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Colouring exact distance graphs of chordal graphs

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ABSTRACT

For a graph $G = (V, E)$ and positive integer p, the *exact distance-p graph* $G^{[\natural p]}$ is the graph with vertex set V and with an edge between vertices x and y if and only if x and y have distance p. Recently, there has been an effort to obtain bounds on the chromatic number $\chi(G^{[\natural p]})$ of exact distance-p graphs for G from certain classes of graphs. In particular, if a graph G has tree-width t, it has been shown that $\chi(G^{[\natural p]}) \in \mathcal{O}(p^{t-1})$ for odd p, and $\chi(G^{[\natural p]}) \in \mathcal{O}(p^t \Delta(G))$ for even p. We show that if G is chordal and has tree-width t, then $\chi(G^{[\natural p]}) \in \mathcal{O}(p\,t^2)$ for odd p, and $\chi(G^{[\natural p]}) \in \mathcal{O}(p\,t^2 \Delta(G))$ for even p.

If we could show that for every graph H of tree-width t there is a chordal graph G of tree-width t which contains H as an *isometric subgraph* (i.e., a distance preserving subgraph), then our results would extend to all graphs of tree-width t. While we cannot do this, we show that for every graph H of genus g there is a graph G which is a triangulation of genus g and contains H as an isometric subgraph.

1. Introduction

All graphs in this paper are assumed to be finite, undirected, simple and without loops. For a graph $G = (V, E)$ and vertices $u, v \in V$, we denote by $d_G(u, v)$ (or $d(u, v)$ when there is no danger of ambiguity) the distance between u and v, that is, the number of edges in a shortest path between u and v.

For a positive integer p, the *pth power* graph $G^p = (V, E^p)$ of G has the same vertex set as G, and the pair uv belongs to E^p if and only if $d_G(u, v) \leq p$.

Problems related to the chromatic number of graph powers were first considered by Kramer and Kramer [11,12] in 1969 and have enjoyed significant attention ever since. It is clear that for $p \geq 2$ any power of a star is a clique. Hence, in order to obtain bounds on $\chi(G^p)$ we need to use the maximum degree $\Delta(G)$ of G. One can easily see that any graph G with $\Delta(G) \geq 3$ satisfies

$$\chi(G^p) \leq 1 + \Delta(G^p) \leq 1 + \Delta(G) \cdot \sum_{i=0}^{p-1} (\Delta(G) - 1)^i \in \mathcal{O}(\Delta(G)^p).$$

However, there are many classes of graphs for which much better bounds can be obtained. Recall that a graph is *k-degenerate* if every subgraph of G contains a vertex of degree at most k. Parametrising in terms of the degeneracy, Agnarsson and Halldórsson [1] gave upper bounds for many classes of graphs.

Theorem 1.1 (*Agnarsson and Halldórsson [1]*). *Let k and p be positive integers. There exists $c = c(k, p)$ such that for every k-degenerate graph G we have $\chi(G^p) \leq c \cdot \Delta(G)^{\lfloor p/2 \rfloor}$.*

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Note that the exponent on $\Delta(G)$ in this result is best possible, even for the class of trees, as the complete Δ-ary tree with radius $\lfloor p/2 \rfloor$ attests [1].

For some classes of graphs it is possible to obtain similar bounds without parametrising in terms of the degeneracy. Recall that a graph G is chordal if every cycle of G has a chord, i.e., if every induced cycle is a triangle. In [10], Král' proved that every chordal graph G with maximum degree Δ satisfies $\chi(G^p) \in \mathcal{O}(\sqrt{p}\Delta^{(p+1)/2})$ for even p, and $\chi(G^p) \in \mathcal{O}(\Delta^{(p+1)/2})$ for odd p. Král' also showed that this upper bound for odd p is tight. It is worth mentioning that, in order to obtain this tight upper bound, Král' gave a simple proof of the already known fact that odd powers of chordal graphs are also chordal [2,7].

Given that graphs with tree-width at most t have degeneracy at most t, Theorem 1.1 gives us an upper bound on $\chi(G^p)$ when G belongs to a graph class with bounded tree-width. Although tree-width is usually defined in terms of tree-decompositions, an equivalent definition can be given in terms of chordal graphs, as follows.

Definition 1.2. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the smallest integer t such that G is a subgraph of a chordal graph with clique number $t + 1$.

