Nilpotent Jacobians and Almost Global Stability

Álvaro Castañeda
University of Chile

NILPOTENT JACOBIANS AND ALMOST GLOBAL STABILITY

ÁLVARO CASTAÑEDA AND MAXIMILIANO MACHADO–HIGUERA

Abstract. In this article we study maps with nilpotent Jacobian in $\mathbb{R}^n$ distinguishing the cases when the rows of JH are linearly dependent over $\mathbb{R}$ and when they are linearly independent over $\mathbb{R}$. In the linearly dependent case, we show an application of such maps on dynamical systems, in particular, we construct a large family of almost Hurwitz vector fields for which the origin is an almost global attractor. In the linearly independent case, we show explicitly the inverse maps of the counterexamples to Generalized Dependence Problem and proving that this inverse maps also have nilpotent Jacobian with rows linearly independent over $\mathbb{R}$.

1. Introduction

A strong relation there exists between the global stability problem (Markus–Yamabe Problem) in [10] and the Jacobian Conjecture since C. Olech in [11] showed that the global stability problem (in dimension two) for the system

$$\dot{x} = F(x) \tag{1}$$

is equivalent to prove the injectivity of the map F. Moreover in the remarkable works about the Jacobian Conjecture of H. Bass *et al.* [1] and A.V. Yagzhev [14], established that in order to show this conjecture it is sufficient to focus on maps of the form $X + H$ where the Jacobian matrix JH is nilpotent. This fact helps to A. Cima *et al.* in [3] to find a counterexample to Markus–Yamabe Problem in dimension larger than three, namely,

$$F(x_1, \ldots, x_n) = (-x_1 + x_3(x_1 + x_2x_3)^2, -x_2 - (x_1 + x_2x_3)^2, -x_3, \ldots, -x_n).$$

Recall that this problem of global stability, which is true when $n \leq 2$, (see [7, 8, 9] for details of proof in dimension two) is asked when the system (1) has to the origin as a global attractor, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 with $F(0) = 0$ and for any $x \in \mathbb{R}^n$ all the eigenvalues of $JF(x)$ have negative real part. The vector fields that have the condition of negativeness over the eigenvalues of the Jacobian matrix are called Hurwitz vector fields. If this condition of the negativeness is satisfied except for a set in $\mathbb{R}^n$ with Lebesgue measure zero, these vector fields are known as almost Hurwitz vector fields. In [12] B. Pires and R. Rabanal have studied in dimension two this kind of vector fields proving that almost Hurwitz vector fields with the origin as an hyperbolic singular point are all topologically equivalents to the radial vector field.

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In the second section of this article we show a large familiy of almost Hurwitz vector fields in large dimension which are constructed by using polynomial maps with nilpotent Jacobian with rows linearly dependent over $\mathbb{R}$. Additionally, we prove that this vector fields of above family support density functions (a formal definition will give later) and therefore the origin is an almost global attractor (global attractor except for a set of initial states with Lebesgue measure zero) for the associated system (1).

In the third section, we construct a large family of examples to weak Markus–Yamabe Conjecture and Jacobian Conjecture on $\mathbb{R}^n$. The construction of this family is based in the counterexamples to Generalized Dependence Problem given by A. van den Essen in [4] for dimension $n \geq 4$. This counterexamples are maps of the form $X + H$ with JH nilpotent and rows linearly independent over $\mathbb{R}$. We show explicitly the inverse map for such maps proving that the properties of nilpotency and linear independence is preserved for the inverse maps.

2. Almost Hurwitz vector fields

This section is devoted to construct almost Hurwitz vector fields for dimension $n \geq 2$ in terms of nilpotent Jacobian with rows linearly dependent over $\mathbb{R}$. Recall that an almost Hurwitz field vector field it is a Hurwitz vector field except for a Lebesgue measure zero set.

In [5, Theorem 7.2.25] is shown that nilpotent Jacobians with rows linearly dependent over $\mathbb{R}$ has the following structure.

Proposition 1. *Let* $H = (u_1(x_1, x_2), u_2(x_1, x_2), \ldots, u_{n-1}(x_1, \ldots, x_n), u_n(x_1, \ldots, x_n))$. *If* JH *is nilpotent, then* H *has coordinates* $(H_1, \ldots, H_n)$ *of the following form*

