Mathematical Programming
https://doi.org/10.1007/s10107-020-01544-8

FULL LENGTH PAPER

Series A

Prophet secretary through blind strategies

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Received: 11 March 2019 / Accepted: 10 July 2020

Abstract

In the classic prophet inequality, a well-known problem in optimal stopping theory, samples from independent random variables (possibly differently distributed) arrive online. A gambler who knows the distributions, but cannot see the future, must decide at each point in time whether to stop and pick the current sample or to continue and lose that sample forever. The goal of the gambler is to maximize the expected value of what she picks and the performance measure is the worst case ratio between the expected value the gambler gets and what a prophet that sees all the realizations in advance gets. In the late seventies, Krengel and Sucheston (Bull Am Math Soc 83(4):745–747, 1977), established that this worst case ratio is 0.5. A particularly interesting variant is the so-called prophet secretary problem, in which the only difference is that the samples arrive in a uniformly random order. For this variant several algorithms are known to achieve a constant of $1 - 1/e \approx 0.632$ and very recently this barrier was slightly improved by Azar et al. (in: Proceedings of the ACM conference on economics and computation, EC, 2018). In this paper we introduce a new type of multi-threshold strategy, called *blind strategy*. Such a strategy sets a nonincreasing sequence of thresholds that depends only on the distribution of the maximum of the random variables, and the gambler stops the first time a sample surpasses the threshold of the stage. Our main result shows that these strategies can achieve a constant of 0.669 for the prophet secretary problem, improving upon the best known result of Azar et al. (in: Proceedings of the ACM conference on economics and computation, EC, 2018), and even that of Beyhaghi et al. (Improved approximations for posted price and second price mechanisms. CoRR arXiv:1807.03435, 2018) that works in the case in which the gambler can select the order of the samples. The crux of the result is a very precise analysis of the underlying stopping time distribution for the gambler's strategy that

Partially supported by CONICYT-Chile through grant FONDECYT 1190043, by ECOS-CONICYT through grant C15E03, and by a Google Research for Latin America Award. A preliminary version of this paper appeared in the Proceedings of the 30th ACM-SIAM Symposium on Discrete Algorithms (SODA 2019).

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is inspired by the theory of Schur-convex functions. We further prove that our family of blind strategies cannot lead to a constant better than 0.675. Finally we prove that no algorithm for the gambler can achieve a constant better than $\sqrt{3}-1 \approx 0.732$, which also improves upon a recent result of Azar et al. (in: Proceedings of the ACM conference on economics and computation, EC, 2018). This implies that the upper bound on what the gambler can get in the prophet secretary problem is strictly lower than what she can get in the i.i.d. case. This constitutes the first separation between the prophet secretary problem and the i.i.d. prophet inequality.

Mathematics Subject Classification 60G40

1 Introduction

Prophet Inequalities. For fixed $n > 1$, let $V_1, \dots, V_n$ be non-negative, independent random variables and T_n the set of stopping times associated with the filtration generated by $V_1, \dots, V_n$. A classic result of Krengel and Sucheston [25,26] asserts that $\mathbb{E}(\max\{V_1, \dots, V_n\}) \leq 2 \cdot \sup\{\mathbb{E}(V_T) : T \in T_n\}$, and that 2 is the best possible bound. The interpretation of this result says that a *gambler*, who only knows the distribution of the random variables and that looks at them sequentially, can select a stopping rule that guarantees her half of the value that a *prophet*, who knows all the realizations, could get. The study of this type of inequalities, known as *prophet inequalities*, was initiated by Gilbert and Mosteller [18] and attracted a lot of attention in the eighties [21,22,30]. In particular Samuel-Cahn [30] noted that rather than looking at the set of all stopping rules one can (quite naturally) only look at threshold stopping rules in which the decision to stop depends on whether the value of the currently observed random variable is above a certain threshold (and possibly on the rest of the history). In the last decade the theory of prophet inequalities has resurged as an important problem due to its connections to *posted price mechanisms* (PPMs) which are frequently used in online sales (see [7,12]). The way these mechanisms work is as follows. Suppose a seller has an item to sell. Consumers arrive one at a time and the seller proposes to each consumer a take-it-or-leave-it offer. The first customer accepting the offer pays that price and takes the item. This is again a stopping problem, and we refer the reader to [2,7,9,20,27] for the connection between this stopping problem and Prophet inequalities.

