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MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA,
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CONTRIBUCIONES AL ESTUDIO DE LA DISTORSIÓN EN GRUPOS DE AUTOMORFISMOS

En este trabajo estudiamos el rol que la distorsión cumple en el grupo de automorfismos de los sistemas expansivos. Comenzamos generalizando resultados para subshifts ligando distorsión y no-expansividad a sistemas expansivos arbitrarios, además de explorar el subconjunto de automorfismos simétricamente distorsionados. Debido a la generalización, podemos determinar que un automorfismo expansivo nunca puede ser distorsionado. Luego, para el caso de un subshift, presentamos un marco general para el estudio de exponentes de Lyapunov puntuales, lo cual nos permite ver la forma en que los automorfismos distorsionados actuan sobre configuraciones específicas y en distintas direcciones de su espacio-tiempo. Finalmente, caracterizamos los automorfismos que tienen una única dirección de no-expansividad con pendiente racional. Esto genera condiciones necesarias sobre subshifts minimales y transitivos para que estos puedan tener automorfismos con direcciones de no-expansividad únicas y de pendiente irracional.

CONTRIBUTIONS TO THE STUDY OF DISTORTION IN AUTOMORPHISM GROUPS

In this work we study the role distortion plays on automorphisms groups of expansive systems. We begin by generalizing results on subshifts linking distortion and non-expansivity to arbitrary expansive systems and explore the subset of symmetrically distorted automorphisms. Due to the generalization, we are able to determine that expansive automorphisms can never be distorted. Next, in the case of subshifts, we introduce a generalized framework for the study of point-wise Lyapunov exponents, which allows us to look at the way distortion behaves on individual configurations and in different space-time directions. We derive new upper-bounds to the directional metric entropy of automorphisms. Finally, we characterize automorphisms with unique non-expansive directions of rational slope. This creates necessary conditions on minimal and transitive subshifts for them to have automorphisms with unique non-expansive directions of irrational slope.

“For instance,” said the boy again, ”if Christmas trees were people and people were Christmas trees, we’d all be chopped down, put up in the living room, and covered in tinsel, while the trees opened our presents.” ”What does that have to do with it?” asked Milo. ”Nothing at all,” he answered, ”but it’s an interesting possibility, don’t you think?” — Norton Juster, The Phantom Tollbooth

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Contents

Introduction

In this thesis we will look at the role different notions of distortion play in the study of the automorphism group. Specifically, we explore the connections between discrete Lyapunov exponents and non-expansive directions when some notion of distortion is involved.

An automorphism of a topological dynamical system is a homeomorphism of a metric compact space that commutes with its action. A classic problem in dynamical systems is characterizing the automorphisms of a given system, describing the algebraic structure of the group of all automorphisms and studying its relationship to the underlying dynamical system. In the general context, this problem remains largely open. Throughout this work, we will restrict ourselves to the case of expansive systems, due to the fact that these have greater structure and rigidity, and in particular contain subshifts.

Traditional studies of the automorphism groups of subshifts consisted on combinatorial properties of the subshift such as marker methods and restrictions imposed by word complexity. These studies where jump started by the seminal works by G. Hedlund in which it is shown that every automorphism of a shift is fully determined by a sliding block code, also known as a Cellular Automata. Later, cornerstone advances were made by Boyle, Lind, Rudolph, Kim and Roush. In particular, they showed that studying the automorphism group through its subgroup structure is a futile endeavour for a broad class of subshifts, known as subshifts of finite type.

This work stems from recent developments that relate two different aspects of the study of shift automorphisms. The first of these is the generalization of the concept of blocking words to higher dimensional actions by Boyle and Lind. They introduced the notions of expansive and non-expansive directions for the study of the directional dynamics of $\mathbb{Z}^d$ actions. These concepts facilitated the discovery of bounds on the entropy of these systems. By viewing automorphisms as $\mathbb{Z}^2$-systems through the use of space-time approaches, we can apply these concepts to the study of automorphisms. The second aspect is the study of the asymptotic dynamics of Cellular Automata systems such as the introduction of discrete Lyapunov Exponents by Shereshevsky, proved fruitful for the generation of entropy inequalities. These were later expanded upon by the likes of Bressaud, Courbage, Kamińsky and Tisseur.

These two aforementioned lines of inquiry where unified in the work of Cyr, Kra and Franks, where the concept of distorted automorphism is introduced, one that is asymptotic in nature. The crucial fact is that certain distorted automorphism have unique non-expansive directions. This connections where further expanded upon by Guillon and Salo, where distorted automorphisms where related to aperiodic Turing Machines.

Starting from these results, this work seeks to further develop the connection between distortion and non-expansivity to the realm of expansive dynamical systems. We do this through to distinct approaches. On the one hand, we generalize established results concerning range distortion to the realm of general expansive systems, including their geometrical consequences. These consecuences include the ability to characterize automorphisms with unique rational non-expansive directions.

On the other hand, in the case of subshifts, we look at the asymptotics of the automorphisms through the lens of Lyapunov exponents on individual configurations. This allows us to study the effects of an automorphism at the configuration level and, in particular, allows us to develop new upper bounds of the automorphisms metric and topological entropy based on the entropy of the subshift.

The structure of this work is divided into five chapters. The first of these introduces all the mathematical background needed. Next, the concept of range distortion is generalized to the context of arbitrary expansive systems, in hopes of broadening the search for a system with unique irrational non-expansive direction. We also introduce the notion of Turing machines as dynamical systems as a tool to create distorted automorphisms. The next section explores how different discrete Lyapunov exponents relate between each other, and what geometrical consequences this has. This is followed by a section that details the relationship between entropy and distortion. The final sections presents the consequences of having automorphisms with unique non-expansive directions.

Chapter 1

Preliminaries

We begin by presenting the mathematical background for the work developed in the thesis. First, the general notions of dynamical systems are introduced. Next, we look at the particular case of symbolic systems. Later, we present subshifts of finite type along with crucial results about their automorphism groups. Finally, we introduce the concepts of group distortion and expansivity.

1.1 Dynamical systems

Given a compact metric space (X, ρ) and a group G, we define a topological dynamical system as the pair (X, G) where G acts on X through homeomorphisms. For an in depth explanation of the topic, see Chapter 7 on [KL16].

To better undestand the action $G \curvearrowright X$, it is useful to understand the behaviour of the points within the space.

Definition 1.1 *Let (X, G) be a dynamical system. Given $x \in X$, we define its orbit under the action of G as the set:*

$$\text{orb}_G(x) = \{g(x) : g \in G\} \subseteq X.$$

Also, we define the stabilizer subgroup as:

$$\text{stab}_G(x) = \{g \in G : g(x) = x\} \leq G.$$

Finally, for the case where $G = \mathbb{Z}$, we will define the ω-limit set of $x \in X$ by

$$\omega(x) = \bigcap_{n \geq 0} \overline{\{T^k x : k \geq n\}}.$$

In this way, it is possible to distinguish the amount of redundancy present on the action on the set. This is achieved through the following notions:

Definition 1.2 *We say the action $G \curvearrowright X$ is:*

- *Faithful if the map $G \to \text{Homeo}(X)$ in inyective.*
- *Free if $\text{stab}_G(x) = \{e_G\}$ for all $x \in X$.*

On the other hand, we have notions regarding the amount of mixing a system undergoes through the action:

Definition 1.3 *Let (X, G) be a dynamical system. We say the system is:*

- *(Topologically) transitive if for every pair of non-empty open sets $U, V \subseteq X$, there exists $g \in G$ such that $gU \cap V \neq \emptyset$.*
- *Minimal if X does not strictly contain a closed non-empty G-invariant subset.*

These properties can be observed from a metric perspective through the density of the orbits of the points within the space.

Proposition 1.4 *Let (X, G) be a dynamical system. Then,*

1. *(X, G) is transitive if and only if there exists $x \in X$ with a dense orbit.*
2. *(X, G) is minimal if and only if every $x \in X$ has a dense orbit.*

Remark 1.5 *In the $G = \mathbb{Z}$ case, we have the following dicothomy of minimal actions: X is either finite or infinite and the action is free.*

To compare two dynamical systems, we look for maps that preserve the structure at hand.

- We say that (X, G) is *(topologically) isomorphic* or *conjugated* to (Y, G) if there is a homeomorphism $\phi : X \to Y$ such that
$$\phi \circ g = g \circ \phi, \ \forall g \in G$$
- We say (Y, G) is a *(topological) factor* of (X, G), or (X, G) in an extension of (Y, G) if there is a continuos surjective function $\phi : X \to Y$ such that
$$\phi \circ g = g \circ \phi, \ \forall g \in G$$

We say $\varphi \in \text{Homeo}(X)$ is an automorphism of (X, G), if $\varphi \circ g = g \circ \varphi$, $\forall g \in G$. We denote the group of automorphisms by $\text{Aut}(X, G)$. In the case of one dimensional shifts, where the group action is straightforward, we will denote this group by $\text{Aut}(X)$.

1.2 Symbolic dynamics

An important case of dynamical systems are symbolic systems. Given a finite alphabet, $\mathcal{A}$, we define the set $\mathcal{A}^G := \{x : G \to \mathcal{A}\}$. This set can be viewed as that colorings of group G with the elements of the alphabet. We imbue this set with the left group action, $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$, through

$$\sigma^g(x)_h = x_{g^{-1}h}$$

Imbuing $(\mathcal{A}^G, \sigma)$ with the product topology, by Tychonoff's Theorem, $\mathcal{A}^G$ is a compact set with a clopen sub-base given by the cylinders $[a]_g = \{x \in \mathcal{A}^G : x_g = a\}$.

We call every finite subset $F \subseteq G$ a support. This way, a pattern P supported in F is an element of the set $\mathcal{A}^F$. We denote $F = \text{supp}(P)$. Furthermore, we can define the cylinder generated by a pattern P as the set $[P]_g = \bigcap_{h \in F} [p_h]_{gh}$ and $[P] = [P]_{e_G}$. These cylinders are the base of the product topology.

When G is finitely generated, by a set S for instance, the product topology is generated by the metric:

$$d(x, y) = 2^{-\inf\{|g|_S :\ x_g \neq y_g\}},$$

where $|\cdot|_S$ is the word metric associated to the generating set S.

Definition 1.6 *A G-subshift (or simply subshift when the group is evident) is a closed subset $X \subseteq \mathcal{A}^G$, invariant under the action of G.*

In particular, subshifts have combinatorial characterizations through the patterns that appear on its configurations.

Proposition 1.7 *Let $X \subseteq \mathcal{A}^G$. The following are equivalent*

1. *X is a subshift,*

2. *There exists a set of patterns $\mathcal{F}$ such that*

$$X = \mathcal{A}^G \setminus \bigcup_{P \in \mathcal{F},\ g \in G} [P]_g,$$

3. *There exists a set of patters $\mathcal{F}$ such that $X = \{x \in \mathcal{A}^G : P \sqsubseteq x \implies P \notin \mathcal{F}\}$.*

   We call set $\mathcal{F}$ the set of forbidden patterns. If $\mathcal{F}$ is finite, we say X is a subshift of finite type.

Definition 1.8 *The language of a G-subshift X, $\mathcal{L}(X)$ is the set of all patterns appearing in configurations belonging to X, that is, $\mathcal{L}(X) := \{P : [P]_e \cap X \neq \emptyset\}$.*

Also, given $F \subseteq G$, we define the language supported by F as $\mathcal{L}_F(X) := \mathcal{L}(X) \cap \mathcal{A}^F$.

1.2.1 Morphisms

We now present the functions that preserve the structure of shift spaces. Let X and Y be to G-subshifts. We say ϕ is shift-conmutative if $\phi \circ \sigma^g = \sigma^g \circ \phi$ for all $g \in G$.

Definition 1.9 *We say $\phi : X \to Y$ is a morphism between two subshifts if it is a continuous shift commuting map.*

This definition of morphism is a special case of the one presented on Section 1. We introduce a type of map that is particular to the case of subshifts.

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. We say that $\phi : \mathcal{A}^G \to \mathcal{B}^G$ is a sliding block code (s.b.c.) if there exists a support $F \subset G$ and a function $\Phi : \mathcal{A}^F \to \mathcal{B}$ such that $\phi(x)_g = \Phi(\sigma^{g^{-1}}(x)|_F)$.

In the case where $G = \mathbb{Z}$, we can choose the support to be symmetric around 0, that is, there exists $R > 0$ such that for all $x, y \in X$ with $x|_{[-R,R]} = y|_{[-R,R]}$, we have that $\phi(x)_0 = \phi(y)_0$. The smallest R such this property holds is called the range of ϕ, and we denote it by $\text{range}(\phi) := R$.

Both of these notions are related by the classic Curtis-Hedlund-Lyndon Theorem.

Theorem 1.10 ([Hed69]) *Let $X \subseteq \mathcal{A}^G$ and $Y \subseteq \mathcal{B}^G$ be subshifts, and $\phi : X \to Y$. Then, ϕ is a sliding block code if and only if it is a morphism.*

For a proof a this general case see [CSC10] (the original Theorem is for the case where $G = \mathbb{Z}$).

Again, when $G = \mathbb{Z}$, this means that $\text{Aut}(X)$ is always countable and that $\text{Aut}(X)$ is a discrete subgroup of $\text{Homeo}(X)$ for the uniform topology.

1.3 Subshifts of finite type

From this point onward we will work with $G = \mathbb{Z}$. See [LM95] for a more detailed description of this case.

As mentioned on Proposition 1.7, we say a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a subshift of finite type (SFT) if its set of forbidden words is finite.

Definition 1.11 *A subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is said to be of finite type if there exists a finite set of words $\mathcal{F}$ such that*

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} : w \sqsubseteq x \implies w \notin \mathcal{F}\}.$$

An important property of this class of subshifts is that they can be associated to bi-infinite walks on graphs. In addition, through the adjacency matrix of the graph, it is possible to to study SFT's through the algebraic properties of non-negative integer matrices.

Graphs are related to the matrices trhough the concept of adjancency matrices.

Definition 1.12 *A graph $\Gamma = (V, E)$ consists of a finite set of vertices V, and a finite set of edges E. Each edge $e \in E$ starts from a vertex denoted $\mathfrak{i}(e) \in V$ and finishes on a vertex, denoted by $\mathfrak{t}(e) \in V$.*

Definition 1.13

- *Let Γ be a graph of vertices V. For two vertices $i, j \in V$, we denote the number of edges with initial vertex i and termianl vextes j by A_{ij}. Then, the adjacency matrix of Γ is given by $A_\Gamma = (A_{ij})$.*
- *Let A be an $n \times n$ matrix with non-negative integer entries. The graph defined by A, Γ_A, has a set of vertices given by $V_A = \{1, ..., n\}$ with A_{ij} distinct edges in E_A starting in i and ending in j.*

Definition 1.14 *Let Γ be a graph with adjacency matrix A. The edge shift X_A is given by*

$$X_A = \{(e_i)_{i\in\mathbb{Z}} \in E^{\mathbb{Z}} : \mathfrak{t}(e_i) = \mathfrak{i}(e_{i+1}),\ \forall i \in \mathbb{Z}\}.$$

The main is result is that al shifts of finite type are congruent to an edge shift.

Theorem 1.15 *Let X be a shift of finite type, then there is a graph Γ_A such that $X_A \cong X$.*

1.4 Complexity of subshifts

For a general subshift (X, σ), the map $P_X : \mathbb{N} \to \mathbb{N}$ defined by $P_X(n) = |\mathcal{L}_n(X)|$, is called the complexity function of (X, σ). As we will later see, the growth rate of this function on some cases imposes restrictions on the properties of the automorphisms group.

Remark 1.16 *It is easy to see that the complexity function is non-decreasing and multiplicative, that is*

$$P_X(n + m) \leq P_X(n)P_X(m),\ \forall n, m \in \mathbb{N}.$$

With the purpose of studying the behaviour of the complexity function, we present the asymptotic notation to compare growth functions.

Definition 1.17 *Let $f, g : \mathbb{N} \to \mathbb{N} \setminus \{0\}$. We write $f(n) = O(g(n))$ if there exists a constant K such that $f(n) \leq Kg(n)$ for all sufficiently large n. Furthermore, we write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Finally, we say f has*

- *polynomial growth, if there exists an integer* $d \geq 1$ *such that* $f(n) = O(n^d)$. *We say the growth is linear if* $d = 1$, *and that it is quadratic if* $d = 2$,

- *super-logarithmic growth if* $\lim_{n\to\infty} \frac{f(n)}{\log(n)} = +\infty$,

- *super-linear growth if* $\lim_{n\to\infty} \frac{f(n)}{n} = +\infty$,

- *sub-lineal (sub-quadratic) if* $\liminf_{n\to\infty} \frac{f(n)}{n^d} = 0$ *for* $d = 1$ *(resp.* $d = 2$*),*

- *sub-exponential if* $\lim_{n\to\infty} \frac{f(n)}{\alpha^n} = 0$, *for all* $\alpha > 1$.

