\mathcal{D}$ in the Hilbert $\mathcal{H}$ are defined as the poles of some meromorphic continuation of $\langle f, (H-z)^{-1}g\rangle$, $f, g \in \mathcal{D}$, from $\mathbb{C}^+$ to $\overline{\mathbb{C}^-}$. See [25].
- A resonance energy is a complex number $\lambda - i\Gamma$ with $\Gamma > 0$ if $|\langle\varphi, e^{-iHt}\varphi\rangle|^2$ behaves approximately as $|e^{-i(\lambda-i\Gamma)t}|^2 = e^{-2\Gamma t}$ for some state $\varphi \in \mathcal{H}$.

The first definition is mathematically more precise while the second relates directly with the system's dynamics, which will be the point of view of the present thesis. Note that that exact exponential decay is impossible for many models [18]. As a corollary of the dynamic definition the manifestation of a resonant behavior is a large Sojourn Time and as we show in this work a large Dwell Time.

The Sojourn Time 1.3 has been studied in many setting as a tool for finding resonances as it is large but finite when it is associated to resonant states. See [4], [8], [12], [13]. The purpose of this thesis is to show that the Dwell Time can also be used as an instrument to study resonant behaviours. In particular we show one general model where both quantities are asymptotically similar under resonant regimes. Furthermore, in this case the Dwell Time is an upper bound for the Sojourn Time, but mostly concentrating on it, in the sense that all the singularities of the Dwell Time are also present in the Sojourn Time.

Let us describe now the contests of this thesis: we present two perturbative models. In the first the perturbation is singular but explicit. In the second model the perturbation is regular and encompasses a pretty wide family of Hamiltonians.

In Chapter 2 we consider the singular model which corresponds to a particle trapped by a delta barrier in one dimension. Its time evolution is governed, formally, by the Hamiltonian

$$H_\omega = -\Delta + \omega|\delta_a >< \delta_a|$$

in $L^2[0,\infty)$ with Dirichlet Boundary conditions at $x = 0$ where $|\delta_a >< \delta_a|(\psi) = \psi(a)\delta_a$. This is a self-adjoint operator from $\mathcal{H}^1 \subset L^2(\mathbb{R}^+)$ to $\mathcal{H}^{-1}$ with absolutely continuous spectrum for $\omega > 0$. The operator H_ω reads the motion of a quantum particle through the positive half-line with a delta-barrier at $x = a$ of strength ω. Intuitively if $\omega \to \infty$, the barrier breaks the positive half-line at $x = a$ in two disjoints regions, trapping the particle in the region $[0, a]$. So, for $\omega < +\infty$ large enough we expect to find a resonance. We use the dynamic definition of it in order to modelate the almost exponential decay of the Survival

Probability Amplitude 1.2. In the Section 2.4, we show the existence of resonance energies using the analytic definition as poles of the meromorphic continuation of the resolvent from the upper half-plane to the lower half-plane. This problem is based on the research of Fernández, Prado and Palma [13] in which we find out an error of calculus in the estimate of the remainder term in Theorem 5.8 where they exhibit a resonant state. Our Theorem 4 corrects that estimate and extends that result.

In Chapter 3 we review a regular perturbation case. Consider a Hamiltonian

$$H_\beta = H_0 + \beta V$$

where H_0 has an eigenvalue ψ embedded in some absolutely continuous spectrum. In order to get nice propagation properties under the βV perturbation we ask for the existence of multiple commutators and the so-called Mourre estimate [22] which implies the Limiting Absorption Principle. This tool give us the necessary regularity for our calculus. In Section 3.3, we show in Theorem 8 that the localized survival probability amplitude $\langle \varphi, Qe^{-iH_\beta t} g(H_\beta)\psi\rangle$, for Q some projector, admits the representation

$$\langle \varphi, Qe^{-iH_\beta t} g(H_\beta)\psi\rangle = (1+\beta)e^{-iz_\beta t} + \beta^3$$

up some constants. Moreover, the Sojourn Time and the Dwell Time are asymptotically of the same order with respect to β as it is stated in Theorem 9.

Chapter 2

Perturbation by a delta potential

In this Chapter we will study the motion of a particle moving through the positive real line trapped by a wall at the origin and a singular barrier given by a Dirac delta at $x = a$, $a \neq 0$. The energy of the system is represented by the Delta Hamiltonian $-\Delta + \omega|\delta_a\rangle\langle\delta_a|$. We define it as a distribution-valued operator H_ω. In Section 2.2 we introduce an analytic expression for the resolvent of the perturbed Hamiltonian proved in ([13]). The first usage of this formula is to show that the Schrödinger operator H_ω converges to an operator H_∞ in the strong resolvent sense, where H_∞ acts as $-\Delta \oplus -\Delta$ on $L^2(0,a) \oplus L^2(a,\infty)$. This limiting operator has eigenvalues ψ_m supported on $(0,a)$ for each $m \in \mathbb{N}$. The main result is Theorem 4 where we show that for ω big enough, the probability amplitude $\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle$ admits a quasi-exponential representation plus an error term of order $1/\omega$. Moreover, this quantity is negligible for $n \neq m$ being the diagonal term the dominant one. Two essential facts in this proof are the following: with the formula given in Theorem 3, the Resolvent Operator can be analytically continued to the positive real line from above. The second is that the set $\{\psi_n\}$ of eigenvectors is an orthonormal basis for $L^2(0,a)$. This will be specially important in order to prove Corollary 5 using Parseval's Identity.

2.1 Delta Operator

We shall use the notation $C_c^1(\mathbb{R}_+)$ for the space of the once differentiable functions with continuous derivatives with compact support contained in $\mathbb{R}_+$ and $\mathcal{H}^1$ the Sobolev space consisting of all the clases of functions $f \in L^2(\mathbb{R}_+)$ for which there exist $g \in L^2(\mathbb{R}_+)$ such that $\int_0^\infty f\varphi' = -\int_0^\infty g\varphi$ for all $\varphi \in C_c^1(\mathbb{R}_+)$. $\mathcal{H}^1$ is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}^1} = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}.$$

Let $\mathcal{H}_0^1$ be the closure of $C_c^1(\mathbb{R}_+)$ in $\mathcal{H}^1$ and $\mathcal{H}^{-1}$ the dual space of $\mathcal{H}_0^1$. We have the embedings $\mathcal{H}_0^1 \subset L^2(\mathbb{R}_+) \subset \mathcal{H}^{-1}$ which are continuous with dense ranges. We also recall that for every $f \in \mathcal{H}_0^1$ there is a continuous function φ (in the equivalence class of f) such that $f = \varphi$ a.e..

Definition 1. *Given $a \in \mathbb{R}_+$, we define the functional ν_a as $\nu_a(\varphi) = \varphi(a)$, with domain $D(\nu_a) = \mathcal{H}_0^1$ in $L^2(\mathbb{R}_+)$.*

Lemma 2. *The functional ν_a satisfies the following properties:*

1. ν_a *is a bounded functional on* $\mathcal{H}_0^1$.

2. *If* δ_a *denotes the delta distribution of* $\mathcal{H}^{-1}$, *that is* $\delta_a(\varphi) = \varphi(a)$, *then the transpose functional* $\nu_a^* : \mathbb{C} \to \mathcal{H}^{-1}$ *is given by* $\nu_a^*(z) = z\delta_a$.

3. *The operator* $\nu_a^*\nu_a : \mathcal{H}_0^1 \subset L^2(\mathbb{R}_+) \to \mathcal{H}^{-1}$ *is given by* $\nu_a^*\nu_a(\varphi) = \varphi(a)\delta_a$.

Proof. Observe that there is a constant C such that $|\varphi(a)| \leq ||\varphi||_\infty \leq C||\varphi||_{\mathcal{H}_0^1}$ and the rest is direct from definition. $\square$

The operator $\nu_a^*\nu_a$ defined in Lemma 2 shall be called the *Delta Operator* and we shall denote it by $|\delta_a\rangle\langle\delta_a|$. Moreover, we introduce the perturbed Hamiltonian H_ω on $L^2(\mathbb{R}_+)$ by

$$H_\omega = -\Delta + \omega|\delta_a\rangle\langle\delta_a|, \qquad (\omega \in \mathbb{R}), \tag{2.1}$$

with the domain $D(H_\omega) = \{\varphi \in L^2(\mathbb{R}_+) : \varphi \in \mathcal{H}_0^1 \text{ and } \varphi(0) = 0\}$ and taking values in $\mathcal{H}^{-1}$ (note that the free Hamiltonian $H_0 := -\Delta$, with domain $\mathcal{H}_0^1$, acts on the same spaces as the Delta Operator) and consider its self-adjoint realization over the positive real line.

2.2 Resolvent representation for delta perturbation

Let us denote the resolvent set of H_ω by $\rho(H_\omega)$, $\omega \geq 0$. For $z \in \rho(H_\omega)$, the resolvent operator shall be denote by $R_\omega(z) = (H_\omega - z)^{-1}$. For the second resolvent identity, we have

$$R_\omega(z) - R_0(z) = -\omega R_\omega(z)\nu_a^*\nu_a R_0(z) \tag{2.2}$$

which implies

$$R_\omega(z)\nu_a^*[1 + \omega\nu_a R_0(z)\nu_a^*] = R_0(z)\nu_a^*$$

and then

$$R_\omega(z)\nu_a^* = k(z)R_0(z)\nu_a^* \tag{2.3}$$

where $k(z) : \mathbb{C} \to \mathbb{C}$, $k(z) = (1 + \omega\nu_a R_0(z)\nu_a^*)^{-1}$. Substituting equation 2.3 in 2.2, we have

$$\begin{aligned} R_\omega(z) &= R_0(z) - \omega k(z)R_0(z)\nu_a^*\nu_a R_0(z) \\ &= R_0(z) - \omega k(z)R_0(z)|\delta_a\rangle\langle\delta_a|R_0(z). \end{aligned} \tag{2.4}$$

Note that in order to obtain equation (2.4) we only used that the perturbation of H_0 is of the form A^*B where A, B are elements of the dual of $L^2(\mathbb{R}_+)$. Taking a step further, in [13] the authors prove the following result

Theorem 3. *For all* $z \in \rho(H_\omega)$ *and* $\psi \in L^2[0,\infty)$,

$$R_\omega(z)\psi(x) = \int_0^\infty \left[G(x,y;z) - \frac{\omega G(a,y;z)G(x,a;z)}{1 + \omega G(a,a;z)}\right]\psi(y)dy$$

where

$$G(x,y;z) = \frac{1}{2i\sqrt{z}}(e^{i\sqrt{z}|x+y|} - e^{i\sqrt{z}|x-y|})$$

for $x, y > 0$ and $Im\sqrt{z} > 0$, is the Green function associated with the solution of the Schrödinger equation

$$\begin{cases} -u''(x) - zu(x) = \psi(x), x \geq 0. \\ u(0) = 0, u \in L^2[0,\infty). \end{cases}$$