Although the situation for the standard prophet inequality just described is well understood, there are variants of the problem, which are particularly relevant given the connection to PPMs, for which the situation is very different. In what follows we describe three important variants that are connected to each other and constitute the main focus of this paper.

- *Order selection (or free order)* In this version the gambler is allowed to select the order in which she examines the random variables. For this version, Chawla et al. [7] improved the bound of $1/2$ (of the standard prophet inequality) to $1 - 1/e \approx 0.6321$. This bound remained the best known for quite some time until Azar et al. [3] improved it to $1 - 1/e + 1/400 \approx 0.6346$. Interestingly, the bound

of Azar et al. actually applies to the random order case described below. Very recently, Beyhaghi et al. [5], used order selection to further improve the bound to $1 - 1/e + 0.022 \approx 0.6541$.
- *Prophet secretary (or random order)* In this version the random variables are shown to the gambler in random order, as in the secretary problem. This version was first studied by Esfandiari et al. [13] who found a bound of $1 - 1/e$. Their algorithm defines a nonincreasing sequence of n thresholds $\tau_1, \dots, \tau_n$ that only depend on the expectation of the maximum of the V_i's and on n. The gambler at time-step i stops if the value of V_{σ_i} (the variable shown at step i) surpasses τ_i. Later, Correa et al. [8] proved that the same factor of $1 - 1/e$ can be obtained with a personalized but nonadaptive sequence of thresholds, that is thresholds $\tau_1, \dots, \tau_n$ such that whenever variable V_i is shown the gambler stops if its value is above τ_i. In recent work, Ehsani et al. [14] show that the bound of $1 - 1/e$ can be achieved using a single threshold (having to randomize to break ties in some situations). This result appears to be surprising since without the ability of breaking ties at random, $1/2$ is the best possible constant and this insight turns out to be the starting point of our work. Shortly after the work of Ehsani et al., Azar et al. [3] improved it to $1 - 1/e + 1/400 \approx 0.6346$ through an algorithm that relies on some subtle case analysis.
- *IID prophet inequality* Finally, we mention the case when the random variables are identically distributed. Here, the constant $1/2$ can also be improved. Hill and Kertz [21] provided a family of "bad" instances from which Kertz [22] proved the largest possible bound one could expect is $1/\beta \approx 0.7451$, where β is the unique solution to $\int_0^1 \frac{1}{y(1-\ln(y))+(\beta-1)} dy = 1$. Quite surprisingly, $1/\beta$ is still the best known upper bound for the order selection problem, and was the best known upper bound for the random order case, prior to our work. Regarding algorithms, Hill and Kertz also proved a bound of $1-1/e$ which was improved by Abolhassani et al. [1] to 0.7380. Finally Correa et al. [8] proved that $1/\beta = 0.7451$ is a tight value. To this end they exhibit a quantile strategy for the gambler in which some quantiles $q_1 < \cdots < q_n$, that only depend on n (and not on the distribution), are precomputed and then translated into thresholds such that if the gambler gets to step i, she will stop with probability q_i.

Prophet inequalities have also been studied for combinatorial structures such as matroids, bipartite matchings, matroid intersection, general matchings, and combinatorial auctions [10,11,15–17,19,23,24]. Furthermore, data-driven versions of the problem where the gambler does not know the underlying distribution, but only has access to samples from them, have also been studied in recent years [4,6,29].

1.1 Our contribution

In this paper we study the prophet secretary problem and propose improved algorithms for it. In particular our work improves upon the recent work of Ehsani et al. [14], Azar et al. [3], and Beyhaghi et al. [5] by providing an algorithm that guarantees the gambler a fraction of $0.669 \approx 1 - 1/e + 1/27$ in the prophet secretary setting. Our main contribution however is not the actual numerical improvement but rather the way in which this is obtained. In addition, we provide an example that shows that no algorithm can achieve a factor better than $\sqrt{3}-1 \approx 0.732$ for the prophet secretary setting.

