A MINI element over star convex polytopes

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ABSTRACT

In this paper, we extend the concept of MINI element over triangles to star convex arbitrary polytopes. This is achieved by employing the volume averaged nodal projection (VANP) method over polytopes in combination with the strain smoothing technique. Within this framework, the dilatation strain is projected onto the linear approximation space, thus resulting in a purely displacement based formulation. The stability is ensured by enhancing the displacement field with bubble basis functions. The salient features of the proposed method are two fold: the VANP alleviates the locking phenomenon and the strain smoothing suppresses the need to compute the derivative of the basis functions, thus reducing the computational burden. Various benchmark problems in two and three dimensions are numerically solved to demonstrate the robustness, accuracy and the convergence properties of the proposed framework.

1. Introduction

The introduction of elements with arbitrary edges and faces in two and three dimensions, respectively, has revolutionized and generalized the finite element method (FEM) and has led to the development of the polygonal FEM [1–4]. Some of the salient features of the PFEM are: (a) relaxes the restriction on element topology; (b) can work with non-conforming elements and quadtree/octree decomposition without splitting into simplex elements or using constraint equations [5] and (c) offers greater flexibility in meshing complex geometries [3,6,7] and in fracture problems [8]. However, one of the challenges faced with the PFEM is in evaluating the integral of the weak form exactly, which affects the accuracy of the results. The issue is basically due to lack of polynomial consistency of the discrete system causing a limit on the convergence of the finite element solutions [9,10]. Thus, various efforts have been put to develop the quadrature scheme for the PFEM [10–12]. Inspired by the nodal integration scheme proposed for the meshfree methods [13–15] and later extended to the finite element method by

Liu et al. [16], Francis and Natarajan [17–19] employed the cell-based strain smoothing scheme for numerically integrating the terms in the bilinear and linear form.

Due to aforementioned properties, in the recent past, considerable effort has been made to develop methods with polygonal discretizations, viz., virtual element method [20–23], virtual node method [24], smoothed finite element method [18,25], scaled boundary finite element method [17,26,27] and the discontinuous Galerkin method [28]. Since its inception, the PFEM has been employed to solve problems involving large deformations [29], contact problems [30] and fracture mechanics [8]. However, to the best of author's knowledge, only a few papers have employed polygonal finite element method for nearly incompressible elasticity [10,31] and fluid flow [11]. Talischi et al. [11], employed a linearly complete barycentric coordinates with piecewise constant pressure interpolation over Voronoi type meshes. It was shown that the formulation was unconditionally stable without any additional treatment. In their formulation, pressure was treated as additional variable, moreover, the pressure is discontinuous across the element boundaries.
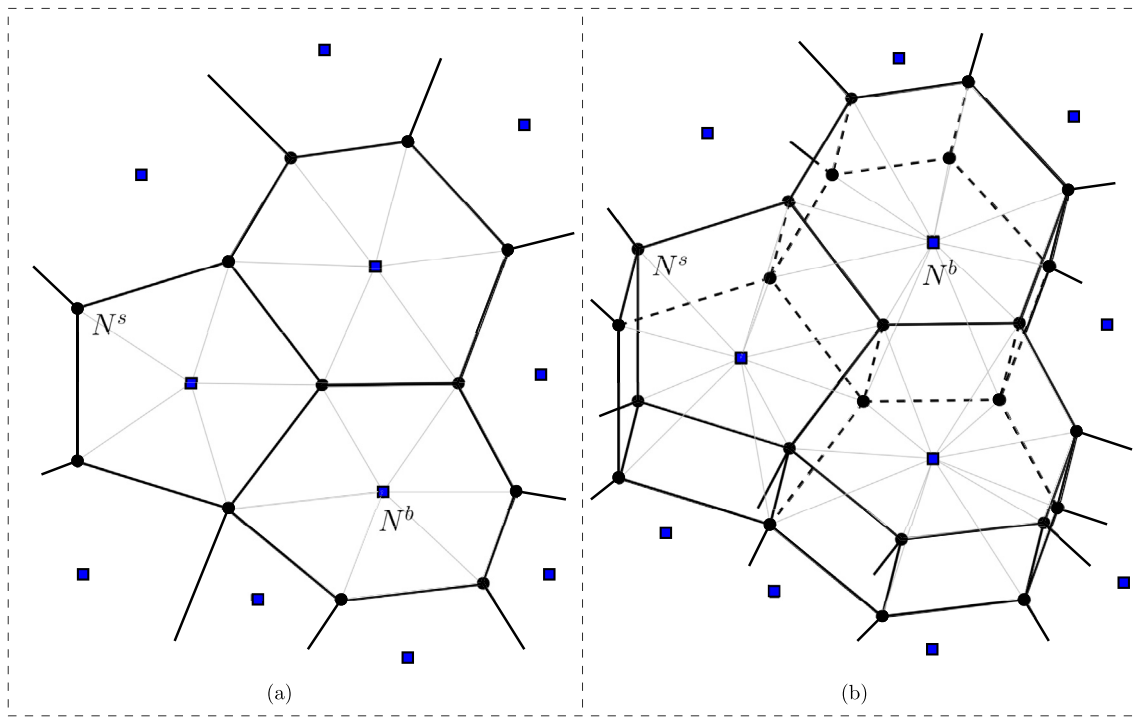
Fig. 1. Schematic representation of enhanced node set N^+ for both the two dimensional and three dimensional formulation, respectively. The bubble node set N^b is shown by the filled square inside the element, located at the geometric center of each arbitrary polytope.

The main objectives of the paper are: (a) to extend the recently proposed '*v*olume averaged nodal projection' (VANP) method to arbitrary polytopes (in both two and three dimensions) and (b) employ the cell-based strain smoothing technique to integrate the terms in the bilinear linear form. The VANP is a purely-displacement based formulation introduced by Ortiz et al., [32–34]. Unlike the previous work, herein we present a pure displacement formulation using the VANP technique and consider both polygonal and polyhedral finite elements. Within this framework, the displacement field is approximated over the enhanced nodes with bubble basis function ensuring stability and the linearly complete pressure field is approximated over the standard nodes using Wachspress basis functions. The locking-free behavior of the VANP

approach is rendered by projecting the dilatation strain onto the linear approximation space. Few other approaches of stabilized mixed formulation with lower-order elements and equal order interpolants using local L^2 projection for pressure are discussed in Refs. [35,36] and the projection of the pressure field onto a lower space through a least-squares based projection technique are given in Refs. [37,38]. The efficacy and robustness of the arbitrary polytopes in nearly incompressible regime are presented by solving few benchmark problems. The salient features of the work are:

- the VANP method alleviates locking phenomenon and is purely a displacement based formulation. The pressure is continuous across
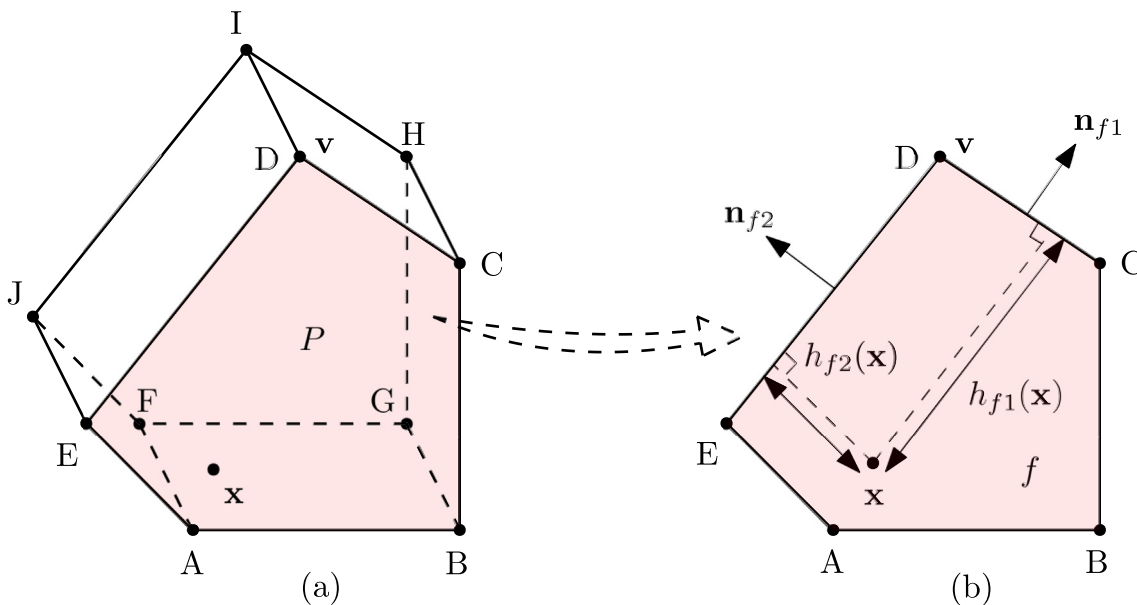

Fig. 2. Construction for the Wachspress basis function.

the element boundaries and computed in the post-processing step.
- the strain smoothing suppresses the need for isoparametric mapping and hence the computation of the Jacobian is eliminated.

The rest of the manuscript is organized as follows: Section 2 summarizes the formulation consisting of the strong form for the incompressible linear elastic isotropic material, the u-*p* mixed weak form, the construction of Wachspress basis functions and bubble basis functions and the VANP method. Section 4 consists of the two dimensional and three dimensional numerical examples to prove the efficiency and the convergence properties of the proposed framework, followed by the conclusions and future work.

2. Formulation

2.1. Strong form

Consider an elastic body in $d = \{2, 3\}$ dimensional space defined by an open domain $\Omega \subset \mathbb{R}^d$, bounded by the d - 1 dimensional surface Γ such that $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. The governing equation with boundary conditions using mixed formulation for both incompressible and nearly-incompressible linear elastic isotropic material is described below:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \text{ in } \Omega \tag{1a}$$

$$\nabla \cdot \mathbf{u} + \frac{p}{\lambda} = 0 \text{ in } \Omega \tag{1b}$$

$$\mathbf{u} = \overline{\mathbf{u}} \text{ on } \Gamma_u \tag{1c}$$

$$\boldsymbol{\sigma}.\mathbf{n} = \overline{\mathbf{t}} \text{ on } \Gamma_t \tag{1d}$$

where $\mathbf{u}$ describes the nodal displacement of the elastic body subjected to external tractions $\overline{\mathbf{t}}$ and body force $\mathbf{b}$. The Γ_u is the Dirichlet boundary and Γ_t is the Neumann boundary. The Cauchy stress tensor $\boldsymbol{\sigma}$ is related to the small strain tensor $\boldsymbol{\epsilon}$ and the pressure parameter p by the following constitutive relation:

$$\boldsymbol{\sigma}(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\nabla\mathbf{u} \tag{2}$$

where λ and μ are the Lame's constant which are defined as:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{3}$$