Thus we can obtain a limit resolvent operator using the earlier Theorem. Denote by

$$(H_\infty - z)^{-1}\psi(x) = \int_0^\infty \left[G(x,y;z) - \frac{G(a,y;z)G(x,a;z)}{G(a,a;z)} \right] \psi(y)dy$$

So, if we write

$$\varphi = (H_\infty - z)^{-1}\psi := \lim_{\omega\to\infty} (H_\omega - z)^{-1}\psi,$$

with $\psi \in L^2(\mathbb{R}_+)$, then is not difficult to see that

$$(-\Delta - z)\varphi = \psi, \text{ for } x \in (0,a) \cup (a,\infty).$$

Also, $(H_\infty - z)^{-1}\psi(a) = 0$ and so $\varphi(0) = \varphi(a) = 0$. In conclusion, if we define the operator H_∞ on $L_0^2(0,a) \oplus L_0^2(a,\infty)$ with domain $D(H_\infty) = D_{(0,a)} \oplus D_{(a,\infty)}$ where

$$D_{(0,a)} = \{\varphi \in L^2(0,a) : \varphi \in \mathcal{H}_0^1(0,a) \text{ and } \varphi(0) = \varphi(a) = 0\}$$

$$D_{(a,\infty)} = \{\varphi \in L^2(a,\infty) : \varphi \in \mathcal{H}_0^1(a,\infty) \text{ and } \varphi(a) = 0\}$$

such as $H_\infty = -\Delta_{(0,a)} \oplus -\Delta_{(a,\infty)}$, with $-\Delta_{(0,a)}$ and $-\Delta_{(a,\infty)}$ acting as the Laplacian operator on $D_{(0,a)}$ and $D_{(a,\infty)}$, respectively, then H_ω converges in the strong resolvent sense to H_∞.

Note that the spectrum of H_∞ is given by $\{(\frac{n\pi}{a})^2 : m \in \mathbb{N}\} \cup [0,\infty)$, i.e., it has a discrete part and an absolutely continuous part, unlike H_ω that is an absolutely continuous operator. The corresponding normalized eigenfunctions for the eigenvalues are given by $\psi_n(x) = \sqrt{\frac{2}{a}}\chi_{[0,a]}(x)\sin\left(\frac{n\pi}{a}x\right)$.

2.3 A quasi-exponential decay law

Now we will show the main theorem for this section. Here, we present a quasi-exponential decay law for the survival probability amplitude and how it relates to the initial and final quantum states.

Theorem 4. *Let us fix λ_m eigenvalue of H_∞ and ψ_m its corresponding eigenvector. Define $\varepsilon = (\omega a)^{-1}$. For $\omega > 0$ large enough, it holds that there exists some complex numbers $z_{\varepsilon,m}$, $C_{n,m}(\varepsilon)$ and $E_{n,m}$ such that*

$$\langle \psi_n, e^{-iH_\omega t}\psi_m \rangle = C_{n,m}(\varepsilon)e^{-iz_{\varepsilon,m}t} + \varepsilon E_{n,m}, \text{ for all } n \in \mathbb{N}.$$

Moreover, for ε small $C_{n,m}(\varepsilon)$ is of order ε for $n \neq m$ and of order 1 if $n = m$.

We observe that the error $|E_{n,m}|$ is a uniform bound in t for the quasi-exponential decay. However, we think it is possible to obtain a time decay for this term using, for example, integration by parts (in a similar way as the main theorem of the next Chapter). This will be part of a future work.

Theorem 4 gives us an expression for a "Generalized" Survival Probability Amplitude $\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle$. Now we use it in order to prove an asymptotic behavior for the probabilities

$$|\langle\psi_m, e^{-iH_\omega t}\psi_m\rangle|^2$$

and

$$||\chi_{[0,a]}e^{-iH_\omega t}\psi_m||^2$$

defined earlier in Chapter 1. The first one corresponds to the probability that the particle remains in its initial state at time t, and the second one to the probability of finding the particle in the region $[0,a]$ at time t, given an initial state.

Corollary 5. *Under the same hypothesis of Theorem 4, if ω is large enough, then for all $t>0$:*

$$\left|||\chi_{[0,a]}e^{-iH_\omega t}\psi_m||^2 - |\langle\psi_m, e^{-iH_\omega t}\psi_m\rangle|^2\right| < o\left(\frac{1}{\omega}\right).$$

First, we prove the Theorem 4.

Proof of Theorem 4. By the Stone Formula, since H_ω has absolutely continuos spectrum we have

$$\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle = \frac{1}{\pi}\lim_{\delta\to 0^+}\int_0^\infty e^{-i\lambda t}\langle\psi_n, \Im R_\omega(\lambda+i\delta)\psi_m\rangle d\lambda. \tag{2.5}$$

By Theorem 3, after taking the limit $\delta\downarrow 0$ in $\lambda+i\delta$, for $\lambda\neq\lambda_m$, we have:

$$\begin{aligned}\Im R_\omega(\lambda)\psi(x) &= \int_0^\infty \Im\left[K(x,y;\lambda) - \frac{\omega K(a,y;\lambda)K(x,a;\lambda)}{1-\omega K(a,a;\lambda)}\right]\psi_m(y)dy \\ &= \left(\sqrt{\lambda} - \Im\left[\frac{2i\sqrt{\lambda}e^{i2a\sqrt{\lambda}}}{-1+i2a\varepsilon\sqrt{\lambda}+e^{i2a\sqrt{\lambda}}}\right]\right)\left(\int_0^a \frac{\sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}}\psi_m(y)dy\right),\end{aligned}$$

where $\varepsilon = (a\omega)^{-1}$. Let us define

$$r_{n,m}(\lambda) := \sqrt{\lambda}\left(\int_0^a \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\psi_n(x)dx\right)\left(\int_0^a \frac{\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}}\psi_m(y)dy\right).$$

Then, resolving the integrals, we obtain

$$r_{n,m}(\lambda) = \frac{2(-1)^{n+m}\sqrt{\lambda_n}\sqrt{\lambda_m}\sin^2(a\sqrt{\lambda})}{a\sqrt{\lambda}(\lambda-\lambda_n)(\lambda-\lambda_m)}. \tag{2.6}$$

Let us notice that for $n\neq m$, $r_{n,m}$ is arbitrary small when λ tends to a λ_m, and if $n=m$ it is of order 1. Thus,

$$\langle\psi_n, \Im R_\omega(\lambda)\psi_m\rangle = \left[1 - \Im\left\{\frac{2ie^{2ia\sqrt{\lambda}}}{f_\varepsilon(\lambda)}\right\}\right]r_{n,m}(\lambda). \tag{2.7}$$

where

$$f_\varepsilon(\lambda) = -1 + 2ai\varepsilon\sqrt{\lambda} + e^{2ai\sqrt{\lambda}}. \tag{2.8}$$

Since $f_\varepsilon(\lambda)$ does not have real zeros, then the quantity

$$\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle = \frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left[1 - \Im\left\{\frac{2ie^{2ia\sqrt{\lambda}}}{f_\varepsilon(\lambda)}\right\}\right] r_{n,m}(\lambda)d\lambda$$

is a well-defined integral.

Also

$$\Im\left\{\frac{2ie^{2ai\sqrt{\lambda}}}{-1+e^{2ia\sqrt{\lambda}}}\right\} = 1$$

so we can write

$$\begin{aligned}
\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle &= \frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{\frac{2ie^{2ai\sqrt{\lambda}}}{-1+e^{2ia\sqrt{\lambda}}} - \frac{2ie^{2ia\sqrt{\lambda}}}{f_\varepsilon(\lambda)}\right\} r_{n,m}(\lambda)d\lambda \\
&= \frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{\frac{-4e^{2ia\sqrt{\lambda}}a\sqrt{\lambda}\varepsilon}{(-1+e^{2ia\sqrt{\lambda}})f_\varepsilon(\lambda)}\right\} r_{n,m}(\lambda)d\lambda \\
&= \frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{\frac{h(\lambda)\varepsilon}{f_\varepsilon(\lambda)}\right\} r_{n,m}(\lambda)d\lambda
\end{aligned}$$

with

$$h(\lambda) = \frac{-4e^{2ia\sqrt{\lambda}}a\sqrt{\lambda}}{-1+e^{2ia\sqrt{\lambda}}}.$$

Note that

$$f_\varepsilon(\lambda) = -2a^2(\sqrt{\lambda}-\sqrt{\lambda_m})^2 + o((\sqrt{\lambda}-\sqrt{\lambda_m})^2) + i(2a\varepsilon\sqrt{\lambda} + \sin(2a\sqrt{\lambda}))$$

so there exists an analityc function $D(\lambda)$ such that

$$f_\varepsilon(\lambda) - D(\lambda) = -2a^2(\sqrt{\lambda}-\sqrt{\lambda_m})^2 + 2ia\varepsilon\sqrt{\lambda}.$$

Define

$$\begin{aligned}
p_\varepsilon(\lambda) &= \frac{h(\lambda)\varepsilon}{-2a^2(\sqrt{\lambda}-\sqrt{\lambda_m})^2 + 2ia\varepsilon\sqrt{\lambda_m}} \\
&= \frac{h(\lambda)\varepsilon}{-2a^2[(\sqrt{\lambda}-\sqrt{\lambda})^2 - 2i\Gamma^2_{\varepsilon,m}]}
\end{aligned}$$

with

$$\Gamma^2_{\varepsilon,m} = \frac{\pi m}{2a^2}\varepsilon.$$

Hence

$$\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle = G_1(\varepsilon,t) + G_2(\varepsilon,t)$$

where

$$G_1(\varepsilon,t)=\frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{p_\varepsilon(\lambda)\right\}r_{n,m}(\lambda)d\lambda$$

and

$$G_2(\varepsilon,t)=\frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{h(\lambda)\varepsilon\left(\frac{1}{f_\varepsilon(\lambda)}-\frac{1}{-2a^2[(\sqrt{\lambda}-\sqrt{\lambda})^2-2i\Gamma_{\varepsilon,m}^2]}\right)\right\}r_{n,m}(\lambda)d\lambda.$$

First we compute $G_1(\varepsilon,t)$.