From a conceptual viewpoint we introduce a class of algorithms which we call *blind strategies*, that are very robust in nature. This type of algorithm is a clever generalization of the single threshold algorithm of Ehsani et al. to a multi-threshold setting. In their algorithm Ehsani et al. first compute a threshold τ such that $\mathbb{P}(\max\{V_1,\dots,V_n\} \leq \tau) = 1/e$ and then use this τ as a single threshold strategy, so that the gambler stops the first time in which the observed value surpasses τ. They observe that this strategy only works for random variables with continuous distributions, however they also note that by allowing randomization the strategy can be extended to general random variables. Rather than fixing a single probability of acceptance we fix a function $\alpha : [0,1] \to [0,1]$ which is used to define a sequence of thresholds in the following way. Given an instance with n continuous distributions, we draw uniformly and independently n random values in $[0,1]$, and reorder them as $u_{[1]} < \cdots < u_{[n]}$. Then we set thresholds $\tau_1,\dots,\tau_n$ such that $\mathbb{P}(\max\{V_1,\dots,V_n\} \leq \tau_i) = \alpha(u_{[i]})$ and the gambler stops at time i if $V_{\sigma_i} > \tau_i$. Note that if the function α is nonincreasing, this leads to a nonincreasing sequence of thresholds.

The idea of blind strategies comes from the i.i.d. case mentioned above. In that setting the blind strategies are indeed best possible as shown by Correa et al. [8]. What makes blind strategies attractive is that although decisions are time dependent, this dependence lies completely in the choice of the function α, which is independent of the instance. This independence significantly simplifies the analysis of multi-threshold strategies. Again, when facing discontinuous distributions we also require randomization for our results to hold.

From a technical standpoint our analysis introduces the use of Schur-convexity [28] in the prophet inequality setting. We start our analysis by revisiting the single threshold strategy of Ehsani et al., which corresponds to a constant blind strategy $\alpha(\cdot) = 1/e$. We exhibit a new analysis of this strategy, which allows us to prove a stochastic dominance type result. Indeed we prove that the probability that the gambler gets a value of more than t is at least that of the maximum being more than t, rescaled by a factor $1-1/e$. This result uses Schur-convexity to deduce that if there is a value above the threshold τ, then it is chosen by the gambler with probability at least $1-1/e$. Then we extend this analysis to deal with more general functions α which require precise bounds on the distribution of the stopping time corresponding to a function α. These bounds make use of results of Esfandiari et al. [13] and Azar et al. [3] and of newly derived bounds that follow from the core ideas in Schur-convexity theory.

Again in this more general setting we find an appropriate stochastic dominance type bound on the probability that the gambler obtains at least a certain amount with respect to the probability that the prophet obtains the same amount. Interestingly we manage to make the bound solely dependent on the blind strategy by controlling the implied stopping time distribution (patience of the gambler). Then optimizing over blind strategies leads to the improved bound of 0.669. Through the paper, we show two lower bounds on the performance of a blind strategy, the second more involved than the first one. In the first case, there is a natural way to optimize over the choice of α solving an ordinary differential equation, leading to a guarantee of 0.665. In the second case, using a refined analysis, we derive the stated bound of 0.669. Although it may seem that our general approach still leaves some room for improvement, we prove that blind strategies cannot lead to a factor better than 0.675. This bound is obtained by taking two carefully chosen instances and proving that no blind strategy can perform well in both.

Finally, we prove an upper bound on the performance of any algorithm: we construct an instance (which is not i.i.d.) in which no algorithm can perform better than $\sqrt{3}-1 \approx 0.732$. This improves upon the best possible bound known of 0.745 which corresponds to the i.i.d. case and was proved by Hill and Kertz [21]. Furthermore it improves and generalizes a recent bound of $11/15 \approx 0.733$ of Azar et al. [3] for the more restricted class of *deterministic distribution-insensitive algorithms*. Prior to our work, no separation between prophet secretary and i.i.d. prophet inequality was known.