$$\begin{aligned}
H_1 &= a_2 f(a_1x_1 + a_2x_2) + b_1,\\
H_2 &= -a_1 f(a_1x_1 + a_2x_2) + b_2,\\
H_3 &= a_4(x_1, x_2) f(a_3(x_1, x_2)x_3 + a_4(x_1, x_2)x_4) + b_3(x_1, x_2),\\
H_4 &= -a_3(x_1, x_2) f(a_3(x_1, x_2)x_3 + a_4(x_1, x_2)x_4) + b_4(x_1, x_2),\\
&\vdots\\
H_{n-1} &= a_n(x_1, \ldots, x_{n-2}) f(a_{n-1}(x_1, \ldots, x_{n-2})x_{n-1} + a_n(x_1, \ldots, x_{n-2})x_n)\\
&\quad + b_{n-1}(x_1 \ldots, x_{n-2}),\\
H_n &= -a_{n-1}(x_1, \ldots, x_{n-2}) f(a_{n-1}(x_1, \ldots, x_{n-2})x_{n-1} + a_n(x_1, \ldots, x_{n-2})x_n)\\
&\quad + b_n(x_1, \ldots, x_{n-2}),
\end{aligned}$$

with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, $a_3, a_4, b_3, b_4 \in \mathbb{R}[x_1, x_2], \ldots, a_{n-1}, a_n, b_{n-1}, b_n \in \mathbb{R}[x_1, \ldots, x_{n-2}]$ *and* $f \in \mathbb{R}[t]$.

The next result shows a family of almost Hurwitz vector fields which are constructed by using above classification on nilpotent Jacobians with rows linearly dependent over $\mathbb{R}$ for even dimension $n \geq 2$.

We emphasize that throughout this section we will consider $b_i \equiv 0$, $i = 1, \ldots, n$ in order to simplify the calculations.

Theorem 1. *Let* $s \geq 1$ *and* $f \in \mathbb{R}[T]$ *such that*

$$f(T) = \sum_{i=0}^{s} A_{2i+1} T^{2i+1} \quad \text{with} \quad A_{2i+1} < 0,\ i = 0, \ldots, s,$$

and the polynomial

$$R(x_{n+1}) = \sum_{l=1}^{k} d_{2l} x_{n+1}^{2l} \quad \text{with} \quad d_{2l} > 0,\ l = 1, \dots, k.$$

Then the vector field

$$\begin{aligned} F(x_1, \dots, x_{n+1}) &= (-x_2, x_1, -x_4, x_3, \dots, -x_n, x_{n-1}, -x_{n+1}R(x_{n+1})) \\ &\quad + R(x_{n+1})(\lambda I x + H(x), 0), \end{aligned} \tag{2}$$

with $x = (x_1, \dots, x_n), \lambda < 0$ *and* H *as in Proposition* (1)*, is an almost Hurwitz vector field.*

Proof. The eigenvalues of Jacobian matrix JF are $\beta_{n+1} = -(R(x_{n+1}) + x_{n+1}R'(x_{n+1}))$, and

$$\begin{aligned} \beta_1^{\pm} &= \lambda R(x_{n+1}) \pm \sqrt{-1 + (a_1^2 + a_2^2)R(x_{n+1})f'(a_1x_1 + a_2x_2)}, \\ \beta_{2j-1} &= \lambda R(x_{n+1}) + \sqrt{-1 + (a_{j-1}^2 + a_j^2)R(x_{n+1})f'(a_{j-1}x_{j-1} + a_jx_j)}, \\ \beta_{2j} &= \lambda R(x_{n+1}) - \sqrt{-1 + (a_{j-1}^2 + a_j^2)R(x_{n+1})f'(a_{j-1}x_{j-1} + a_jx_j)}, \end{aligned}$$

with $a_{2j-1} = a_{2j-1}(x_{j-1}, x_j)$ and $a_{2j} = a_{2j}(x_{j-1}, x_j)$ for $j = 2, \dots, n/2$.

Therefore, for $x_{n+1} \neq 0$ (resp. $x_{n+1} = 0$), we have $\beta_{n+1} < 0$ (resp. $\beta_{n+1} = 0$), and $\beta_1^{\pm}, \beta_{2j-1}$ and β_{2j} have negative real part (resp. null real part). Thus, F verifies the Hurwitz condition except in the invariant plane $x_{n+1} = 0$. □

In order to state the main result of this section, we recall the concept of density function (dual of a Lyapunov function) introduced by A. Rantzer in [13].

Definition 1. *A density function of* (1) *is a* C^1 *map* $\rho \colon \mathbb{R}^d \setminus \{0\} \to [0, +\infty)$*, integrable outside a ball centered at the origin that satisfies*

$$[\nabla \cdot \rho F](x) > 0$$

almost everywhere with respect to $\mathbb{R}^d$*, where*

$$\nabla \cdot [\rho F] = \nabla \rho \cdot F + \rho [\nabla \cdot F],$$

and $\nabla \rho$*,* $\nabla \cdot F$ *denote respectively the gradient of* ρ *and the divergence of* F*.*

The result of A. Rantzer [13, Theorem 1] establishes a relation between density functions and almost global stability.