An important notion related to the complexity of a subshift is the concept of special word. A word $w \in \mathcal{L}(X)$ is said to be left special if there exists at least two different letter $a, b \in \mathcal{A}$ such that $wa, wb \in \mathcal{L}(X)$. A right special word is defined analogously.

The relationship between this notion and the complexity is subtly presented in a stronger version of the famous Morse-Hedlund Theorem.

Theorem 1.18 ([MH40]) *Let* X *be an infinite subshift. Then, for all* $n \in \mathbb{N}$ *there exists a left special (resp. right special) of length* n.

Remark 1.19 *The original statement of the Morse-Hedlund Theorem can be obtained easily from the previous Theorem: a subshift generated by an aperiodic point verifies* $P_X(n) > n$ *for all* $n \in \mathbb{N}$.

1.5 Automorphism groups

As we stated before, an automorphism of any dynamical system is a self-conjugacy of that system. The study of these groups consists on finding constraints from dynamical properties.

The classification begins by observing two fundamental facts about automorphisms. First, it is easy to see that $\mathrm{Aut}(X)$ is finite if and only if X is finite. Secondly, every automorphism maps periodic points into periodic points of the same period. This allows us to present a complete description of finite subshifts:

Proposition 1.20 ([SS]) *Let* $X \subseteq \mathcal{A}^Z$ *be a finite subshift. Then, the shift is conjugated to a permutation* $\pi \in S_{|X|}$. *Let*

$$\pi = \prod_{i=1}^{I} \prod_{j=1}^{J_i} \pi_{i,j},$$

be the cycle decomposition of π *where each* $\pi_{i,j}$ *is an* i*-cycle. Then,*

$$\mathrm{Aut}(X) \cong \prod_{i=1}^{I} (\mathbb{Z}/i\mathbb{Z})^{J_i} \rtimes S_{J_i}.$$

For infinite subshifts, although the automorphisms group is countable, it is nonetheless quite complicated. For instance, when (X, σ) is a shift of finite type, the group is quite large as evidenced by the following two results:

Theorem 1.21 *Let* (X_A, σ_A) *be a shift of finite type where* A *is a primitive matrix.*

1. *([BLR88]) The group* $\mathrm{Aut}(X_A)$ *contains isomorphic copies of each of the following groups:*

   (a) Any finite group,

   (b) $\bigoplus_{i=1}^{\infty} \mathbb{Z}$*,*

   (c) The free group on two generators F_2*.*

2. *([KR+90]) For any* $n \geq 2$*, let* (X_n, σ_n) *be the full shift on* n *symbols. Then,* $\mathrm{Aut}(X_n)$ *is isomorphic to a subgroup of* $\mathrm{Aut}(X_A)$*.*

Both of these results tell us that the subgroup structure of the automorphisms group is a useless strategy to distinguish them. They say that the groups for mixing SFT's have roughly the same structure.

We note that 1 tells us that $\mathrm{Aut}(X_A)$ is never amenable. Also, through 2 we can see that the types of groups that can appear as subgroups of $\mathrm{Aut}(X_n)$ are independent of n.

The strategies used to show part 1. of the theorem use the so-called marker methods. These marker automorphisms are part of a broader class first introduced by Nasu [Nas88] called simple automorphisms.

Let A be a square matrix over $\mathbb{Z}_+$, and let Γ_A be the corresponding directed graph. A simple graph automorphism of Γ_A is a graph automorphism of Γ_A which fixes all vertices. This induces a automorphism $\gamma \in \mathrm{Aut}(X_A)$ given by a 0-block code.

Definition 1.22 *We call an automorphism* $\alpha \in \mathrm{Aut}(X_A)$ *simple if* $\alpha = \Psi^{-1}\gamma\Psi$*, where* $\Psi : (X_A, \sigma_A) \to (X_B, \sigma_B)$ *is a conjugacy to some shift of finite type* (X_B, σ_B) *and* $\gamma \in \mathrm{Aut}(X_B)$ *is induced by a simple graph automorphism of* Γ_B*.*
We define,

$$\mathrm{Simp}(X_A) = \langle \{\alpha \in \mathrm{Aut}(X_A) : \alpha \text{ is a simple automorphism.}\} \rangle \trianglelefteq \mathrm{Aut}(X_A).$$

We also have a fundamental result describing the underlying structure of these automorphisms groups

Theorem 1.23 ([Rya72]) *If A is primitive, the center of* $\mathrm{Aut}(X_A)$ *is generated by* σ_A.

Besides the results presented so far in this section, several straightforward questions remain unanswered due to the lack of consistent or standard techniques. In particular, Boyle, Lind and Rudolph [BLR88] have the fundamental question of distinguishing between the automorphism groups of full-shifts. It is not known whether $\mathrm{Aut}(X_2)$ is isomorphic to $\mathrm{Aut}(X_3)$ or not. In general,

Question 1.24 *Is* $\mathrm{Aut}(X_n)$ *isomorphic to* $\mathrm{Aut}(X_m)$, *for* $m, n \geq 2$?

Although specific cases have been shown not to be isomorphic, the question for general m and n's remains open.

1.5.1 Automorphism groups of subsfhifts with low complexity

It is interesting to see how the complexity of subshifts restricts the structure of its automorphism group. With added hypothesis such as transitivity or minimality, it is possible to establish results about the groups.

Theorem 1.25 ([Don+16]) *Let (X, σ) be a subshift such that,*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < \infty.$$

Suppose that there is a point $x_0 \in X$ such that $\omega(x_0) = X$ that is asymptotic to a different point. Then, the following are true:

1. $\mathrm{Aut}(X)/\langle\sigma\rangle$ *is finite*
2. *If (X, σ) is minimal, the quotient* $\mathrm{Aut}(X)/\langle\sigma\rangle$ *is isomorphic to a finite subgroup of permutations without fixed points and* $|\mathrm{Aut}(X)/\langle\sigma\rangle|$ *divides the number of asymptotic components of (X, σ).*

Theorem 1.26 ([CK16]) *Let (X, σ) be a transitive subshift of sublinear complexity. Then, the group* $\mathrm{Aut}(X)/\langle\sigma\rangle$ *is periodic.*

A group is said to be periodic if every element has finite order.

Theorem 1.27 ([CK15b]) *If (X, σ) is a minimal subshift such there exists $d \in \mathbb{N}$ with*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n^d} = 0,$$

then $\mathrm{Aut}(X)$ *is amenable. Furthermore, any finitely generated torsion-free subgroup of* $\mathrm{Aut}(X)$ *has polynomial growth of degree at most* $d-1$*.*

Theorem 1.28 *If* (X, σ) *is a minimal subshift such that there exists* $\beta < 1/2$ *with*

$$\limsup_{n\to\infty} \frac{\log(P_X(n))}{n^\beta} = 0,$$

then $\mathrm{Aut}(X)$ *is amenable. In addition, the growth rate of any finitely generated torsion-free subgroup of* $\mathrm{Aut}(X)$ *is given by* $O(\exp(n^{\beta/(1-\beta)}))$*.*

1.6 The dimension representation

Let A be a $N \times N$ matrix over $\mathbb{Z}_+$. The Eventual Range of A is the subspace given by

$$R(A) = \bigcap_{n=1}^{\infty} \mathbb{Q}^N A^n = \mathbb{Q}^N A^N,$$

consisting of row vectors. An alternative definition for the eventual range is as the largest subspace where matrix A is invertible.

The dimension triple associated with A is the following:

- An abelian group

$$\mathcal{G}_A := \{v \in R(A) : vA^k \in \mathbb{Z}^N \text{ for some } k \geq 0\}.$$

- A semi-group within $\mathcal{G}_A$:

$$\mathcal{G}_A^+ := \{v \in R(A) : vA^k \in (\mathbb{Z}_+)^N \text{ for some } k \geq 0\}.$$

- An automorphism $\delta_A(v) = vA$.

The details of this dimension representation, $(\mathcal{G}_A, \mathcal{G}_A^+, \delta_A)$, can be found in [LM95]. While this representation relies on the matrix A, there is an alternative construction for an equivalent dimension representation due to Krieger, which is built from the system (X_A, σ_A). Let us assume that A is irreducible and that the SFT (X_A, σ_A) has positive entropy.

We define an m-ray as the subset

$$R_m^+(x) := \{y \in X_A : y_{(-\infty,m]} = x_{(-\infty,m]}\},$$

where $x \in X_A$ and $m \in \mathbb{Z}$. We also refer to a finite union of m-rays as an m-beam. It is easy to see that two m-rays are either disjoint or the same. In addition, if U is an m-beam, it is also an n-beam for all $n \geq m$.

Let U be an m-beam represented by

$$U = \bigcup_{i=1}^{I} R_m^+(x^{(i)}).$$

We define the vector $v_{U,m} \in \mathbb{Z}^k$ by:

$$(v_{U,m})_J = |\{x^{(i)} \in U : \mathfrak{t}(x_m^{(i)}) = J\}|.$$

It is possible to show that

$$v_{U,m+1} = v_{\sigma(U),m} \text{ and } v_{U,m+k} = v_{U,m}A^k.$$

We will say two beams, U and V, are equivalent if $v_{U,m} = v_{V,m}$ for some $m \in \mathbb{Z}$. We denote by $[U]$ the equivalence class of beam U. Because A is irreducible and $\log(\lambda_A) > 0$, the directed graph represented by A has a cycle with an incoming edge which is not in the cycle. This means that given two beams U and V, it is always possible to find two equivalent beams U' and V' such that $U' \cap V' = \emptyset$.

With this, we define the semi-group D_A^+ by the operation

$$[U] + [V] = [U' \cup V'].$$

By adding the formal difference of elements in D_A^+, we arrive at its group completion, D_A. Then, the map induced by $d_A([U]) = [\sigma_A(U)]$ finishes the description of Krieger's dimension triple (D_A, D_A^+, d_A).

Finally, we can see that the two dimensión representations are equivalent through the following semi-group homomorphism:

$$\theta([U]) = \delta^{-N-n}(v_{U,n}A^N),$$

where U is an N beam.

Proposition 1.29 ([LM95], Theorem 7.5.13) *The map $\theta : D_A^+ \to \mathcal{G}_A^+$ satisfies $\theta(D_A^+) = \mathcal{G}_A^+$, and induces an isomorphism $\theta : D_A \to \mathcal{G}_A$ such that $\theta \circ d_A = \delta_A \circ \theta$. In this way, θ is an isomorphism of triples, $\theta : (\mathcal{G}_A, \mathcal{G}_A^+, \delta_A) \to (D_A, D_A^+, d_A)$.*

Automorphisms and the dimension group

By using Kreiger's dimension triple, it is easy to induce an automorphism of the triple from an automorphism of the shift.

We define the group of automorphisms of the dimension triple, $\mathrm{Aut}(\mathcal{G}_A)$ as the set of all group automorphisms $\Phi : \mathcal{G}_A \to \mathcal{G}_A$ such that $\Phi(\mathcal{G}_A^+) = \mathcal{G}_A^+$ and $\Phi \circ \delta_A = \delta_A \circ \Phi$.

An automorphism of the shift $\phi \in \mathrm{Aut}(X_A)$, induces an automorphism of the equivalent dimension triple $\phi^* : (D_A, D_A^+, d_A) \to (D_A, D_A^+, d_A)$ by

$$\phi^*([U]) = [\phi(U)].$$

With this at hand, let $S_\phi \in \mathrm{Aut}(\mathcal{G}_A)$ be the automorphism for which the diagram

$$\begin{array}{ccc} D_A & \xrightarrow{\theta} & \mathcal{G}_A \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle S_\phi} \\ D_A & \xrightarrow{\theta} & \mathcal{G}_A \end{array}$$

commutes. We define the dimension representation of $\mathrm{Aut}(X_A)$ as the homomorphism

$$\begin{aligned} \pi_A : \mathrm{Aut}(X_A) &\to \mathrm{Aut}(\mathcal{G}_A) \\ \phi &\longmapsto S_\phi \end{aligned}$$

It is an open question whether the dimension representation is surjective. From this fact, we have the following specific open question:

Question 1.30 *Given an SFT, what is the range of the dimension representation?*

In the case of full-shifts, we have the following result,

Proposition 1.31 *If n has j distinct prime divisors, then* $\mathrm{Aut}(\mathcal{G}_n) \cong \mathbb{Z}^j$ *and the map* $\pi_A : \mathrm{Aut}(X_n) \to \mathrm{Aut}(\mathcal{G}_n)$ *is surjective.*

We call an automorphism $\phi \in \mathrm{Aut}(X_A)$ inert if it belong to the kernel of π_A. We denote the subgroup of inert automorphisms by $\mathrm{Inert}(X_A) := \ker(\pi_A)$. This subgroup is of fundamental importance to the study of automorphism groups on subshifts of finite type. In fact, it is posible to see that this subgroup is indeed very large:

Proposition 1.32 *For any shift of finite type* (X_A, σ_A), $\mathrm{Simp}(X_A) \subseteq \mathrm{Inert}(X_A)$.

Finally, we mention two important conjectures related to the study of the automorphisms group.

Simple finite order generation conjecture (SFOG):

For any shift of finite type (X_A, σ_A), there is an equality, $\mathrm{Inert}(X_A) = \mathrm{Simp}(X_A)$.

Finite order generation conjecture (FOG):

For any shift of finite type (X_A, σ_A), $\mathrm{Inert}(X_A)$ is generated by elements of finite order.

Note that every simple automorphism is of finite order, therefore SFOG implies FOG. Both of these conjectures were shown to be false for certain examples [KR91; KRW+00].

1.6.1 Measures on unstable sets

For a point $x \in X_A$, we define its unstable set by

$$W^u(x) = \bigcup_{n \in \mathbb{Z}} R_n^+(x).$$

We make the set $W^u(x)$ σ-compact by endowing it with a basis for its topology given by

$$\{R_m^+(y) : y \in W^u(x), m \in \mathbb{Z}\}.$$

We denote by λ_A the Perron-Frobenius eigenvalue of A, and let w be a right eigenvector for λ_A.

On each $W^u(x)$ we define a σ-finite measure, μ_u^x by

$$\mu_u^x(R_m^+(y)) = \lambda_A^{-m} \cdot w_{\mathfrak{t}(y_m)},$$

where $\mathfrak{t}(y_m)$ denotes the terminal node for the edge corresponding to y_m.

The collection of measures $\{\mu_u^x\}_{x \in X_A}$ satisfies the following two properties:

1. $\mu_u^x(R_m^+(x)) = \lambda_A^{-1}\mu_u^{\sigma(x)}(R_{m-1}^+(\sigma(x)),$
2. there exists $N \in \mathbb{N}$ such that if $x_{[0,N]} = y_{[0,N]}$, then $\mu_u^x(R_N^+(x)) = \mu_u^y(R_N^+(y))$.

This collection of measures is unique except for a multiplicative constant:

Proposition 1.33 ([Boy86], Proposition 3.2) *Let $\{\nu_u^x\}_{x \in X_A}$ be a collection of measures that satisfy properties* (a) *and* (b)*. Then, there exists a constant K such that $\nu_u^x = K\mu_u^x$ for all $x \in X_A$.*

With this proposition, we define the map $\tau_\mu : D_A^+ \to \mathbb{R}_+$ by

$$\tau_\mu(R_m^+(x)) = \mu_u^x(R_m^+(x)),$$

which induces a group homomorphism $\tau : D_A \to \mathbb{R}$ such that $\tau(D_A^+) \subseteq \mathbb{R}_+$. Due to Proposition 1.33, for any $\phi \in \mathrm{Aut}(X_A)$ there exists a unique λ_ϕ such that:

$$\tau(\phi^*(W)) = \lambda_\phi \tau(W) \text{ for all } W \in D_A.$$

This allows us to define the homomorphism $\Psi : \mathrm{Aut}(X_A) \to \mathbb{R}_+^*$ by $\Psi(\phi) = \lambda_\phi$. An important property of this map is that when $\det(I - tA)$ is an irreducible polinomial, Ψ is inyective ([BMT87], Corollary 5.11).

Finally, we note that λ_ϕ is in fact an eigenvalue of S_ϕ.

1.7 Distortion

Group distortion

Definition 1.34 *Let G be a finitely generated group, and S a symmetric generating set. Given $g \in G$, we define the length of g with respect to S, $\ell_S(g)$, as the smallest n such that g can be written as a product of n elements of S. We write,*

$$\ell_S(g) = n.$$

By convention, we use that $\ell_S(e) = 0$.

It is easy to see that ℓ_S is a symmetric and sub-additive function. This motivates the definition of the length function of the group.

Definition 1.35 *Let G de a group. A length function of G is a function $L : G \to \mathbb{R}_+$ that is symmetric, sub-additive and satisfies $L(e) = 0$.*

These functions are useful to show that an element of the group is not distorted. Next, we see that the function ℓ_S depends on the generating set S only by a multiplicative constant.