1.2 Preliminaries and statement of results

Given nonnegative independent random variables $V_1, \ldots, V_n$ and a random permutation $\sigma : [n] \to [n]$,[1] in the prophet secretary problem a gambler is presented with the random variables in the order given by σ, i.e., at time j she sees the realization of V_{σ_j}. The goal of the gambler is to find a stopping time T such that $\mathbb{E}(V_{\sigma_T})$ is as large as possible. In particular we want to find the largest possible constant $c \in [0, 1]$ such that

$$\sup\{\mathbb{E}(V_{\sigma_T}) : T \in \mathcal{T}_n\} \geq c \cdot \mathbb{E}(\max\{V_1, \ldots, V_n\}),$$

where $\mathcal{T}_n$ is the set of stopping times. Note that the optimal stopping time can be computed recursively on n (*dynamic programming principle*). Nonetheless, in general this does not give an explicit expression of the reward given by such stopping time, and thus another approach is required to solve the above problem.

Throughout this paper we denote by $F_1, \ldots, F_n$ the underlying distributions of $V_1, \ldots, V_n$, which we assume to be continuous. All our results apply unchanged to the case of general distributions by introducing random tie-breaking rules (this is made precise in Sect. 8). To see why random tie-breaking rules are actually needed, consider the single threshold strategy of Ehsani et al. [14]. Recall that they compute a threshold τ such that $\mathbb{P}(\max\{V_1, \ldots, V_n\} \leq \tau) = 1/e$ and then use this τ as a single threshold strategy, which, by allowing random tie-breaking, leads to a performance of $1 - 1/e$. However, if random tie-breaking is not allowed, a single threshold strategy cannot achieve a constant better than $1/2$. Indeed, consider the instance with $n - 1$ deterministic random variables equal to 1 and one random variable giving n with probability $1/n$ and zero otherwise. Now, for a fixed threshold $\tau < 1$ the gambler gets n with probability $1/n^2$ and 1 otherwise so that she gets approximately 1, whereas if $\tau \geq 1$ the gambler gets n with probability $1/n$, leading to an expected value of 1. Noting that the expectation of the maximum in this instance is approximately 2, we conclude that fixed thresholds cannot achieve a guarantee better than $1/2$, for n tending to infinity. However, as Ehsani et al. note, if in this instance the gambler accepts a deterministic random variable with probability $1/n$, then her expected value

[1] Here $[n]$ denotes the set $\{1, \ldots, n\}$.

approaches $(1 + 1/e)$, which leads to a guarantee of $(1 + 1/e)/2 \simeq 0.68$. In Sect. 8 we extend this idea to the more general multi-threshold strategies.

The main type of stopping rules we deal with in this paper use a nonincreasing threshold approach. This is quite a natural idea, since Esfandiari et al. [13] already use such an approach to derive a guarantee of $1 - 1/e$. Interestingly, the analysis of multi-threshold strategies becomes rather difficult when trying to go beyond this bound. This is evident from the fact that the more recent results take a different approach. In this paper we use a rather restrictive class of multi-threshold strategies that we call *blind strategies*. These are simply given by a nonincreasing function $\alpha : [0, 1] \to [0, 1]$ which is turned into an algorithm as follows: given an instance $F_1, \dots, F_n$ of continuous distributions, we independently draw $u_1, \dots, u_n$ from a uniform distribution on [0, 1] and find thresholds $\tau_1, \dots, \tau_n$ such that

$$\mathbb{P}(\max_{i\in[n]}\{V_i\} \le \tau_i) = \alpha(u_{[i]}),$$

where $u_{[i]}$ is the ith order statistic of $u_1, \dots, u_n$. Then the algorithm for the gambler stops at the first time in which a value exceeds the corresponding threshold. In other words, the gambler applies the following algorithm:

Algorithm 1 Time Threshold Algorithm ($TTA(\tau_1, \dots, \tau_n)$)

```
for i = 1, ..., n do
    if V_{σ_i} > τ_i then
        Take V_{σ_i}
    end if
end for
```

Note that a blind strategy is uniquely determined by the choice of function α, independent of the actual distributions or even size of the instance. This justifies that we may simply talk about strategy α. Our goal is thus to find a *good* function α such that the previous algorithm performs well against any instance.