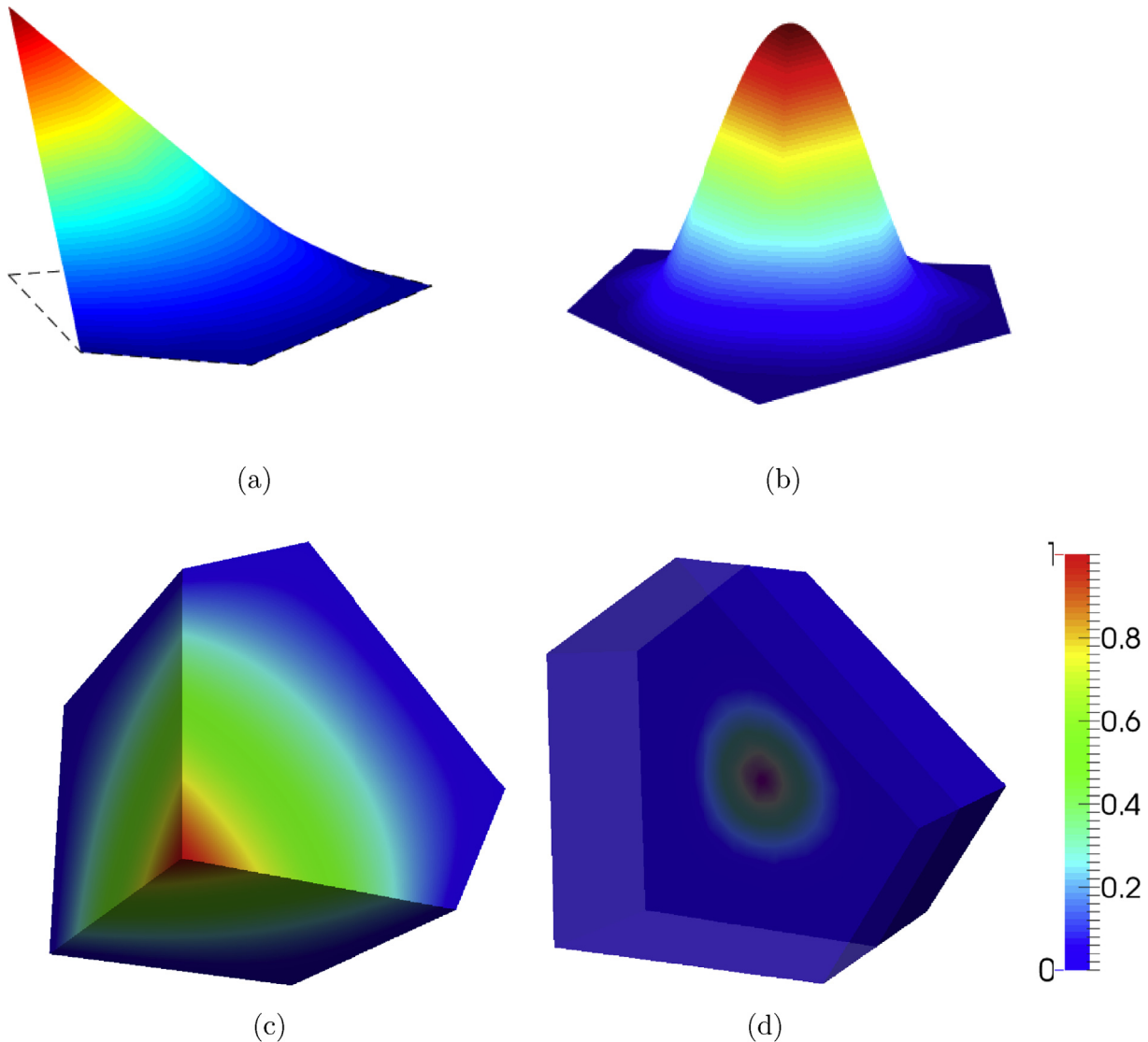$$\mu = \frac{E}{2(1+\nu)} \tag{4}$$

Fig. 3. Wachspress basis function: (a) standard node of a pentagon, (b) enhanced/bubble node of a pentagon, (c) standard node of a polyhedra and (d) bubble node of a polyhedra.

where ν is the Poisson's ratio and E is the Young's modulus of the material. The kinematic relation between the small strain tensor ε and the displacement vector $\mathbf{u}$ is:

$$\varepsilon = \nabla \mathbf{u} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \tag{5}$$

For nearly-incompressible linear elastic body, as ν tends to 0.5, the volumetric strain is zero, therefore we can say $\nabla.\mathbf{u} = \varepsilon^{vol} = \text{tr}(\varepsilon) \approx 0$ Thus, the hydrostatic pressure field p can be approximate as:

$$p = -\lambda \nabla.\mathbf{u} = -\lambda \text{tr}(\varepsilon) = -\lambda \varepsilon^{vol} \tag{6}$$

2.2. Weak form

Let u and V be the trial and the test spaces for the displacement field, respectively, such that

$$U := \{\mathbf{u} : \mathbf{u} \in H^1(\Omega), \mathbf{u} = \overline{\mathbf{u}} \text{ on } \Gamma_u\}$$

$$V := \{\delta \mathbf{u} : \delta \mathbf{u} \in H^1(\Omega), \delta \mathbf{u} = \mathbf{0} \text{ on } \Gamma_u\}$$

where $H^1(\Omega)$ denotes the Sobolev space of order one. Since, only nodal variable pressure is required, let p and δp be the trial and the test functions for the pressure variable, respectively. Such that $p \in P$ and $\delta p \in P$, we define P by

$$P := \{p : p \in L^2(\Omega), \int_\Omega p \, d\Omega = 0\}$$

where $L^2(\Omega)$ denotes the Sobolev space of order zero. The u $-$ p mixed weak form is given by

$$2\mu \int_\Omega \nabla \mathbf{u} : \nabla \delta \mathbf{u} \, d\Omega - \int_\Omega p \nabla.\delta \mathbf{u} \, d\Omega = \int_\Omega \mathbf{b}.\delta \mathbf{u} \, d\Omega + \int_{\Gamma_t} \overline{\mathbf{t}}.\delta \mathbf{u} \, d\Gamma \; \forall \delta \mathbf{u} \in V \tag{7a}$$

$$\int_\Omega \delta p \left(\nabla.\mathbf{u} + \frac{p}{\lambda} \right) d\Omega = 0 \; \forall \delta p \in P \tag{7b}$$

The discretization of the weak form leads to a system of linear equations. In the standard u $-$ p mixed formulation both the displacement and the pressure are independent nodal variables. The pressure and the displacement field in Equation (2.2) is discretized using

$$p_h(\mathbf{x}) = \sum_{I=1}^{N^s} \phi_I(\mathbf{x}) p_I \tag{8a}$$

$$\delta p_h(\mathbf{x}) = \sum_{I=1}^{N^s} \phi_I(\mathbf{x}) \delta p_I \tag{8b}$$

$$u_h(\mathbf{x}) = \sum_{I=1}^{N^+} \phi_I(\mathbf{x}) u_I \tag{9a}$$

$$\delta u_h(\mathbf{x}) = \sum_{I=1}^{N^+} \phi_I(\mathbf{x}) \delta u_I \tag{9b}$$

where p is the nodal pressure, $\mathbf{u}$ is the nodal displacement and ϕ_I are the associated basis functions. In order to ensure the stability of the method, the displacement field is enhanced with bubble basis functions by adding an additional bubble node (denoted by N^b) at the geometric center of each element as shown in Fig. 1. The resulting enhanced node set is denoted by N^+ and the standard node set is denoted by N^s, such that $N^+ = N^s \cup N^b$. From Equations (2.2)-(2.2), we can observe that the pressure field is approximated over the standard node set N^s, whereas, the displacement field is approximated over the enhanced node set N^+.

In this paper, we discretize the domain with arbitrary polygons and polyhedra in two and three dimensions, respectively. While there are different ways to represent the basis functions over arbitrary polytopes [2], we choose the Wachspress interpolants to describe the unknown fields [39]. These functions are rational polynomials and the construction of the Wachspress basis function is given as: Let $P \subset I\!R^3$ be a simple convex polyhedron with facets F and vertices V as shown in Fig. 2. For each facet $f \in F$, let $\mathbf{n}_f$ be the unit outward normal and for any $\mathbf{x} \in P$, let $h_f(\mathbf{x})$ denote the perpendicular distance of $\mathbf{x}$ to f, which is given by

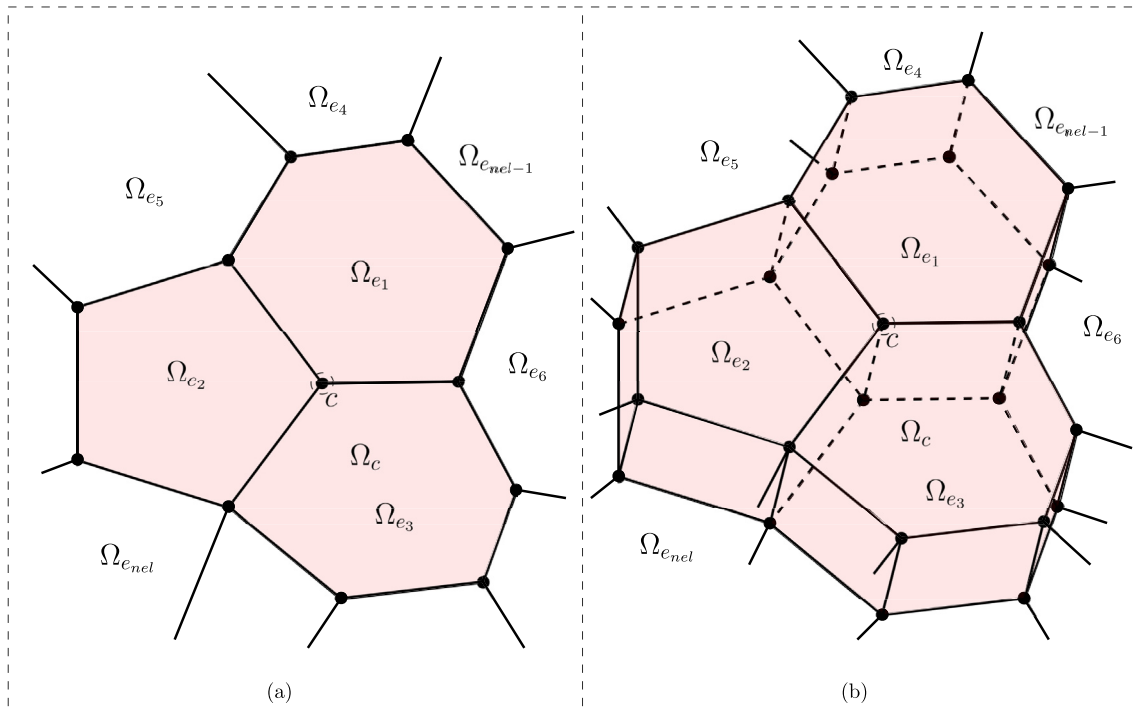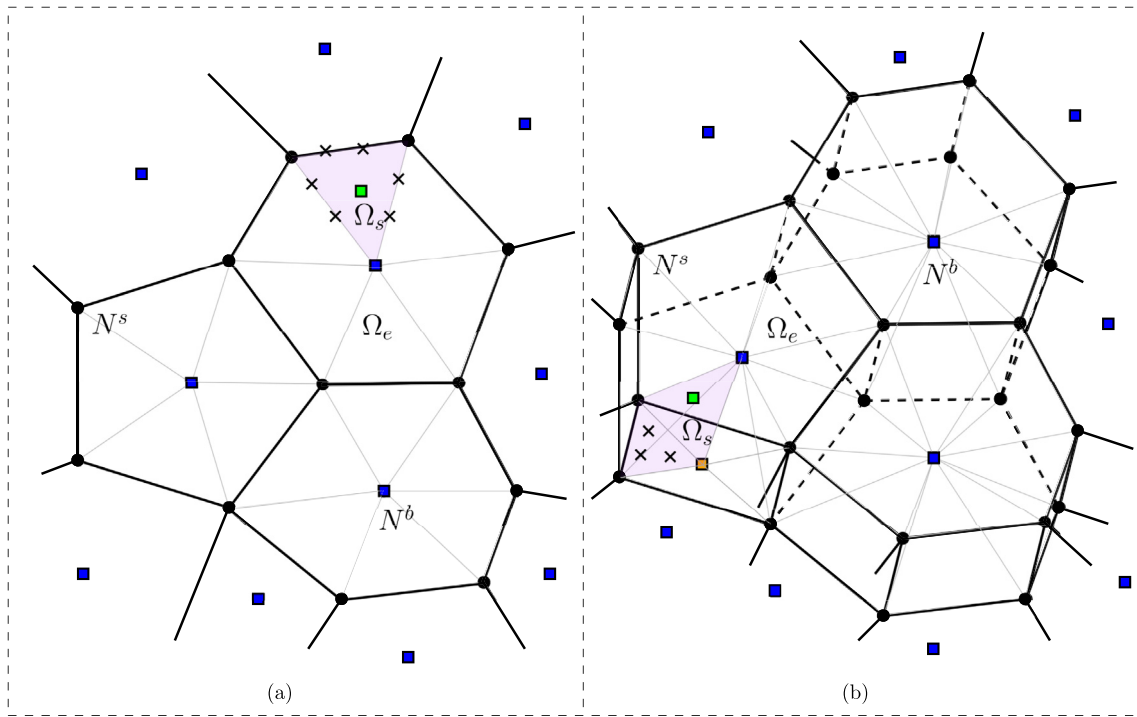$$h_f(\mathbf{x}) = (\mathbf{v} - \mathbf{x}) \cdot \mathbf{n}_f \tag{10}$$

Fig. 4. Schematic representation of two and three dimensional integration cell Ω_c, used for the computation of the VANP operator over a current node 'c'.