The (simple) poles of $p_\varepsilon(z)$ are $z_{\varepsilon,m}=(\sqrt{\lambda_m}-\Gamma_{\varepsilon,m}-i\Gamma_{\varepsilon,m})^2$, $\overline{w_{\varepsilon,m}}=(\sqrt{\lambda_m}+\Gamma_{\varepsilon,m}+i\Gamma_{\varepsilon,m})^2$ and λ_m (pole of $h(\lambda)$). Note that if $\varepsilon<4a$ then

$$\Im z_{\varepsilon,m}=-2\Gamma_{\varepsilon,m}\sqrt{\lambda_m}-\Gamma_{\varepsilon,m}^2<0 \tag{2.9}$$

and then $z_{\varepsilon,m}$ lives on the lower half-plane, $\overline{w_{\varepsilon,m}}$ on the upper half-plane and λ_m on the real axis. For $\delta>0$ small and $R>0$ large let us define the curve γ as

$$\gamma=[0,\lambda_m-\delta]\cup\gamma_\delta\cup[\lambda+\delta,R]\cup\gamma_R$$

where $\gamma_\delta(s)=\delta e^{is}+\lambda_m$ and $\gamma_R(s)=R(e^{-is}+1)$ for $s\in[0,\pi]$.

For R large enough, γ is a closed curve that contains z_m. Thus,

$$\frac{1}{\pi}\int_\gamma e^{-izt}p_\varepsilon(z)r_{n,m}(z)dz=2ir_{n,m}(z_{\varepsilon,m})C_1(\Gamma_{\varepsilon,m})e^{-iz_{\varepsilon,m}t}$$

where

$$C_1(\Gamma_{\varepsilon,m})=\frac{8a^3z_{\varepsilon,m}\Gamma_{\varepsilon,m}}{\pi m(1+i)(1-e^{2ai(1+i))\Gamma_{\varepsilon,m}})}\overset{\Gamma_\varepsilon\to 0}{\longrightarrow}2m\pi.$$

Similarly, we have

$$\frac{1}{\pi}\int_\gamma e^{-izt}\overline{p_\varepsilon}(z)r_{n,m}(z)dz=-2ir_{n,m}(w_{\varepsilon,m})C_2(\Gamma_{\varepsilon,m})e^{-iw_mt}$$

where

$$C_2(\Gamma_{\varepsilon,m})=\frac{-8a^3w_{\varepsilon,m}\Gamma_{\varepsilon,m}}{\pi m(1-i)(1-e^{2ai(1-i)\Gamma_{\varepsilon,m}})}\overset{\Gamma_\varepsilon\to 0}{\longrightarrow}2m\pi.$$

Hence,

$$\frac{1}{\pi}\int_\gamma e^{-izt}\Im\{p_\varepsilon(z)\}r_{n,m}(z)dz=C_3(\Gamma_{\varepsilon,m})e^{-iz_{\varepsilon,m}t}$$

where

$$C_{n,m}(\Gamma_{\varepsilon,m})=r_{n,m}(z_{\varepsilon,m})C_1(\Gamma_{\varepsilon,m})+r_{n,m}(w_{\varepsilon,m})C_2(\Gamma_{\varepsilon,m})e^{-i4(\sqrt{\lambda_m}\Gamma_{\varepsilon,m}-i\Gamma_{\varepsilon,m}^2)t}.$$

It is customary to prove that the contour integrals over γ_δ and γ_R converges to zero when δ tends to zero and R tends to infinity, respectively. In order to do this, for γ_δ it is sufficient to see that

$$\lim_{\delta\to 0}\int_{\gamma_\delta} e^{-izt}p_\varepsilon(z)r_{n,m}(\lambda)dz = \lim_{\delta\to 0}\int_{\gamma_\delta} e^{-izt}\overline{p_\varepsilon(z)}r_{n,m}(\lambda)dz.$$

For γ_R we write

$$\int_{\gamma_R} e^{-izt}p_\varepsilon(z)r_{n,m}(z)dz = \int_0^\pi e^{-i\gamma_c(s)t}\phi(s)ds \tag{2.10}$$

where $e^{-i\gamma_c(s)t}$ (of order e^{-R}) dominate the factor $\phi(s) = p_\varepsilon(\gamma_c(s))r_{n,m}(\gamma_c(s))\gamma_c'(s)$, which is of order R. The same occurs for $\int \gamma_R e^{-izt}\overline{p_\varepsilon}(z)r_{n,m}(z)dz$.

All this implies

$$G_1(\varepsilon,t) = C_{n,m}(\Gamma_{\varepsilon,m})e^{-iz_{\varepsilon,m}t}$$

Moreover, it holds

$$\lim_{\varepsilon\to 0} C_{n,m}(\Gamma_{\varepsilon,m}) = 4m\pi r_{n,m}(\lambda_m).$$

On the other hand,

$$\begin{aligned} G_2(\varepsilon,t) &= \frac{1}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{h(\lambda)\varepsilon\left(\frac{1}{f_\varepsilon(\lambda)} - \frac{1}{-2a^2(\sqrt{\lambda}-\sqrt{\lambda_m})^2 + 2ia\varepsilon\sqrt{\lambda_m}}\right)\right\} r_{n,m}(\lambda)d\lambda \\ &= \frac{\varepsilon}{\pi}\int_0^\infty e^{-i\lambda t}\Im\left\{h(\lambda)g_\varepsilon(\lambda)\right\} r_{n,m}(\lambda)d\lambda \end{aligned}$$

where

$$g_\varepsilon(\lambda) = \frac{-2ai\varepsilon(\sqrt{\lambda_m}-\sqrt{\lambda}) + D(\lambda)}{f_\varepsilon(\lambda)(-2a^2(\sqrt{\lambda}-\sqrt{\lambda_m})^2 + 2ia\varepsilon\sqrt{\lambda_m})}$$

Since the functions $h(\lambda)$, g_ε and $r_{n,m}$ are explicit, it is not difficult to see that the integral is well-defined. In conclusion, $G_2(\varepsilon,t)$ is of order ε and the theorem follows from defining $E_{n,m}(\varepsilon)$ as the previous integral.

□

Now we prove the Corollary 5,

Proof of Corollary 5. From (1.5) and since $\{\psi_n : n\in\mathbb{N}\}$ is an orthonormal basis for states supported on the interval $(0,a)$, we have

$$\begin{aligned} \big| ||\chi_{[0,a]}e^{-iH_\omega t}\psi_m||^2 - |\langle\psi_m, e^{-iH_\omega t}\psi_m\rangle|^2 \big| &< \sum_{n\neq m} |\langle\psi_n, e^{-iH_\omega t}\psi_m\rangle|^2 \\ &< \left(\sum_{n\neq m}|C_{n,m}(\Gamma_{\varepsilon,m})|^2\right) e^{2\Im z_{\varepsilon,m}t} + \varepsilon^2\left(\sum_{n\neq m}|E_{n,m}(\varepsilon)|^2\right) \end{aligned} \tag{2.11}$$

Now we note that for $\lambda\neq\lambda_m$, the sum $\sum_{n\neq m}|r_{n,m}(\lambda)|^2$ always converges. Also, if ε is small, for $n\neq m$, then $C_{n,m}(\Gamma_{\varepsilon,m})$ is of the same order than $\sin(a\sqrt{z_{\varepsilon,m}})$ which is of order $\Gamma_{\varepsilon,m} = o(\sqrt{\varepsilon})$. This two observations implies that the first sum of the right-hand side of equation 2.11 is of order ε and the second of order 1, and the corollary follows. □

2.4 Resonances as poles of the resolvent

As we discussed in Chapter 1, there are different approaches when it comes to define resonances. If we seek for a more classical-mathematical definition, we can take a path a bit different. By equation 2.7, we can look for the zeros in $\mathbb{C}$ of the function f_ε. Given $\varepsilon > 0$ let us consider the complex function

$$f_\varepsilon(z) = -1 + 2ia\varepsilon z + e^{2iaz}$$

with $z = x + iy$, $x, y \in \mathbb{R}$. So we have

$$f_\varepsilon(z) = 0 \Leftrightarrow \begin{cases} u(x, y, \varepsilon) := -1 - 2a\varepsilon y + e^{-2ay}\cos(2ax) = 0. \\ v(x, y, \varepsilon) := 2a\varepsilon x + e^{-2ay}\sin(2ax) = 0. \end{cases}$$

The triple $(x, y, \varepsilon) = \left(\frac{m\pi}{a}, 0, 0\right)$ is a solution of the system. Besides, the Jacobian $|\partial(u, v)/\partial(x, y)|$ for that point is $4a^2 \neq 0$. The Implicit Function Theorem says that there exists unique continuously differentiables functions x_ε and y_ε such that $x_0 = \frac{m\pi}{a}$, $y_0 = 0$ and

$$\begin{cases} u(x_\varepsilon, y_\varepsilon, \varepsilon) = 0 \\ v(x_\varepsilon, y_\varepsilon, \varepsilon) = 0. \end{cases}$$

Derivating with respect to ε we have

$$\begin{cases} -y'_\varepsilon h_1(x_\varepsilon, y_\varepsilon, \varepsilon) + x'_\varepsilon h_2(x_\varepsilon, y_\varepsilon, \varepsilon) - y_\varepsilon = 0 \\ x'_\varepsilon h_1(x_\varepsilon, y_\varepsilon, \varepsilon) - y'_\varepsilon h_2(x_\varepsilon, y_\varepsilon, \varepsilon) + x_\varepsilon = 0 \end{cases} \tag{2.12}$$

where

$$\begin{cases} h_1(x_\varepsilon, y_\varepsilon, \varepsilon) = \varepsilon + e^{-2y_\varepsilon a}\cos(2ax) \\ h_1(x_\varepsilon, y_\varepsilon, \varepsilon) = e^{-2y_\varepsilon a}\sin(2ax_\varepsilon) \end{cases}$$

This implies that

$$x'_0 = -\frac{m\pi}{a}, \quad y'_0 = 0.$$

Derivating again and evaluating in $(x_0, y_0, 0)$ we obtain

$$x''_0 = \frac{2m\pi}{a}, \quad y''_0 = \frac{-2m^2\pi^2}{a}.$$

Thus, the zeros of f_ε admit the following representation as Taylor Series around $\varepsilon = 0$:

$$\begin{aligned} z &= \frac{m\pi}{a} - \frac{m\pi}{a}\varepsilon + \left(\frac{2m\pi}{a} - i\frac{2m^2\pi^2}{a}\right)\varepsilon^2 + o(\varepsilon^2) \\ &= \sqrt{\lambda_m} - \sqrt{\lambda_m}\varepsilon + 2\sqrt{\lambda_m}(1 - im\pi)\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

We summarize this in the following theorem:

Theorem 6. *For $n, m \in \mathbb{N}$, if ε is small enough, then the poles of $\langle \psi_n, R_\omega(\lambda)\psi_m \rangle$ are of the form*

$$v_{\varepsilon,m} = \sqrt{\lambda_m} - \sqrt{\lambda_m}\varepsilon + 2\sqrt{\lambda_m}(1 - im\pi)\varepsilon^2 + o(\varepsilon^2)$$

Physicist usually call *lifetime* of the resonant energy to the value $\Im v_{\varepsilon,m}$. In this case its dominant term with respect to ε is $2\sqrt{\lambda_m} m\pi\varepsilon$ which is of order $1/\omega^2$. From Theorem 4 we have that $\Im z_{\varepsilon,m} = o\left(\frac{1}{\sqrt{\omega}}\right)$ (See equation 2.9). So the resonance just computed as a pole of the resolvent approach the eigenvalue *much faster* than the one in the previous section. This difference is a direct consequence of the approximation used in Theorem 4.