For a blind strategy α and an instance $F_1, \dots, F_n$, we will be interested in the underlying stopping time T which is the random variable defined as $T := \inf\{i \in [n] : V_{\sigma_i} > \tau_i\}$, where the $\tau_1, \dots, \tau_n$ are the corresponding thresholds. In particular the reward of the gambler is $V_{\sigma_T}\mathbb{1}_{T<\infty}$, which we simply denote by V_{σ_T}.
Our main result is the following:

Theorem 1.1 *There exists a nonincreasing function* $\alpha : [0, 1] \to [0, 1]$ *such that for any instance* $F_1, \dots, F_n$

$$\mathbb{E}(V_{\sigma_T}) \ge 0.669 \cdot \mathbb{E}(\max_{i\in[n]}\{V_i\}),$$

where T *is the stopping time of the blind strategy* α.

In addition, we prove the following upper bound on the performance of blind strategies:

Theorem 1.2 *No blind strategy can guarantee, for all instances, a constant better than 0.675.*

Theorem 1.3 *No strategy can guarantee a constant better than* $\sqrt{3}-1 \approx 0.732$*, as n goes to infinity.*

The rest of the paper is organized as follows. Section 2 presents an alternative simple proof that single threshold strategies guarantee a constant $1-1/e$, that will help the reader to understand the proof of Theorem 1.1. Section 3 explains how we analyze blind strategies. Section 4 proves that blind strategies guarantee a constant 0.665. Section 5 sharpens the analysis of Sect. 4 to prove Theorem 1.1. Sections 6 and 7 prove the upper bounds of Theorems 1.2 and 1.3. Section 8 explains how to deal with discontinuous distributions.

2 Single threshold

In the rest of the paper we assume for simplicity that $F_1, F_2, \ldots, F_n$ are continuous (see Sect. 8 for an explanation on how to extend the results to the discontinuous case). As a warm-up exercise, we illustrate the main ideas in this paper by providing an alternative proof of a recent result by Eshani et al. [14].

Proposition 2.1 *Consider the blind strategy given by* $\alpha \equiv 1/e$*, which corresponds exactly to the single threshold algorithm of Eshani et al.* [14]. *Then*

$$\mathbb{E}(V_{\sigma_T}) \geq \left(1-\frac{1}{e}\right)\cdot \mathbb{E}(\max_{i\in[n]}\{V_i\}).$$

Notice that, for a nonnegative random variable V we have that $\mathbb{E}(V)=\int_0^\infty \mathbb{P}(V>t)dt$. Thus, having a stopping time T such that $\mathbb{P}(V_{\sigma_T}>t)\geq c\cdot\mathbb{P}(\max_{i\in[n]}\{V_i\}>t)$, implies that c is a lower bound for the prophet inequality. Therefore, we will focus on deriving stochastic dominance type results. The general structure of our approach follows three steps: first, we express $\mathbb{P}(V_{\sigma_T}>t)$ in terms of a weighted sum of $(\mathbb{P}(V_i>t))_{1\leq i\leq n}$. In a second step, we bound from below the weights by an expression involving only α. In a third step, we optimize over the choice of α, to obtain the desired stochastic dominance inequality.

Step 1: decomposition Recall that given an instance $V_1,\ldots,V_n$, the blind strategy $\alpha\equiv p\in[0,1]$ first computes τ such that $\mathbb{P}(\max_{i\in[n]}\{V_i\}\leq\tau)=p$ and then uses $TTA(\tau_1=\tau,\ldots,\tau_n=\tau)$, which simply stops the first time a value above τ is observed.