Polygonal/Polyhedral finite elements (Ω_e)

Smoothing cell (Ω_s)

● Field Nodes

■ Centriod of the element/Bubble node

□ Centriod of the element face

□ Centriod of the smoothing cell

× Integration points on the edge/face of the elements

Fig. 5. Schematic representation of the integration points within the domain shown by the centroid of the smoothing cell and on the edges/faces of the subcell Ωs shown by the cross for constant smoothing, where the filled square node (i.e the centroid of the element) is to discretize the element into the smoothing cells.

for any vertex $\mathbf{v} \in V$ that belongs to f. For each vertex $\mathbf{v} \in V$, let f_1, f_2, f_3 be the three faces incident to $\mathbf{v}$ and for $\mathbf{x} \in P$, let

$$w_{\mathbf{v}}(\mathbf{x}) = \det(\mathbf{p}_{f_1}, \mathbf{p}_{f_2}, \mathbf{p}_{f_3}) \tag{11}$$

where, $\mathbf{p}_f := \mathbf{n}_f / h_f(\mathbf{x})$ is the scaled normal vector, $f_1, f_2, \ldots, f_d$ are the d faces adjacent to $\mathbf{v}$ listed in an counter-clockwise ordering around $\mathbf{v}$ as seen from outside P (see Fig. 2) and *det* denotes the regular vector determinant in $\mathbb{R}^d$. The shape functions for $\mathbf{x} \in P$ is then given by

$$\phi_{\mathbf{v}}(\mathbf{x}) = \frac{w_{\mathbf{v}}(\mathbf{x})}{\sum_{\mathbf{u} \in V} w_{\mathbf{u}}(\mathbf{x})}. \tag{12}$$

The Wachspress basis functions are the lowest order shape functions that satisfy boundedness, Kronecker delta property, linearity and linear consistency over arbitrary convex polytopes as shown in Fig. 3. The bubble basis functions denoted by ϕ_b are zero along the boundaries of the element and at the standard nodes N^s and one at the bubble node N^b of the element as shown in Fig. 3. Mathematically, the bubble basis functions is given by;

$$\phi_{b_e} = (N_e^s)^2 \prod_{I=1}^{N_e^s} \phi_I \tag{13}$$

where, N_e^s, ϕ_{b_e} are the standard node set and bubble node of an element Ω_e as shown in Fig. 3.

2.3. Volume-average nodal projection (VANP) operator

The VANP method inspired by the B-bar method [40], was introduced by Ortiz et al. [32–34], to alleviate volumetric locking for nearly incompressible media in the meshfree methods. The key advantage of the proposed method is that the displacement field and the pressure field are coupled. Such that, the classical u − p mixed formulation can be translated into a purely displacement based formulation using the VANP method. Such that, the classical u − p mixed formulation can be translated into a purely displacement based formulation using the VANP method, which was originally proposed in the finite element pressure projection [37] and RKPM meshfree pressure projection [38] for nearly incompressible hyperelasticity. This implies that the displacement field is the only nodal solution and the pressure field is obtained as post processing.

In this technique, for each current node 'c' of an element (*n*-sided polygon or polyhedron) an integration cell denoted by Ω_c is created, as shown in Fig. 4. All the elements attached to the current node 'c'
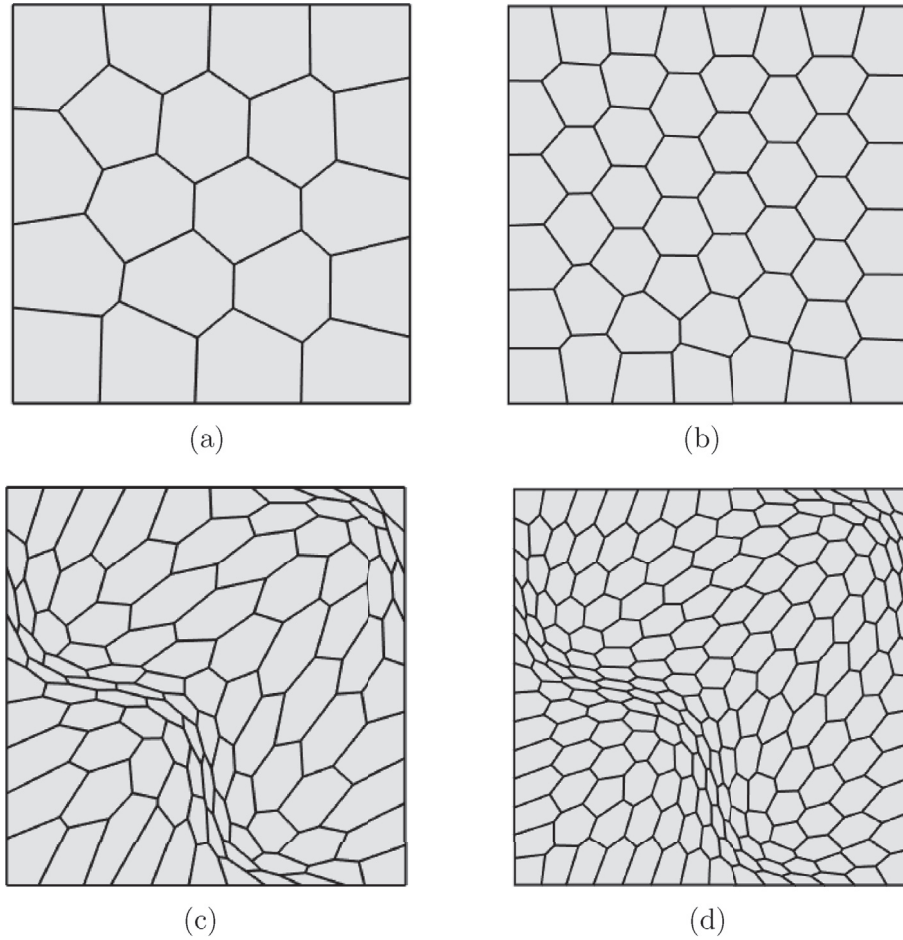
Fig. 6. Square domain: the sample meshes showing domain discretization with polygonal element.

Table 1
Relative error in the L^2 norm and H^1 seminorm for the two-dimensional linear patch test.

Mesh	PFEM		VANP-CS	
	L^2	H^1	L^2	H^1
(a)	1.24×10^{-06}	6.20×10^{-06}	4.67×10^{-15}	1.89×10^{-14}
(b)	1.94×10^{-06}	7.26×10^{-06}	1.62×10^{-14}	2.95×10^{-14}
(c)	1.54×10^{-05}	7.71×10^{-05}	2.61×10^{-14}	7.81×10^{-14}
(d)	3.47×10^{-05}	1.10×10^{-04}	2.91×10^{-14}	1.10×10^{-13}

becomes the part of the integration cell Ω_c as shown in Fig. 4. The integration cell Ω_c created is used to evaluate the projection operator denoted by π, which is later used to calculate the modified strain denoted by $\overline{\varepsilon}_h(\mathbf{u}_h)$ to alleviate locking phenomenon. In this process, an additive decomposition is applied to the standard total strain, which leads to:

$$\varepsilon_h(\mathbf{u}_h) = \varepsilon_h^{dev}(\mathbf{u}_h) + \varepsilon_h^{vol}(\mathbf{u}_h)$$

$$= \left(\varepsilon_h(\mathbf{u}_h) - \frac{1}{3}\text{tr}\,\varepsilon_h(\mathbf{u}_h)\mathbf{I}\right) + \frac{1}{3}\text{tr}\,\varepsilon_h(\mathbf{u}_h)\mathbf{I} \tag{14}$$

In the present method, the dilatational part is projected onto the another space which helps to alleviate the volumetric locking. The projection operator denoted by π operates over the dilatational strain $\varepsilon_h^{vol}(\mathbf{u}_h)$. Thus, the projected dilatational strain is denoted by $\overline{\varepsilon}_h^{vol}(\mathbf{u}_h)$. The total modified strain is called as a discrete modified strain denoted by $\overline{\varepsilon}_h(\mathbf{u}_h)$ is evaluated as:

$$\overline{\varepsilon}_h(\mathbf{u}_h) = \varepsilon_h^{dev}(\mathbf{u}_h) + \pi\left[\varepsilon_h^{vol}(\mathbf{u}_h)\right] = \varepsilon_h^{dev}(\mathbf{u}_h) + \overline{\varepsilon}_h^{vol}(\mathbf{u}_h) \tag{15}$$

The pressure constraint is obtained by substituting Equations (2.2)-(2.2) into Equation (2.2). It can be shown that after relying on the arbitrariness of the nodal pressure test functions the discrete pressure term leads to the following:

$$\int_\Omega \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\delta p_I \left\{\nabla.\mathbf{u}_h + \frac{1}{\lambda}\sum_{I=1}^{N^s}\phi_I(\mathbf{x})p_I\right\} d\Omega = 0 \;\; \forall \delta p \in P \tag{16a}$$

$$\int_\Omega \phi_I(\mathbf{x})\nabla.\mathbf{u}_h \, d\Omega + \frac{1}{\lambda}\sum_{I=1}^{N^s}\int_\Omega \phi_I(\mathbf{x})\phi_I(\mathbf{x})p_I \, d\Omega = 0 \tag{16b}$$