Chapter 3

Perturbation of an eigenvalue embedded in the continuum

3.1 Context and preliminaries

Let us consider the case where Q is some arbitrary projector on the Hilbert Space $\mathcal{H}$ and $\psi \in Ran(Q)$ is unitary. Then, for the Schrödinger operator H we have

$$|\langle\psi, e^{-iHt}\psi\rangle|^2 = |\langle Q\psi, Qe^{-iHt}\psi\rangle|^2 \leq ||Q\psi||^2||Qe^{-iHt}\psi||^2 = ||Qe^{-iHt}\psi||^2$$

thus

$$\int_{\mathbb{R}} |\langle\psi, e^{-iHt}\psi\rangle|^2 dt \leq \int_{\mathbb{R}} ||Qe^{-iHt}\psi||^2 dt$$

In this case, the Sojourn Time 1.3 is a lower bound for the Dwell Time 1.4. But we want to take it a step further. We claim that, in a suitable way, the Sojourn Time behaves like the Dwell Time. We formalize this idea as follows.

Let H_0 be a self-adjoint operator defined on a Hilbert Space $\mathcal{H}$ with a simple eigenvalue λ_0 embedded on an absolutely continuous spectrum and ψ its corresponding normalized eigenvector. Denote $P = |\psi\rangle\langle\psi|$ the rank one eigenspace projection and denote $H^\perp = P^\perp H P^\perp$ for any operator H such that $\psi \in Dom(H)$. Consider now the perturbed Hamiltonian

$$H_\beta = H_0 + \beta V, \quad \beta \neq 0$$

where V is a real valued H_0-bounded function with H_0-bound smaller than 1. As in the previous chapter, we shall denote by $R_\beta(z) := (H_\beta - z)^{-1}$ for z in the resolvent set of H_β and $\beta \geq 0$.

Let $g \in C_c^\infty(\mathbb{R})$ be a real valued function compactly supported on some interval (a, b) such as $0 \leq g \leq 1$ with $g \equiv 1$ on $[a_0, b_0]$ where $a < a_0 < \lambda_0 < b_0 < b$.

Denote by $E_\Omega(H)$ the spectral projection of a self-adjoint operator H on a Borel set Ω, $ad_A(B) = [A, B] = AB - BA$ the commutator of A and B and $ad_A^k(B) = ad_A^{k-1}(ad_A(B))$ its iterations. Note that the commutator $[A, B]$ is defined in the sense of sesquilinear forms [27]. Assume there exist a self-adjoint operator A and an open interval I such that following set of hypothesis satisfies:

(H1) $e^{-isA}D(H_0) \subset D(H_0)$ for all $s \in \mathbb{R}$;

(H2) a Mourre estimates holds:

$$E_I(H_0)i[H_0, A]E_I(H_0) \geq cE_I(H_0) + K$$

for some $c > 0$ and K compact;

(H3) $ad_A^k(H_0)$ are H_0-bounded for $k \in \{1, 2, 3\}$;

(H4) $ad_A^k(V)$ are H_0-bounded for $k \in \{1, 2, 3\}$.

Now, given the states $\varphi, \psi \in \mathcal{H}$ the most amount of work will be dedicate to estimate the quantity

$$\langle \varphi, Qe^{-iH_\beta t}g(H_\beta)\psi\rangle, \quad t \in \mathbb{R}. \tag{3.1}$$

The main idea is to consider the survival probability amplitude $\langle \varphi, Qe^{-iH_\beta t}g(H_\beta)\psi\rangle$ and write it as the sum of two terms, $A_{\varphi,\psi}(t, \beta)$ and $B_{\varphi,\psi}(t, \beta)$ being $A_{\varphi,\psi}$ the projection onto the eigenspace of ψ and B the projection onto the complement. In [14] the authors proved that $A_{\varphi,\psi}(t, \beta)$ admits a quasi-exponential representation which delivers a Sojourn Time of order $1/\beta^2$. The main result is enunciated in Theorem 8 (with all the technical arguments in Proposition 11). Here we show that $B_{\varphi,\psi}$, as well as $itB_{\varphi,\psi}$, has a β-ponderate exponential representation plus a β^3-order error term (respectively, $\beta^{\frac{11}{9}}$-order for $itB_{\varphi,\psi}$) if an extra reflectionless condition is assumed over H_β. Immediately after we explain that this particular condition is actually not necessary if we modify our energy locator function g in a suitable fashion. However, this alternative approach does not allow us to obtain bounds for $itB_{\varphi,\psi}$. Finally, in Theorem 9, we conclude that the asymptotically behavior of $A_{\varphi,\psi}$ in β is similar to that of $B_{\varphi,\psi}$. This means that the Sojourn Time and the Dwell Time are of the same order.

3.2 Reduction of the problem

By the Spectral Theorem and the Stone formula we have

$$\begin{aligned}
\langle \varphi, Qe^{-iH_\beta t}g(H_\beta)\psi\rangle &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}g(\lambda)\langle Q\varphi, \Im R_\beta(\lambda + i0)\psi\rangle d\lambda \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}g(\lambda)\langle Q\varphi, P\Im R_\beta(\lambda + i0)g(\lambda)\psi\rangle d\lambda \\
&\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}g(\lambda)\langle Q\varphi, P^\perp\Im(R_\beta(\lambda + i0)g(\lambda)\psi\rangle d\lambda \qquad (3.2)\\
&= \langle Q\varphi, \psi\rangle\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}g(\lambda)\langle \psi, \Im R_\beta(\lambda + i0)\psi\rangle d\lambda \\
&\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}g(\lambda)\langle Q\varphi, P^\perp\Im R_\beta(\lambda + i0)P\psi\rangle d\lambda.
\end{aligned}$$

We denote the two terms of the right-hand side of the last equality of 3.2 by

$$A_{\varphi,\psi}(t,\beta) := \langle Q\varphi,\psi\rangle \int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda)\langle\psi, \Im R_\beta(\lambda+i0)\psi\rangle d\lambda, \tag{3.3}$$

and

$$B_{\varphi,\psi}(t,\beta) := \int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda)\langle Q\varphi, P^\perp \Im R_\beta(\lambda+i0)P\psi\rangle d\lambda. \tag{3.4}$$

So, in order to estimate 3.1, we need to estimate these two integrals. Equation 3.3 corresponds to the eigenprojection P of the resolvent, and Equation 3.4 is the projection onto $P^\perp$. Note that, after taking square modulus, the first term corresponds to the survival probability defined in equation 1.2.

3.3 Main results

Definition 7. *We say that a self-adjoint Schrödinger operator H is reflectionless at φ if for all $\lambda \in \sigma_{ess}(H)$ it holds*

$$\lim_{\epsilon\downarrow 0} \Re\langle\varphi, (H-\lambda-i\epsilon)^{-1}\varphi\rangle = 0$$

Reflectionless operators have been investigated in [19] and [20], for example.

The first important result of this chapter is the following.

Theorem 8. *Assume hypothesis $(H1)$ to $(H4)$. Also assume*

(H5) $\quad \Im\mathcal{F}(\lambda_0, 0) > 0.$

Fix β sufficiently small. We have that for all $t \in \mathbb{R}$, $B_{\varphi,\psi}(t,\beta)$ admits a quasi-exponential representation with an error term E of order β and itE also of order β. Moreover, if the operator H_β is reflectionless at ψ, then E is of order β^3 and itE is of order $\beta^{\frac{11}{9}}$. Thus, we have the following estimates:

- *General case:*

$$\langle\varphi, Qe^{-iH_\beta t}g(H_\beta)\psi\rangle = A_{\varphi,\psi}(t,\beta) + c\beta e^{-i\zeta_\beta t} + c' \begin{cases} o(\beta) \text{ for } |t|<1 \\ \frac{1}{t}o(\beta) \text{ for } |t|>1 \end{cases}.$$

- *H_β reflectionless at ψ:*

$$\langle\varphi, Qe^{-iH_\beta t}g(H_\beta)\psi\rangle = A_{\varphi,\psi}(t,\beta) + d\beta e^{-i\zeta_\beta t} + d' \begin{cases} o(\beta^3) \text{ for } |t|<1 \\ \frac{1}{t}o(\beta^{\frac{11}{9}}) \text{ for } |t|>1 \end{cases}.$$

where $c = c(\varphi,\psi)$, $c' = c'(\varphi,\psi)$, $d = d(\varphi,\psi)$ and $d' = d'(\varphi,\psi)$ are some constants and ζ_β is a complex number with $\Im\zeta_\beta < 0$ of order β^2.

The reflectionless condition is not necessary but sufficient. A way to obtain a β^2-order error term E without assuming this hypothesis is to operate just as in the proof of Theorem 8 with the difference that the energy-locator function g we will consider with variable support $[a_\beta, b_\beta]$ such that $a_\beta < \lambda_0 < b_\beta$, with $a_\beta := \lambda_0 - |\beta|||V||$, $b_\beta := \lambda_0 + |\beta|||V||$ and $g \equiv 1$ on $[\frac{1}{2}a_\beta, \frac{1}{2}b_\beta]$. The weakness of this approach is due to the following fact: At some point we derivate g in order to get an estimate for itE but the smoothness of g implies that g' does not exist in the limit. Since we need to obtain an integrable decay over time, we choose to introduce the reflectionless hypothesis. Whatever the chosen path, the following result holds.

Using Theorem 8 we can assure that the Sojourn Time (1.3) and the Dwell Time (1.4) have singularities of equal order with respect to β. We now enunciate the main result of this chapter:

Theorem 9. *Assume hypothesis $(H1)$ to $H(5)$ of Theorem 8. If β is small enough, then for $t \in \mathbb{R}$*

$$\left| ||Qe^{-iH_\beta t} g(H_\beta)\psi||^2 - ||Q\psi||^2 |\langle \psi, e^{-iH_\beta t}\psi\rangle|^2 \right| \leq \beta^2.$$

Furthermore

$$\left| \mathfrak{D}_{H_\beta}(\psi) - ||Q\psi||^2 \mathfrak{S}_{H_\beta}(\psi) \right| \leq o(1).$$

3.4 Proofs

Now, in order to prove Theorem 8, first we need to control the term $B_{\varphi,\psi}(t,\beta)$ (3.4).