A notion related to graph powers is that of exact distance graphs. For a positive integer p, the *exact distance-p graph* $G^{[\natural p]} = (V, E^{[\natural p]})$ of G has the same vertex set as G, and the pair uv belongs to $E^{[\natural p]}$ if and only if $d_G(u, v) = p$. Clearly, $E^{[\natural p]}$ is a subset of the edge set of G^p, which means that $\chi(G^{[\natural p]}) \leq \chi(G^p)$. We immediately see that Theorem 1.1 also gives an upper bound for the chromatic number of $G^{[\natural p]}$ in terms of the degeneracy and the maximum degree of G. However, when considering exact distance graphs, this upper bound is far from best possible. This is attested, for instance, by the following recent result of Van den Heuvel, Kierstead and Quiroz [9].

Theorem 1.3 (*Van den Heuvel et al. [9]*)**.**

(a) Let p be an odd integer. For every graph G with tree-width at most t we have $\chi(G^{[\natural p]}) \leq t \cdot \binom{p+t-1}{t} + 1 \in \mathcal{O}(p^{t-1})$.

(b) Let p be an even integer. For every graph G with tree-width at most t we have $\chi(G^{[\natural p]}) \leq \left(t \cdot \binom{p+t}{t} + 1\right) \cdot \Delta(G) \in \mathcal{O}(p^t \cdot \Delta(G))$.

In the following sense, this result actually extends to all classes excluding a fixed minor. Let $\mathcal{K}$ be a graph class excluding a fixed minor. It was first shown in [14] that for every odd integer p, there exists a constant $N = N(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{[\natural p]}) \leq N$. In [9] it is shown that for every even p there exists a constant $N' = N'(\mathcal{K}, p)$ such that for every graph $G \in \mathcal{K}$ we have $\chi(G^{[\natural p]}) \leq N' \cdot \Delta(G)$. Moreover, these results remain true if we take $\mathcal{K}$ to be any class with bounded expansion.

Our main result is a significant improvement on the bounds of Theorem 1.3 for chordal graphs.

Theorem 1.4. *Let G be a chordal graph with clique number $t \geq 2$.*

(a) For every odd integer $p \geq 1$ we have $\chi(G^{[\natural p]}) \leq \binom{t}{2} \cdot (p+1)$.

(b) For every even integer $p \geq 2$ we have $\chi(G^{[\natural p]}) \leq \binom{t}{2} \cdot \Delta(G) \cdot (p+1)$.

This result is implied by the more general Theorem 1.5 below.

Although Definition 1.2 tells us that every graph of tree-width t is a subgraph of a chordal graph with clique number $t + 1$, Theorem 1.4 does not extend to all graphs with tree-width at most t. We shall say more about this at the end of this section. Before that, let us state the full generality of our results.

For two (labelled) graphs $G = (V, E)$ and $G' = (V, E')$ on the same vertex set, define $G \cup G' = (V, E \cup E')$. For a fixed positive integer p, Theorem 1.4 trivially gives $\chi(G^{[\natural p_1]} \cup G^{[\natural p_2]} \cup \cdots \cup G^{[\natural p_s]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)^s$ for any subset $\{p_1, p_2, \ldots, p_s\}$ of $\{1, 2, \ldots, p\}$ with q even elements. Notice that if we take $\{p_1, p_2, \ldots, p_s\} = \{1, 2, \ldots, p\}$, then we have $G^{[\natural p_1]} \cup G^{[\natural p_2]} \cup \cdots \cup G^{[\natural p_s]} = G^p$, meaning that Theorem 1.4 implies a version of Theorem 1.1 for chordal graphs. Taking a subset of even integers turns out to be quite different from taking a subset of odd integers. For even p, we note that the complete Δ-ary tree of radius $\lfloor p/2 \rfloor$, $T_{\Delta,p}$, shows that $\chi(T_{\Delta,p}^{[\natural 2]} \cup T_{\Delta,p}^{[\natural 4]} \cup \ldots \cup T_{\Delta,p}^{[\natural p]}) \in \Omega(\Delta^{p/2})$. Hence, the bound of Theorem 1.1 gives again the right exponent on $\Delta(G)$. In contrast, we see that for odd p we obtain an upper bound on $\chi(G^{[\natural 1]} \cup G^{[\natural 3]} \cup \cdots \cup G^{[\natural p]})$ which does not depend on $\Delta(G)$. However, these trivial upper bounds stop being linear in p, even if we simply consider $\chi(G^{[\natural (p-2)]} \cup G^{[\natural p]})$.