The above integration is performed over an integration cell Ω_c to evaluate pressure field at any node for example say node 'c' (see Fig. 4) is given by

$$p_c(\mathbf{x}) = -\lambda \sum_{I=1}^{N^s}\phi_I(\mathbf{x})\frac{\int_{\Omega_c}\phi_c(\mathbf{x})\nabla.\mathbf{u}_h \, d\Omega}{\int_{\Omega_c}\phi_c(\mathbf{x}) \, d\Omega}$$
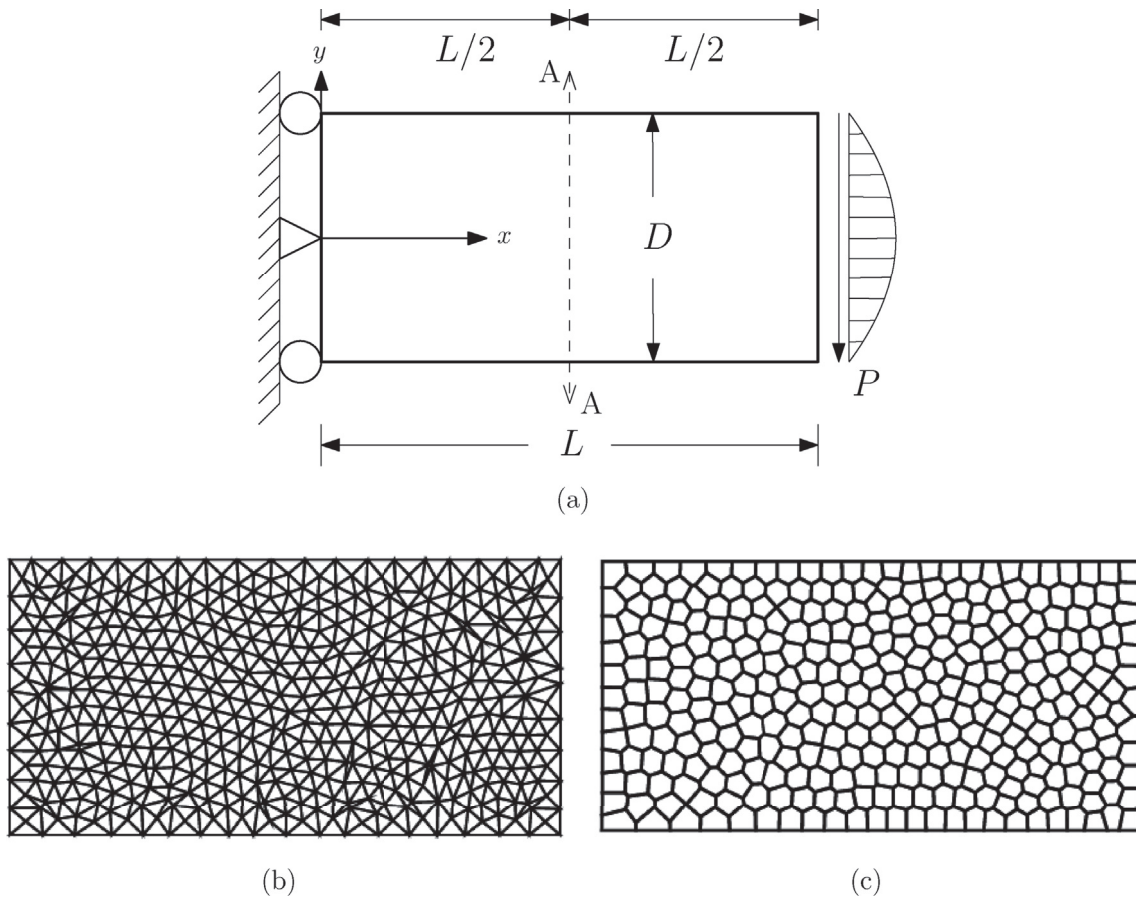
Fig. 7. Cantilever beam: (a) geometry and boundary conditions, (b–c) representative triangular (901 elements) and polygonal mesh (320 elements), respectively.
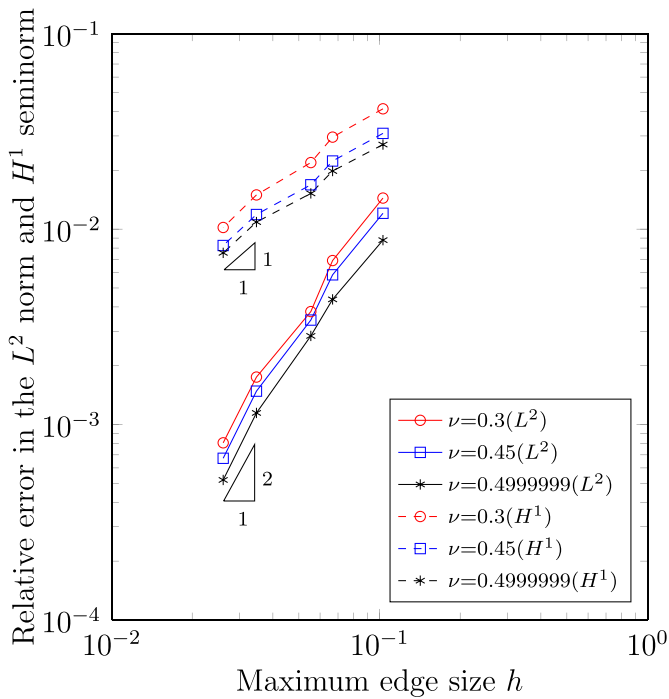

Fig. 8. Cantilever beam: Convergence rate for the L^2 norm and H^1 seminorm using polygonal element is optimal and accurate with VANP-CS for different Poisson's ratio i.e for both the compressible and nearly-incompressible problem.

$$= -\lambda \sum_{I=1}^{N^s} \phi_I(\mathbf{x}) \frac{\int\limits_{\Omega_c} \phi_c(\mathbf{x})\varepsilon_h^{vol} \, d\Omega}{\int\limits_{\Omega_c} \phi_c(\mathbf{x}) \, d\Omega} \tag{16c}$$

which is called the *volume − averaged nodal pressure*. The nodal pressure is evaluated as post processing using only the standard node set N^s. From Equation (2.2), the bar operator (π_c) is given by

$$\pi_c[.] = \frac{\int\limits_{\Omega_c} \phi_c(\mathbf{x})[.] \, d\Omega}{\int\limits_{\Omega_c} \phi_c(\mathbf{x}) \, d\Omega} \tag{17}$$

The VANP operator denoted $\pi[.]$ is evaluated by the linear combination over the bar operator given as:

$$\pi[.] = \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\pi_c[.] \tag{18}$$

Thus, the projected dilatational strain $\overline{\epsilon}_h^{vol}$ is computed as follows:

$$\overline{\epsilon}_h^{vol} = \pi\left[\varepsilon_h^{vol}\right] = \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\pi_c\left[\varepsilon_h^{vol}\right] = \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\left\{ \frac{\int\limits_{\Omega_c} \phi_c(\mathbf{x})\varepsilon_h^{vol} \, d\Omega}{\int\limits_{\Omega_c} \phi_c(\mathbf{x}) \, d\Omega} \right\} \tag{19}$$

2.4. Discrete equations

The domain Ω is partitioned into n_{el} non-overlapping polygonal elements with straight edges and polyhedral elements with planar/non-planar faces in two dimensions and three dimensions, respectively. The proposed method is purely the displacement based method as the pressure field is derived in terms of the volumetric strain. In order to ensure

stability of the solution, the displacement field is computed over the enhanced node set N^+. As in finite element the discrete strain relation is given by:

$$\varepsilon_h(\mathbf{u}_h) = \sum_{I=1}^{N^+} B_I(\mathbf{x})\mathbf{u}_I \tag{20a}$$

$$\varepsilon_h(\delta\mathbf{u}_h) = \sum_{I=1}^{N^+} B_I(\mathbf{x})\mathbf{u}_I \tag{20b}$$

$$\overline{\varepsilon}_h^{vol}(\mathbf{u}_h) = \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\pi_c\left[\mathbf{m}^T \sum_{I=1}^{N^+} B_I\right]\mathbf{u}_I \tag{20c}$$

$$\overline{\varepsilon}_h^{vol}(\delta\mathbf{u}_h) = \sum_{I=1}^{N^s} \phi_I(\mathbf{x})\pi_c\left[\mathbf{m}^T \sum_{I=1}^{N^+} B_I\right]\delta\mathbf{u}_I \tag{20d}$$

where $\mathbf{m}$ is the linear operator and $\mathbf{B}_I(\mathbf{x})$ is the strain displacement matrix for enhanced node N^+, given by:

$$\mathbf{m} = [1\ 1\ 0]^T \tag{21a}$$

$$B_I = \begin{bmatrix} \phi_{I,x} & 0 \\ 0 & \phi_{I,y} \\ \phi_{I,y} & \phi_{I,x} \end{bmatrix} \tag{21b}$$

for two dimensions and in three dimensions

$$\mathbf{m} = [1\ 1\ 1\ 0\ 0\ 0]^T \tag{22a}$$

$$B_I = \begin{bmatrix} \phi_{I,x} & 0 & 0 \\ 0 & \phi_{I,y} & 0 \\ 0 & 0 & \phi_{I,z} \\ \phi_{I,y} & \phi_{I,x} & 0 \\ \phi_{I,z} & 0 & \phi_{I,x} \\ 0 & \phi_{I,z} & \phi_{I,y} \end{bmatrix} \tag{22b}$$

where $\phi_{I,j}(j = x, y, z)$ is the derivative of shape functions. The above strain is computed over the displacement degrees of freedom i.e. over the enhanced node set. These discrete quantities are substituted into weak form which leads to the following system of equations:

$$\left(\mathbf{K}^{dev} + \mathbf{K}^{vol}\right)\mathbf{u} = \mathbf{f} \tag{23}$$

where $\mathbf{u}$ is the column vector of nodal coefficients. The external force vector, $\mathbf{f}$ and the stiffness matrix are given by:

$$\mathbf{K}^{dev} = \int_\Omega B^T \mathbf{C}_\mu B\ d\Omega \tag{24a}$$

$$\mathbf{K}^{vol} = \lambda \int_\Omega B^T \mathbf{m} \sum_{I=1}^{N^s} \phi_I \pi_c\left[\mathbf{m}^T B\right]\ d\Omega \tag{24b}$$

$$\mathbf{f} = \int_\Omega \phi_I \mathbf{b}\ d\Omega + \int_{\Gamma_t} \phi_I \overline{\mathbf{t}}\ d\Gamma \tag{24c}$$

and the constitutive matrix is given by:

$$\mathbf{C}_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{25}$$

in two dimensions, and

$$\mathbf{C}_\mu = \begin{bmatrix} 2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{26}$$

in three dimensions. The nodal coefficients i.e. nodal displacements are obtained by solving Equation (2.2).

3. Constant smoothing

In this paper, the constant smoothing method is used for computing the terms in Equation (2.2). For the purpose of numerical integration, the elements are divided into subcells denoted by Ω_s as shown in Fig. 5. In this study, we use triangular subcells in the two dimensional space and tetrahedral subcells in the three dimensional space as shown in Fig. 5. The strain smoothing technique is then applied within each subcell to evaluate the modified strain. For simplicity of the notation, the derivation of the smoothing scheme is given in detail only for the two-dimensions. The Cartesian coordinate system is chosen, where for convenience $x \equiv x_1$ and $y \equiv x_2$. In addition, n_j $(j = 1, 2)$ is the j-th component of the unit outward normal to a cell edge in the Cartesian coordinate system. The discrete strain field $\widetilde{\varepsilon}_{ij}^h$ that yields the modified strain-displacement matrix $\widetilde{\mathbf{B}}$ that is used to build the stiffness matrix is computed by a weighted average of the standard strain field ε_{ij}^h in each subcell Ω_s^h, as follows:

$$\widetilde{\varepsilon}_{ij}^h = \int_{\Omega_s^h} \varepsilon_{ij}^h(\mathbf{x})f(\mathbf{x})d\Omega, \tag{27}$$

where $f(\mathbf{x})$ is a smoothing function. On writing Equation (2.2) at the basis functions derivatives level, its right-hand side can be expressed in terms of the divergence theorem, as follows:

$$\int_{\Omega_s^h} \phi_{I,j}f(\mathbf{x})\ d\Omega = \int_{\Gamma_s^h} \phi_I f(\mathbf{x})n_j\ d\Gamma - \int_{\Omega_s^h} \phi_I f_j(\mathbf{x})\ d\Omega. \tag{28}$$

where Ωs is the domain of the smoothing cell bounded by Γs and n_j are the outward normals. The ϕ_I are the Wachspress shape functions at the cartesian coordinates and $\phi_{I,j}$ are the modified derivative to be evaluated for the element. The above Equation (2.2) was coined as *divergence consistency* in Duan et al. [41], where it was introduced to correct integration errors in second- and third-order meshfree approximations. This divergence consistency was later used to correct integration errors in the meshfree method [42,43]. The above Equation (2.2), originated by the so called "integration constraint" in Chen et al. [13] in the form of

Table 2
The convergence of the VANP-CS method with mesh refinement for different Poisson's ratio.