Lemma 1.36 ([CF06], Lema 2.4) *If S_1 and S_2 are two generating sets of G, then there exists a constant $c \geq 1$ such that*

$$\frac{1}{c}\ell_{S_2}(g) \leq \ell_{S_1}(g) \leq c\ell_{S_2}(g) \ \forall g \in G.$$

Definition 1.37 *Let G be a finitely generated group and S a symmetric generating set. The transition length of an element $g \in G$ is defined as the limit:*

$$\|g\|_S := \lim_{n\to\infty} \frac{\ell_S(g^n)}{n}.$$

We say g is a distorted element if $\|g\|_S = 0$.

Remark 1.38 *It is important to note that due to Lemma 1.36, the property of being distorted is independent of the generating set.*

We are also interested in comparing the growth rates of different elements.

Definition 1.39 *Given $g : \mathbb{N} \to \mathbb{N}$, we say that the distortion (with respect to a generating set S) of $h \in G$ grows faster than g if there exists a sequence $n_i \to \infty$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) > g(n)$ for all sufficiently large n and*

$$\ell_S(h^{f(n_i)}) \leq n_i.$$

For example, we say that h has quadratic distortion if $g(n) = n^2$ or that it has exponential distortion if $g(n) = e^n$.

Finally, the notion of distortion can be generalized to non-finitely generated groups.

Definition 1.40 *Let G be a group. An element $g \in G$ is distorted if there exists a finitely generated subgroup $H \leq G$, with $g \in H$, such that g is distorted in H.*

There are some algebraic tools to check the distortion of an element.

Proposition 1.41 ([CF06])
1. *If $\phi : G \to H$ is a homomorphism and $\phi(g)$ is not distorted in H, then g is not distorted in G.*
2. *If $L : G \to \mathbb{R}_+$ is a length function and*
$$\lim_{n\to\infty} \frac{L(g^n)}{n} > 0,$$
then g is not distorted.
3. *A quasi-morphism is a map $\phi : G \to \mathbb{R}$ such that there exists a constant $c > 0$ for which*
$$|\phi(g) + \phi(h) - \phi(gh)| \leq c, \ \forall g, h \in G.$$
If $|\phi(g)| > c$, then g is not distorted.

Let's see some examples of distorted elements on different groups.

Example 1.42
1. *In any group, torsion elements are distorted.*
2. *On free groups or free abelian groups, the only distorted element is the identity.*
3. *For the Heisenberg group, defined by*
$$\mathcal{H} := \mathcal{U}_3(\mathbb{Z}) = \langle x, y, z \mid [x, z], [y, z], [x, y]z^{-1} \rangle,$$
z has quadratic distorsion,
$$z^{n^2} = [x^n, y^n] \quad \forall n \in \mathbb{Z}.$$
4. *For the Baumslag-Solitar, group defined by*
$$\mathrm{BS}(1, p) = \langle a, b \mid bab^{-1} = a^p \rangle,$$
the element a has exponential distortion. Similarly, in the group defined by
$$\langle a, b, c \mid bab^{-1} = a^2, \ aca^{-1} = c^2 \rangle,$$
the element c has doubly exponential distortion.
5. *Let $G = \mathrm{GL}(n, \mathbb{C})$. We define $L : G \to \mathbb{R}_+$ by:*
$$L(A) = \log\left(\max\left\{\|A\|, \|A^{-1}\|\right\}\right).$$
It is clear that L is a length function. As such, if A has an eigenvalue whose absolute value is not 1, then A is not distorted.

Lemma 1.43 *Let $L : G \to \mathbb{R}_+$ be a subadditive function such that $L(e) = 0$. Then, the function defined by*

$$\mathbb{L}(g) = \max\left\{L(g), L(g^{-1})\right\},$$

is a length function of G.

1.8 Expansivity

Let (X, ρ) be a compact metric space which we asume to be infinite. A $\mathbb{Z}^d$-action Ψ on X is a homomorphism of the additive group $\mathbb{Z}^d$ to the group $\text{Homeo}(X)$. Given a subset $F \subseteq \mathbb{R}^d$ we define:

$$\rho_\Psi^F(x, y) = \sup\{\rho(\Psi^n(x), \Psi^n(y)) : n \in F \cap \mathbb{Z}^d\}.$$

If $F \cap \mathbb{Z}^d = \emptyset$, we write $\rho_\Psi^F(x, y) = 0$.

Definition 1.44 *A $\mathbb{Z}^d$-action Ψ on X is expansive if there exists $c > 0$ such that*

$$\rho_\Psi^{\mathbb{R}^d}(x, y) \leq c \implies x = y.$$

In such a case, c is called the expansivity constant of Ψ.

In the case of $d = 1$, we say Ψ is positively expansive if if there exists $c > 0$ such that

$$\rho_\Psi^{\mathbb{R}_+}(x, y) \leq c \implies x = y.$$

For a subset $F \subseteq \mathbb{R}^d$ y $v \in \mathbb{R}^d$, we define

$$\text{dist}(v, F) = \inf\{\|v - w\| : w \in F\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. For $t > 0$ we define the thickening of F by t as $F^t = \{v \in \mathbb{R}^d : \text{dist}(v, F) \leq t\}$.

Definition 1.45 *Let Ψ be a $\mathbb{Z}^d$-action on X and $F \subseteq \mathbb{R}^d$. Then, F is expansive for Ψ if there exists $\varepsilon > 0$ and $t > 0$ such that*

$$\rho_\Psi^{F^t}(x, y) \leq \varepsilon \implies x = y.$$

If F does not satisfy this condition, it is said to de non-expansive.

Definition 1.46 *Let $E, F \subseteq \mathbb{R}^d$, and Ψ an expansive $\mathbb{Z}^d$-action on X. We say E codifies F if for all $v \in \mathbb{R}^d$,*

$$\rho_\Psi^{E+v}(x, y) \leq c \implies \rho_\Psi^{F+v}(x, y) \leq c.$$

When the action is expansive, the following Lemma allows us to consider a uniform ε in the previous definition.

Lemma 1.47 ([BL97], Lema 2.3) *Let Ψ be an expansive $\mathbb{Z}^d$-action on X, with expansivity constant c. Then, for each expansive subset $F \subseteq \mathbb{R}^d$ there exists $s > 0$ such that*

$$\rho_{\Psi}^{F^s}(x, y) \le c \implies x = y.$$

We are interested in defining a notion of distortion for an automorphism on an arbitrary expansive system.

Given $\phi \in \mathrm{Aut}(X, \Psi)$ and $\varepsilon > 0$, because its domain is compact and ϕ is a continuous function:

$$\exists \delta > 0 : \forall x, y \in X,\ \rho(x, y) \le \delta \implies \rho(\phi(x), \phi(y)) \le \varepsilon$$

In particular, we call $\delta(\phi)$ the δ obtained when taking $\varepsilon = c$.

Lemma 1.48 *Let Ψ be an expansive $\mathbb{Z}^d$-action on X, with expansivity constant c. Then, for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $\forall x, y \in X$,*

$$\rho_{\Psi}^{B_\infty(0,M)}(x, y) \le c \implies \rho(x, y) \le \varepsilon.$$

Proof. We proceed by contradiction. Let ε be such that for all $m \in \mathbb{N}$, there are $x_m, y_m \in X$ such that $\rho_{\Psi}^{B_\infty(0,m)}(x_m, y_m) \le c$ and $\rho(x_m, y_m) > \varepsilon$.

Because X is compact, we have a subsequence $(m_i)_{i\in\mathbb{N}}$ such that the sequences $(x_{m_i})_{i\in\mathbb{N}}$ and $(y_{m_i})_{i\in\mathbb{N}}$ converge to $\bar{x}$ and $\bar{y}$ respectively.

Let us have $\eta > 0$, $n \in \mathbb{Z}^d$ and $I \in \mathbb{N}$ such that $\forall i \ge I$, $m_i \ge \max\{\|n\|_\infty, m_I\}$

$$\rho(\Psi^n x_{m_i}, \Psi^n \bar{x}) \le \frac{\eta}{2} \quad \text{and} \quad \rho(\Psi^n y_{m_i}, \Psi^n \bar{y}) \le \frac{\eta}{2}$$

Then,

$$\begin{aligned}\rho(\Psi^n \bar{x}, \Psi^n \bar{y}) &\le \rho(\Psi^n x_{m_i}, \Psi^n \bar{x}) + \rho(\Psi^n x_{m_i}, \Psi^n y_{m_i}) + \rho(\Psi^n y_{m_i}, \Psi^n \bar{y}) \\ &\le \eta + c.\end{aligned}$$

By the previous argument, we take $\eta \to 0$ (m_i is always greater than n) to obtain

$$\rho(\Psi^n \bar{x}, \Psi^n \bar{y}) \le c,\ \forall n \in \mathbb{Z}^d.$$

Because Ψ is expansive, this means that $\bar{x} = \bar{y}$. Therefore, for a sufficiently large i

$$\rho(x_{m_i}, y_{m_i}) \le \rho(x_{m_i}, \bar{x}) + \rho(y_{m_i}, \bar{y})$$

$$\leq \varepsilon$$

which is a contradiction. □

By setting $\varepsilon = \delta(\phi)$ on the previous lemma, we are allowed to make the following definition.

Definition 1.49 *Let Ψ be the $\mathbb{Z}^2$-action defined by T and $\phi \in \text{Aut}(X,T)$. We call the range of ϕ the minimum $M \in \mathbb{N}$ such that $\forall x, y \in X$,*

$$\rho_{\Psi}^{[-M,M]}(x,y) \leq c \implies \rho(\phi(x),\phi(y)) \leq c,$$

and we denote it by $\text{range}(\phi)$*.*

1.8.1 Range distortion

Definition 1.50 *The asymptotic range of an automorphism $\phi \in \text{Aut}(X,T)$ is defined by*

$$\text{range}_{\infty}(\phi) := \lim_{n\to\infty} \frac{\text{range}(\phi^n)}{n}.$$

If $\text{range}_{\infty}(\phi) = 0$, ϕ is said to be range distorted.

The asymptotic range is well defined due to the fact that the sequence $(\text{range}(\phi^n))_n$ is subadditive.

It is easy to see that both notions of distortion are in fact related:

Proposition 1.51 ([Cyr+16], Prop 3.4) *Let G be a finitely generated subgroup of $\text{Aut}(X)$. If $g \in G$ is distorted, then g is range distorted and $h_{\text{top}}(g) = 0$.*

Furthermore, by Lemma 1.43 we can see that range : $\text{Aut}(X) \to \mathbb{N}$ can be seen as a length function of the automorphism group.

The following questions arise naturally from the relationship of the two concepts:

Question 1.52 *¿For $\phi \in \text{Aut}(X)$ that is range distorted, is it true that ϕ^{-1} is range distorted?*

Question 1.53 *¿For $\phi \in \text{Aut}(X)$ that is range distorted, is it true that ϕ is a distorted element of* $\text{Aut}(X)$*?*

Chapter 2

Range distorsion on expansive systems

Since Boyle and Lind introduced the notions of expansive and non-expansive directions for the study of directional dynamics of an action, there has been one persistent question: which sets can occur as sets of non-expansive directions?

In [BL97] they showed that this set is closed and, if the domain is infinite, non-empty. Furthermore, they showed that any closed set of directions, with two or more elements is the set of non-expansive directions for some action. Later, Hochman showed in [Hoc11] that for every direction, there exists an automorphism of a subshift such that its unique non-expansive direction is the selected one, effectively solving the realization problem. Nevertheless, the domain he constructs to achieve this lacks many natural properties one expects from subshifts, such as being transitive or an SFT. This motives him to ask the following, still open, question:

Question 2.1 ([Hoc11], Problem 1.3) *Does any closed non-empty set of directions arise as the set of non-expansive direction of a* $\mathbb{Z}^2$*-action that is transitive or minimal?*

To approach this problem, we want to expand the connection that exists between distorted automorphisms and a unique non-expansive direction (as shown in [CFK19]) to the realm of general expansive systems.

As a first part of this chapter, we begin by establishing this connection, we generalize the notion of range present on morphisms of shift spaces to expansive systems, allowing us to introduce the concept of distorted automorphisms to this generalized setting. Then, we continue by generalizing the Lyapunov exponents introduced by Cyr et al in [CFK19]. Finally we look at some properties of the set of distorted automorphisms whose inverse is also distorted.

The second part of this chapter is concerned with the connection between geometry and distortion through the various Lyapunov exponents defined up to that point. We begin by generalizing the results obtained by Cyr et al. in [CFK19] to the context of expansive systems. These results establish a definitive connection between Lyapunov exponents and non-expansive directions. Then, we present the notion of Prediction Shapes introduced by

Hochman in [Hoc11] which provides an alternative geometrical view of distortion.

2.1 Generalizing range

Using the terminology introduced in the previous section, when $d = 2$ the range of ϕ can be understood as the minimum $M \in \mathbb{N}$ such that $[-M, M] \times \{0\}$ codifies $\{(0,1)\}$ on the $\mathbb{Z}^2$-system (X, T, ϕ). It is clear that if the system is a subshift, the previous definition coincides with the usual notion of the radius of an automorphism.

Lemma 2.2 *Let* $\phi, \psi \in \mathrm{Aut}(X, T)$. *Then,*

$$\mathrm{range}(\phi \circ \psi) \leq \mathrm{range}(\phi) + \mathrm{range}(\psi).$$

In particular, the sequence $(\mathrm{range}(\phi^n))_{n\in\mathbb{N}}$ *is subadditive.*

Proof. Let $M = \mathrm{range}(\phi)$ and $N = \mathrm{range}(\psi)$. If we have $\rho_T^{[-(M+N),M+N]}(x, y) \leq c$, then

$$\forall t \in [-(M + N), M + N] : \ \rho(T^t x, T^t y) \leq c$$

If we fix $m \in [-M, M]$ and define $\bar{x} = T^m x$, $\bar{y} = T^m y$, from the previous inequality we obtain that:

$$\forall n \in [-N, N] : \ \rho(T^n \bar{x}, T^n \bar{y}) \leq c.$$

By definition of radius, this means that $\rho(\psi(\bar{x}), \psi(\bar{y})) \leq c$. Because this is posible for any $m \in [-M, M]$ we have:

$$\forall m \in [-M, M] : \ \rho(T^m \psi(x), T^m \psi(y)) \leq c,$$

which implies that $\rho(\phi \circ \psi(x), \phi \circ \psi(y)) \leq c$, and therefore, $\mathrm{range}(\phi \circ \psi) \leq N + M$. □

Because $(\mathrm{range}(\phi^n))_{n\in\mathbb{N}}$ is a subadditive sequence, by Fekete's Lemma we have that the limit $\lim_{n\to\infty} \frac{\mathrm{range}(\phi^n)}{n}$ exists.

Definition 2.3 *The asymptotic range of* $\phi \in \mathrm{Aut}(X, T)$ *is defined by*

$$\mathrm{range}_\infty(\phi) := \lim_{n\to\infty} \frac{\mathrm{range}(\phi^n)}{n}$$

If $\mathrm{range}_\infty(\phi) = 0$ *we say* ϕ *is range distorted, and denote the set of all range distorted automorphisms by* $RD(X, T)$.

Proposition 2.4 *Let* $\phi, \psi \in \mathrm{Aut}(X, T)$. *We have,*

1. $\text{range}_\infty(\psi \circ \phi \circ \psi^{-1}) = \text{range}_\infty(\phi)$,

2. $\text{range}_\infty(\phi^p) = p \cdot \text{range}_\infty(\phi)$ *for* $p \in \mathbb{N}$,

3. *if* ψ *and* ϕ *commute, then* $\text{range}_\infty(\psi \circ \phi) \leq \text{range}_\infty(\psi) + \text{range}_\infty(\phi)$.

2.2 Alternative notion of distortion

The definition of asymptotic range concerns the average evolution of the symmetric window with which the automorphism is computed. To complement this analysis, we introduce an alternative notion of distortion through the use of Lyapunov exponents first presented by Cyr et al. in [CFK19]. These exponents study the average speed at which information propagates asymptotically through the automorphism. These notion will later be shown to be very important because of their connection to the geometry of the automorphism.

Lemma 2.5 *Let* (X, T) *be an expansive system and* $\phi \in \text{Aut}(X, T)$*. Then,*

$$\rho_T^{[0,+\infty)}(x, y) \leq c \implies \rho_T^{[\text{range}(\phi),+\infty)}(\phi(x), \phi(y)) \leq c.$$

That is, $[0, +\infty) \times \{0\}$ *codifies* $[\text{range}(\phi), +\infty) \times \{1\}$ *in* (X, T, ϕ).

For $\phi \in \text{Aut}(X, T)$, we say $A \subseteq \mathbb{Z}$ ϕ-codifies $B \subseteq \mathbb{Z}$ if, in (X, T, ϕ), $A \times \{0\}$ codifies $B \times \{1\}$. We consider the following sets:

$$C^-(\phi) = \{k \in \mathbb{Z} : (-\infty, 0] \ \phi\text{-codifies } (-\infty, k]\},$$

$$C^+(\phi) = \{k \in \mathbb{Z} : [0, \infty) \ \phi\text{-codifies } [k, \infty)\}.$$

Due to Lemma 2.5, both sets are non-empty. This allows us to define the quantities:

$$W^-(n, \phi) = \sup C^-(\phi^n).$$
$$W^+(n, \phi) = \inf C^+(\phi^n).$$

By definition, we have that for $n \geq 1$, $W^\pm(n, \phi) = W^\pm(1, \phi^n)$.