We prove Theorem 1.4 by proving the following stronger result which gives upper bounds on the chromatic number of all these gradations between $G^{[\natural p]}$ and G^p. For instance, these upper bounds are linear in p if the size of the subsets of $\{1, 2, \ldots p\}$ considered does not grow with p.

Theorem 1.5. *Let G be a chordal graph with clique number $t \geq 2$. Let p be a positive integer, $S = \{p_1, p_2, \ldots, p_s\} \subseteq \{1, 2, \ldots, p\}$ and q be the number of even integers in S.*

(a) If $1 \notin S$, then we have $\chi(G^{[\natural p_1]} \cup G^{[\natural p_2]} \cup \cdots \cup G^{[\natural p_s]}) \leq \binom{t}{2}^s \cdot \Delta(G)^q \cdot (p+1)$.

(b) If $1 \in S$, then we have $\chi(G^{[\natural p_1]} \cup G^{[\natural p_2]} \cup \cdots \cup G^{[\natural p_s]}) \leq t \cdot \binom{t}{2}^{s-1} \cdot \Delta(G)^q \cdot (p+1)$.

Of course, if $S = \{1\}$ then we have that $\chi(G^{[\natural p_1]} \cup G^{[\natural p_2]} \cup \cdots \cup G^{[\natural p_s]}) = \chi(G) = t$, given the well known fact that chordal graphs are perfect and hence satisfy $\chi(G) = \omega(G)$.

We obtain Theorem 1.5 by partitioning the graph G into levels. We fix a vertex $x \in V(G)$ and we define the level ℓ as the set of vertices having distance ℓ with x. We bound the number of colours needed to colour one level and then give different colours to levels which are at distance at most p. Apart from being natural in the context of exact distance graphs, this simple levelling argument is regularly used in colouring problems related to perfect graphs. (The real problem is, of course, in the analysis of each level.) Kündgen and Pelsmajer [13] used level partitions of chordal graphs to find an upper bound on the number of colours needed in a nonrepetitive colouring of a graph with tree-width t. Level partitions also play a key role in a series of papers of Chudnovsky, Scott, Seymour and Spirkl which starts off by proving a conjecture of Gyárfás stating that there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G with no odd hole [16].

Together with level partitions, the notion of an adjacent-cliques graph is key in our proof of Theorem 1.5. For a graph G and two cliques K and K^* in G, we say that K and K^* are *adjacent* if they are vertex-disjoint and there are vertices $x \in K$, $y \in K^*$ with $xy \in E(G)$. The *adjacent-cliques graph* $AC(G)$ of a graph G has a vertex for each clique of G, and two vertices K and K^* of $AC(G)$ are adjacent if and only if their corresponding cliques in G are adjacent. We prove the following result for the chromatic number of $AC(G)$ when G is chordal.

Theorem 1.6. *Let G be a chordal graph with clique number at most t. We have $\chi(AC(G)) \leq \binom{t+1}{2}$.*

We denote the line graph of a graph G by $L(G)$. It is easy to see that $AC(G)$ contains G and $L(G)^{[\natural 2]}$ as subgraphs. Hence, Theorem 1.6 tells us that for all chordal graphs G with clique number t there is a constant $c(t)$ such that $\chi(L(G)^{[\natural 2]}) \leq c(t)$. Contrast this with the fact that even for trees, $L(G)^{[\natural p]}$ can have arbitrarily large cliques if p is odd (consider stars and subdivided stars). Whether or not there are similar constant upper bounds on $\chi(L(G)^{[\natural p]})$ for all even $p \geq 4$, we leave as an open problem. Also, in light of known results about the strong chromatic index [5], we conjecture that for every k there is a constant $c(k)$, such that $\chi(L(G)^{[\natural 2]}) \leq c(k)$ for every k degenerate graph G.

Let H be a subgraph of a graph G. We say H is an *isometric subgraph* of G if $d_H(u, v) = d_G(u, v)$ for every $u, v \in V(H)$. Note that if H is an isometric subgraph of G, then $\chi(H^{[\natural p]}) \leq \chi(G^{[\natural p]})$ for every positive integer p. Thus, if we could show that for every graph H of tree-width t there is a chordal graph G of tree-width t which contains H as an isometric subgraph, then we could extend Theorems 1.4 and 1.5 to all graphs of tree-width t. While we cannot do this we prove the following.