Proposition 2. *Given the differential system*

$$\dot{x} = F(x),$$

where the map $F \in C^1(\mathbb{R}^d)$ *and* $F(0) = 0$*. Suppose there exists a density function* $\rho : \mathbb{R}^d \setminus \{0\} \to [0, +\infty)$ *such that* $\rho(x)F(x)/\parallel x \parallel$ *is integrable on* $\{x \in \mathbb{R}^d : \parallel x \parallel \geq 1\}$*. Then, almost all trajectories converge to the origin, i.e., the origin is almost globally stable.*

The following theorem shows that the system (1) with F as in (2) under suitable conditions support a density function.

Theorem 2. *Let consider F as in* (2) *with*

$$a_{2j-1} = \pm a_{2j},\ j = 1, \dots, n/2 \qquad \text{and} \qquad A_{2i+1} = 0,\ i = 1, \dots, s.$$

Then the function

$$\rho(x_1, \dots, x_{n+1}) = \frac{1}{(x_1^2 + \cdots + x_n^2 + R(x_{n+1}))^\alpha} \tag{3}$$

is a density function of the system (1) *where*

$$\alpha > \max\left\{2, \frac{3 - n\lambda}{2}, \frac{2k + 1 - n\lambda}{2(a_{2j-1}^2 A_1 - \lambda)}\right\} \text{ with } a_{2j-1}^2 < \lambda / A_1,\ j = 1, \dots, n/2. \tag{4}$$

Proof. Throughout this proof, we will consider the function

$$\mathcal{P}(x_1, \dots, x_{n+1}) = \mathcal{P} := \frac{1}{(x_1^2 + \cdots + x_n^2 + R(x_{n+1}))}.$$

Since we have the condition $a_{2j-1} = a_{2j}$, $j = 1, \dots, n/2$ (the case $a_{2j-1} = -a_{2j}$ is similar) and $A_{2i+1} = 0$, $i = 1, \dots, n$, the vector field F becomes

$$\begin{aligned} F(x_1, \dots, x_{n+1}) \;=\; & (-x_2, x_1, -x_4, x_3, \dots, -x_n, x_{n-1}, -x_{n+1} R(x_{n+1})) \\ & + R(x_{n+1}) \left\{(\lambda + a_1^2 A_1) x_1 + a_1^2 A_1 x_2, -a_1^2 A_1 x_1 + (\lambda - a_1^2 A_1) x_2\right\} \\ & + R(x_{n+1}) \left\{(\lambda + a_3^2 A_1) x_3 + a_3^2 A_1 x_4, -a_3^2 A_1 x_3 + (\lambda - a_3^2 A_1) x_4\right\} \\ & \vdots \\ & + R(x_{n+1}) \left\{(\lambda + a_{n-1}^2 A_1) x_{n-1} + a_{n-1}^2 A_1 x_n\right\} \\ & + R(x_{n+1}) \left\{-a_{n-1}^2 A_1 x_{n-1} + (\lambda - a_{n-1}^2 A_1) x_n\right\}. \end{aligned} \tag{5}$$

Notice that $\lambda + a_{2j-1}^2 A_1 < 0$, and $\lambda - a_{2j-1}^2 A_1 < 0$, $j = 1, \dots, n/2$. Now, we will prove that the function ρ of (3) with α as in (4) is a density function for the system (1). Indeed, the condition $\alpha > 2$ ensures the integrability of ρ outside the ball centered at the origin of radius one.

It remains to prove that $\nabla \cdot (\rho F)(x_1, \dots, x_{n+1})$ is positive almost everywhere in $\mathbb{R}^{n+1}$. We have

$$\triangledown \rho(x_1, \dots, x_{n+1}) = -\alpha \mathcal{P}^{\alpha+1} \, (2x_1, \dots, 2x_n, R'(x_{n+1})),$$

and

$$[\triangledown \cdot F](x_1, \dots, x_{n+1}) = (n\lambda - 1) R(x_{n+1}) - x_{n+1} R'(x_{n+1}).$$