Lemma 2.6 *Let* $\phi, \psi \in \text{Aut}(X, T)$*. Then,*

$$W^+(n, \phi\psi) \leq W^+(n, \phi) + W^+(n, \psi) \text{ and } W^-(n, \phi\psi) \geq W^-(n, \phi) + W^-(n, \psi).$$

In particular, the sequences $(W^+(n, \phi))_{n \in \mathbb{N}}$ *and* $(-W^-(n, \phi))_{n \in \mathbb{N}}$ *are subadditive.*

Again, Fekete's Lemma allows us to make the following definition:

Definition 2.7 *Given $\phi \in \mathrm{Aut}(X)$, we define the exponents:*

$$\alpha^-(\phi) = \lim_{n\to\infty} \frac{W^-(n,\phi)}{n}$$

$$\alpha^+(\phi) = \lim_{n\to\infty} \frac{W^+(n,\phi)}{n}$$

Definition 2.8 *We say an automorphism $\phi \in \mathrm{Aut}(X,T)$ is α-distorted if $\alpha^\pm(\phi) = 0$. We denote the set of all α-distorted automorphisms by $AD(X,T)$.*

These exponents satisfy some very useful properties:

Proposition 2.9 *For $\phi \in \mathrm{Aut}(X,T)$, we have the following properties:*

1. *For all $k \in \mathbb{Z}$, $\alpha^\pm(T^k\phi) = \alpha^\pm(\phi) + k$.*
2. *For all $m \in \mathbb{N}$, $\alpha^\pm(\phi^m) = m\alpha^\pm(\phi)$.*
3. *If $\psi \in \mathrm{Aut}(X,T)$ conmutes with ϕ, then:*

$$\alpha^+(\phi\psi) \leq \alpha^+(\phi) + \alpha^+(\psi) \text{ and } \alpha^-(\phi\psi) \geq \alpha^-(\phi) + \alpha^-(\psi).$$

4. *$\alpha^+(\phi) + \alpha^+(\phi^{-1}) \geq 0$ and $\alpha^-(\phi) + \alpha^-(\phi^{-1}) \leq 0$,*
5. *If X is an infinite subshift, then $\alpha^-(\phi) \leq \alpha^+(\phi)$.*

Using the following Lemma we can see that α-distortion is weaker than range distortion. We denote the interval $[-W^+(n,\phi), -W^-(n,\phi)]$ by $I(n,\phi)$.

Lemma 2.10 *Let ϕ be an automorphism of (X,T). If J is an interval that ϕ^n-codes $\{0\}$, then $I(n,\phi) \subseteq J$.*

Proof. Let $J = [a,b]$ be an interval that ϕ^n-codes $\{0\}$. Then, $(-\infty, 0]$ must ϕ^n-code $(-\infty, -b]$ and $[0,\infty)$ must ϕ^n-code $[-a,\infty)$. We conclude using the definition of $I(n,\phi)$. □

Lemma 2.11 *For $\phi \in \mathrm{Aut}(X,T)$, $\mathrm{range}_\infty(\phi) \geq \max\left\{\alpha^+(\phi), -\alpha^-(\phi)\right\}$.*

Proof. The result follows from the previous lemma by noting that the interval $[-\mathrm{range}(\phi^n), \mathrm{range}(\phi^n)]$ ϕ^n-codes $\{0\}$. □

Proposition 2.12 *Let (X,σ) be an infinite subshift. Then, $RD(X) \subseteq AD(X)$.*

Proof. For $\phi \in \text{Aut}(X)$, due to (5) on Proposition 2.9 and Lemma 2.11, we know that

$$\text{range}_\infty(\phi) \geq \alpha^+(\phi) \geq \alpha^-(\phi) \geq -\text{range}_\infty(\phi),$$

which concludes the proof. □

We can see that in the context of SFTs, the two notions are in fact equivalent. To see this, we first need an auxiliary result:

Lemma 2.13 ([CFK19], Lema 3.21) *Let (X,T) be an SFT and $\phi \in \text{Aut}(X)$. Then, there is a constant $C(\phi)$ such that*

$$\frac{|I(n,\phi)|-1}{2} \leq \text{range}(\phi^n) \leq |I(n,\phi)| + C(\phi).$$

If X is a full-shift we can take $C(\phi) = 0$.

Theorem 2.14 *Let (X,σ) be an SFT. Then, $AD(X) = RD(X)$.*

Proof. By diving by n and taking limit on the expression given by Lema 2.13, we conclude. □

2.3 Symmetric distorsion subset

We denote the subset of range distorted automorphisms, with a range distorted inverse by:

$$\mathfrak{D}(X,T) = \{f \in \text{Aut}(X) : \text{range}_\infty(f) = \text{range}_\infty(f^{-1}) = 0\},$$

and the subgroup of distorted elements (in the algebraic sense) by $GD(X,T)$.

Proposition 2.15 *$GD(X,T) \subseteq \mathfrak{D}(X,T)$.*

Proof. Let g be a distorted element of $\text{Aut}(X,T)$. This means that there exists a finitely generated subgroup G such that $\|g\|_S = 0$, for a symmetric generating set S. Then, by Proposition 2.2,

$$\text{range}(g^n) \leq \ell_S(g^n) \cdot \max_{s\in S}\{\text{range}(s)\}.$$

Dividing the expresion by n and taking limit, g is range distorted. We conclude by noting that if g is group distorted, its inverse also is.

Note that this proposition can also be proved by taking into account that through Lema 1.43, $\text{range}(\cdot)$ is a length function for the automorphism group. □

Lemma 2.16 *Let $f \in \text{Aut}(X,T)$ be equicontinuous, where (X,T) is an expansive system of constant c. Then $f \in \mathfrak{D}(X,T)$.*

Proof. Because f is equicontinuous, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(x, y) \leq \delta \implies \rho(f^n(x), f^n(y)) \leq \varepsilon, \quad \forall n \in \mathbb{Z}.$$

By picking $\varepsilon = c$, Lemma 1.48 tells us that there exists $M > 0$ such that

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(x, y) \leq \delta.$$

This implies that,

$$\rho_T^{[-M,M]}(x, y) \leq c \implies \rho(f^n(x), f^n(y)) \leq c, \quad \forall n \in \mathbb{Z},$$

that is, $\mathrm{range}(f^n) \leq M$ for all $n \in \mathbb{Z}$. We conclude that $\mathrm{range}_\infty(f) = 0$.

□

Remark 2.17 *As a consequence of the Arzelà-Ascoli Theorem, any compact subgroup, K, of $\mathrm{Aut}(X, T)$ satisfies $K \subseteq \mathfrak{D}(X, T)$.*

We are interested in understanding the structure of $\mathfrak{D}(X, T)$. In the general setting, this set is not a subgroup of $\mathrm{Aut}(X, T)$, as is shown in Example 4.2, where we show an automorphism which is not distorted, but is a composition of two distorted automorphisms.

The next Lemma follows directly from Prop. 2.4:

Lemma 2.18 *Let $f, g \in \mathfrak{D}(X, T)$. Then, we have the following,*

1. *If $[f, g] = \mathrm{id}$, then $f \circ g \in \mathfrak{D}(X, T)$.*
2. *For all $h \in \mathrm{Aut}(X, T)$, $h \circ f \circ h^{-1} \in \mathfrak{D}(X, T)$.*
3. *$f^p \in \mathfrak{D}(X, T)$, for all $p \in \mathbb{N}$.*

Proposition 2.19 *Let (X_A, σ_A) be an SFT, where A is primitive. Then, $\mathfrak{D}(X_A)$ contains an isomorphic copy of every finite group.*

Proof. This result follows from the fact that every finite order automorphism is equicontinuous and Theorem 1.21. □

Finally, let us generalize the fact that the subgroup generated by the action T has a trivial intersection with $\mathfrak{D}(X, T)$.

Lemma 2.20 *Let (X, T) be an expansive system of expansive constant $c > 0$. If X is infinite, then $\mathrm{range}_\infty(T) = 1$.*

To prove this result we make use of a result from Schwartzman about infinite systems.

Theorem 2.21 ([BL97], Theorem 3.9) *Let T be a homeomorphism of an infinite compact metric space (X, ρ) and $\delta > 0$. Then, there exists two distinct $x, y \in X$ such that $\rho(T^n x, T^n y) \leq \delta$ for all $n \geq 0$.*

Proof of 2.20. It is evident that $\text{range}(T^n) \leq n$. To obtain the other bound, by applying Theorem 2.21 to T^{-1}, we obtain two distinct points $x, y \in X$ such that $\rho_T^{(-\infty,0]}(x, y) \leq c$. Given that T is expansive and the points are different, we choose the smallest $m > 0$ such that $\rho(T^m x, T^m y) > c$.

For $n \in \mathbb{N}$, we define $\bar{x} = T^{m-n}x$ and $\bar{y} = T^{m-n}y$. Then, $\rho_T^{[-n+1,n-1]}(\bar{x}, \bar{y}) \leq c$, with $\rho(T^n \bar{x}, T^n \bar{y}) > c$. This means that $\text{range}(T^n) > n - 1$, and as a consequence $\text{range}(T^n) = n$. □

Corollary 2.22 *Let (X, T) be an expansive dynamical system of constant $c > 0$. If X is infinite, then $\mathfrak{D}(X, T) \cap \langle T \rangle = \{\text{id}\}$.*

2.4 Examples

Non-trivial examples of infinite order distorted automorphisms are hard to come by. This is especially true for subshifts of higher rigidity such as transitive ones. In fact, it is still an open question whether one exists on minimal subshifts at all. Amongst the first examples is the one constructed by Hochman [Hoc11] - the problem being that the domain on which it is defined is highly specific.

In spite of these complications, Guillon and Salo established a connection between aperiodic Turing Machines and distorted automorphisms [GS17]. Through the use of conveyor belt techniques, it is possible to construct examples for fullshifts.

In this section we present one of the dynamical models for Turing Machines, moving head machines, introduced by Kürka. This machines are then connected to distorted automorphisms, and through conveyor belts, are shown to define them on the full-shift.

2.4.1 Turing machines as dynamical systems

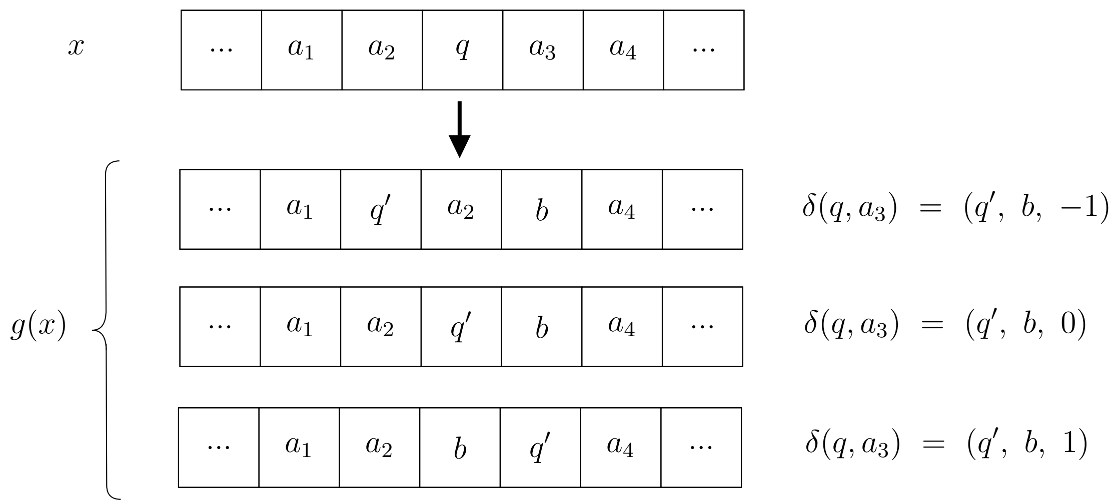
In the context of dynamical systems, there are two ways of representing a Turing Machine (TM) as a dynamical system: one where the tape moves and one where the head does. In this section we will use the second model, presented in [K97̊].

We will denote the set of states of the machine by Q, the alfabet by A and $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$ its transition function. For $n \geq 0$ we define the subshift,

$$X_n = \{x \in (Q \cup A)^{\mathbb{Z}} : |\{i \in \mathbb{Z} : x_i \in Q\}| \leq n\}.$$

It is posible to show that X_n is a sofic subshift.

Definition 2.23 *We define a moving head Turing Machine (TMH) as $g \in \mathrm{End}(X_1)$ where the head (given by the coordinate $x_i \in Q$) points to the site in its right and g executes the machine given by the transition function $\delta : Q \times A \to Q \times A \times \{-1, 0, 1\}$.*



It is easy to see that for all TMH $\mathrm{range}(g) = 2$.

Definition 2.24 *The position function $\mathfrak{p} : X_1 \to \mathbb{Z} \cup \{\infty\}$ of a TMH g is defined by $\mathfrak{p}(x) = n$ if $x_n \in Q$ and $\mathfrak{p}(x) = \infty$ on the other case.*

The furtherest site the machine visits up to time t by the machine on configuration $x \in X_1$ as:

$$s_t(x) := \max\{|\mathfrak{p}(g^s(x))| : 0 \leq s \leq t\}.$$

Then, we define the movement function of the machine at time t as:

$$m(t) = \max_{x \in X_1} s_t(x).$$

It is clear that $\mathrm{range}(g^t) = m(t)$. Also, there exists a tricotomy with respect to the velocity which machines can have,

Theorem 2.25 ([GS17], Theorem 1) *Let g be a TMH with movement function m. Then, exactly one of the following holds:*

- *m is bounded,*

- $m(t) = \Omega(\log(t))$ *y* $m(t) = O(t/\log(t))$,
- $m(t) = \Theta(t)$.

In addition, it is posible to establish a connection between the periodicity of the function and its asymptotic speed rate.

Theorem 2.26 ([GS17], Theorem 2) *Every TMH with no weakly periodic configurations on* $X_1 \setminus X_0$ *is range distorted.*

An example of an aperiodic machine is constructed in [COTA17]. Called the SMART machine, this reversible TMH is among other properties, aperiodic, which given the previous Theorem implies that it is distorted. To find distorted automorphisms on the fullshift, we can embed this and other TMH's into its automorphism group through the use of conveyor belts. By slightly modifying Lemma 3 from [GS17] we obtain the following result:

Proposition 2.27 *Let* g *be a TMH. Then, by defining*

$$\Gamma = (\Sigma^2 \times \{<, >\}) \cup (Q \times \Sigma) \cup (\Sigma \times Q),$$

there exists an endomorphism $f : \Gamma^{\mathbb{Z}} \to \Gamma^{\mathbb{Z}}$ *such that if* $m : \mathbb{N} \to \mathbb{N}$ *is the movement funciton of* g*, then* $\mathrm{range}(f^t) \leq m(t)$ *for all* $t \in \mathbb{N}$*. Furthermore,* f *is reversible if and only if* g *is.*

Because the conveyor belt method allows us to see every reversible TMH within the automorphism group of a fullshift, we can conclude the following:

Theorem 2.28 *Let* (X_A, σ_A) *be an SFT such that* A *is primitive. Then, the set of reversible TMH is contained in* $\mathrm{Aut}(X_A)$*. In particular, it contains an infinite order distorted automorphism.*

The proof of this fact follows directly from the previous Proposition and Theorem 1.21.

2.5 Geometry and distortion

In [CFK19], Cyr, Franks and Kra showed that there is a connection between discrete Lyapunov exponents and the geometry of the $\mathbb{Z}^2$-system (X, σ, ϕ) where $\phi \in \mathrm{Aut}(X)$. This connection was first explored by Hochman in [Hoc11] through the notion of prediction shapes. We generalize these result to the context of expansive systems (X, T). Finally, in the context of subshifts, we connect the newly introduced direction exponents to the standard ones. We relate the fact of having these exponents be equal to zero to having non-expansive directions.

Theorem 2.29 *Let* (X, T) *be an expansive dynamical system and* $\phi \in \mathrm{Aut}(X, T)$*. Then, the line defined by* $x = \alpha^+(\phi)y$ *is not left-expansive. Analogously, the line defined by* $x = \alpha^-(\phi)y$ *is not right-expansive.*

The proof of this fact is very technical and can be retraced step by step from [CFK19].

Theorem 2.30 *Let (X,T) be an expansive dynamical system, $\phi \in \mathrm{Aut}(X,T)$ and L a line in $\mathbb{R}^2$ given by $x = my$, if $m > \max\{\alpha^+(\phi), -\alpha^-(\phi^{-1})\}$ or $m < \min\{\alpha^-(\phi), -\alpha^+(\phi^{-1})\}$, then L is expansive.*

Proof. Let us first show that if $m > \alpha^+(\phi)$, then L is left-expansive.