Proposition 1.7. *Let H be a graph with genus $g \geq 0$. There is a graph G which is a triangulation of genus g and contains H as an isometric subgraph.*

While there are results showing, for instance, that every graph is an isometric subgraph of a vertex-transitive graph [4], Proposition 1.7 is new as far as we are aware.

By Proposition 1.7 we have, for instance, that an upper bound on $\chi(G^{[\natural p]})$ for all planar triangulations implies an upper bound on $\chi(G^{[\natural p]})$ for all planar graphs. In [9], Van den Heuvel et al. prove that $\chi(G^{[\natural 3]}) \leq 105$ for all planar graphs G. While this represents a major improvement on previous upper bounds (a bound of $5 \cdot 2^{20,971,522}$ is implied by [14], and one of $70 \cdot 2^{70}$ by [17]), it is still far from the lower bound of 7, given also in [9]. Proposition 1.7 opens the gates for classical techniques for colouring triangulations of planar graphs to be applied on this problem.

The rest of the paper is organised as follows. In Section 2 we study the properties of level partitions of chordal graphs which will be essential for the proof of Theorem 1.5. In Section 3 we prove Theorem 1.6, and in Section 4 we complete the proof of Theorem 1.5. In Section 5 we prove Proposition 1.7. We conclude with a short discussion on lower bounds and by mentioning some open problems.

2. Level partitions of chordal graphs

Let G be a graph and x be a fixed vertex of G. For any positive integer ℓ, set $N^\ell(x) = \{v \in V(G) \mid d(v, x) = \ell\}$. We call $N^\ell(x)$ the *ℓ-th level* of G with respect to x, and if we set $N^0(x) = \{x\}$ we get that these levels partition the vertices of the connected component of G containing x. We also set $N^{<\ell}(x) = \bigcup_{i<\ell} N^i(x)$ and $N^{>\ell}(x) = \bigcup_{i>\ell} N^i(x)$.

Let G_ℓ, $G_{<\ell}$ and $G_{>\ell}$ be the graphs induced by $N^\ell(x)$, $N^{<\ell}(x)$ and $N^{>\ell}(x)$, respectively. Define the *ℓ-shadow* of a subgraph H of G as the set of vertices in $N^\ell(x)$ which have a neighbour in $V(H)$. We say that G is *shadow complete* (with respect to x) if for every non-negative integer ℓ, the ℓ-shadow of every connected component of $G_{>\ell}$ induces a complete graph.

Using a well-known theorem of Dirac [6] which characterises chordal graphs in terms of their minimal vertex cut sets, Kündgen and Pelsmajer [13] proved that connected chordal graphs are shadow complete with respect to any vertex.

Lemma 2.1 (*Kündgen and Pelsmajer [13]*)**.** *Let G be a connected chordal graph with clique number $t \geq 2$ and let x be any vertex in $V(G)$. Then G is shadow complete with respect to x and every G_ℓ is a chordal graph with clique number strictly smaller than t.*

Before we start to see some implications of this lemma, let us state one additional definition. We say that a vertex $v \in N^{\ell}(x)$, is an *ancestor* (with respect to x) of a vertex $u \in N^{m}(x)$, $\ell < m$, if there is a path between u and v of length $m - \ell$. Clearly, any such path has exactly one vertex in each level $N^{\ell}(x), N^{\ell+1}(x), \ldots, N^{m}(x)$.

The following result follows directly from Lemma 2.1.

Corollary 2.2. *Let G be a connected chordal graph, $x \in V(G)$, and $u, v \in N^{\ell}(x)$ for some positive integer ℓ. If u and v are both ancestors of some $y \in N^{>\ell}(x)$, then u and v are neighbours.*

With a bit more care we can prove that if two vertices are at the same level ℓ and are at distance p, then their ancestors at level $\ell - \lfloor p/2 \rfloor$ form cliques which either intersect or are adjacent.

Lemma 2.3. *Let G be a connected chordal graph, $x \in V(G)$, and $u, v \in N^{\ell}(x)$ for some positive integer ℓ. Suppose $d(u, v) = p \geq 2$ and let K_u, K_v be the (complete) graphs induced by the ancestors of u and v in $N^{\ell-\lfloor p/2 \rfloor}(x)$, respectively. We have that*

(a) if p is odd, then K_u and K_v are adjacent;
(b) if p is even, then K_u and K_v are adjacent or $K_u \cap K_v \neq \varnothing$.