Then

$$
\begin{aligned}
[\nabla \cdot \rho F] &= (\nabla\rho \cdot F)(x_1,\dots,x_{n+1}) + \rho(x_1,\dots,x_{n+1})\,[\nabla \cdot F](x_1,\dots,x_{n+1}) \\
&= -\alpha\, R(x_{n+1})\mathcal{P}^{\alpha+1}\Big\{2x_1((\lambda + a_1^2A_1)x_1 + a_1^2A_1x_2) \\
&\qquad +2x_2(-a_1^2A_1x_1 + (\lambda - a_1^2A_1)x_2) \\
&\qquad +2x_3((\lambda + a_3^2A_1)x_3 + a_3^2A_1x_4) \\
&\qquad +2x_4(-a_3^2A_1x_3 + (\lambda - a_3^2A_1)x_4) \\
&\qquad \vdots \\
&\qquad +2x_{n-1}((\lambda + a_{n-1}^2A_1)x_{n-1} + a_{n-1}^2A_1x_n) \\
&\qquad +2x_n(-a_{n-1}^2A_1x_{n-1} + (\lambda - a_{n-1}^2A_1)x_n) - x_{n+1}\,R'(x_{n+1})\Big\} \\
&\quad +\mathcal{P}^{\alpha}\,[(n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&= \mathcal{P}^{\alpha+1}\Big\{x_1^2[(-2\alpha(\lambda + a_1^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad +x_2^2[(-2\alpha(\lambda - a_1^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad +x_3^2[(-2\alpha(\lambda + a_3^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad +x_4^2[(-2\alpha(\lambda - a_3^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad \vdots \\
&\qquad +x_{n-1}^2[(-2\alpha(\lambda + a_{n-1}^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad +x_n^2[(-2\alpha(\lambda - a_{n-1}^2A_1) + n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})] \\
&\qquad +R(x_{n+1})\,[\alpha x_{n+1}R'(x_{n+1}) + (n\lambda - 1)R(x_{n+1}) - x_{n+1}R'(x_{n+1})]\Big\}.
\end{aligned}
$$

Since $R(x_{n+1}) = \sum_{l=1}^{k} d_{2l}x_{n+1}^{2l}$ and $x_{n+1}R'(x_{n+1}) = \sum_{l=1}^{k} 2l d_{2l}x_{n+1}^{2l}$ with $d_{2l} > 0$ for $l = 1,\dots,k$, we obtain $[\nabla \cdot \rho F](x_1,\dots,x_{n+1}) > 0$ for $x_{n+1} \neq 0$, if for each $j = 1,\dots,n/2$ we have that

$$-2\alpha(\lambda - a_{2j-1}^2A_1) + n\lambda - 1 - 2l > 0\ , l = 1,\dots,k,$$

and

$$2l(\alpha - 1) + n\lambda - 1 > 0\ , l = 1,\dots,k.$$

Moreover, $[\nabla \cdot \rho F](x_1,\dots,x_{n+1}) = 0$ for $x_{n+1} = 0$. Therefore, the desired result follows from (4). □

Now, we can show the origin is an almost global attractor of system (1) where F is given by (2). In fact, we have the following corollary.

Corollary 1. *The system* (1) *where* F *as in previous Theorem has the origin as an almost global attractor which is not locally asymptotic stable.*

Proof. By Theorem 2, the function $\rho(x_1,\dots,x_{n+1})$ from (3) with α as in (4) is a density function for the system (1) with F as in (2) if

$$a_{2j-1} = \pm a_{2j},\ j = 1,\dots,n/2 \qquad \text{and} \qquad A_{2i+1} = 0,\ i = 1,\dots,n,$$

thus, the vector field F becomes (5).

We have $F(x_1,\dots,x_n,0) = (-x_2,x_1,\dots,-x_n,x_{n-1}0)$, then the origin is not locally asymptotically stable.

On the other hand, to prove that the origin is almost global attractor we use Rantzer's result (Theorem 2). Then it is sufficient to show that the condition $\alpha > 2$ ensures the integrability of

$$\rho(x_1, \ldots, x_{n+1}) F(x_1, \ldots, x_{n+1}) / \| (x_1, \ldots, x_{n+1}) \| \tag{6}$$

outside the ball centered at the origin of radius one.

In fact, if we consider constants $M_0 = \max\{1, d_2\}$, and for $j = 1, \ldots, n/2$,

$$M_{2j-1} = \max\{1, (\lambda - a_{2j-1}^2 A_1)^2\} \text{ and } N_{2j-1} = -2A_1 a_{2j-1}^2,$$