For z complex number with nonzero imaginary part and $\beta > 0$ define

$$\begin{cases} \mathcal{F}(z,\beta) = \langle \psi, VP^\perp (H_\beta^\perp - z)^{-1} P^\perp V\psi\rangle, \\ \mathcal{J}(z,\beta) = \langle \varphi, QP^\perp R_\beta^\perp(z) P^\perp VP\psi\rangle. \end{cases} \tag{3.5}$$

For $\beta \geq 0$ denote $R_\beta^\perp(z) = P^\perp (H_\beta^\perp - z)^{-1} P^\perp$ and $\lambda_\beta = \lambda_0 + \beta\langle\psi, V\psi\rangle$.

Proposition 10. *Assume hypothesis $(H1)$ to $(H4)$. Also assume*

(H5) $\Im \mathcal{F}(\lambda_0, 0) > 0$.

Then, for β sufficiently small it holds that there exists $C > 0$ such that

$$|B_{\varphi,\psi}(t,\beta)| \leq C\beta.$$

The hypothesis $(H5)$ is the so-called Fermi Golden Rule.

Proof. Proposition 14 of Section 3.5 implies that

$$\begin{cases} PR_\beta(z)P = P(P(H_\beta - z)P - PH_\beta P^\perp (H_\beta - z)^{-1} P^\perp H_\beta P)^{-1} P = \dfrac{1}{\lambda_\beta - \overline{z} - \beta^2 \mathcal{F}(z,\beta)} P \\ P^\perp R_\beta(z) P = \dfrac{-\beta P^\perp R_\beta^\perp(z) P^\perp VP}{\lambda_\beta - \overline{z} - \beta^2 \mathcal{F}(z,\beta)}. \end{cases} \tag{3.6}$$

Moreover, if we denote

$$G(z,\beta) = \frac{1}{\lambda_\beta - \overline{z} - \beta^2 \mathcal{F}(z,\beta)} \tag{3.7}$$

then we can write

$$\begin{aligned} B_{\varphi,\psi}(t,\beta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda) \frac{1}{2i} \langle Q\varphi, P^\perp [R_\beta(\lambda+i0) - R_\beta(\lambda - i0)] P\psi\rangle d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda) \langle Q\varphi, P^\perp R_\beta(\lambda+i0) P\psi\rangle d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda) \langle Q\varphi, P^\perp R_\beta(\lambda-i0) P\psi\rangle d\lambda \\ &= \frac{-\beta}{2\pi i}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda+i0,\beta) G(\lambda+i0,\beta) d\lambda \\ &\quad + \frac{-\beta}{2\pi i}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda-i0,\beta) G(\lambda-i0,\beta) d\lambda. \end{aligned} \tag{3.8}$$

Define

$$D(\lambda,\beta) = G(\lambda,\beta)^{-1} = \lambda_\beta - \lambda - \beta^2 \mathcal{F}(\lambda,\beta)$$

for $(\lambda,\beta) \in [a,b] \times [-\beta_0,\beta_0]$. Take $0 < \delta_1 < \min\{|a_0 - \lambda_0|, |\lambda_0 - b_0|, 1\}$. We can pick $0 < \beta_1 < \beta_0$ such that for any $|\beta| < \beta_1$ it holds that

$$\begin{cases} Ran\left(\lambda_\beta - \beta^2 \Re\mathcal{F}(\cdot,\beta)\right) \subset [a_0+\delta_1, b_0-\delta_1] \quad \text{and} \\ \beta_1^2 \sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_0,\beta_0]} |\mathcal{F}'(\lambda,\beta)| < 1-\delta_1. \end{cases} \tag{3.9}$$

By continuity, in addition with the Fermi Golden Rule $(H5)$, we can assume that

$$\Im\mathcal{F}(\lambda,\beta) > 0$$

for $\lambda \in [a_0,b_0]$ and $0 < |\beta| < \beta_1$. Therefore, for $0 < |\beta| < \beta_1$, this entails, in the first place, that

$$|D(\lambda,\beta)| \geq \beta^2 \Im\mathcal{F}(\lambda,\beta) > 0$$

if $\lambda \in [a_0,b_0]$. Also, for any $\lambda \in [a,a_0] \cup [b_0,b]$,

$$|D(\lambda,\beta)| > \left|\lambda_\beta - \lambda - \beta^2 \Re\mathcal{F}(\lambda,\beta)\right| \geq \delta_1.$$

This gives a sense for the two integrals in the right-hand side of 3.8 for $0 < |\beta| < \beta_1$.

□

Now we want to go a step further and show that $B_{\varphi,\psi}$ admits a quasi-exponential representation (smaller than the one of $A_{\varphi,\psi}$) plus a controllable error term.

Note that $B_{\varphi,\psi}(t,\beta)$ is equal to

$$\begin{aligned} B_{\varphi,\psi}(t,\beta) =& \frac{-\beta}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda)\mathcal{J}(\lambda+i0,\beta)\Im G(\lambda+i0,\beta)d\lambda \\ &+ \frac{-\beta}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} g(\lambda)\widehat{\mathcal{J}}(\lambda-i0,\beta)G(\lambda-i0,\beta)d\lambda. \end{aligned} \tag{3.10}$$

where

$$\widehat{\mathcal{J}}(\lambda-i0,\beta) = \langle Q\varphi, \Im(R_\beta^\perp(\lambda-i0))V\psi\rangle.$$

In the following, we will prove that the first integral of the right-hand side of equation 3.10, which we denote by $\mathcal{I}(t,\beta)$, can be approximated by another integral which integrand is a Lorentzian-like function.

Proposition 11. *Under the same hypothesis of Proposition 10, there exist some constant C and a complex number ζ_β in the lower half-plane, which tends to λ_0 as β tends to 0, such that if β is sufficiently small, then*

$$\mathcal{I}(t,\beta) = C\beta e^{-i\zeta_\beta t} + R(t,\beta)$$

where the error term satisfies

a) $R(t,\beta) = o(\beta^3)$

b) $itR(t,\beta) = o(\beta^{\frac{11}{9}})$.

First, we prove the following technical lemma.

Lemma 12. *Given any $\beta \in [-\beta_1,\beta_1]$, there is a unique solution to the equation $\lambda = \lambda_\beta - \beta^2\Re\mathcal{F}(\lambda,\beta)$ in $[a_0,b_0]$. Also, if λ_β^∞ denotes the solution, we have that $\lambda_\beta^\infty \in [a_0+\delta_1, b_0-\delta_1]$ and $|\lambda_\beta^\infty - \lambda_\beta| \le \beta^2 \sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_1,\beta_1]} |\mathcal{F}(\lambda,\beta)|$.*

Proof. Equations 3.9 implies that function $\lambda \to \lambda_\beta - \frac{\beta^2}{2}\Re\mathcal{F}(\lambda,\beta)$ maps $[a_0+\delta_1, b_0-\delta_1]$ on itself and is strictly contractive. By the Banach Fixed Point Theorem the lemma follows. □

Proof of Proposition 11. Define

$$\begin{aligned} \widehat{D}(\lambda,\beta) &= \widehat{G}(\lambda,\beta)^{-1} = \lambda_\beta - \lambda - \beta^2\mathcal{F}(\lambda_\beta^\infty,\beta) \\ D_1(\lambda,\beta) &= G_1(\lambda,\beta)^{-1} = \lambda_\beta - \lambda - \beta^2\mathcal{F}(\lambda_\beta^\infty,\beta) - \beta^2\mathcal{F}'(\lambda_\beta^\infty,\beta)(\lambda-\lambda_\beta^\infty) \\ &= \lambda_\beta^\infty - \lambda - i\beta^2\Im\mathcal{F}(\lambda,\beta) - \beta^2\mathcal{F}'(\lambda_\beta^\infty,\beta)(\lambda-\lambda_\beta^\infty) \end{aligned}$$

for $(\lambda,\beta) \in [a_0,b_0]\times[-\beta_1,\beta_1]$. By hypothesis $(H5)$, $|\widehat{D}(\lambda,\beta)| > 0$ for $\beta \neq 0$. This allow us to define $\widehat{G}(\lambda,\beta) = \widehat{D}(\lambda,\beta)^{-1}$ for $\lambda \in [a_0,b_0]$ and $0 < |\beta| < \beta_1$.

For $\lambda \in [a,b]$ and $|\beta| < \beta_1$, define the Lorentzian-like function $G_1(\lambda,\beta)$ by

$$G_1(\lambda,\beta) := \frac{1}{\lambda_\beta - \lambda - \beta^2\mathcal{F}(\lambda_\beta^\infty,\beta) - \beta^2\mathcal{F}'(\lambda_\beta^\infty,\beta)(\lambda-\lambda_\beta^\infty)}$$

and

$$\mathcal{J}_1(\lambda,\beta) := \mathcal{J}(\lambda_\beta^\infty,\beta) + \mathcal{J}'(\lambda_\beta^\infty,\beta)(\lambda-\lambda_\beta^\infty).$$

where $\mathcal{J}'$ is the derivative of $\mathcal{J}$ with respect to λ. The function G_1 is well-defined since

$$\begin{aligned}|D_1(\lambda,\beta)| &\geq |\lambda_\beta^\infty - \lambda - i\beta^2\mathcal{F}(\lambda_\beta,\beta)| - \beta^2\left(\sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_0,\beta_0]}|\mathcal{F}'(\lambda,\beta)|\right)|\lambda_\beta^\infty-\lambda| \\ &\geq \left(1-\beta^2\sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_0,\beta_0]}|\mathcal{F}'(\lambda,\beta)|\right)|\widehat{D}(\lambda,\beta)| \geq \delta_1|\widehat{D}(\lambda,\beta)|.\end{aligned}$$

This implies that if $\lambda\in[a_0,b_0]$ and $0<|\beta|\geq\beta_1$ then

$$|G_1(\lambda,\beta)| \leq \delta_1^{-1}|\widehat{G}(\lambda,\beta)|. \tag{3.11}$$

Also

$$D(\lambda,\beta) = \lambda_\beta^\infty - \lambda - i\beta^2\Im\mathcal{F}(\lambda_\beta^\infty,\beta) - \beta^2(\mathcal{F}(\lambda,\beta)-\mathcal{F}(\lambda_\beta^\infty,\beta)).$$

For $\lambda\in[a_0,b_0]$ and $|\beta|<\beta_1$, by the Mean Value Theorem,

$$\begin{aligned}|D(\lambda,\beta)| &\geq |\lambda_\beta^\infty - \lambda - i\tfrac{\beta^2}{2}\mathcal{F}(\lambda_\beta,\beta)| - \frac{\beta^2}{2}\left(sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_0,\beta_0]}|\mathcal{F}'(\lambda,\beta)|\right)|\lambda_\beta^\infty-\lambda| \\ &\geq \left(1-\frac{\beta^2}{2}\sup_{(\lambda,\beta)\in[a_0,b_0]\times[-\beta_0,\beta_0]}|\mathcal{F}'(\lambda,\beta)|\right)|\widehat{D}(\lambda,\beta)| \geq \delta_1|\widehat{D}(\lambda,\beta)|\end{aligned}$$

therefore

$$|G(\lambda,\beta)| \leq \delta_1^{-1}|\widehat{G}(\lambda,\beta)|. \tag{3.12}$$