Element size h	L^2 norm			H^1 seminorm		
	$v = 0.3$	$v = 0.45$	$v = 0.4999999$	$v = 0.3$	$v = 0.45$	$v = 0.4999999$
0.1030	0.0144	0.0120	0.0087	0.0412	0.0309	0.0270
0.0668	0.0069	0.0058	0.0043	0.0295	0.0223	0.0198
0.0554	0.0037	0.0034	0.0028	0.0219	0.0168	0.0152
0.0347	0.0017	0.0014	0.0011	0.0149	0.0118	0.0108
0.0261	0.0008	0.0006	0.0005	0.0101	0.0082	0.0075
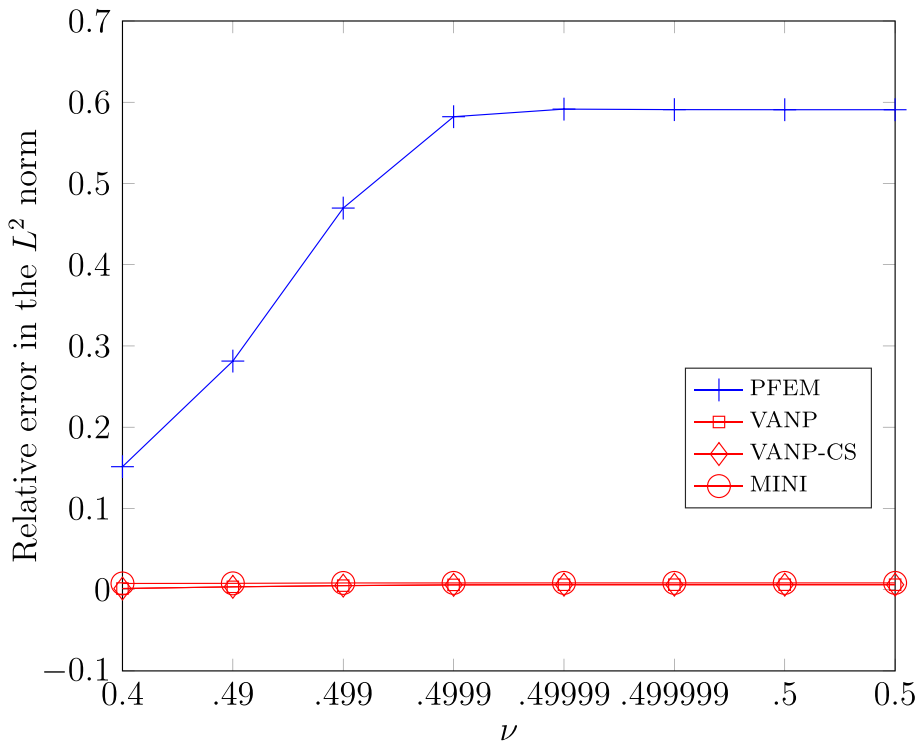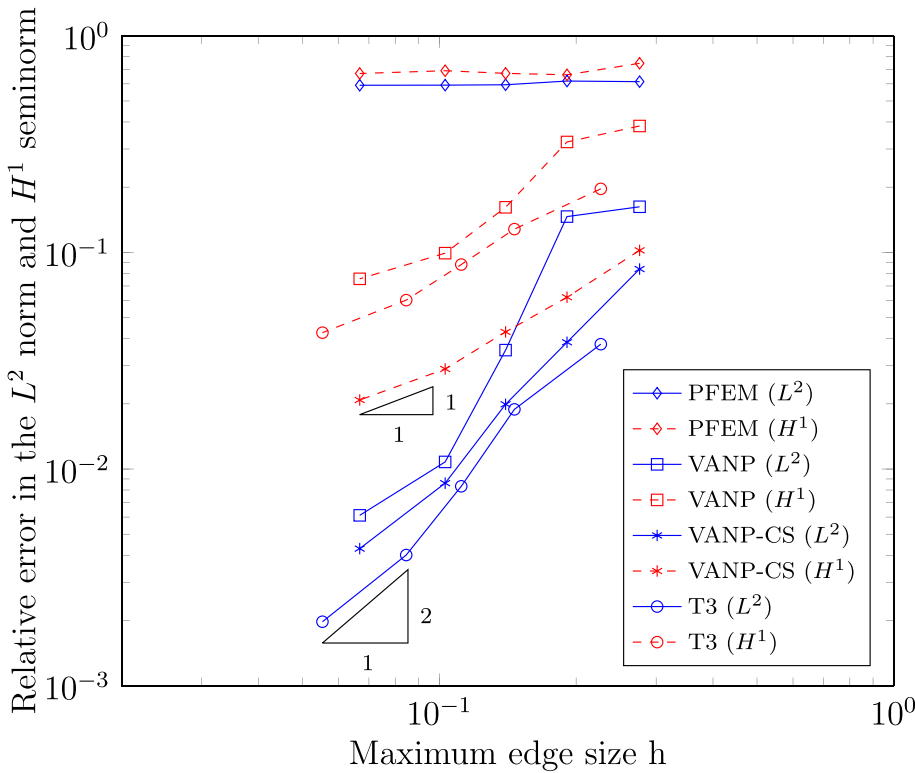
Fig. 9. Cantilever beam: (a) Relative error in the L^2 norm for different Poisson's ratio using polygonal elements (PFEM, VANP, VANP-CS) (160 polygonal elements with 638° of freedom) and triangular elements (MINI) (217 triangular elements with 678° of freedom). (b) Convergence rate for the relative error in L^2 norm and H^1 seminorm for $\nu = 0.4999999$ using different methods.

divergence condition, was coined as *divergence consistency* in Duan et al. [41]. The extension of integration constraint to higher order meshfree approximation has been introduced in Refs. [14,42,43].In the constant

smoothing method, the smoothing function is chosen to be a constant, that is,
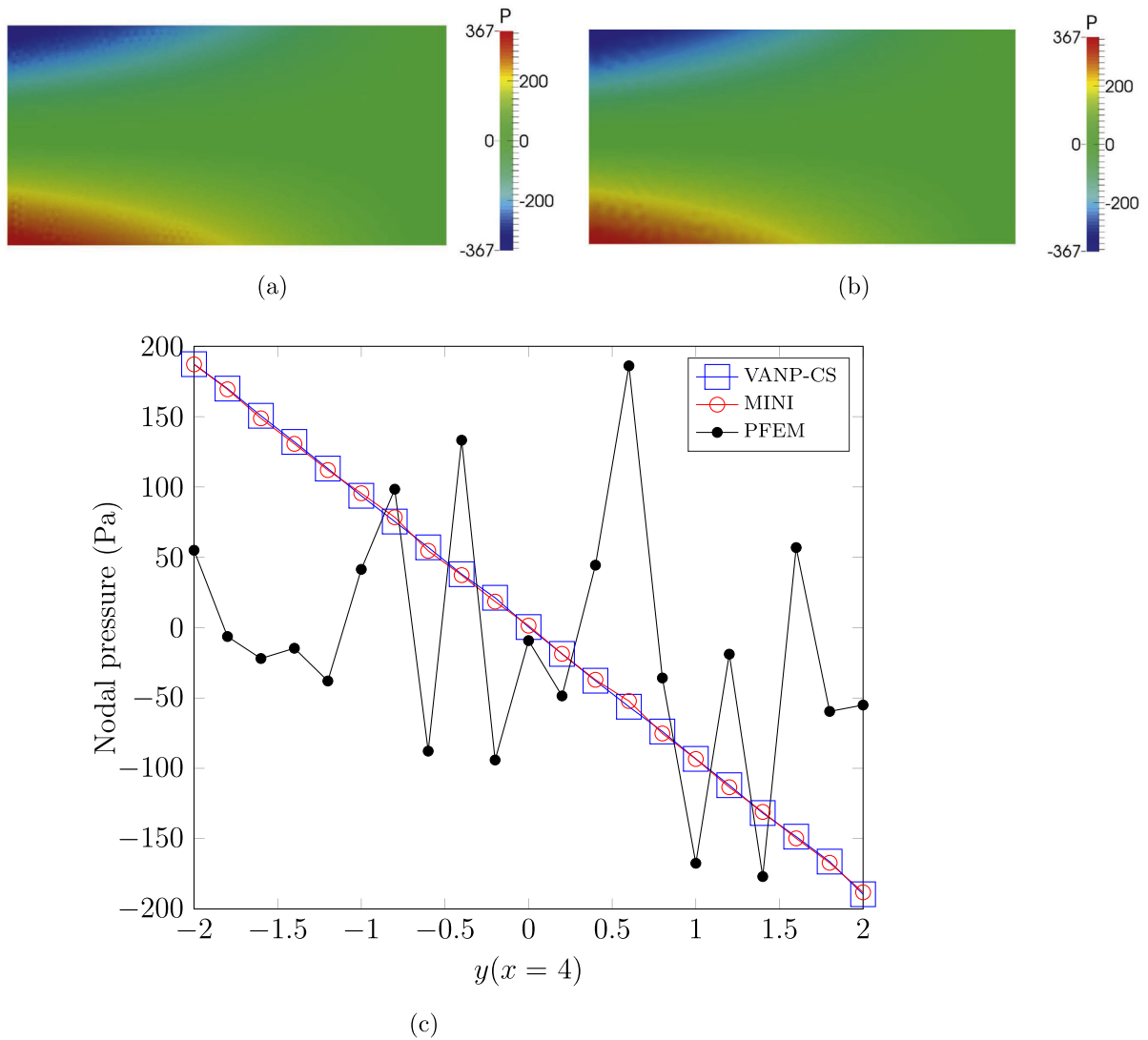
$$f(\mathbf{x}) = 1 \tag{29}$$

(a)

(b)

(c)

Fig. 10. Cantilever beam discretized with triangular and polygonal elements with 11274° of freedom for both the elements: (a) nodal pressure for 1890 polygonal elements (b) nodal pressure for 3724 triangular elements (c) nodal pressure (Pa) values along a section at $x = 4$ m using VANP and MINI over the given discretization.

The divergence consistency equation reduces to,

$$\int_{\Omega_s^h} \phi_{I,j}\, d\Omega = \int_{\Gamma_s^h} \phi_I n_j\, d\Gamma \tag{30}$$

The above equation is the strain smoothing of the stabilized conforming nodal integration (SCNI) [13] and can be used to obtain the modified derivative, $\phi_{I,j}$. On numerical integration of the above equation over Ωs. We get,

$$\sum_{m=1}^{mg} W_m \phi_{I,j}(x_m) = \sum_{s=1}^{sl} \sum_{g=1}^{gl} \phi_I(x_g) n_j^L W_g \tag{31}$$

where mg and W_m are number of Gauss points and corresponding weights per smoothing cell, respectively. Only one Gauss point (mg) is required for the numerical integration of the smoothing cell as shown in Fig. 5. Following the sub-domain stabilized conforming nodal integration originally proposed in Ref. [15], only one Gauss point (mg) is required for the numerical integration of the smoothing cell as shown in Fig. 5. Similarly, gl and W_g are Gauss points and corresponding weights along the boundary (edges) of the subcell, respectively. The Gauss points required for the numerical integration along the boundaries of the smoothing cell are: two (i.e. $gl = 2$) for two dimensions and three (i.e. $gl = 3$) for the three dimensions, as shown in Fig. 5.

While sl represents the number of edges/faces. The modified derivatives of the shape functions thus obtained using cell based smoothing technique are used to compute the strain displacement matrix i.e B_I as shown in Equations (2.2) and (2.4), respectively.