we have that

$$\begin{aligned}
\| F\|^2 &= x_1^2 + \cdots + x_n^2 \\
&\quad + R^2(x_{n+1}) \left\{ [(\lambda + a_1^2 A_1)^2 + a_1^4 A_1^2] x_1^2 \right. \\
&\quad + [(\lambda - a_1^2 A_1)^2 + a_1^4 A_1^2] x_2^2 + 4 a_1^4 A_1^2 x_1 x_2 \\
&\quad + \cdots + [(\lambda + a_{n-1}^2 A_1)^2 + a_{n-1}^4 A_1^2] x_{n-1}^2 \\
&\quad \left. + [(\lambda - a_{n-1}^2 A_1)^2 + a_{n-1}^4 A_1^2] x_n^2 + 4 a_{n-1}^4 A_1^2 x_{n-1} x_n + x_{n+1}^2 \right\} \\
&\quad - 2A_1 R(x_{n+1}) \left\{ a_1^2 (x_1 + x_2)^2 + a_3^2 (x_3 + x_4)^2 + \cdots + a_{n-1}^2 (x_{n-1} + x_n)^2 \right\} \\
&\leq x_1^2 + \cdots + x_n^2 \\
&\quad + R^2(x_{n+1}) \Big\{ [(\lambda - a_1^2 A_1)^2 + a_1^4 A_1^2](x_1^2 + x_2^2) + 2 a_1^4 A_1^2 (x_1^2 + x_2^2) + \cdots + \\
&\quad + [(\lambda - a_{n-1}^2 A_1)^2 + a_{n-1}^4 A_1^2](x_{n-1}^2 + x_n^2) + 2 a_{n-1}^4 A_1^2 (x_{n-1}^2 + x_n^2) + x_{n+1}^2 \Big\} \\
&\quad - 2A_1 R(x_{n+1}) \left\{ a_1^2 (x_1 + x_2)^2 + a_3^2 (x_3 + x_4)^2 + \cdots + a_{n-1}^2 (x_{n-1} + x_n)^2 \right\} \\
&\leq x_1^2 + \cdots + x_n^2 + x_{n+1}^2 \\
&\quad + \sum_{j=1}^{n/2} M_{2j-1} (x_1^2 + \cdots + x_n^2 + R(x_{n+1}))^2 (x_1^2 + \cdots + x_{n+1}^2) \\
&\quad + \sum_{j=1}^{n/2} N_{2j-1} (x_1^2 + \cdots + x_n^2 + R(x_{n+1})) (x_1^2 + \cdots + x_{n+1}^2).
\end{aligned}$$

and that $x_1^2 + \cdots + x_n^2 + R(x_{n+1}) \geq M_0$. Thus, this facts combined with the assumption over α imply that

$$\begin{aligned}
\frac{\| F\|^2 \rho^2}{\| (x_1, \ldots, x_{n+1}) \|^2} &\leq \frac{1}{(x_1^2 + \cdots + x_n^2 + R(x_{n+1}))^{2\alpha}} \\
&\quad + \sum_{j=1}^{n/2} \frac{M_{2j-1}}{(x^2 + \cdots + x_n^2 + R(x_{n+1}))^{2\alpha-2}} \\
&\quad + \sum_{j=1}^{n/2} \frac{N_{2j-1}}{(x_1^2 + \cdots + x_n^2 + R(x_{n+1}))^{2\alpha-1}} .
\end{aligned}$$

Therefore (6) is integrable outside the ball centered at the origin of radius one, and result follows. □

Remark 1. *This Corollary shows a large family of Hurwitz vector fields in dimension* $n+1$ *with the origin as an almost global attractor generalizing three dimensional results from* [2, Corollary 3.5].

3. Injectivity

This section is devoted to show a large family of example to the next two conjectures:
Jacobian Conjecture on $\mathbb{R}^n$. Every polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det JF \equiv 1$ is a bijective map with a polynomial inverse, and
Weak Markus–Yamabe Conjecture (WMYC): If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-Hurwitz map, then F is injective.

It is known that Jacobian Conjecture is open for $n \geq 2$ and **WMYC** is true when $n \leq 2$ and, to the best of our knowledge, it has been proved in dimension $n \geq 3$ for C^1 Lipschitz Hurwitz maps by A. Fernandes *et al.* in [6, Corollary 4].

We carry out this task of show examples to this problems determining the inverse of the maps $F = \lambda I + H$ in dimension $n \geq 4$, where $\lambda < 0$ and H are counterexamples to Generalized Dependence Problem (see [5, Proposition 7.1.9].

Proposition 3. *The polynomial maps* $F = \lambda I + H : \mathbb{R}^n \to \mathbb{R}^n$ *with* $n \geq 4, \lambda \neq 0$ *and* $H = (H_1, \ldots, H_n)$ *where*

$$\begin{aligned} H_1(x_1,\ldots,x_n) &= g(x_2 - a(x_1))\,, \\ H_i(x_1,\ldots,x_n) &= x_{i+1} + \frac{(-1)^i}{(i-1)!}\, a^{(i-1)}(x_1)\, g(x_2 - a(x_1))^{i-1}\,,\ \textit{if}\ \ 2 \leq i \leq n-1\,, \\ H_n(x_1,\ldots,x_n) &= \frac{(-1)^n}{(n-1)!}\, a^{(n-1)}(x_1)\, g(x_2 - a(x_1))^{n-1} \end{aligned}$$

where $a(x_1) \in \mathbb{R}[x_1]$ *with* $\deg a = n-1$ *and* $g(t) \in \mathbb{R}[t], g(0) = 0$ *and* $\deg_t g(t) \geq 1$, *have polynomial inverse are examples to Jacobian Conjecture on* $\mathbb{R}^n$ $(\lambda = 1)$ *and Weak Markus–Yamabe Conjecture* $(\lambda < 0)$.