We take $x, y \in X$ such that:

$$\rho(T^n\phi^k(x), T^n\phi^k(y)) \leq c, \quad \forall (n,k) \in \mathbb{Z}^2 \text{ such that } n > mk.$$

Because $m > \alpha^+(\phi)$, vector $(\alpha^+(\phi), 1)$ is not parallel to L and points from its right to its left. By Definition 2.7, for sufficiently large n, vector $(W^+(n), n)$ is not parallel to L.

Next, let us have (u_0, v_0), an arbitrary point to the left of L (that is, $u_0 < mv_0$). There exists $n_0 > 0$ such that if $u_1 = u_0 - W^+(n_0)$ and $v_1 = v_0 - n_0$, then (u_1, v_1) is to the right of L. Therefore, the line given by $\{(t, v_1) : u_1 \leq t\}$ is to the right of L and codifies (u_0, v_0) by definition of $W^+(n_0)$. This shows that L is left expansive.

Analogously, if $m < \alpha^-(\phi)$ then L is right expansive.

Lastly, we can see that the transformation $r(x,y) = (x, -y)$ allows us to move between the $\mathbb{Z}^2$-systems (X, T, ϕ) and (X, T, ϕ^{-1}). Consequently, L is right-expansive (left) on the first system if and only if $r(L)$ is left-expansive (right) one the second one. This fact concludes the proof □

By combining these results, we arrive at the fundamental connection between distortion and non-expansive subspaces.

Corollary 2.31 *Let (X,T) be an expansive dynamical system and $f \in \mathrm{Aut}(X,T)$. Then, $f, f^{-1} \in AD(X,T)$ if and only if $x = 0$ is the only non-expansive direction of f.*

A first consequence of this connection is the fact that expansive automorphism can't be α-distorted.

Theorem 2.32 *Let (X,T) be an expansive system of constant c and $\phi \in \mathrm{Aut}(X,T)$ an expansive automorphism of constant δ. Then, $\phi \notin AD(X)$.*

Proof. Let us call Ψ the joint $\mathbb{Z}^2$-action of T and ϕ. Due to Lemma 1.48, we know that there exists $M \in \mathbb{N}$ such that:

$$\forall x, y \in X : \ \rho_T^{[-M,M]}(x,y) \leq c \implies \rho(x,y) \leq \delta.$$

Now, let us see that L_0, defined by $x = 0$, is an expansive direction. If we have $x, y \in X$ such that

$$\rho_\Psi^{L_0^M}(x, y) \le c,$$

in particular we have that,

$$\forall n \in \mathbb{Z}: \ \rho_T^{[-M,M]}(\phi^n(x), \phi^n(y)) \le c.$$

This implies that,

$$\forall n \in \mathbb{Z}: \ \rho(\phi^n(x), \phi^n(y)) \le \delta.$$

Given that ϕ is expansive, this means that $x = y$. We conclude by Theorem 2.29. □

2.5.1 Prediction Shapes

To further understand distorsion through geometrical lens, we make use of the notion of prediction shapes introduced by Hochman in [Hoc11]. The generalization is straight forward from the one presented in the aforementioned article.

Definition 2.33 *Let (X, T) be an expansive system of constant $c > 0$ and $\phi \in \mathrm{Aut}(X, T)$. A convex, open subset $\Lambda \subseteq \mathbb{R}^2$ is said to be a prediction shape for ϕ if $(0, 1) \times \{0\} \subseteq \Lambda$ and for every compact set $\Lambda_0 \subseteq \Lambda$, and all n large enough, if $x, y \in X$ satisfy*

$$\rho_T^{[-n,n]}(x, y) \le c,$$

then

$$\rho_{(T,\phi)}^{n\Lambda_0}(x, y) \le c.$$

Remark 2.34 *If $\mathrm{range}(\phi), \mathrm{range}(\phi^{-1}) \le R$ then the diamond shaped region with vertices at $(-1, 0)$ and $(1, 0)$ with sides of slopes $\pm R^{-1}$ is a prediction shape for ϕ due to the definition of range for automorphisms of expansive systems.*

The importance of these notions comes from two results that reveal the geometric structure of distorted elements.

Proposition 2.35 ([Hoc11], Prop. 6.1) *Let X be an infinite subshift, $\phi \in \mathrm{Aut}(X)$ and Λ a prediction shape for ϕ. Let θ^+ and θ^- denote the slopes of the right-tangent and left-tangent rays of $\partial\Lambda$ at $(-1, 0)$ and $(1, 0)$, respectively. Then, $\alpha^+(\phi) \le \dfrac{1}{\theta^+}$ and $\alpha^-(\phi) \ge \dfrac{1}{\theta^-}$.*

Corollary 2.36 ([Hoc11], Corolary 6.2) *If the strip $\Lambda = \{(x, y) : |x| < 1\}$ is a prediction shape for ϕ, then $\phi \in AD(X)$.*

2.6 Non-shift examples

Let us look at the range of an automorphism that is not from a shift space.

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. We endow this space with a metric induced by the 2-norm on $\mathbb{R}^n$:

$$\rho(x, y) = \inf_{k\in\mathbb{Z}^n} \|x - y - k\|_2.$$

To find expansive homeomorphisms, we use the following result about automorphisms of the torus.

Proposition 2.37 *Let T_A be an automorphism of the n-torus, with $n \geq 2$ and A its corresponding matrix on $GL(n, \mathbb{Z})$ over $\mathbb{R}^2$. Then, the following statements are equivalent:*

1. *T_A is expansive,*

2. *$A \in GL(n, \mathbb{Z})$ is expansive in $\mathbb{R}^n$,*

3. *A has no eigenvalue of modulus 1.*

We note that n must be grater or equal than two, due to the fact that there are no expansive automorphisms on the 1-torus. The proof of the proposition is outlined in [Wal00]. By following its procedure, we can obtain the following Lemma:

Lemma 2.38 *Let T_A be an expansive automorphism of the n-torus. Then, if we define $L'(A) = \max\{\|A\|, \|A^{-1}\|\}$, the expansive constant for the automorphism is given by*

$$c = \min\left\{\frac{1}{2L'(A)}, \frac{1}{4}\right\}.$$

Proof. Due to the previous Proposition, we know that if T_A is expansive, A is expansive. This in turn means that the set $\{\|A^m x\| : m \in \mathbb{Z}\}$ is unbounded.

Because T_A is linear, we only have to prove the following: for $x \in \mathbb{T}^n$ such that $x \neq 0$, then there exists $m \in \mathbb{Z}$ such that $\rho(T_A^m x, 0) > c$. We do this in two cases. If $\|x\| > c$, it is evident that:

$$\rho(T_A^0 x, 0) = \|x\|_2 > c$$

If $\|x\| \leq c$, due to the aforementioned set being unbounded we can define:

$$k = \inf\{|m| : \|A^m x\| > c,\ m \in \mathbb{Z}\}.$$

Let us suppose without loss of generality that $\|A^k x\| > c$. Then we have that

$$c < \|A^k x\| \leq \|A\|\|A^{k-1}x\| \leq L'(A)c \leq \frac{1}{2},$$

which means that $A^k x \in (-1, 1)^n$. Finally, $\rho(T_A^k x, 0) = \|A^k x\| > c$.

□

Let us see how everything works by taking the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

For simplicity's sake we will use the same notation for A and T_A.

Its eigenvalues are $\lambda_1 = 1+\sqrt{2}$ and $\lambda_2 = 1-\sqrt{2}$, which means that it defines an expansive homeomorphism. It is possible to see that its expansive constant is in fact $c = \frac{1}{4}$.

Furthermore, by examining at the matrices that conmunte with A, we find that

$$\mathrm{Aut}(\mathbb{T}^2, A) = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a^2 \neq 2b^2,\ a, b \in \mathbb{Z} \right\}.$$

It is posible to observe that a matrix such as

$$M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathbb{T}^2, A)$$

satisfies range$(M) = 1$, due to the fact that for $x, y \in \mathbb{T}^2$

$$\rho(Ax, Ay) \leq \frac{1}{4} \implies \rho(Mx, My) \leq \frac{1}{4}.$$

Finally, we notice that the eigenvalues of matrices in $\mathrm{Aut}(\mathbb{T}^2, A)$ are given by $\lambda_1 = a+\sqrt{2}b$ and $\lambda_2 = a - \sqrt{2}b$. By Theorem 2.32 we have that

$$AD(\mathbb{T}^2, A) = \{I, -I\},$$

where I is the identity matrix.

2.7 Open Questions

Having introduced different classes of distorted automorphisms, we procede to re-state the open questions related to the difference between group distortion and range distortion.

The first of these asks if the distortion of an automorphism forces the distortion of its inverse:

Question 2.39 *$RD(X, T) = \mathfrak{D}(X, T)$?*

Because the inverse of a group distorted element is group distorted, a negative answer to Question 2.39 would imply a negative answer to the fundamental question:

Question 2.40 *$RD(X, T) = GD(X, T)$?*

This fact also motivates us to ask the following question:

Question 2.41 $GD(X,T) = \mathfrak{D}(X,T)$*?*

Chapter 3

Lyapunov exponents and geometrical aspects of distortion

To refine our study of distortion, we would like to look at the way an automorphism propagates information on individual configurations. For this purpose, we use the notion of discrete Lyapunov exponents. The first appearance of these "point-wise" Lyapunov exponents can be traced to Shereshevsky [She92], and later expanded upon by Tisseur, Bressaud [Tis00; BT06], Courbage and Kamińsky [CK15a; CK06]. We will present a modified version of these exponents to better connect them to the ones defined on the previous chapter. Next, we modify the directional Lyapunov exponents introduced by Courbage and Kamińsky as we did with the conventional exponents. The directional exponents quantify the propagation of information through specific directions of the automorphism's space-time. Finally, we relate zero-valued directional Lyapunov exponents to directions of non-expansiveness.

3.1 Lyapunov exponents

Let (X, σ) be a subshift, and $\phi \in \mathrm{Aut}(X)$. As explained before, we introduce modifications to the traditional definitions of discrete Lyapunov exponents for them to better work with the exponents, $\alpha^{\pm}(\phi)$, introduced earlier. We define the sets,

$$R_s^+(x) = \{y \in X : y_{[s,\infty)} = x_{[s,\infty)}\},$$
$$R_s^-(x) = \{y \in X : y_{(-\infty,s]} = x_{(-\infty,s]}\},$$

where $x \in X$ and $s \in \mathbb{N}$. It is easy to see that the following properties are held by these sets:

$$\sigma^a R_c^{\pm}(\sigma^b x) = R_{c-a}^{\pm}(\sigma^{a+b} x),\ \forall a, b, c \in \mathbb{Z},\ x \in X.$$

Due to the fact that ϕ is a sliding block code, we can see that,

$$\phi^n(R_0^+(x)) \subseteq R_{nr}^+(\phi^n(x)) \text{ and } \phi^n(R_0^-(x)) \subseteq R_{-nr}^-(\phi^n(x)),$$

where $r = \text{range}(\phi)$.

This allows us to introduce the following definitions,

$$\tilde{\Lambda}_n^+(x) := \inf\{s : \phi^n(R_0^+(x)) \subseteq R_s^+(\phi^n(x))\},$$

$$\tilde{\Lambda}_n^-(x) := \sup\{s : \phi^n(R_0^-(x)) \subseteq R_s^-(\phi^n(x))\},$$

and

$$\Lambda_n^+(x) := \sup_{j\in\mathbb{Z}} \tilde{\Lambda}_n^+(\sigma^j x),$$

$$\Lambda_n^-(x) := \inf_{j\in\mathbb{Z}} \tilde{\Lambda}_n^-(\sigma^j x).$$

By following the procedures analogous to the ones outlined in [CK15a] we can finally introduce the notion of Lyapunov exponents.

Definition 3.1 *Let (X, σ) be a subshift and $\phi \in \text{Aut}(X)$. We define the right (resp left) Lyapunov exponent at $x \in X$ as:*

$$\lambda^\pm(x) = \lim_{n\to\infty} \frac{\Lambda_n^\pm(x)}{n}.$$

Remark 3.2 *It is easy to see that both functions $\lambda^\pm$ are σ and ϕ invariant.*

Let us define a new notion of distortion focused on the propagation of errors by the automorphism on individual configurations.

Definition 3.3 *We say a configuration $x \in X$ is distorted if $\lambda^\pm(x) = 0$. We denote the set of distorted configurations of ϕ by $\text{Dist}(X, \phi)$.*

We say ϕ is Lyapunov distorted if $\text{Dist}(X, \phi) = X$, and denote the set of all Lyapunov distorted automorphisms by $LD(X)$.

We can relate these new functions with the previously discussed Lyapunov exponents associated to ϕ.

Proposition 3.4 *Let (X, σ) be a subshift and $\phi \in \text{Aut}(X)$. Then, for all $x \in X$ we have*

$$\lambda^+(x) \le \alpha^+(\phi) = \lim_{n\to\infty} \sup_{z\in X} \frac{\Lambda_n^+(z)}{n},$$

$$\lambda^-(x) \ge \alpha^-(\phi) = \lim_{n\to\infty} \inf_{z\in X} \frac{\Lambda_n^-(z)}{n}.$$

PROOF. Let n be a natural number. Then, by definition of the quantity $W^+(n,\phi)$ we have that

$$\tilde{\Lambda}_n^+(x) \le W^+(n,\phi),\ \forall x \in X.$$

In particular, it is evident that,

$$\Lambda_n^+(x) \le W^+(n,\phi),\ \forall x \in X.$$

Dividing by n and taking the limit, we obtain $\lambda^+(x) \le \alpha^+(\phi)$.

For the second inequality, we begin by defining

$$S_n := \sup_{x\in X} \Lambda_n^+(x).$$

For any pair of configurations $x, y \in X$, where $x_{[0,+\infty)} = y_{[0,+\infty)}$, we have $\phi^n(x)_{[S_n,+\infty)} = \phi^n(y)_{[S_n,+\infty)}$. As a consequence, $W^+(n,\phi) \le S_n$. Once again, dividing by n and taking limit we conclude. The procedure for α^- is analogous. □

Lemma 3.5 *If X is an infinite subshift, there exists $k(n,x) > 0$ such that, if $x_{[0,k(n,x)]} = y_{[0,k(n,x)]}$ then $\phi^n(x)_{\Lambda_n^+(x)} = \phi^n(y)_{\Lambda_n^+(x)}$, and if $x_{[-k(n,x),0]} = y_{[-k(n,x),0]}$ then $\phi^n(x)_{\Lambda_n^-(x)} = \phi^n(y)_{\Lambda_n^-(x)}$.*

PROOF. Let us prove this lemma by contradiction. This means that for every $k \in \mathbb{N}$ there exists $y^{(k)} \in X$ such that

$$x_{[0,k]} = y^{(k)}_{[0,k]} \text{ and } \phi^n(x)_{\Lambda_n^+(x)} \ne \phi^n(y^{(k)})_{\Lambda_n^+(x)}.$$

Due to compactness, there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ and $y \in X$ such that

$$y^{(k_j)} \to y.$$

We clearly have that $x_{[0,+\infty)} = y_{[0,+\infty)}$ and $\phi^n(x)_{\Lambda_n^+(x)} \ne \phi^n(y)_{\Lambda_n^+(x)}$. This is a contradiction with the fact that if $y \in R_0^+(x)$ then $\phi^n(x)_{[\Lambda_n^+(x),+\infty)} = \phi^n(y)_{[\Lambda_n^+(x),+\infty)}$. The case for $\Lambda_n^-(x)$ is analogous. □

We now introduce an important concept to relate previous notions of distortion to Lyapunov distortion.

Definition 3.6 *Let (X,σ) be a subshift and $\phi \in \mathrm{Aut}(X)$. We say a configuration $x \in X$ is weakly periodic if there are $m, n \in \mathbb{Z}$ such that $\phi^m(x) = \sigma^n(x)$. We say ϕ is aperiodic if it has no weakly periodic configurations.*

Proposition 3.7 *Let ϕ be an aperiodic automorphism. Then, for every $x \in X$, $\lambda^+(x) \ge \lambda^-(x)$.*

Proof. Let us procede by contradiction. If there exists $x \in X$ such that $\lambda^+(x) < \lambda^-(x)$. We begin by defining

$$A = \{\phi^m(\sigma^n x) : m, n \in \mathbb{Z}\},$$

with its associated complexity function

$$P_A(k) = |\{w \in \mathcal{L}_k(X) : \exists m, n \in \mathbb{Z} : \ w \sqsubseteq \phi^m(\sigma^n x)\}|.$$

It easy to see that this is a non-decreasing function.