According to this, we can write

$$\mathcal{I}(t,\beta) = \frac{-\beta}{\pi}\left(\mathcal{I}_1(t,\beta)+\mathcal{I}_2(t,\beta)+\mathcal{I}_3(t,\beta)+\mathcal{I}_\partial(t,\beta)\right)$$

where

$$\begin{aligned}\mathcal{I}_1(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t}\mathcal{J}_1(\lambda,\beta)\Im G_1(\lambda,\beta)d\lambda \\ \mathcal{I}_2(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t}[\mathcal{J}(\lambda,\beta)-\mathcal{J}_1(\lambda,\beta)]\Im G(\lambda,\beta)d\lambda \\ \mathcal{I}_3(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t}\mathcal{J}_1(\lambda,\beta)\Im[G(\lambda,\beta)-G_1(\lambda,\beta)]d\lambda \\ \mathcal{I}_\partial(t,\beta) &= \int_{a}^{a_0} e^{-i\lambda t}g(\lambda)\mathcal{J}(\lambda,\beta)\Im G(\lambda,\beta)d\lambda + \int_{b_0}^{b} e^{-i\lambda t}g(\lambda)\mathcal{J}(\lambda,\beta)\Im G(\lambda,\beta)d\lambda\end{aligned}$$

We shall perform now the analysis of each term.

Step 1: We start with the term $\mathcal{I}_1(t,\beta)$. We write for $\lambda\in[a_0,b_0]$, $0<|\beta|<\beta_1$,

$$\Im G_1(\lambda, \beta) = \frac{1}{2i}\left(\frac{1}{b_\beta - a_\beta \lambda} - \frac{1}{\overline{b_\beta} - \overline{a_\beta}\lambda}\right)$$

where $a_\beta = 1 + \beta^2 \mathcal{F}'(\lambda_\beta^\infty, \beta)$, $b_\beta = \lambda_\beta^\infty a_\beta - i\beta^2 \mathcal{F}(\lambda_\beta^\infty, \beta)$, and consider the function

$$g_1(z, \beta) = \frac{1}{2i}(\mathcal{J}(\lambda_\beta^\infty, \beta) + \mathcal{J}'(\lambda_\beta^\infty, \beta)(z - \lambda_\beta^\infty))\left(\frac{1}{b_\beta - a_\beta z} - \frac{1}{\overline{b_\beta} - \overline{a_\beta} z}\right)$$

which is a meromorphic function in the complex plane with two simple poles ζ_β and $\overline{\zeta_\beta}$ with

$$\zeta_\beta = \lambda_\beta^\infty - i\beta^2 \frac{\Im \mathcal{F}(\lambda_\beta^\infty, \beta)}{a_\beta}.$$

In particular, for $\lambda \in [a_0, b_0]$, $0 < |\beta| < \beta_1$, $g_1(\lambda, \beta) = \mathcal{J}_1(\lambda, \beta)\Im G_1(\lambda, \beta)$

In [14], in the proof of Th. 2.1, it is shown that

$$\Re \zeta_\beta \in [a_0 - (\delta_1 - \delta_1'), b_0 - (\delta_1 - \delta_1')]$$

and

$$\Im \zeta_\beta < 0.$$

Let γ be a fixed smooth curve in the lower half plane, joining the endpoints of the interval $[a_0, b_0]$ and staying at positive distance from the closure of the bounded set $\{z_\beta : 0 < |\beta| < \beta_1\}$. Then, there exist $\beta_1' > 0$ such as for $0 < |\beta| < \beta_1'$, the closed curve $[a_0, b_0] \cup \gamma^-$ enclose only the pole ζ_β and so,

$$\oint_{[a_0,b_0]\cup\gamma^-} e^{-izt} g_1(z, \beta) dz = c(\varphi) e^{-i\zeta_\beta t} \tag{3.13}$$

where

$$c(\varphi) = c_1 \mathcal{J}(\lambda_\beta^\infty, \beta) + c_2 \mathcal{J}'(\lambda_\beta^\infty, \beta)$$

with $c_1 = -\frac{1}{a_\beta}$ and $c_2 = -\frac{1}{a_\beta}(\zeta_\beta - \lambda_\beta^\infty)$. Therefore,

$$\mathcal{I}_1(t, \beta) = c(\varphi) e^{-i\zeta_\beta t} + \int_\gamma e^{-izt} g_1(z, \beta) dz. \tag{3.14}$$

Now, for all $z \in \gamma$, $g_1(z, \beta) = \beta^2 h_1(z, \beta)$ where

$$h_1(z, \beta) = \frac{(\mathcal{J}(\lambda_\beta^\infty, \beta) + \mathcal{J}'(\lambda_\beta^\infty, \beta)(z - \lambda_\beta^\infty))(p_\beta z + q_\beta)}{|a_\beta|^2 (z - \zeta_\beta)(z - \overline{\zeta_\beta})}.$$

where $p_\beta = \Im \mathcal{F}'(\lambda_\beta^\infty, \beta)$, $q_\beta = \Im \mathcal{F}(\lambda_\beta^\infty, \beta) - \lambda_\beta^\infty \Im \mathcal{F}'(\lambda_\beta^\infty, \beta)$.

By construction, $\inf_{z\in\gamma, 0<|\beta|<\beta_1'} |z - \zeta_\beta| > 0$ and $\inf_{z\in\gamma, 0<|\beta|<\beta_1'} |z - \overline{\zeta_\beta}| > 0$, so the functions $h_1(\cdot, \beta)$ are analytic in some fixed open region containing γ for any $0 < |\beta| < \beta_1'$. This entails that $\sup_{z\in\gamma, 0<|\beta|<\beta_1'} |h_1(z, \beta)|$ and $\sup_{z\in\gamma, 0<|\beta|<\beta_1'} |h_1'(z, \beta)|$ are finite.

Since γ is contained in the lower half-plane, then $|e^{-izt}| \leq 1$ for $z \in \gamma$ and $t \geq 0$. Thus, for $t \geq 0$ and $0 < |\beta| < \beta_1'$, we have

$$\left| \int_\gamma e^{-izt} g_1(z,\beta) dz \right| \leq C\beta^2, \tag{3.15}$$

and

$$\left| \int_\gamma e^{-izt} g_1'(z,\beta) dz \right| \leq C\beta^2. \tag{3.16}$$

for some $C > 0$.

Step 2:
Therefore, if we define

$$R(t,\beta) = \frac{-\beta}{\pi} \left(\int_\gamma e^{-izt} g_1(z,\beta) dz + \mathcal{I}_2(t,\beta) + \mathcal{I}_3(t,\beta) + \mathcal{I}_\partial(t,\beta) \right), \tag{3.17}$$

we have that

$$\mathcal{I}(t,\beta) = \beta c_1(\varphi) e^{-i\zeta_\beta t} + R(t,\beta).$$

According to 3.15, Proposition 19 and Proposition 20, the error term is of order β^3 (for β small enough). This proves item (a).

Note that by using integration by parts, and since $g(a) = 0 = g(b)$ and $g(a_0) = 1 = g(b_0)$, we have the following four equations

$$\begin{aligned}
it \int_\gamma e^{-izt} g_1(z,\beta) dz &= \int_\gamma e^{-izt} g_1'(z,\beta) dz \\
&\quad - e^{-ib_0 t} \mathcal{J}_1(b_0,\beta) \Im G_1(b_0,\beta) + e^{-ia_0 t} \mathcal{J}_1(a_0,\beta) \Im G_1(a_0,\beta) \\
it\mathcal{I}_2(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t} [(\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta)) \Im G(\lambda,\beta)]' d\lambda \\
&\quad - e^{-ib_0 t} (\mathcal{J}(b_0,\beta) - \mathcal{J}_1(b_0,\beta)) \Im G(b_0,\beta) \\
&\quad + e^{-ia_0 t} (\mathcal{J}(a_0,\beta) - \mathcal{J}_1(a_0,\beta)) \Im G(a_0,\beta) \\
it\mathcal{I}_3(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t} [\mathcal{J}_1(\lambda,\beta) \Im (G(\lambda,\beta) - G_1(\lambda,\beta))]' d\lambda \\
&\quad - e^{-ib_0 t} \mathcal{J}_1(b_0,\beta) \Im (G(b_0,\beta) - G_1(b_0,\beta)) \\
&\quad + e^{-ia_0 t} \mathcal{J}_1(a_0,\beta) \Im (G(a_0,\beta) - G_1(a_0,\beta)) \\
it\mathcal{I}_\partial(t,\beta) &= \int_a^{a_0} e^{-i\lambda t} [g(\lambda) \mathcal{J}(\lambda,\beta) \Im G(\lambda,\beta)]' d\lambda \\
&\quad + \int_{b_0}^{b} e^{-i\lambda t} [g(\lambda) \mathcal{J}(\lambda,\beta) \Im G(\lambda,\beta)]' d\lambda \\
&\quad - e^{-ia_0 t} \mathcal{J}(a_0,\beta) \Im G(a_0,\beta) + e^{-ib_0 t} \mathcal{J}(b_0,\beta) \Im G(b_0,\beta)
\end{aligned}$$

which implies that

$$itR(t,\beta) = \frac{-\beta}{\pi}\left(K_1(t,\beta) + K_2(t,\beta) + K_3(t,\beta) + K_\partial(t,\beta)\right)$$

where

$$\begin{aligned}
K_1(t,\beta) &= \int_\gamma e^{-izt} g_1'(z,\beta)dz \\
K_2(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t}[(\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta))\Im G(\lambda,\beta)]' d\lambda \\
K_3(t,\beta) &= \int_{a_0}^{b_0} e^{-i\lambda t}[\mathcal{J}_1(\lambda,\beta)\Im(G(\lambda,\beta) - G_1(\lambda,\beta))]' d\lambda \\
K_\partial(t,\beta) &= \int_{a}^{a_0} e^{-i\lambda t}[g(\lambda)\mathcal{J}(\lambda,\beta)\Im G(\lambda,\beta)]' d\lambda + \int_{b_0}^{b} e^{-i\lambda t}[g(\lambda)\mathcal{J}(\lambda,\beta)\Im G(\lambda,\beta)]' d\lambda
\end{aligned} \tag{3.18}$$

By equation 3.16 and Proposition 20, K_1 and K_∂ are of order β^2. By Proposition 21, K_2 and K_3 are of order $\beta^{\frac{2}{9}}$. This completes the proof.