4. Numerical examples

In this section, we demonstrate the accuracy and the convergence properties of the proposed VANP method with constant smoothing over arbitrary polytopes for nearly-incompressible problems in two and three dimensions. The discretization is based on centroid Voronoi tessellation. The two dimensional region is discretized with polygons using Polymesher [3], a MATLAB based meshing tool. For three dimensional problems, we employ the approach proposed in Ref. [19] to generate polyhedral meshes. The results from the proposed method are compared with MINI [44] and analytical solutions where available. The relative error in L^2 norm and H^1 seminorm is used to assess the accuracy and the convergence rates of the proposed framework. Whilst discussing the results, the following convention is adopted:

- VANP–CS– for the proposed VANP method with constant smoothing over arbitrary polytopes.
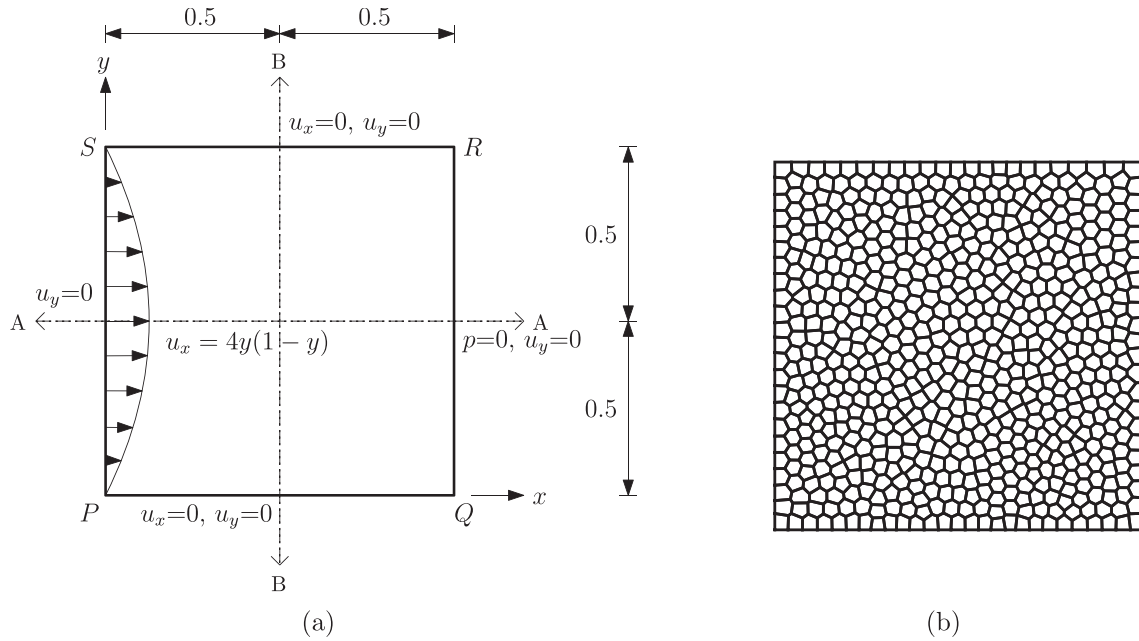- VANP- for the VANP method over arbitrary polytopes.

Fig. 11. Poiseuille flow:(a) Geometry and boundary condition (b) representative mesh of 640 polygonal elements.
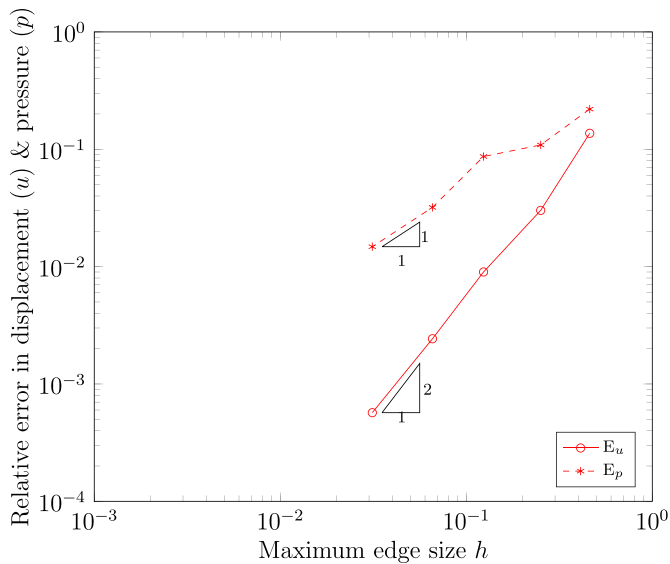

Fig. 12. Poiseuille flow: Relative error in the displacement (u) and pressure (p) converges well with the mesh refinement with an optimal convergence rate for $v = 0.4999999$.

- PFEM- for the conventional polygonal finite element method.
- MINI- for the triangular and tetrahedral finite element.

4.1. Applications to two dimensional problems

We first study the convergence and the accuracy properties of the proposed VANP-CS over arbitrary polytopes in two dimensions. Three problems are considered: (a) patch test, (b) cantilever beam with end shear load and (c) driven cavity flow.

Patch test. In the first example, the convergence and the accuracy properties are demonstrated with a linear patch test and the results are compared with the classical PFEM. For this, consider a unit square,

$\Omega \in (0,1)^2$ subjected to the following conditions on the boundary, Γu:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix} \tag{32}$$

Fig. 6 shows four different polygonal meshes employed for this study. A state of plane strain condition is assumed with the following material properties: Young's modulus $E = 3 \times 10^7$ MPa and Poisson's ratio $v = 0.499$ [45]. The relative error in the L^2 norm and H^1 semi-norm are given in Table 1. It can be inferred from the results that the patch test is satisfied up-to machine precision for the nearly incompressible material behavior with the proposed VANP-CS method over the arbitrary polygons in two dimensions.

Cantilever beam. In this two dimensional problem, consider a cantilever beam of length $L = 8$ m and height $D = 4$ m, subjected to a parabolic shear load, $P = 250$ N at the free end as shown in Fig. 7(a).The exact solution for this problem is given by Ref. [46]:

$$u(x,y) = \frac{Py}{6\bar{E}I} \left[(6L - 3x)x + (2 + \bar{v})\left(y^2 - \frac{D^2}{4}\right) \right]$$

$$v(x,y) = -\frac{P}{6\bar{E}I} \left[3\bar{v}y^2(L-x) + (4 + 5\bar{v})\frac{D^2x}{4} + (3L-x)x^2 \right] \tag{33}$$

where $I = \frac{D^3}{12}$ is the moment of inertia, $\bar{E} = E/(1-v^2)$, $\bar{v} = v/(1-v)$ for plane strain condition. Fig. 7(c) shows a representative polygonal mesh employed in this example. The convergence of the relative error in the L^2 norm and H^1 seminorm with mesh refinement with the proposed framework is shown in Fig. 8 and the values are tabulated in Table 2. The influence of different Poisson's ratio ($v = 0.3, 0.45, 0.4999999$) is also shown. It can be inferred that the proposed framework yields accurate results for all values of Poisson's ratio and converges at the optimal convergence rate. It can be further opined that the VANP-CS suppresses the volumetric locking.

Next, the convergence and the accuracy of the proposed VANP-CS is demonstrated by comparing the results with conventional PFEM[1] and with MINI element. For comparison with the MINI element, the domain is discretized with triangular elements (see Fig. 7(b) for a representative

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[1] without nodal projection (i.e. PFEM) and without strain smoothing (i.e., VANP) to compute the terms in the bilinear/lienar form.
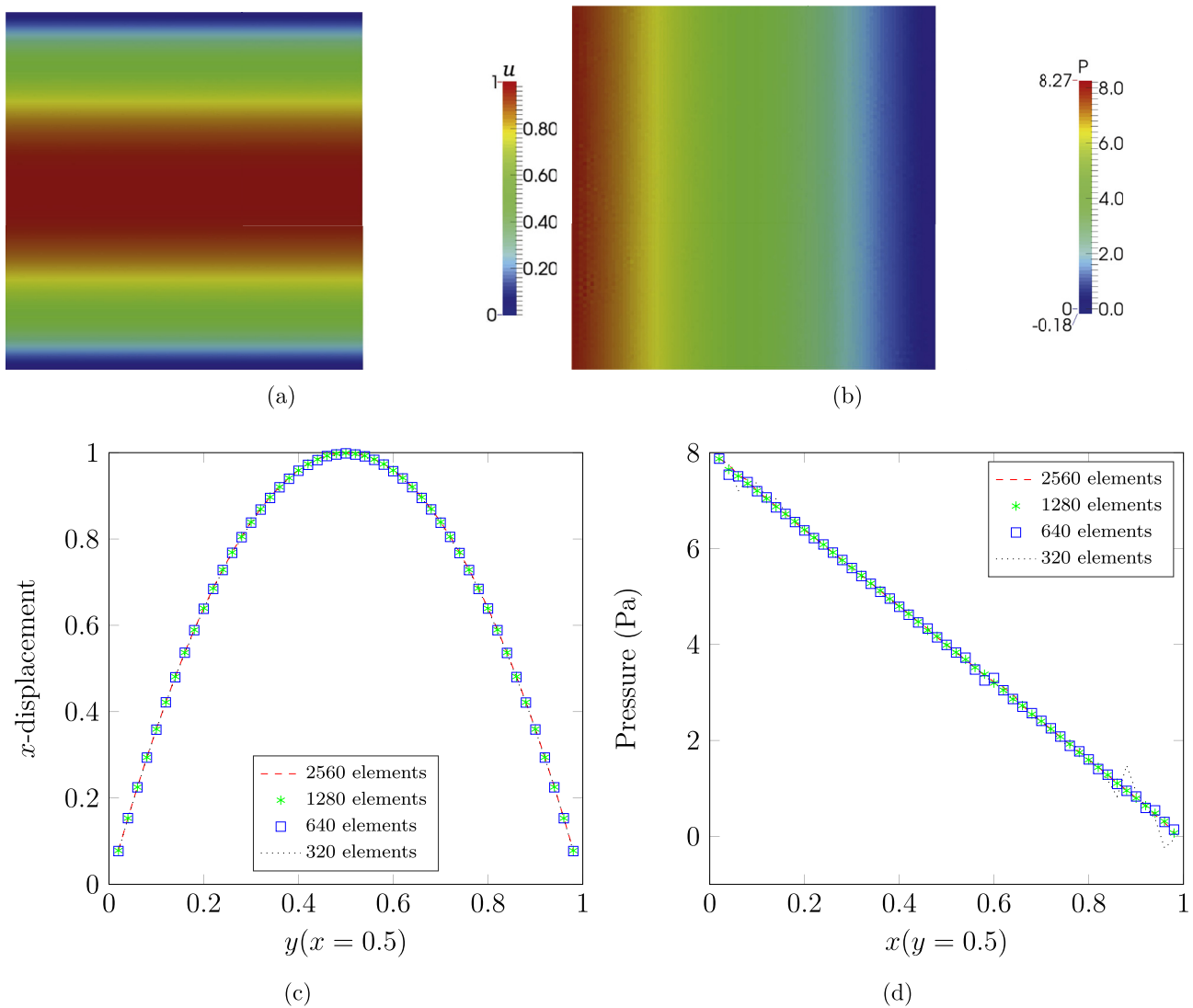
Fig. 13. Poiseuille flow: (a) Displacement in *x* direction contour plot, (b) Nodal pressure (Pa) profile using polygonal elements, (c) Displacement in *x* direction along section B-B, for different polygonal discretization and (d) Pressure along section A-A, for different polygonal discretization.

mesh). Fig. 9(a) shows the relative error in the L^2 norm as a function of Poisson's ratio for different approaches for a mesh of 160 polygonal elements and 217 triangular elements. For comparison, the dofs are kept almost the same for comparison. It is clear that the conventional PFEM suffers from volumetric locking in the near incompressibility limit, whilst all other approaches are free from locking. Note that VANP over polygons and VANP-CS both alleviate locking syndrome, however, VANP over polygons requires 'many' integration points to compute the bilinear and linear form. The strain smoothing on the other hand, reduces the computational burden. The convergenece of the relative error in the L^2 norm and H^1 seminorm with mesh refinement for different approaches for the Poisson's ratio $v = 0.4999999$ is shown in Fig. 9. It can be opined that the proposed framework VANP with and without strain smoothing alleviates volumetric locking, yields accurate results and converges at optimal convergence rate.

Fig. 10 shows the pressure contour for the cantilever beam when meshed with triangular and poylgonal elements. From Fig. 10(a)-10(b), it can be seen that the nodal pressure solution does not show any oscial-

Table 3
Values of γ_h in the numerical inf-sup tests. The domain is discretized with arbitrary polytopes.