Now for sufficiently large n, we will have that $\Lambda_n^+(x) < \Lambda_n^-(x)$. Due to Lemma 3.5, we have that there exists $R > 0$ such that

$$x_{[0,R]} = y_{[0,R]} \ \Rightarrow \ \phi^n(x)_{\Lambda_n^+(x)} = \phi^n(y)_{\Lambda_n^+(x)},$$

and there exists $R' > 0$ such that

$$x_{[-R',0]} = y_{[-R',0]} \ \Rightarrow \ \phi^n(x)_{\Lambda_n^-(x)} = \phi^n(y)_{\Lambda_n^-(x)}.$$

Therefore, we have for $t > 0$

$$x_{[0,R+t]} = y_{[0,R+t]} \ \Rightarrow \ \phi^n(x)_{[\Lambda_n^+(x),\ \Lambda_n^+(x)+t-1]} = \phi^n(y)_{[\Lambda_n^+(x),\ \Lambda_n^+(x)+t-1]},$$

$$x_{[0,R'+t]} = y_{[0,R'+t]} \ \Rightarrow \ \phi^n(x)_{[\Lambda_n^-(x)+R',\ \Lambda_n^-(x)+R'+t]} = \phi^n(y)_{[\Lambda_n^-(x)+R',\ \Lambda_n^-(x)+R'+t]}.$$

If we take $t = R' + r$, where $r > \Lambda_n^-(x) - \Lambda_n^+(x) + 1$, by combining the two previous properties, we will have overlapping intervals, arriving at

$$x_{[0,R+R'+r]} = y_{[0,R+R'+r]} \ \Rightarrow \ \phi^n(x)_{[\Lambda_n^+(x),\ \Lambda_n^-(x)+R'+r]} = \phi^n(y)_{[\Lambda_n^+(x),\ \Lambda_n^-(x)+R'+r]},$$

Because, both $\Lambda_n^+(x)$ and $\Lambda_n^-(x)$ are σ invariant, we have that

$$P_A(R + R' + r) \geq P_A(\Lambda_n^-(x) - \Lambda_n^+(x) + R + R' + r).$$

Because of our assumption that $\Lambda_n^+(x) < \Lambda_n^-(x)$ and the fact that P_A is a non-decreasing function we have

$$P_A(R + R' + r) = P_A(\Lambda_n^-(x) - \Lambda_n^+(x) + R + R' + r).$$

This means that any word in A of length $R + R' + r$ can be uniquely extended to the right to a word in A of length $\Lambda_n^-(x) - \Lambda_n^+(x) + R + R' + r$, and therefore can be extended to a ray to the right. By an analogous procedure it can be extended uniquely and indefinitely to the left. Hence, A is finite, which contradicts the aperiodicity of ϕ. □

Theorem 3.8 *Let ϕ be an aperiodic automorphism in $AD(X)$. Then, $\phi \in LD(X)$.*

PROOF. By virtue of Propositions 3.4 and 3.7, it is possible to see that if $\phi \in AD(X)$, then $\text{Dist}(X, \phi) = X$. □

We conjecture that in fact $AD(X) \subseteq LD(X)$. A path to prove this fact is obtained through the following result:

Lemma 3.9 ([Cyr+16], Lemma 4.3) *Let (X, σ) be a subshift and let $\phi \in \text{Aut}(X)$. If there exists $n, m \in \mathbb{Z} \setminus \{0\}$ and an aperiodic $x \in X$ such that $\phi^n(x) = \sigma^m(x)$, then*

$$\text{range}(\phi^{kn}) \geq |m|k, \ \forall k \in \mathbb{N}.$$

In particular, ϕ is not range distorted.

We can rewrite the previous Lemma to say that no range distorted automorphism can be weakly periodic on an aperiodic configuration.

This creates the following dichotomy:

Proposition 3.10 *Let (X, σ) be a subshift and let $\phi \in AD(X)$. Then, for each $x \in X$ one of the following holds:*

- *ϕ is not weakly periodic on x*
- *There exists $m \in \mathbb{N}$ such that $\phi^m(x) = x$.*

By this proposition, to obtain the aforementioned conclusion we would need to prove that for $x \in X$ where there exists $m \in \mathbb{N}$ such that $\phi^m(x) = x$, either $\lambda^+(x) \geq \lambda^-(x)$ or $\lambda^{\pm}(x) = 0$.

Through the following example, we can see that the too classes can never be the same. In other words, we can see that local distortion does not imply global distortion:

Example 3.11 *We begin by denoting the binary expression of a number $n \in \mathbb{N}$ by $[n]_2$, and $|[n]_2| = b_n$. For this example, we use the alphabet $\Sigma = \{0, 1, *\}$. Let $R > 0$. For $n \in \mathbb{N}$ and $k \in \{1, ..., n\}$ we define the following words:*

$$w_n^k = [n]_2 \ 0^{b_n - b_k} \ [k]_2 \ [n]_2$$

We define the configurations:

$$x_n^k = 0^\infty \ w_n^k \ 0^\infty,$$
$$x_n^* = 0^\infty \ * \ 0^{R \cdot n} \ w_n^n \ 0^{R \cdot n} \ * \ 0^\infty$$

With this, we can define the subshift:

$$X = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{k \in \{1, \dots, n\}} \overline{\text{orb}(x_n^k)} \ \cup \ \overline{\text{orb}(x_n^*)} \right).$$

Next, we create the automorphism φ *of* X *through the local rule:*

$$\begin{aligned} 0\ w_n^k\ 0 &\longrightarrow 0\ w_n^{k+1}\ 0, \quad \forall k \in \{1, ..., n-1\}, \\ 0^{R\cdot n+2}\ w_n^n\ 0^{R\cdot n+2} &\longrightarrow 0\ *\ 0^{R\cdot n}\ w_n^n\ 0^{R\cdot n}\ *\ 0, \\ 0\ *\ 0^{R\cdot n}\ w_n^n\ 0^{R\cdot n}\ *\ 0 &\longrightarrow 0^{R\cdot n+2}\ w_n^1\ 0^{R\cdot n+2}, \end{aligned}$$

and acts as the identity on all other words. It is easy to see that $\varphi(x_n^k) = x_n^{k+1}$ *if* $k < n$, $\varphi(x_n^n) = x_n^*$ *and* $\varphi(x_n^*) = x_n^1$.

In addition, we note that

$$\Lambda_m^{\pm}(x_n^0) = \begin{cases} 2b_n + 2 \text{ if } m < n \\ 2b_n + Rn + 2 \text{ if } m \geq n \end{cases},$$

which implies that $\lambda^{\pm}(x_n^0) = 0$. *Analogously, one can see that* $\lambda^{\pm}(x) = 0$ *for all* $x \in X$. *Nevertheless,*

$$\sup_{x\in X} \frac{\Lambda_n^{\pm}(x)}{n} \geq \frac{\Lambda_n^{\pm}(x_n^0)}{n} > R > 0,$$

and as a consequence

$$\alpha^+(\varphi) = \lim_{n\to\infty} \sup_{x\in X} \frac{\Lambda_n^+(x)}{n} \geq R > 0 = \sup_{x\in X} \lambda^+(x).$$

Example 3.12 *Let us see an example of how the exponents work through a family of endomorphisms. We say an endomorphism* φ *with* $\mathrm{range}(\varphi) = R$ *and local function given by* Φ *is left-permutative if for all words* $w \in \mathcal{L}_{2R}(X)$ *the map* $a \in \mathcal{A} \mapsto \Phi(aw)$ *is biyective.*

Let us see that for a left-permutative endomorphism $\mathrm{range}_\infty(\varphi) = R$. *Indeed, let* $x, y \in X$ *be two distinct points such that* $a = x_{-1} \neq y_{-1} = b$. *If we call* $w = x_{[0,2R]} = y_{[0,2R]}$, *because* φ *is left permutative, we will have that* $\varphi(x)_{R-1} = \Phi(aw) \neq \Phi(bw) = \varphi(y)_{R-1}$. *As a consequence,* $R \leq \Lambda_1^+(x)$. *By iterating this process for the points obtained by succesively applying the endomorphism, we obtain that* $Rn \leq \Lambda_n^+(x)$, *and thus* $R \leq \lambda^+(x)$.

Finally, due to the inequality

$$\sup_{x\in X} \lambda^+(x) \leq \alpha^+(\varphi) \leq \mathrm{range}_\infty(\varphi) \leq R,$$

we conclude.

By using this established notation, we can re-write Bressaud and Tisseur's conjecture:

Conjecture 3.13 ([BT06], Conjecture 3) *If an endomorphism has no equicontinuity points, then there exists a point* x *such that* $\lambda^+(x) > 0$ *or* $\lambda^-(x) < 0$.

This conjecture has been shown to be false through a construction by Hochman [Hoc11], creating the following proposition.

Proposition 3.14 ([Hoc11], Prop. 1.4) *There exists an endomorphism f such that, for every r, there is a subshift X_α on which f acts as an automorphism without equicontinuity points and such that $\alpha^+(f) = r$, $\alpha^-(f) = r$.*

In a similar fashion, re-structuring of the definitions involved in the proof of Proposition 2 in [BT06], allow us to state the following.

Proposition 3.15 *Let (X, σ) be a subshift and $\varphi \in \mathrm{End}(X)$ positively expansive. Then, there exists a constant $\Lambda > 0$ such that for all $x \in X$ we have $\lambda^+(x) \geq \Lambda$ and $\lambda^-(x) \leq -\Lambda$.*

As stated before, the consequence of this proposition is that no positively expansive endomorphism can be a distorted automorphism.

3.1.1 Directional Lyapunov exponents

We want to extend the previously obtained results to the notion of directional exponents. As stated earlier, these exponents where first introduced by Courbage and Kaminsky [CK06] to study the propagation of information along distinct directions. We introduce modifications to the original definition in the same spirit as the ones in the previous section.

Let $\phi \in \mathrm{Aut}(X)$, and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. We define

$$\alpha(t) = [at], \;\; \beta(t) = [bt], \;\; t \in \mathbb{N},$$

where $[\cdot]$ is the integer part function.

Based on Proposition 1 and Corollary 2 on [CK06], we define the directional Lyapunov exponents as follows.

Definition 3.16 *For $\phi \in \mathrm{Aut}(X)$ and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$, we define*

$$\tilde{\Lambda}^+_{\vec{v},t}(x) = \inf \left\{ s : \; \sigma^{\alpha(t)}\phi^{\beta(t)} \left(R^+_0(x) \right) \subseteq R^+_s \left(\sigma^{\alpha(t)}\phi^{\beta(t)}(x) \right) \right\},$$

$$\tilde{\Lambda}^-_{\vec{v},t}(x) = \sup \left\{ s : \; \sigma^{\alpha(t)}\phi^{\beta(t)} \left(R^-_0(x) \right) \subseteq R^-_s \left(\sigma^{\alpha(t)}\phi^{\beta(t)}(x) \right) \right\},$$

and

$$\Lambda^+_{\vec{v},t}(x) = \sup_{j \in \mathbb{Z}} \tilde{\Lambda}^+_{\vec{v},t}(\sigma^j x), \quad \Lambda^-_{\vec{v},t}(x) = \inf_{j \in \mathbb{Z}} \tilde{\Lambda}^-_{\vec{v},t}(\sigma^j x).$$

Remark 3.17 *It is direct from the definition that*

$$\Lambda^+_{\vec{v},t}(x) \leq \beta(t) \cdot \mathrm{range}(\phi) - \alpha(t),$$

$$\Lambda^-_{\vec{v},t}(x) \geq -\beta(t) \cdot \mathrm{range}(\phi) - \alpha(t).$$

Definition 3.18 *We define the left and right directional Lyapunov exponents for ϕ at $x \in X$ by*

$$\lambda^{\pm}_{\vec{v}}(x) = \liminf_{t\to\infty} \frac{\Lambda^{\pm}_{\vec{v},t}(x)}{t}.$$

We note that Remark 3.17 gives the following

$$\lambda^{+}_{\vec{v}}(x) \leq b \cdot \mathrm{range}(\phi) - a,$$

$$\lambda^{-}_{\vec{v}}(x) \geq -a - b \cdot \mathrm{range}(\phi).$$

These Lyapunov exponents behave properly under continuous changes of the chosen direction, as evidenced by the following results.

Lemma 3.19 ([CK06], Lemma 2) *The function $\vec{v} \to \lambda^{\pm}_{\vec{v}}$ is positively homogeneous, that is, for every $c \in \mathbb{R}^+$,*

$$\lambda^{\pm}_{c\vec{v}} = c\lambda^{\pm}_{\vec{v}}$$

Proposition 3.20 ([CK06], Prop. 3) *The directional Lyapunov exponents are continuous as a function of $\vec{v}$. Specifically, $\lim_{n\to\infty} \lambda^{\pm}_{\vec{v}_n}(x) = \lambda^{\pm}_{\vec{v}}(x)$, for $\vec{v}_n \to \vec{v}$.*

Inspired by Proposition 3.4, we will introduce new Lyapunov exponent that will play the role that $\alpha^{\pm}$ plays for $\lambda^{\pm}$.

Definition 3.21 *Let $\phi \in \mathrm{Aut}(X)$ and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$, we define*

$$W^{+}_{\vec{v}}(n, \phi) := \sup_{x\in X} \Lambda^{+}_{\vec{v},t}(x),$$

$$W^{-}_{\vec{v}}(n, \phi) := \inf_{x\in X} \Lambda^{-}_{\vec{v},t}(x).$$

We define the directional α-exponents as

$$\alpha^{\pm}_{\vec{v}}(\phi) = \lim_{n\to\infty} \frac{W^{\pm}_{\vec{v}}(n, \phi)}{n}.$$

Remark 3.22 *It is evident that the following inequalities hold: $\lambda^{+}_{\vec{v}}(x) \leq \alpha^{+}_{\vec{v}}(\phi)$ and $\lambda^{-}_{\vec{v}}(x) \geq \alpha^{-}_{\vec{v}}(\phi)$ for all $x \in X$. We can also see that if $\vec{v} = (0, 1)$, we obtain the previously defined exponents.*

As in Lemma 3.19, it follows directly from the definition that the new exponent is positively homogeneous.

Lemma 3.23 *For $c \in \mathbb{R}^+$, $\alpha^{\pm}_{c\vec{v}}(\phi) = c\alpha^{\pm}_{\vec{v}}(\phi)$.*

In fact, the function mapping $\vec{v} \to \alpha^{\pm}_{\vec{v}}(\phi)$ is also continuous.

Proposition 3.24 *The exponents $\alpha^{\pm}_{\vec{v}}(\phi)$ are continuous as a function of $\vec{v}$. Specifically, $\lim_{n\to\infty} \alpha^{\pm}_{\vec{v}_n}(\phi) = \alpha^{\pm}_{\vec{v}}(\phi)$, for $\vec{v}_n \to \vec{v}$.*

Proof. We will give a proof analogous to the one provided in [CK06] for the continuity of the directional exponents.

We begin by showing that for $\vec{v} = (a, b)$ and $\vec{w} = (a', b')$,

$$b\alpha^{+}_{\vec{w}}(\phi) \leq b'\alpha^{+}_{\vec{v}}(\phi) - (a'b - ab').$$

Due to homogeneity, it suffices to study the case where $b = b' = 1$. Here, we will have $\beta(t) = \beta'(t) = t$. We must also define $d(t) := \alpha'(t) - \alpha(t)$.

$$\begin{aligned}
\sigma^{\alpha'(t)}\phi^{\beta'(t)}\left(R^{+}_0(x)\right) &= \sigma^{\alpha(t)+d(t)}\phi^{\beta(t)}\left(R^{+}_0(x)\right) \\
&= \sigma^{d(t)}\sigma^{\alpha(t)}\phi^{\beta(t)}\left(R^{+}_0(x)\right) \\
&\subseteq R^{+}_{\Lambda^{+}_{\vec{v},t}(x)-d(t)}\left(\sigma^{\alpha'(t)}\phi^{\beta'(t)}(x)\right).
\end{aligned}$$

This means that,

$$\Lambda^{+}_{\vec{w},t}(x) \leq \Lambda^{+}_{\vec{v},t}(x) - d(t).$$

By taking supremum over X,

$$W^{+}_{\vec{w}}(t, \phi) \leq W^{+}_{\vec{v}}(t, \phi) - d(t),$$

and therefore,

$$\alpha^{+}_{\vec{w}}(\phi) \leq \alpha^{+}_{\vec{v}}(\phi) - (a' - a).$$

Now, let $\vec{v} = (a, b), b > 0$ be fixed and $\vec{v}_n = (a_n, b_n)$ such that $\vec{v}_n \to v$. Using the previous inequality we obtain,

$$\limsup_{n\to\infty} \alpha^{+}_{\vec{v}_n}(\phi) \leq \alpha^{+}_{\vec{v}}(\phi),$$

$$\alpha^{+}_{\vec{v}}(\phi) \leq \liminf_{n\to\infty} \alpha^{+}_{\vec{v}_n}(\phi),$$

and thus, $\lim_{n\to\infty} \alpha^{+}_{\vec{v}_n}(\phi) = \alpha^{+}_{\vec{v}}(\phi)$. The case for $\alpha^{-}_{\vec{v}}(\phi)$ is analogous. □

3.2 Directional exponents and geometry

By using directional exponents as introduced in the previous section, we can observe the asymptotic behaviour of an automorphism on its directions of non-expansiveness. We shall see this through the following results:

Theorem 3.25 *Let $\phi \in \text{Aut}(X)$ and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. Then, $\alpha^{\pm}_{\vec{v}}(\phi) = a + b \cdot \alpha^{\pm}(\phi)$.*

Proof. We begin by noting that if $\vec{v} = (p, q) \in \mathbb{Z} \times \mathbb{N}$, then $\alpha^{\pm}_{\vec{v}}(\phi) = \alpha^{\pm}(\sigma^p \phi^q)$. By vitue of Proposition 2.9, we then have that, $\alpha^{\pm}_{\vec{v}}(\phi) = p + q\alpha^{\pm}(\phi)$.