Proof. Let $u_i = \lambda x_i + H_i(x_1, \ldots, x_n)$ for $1 \leq i \leq n$. Then

$$\sum_{i=2}^{n} \frac{(-1)^i}{\lambda^{i-1}} u_i - a\left(\frac{1}{\lambda} u_1\right) = x_2 - a(x_1). \tag{7}$$

In fact, w have that

$$\begin{aligned}
\sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}u_i &= \sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}\left[\lambda x_i + x_{i+1} + \frac{(-1)^i}{(i-1)!}a^{(i-1)(x_1)g(x_2-a(x1))^{i-1}}\right]\\
&= \sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-2}}x_i + \sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}x_{i+1}\\
&\quad + \sum_{i=2}^{n}\frac{(-1)^i}{(i-1)!\lambda^{i-1}}a^{(i-1)}(x_1)g(x_2-a(x1))^{i-1}\\
&= x_2 + \frac{(-1)^{n+1}}{\lambda^{n-2}}x_n + \sum_{i=2}^{n}\frac{(-1)^i}{(i-1)!\lambda^{i-1}}a^{(i-1)}(x_1)g(x_2-a(x_1))^{i-1}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}u_i &= x_2 + \frac{(-1)^{n+1}}{\lambda^{n-2}}x_n + \sum_{i=2}^{n}\frac{(-1)^i}{(i-1)!\lambda^{i-1}}a^{(i-1)}(x_1)g(x_2-a(x1))^{i-1}\\
&\quad + \frac{(-1)^n}{\lambda^{n-1}}\left[\lambda x_n + \frac{(-1)^n}{\lambda^{n-1}!}a^{(n-1)}(x_1)g(x_2-a(x_1))^{n-1}\right]\\
&= x_2 + \sum_{i=2}^{n}\frac{1}{(i-1)!\lambda^{i-1}}a^{(i-1)}(x_1)g(x_2-a(x_1))^{i-1}\\
&= x_2 - a(x_1) + \sum_{i=1}^{n}\frac{1}{(i-1)!\lambda^{i-1}}a^{(i-1)}(x_1)g(x_2-a(x_1))^{i-1}\\
&= x_2 - a(x_1) + a\left(x_1 + \frac{1}{\lambda}g(x_2-a(x_1))\right)\\
&= x_2 - a(x_1) + a\left(\frac{1}{\lambda}u_1\right),
\end{aligned}$$

which show (7). Moreover, putting $\Psi(u) = g\Big(\sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}u_i - a\left(\frac{1}{\lambda}u_1\right)\Big)$ we have that

$$\begin{aligned}
x_1 &= \tfrac{1}{\lambda}(u_1 - \Psi(u)),\\
x_2 &= a(x_1) - a\left(\tfrac{1}{\lambda}\right) + \sum_{i=2}^{n}\frac{(-1)^i}{\lambda^{i-1}}u_i,\\
x_3 &= -\lambda\Big[a(x_1) + a'(x_1)\tfrac{\Psi(u)}{\lambda}\Big] + \lambda a\left(\tfrac{1}{\lambda}u_1\right) - \sum_{i=3}^{n}\frac{(-1)^i}{\lambda^{i-2}}u_i,\\
x_4 &= \lambda^2\Big[a(x_1 + a'(x_1)\tfrac{\Psi(u)}{\lambda} + \tfrac{1}{2}a''(x_1)\tfrac{\Psi(u)^2}{\lambda^2})\Big] - \lambda^2 a\left(\tfrac{1}{\lambda}u_1\right) + \sum_{i=4}^{n}\frac{(-1)^i}{\lambda^{i-3}}u_i,
\end{aligned}$$

and, in general, for $3 \leq k \leq n$, we have

$$x_k = (-1)^k \Big[\lambda^{k-2} \sum_{i=0}^{k-2} \frac{1}{i!} \frac{\Psi(u)^k}{\lambda^k} a^i(x_1) - \lambda^{k-2} a \left(\frac{1}{\lambda} u_1 \right) + \sum_{i=k}^{n} \frac{(-1)^i}{\lambda^{i-k-1}} u_i \Big],$$

or

$$x_k = \frac{1}{\lambda} u_k + G_k(u_1, \dots, u_n),$$

with

$$G_k(u_1, \dots, u_n) = (-1)^k \Big[\lambda^{k-2} \sum_{i=0}^{k-2} \frac{1}{i!} \frac{\Psi(u)^k}{\lambda^k} a^i(x_1) - \lambda^{k-2} a \left(\frac{1}{\lambda} u_1 \right) + \sum_{i=k}^{n} \frac{(-1)^i}{\lambda^{i-k-1}} u_i \Big]$$

and the result follows. □

The following examples to the conjectures are base in a counterexample to Generalized Dependence Problem which does not belong to the family of Proposition 3 in dimension 4, which is a generalization of the map H in [5, pp. 302].