Lemma 2.1 *Let* $t\geq 0$.

$$\mathbb{P}(V_{\sigma_T}>t)=\begin{cases}1-p; & t\leq\tau\\ \sum_{i\in[n]}\mathbb{P}(V_i>t)\mathbb{P}(\sigma_T=i|V_i>\tau); & t>\tau\,.\end{cases}$$

Proof Note that for $t\leq\tau$, we have that $\mathbb{P}(V_{\sigma_T}>t)=\mathbb{P}(\max_{i\in[n]}\{V_i\}>\tau)=1-p$.

On the other hand, for $t > \tau$, we have that

$$\begin{aligned}\mathbb{P}(V_{\sigma_T} > t) &= \sum_{i\in[n]} \mathbb{P}(V_i > t|\sigma_T = i)\mathbb{P}(\sigma_T = i)\\ &= \sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{\mathbb{P}(V_i > \tau)}\mathbb{P}(\sigma_T = i)\\ &= \sum_{i\in[n]} \mathbb{P}(V_i > t)\mathbb{P}(\sigma_T = i|V_i > \tau).\end{aligned}$$

The second equality stems from the independence of the V_i. □

Step 2: Lower bounds and Schur-convexity The novelty of this lower bound relies on the use of Schur-convexity.

Lemma 2.2 *Consider $V_1, \ldots, V_n$ independent random variables and σ an independent random uniform permutation of $[n]$. Consider $\alpha \equiv p \in (0, 1)$ and let τ be defined by $\mathbb{P}(\max_{i\in[n]}\{V_i\} \leq \tau) = p$. Then, with T the stopping time of $TTA(\tau_1 = \tau, \ldots, \tau_n = \tau)$, we have that for all i such that $\mathbb{P}(V_i > \tau) > 0$*

$$\mathbb{P}(\sigma_T = i|V_i > \tau) \geq \frac{1-p}{-\ln p}.$$

Proof Denoting the distribution of V_j by F_j, using the fact that σ is a uniform random order and the definition of τ, we get the following:

$$\begin{aligned}\mathbb{P}(\sigma_T = i|V_i > \tau) &= \sum_{S\subseteq[n]\setminus\{i\}} \mathbb{P}(\sigma_T = i, \{j : V_j > \tau\} = S \cup \{i\}|V_i > \tau)\\ &= \sum_{S\subseteq[n]\setminus\{i\}} \mathbb{P}(\{j : V_j > \tau\} = S \cup \{i\}, \sigma_i \leq \min_{j\in S} \sigma_j|V_i > \tau)\\ &= \sum_{S\subseteq[n]\setminus\{i\}} \mathbb{P}(\{j : V_j > \tau\} = S \cup \{i\}, \sigma_i \leq \min_{j\in S} \sigma_j)\\ &= \sum_{S\subseteq[n]\setminus\{i\}} \frac{1}{|S|+1} \prod_{j\in S} 1 - F_j(\tau) \prod_{j\in[n]\setminus(S\cup\{i\})} F_j(\tau)\\ &= \prod_{j\in[n]\setminus\{i\}} F_j(\tau) \sum_{S\subseteq[n]\setminus\{i\}} \frac{1}{|S|+1} \prod_{j\in S} \frac{1-F_j(\tau)}{F_j(\tau)}\\ &= \frac{p}{F_i(\tau)} \sum_{S\subseteq[n]\setminus\{i\}} \frac{1}{|S|+1} \prod_{j\in S} \frac{1-F_j(\tau)}{F_j(\tau)}.\end{aligned}$$