Let now $\vec{v} = \left(\frac{a}{b}, \frac{c}{d}\right)$, where $a \in \mathbb{Z}$ and $b, c, d \in \mathbb{N}$. Due to Lemma 3.23,

$$\alpha^{\pm}_{bd\vec{v}}(\phi) = bd \cdot \alpha^{\pm}_{\vec{v}}(\phi).$$

Also,

$$\alpha^{\pm}_{bd\vec{v}}(\phi) = \alpha^{\pm}(\sigma^{ad}\phi^{bc}) = ad + bc \cdot \alpha^{\pm}(\phi).$$

Joining both statements, we arrive at

$$\alpha^{\pm}_{\vec{v}}(\phi) = \frac{a}{b} + \frac{c}{d} \cdot \alpha^{\pm}(\phi).$$

Finally, let $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$ and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$, $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}^+$ such that $a_n \to a$ and $b_n \to b$. If we denote $\vec{v}_n = (a_n, b_n)$ we now have that,

$$\alpha^{\pm}_{\vec{v}_n}(\phi) = a_n + b_n \cdot \alpha^{\pm}(\phi), \ \forall n \in \mathbb{N}.$$

By taking limit at both sides, we have

$$\alpha^{+}_{\vec{v}}(\phi) = \lim_{n\to\infty} \alpha^{\pm}_{\vec{v}_n}(\phi) = a + b \cdot \alpha^{\pm}(\phi),$$

where the first equality is due to Proposition 3.24. □

Corollary 3.26 *Let $\phi \in \text{Aut}(X)$ and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. If either $\alpha^{+}_{\vec{v}}(\phi) = 0$ or $\alpha^{-}_{\vec{v}}(\phi) = 0$, then the line defined by $ay + bx = 0$ is non-expansive.*

Proof. Let us assume without loss of generality, that $\alpha^{+}_{\vec{v}}(\phi) = 0$. By Theorem 3.25, we have that $0 = a + b \cdot \alpha^{+}(\phi)$. Therefore, we have that $\alpha^{+}(\phi) = -\frac{a}{b}$.

By Theorem 2.29, the line $x = -\frac{a}{b}y$ is a non-expansive direction for ϕ. □

Corollary 3.27 *Let $\phi \in \text{Aut}(X)$ and $\vec{v} = (a, b) \in \mathbb{R} \times \mathbb{R}^+$. The line L defined by the equation $ay + bx = 0$ is the only non-expansive direction for ϕ if and only if $\alpha^{\pm}_{\vec{v}}(\phi) = \alpha^{\pm}_{\vec{v}}(\phi^{-1}) = 0$.*

Chapter 4

Entropy and distortion

One of the fundamental aspects to explore when working with dynamical systems is entropy. With this lens, a fundamental property of distorted automorphisms is to have zero entropy. This fact is derived as a direct consequence from the following result from [CFK19], which relates the asymptotic range of an automorphism to its entropy.

Theorem 4.1 ([CFK19], Teorema 5.13) *Let* (X, σ) *be a subshift and* $\phi \in \mathrm{Aut}(X)$*. Then,*

$$h_{\mathrm{top}}(\phi) \leq 2 \ \mathrm{range}_{\infty}(\phi) h_{\mathrm{top}}(\sigma).$$

Nevertheless, this does not characterize distorted automorphisms, as shown in the following expample due to Schmieding [Sch19]:

Example 4.2 *Let* $X = \{0, 1, 2\}^{\mathbb{Z}}$ *be the fullshift on 3 symbols. Let* ϕ_1 *be the marker automorphism that permutes 000111 with 002111, and* ϕ_2 *the marker automorphism that permutes 000111 with 002111 (marker automorphisms are presented in great detail in [BLR88]). If we define* $\phi = \phi_2 \circ \phi_1$*, we have that* $h_{\mathrm{top}}(\phi) = 0$*, but* ϕ *is not distorted.*

In this chapter, using the definition of directional entropy introduced by Milnor and expanded upon by Boyle and Lind [BL97], we will generalize the zero entropy result for any expansive system where $\phi, \phi^{-1} \in AD(X, T)$.

Then, from the entropy inequalities obtained by Courbage and Kaminsky, we can further generalize the zero entropy result to Lyapunov distorted automorphisms on shift, and obtain a new upper bound for its directional entropy.

Finally, we explore how distortion can restrict the types of subgroups that can embed into the automorphism groups and which embedding questions still remain unanswered.

4.1 Directional entropy

Let α be a $\mathbb{Z}^2$-action over a compact metric space (X, ρ). Also, let $E \subseteq \mathbb{R}^2$ be a compact subset and $\delta > 0$. Let us define the quantity $N_\alpha(E, \delta)$ as the cardinality of the smallest finite subset $F \subseteq X$ such that for all $x \in X$ there is a $y \in F$ that satisfies $\rho_\alpha^E(x, y) < \delta$.

Definition 4.3 *Let $E \subseteq \mathbb{R}^2$ be a compact subset and $\delta > 0$. We define*

$$\eta_1(E, \delta) = \sup_{t>0} \limsup_{s\to\infty} \frac{N_\alpha((sE)^t, \delta)}{s^2}.$$

With this, the one-dimensional topological entropy of E with respect to α is defined as

$$\eta_1(E) := \lim_{\delta\to 0} \eta_1(E, \delta).$$

For $\vec{v} \in \mathbb{R}^2$, we define the direction topological entropy of α on direction $\vec{v}$ as

$$h_{\vec{v}}(\alpha) = \eta_1([0, \vec{v}]).$$

Remark 4.4 *Due to the definition, it is evident that for a system (X, T), $\phi \in \mathrm{Aut}(X, T)$ and $\vec{v} = (n, m) \in \mathbb{Z}^2$, the directional entropy satisfies:*

$$h_{\vec{v}}(\alpha) = h_{\text{top}}(T^n \phi^m).$$

where α is the $\mathbb{Z}^2$ action defined by T and ϕ. (For the directional entropy on $\mathbb{Z}^d$-systems we refer the reader to [BL97]).

In particular, Boyle and Lind give a complete description of the directional entropy for actions with a unique non-expansive direction.

Proposition 4.5 ([BL97], Prop. 6.27) *Let α be a $\mathbb{Z}^2$-action with a unique non-expansive direction given by L. Then, there exists a linear function $f : \mathbb{R}^2 \to \mathbb{R}$ whose kernel contains L and is such that $h_{\vec{v}}(\alpha) = |f(\vec{v})|$ for all $\vec{v} \in \mathbb{R}^2$.*

With this result we are able to generalize Proposition 4.1.

Proposition 4.6 *Let (X, T) be an expansive system and an automorphism ϕ such that $\phi, \phi^{-1} \in AD(X, T)$. Then, $h_{\text{top}}(\phi) = 0$.*

Proof. We will use the $\mathbb{Z}^2$-system (X, T, ϕ) and denote the action by α. By Corollary 2.31, the system has a unique non-expansive direction given by $L_0 = \mathbb{R} \cdot \vec{e}_2$. Hence, by Proposition 4.5, there exists a linear function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $h_{\vec{v}}(\alpha) = |f(\vec{v})|$ for all $\vec{v} \in \mathbb{R}^2$, and in addition $f(L_0) = 0$. Particularly, we have that

$$h_{\text{top}}(\phi) = h_{\vec{e}_2}(\alpha) = 0.$$

□

4.1.1 Entropy and Lyapunov Exponents

There is an analogous definition of directional entropy for the metric case. This definition is also due to Milnor, but we will present a slight modification introduced in [CK15a].

Let $\phi \in \mathrm{Aut}(X,T)$, and let us call Ψ the the $\mathbb{Z}^2$-action defined by ϕ and T. For every $u=(n,m)\in\mathbb{Z}^2$, we have $\Psi^u = T^n\phi^m$. If μ is a ϕ and σ invariant probability measure and $\mathcal{P}$ is the family of all finite measurable partitions of X, for any $P\in\mathcal{P}$ and bounded set $A\subseteq\mathbb{R}^2$ we put

$$P(A) = \bigvee_{u\in A\cap\mathbb{Z}^2} \Psi^u(P).$$

For any $P\in\mathcal{P}$ we denote the entropy of P with respect to μ as $H_\mu(P)$. For a vector $\vec{v}\in\mathbb{R}^2$

$$h^\mu_{\vec{v}}(\phi,P) = \sup_A \limsup_{t\to\infty} \frac{1}{t} H_\mu(P(A+[0,t)\vec{v})),$$

where the supremum is taken over all bounded A.

The directional entropy of ϕ in the direction $\vec{v}$ is defined by,

$$h^\mu_{\vec{v}}(\phi) = \sup_{P\in\mathcal{P}} h^\mu_{\vec{v}}(\phi,P).$$

This definition gives us a generalization of the classic (Kolmogorov-Sinai) metric entropy in the sense that for $\vec{v}=(n,m)\in\mathbb{Z}^2$ we have $h^\mu_{\vec{v}}(\phi) = h_\mu(T^n\phi^m)$.

As was the case with $\lambda^\pm_{\vec{v}}$ and $\alpha^\pm_{\vec{v}}$, the function mapping $\vec{v}\mapsto h^\mu_{\vec{v}}(\phi)$ is positively homogenous and continuous (this last property is what was known as the Milnor Problem, and was solved by K.K. Park [Par99]).

We have the following connection between directional Lyapunov exponents and directional entropy.

Theorem 4.7 ([CK06], Theorem 1) *For any $\vec{v}=(a,b)\in\mathbb{R}\times\mathbb{R}^+$ and any Ψ invariant measure μ we have*

$$h^\mu_{\vec{v}}(\phi) \le \int_X h_\mu(\sigma,x)\left(\max\{\lambda^+_{\vec{v}}(x),0\} + \max\{-\lambda^-_{\vec{v}}(x),0\}\right)d\mu,$$

where $h_\mu(\sigma,x)$ is the local entropy of σ at the point x.

For a brief overview of the concept of local entropy, let (X,ρ) be a compact metric space, μ a probability measure in X and $T: X\to X$ a continuous μ-invariant map.

For $x\in X$ and $\varepsilon>0$ we define,

$$B_n(T,x,\varepsilon) = \{y\in X : \rho(T^kx,T^ky)<\varepsilon,\ \text{for } 0\le k\le n\}.$$

The local entropy of T at x (with respect to μ) is given by

$$h_\mu(T,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(B_n(T,x,\varepsilon)\right).$$

Brin and Katok [BK83] showed that,

$$h_\mu(T) = \int_X h_\mu(T,x)d\mu.$$

As a consequence of this Theorem we extend the zero entropy property to Lyapunov distorted automorphisms.

Corollary 4.8 *Let ϕ be in $LD(X)$. Then, $h_{\text{top}}(\phi) = 0$.*

Proof. We obtain the result by taking $\vec{v} = (0,1)$ and applying the Variational Principle. □

By adding the newly defined directional Lyapunov exponents, we are able to obtain a new inequality for the directional entropy.

Theorem 4.9 *For any $\vec{v} = (a,b) \in \mathbb{R}\times\mathbb{R}^+$ and any Ψ invariant measure μ we have*

$$h^\mu_{\vec{v}}(\phi) \le (\max\{a + b\cdot\alpha^+(\phi), 0\} - \min\{a + b\cdot\alpha^-(\phi), 0\})h_\mu(\sigma).$$

Proof. By Theorem 4.7, we have that

$$h^\mu_{\vec{v}}(\phi) \le \int_X h_\mu(\sigma,x)\left(\max\{\lambda^+_{\vec{v}}(x), 0\} + \max\{-\lambda^-_{\vec{v}}(x), 0\}\right) d\mu.$$

Because, $\lambda^+_{\vec{v}}(x) \le \alpha^+_{\vec{v}}(\phi)$ and $\lambda^-_{\vec{v}}(x) \ge \alpha^-_{\vec{v}}(\phi)$ for all $x \in X$, using Brin and Katok's formula we get

$$\begin{aligned} h^\mu_{\vec{v}}(\phi) &\le \left(\max\{\alpha^+_{\vec{v}}(\phi), 0\} + \max\{-\alpha^-_{\vec{v}}(\phi), 0\}\right) \int_X h_\mu(\sigma,x)d\mu \\ &= \left(\max\{\alpha^+_{\vec{v}}(\phi), 0\} + \max\{-\alpha^-_{\vec{v}}(\phi), 0\}\right) h_\mu(\sigma). \end{aligned}$$

We conclude by applying Theorem 3.25 to the inequality. □

Corollary 4.10 *Let ϕ be in $AD(X)$. Then, $h_{\text{top}}(\phi) = 0$.*

4.2 Embedding restrictions

The existence of exponentially distorted automorphisms restricts the values that can take the entropy of the subshift.

Theorem 4.11 ([Cyr+16], Theorem 3.8) *Let* (X, σ) *be a subshift and* $\phi \in \mathrm{Aut}(X)$ *of infinite order such that* $\mathrm{range}(\phi^m) \leq \mathcal{R}\log(m)$ *for some* $\mathcal{R} > 0$*. Then,*

$$h_{\mathrm{top}}(\sigma) \geq \frac{1}{2\mathcal{R}}.$$

As a consequence of this theorem, groups with exponentially distorted elements cannot embed into the automorphism group of a zero entropy subshift.

Corollary 4.12 ([Cyr+16], Corolario 3.10) *Let* (X, σ) *be a zero entropy subshift. Let* G *be a group with an exponentially distorted element* g*. Then, if* $\Phi : G \to \mathrm{Aut}(X)$ *is a homomorphism,* $\Phi(g)$ *has finite orden in* $\mathrm{Aut}(X)$*. Moreover, if* G *is almost simple, then* $\Phi(G)$ *is a finite group.*

The best conclusion that can be derived from this result is that groups with exponentially distorted elements, such as $\mathrm{BS}(1, n)$ and $\mathrm{SL}(k, \mathbb{Z})$, are not allowed to appear as subgroups of $\mathrm{Aut}(X)$ when X has zero entropy.

Although the previous mentioned groups have been ruled out, there remain big open questions about which groups can be embedded:

Question 4.13 *Does the Heisenberg group* $\mathcal{H}$ *embed into the automorphisms group of a shift? More generally, does it contain a distorted element of infinite order?*

Question 4.14 *Does a group with exponentially distorted elements embed into the automorphisms group of a subshift with positive entropy?*

Chapter 5

Unique non-expansive direction

To obtain a unique non-expansive direction of irracional slope, we study the structure of automorphisms with unique non-expansive directions. First, we give a characterization of automorphism with a unique non-expansive direction of rational slope, showing that it can be decomposed into a distorted automorphism and a power of the shift.

Next, we see that restrictions on the structure of the quotient $\mathrm{Aut}(X,T)/\langle T\rangle$ force automorphisms to have a unique non-expansive rational direction. In the case of subshifts, due to results that restrict the structure of the aforementioned quotient from the complexity of subshifts ([Don+16], [CK15b], [CK16]), we can show that some minimal subshifts of low complexity, do not allow for unique directions of irrational slope. A similat result can be also obtained for general expansive systems when looking at their number of asymptotic components.

Finally, for the case of SFTs, we study when automorphisms with unique non-expansive rational directions are conjugated to an SFT, showing that this is always the case under certain conditions of irreducibility.

For simplicity's sake, we will define

$$\mathfrak{A}(X,T) = \{\phi \in \mathrm{Aut}(X,T) : \phi, \phi^{-1} \in AD(X,T)\}.$$

5.1 Rational directions

Lemma 5.1 *Let (X,T) be an expansive dynamical system and $\phi \in \mathrm{Aut}(X,T)$ with a unique non-expansive direction given by $x = my$. Then, ϕ^{-1} has a unique non-expansive direction given by $x = -my$.*

Proof. As seen on the proof of 2.30, the system (X,T,ϕ^{-1}) is a rotation around the x-axis of the $\mathbb{Z}^2$-sistem (X,T,ϕ). In addition, due to Theorems 2.29 and 2.30, It suffices to show that $\alpha^+(\phi^{-1}) = \alpha^-(\phi^{-1}) = -m$.

By Proposition 2.9, $\alpha^+(\phi^{-1}) \geq -m \geq \alpha^-(\phi^{-1})$. Si $\alpha^+(\phi^{-1}) > -m$, and by Theorem 2.29, the line $x = \alpha^+(\phi^{-1})y$ is a non-expansive direction for ϕ^{-1}. Due to the relation between the systems, the line $x = -\alpha^+(\phi^{-1})y$ is a non-expansive direction for ϕ, which contradicts uniqueness. The case for $-m > \alpha^-(\phi^{-1})$ is analogous. □

We would like to characterize automorphisms of the form $f \circ T^\ell$ where f is distorted and $\ell \in \mathbb{N}$. This can be achieved assuming that not only f is distorted, but its inverse also is.