Proposition 4. *Consider a polynomial map of the form*

$$\begin{array}{rcl} H(x,y,z,w) & = & f(t)(-1, b_1 + 2v_1\alpha x, -\alpha f(t), \lambda(b_1 + 2v_1\alpha x)) \\ & & +(0, \lambda(b_1 x + 2v_1 \alpha x^2) - v_1 z + w, 0, -\lambda v_1 z) \end{array}$$

such that $f \in \mathbb{R}[t]$ where $t = \lambda(y + b_1 x + v_1 \alpha x^2)$ and $v_1\alpha \neq 0$. Then JH is nilpotent and the rows of JH are linearly independent of $\mathbb{R}$. Moreover, $\lambda I + H$ is a example to Weak Markus–Yamabe Conjecture (resp. Jacobian Conjecture) with $\lambda < 0\,(\text{resp.}\, \lambda = 1)$.

Proof. It is easy to see that the trace of JH is zero. Furthermore, note that the determinant of JH is zero due to $|\,\partial(H_1, H_3)/\partial(x,z)| \equiv 0$. By using a algebraic manipulator it is straightforward see that the principal minors of the order 2 and 3 are zero. On the other hand, to prove the linearly independence of rows of JH over $\mathbb{R}$ we write $\displaystyle\sum_{i=1}^{4} \gamma_i H_i = 0$ for some $\lambda_i \in \mathbb{R}$ and we will show that $\gamma_i = 0$, $i = 1, \dots, 4$. In fact, it is easy to see that $\gamma_2 = 0$ due to that w only appears in H_2 and as a consequence we have that $\gamma_4 = 0$, and finally $\gamma_1 = \gamma_2 = 0$ noticing $y-$degree of H_1 and H_3 respectively.

On the other hand, we consider

$$\begin{array}{rcl} u_1 & = & \lambda x - f(t) \\ u_2 & = & \lambda y + \lambda(b_1 x + v_1 \alpha x^2) - v_1 z + (b_1 + 2v_1\alpha x) f(t) + w \\ u_3 & = & \lambda z - \alpha(f(t))^2 \\ u_4 & = & \lambda w + \lambda(b_1 + 2v_1 \alpha x) f(t) + \lambda v_1 z. \end{array}$$

It is easy to see that $\Phi = \lambda(y + b_1x + v_1\alpha x^2) = u_2 - \frac{1}{\lambda}u_4$, thus we have that the inverse of map F is $F^{-1} = \gamma(x, y, z, w) + (Q_1, Q_2, Q_3, Q_4)$ where

$$\begin{aligned}
Q_1 &= \gamma f(\Phi)\\
Q_2 &= -\gamma(b_1x + \gamma v_1\alpha x^2 + \gamma w) + \gamma(\gamma^2(v_1\alpha x + \tfrac{1}{\gamma}b_1) - 2b_1 - 4\gamma v_1\alpha x)f(\Phi)\\
&\quad +\gamma^2(\tfrac{1}{\gamma^2}(b_1 - v1\alpha)2v_1\alpha(\gamma x - 1))(f(\Phi))^2 + \gamma^3 v_1\alpha(f(\Phi))^3\\
Q_3 &= \gamma\alpha(f(\Phi))^2\\
Q_4 &= \gamma v_1 z - \gamma^2(v_1\alpha x + \tfrac{1}{\gamma}b_1)f(\Phi)\\
&\quad -\gamma^2(2v_1\alpha x + \tfrac{1}{\gamma}(b_1 - v_1\alpha))(f(\Phi))^2 - \gamma^2 v_1\alpha(f(\Phi))^3,
\end{aligned}$$

with $\gamma = 1/\lambda$. □

A similar result can be obtained if we consider the map H of Proposition 1. Indeed,

Proposition 5. *The maps $\lambda I + H$ with $\lambda \neq 0$ and H as in Proposition 1 are examples to Weak Markus–Yamabe Conjecture (resp. Jacobian Conjecture) with* $\lambda < 0$ (resp. $\lambda = 1$).

Proof. We can calculated the inverse explicitly. Indeed,

$$\begin{aligned}
F_1^{-1} &= \gamma x_1 - \gamma a_2 f(\gamma(a_1x_1 + a_2x_2)) - b_1,\\
F_2^{-1} &= \gamma x_2 + \gamma a_1 f(\gamma(a_1x_1 + a_2x_2)) - b_2,\\
F_3^{-1} &= \gamma x_3 - \gamma a_4(x_1, x_2) f(\gamma(a_3(x_1, x_2)x_3 + a_2(x_1, x_2)x_4)) - b_3(x_1, x_2),\\
F_4^{-1} &= \gamma x_4 + \gamma a_3(x_1, x_2) f(\gamma(a_3(x_1, x_2)x_3 + a_2(x_1, x_2)x_4)) - b_4(x_1, x_2),\\
&\vdots\\
F_{n-1}^{-1} &= \gamma x_{n-1}\\
&\quad -\gamma a_n(x_1, \ldots, x_{n-2}) f(\gamma(a_{n-1}(x_1, \ldots, x_{n-2})x_{n-1} + a_n(x_1, \ldots, x_{n-2})x_n))\\
&\quad -b_{n-1}(x_1 \ldots, x_{n-2})\\
F_n^{-1} &= \gamma x_n\\
&\quad +\gamma a_{n-1}(x_1, \ldots, x_{n-2}) f(\gamma(a_{n-1}(x_1, \ldots, x_{n-2})x_{n-1} + a_n(x_1, \ldots, x_{n-2})x_n))\\
&\quad -b_n(x_1, \ldots, x_{n-2})
\end{aligned}$$