□

Now we prove the Theorem 8.

Proof of Theorem 8. By Proposition 11, we have that for β small enough,

$$B_{\varphi,\psi}(t,\beta) = C\beta e^{-i\zeta_\beta t} + E(t,\beta),$$

with

$$E(t,\beta) = R(t,\beta) + \widehat{\mathcal{I}}(t,\beta)$$

where $\widehat{\mathcal{I}}(t,\beta)$ is the second integral of the right-hand side of 3.10. Note that this term converges and is of order β by Proposition 10. Using a completely analogous argument as used in Proposition 11(b) we see that $it\widehat{\mathcal{I}}(t,\beta)$ is also of order β. The only difference is that in the first integral (namely $\widehat{\mathcal{I}_1}(t,\beta)$), in the previous Proposition we got an exponencial contribution but now we can skip the pole of $G(\lambda - i0)$ defining a convenient path γ. This proves the first part of the Theorem.

Now, if H_β is reflectionless at ψ, then we have that the function $G(\lambda - i0)$ is purely imaginary so, as before, we can use the exact same approximation argument used for $\mathcal{I}(t,\beta)$ in Proposition 11(a) and show that $\widehat{\mathcal{I}}(t,\beta)$ has also a quasi-exponential representation plus a β^3-order error term. □

Proof of Theorem 9. By equations 3.2, we see that

$$\left| ||Qe^{-iH_\beta t} g(H_\beta)\psi||^2 - |\langle \psi, e^{-iH_\beta t}\psi\rangle|^2 \right| \le \sup_{||\varphi||=1} |B_{\varphi,\psi}(t,\beta)|^2$$

and equation 3.10 says

$$\sup_{||\varphi||=1} |B_\varphi(t,\beta)|^2 \leq \sup_{||\varphi||=1} |\mathcal{I}(t,\beta)|^2 + \sup_{||\varphi||=1} |\widehat{\mathcal{I}}(t,\beta)|^2$$

where the constants of each term of $\mathcal{I}(t,\beta)$ are bounded by some linear combination of $\sup_{\lambda\in[a_0,b_0]} |\mathcal{J}(\lambda,\beta)|$ and $\sup_{\lambda\in[a_0,b_0]} |\mathcal{J}'(\lambda,\beta)|$. See equations 3.13, 3.15, 3.16, and Propositions 19, 20 and 21. Similarly, the constants of $\widehat{\mathcal{I}}(t,\beta)$ are bounded by linear combinations of $\sup_{\lambda\in[a_0,b_0]} |\mathcal{J}(\lambda-i0,\beta)|$ and $\sup_{\lambda\in[a_0,b_0]} |\mathcal{J}'(\lambda-i0,\beta)|$. For Lemma 15, this supremums are bounded.

This and Theorem 8 imply

$$\sup_{||\varphi||=1} |B_{\varphi,\psi}(t,\beta)|^2 = C\beta^2 |e^{-i\zeta_\beta t}|^2 + \begin{cases} o(\beta^2) \text{ for } |t|<1 \\ \frac{1}{t^2} o(\beta^2) \text{ for } |t|>1 \end{cases}.$$

which proved the first assertion. The rest of the proof is straightforward from the fact that $\Im\zeta_\beta = o(\beta^2)$.

□

3.5 Technicalities

First, we need the following technical lemmas for the preliminaries of Section 3.3.

Lemma 13. *Let H be a self-adjoint operator and P an orthogonal projection such that $PH \subset HP$. Then, for all $z \in \rho(H)$,*

$$P^\perp (H-z)^{-1} P^\perp = P^\perp (P^\perp H P^\perp - z)^{-1} P^\perp.$$

Proof. See [14], Lemma 3.1. □

Proposition 14. *(Schur-Livsic-Feshbach-Grushin Formula)*
Let $z \in \rho(H)$. The operator

$$P(H-z)P - PHP^\perp (H^\perp - z)^{-1} P^\perp HP$$

is invertible as an operator from $RanP$ to $RanP$. Moreover,

$$P(H-z)^{-1}P = P(P(H-z)P - PHP^\perp (H^\perp - z)^{-1} P^\perp HP)^{-1} P$$

and

$$P^\perp (H-z)^{-1} P = -P^\perp (H^\perp - z)^{-1} P^\perp HP(H-z)^{-1} P.$$

Proof. The first part is proved in [23]. Since $L^2(\mathbb{R}) = PL^2(\mathbb{R}) \oplus P^\perp L^2(\mathbb{R})$, the operators $(H-z)^{-1}$ and $H-z$ can be represented as an operators matrices:

$$\begin{pmatrix} P(H-z)P & P(H-z)P^\perp \\ P^\perp(H-z)P & P^\perp(H-z)P^\perp \end{pmatrix} = (H-z)$$

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} := \begin{pmatrix} P(H-z)^{-1}P & P(H-z)^{-1}P^{\perp} \\ P^{\perp}(H-z)^{-1}P & P^{\perp}(H-z)^{-1}P^{\perp} \end{pmatrix} = (H-z)^{-1}.$$

So

$$\begin{cases} y_{21}P(H-z)P + y_{22}P^{\perp}(H-z)P = 0 \\ y_{21}P(H-z)P^{\perp} + y_{22}P^{\perp}(H-z)P^{\perp} = I_{rang(P^{\perp})}. \end{cases}$$

The second equation implies that

$$y_{22} = P^{\perp}(H-z)^{-1}P^{\perp} - y_{21}PHP^{\perp}(H-z)^{-1}P^{\perp}$$

and substituting in the first equation we obtain

$$y_{21}P(H-z)P + P^{\perp}(H-z)^{-1}P^{\perp}HP - y_{21}PHP^{\perp}(H-z)^{-1}P^{\perp}HP = 0$$

$$\Longrightarrow y_{21} = -P^{\perp}(H-z)^{-1}P^{\perp}HP(H-z)^{-1}P$$

□

The next lemma, known as the Limiting Absorption Principle, gives us regularity enough over the functions $\mathcal{F}$, $\mathcal{J}$ and its derivatives with respecto to λ when we take the limit to the real line. This is critical. Hypothesis $(H1)-(H4)$ are not necessary but sufficient.

Lemma 15. *Assume the set of hypothesis $(H1)$ to $(H4)$. There exists an interval I containing λ_0 and $\beta_0 > 0$ such that for $|\beta| < \beta_0$ and all $\lambda \in I$ the norm limits*

$$\mathcal{F}(\lambda \pm i0, \beta) = \lim_{\epsilon \downarrow 0} \mathcal{F}(\lambda \pm i\epsilon, \beta)$$

$$\mathcal{J}(\lambda \pm i0, \beta) = \lim_{\epsilon \downarrow 0} \mathcal{J}(\lambda \pm i\epsilon, \beta)$$

exists and are bounded uniformly in λ and β. Furthermore, $\mathcal{J}(\lambda \pm i0, \beta)$ and $\mathcal{F}(\lambda \pm i0, \beta)$ are $C^1(I^{\pm} \times [-\beta_0, \beta_0])$ and its derivatives are $\frac{1}{9}$-Hölder continuous on $I^{\pm} \times [-\beta_0, \beta_0]$.

Proof. Under the assumption of $\psi \in D(A^2)$, by a complete analogous proof than Lemma 8.11 in [22], the lemma follows from Theorem 2.2 in [24]. In [21] the authors prove that $\psi \in D(A^2)$ is consequence of the regularity of $ad_A^3(H_0)$.

□

The next technical results are used in order to prove Proposition 11. Lemmas 16, 17 and 18 are borrowed from [14].

Lemma 16. *Let $(\mu, \nu) \in [0, \infty[\times [0, \infty[$, $0 \le |\beta| < \beta_1$ and $z_\beta = \lambda_\beta^\infty - i\beta^2 \Im F(\lambda_\beta^\infty, \beta) = \lambda_\beta - \beta^2 \mathcal{F}(\lambda_\beta^\infty, \beta)$. There exist $C > 0$ and $0 < \beta_2 < \beta_1$, such that for any $0 < |\beta| < \beta_2$,*

$$\int_{a_0}^{b_0} \frac{|\lambda - \Re z_\beta|^\mu}{|\lambda - z_\beta|^\nu} d\lambda \le \begin{cases} C\beta^{2(\mu-\nu+1)} & \textit{if} \quad \mu-\nu+1 < 0 \\ C|\log|\beta|| & \textit{if} \quad \mu-\nu+1 = 0 \\ C & \textit{if} \quad \mu-\nu+1 > 0. \end{cases}$$

Proof. See [14], Lemma 2.2.

□

Lemma 17. *There exist $C > 0$ such that for any $\lambda \in [a_0, b_0]$, $0 < |\beta| < \beta_1$:*

$$|G(\lambda,\beta) - G_1(\lambda,\beta)| \leq C\beta^2|\widehat{G}(\lambda,\beta)|^2|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}$$

Proof. See [14] Lemma 2.3.

□

Lemma 18. *There exists some $C > 0$ such that for any $\lambda \in [a_0, b_0]$ and $0 < |\beta| < \beta_1$:*

$$|G'(\lambda,\beta) - G_1'(\lambda,\beta)| \leq C\beta^2|\widehat{G}(\lambda,\beta)|^2|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}}$$

Proof. See [14] Lemma 2.4. □

Proposition 19. *For β small enough, there exists $C > 0$ such that*

$$|\mathcal{I}_2(t,\beta)| \leq C\beta^2 \quad \textit{and} \quad |\mathcal{I}_3(t,\beta)| \leq C\beta^2.$$

Proof. For $\lambda \in [a_0, b_0]$ and $\beta \leq \beta_1$, it holds that:

$$\begin{aligned}
|\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta)| &= |\mathcal{J}(\lambda,\beta) - \mathcal{J}(\lambda_\beta^\infty,\beta) - \mathcal{J}'(\lambda_\beta^\infty,\beta)(\lambda - \lambda_\beta^\infty)| \\
&= \left|\int_{\lambda_\beta^\infty}^{\lambda} \mathcal{J}'(\mu,\beta) - \mathcal{J}'(\lambda_\beta^\infty,\beta)d\mu\right| \\
&\leq \sup_{\mu\in[\lambda_\beta^\infty,\lambda]} |\mathcal{J}'(\mu,\beta)||\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}
\end{aligned} \tag{3.19}$$

where the last inequality is consequence of Lemma 15.