Number of elements	h	γ_h
40	0.25	0.4087
80	0.18	0.4214
160	0.12	0.4192
320	0.08	0.4222

lation. Fig. 10(c) shows the pressure profile through the thickness of the beam at $x = 4$ for conventional PFEM, MINI and VANP-CS. The conventional PFEM shows pressure oscillations, whilst MINI and VANP-CS does not show any pressure oscillations for same degrees of freedom, as expected. Moreover, we can observe that the performance of coarser polygonal meshes is similar to the performance of the simplest finite element (i.e. triangular elements).
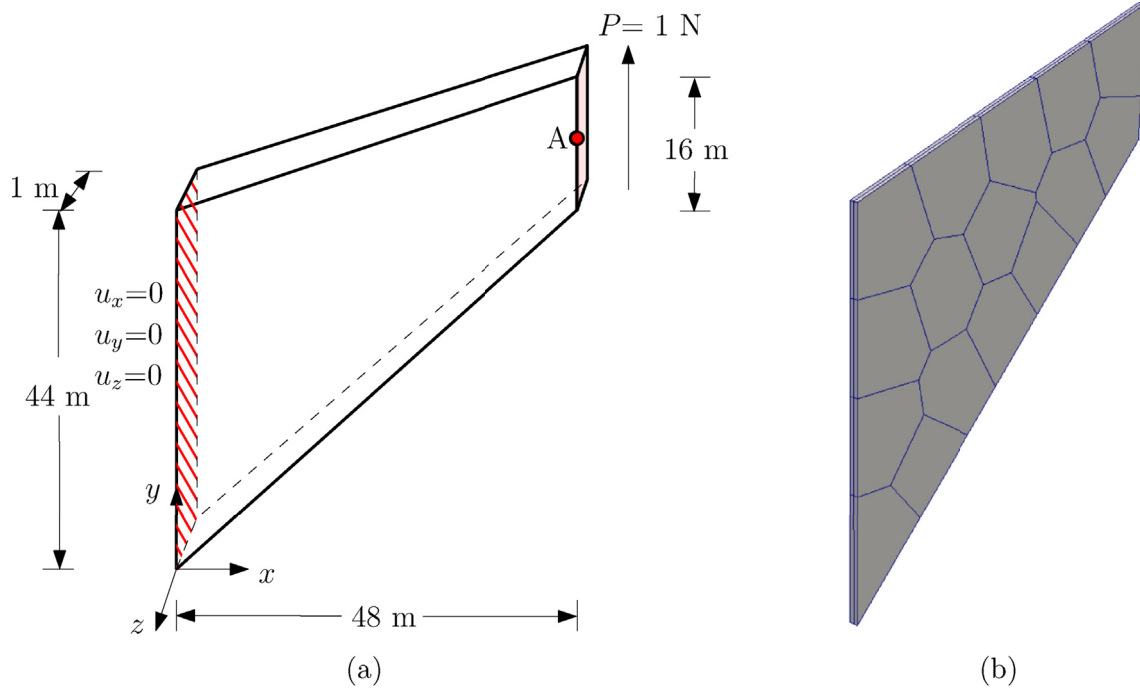
Fig. 14. Cooks membrane: (a) geometry and boundary conditions (b) a sample polyhedral mesh.

Poiseuille flow problem. Next, we consider Poiseuille flow problem. For this example, we consider a two dimensional unit square domain under plane strain condition with a nearly incompressible limit. Fig. 11(a) shows the geometry and the boundary conditions employed in this study. The analytical solution for the displacement and the pressure is given by:

$$u = 4y(1-y); \quad v = 0; \qquad P = 8(1-x). \tag{34}$$

The analytical displacements are prescribed on the boundary of the domain and the left bottom corner is subjected to zero pressure condition (see Fig. 11(a). A representative polygonal mesh used is shown in Fig. 11(b). Fig. 12 shows the convergence of the relative error in the displacement and the pressure with mesh refinement. It can be inferred that the method yields accurate results and converges at optimal convergence rate in both the displacement and pressure. Fig. 13 shows the displacement contours along the x direction and displacement profile along section B-B. It is seen that the proposed method is accurate even on a coarser grid.

The pressure profile for this problem is shown in Fig. 13(b) for the finest mesh. This shows the stability of pressure within the domain for nearly incompressibility limit of $v = 0.4999999$. The stability in the pressure is also examined by evaluating pressure at section A-A for different mesh discretization, see Fig. 11(a). Fig. 13(d) illustrates the stability in the pressure with mesh refinement along a section A-A. Further, the numerical inf-sup value evaluated with mesh refinement is presented in Table 3, where, $\gamma_h = \sqrt{\lambda_k}$ and λ_k is the smallest non-zero eigenvalue. The details of the numerical inf-sup can be found in Ref. [47]. It can be inferred from Table 3 that the numerical inf-sup test value converges to a value that is bounded away from zero with mesh refinement. Therefore, the VANP method passes the numerical inf-sup test and is stable.

### 4.2. Applications to three dimensional problems

Next, we study the convergence properties of the proposed framework over star convex polytopes in three dimensions. Two problems are

considered: (a) Cook's membrane problem and (b) the short cantilever problem. As both these problems do not have analytical solutions, an overkill finite element solution is used as a reference solution. The finite element solution is computed using the commercial software Abaqus, with the domain discretized with tetrahedral elements (C3D10H) or hexahedral elements (C3D8H).

Cook's membrane. Consider the standard bending dominated Cook's membrane. The geometry and the boundary conditions are shown in Fig. 14(a). The right end of the membrane is subjected to an in-plane shear load, $P = 1$ N and the left end is fixed in all the three directions, resulting in a deformation that is dominated by bending. A sample mesh used in the study is shown in Fig. 14(b). The material properties are: Young's modulus $E = 1$ Pa and Poisson ratio $v = 0.4999999$. The reference solution is obtained with a mesh consisting of 12,500 hexahedral elements and the vertical tip displacement at point 'A' is 24.213 mm and the total strain energy is 12.12 Pa.

The convergence of the vertical displacement at point 'A' and the convergence of the total strain energy solution is shown in Fig. 15(a) and (b), respectively. The results from the present formulation is also compared with tetrahedral elements. It can be inferred that the both polyhedral elements and tetrahedral elements yield accurate result and converges to the reference solution asymptotically. The nodal pressure solution is shown in Fig. 16(a) for polyhedral elements using VANP-CS and Fig. 16(b) for tetrahedral elements using MINI, respectively. It is observed that the nodal pressure is stable within the domain for both the MINI and the proposed method.

Short cantilever. As a last example, consider a three dimensional short cantilever subjected to a uniform pressure on its upper face. The displacements (in all three directions) on the adjacent vertical face (hatched region in Fig. 17(a) is constrained. The geometry and the boundary conditions of the problem are shown in Fig. 17(a). A representative polyhedral mesh used in the study is shown in Fig. 17(b). As approximate strain energy of 0.950930 MPa is taken as the reference solution, as reported in Ref. [48] and the reference solution for the vertical displacement at point 'A' (see Fig. 17(a)) is taken as 3.312 mm as reported in Ref. [49]. This is for a Poisson's ratio $v = 0.25$ and
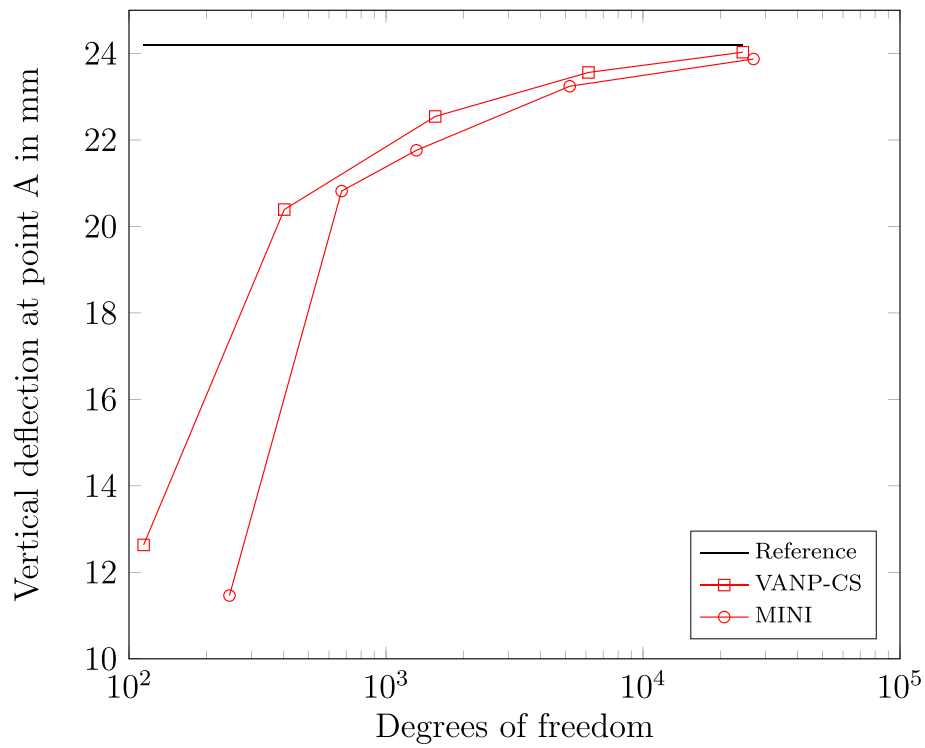
(a)
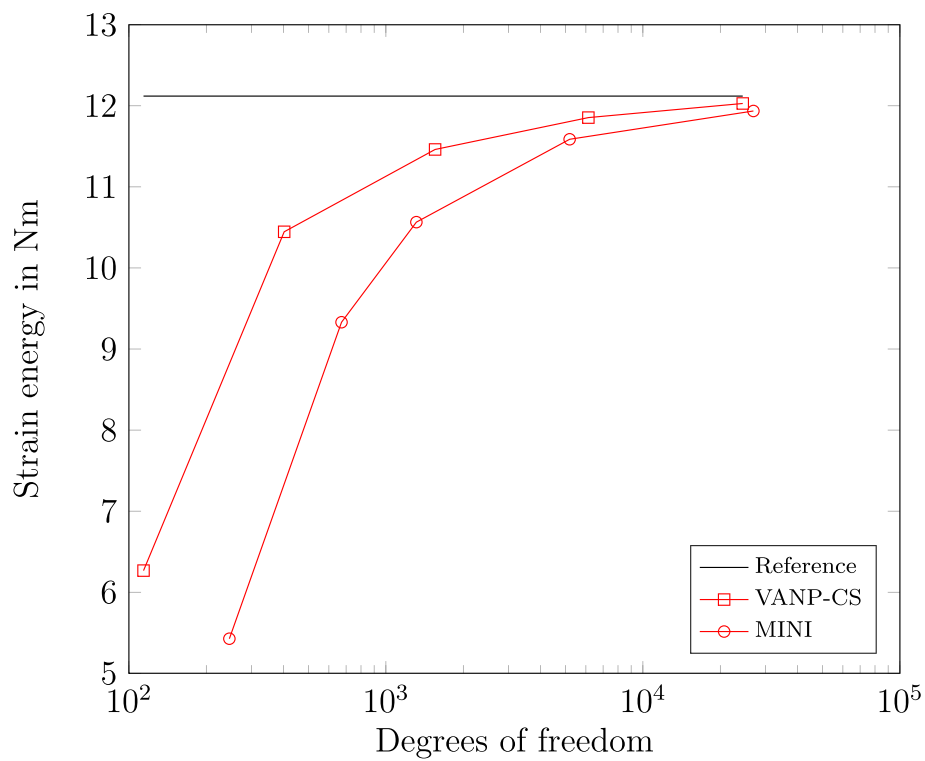
(b)