Proof. Let $k = \lfloor p/2 \rfloor$, and note that we must have $\ell \geq k$ as otherwise there would be a walk from u to v that goes through x and has length $2l < 2k \leq p$, which would contradict $d(u, v) = p$. We prove (a) and (b) simultaneously by considering two possibilities for u and v.

We first consider the case in which u and v are in different components of $G_{>\ell-k}$. In this case it is clear that every path of length p joining u and v must contain a vertex from $G_{\ell-k}$ (and no vertices in $G_{<\ell-k}$). It is also easy to see that if p is even, then every path of length p joining u and v must have exactly one vertex in $G_{\ell-k}$. Since $d(u, v) = p$, this means that $K_u \cap K_v \neq \varnothing$. If p is odd, then every path of length p joining u and v must have exactly two vertices in $G_{\ell-k}$. This implies that K_u and K_v are adjacent.

We are now left to consider the case in which u and v are in the same connected component C of $G_{>\ell-k}$. Let $z \in G_{\ell-k+1}$ be an ancestor of u and let $z' \in G_{\ell-k+1}$ be an ancestor of v. Clearly, z and z' belong to C. We know by Lemma 2.1 that since G is chordal it is shadow complete, and so the neighbours of z and z' in $G_{\ell-k}$ form a clique. This means that either K_u and K_v are adjacent or $K_u \cap K_v \neq \varnothing$. However, if p is odd we cannot have $K_u \cap K_v \neq \varnothing$. $\square$

3. Adjacent-cliques graphs

In this section we prove Theorem 1.6. In order to prove this result we need to recall a specific characterisation of chordal graphs.

A *perfect elimination ordering* of a graph G is a linear ordering L of $V(G)$ such that, for every vertex $v \in V(G)$, the neighbours of v which are smaller than v in L form a clique. The following classical result is proved in [8, Section 7].

Proposition 3.1 (*Fulkerson and Gross [8]*)**.** *A graph is chordal if and only if it has a perfect elimination ordering.*

Proof of Theorem 1.6. By Proposition 3.1 we know G has a perfect elimination ordering. We fix one such ordering L. We say a vertex u is a *predecessor* of a vertex v if $uv \in E(G)$ and $u <_L v$. Moving along the ordering L, we colour the vertices of G in the following way. A vertex v gets a colour $a(v)$ which is different from $a(u)$ if u is a predecessor of v or u is a predecessor of a predecessor of v. Since the clique number of G is at most t and since L is a perfect elimination ordering, each vertex has at most $t-1$ predecessors. Moreover, by choice of L we have that if v has $r \leq t-1$ predecessors, the largest (with respect to L) of its predecessors has at most $t - r$ predecessors which are not already predecessors of v; the second largest predecessor of v has at most $t - (r - 1)$ predecessors which are not already predecessors of v, and so on. Therefore, for any vertex v the set of predecessors and predecessors of a predecessor of v has size at most $r + (t - r) + (t - (r - 1)) + \cdots + (t - 1) \leq (t - 1) + 1 + 2 + \cdots + t - 1 = \binom{t+1}{2} - 1$. And so, the colouring a uses at most $\binom{t+1}{2}$ colours.

We define a colouring c on the vertices of $AC(G)$ in the following way. For every vertex K in $AC(G)$, (i.e., for every clique K in G) we set $\mu(K)$ as the smallest vertex of K with respect to L. Every vertex K is assigned the colour $c(K) = a(\mu(K))$. To prove the theorem, it suffices to show that c is a proper colouring of $AC(G)$.

Let K and K^* be adjacent vertices in $AC(G)$. We must show that we have $a(\mu(K)) \neq a(\mu(K^*))$. Let $u, u' \in K$ and $v, v' \in K^*$ be vertices of G such that $u = \mu(K)$, $v = \mu(K^*)$ and $u'v' \in E(G)$. Without loss of generality we assume that $u' <_L v'$. If $v = v'$, we have that $u'v \in E(G)$. Otherwise, we have that both u' and v are predecessors of v'. Since L is a perfect elimination ordering, we also obtain $u'v \in E(G)$.