Consequence of equation 3.12 is

$$\begin{aligned}
&\left|\int_{a_0}^{b_0} e^{-i\lambda t}(\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta))\Im G(\lambda,\beta)d\lambda\right| \\
&\leq \int_{a_0}^{b_0} \sup_{\mu\in[\lambda_\beta^\infty,\lambda]} |\mathcal{J}'(\mu,\beta)||\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}|\Im G(\lambda,\beta)|d\lambda \\
&= \sup_{\lambda\in[a_0,b_0]} |\mathcal{J}'(\lambda,\beta)|\beta^2 \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}\Im\mathcal{F}(\lambda,\beta)}{|D(\lambda,\beta)|^2}d\lambda \\
&\leq c \sup_{\lambda\in[a_0,b_0]} |\mathcal{J}'(\lambda,\beta)|\beta^2 \sup_{\lambda\in[a_0,b_0]} |\Im\mathcal{F}(\lambda,\beta)| \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\lambda - \lambda_\beta^\infty + i\beta^2\Im\mathcal{F}(\lambda_\beta^\infty,\beta)|^2}d\lambda
\end{aligned} \tag{3.20}$$

where the last integral converges due to Lemma 16 for $0 \leq |\beta| \leq \beta_2$. So,

$$|\mathcal{I}_2(t,\beta)| \leq C\beta^2$$

if $0 < |\beta| < \beta_2$ for some $C > 0$.

By Lemmas 16 and 17, if $0 < |\beta| < \beta_2$ there exist $C > 0$ such that

$$\begin{aligned} |\mathcal{I}_3(t,\beta)| &= \left| \int_{a_0}^{b_0} e^{-i\lambda t} \mathcal{J}_1(\lambda,\beta) \Im(G(\lambda,\beta) - G_1(\lambda,\beta)) d\lambda \right| \\ &\leq \sup_{\lambda \in [a_0,b_0]} |\mathcal{J}(\lambda,\beta)| \int_{a_0}^{b_0} |G(\lambda,\beta) - G_1(\lambda,\beta)| d\lambda \\ &\leq \sup_{\lambda \in [a_0,b_0]} |\mathcal{J}(\lambda,\beta)| \beta^2 \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\lambda - \lambda_\beta^\infty + i\beta^2 \Im\mathcal{F}(\lambda_\beta^\infty,\beta)|^2} d\lambda \\ &\leq C\beta^2. \end{aligned} \tag{3.21}$$

□

Proposition 20. *For β small enough, it holds that*

$$|\mathcal{I}_\partial(t,\beta)| \leq C\beta^2 \quad \textit{and} \quad |K_\partial(t,\beta)| \leq C\beta^2.$$

Proof. By definition of β_1, for all $\lambda \in [a, a_0]$, $|\beta| < \beta_1$, $|D(\lambda,\beta)| > a_0 + \delta_1 - \lambda \geq \delta_1 > 0$. Also, if $\beta \neq 0$,

$$\Im G(\lambda,\beta) = \beta^2 \frac{\Im\mathcal{F}(\lambda,\beta)}{|D(\lambda,\beta)|^2}. \tag{3.22}$$

As $\mathcal{F}$ is bounded, we have that for $0 < |\beta| < \beta_1$,

$$\begin{aligned} &\left| \int_a^{a_0} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda,\beta) \Im G(\lambda,\beta) d\lambda \right| \\ &\leq \beta^2 \sup_{\lambda \in [a_0,b_0]} |\mathcal{F}(\lambda,\beta)| \sup_{\lambda \in [a_0,b_0]} |\mathcal{J}(\lambda,\beta)| \int_a^{a_0} \frac{1}{(a_0 + \delta_1 - \lambda)^2} d\lambda \\ &\leq C\beta^2 \end{aligned} \tag{3.23}$$

for some $C > 0$ and similarly

$$\left| \int_{b_0}^{b} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda,\beta) \Im G(\lambda,\beta) d\lambda \right| \leq C\beta^2$$

and so

$$|\mathcal{I}_\partial(t,\beta)| \leq C\beta^2. \tag{3.24}$$

On the other side, for $\lambda \in [a, a_0]$,

$$\begin{aligned} K_\partial(t,\beta) = &\int_a^{a_0} e^{-i\lambda t} g'(\lambda) \mathcal{J}(\lambda,\beta) \Im G(\lambda,\beta) d\lambda \\ &+ \int_a^{a_0} e^{-i\lambda t} g(\lambda) \mathcal{J}'(\lambda,\beta) \Im G(\lambda,\beta) d\lambda \\ &+ \int_a^{a_0} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda,\beta) \Im G'(\lambda,\beta) d\lambda \end{aligned}$$

By the same argument in equation 3.23, if $0 < |\beta| < \beta_1$, the first and the second integrals are of order β^2. From 3.22, we have

$$\Im G'(\lambda,\beta) = \beta^2 \frac{\Im \mathcal{F}'(\lambda,\beta)}{|D(\lambda,\beta)|^2} - 2\beta^2 \frac{\Re(\overline{D(\lambda,\beta)} D'(\lambda,\beta))\Im \mathcal{F}(\lambda,\beta)}{|D(\lambda,\beta)|^4} \tag{3.25}$$

with $D'(\lambda,\beta) = -1 - \beta^2 \mathcal{F}(\lambda,\beta)$. It follows that for $t \in \mathbb{R}$, $0 < |\beta| < \beta_1$:

$$\begin{aligned}
&\left| \int_a^{a_0} e^{-i\lambda t} g(\lambda) \mathcal{J}(\lambda,\beta) \Im G'(\lambda,\beta) d\lambda \right| \\
&\leq c_4(\varphi)\beta^2 \int_a^{a_0} \frac{g(\lambda)|\Im \mathcal{F}(\lambda,\beta)| + |g'(\lambda)||\Im \mathcal{F}'(\lambda,\beta)|}{(a_0+\delta_1-\lambda)^2} d\lambda \\
&+ 2c_4(\varphi)\beta^2 \int_a^{a_0} \frac{(1+\beta^2|\mathcal{F}'(\lambda,\beta)|)|\Im \mathcal{F}(\lambda,\beta)|}{(a_0+\delta_1-\lambda)^3} d\lambda \\
&\leq C\beta^2
\end{aligned} \tag{3.26}$$

for some $C > 0$. A similar procedure applies for $\lambda \in [b_0, b]$.

□

Proposition 21. *For any small enough β, there exists $C > 0$ such that*

$$|K_2(t,\beta)| \leq \beta^{\frac{2}{9}} \quad \textit{and} \quad |K_3(t,\beta)| \leq \beta^{\frac{2}{9}}.$$

Proof. Let $0 < |\beta| \leq \beta_1$.

After derivate, for K_2 we have

$$\begin{aligned}
|K_2| &\leq \int_{a_0}^{b_0} |\mathcal{J}'(\lambda_\beta^\infty)||\Im(G(\lambda,\beta) - G_1(\lambda,\beta))| d\lambda + \int_{a_0}^{b_0} |\mathcal{J}_1(\lambda)||\Im(G'(\lambda,\beta) - G_1'(\lambda,\beta))| d\lambda \\
&\leq |\mathcal{J}'(\lambda_\beta^\infty)| \int_{a_0}^{b_0} |G(\lambda,\beta) - G_1(\lambda,\beta)| d\lambda \\
&+ \sup_{\lambda \in [a_0,b_0]} |\mathcal{J}(\lambda_\beta^\infty,\beta) + \mathcal{J}'(\lambda_\beta^\infty,\beta)(\lambda - \lambda_\beta^\infty)| \int_{a_0}^{b_0} |G'(\lambda,\beta) - G_1'(\lambda,\beta)| d\lambda
\end{aligned} \tag{3.27}$$

By Lemmas 17 and 18, we have that

$$\int_{a_0}^{b_0} |G(\lambda,\beta) - G_1(\lambda,\beta)| d\lambda \leq \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\lambda - \lambda_\beta^\infty + i\beta^2 \Im \mathcal{F}(\lambda_\beta^\infty,\beta)|^2} d\lambda$$

and

$$\int_{a_0}^{b_0} |G'(\lambda,\beta) - G_1'(\lambda,\beta)| d\lambda \leq \beta^2 \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}}}{|\lambda - \lambda_\beta^\infty + i\beta^2 \Im \mathcal{F}(\lambda_\beta^\infty,\beta)|^2} d\lambda.$$

Now, if $0 < |\beta| < \beta_2$, Lemma 16 implies that the first integral is of order β^2 and the second one is, up to a positive multiplicative constant, bounded by $\beta^{\frac{2}{9}}$.

For K_3 holds that

$$\begin{aligned}|K_3| \leq &\int_{a_0}^{b_0} |\mathcal{J}'(\lambda,\beta) - \mathcal{J}_1'(\lambda,\beta)||\Im G(\lambda,\beta)|d\lambda \\ &+ \int_{a_0}^{b_0} |\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta)||\Im G'(\lambda,\beta)|d\lambda.\end{aligned}$$

By equations 3.19, 3.22 and 3.12 for the first integral we have there exists $C > 0$ such that

$$\begin{aligned}\int_{a_0}^{b_0} |\mathcal{J}'(\lambda,\beta) - \mathcal{J}_1'(\lambda,\beta)||\Im G(\lambda,\beta)|d\lambda &\leq C\beta^2 \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|D(\lambda,\beta)|^2} d\lambda \\ &\leq C\beta^2 \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\widehat{D}(\lambda,\beta)|^2} d\lambda\end{aligned} \tag{3.28}$$

where the last integral converges due to 16 for $0 < |\beta| < \beta_2$. For the second integral, because of equations 3.19 and 3.25 we have there exists some $C > 0$ and $C' > 0$ such that

$$\begin{aligned}\int_{a_0}^{b_0} |\mathcal{J}(\lambda,\beta) - \mathcal{J}_1(\lambda,\beta)||\Im G'(\lambda,\beta)|d\lambda &\leq C\beta^2 \int_{a_0}^{b_0} |\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1} \frac{|\Im\mathcal{F}'(\lambda,\beta)|}{|D(\lambda,\beta)|^2} d\lambda \\ &+ C'\beta^2 \int_{a_0}^{b_0} |\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1} \frac{\Re(\overline{D(\lambda,\beta)}D'(\lambda,\beta))\Im\mathcal{F}(\lambda,\beta)}{|D(\lambda,\beta)|^4} d\lambda \\ &\leq \beta^2 \left(C \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\widehat{D}(\lambda,\beta)|^2} d\lambda + C' \int_{a_0}^{b_0} \frac{|\lambda - \lambda_\beta^\infty|^{\frac{1}{9}+1}}{|\widehat{D}(\lambda,\beta)|^3} d\lambda \right)\end{aligned} \tag{3.29}$$

if $0 < |\beta| < \beta_2$. Consequence of Lemma 16, just like with K_2, with $\mu = \frac{1}{9} + 1$ and $\nu = 3$, is that the second term of the RHS is the dominant term and so, the integral in the LHS, and thus K_3, is bounded, up to a positive multiplicative constant, by $\beta^{\frac{2}{9}}$.

$\square$

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