Fig. 15. 3D Cooks membrane - convergence of the (a) vertical displacement (u_y) at point A and (b) total strain energy with mesh refinement for Poisson's ratio $v = 0.4999999$ to the reference solution.
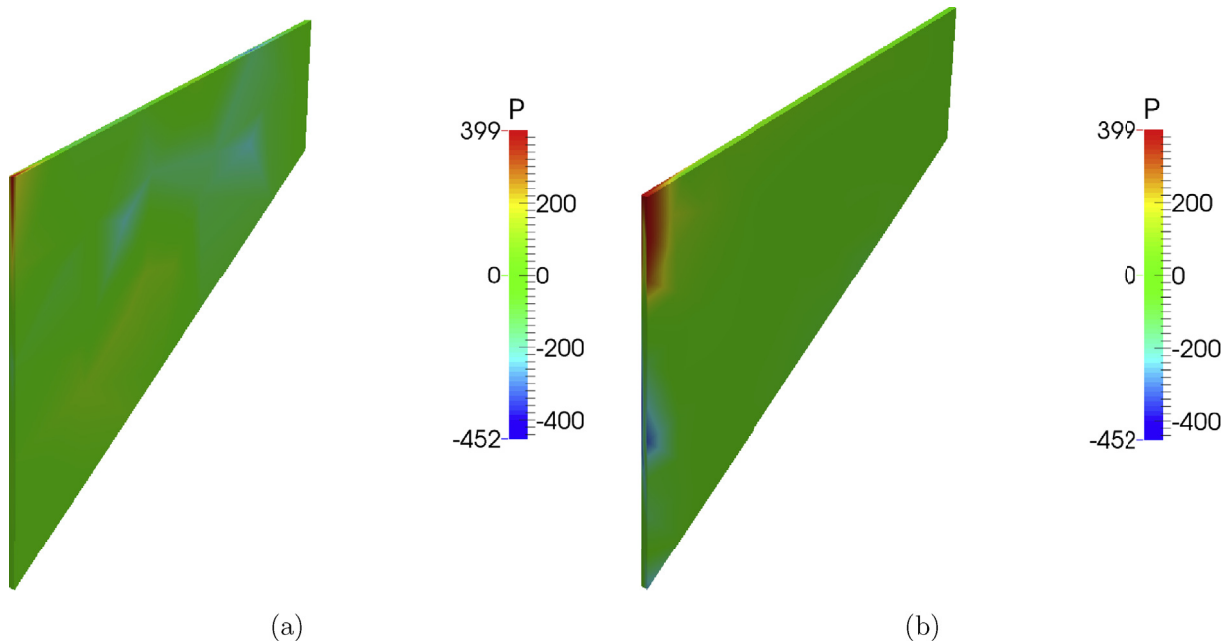
Fig. 16. Cooks membrane discretized with tetrahedral and polyhedral elements, respectively (a) nodal pressure for 224 polyhedral elements with 306° of freedom (b) nodal pressure for 554 tetrahedral elements with 669° of freedom.
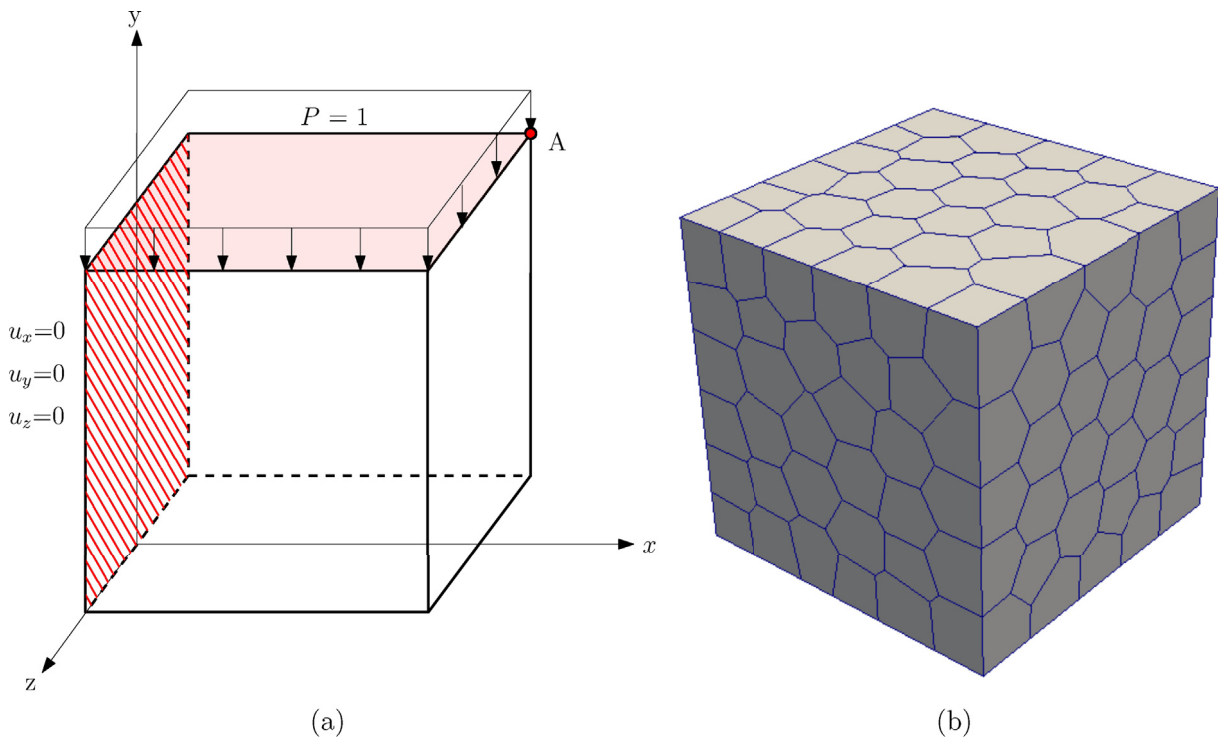

Fig. 17. Short cantilever: (a) Geometry and boundary conditions (b) a sample polyhedral mesh.

Young's modulus, $E = 1$ MPa. The vertical displacement at point 'A' is estimated using Abaqus with 20,675 tetrahedral elements as 3.3 mm, 3.513 mm and 3.32 mm for different Poisson's ratio (i.e $v = 0.25, 0.45, 0.4999999$). Similarly, the strain energy is estimated as 0.9438 MPa, 1.0031 MPa and 0.9518 MPa for different Poisson's ratio. The convergence of the vertical displacement at point 'A' and the total strain energy is shown in Fig. 18. It can be seen that the proposed framework converges to the reference solution asymptotically.

5. Concluding remarks

In this work, we extended the MINI element over triangles to arbitrary polytopes by employing the VANP method. The computational burden is reduced by using the strain smoothing technique. It is noted that both were originally proposed for meshfree methods. The framework presented alleviates volumetric locking and leads to a purely displacement based formulation with the nodal pressure computed at the post-processing stage. The constant strain smoothing technique is used
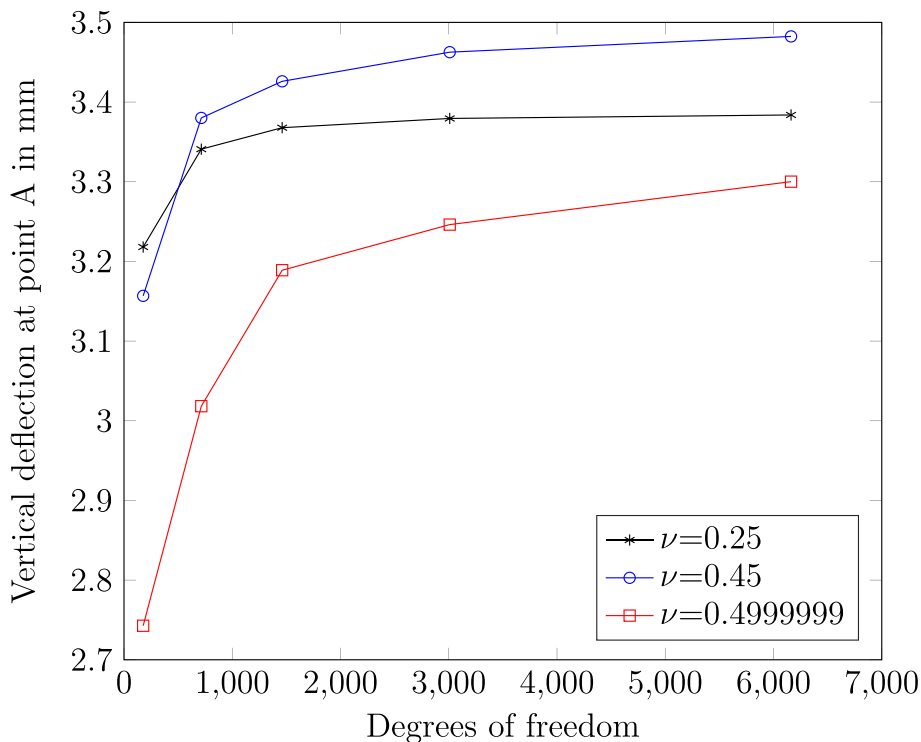
(a)
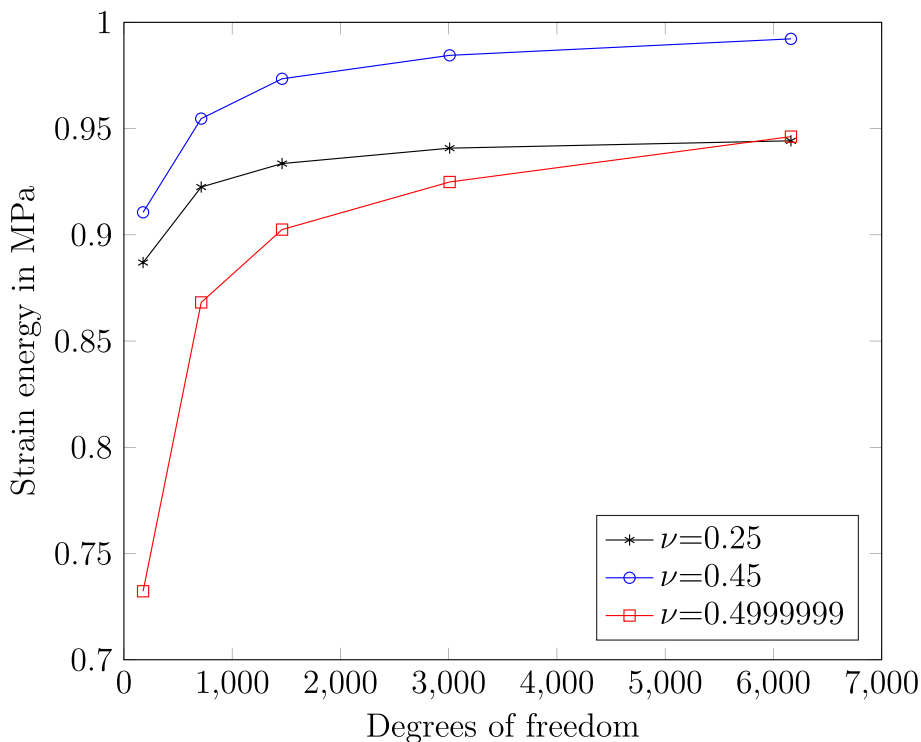
(b)

Fig. 18. Short cantilever: (a) Convergence of the vertical deflection (u_y) at point 'A' and (b) convergence of the total strain energy with the mesh refinement for different Poisson's ratio i.e ν = 0.25, 0.45 and 0.4999999.

to compute the terms in the bilinear and linear form, which significantly reduces the computational burden, without compromising accuracy. It is noted that the VANP over classical simplex element is the well known

MINI element. From the numerical study, it can be inferred that the proposed framework does not suffer from volumetric locking phenomenon. The proposed method also preserves optimal convergence rates in both

the L^2 norm and in the H^1 seminorm for both two and three dimensional problems.

Acknowledgements

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.finel.2019.103368.

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