Since a is a proper colouring of G, if we have $u = u'$ we immediately get that $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$ as desired. So assume $u \neq u'$. If $v <_L u'$, we have that both u and v are predecessors of u', and so $uv \in E(G)$. This again gives us that $a(\mu(K)) \neq a(\mu(K^*))$. Otherwise, if $u' <_L v$ we have that u' is a predecessor of v. Since u is a predecessor of u' we also obtain that $a(\mu(K)) = a(u) \neq a(v) = a(\mu(K^*))$ by definition of a. $\square$

For later use, we note a property of the colouring c we defined in the previous proof.

Lemma 3.2. *Let G be a chordal graph, and K, K^* clique subgraphs of G. Let c and μ be as defined in the proof of Theorem 1.6. If $K \cap K^* \neq \varnothing$ and $c(K) = c(K^*)$, then we have $\mu(K) = \mu(K^*)$.*

Proof. We prove that if we have $K \cap K^* \neq \varnothing$ and $\mu(K) \neq \mu(K^*)$, then we have $c(K) \neq c(K^*)$. We do this by proving that $\mu(K)$ and $\mu(K^*)$ are adjacent in G. Since a is a proper colouring of G, this will tell us that $c(K) = a(\mu(K)) \neq a(\mu(K^*)) = c(K^*)$ which gives us the result.

If $\mu(K)$ and $\mu(K^*)$ are not adjacent, we have that neither of $\mu(K), \mu(K^*)$ belong to $K \cap K^*$. Therefore, the minimum vertex v in $K \cap K^*$ with respect to L is adjacent to $\mu(K)$ and $\mu(K^*)$, and $\mu(K), \mu(K^*)$ are smaller than v in L. This contradicts the choice of L, since the neighbours of v smaller than v in L must form a clique, and thus be pairwise adjacent. $\square$

4. Exact distance graphs of chordal graphs

Theorem 1.5 will follow from the next lemma.

Lemma 4.1. *Let G be a connected chordal graph with clique number $t \geq 2$, let x be a vertex in G, and $p \geq 2$ an integer. For any non-negative integer ℓ, we have that*

(a) if p is odd, then there is a colouring h of $N^\ell(x)$ using at most $\binom{t}{2}$ colours such that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{[\natural p]})$, then $h(u) \neq h(v)$;

(b) if p is even, then there is a colouring h' of $N^\ell(x)$ using at most $\binom{t}{2} \cdot \Delta(G)$ colours such that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{[\natural p]})$, then $h'(u) \neq h'(v)$.

Proof. Let $k = \lfloor p/2 \rfloor$ and note, just as in the proof of Lemma 2.3, that if $u, v \in N^\ell(x)$ satisfy $uv \in E(G^{[\natural p]})$, then we must have $\ell \geq k$. By Lemma 2.1 we know that $G_{\ell-k}$ is a chordal graph with clique number at most $t-1$. By Theorem 1.6 we know that there is a proper colouring c of the vertices of $AC(G_{\ell-k})$ which uses at most $\binom{t}{2}$ colours.

For each vertex $y \in N^\ell(x)$ we consider the set of vertices $K_y \subseteq N^{\ell-k}(x)$ which are ancestors of y. By Corollary 2.2 we have that K_y is a vertex of $AC(G_{\ell-k})$.

(a) Define the colouring h by assigning $h(y) = c(K_y)$ to every vertex $y \in N^\ell(x)$. Let $u, v \in N^\ell(x)$ be such that $uv \in E(G^{[\natural p]})$. By Lemma 2.3(a) we have $K_u K_v \in E(AC(G_{\ell-k}))$. Therefore, we have $h(u) = c(K_u) \neq c(K_v) = h(v)$, as desired.

(b) For each $w \in N^{\ell-k}(x)$ we choose an injective function $b_w : N(w) \to \{1, \dots, \Delta(G)\}$. For every vertex $y \in N^\ell(x)$ we choose an arbitrary vertex $\sigma(y)$ from $N^{k-1}(y) \cap N(\mu(K_y))$. The colouring h' assigns $h'(y) = (c(K_y), b_{\mu(K_y)}(\sigma(y)))$ to every vertex $y \in N^\ell(x)$. Clearly h' uses at most $\binom{t}{2} \cdot \Delta(G)$ colours.