Lemma 5.2 *Let (X, T) be an expansive dynamical system, $\phi \in \mathrm{Aut}(X, T)$ and $\ell \in \mathbb{N}$. Then, ϕ has a unique non-expansive direction given by the line $x = \ell y$ if and only if $\phi = f \circ T^\ell$ with $f \in \mathfrak{A}(X, T)$.*

Proof. ($\Leftarrow$) Let $f \in \mathfrak{A}(X, T)$ and $\phi = f \circ T^\ell$. By Proposition 2.9, $\alpha^\pm(\phi) = \ell$ and $\alpha^\pm(\phi^{-1}) = -\ell$. Due to Theorems 2.29 and 2.30, $x = \ell y$ is the unique non-expansive direction of ϕ.

($\Rightarrow$) Let $\phi \in \mathrm{Aut}(X, T)$ with a unique non-expansive direction given by $x = \ell y$. By using Theorems 2.29 and 2.30, in addition lo Lemma 5.1,we arrive at $\alpha^\pm(\phi) = \ell$ and $\alpha^\pm(\phi^{-1}) = -\ell$. By defining $\psi = \phi \circ T^{-\ell}$, once again by Proposition 2.9, $\alpha^\pm(\psi) = 0 = \alpha^\pm(\psi^{-1})$.

Therefore, $x = 0$ is the unique non-expansive direction of ψ. By Corolary 2.31, $\psi \in \mathfrak{A}(X, T)$. □

The previous Lemma inspires us to propose a characterization of all automorphisms with a unique non-expansive direction of rational slope.

Theorem 5.3 *Let (X, T) be an expansive dynamical system, $\phi \in \mathrm{Aut}(X, T)$, $p \in \mathbb{N}$ and $q \in \mathbb{N}^*$. Then, ϕ has a unique non-expansive direction given by the line $x = \frac{p}{q} y$ if and only if $\phi^q = f \circ T^p$ where $f \in \mathfrak{A}(X, T)$.*

Proof. ($\Rightarrow$) By Prop. 2.9, $\alpha^\pm(\phi^q) = q\alpha^\pm(\phi) = p$. Using Lemma 5.1 on ϕ, we obtain $\alpha^\pm(\phi^{-q}) = -p$. As we have already seen, this implies that ϕ^q has the line $x = py$ as its unique non-expansive direction. Furthermore, due to Lemma 5.2, there exists $f \in \mathfrak{A}(X, T)$ such that $\phi^q = f \circ T^p$.

($\Leftarrow$) Once again, it is possible to see that $\alpha^\pm(\phi^q) = p$ and $\alpha^\pm(\phi^q) = q\alpha^\pm(\phi)$, and therefore $\alpha^\pm(\phi) = \frac{p}{q}$. Analogously, we obtain $\alpha^\pm(\phi^{-1}) = -\frac{p}{q}$. Using Theorems 2.29 and 2.30 we conclude that ϕ has a unique non-expansive direction given by $x = \frac{p}{q} y$. □

Having an automorphism with a unique non-expansive directions guarantees that there are countable automorphisms within the group, which have a unique non-expansive direction, different to the original.

Proposition 5.4 *Let (X,T) be an expansive dynamical system and $\phi \in \mathrm{Aut}(X,T)$ have a unique non-expansive direction given by $x = \alpha y$. Then, $\forall m \in \mathbb{N}$, $\forall n \in \mathbb{Z}$, there exists an automorphism $\psi \in \mathrm{Aut}(X,T)$ with a unique non-expansive direction given by $x = (m\alpha+n)y$.*

Proof. Le us see that the automorphism defined by $\psi := \phi^m \circ \sigma^n$ satisfies what we are looking for. By Prop. 2.9, $\alpha^{\pm}(\phi^m) = m\alpha^{\pm}(\phi)$ and $\alpha^{\pm}(\phi^m \circ \sigma^n) = \alpha^{\pm}(\phi^m) + n$. We conclude by Theorems 2.30 and 2.29. □

5.1.1 Establishing uniqueness

To conclude this section, we present two results that allow us to determine when a direction is indeed the unique non-expansive direction of an automorphism.

Lemma 5.5 *Let (X,σ_1,σ_2) be an infinite $\mathbb{Z}^2$-subshift with metric ρ. Let L be a line such that there exists an $\varepsilon > 0$ and the following holds: for any $x, x' \in X$ such that $x_0 \neq x'_0$ the sets*

$$M_+ = \left\{(n,m) \in \mathbb{Z}^2 : d((n,m),L_+) < \epsilon,\ \rho(x,\sigma_1^n\sigma_2^m x) < \frac{1}{2},\ \rho(x',\sigma_1^n\sigma_2^m x') < \frac{1}{2}\right\},$$

$$M_- = \left\{(n,m) \in \mathbb{Z}^2 : d((n,m),L_-) < \epsilon,\ \rho(x,\sigma_1^n\sigma_2^m x) < \frac{1}{2},\ \rho(x',\sigma_1^n\sigma_2^m x') < \frac{1}{2}\right\},$$

are infinite, where L_+ and L_- are half lines such that $L = L_+ \cup L_-$. Then (X,σ_1,σ_2) has a unique non-expansive direction given by L.

Proof. Let $y = mx$ be the equation that defines L. Let us suppose that there is another non-expansive direction L' given by the equation $y = kx$. Without loss of generality we will asume that $k > m$. We have that there exists $x, x' \in X$ such that

$$x_v = x'_v \ \forall v \in \mathbb{Z}^2 \ \text{ such that } v_2 < kv_1,$$

but there exists $u \in \mathbb{Z}^2$ such that $u_2 > ku_1$ and $x_u \neq x'_u$. Now, because M_+ and M_- are infinite, we can find $r = (r_1,r_2) \in M_+ \cup M_-$ such that $(u_2+r_2) < k \cdot (u_1+r_1)$. Then,

$$x_{u+r} = x_u \neq x'_u = x'_{u+r},$$

which is a contradiction. Finally, from [BL97] we know that, when the domain is infinite, the set of non-expansive directions is always a non-empty closed set, meaning that L is the unique non-expansive direction. □

5.2 Unique non-expansive directions on shifts of low complexity

Because of the structure of automorphisms with a unique non-expansive rational direction, having a simple quotient $\mathrm{Aut}(X,T)/\langle T\rangle$ implies non-expansive directions must all be rational. As many results have established, this happens in particular in shifts of low-complexity.

Theorem 5.6 *Let (X, T) be an expansive system such that* $\mathrm{Aut}(X, T)$ *satisfies one of the following:*

1. *is virtually* $\mathbb{Z}$,

2. $\mathrm{Aut}(X, T)/\langle T\rangle$ *is periodic.*

Then, every automorphism has a unique non-expansive direction of rational slope and

$$GD(X, T) = \mathfrak{D}(X, T) = RD(X, T) = AD(X, T).$$

Proof. Let us have $\phi \in \mathrm{Aut}(X, T)$ different from the identity. If $\mathrm{Aut}(X, T)$ satisfies either (1) or (2), there exists $k \in \mathbb{Z}$ such that $\phi^k \in \langle T\rangle$. If $\phi^k = T^m$, by Theorem 5.3, ϕ has a unique non-expansive direction given by $x = \frac{m}{k}y$. It is clear that ϕ is distorted if and only if $m = 0$ which happens if and only if $\alpha^{\pm}(\phi) = 0$. □

Let us apply this result to the specific case of subshifts of low complexity.

Corollary 5.7 *Let (X, σ) be a subshift that satisfies one of the following:*

- *(X, σ) is transitive and*
$$0 < \limsup_{n\to\infty} \frac{P_X(n)}{n} < \infty,$$

- *(X, σ) is transitive and*
$$\liminf_{n\to\infty} \frac{P_X(n)}{n^2} < \infty,$$

- *(X, σ) is minimal and*
$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < k.$$

Then, every automorphism has a unique non-expansive direction of rational slope and

$$GD(X) = \mathfrak{D}(X) = RD(X) = AD(X).$$

The result stems from the fact that under their respective hypothesis, their automorphism groups are virtually $\mathbb{Z}$ or $\mathrm{Aut}(X)/\langle\sigma\rangle$ is periodic. The proofs of these facts can be found in [Don+16], [CK15b] and [CK16].

This result has an impact on the search for automorphisms with irrational directions, by restricting the word complexity their domains must have:

Theorem 5.8 *Let (X, σ) be a subshift and $\phi \in \mathrm{Aut}(X)$ with a unique non-expansive direction of irrational slope. Then,*

- *if X is minimal, its complexity function is super-linear,*
- *if X is transitive, its complexity function is super-quadratic.*

5.2.1 Restrictions on expansive systems

We can obtain similar results for expansive systems by looking at their asymptotic components structure. By using results obtained by Donoso, Durand, Maass and Petite in [Don+16] we can state the following result:

Theorem 5.9 *Let (X,T) be an expansive system. Suppose that there is a point $x_0 \in X$ such that $\omega(x_0) = X$ that is asymptotic to a different point. If (X,T) has a finite number of asymptotic components, then every automorphism has a unique non-expansive direction of rational slope and*

$$GD(X,T) = \mathfrak{D}(X,T) = RD(X,T) = AD(X,T).$$

Let us start by looking at an arbitrary dynamical system (X,T). We say two points $x, y \in X$ are asymptotic if

$$\lim_{n\to\infty} \rho(T^n x, T^n y) = 0.$$

We say that $\text{orb}(x)$ and $\text{orb}(y)$ are asymptotic if there exists $x' \in \text{orb}(x)$ and $y' \in \text{orb}(y)$ such that x' and y' are asymptotic. We can define an equivalence relation on the orbits from this relationship, which we will denote by $\text{orb}(x)\ \mathcal{AS}\ \text{orb}(y)$.

When an equivalence class of the $\mathcal{AS}$ relation is not a single element, we call it an asymptotic component. We also denote the set of al asymptotic components by $\mathcal{AS}$.

It is possible to see that asymptoticity is preserved through automorphisms, that is, if we have $\phi \in \text{Aut}(X,T)$ and an asymptotic pair $x, y \in X$, then $\phi(x)$ and $\phi(y)$ are asymptotic. This fact is also lifted to orbits. This way, $\phi \in \text{Aut}(X,T)$ induces a permutation $j(\phi)$ on $\mathcal{AS}$. The following group morphism is well defined:

$$\begin{aligned} j : \text{Aut}(X,T) &\to \text{Per}(\mathcal{AS}) \\ \phi &\mapsto j(\phi), \end{aligned}$$

where $\text{Per}(\mathcal{AS})$ denotes the set of permutations of $\mathcal{AS}$.

Proposition 5.10 ([Don+16], Corollary 3.3) *Let (X,T) be a dynamical system. Suppose that there is a point $x_0 \in X$ such that $\omega(x_0) = X$ that is asymptotic to a different point. We have the following exact sequence:*

$$\{1\} \longrightarrow \langle T \rangle \xrightarrow{\text{id}} \text{Aut}(X,T) \xrightarrow{j} \text{Per}(\mathcal{AS})$$

When (X,T) has a finite number of asymptotic components, through the previous Proposition, we can state:

Proposition 5.11 *Let (X,T) be an expansive system. Suppose that there is a point $x_0 \in X$ such that $\omega(x_0) = X$ that is asymptotic to a different point. If (X,T) has a finite number of asymptotic components, then the quotient $\mathrm{Aut}(X,T)/\langle T\rangle$ is finite.*

The proof of Theorem 5.9 is directly obtained from this Proposition and Theorem 5.6.

5.3 Conjugation to SFT's

For this section, we will asume that $X = X_A$ is an SFT given by matrix A. The question we want to answer is, when does is automorphism of the form $f \circ \sigma^\ell$, with f being distorted, conjugated to an SFT? For this purpose, we begin by showing previous results:

Lemma 5.12 ([BK87], Lemma 2.6) *If $\varphi \in \mathrm{Aut}(X_A)$ and there exists $k \in \mathbb{N}$ such that φ^k is conjugated to an SFT, then φ is conjugated to an SFT.*

Theorem 5.13 ([BK87], Teorema 2.17) *Let (X_A, σ) be an irreducible SFT, where $\det(I - tA)$ is an irreducible polinomial and $\varphi \in \mathrm{Aut}(X_A)$ with $M \in \mathbb{N}$ such that $\mathrm{range}(\varphi), \mathrm{range}(\varphi^{-1}) < M$. Then, for all $m \geq 2M+1$, $\varphi \circ \sigma^m$ is conjugated to an SFT and*

$$h_{\mathrm{top}}(\varphi \circ \sigma^m) = \log(\lambda_\varphi) + m h_{\mathrm{top}}(\sigma).$$

Theorem 5.14 ([CFK19], Teorema 5.15) *Let (X_A, σ) be an irreducible SFT, where $\det(I - tA)$ is an irreducle polinomial. Then, $\mathfrak{D}(X_A) \subseteq \mathrm{Inert}(X_A)$.*

With these elements in place, we can see when an automorphism with a unique non-expansive direction of rational slope is conjugated to an SFT.

Theorem 5.15 *Let (X_A, σ) be an irreducible SFT, where $\det(I - tA)$ is an irreducible polinomial and $\varphi \in \mathrm{Aut}(X_A)$ with a unique non-expansive direction given by $x = \dfrac{p}{q} y$. Then, φ is conjugated to a SFT and*

$$h_{\mathrm{top}}(\varphi) = \frac{p}{q}\log(\lambda_A).$$

Proof. By Theorems 5.3 and 2.14, we know that $f := \varphi^q \circ \sigma^{-p} \in \mathfrak{D}(X_A) = \mathfrak{A}(X_A)$. Due to this, for all sufficiently large k ,

$$2 \cdot \mathrm{range}(f^k) + 2 \leq kp \ \text{ and } \ 2 \cdot \mathrm{range}(f^{-k}) + 2 \leq kp$$

Then, by Theorem 5.13, φ^{kq} is conjugated to an SFT, and

$$h_{\text{top}}(\varphi^{kq}) = \log(\lambda_{f^k}) + kp\log(\lambda_A).$$

By Theorem 5.14, we obtain $\lambda_{f^k} = 1$. Therefore:

$$h_{\text{top}}(\varphi) = \frac{1}{kq} h_{\text{top}}(\varphi^{kq}) = \frac{p}{q}\log(\lambda_A).$$

Lastly, by Lemma 5.12, φ is conjugated to an SFT. $\square$

Conclusion

Throughout this work we have seen and developed the connections between distortion and non-expansivity. First, by generalizing the concept of range to general expansive systems, we have seen the aforementioned connection is not exclusive to subshifts. This ultimately led to the fact that expansive automorphisms cannot be distorted. What the obtained results suggest, is that a non expansive direction is one in which the rate at which information propagates with sub-linear speed. This can be seen in Corollary 3.26 and Lemma 5.5. This connection is further cemented by the fact that automorphisms with unique rational non expansive directions are characterized through shifts of distorted automorphisms. Although, the ultimate structure of automorphisms with unique irrational non expansive directions remains hidden, we have been able to identify necessary conditions on the word complexity of their domains, when these are minimal or transitive as stated by Theorem 5.8.

Second, by introducing a general framework for the multiple discrete Lyapunov exponents present in the literature we have seen how distortion behaves locally. These local behaviour may have seemed obvious at first glance, but examples such as 3.11 show that even if an automorphism acts in a distorted way on all configurations, it does not have to be distorted. It is also important to mention, that even through the notions of distortion where expanded upon through the introduction of new Lyapunov exponents, the fact that they have zero topological entropy remains.

Nevertheless, the greatest question concerning distortion on automorphism groups remains open: is every range distorted automorphism group distorted? Even though there seems to be a direct path for solving this question, constructing a Turing machine-like automorphism that is range distorted but not group distorted, it is not clear how the construction of this automorphism can be achieved. It is possible that the study of the group generated by symmetrically distorted automorphisms, $\langle \mathfrak{D}(X) \rangle$, can shed some light on this mystery. It can also be possible to answer the question by studying the group distorted Turing machines on the group of reversible Turing machines presented by Barbieri, Kari and Salo in [BKS16].

There also remains to see if it is possible to have an automorphism with a unique non-expansive direction of irrational slope over a domain which is transitive or minimal. A possible approach consists on codifying a subshift suspension in a way that the non-expansive directions of the suspension are preserved. Lemma 5.5 can be a useful tool on this regard.

Finally, it remains to be seen whether if different properties on domains or automorphisms allow distortion. For instance, we do not know if a minimal subshift can have an infinite order non trivial distorted automorphism in its group. Point-wise Lyapunov exponents seem like the appropriate tool to tackle these problems, especially due to the fact that it takes only one configuration to have a strictly positive Lyapunov exponent to rule out the distortion of an automorphism. This option will be explored in future works.

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