with $\gamma = 1/\lambda$. □

Remark 2. *The Proposition* 3 *in dimension three is Theorem* 2.2 *from* [2]*. Additionally, in the same article, in Remark* 2.4 *we give the inverse of F on explicitly way but without details as was constructed. Finally, notice that the maps F in the Proposition* 3 *have property of that JH is nilpotent with rows linearly independent over* $\mathbb{R}$.

Due to previous examples and remarks we have the following question:

Question: If $(\lambda I + H)^{-1} = \lambda^{-1}I + \overline{H}$ then $J\overline{H}$ preserves the same properties of nilpotency and independence of rows over $\mathbb{R}$ of JH?

To contextualize the answer to this question, we introduce some notations.

Let $F = X + H$ with $N := JH$ nilpotent, *i.e* $N^{d+1} = 0$. Let $\lambda \in \mathbb{R} \setminus \{0\}$ Put

$$F_\lambda := \lambda X + H = \lambda X \circ (X + \lambda^{-1} H).$$

Assume $X + \lambda^{-1} H$ is invertible with inverse $G := X + \overline{H}$. Then

$$F_\lambda^{-1} = (X + \overline{H}) \circ (\lambda^{-1} X) = \lambda^{-1} X + \overline{H}(\lambda^{-1} X) = \lambda^{-1}(X + \lambda \overline{H}(\lambda^{-1} H)).$$

Put $\widetilde{H} := \lambda \overline{H}(\lambda^{-1} X)$. Then $F_\lambda^{-1} = \lambda^{-1}(X + \widetilde{H})$.

Now, we have the following result which gives an affirmative answer to previous question.

Theorem 3. *Under above notations*

i) $J\widetilde{H}$ *is nilpotent.*
ii) *If the rows of* JH *are linearly independent over* $\mathbb{R}$, *then the rows of* $J\widetilde{H}$ *are linearly independent over* $\mathbb{R}$.

Proof.

i) $J\widetilde{H} = \lambda J\overline{H}(\lambda^{-1} X)\lambda^{-1} = J\overline{H}(\lambda^{-1} X)$. Furthermore $JF(G)JG = I$ and $JG = I + J\overline{H}$. Thus, $I + J\overline{H} = (JF)^{-1}(G)$. Observing that $JF = I + N$, we obtain that

$$(JF)^{-1} = I + \sum_{i=1}^{d} (-1)^i N^i,$$

which implies that

$$I + J\overline{H} = (JF)^{-1}(G) = I + \sum_{i=1}^{d} (-1)^i N(G)^i.$$

In consequence

$$J\overline{H} = \sum_{i=1}^{d} (-1)^i N(G)^i$$

which is nilpotent since $N(G)^{d+1} = 0$. Since, as seen above, $J\widetilde{H} = J\overline{H}(\lambda^{-1} X)$, statement i) follows.

ii) Assume that the rows of $J\widetilde{H}$ are linearly dependent over $\mathbb{R}$. Then there exists $0 \neq c \in \mathbb{R}^n$ such that $(J\widetilde{H})^T c = 0$. Hence $(J\overline{H})^T c = 0$ due to $J\widetilde{H} = J\overline{H}(\lambda^{-1} X)$. By (8) we have that $J\overline{H} = N(G)A$, where

$$A := \sum_{i=1}^{d} (-1)^i N(G)^i = -I + N(G) + N(G)^2 + \cdots$$

Since $N(G)$ is nilpotent, A is invertible and hence A^T is invertible also. Since $(J\overline{H})^T c = 0$ which implies that $A^T N(G)^T c = 0$, we obtain $N(G)^T c = 0$, *i.e* $(JH)(G)^T c = 0$. Substituting $X = F$ and using $G(F) = X$ we obtain that $(JH)^T c = 0$. Therefore, the rows of JH are linearly dependent over $\mathbb{R}$, obtaining a contradiction.

□

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Universidad de Chile, Departamento de Matemáticas. Casilla 653, Santiago, Chile

Universidad de Ibagué, Facultad de Ciencias Naturales y Matemáticas, Ibagué, Colombia

E-mail address: castaneda@uchile.cl, maximiliano.machado@unibague.edu.co