Let $u, v \in N^\ell(x)$ be such that $uv \in E(G^{[\natural p]})$. We must show that $h'(u) \neq h'(v)$. Suppose we have $K_u \cap K_v = \varnothing$. By Lemma 2.3(b) we know that K_u and K_v are adjacent. As in part (a) we obtain $c(K_u) \neq c(K_v)$ and so $h'(u) \neq h'(v)$, as desired.

We can thus assume we have $K_u \cap K_v \neq \varnothing$. We also assume $c(K_u) = c(K_v)$, as otherwise we would have $h'(u) \neq h'(v)$. By Lemma 3.2 we obtain that the corresponding cliques K_u and K_v satisfy $\mu(K_u) = \mu(K_v)$. Now notice that $\sigma(u)$ must be different from $\sigma(v)$, as otherwise there would be a walk of length $p-2$ joining u and v and going through $\sigma(u)$, which would contradict $d(u, v) = p$. Since $b_{\mu(K_u)}$ is injective, we obtain $h'(u) \neq h'(v)$, as desired. $\square$

Proof of Theorem 1.5. We may assume that G is connected. As we mentioned earlier, this theorem follows from Lemma 4.1. Here we prove (a) and leave (b) to the reader.

Fix a vertex $x \in V(G)$. Define a function $f : V(G) \to \{0, \dots, p\}$ which satisfies $f(u) = k$ for all $u \in N^\ell(x)$ with $\ell \equiv k \mod (p+1)$. For each level $N^\ell(x)$ and integer $p_i \in \{p_1, p_2, \dots, p_s\}$, we define $g_{\ell,i}$ as the colouring of $N^\ell(x)$ guaranteed by Lemma 4.1, which assigns different colours to vertices of $N^\ell(x)$ having distance p_i. To each vertex $u \in N^\ell$ we assign a colour $F(u) = (f(u), g_{\ell,1}(u), g_{\ell,2}(u) \dots, g_{\ell,s}(u))$, and we do this for all ℓ. Notice that for every $1 \leq i \leq s$, each vertex $u \in N^\ell$ can only have distance p_i with vertices not in $N^{<\ell-p}(x) \cup N^{>\ell+p}(x)$. Hence, this colouring guarantees that, for all $1 \leq i \leq s$, u gets a different colour from v whenever u and v have distance p_i. $\square$

5. Proof of Proposition 1.7

We may assume $V(H) \geq 3$, as otherwise the result is trivial. We may also assume that H is connected.

Fix an embedding of H in a surface of genus g. We first construct from H a graph H' of genus g having the property that all of its faces have a cycle as its boundary. This can be done without altering distances by means of the following operation. Suppose $y \in V(H)$ is a cut vertex. There is an ordering $x_1, x_2, \dots, x_{|N(y)|}$ of the vertices in $N(y)$ such that adding an edge between x_i and x_{i+1} (wherever such an edge does not already exist) would not create crossings. Using this ordering we add a path of length 2 between x_i and x_{i+1} (modulo $|N(y)|$) if there is no edge joining the pair. Clearly y ceases to be a cut vertex after this operation, and no new cut vertices are created. We repeat this operation until there are no cut vertices. It is easy to see that H' satisfies that all of its faces have a cycle as its boundary, and that for every $u, v \in V(H)$ we have $d_H(u, v) = d_{H'}(u, v)$.

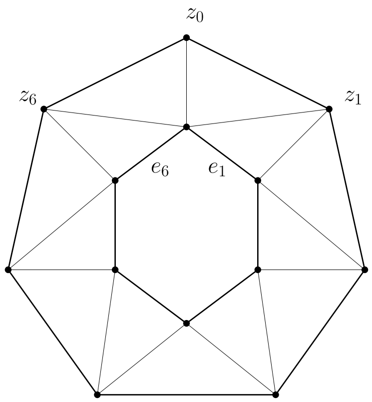

Fig. 1. Drawing a C_6 inside a face of H which has a C_7 as its boundary.

Fig. 2. Chordal graphs G with clique number 3 for which $\omega(G^{odd})$, and hence $\chi(G^{odd})$, can be arbitrarily large.

If H' is not a triangulation of genus g, then there is a face of H having as its boundary a cycle C_k, with vertices $z_0, \dots, z_{k-1}$, for some $k > 3$. Inside this face we draw a cycle C_{k-1} with edges $e_1, \dots, e_{k-1}$. For all $1 \leq i \leq k-1$, we add edges joining z_i with the endvertices of e_i. We also add an edge joining z_0 with the common endvertex of e_1 and e_{k-1}. It is easy to see that this can be done in such a way that no crossings are made, the area between C_k and C_{k-1} is triangulated and C_{k-1} is the boundary of a face. Fig. 1 shows how to do this for $k = 7$. We call the resulting embedded graph F. If F is not a triangulation, we repeat the operation on F, and we do this until we get a graph G which is a triangulation of genus g.

To prove that we have $d_H(u, v) = d_G(u, v)$ for every $u, v \in V(H)$, it is enough to prove that we have $d_F(u, v) = d_H(u, v)$ for every $u, v \in V(H)$. If this was not the case, there would be a pair of vertices $x, y \in V(C_k)$ such that F contains an xy-path of length at most $d_H(x, y) - 1$ with vertices in C_{k-1} and no vertices outside of $C_k \cup C_{k-1}$. Hence, it suffices to show that every pair of vertices $x, y \in V(C_k)$ satisfies $d_{F[C_k \cup C_{k-1}]}(x, y) = d_{C_k}(x, y)$. By the construction inside the face having C_k as its boundary, this is easy to check.

6. Lower bounds and open problems

In [15] the following question from Van den Heuvel and Naserasr is mentioned: Is there a constant C such that for every odd integer p and every planar graph G we have $\chi(G^{[\natural p]}) \leq C$? Very recently Bousquet, Esperet, Harutyunyan, and De Joannis de Verclos [3] gave a negative answer to this question in the following way. They constructed a family of chordal graphs $U_3, U_5, \dots$ with clique number 3 which they proved satisfies $\chi(U_p^{[\natural p]}) \in \Omega(\frac{p}{\log(p)})$. This shows that Theorem 1.4(a) is asymptotically best possible (as p tends to infinity), up to a $\log(p)$ factor.

For any integer $t \geq 1$ and any odd integer $p \geq 1$, it is easy to construct a chordal graph G with clique number t such that $\chi(G^{[\natural p]}) = t$. This can be improved, of course. For instance, Van den Heuvel et al. [9] constructed a chordal graph with clique number 3 such that its exact distance-3 graph has chromatic number 5. But while the upper bound on $\chi(G^{[\natural p]})$ given by Theorem 1.4(a) is quadratic on t, we cannot give any superlinear lower bound on t.

A consequence of Theorem 1.5(a) is that for every odd p, there is a constant $N_{p,t}$ such that $\chi(G^{[\natural 1]} \cup G^{[\natural 3]} \cup \dots \cup G^{[\natural p]}) \leq N_{p,t}$ for all chordal graphs with clique number t. Nešetřil and Ossona de Mendez [15] gave a construction that shows that this constant must grow with p, even for chordal graphs with clique number 3. Fig. 2 gives a simpler construction with the same property. However, we note that the construction given by Nešetřil and Ossona de Mendez can be generalised to show that for every $t \geq 3$ and odd positive p, there is a chordal graph G with clique number t such that $\chi(G^{[\natural 1]} \cup G^{[\natural 3]} \cup \dots \cup G^{[\natural p]}) \in \Omega(t^{\lfloor p/2 \rfloor + 1})$. Meanwhile, Theorem 1.5(a) gives that $\chi(G^{[\natural 1]} \cup G^{[\natural 3]} \cup \dots \cup G^{[\natural p]}) \in \mathcal{O}(t^{2\lfloor p/2 \rfloor + 2})$.

For a graph G, a natural generalisation of $G^{[\natural 1]} \cup G^{[\natural 3]} \cup \dots \cup G^{[\natural p]}$ is the graph G^{odd}, which has the same vertex set as G, and xy is an edge in G^{odd} if and only if x and y have odd distance. Both of the constructions mentioned above tell us that, even for chordal graphs G with clique number 3, the chromatic number of G^{odd} can be arbitrarily large, by witnessing that the clique number $\omega(G^{odd})$ can be arbitrarily large. The fact that these constructions are also planar inspired the following question of Thomassé, which appears in [14] (also [15]).

Problem 6.1 (*[14, Problem 11.2]*)**.** Is there a function f such that for every planar graph G we have $\chi(G^{odd}) \leq f(\omega(G^{odd}))$?

We ask whether there is a function f_t such that for every chordal graph G with clique number t we have $\chi(G^{odd}) \leq f_t(\omega(G^{odd}))$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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