Define $\phi : \mathbb{R}_+^{n-1} \longrightarrow \mathbb{R}$ by

$$\phi(y) := \sum_{S\subseteq[n-1]} \frac{1}{|S|+1} \prod_{j\in S} \frac{1-e^{-y_j}}{e^{-y_j}} = \sum_{S\subseteq[n-1]} \frac{1}{|S|+1} \prod_{j\in S} e^{y_j} - 1$$

and $\beta := -\ln p + \ln F_i(\tau)$. The previous inequality can be written as

$$\mathbb{P}(\sigma_T = i | V_i > \tau) \geq \frac{p}{F_i(\tau)} \phi(y),$$

where $y = (-\ln F_j(\tau))_{j \neq i}$. Because

$$\sum_{j \neq i} -\ln F_j(\tau) = -\ln \prod_{j \neq i} F_j(\tau) + \ln F_i(\tau) = \beta,$$

we deduce that

$$\mathbb{P}(\sigma_T = i | V_i > \tau) \geq \frac{p}{F_i(\tau)} \min_{y \in \mathbb{R}_+^{n-1}} \left\{ \phi(y) : \sum_{j \in [n-1]} y_j = \beta \right\}.$$

Clearly $\phi \in \mathcal{C}^\infty((0,1)^{n-1}; \mathbb{R})$ and is permutation symmetric. Therefore, to check that it is Schur-convex we must simply confirm the following condition, known as the Schur–Ostrowski criterion [28],

$$\forall y \in (0,1)^{n-1} \quad (y_1 - y_2)[\partial_{y_1} \phi(y) - \partial_{y_2} \phi(y)] \geq 0.$$

Straightforward calculations yield

$$\begin{aligned}
\partial_{y_1} \phi(y) &= \sum_{\substack{S \subseteq [n-1] \\ S \ni 1}} \frac{1}{|S|+1} \quad e^{y_1} \prod_{\substack{j \in S \\ j \neq 1}} e^{y_j} - 1 \\
&= e^{y_1}(e^{y_1} - 1)^{-1} \left(\sum_{\substack{S \subseteq [n-1] \\ S \ni 1,2}} \frac{1}{|S|+1} \quad \prod_{j \in S} e^{y_j} - 1 \right) \\
&\quad + \frac{e^{y_1}(e^{y_1} - 1)^{-1}}{(e^{y_2} - 1)} \left(\sum_{\substack{S \subseteq [n-1] \\ S \ni 1,2}} \frac{1}{|S|} \quad \prod_{j \in S} e^{y_j} - 1 \right) \\
&=: \frac{e^{y_1}}{e^{y_1} - 1} a + \frac{e^{y_1}}{(e^{y_1} - 1)(e^{y_2} - 1)} b
\end{aligned}$$

and, by symmetry, $\partial_{y_2} \phi(y) = \frac{e^{y_2}}{e^{y_2}-1} a + \frac{e^{y_2}}{(e^{y_2}-1)(e^{y_1}-1)} b$. Then,

$$
\begin{aligned}
[\partial_{y_1}\phi(y) - \partial_{y_2}\phi(y)] &= a\left[\frac{e^{y_1}}{e^{y_1}-1} - \frac{e^{y_2}}{e^{y_2}-1}\right] \\
&\quad + b\left[\frac{e^{y_1}}{(e^{y_1}-1)(e^{y_2}-1)} - \frac{e^{y_2}}{(e^{y_2}-1)(e^{y_1}-1)}\right] \\
&= (b-a)\left[\frac{e^{y_1}-e^{y_2}}{(e^{y_2}-1)(e^{y_1}-1)}\right].
\end{aligned}
$$

Finally, since $e^{y_j}-1>0$ and $b>a$, we get that $(y_1-y_2)[\partial_{y_1}\phi(y)-\partial_{y_2}\phi(y)]\geq 0$ if and only if $(y_1-y_2)\,(e^{y_1}-e^{y_2})\geq 0$, which holds by monotonicity of the exponential function. Therefore we have proven that ϕ is Schur-convex.

Schur-convexity readily implies that the optimization problem $\min\left\{\phi(y);\ s.t. \sum_{j\in[n-1]} y_j=\beta\right\}$ is solved at $y^*=(\beta/(n-1),\ldots,\beta/(n-1))$. Consequently, for fixed $F_i(\tau)$, and under the constraint that $\prod_{j\in[n]\setminus\{i\}} F_j(\tau)=p/F_i(\tau)$, the quantity $\mathbb{P}(\sigma_T=i|V_i>\tau)$ is minimal when, for all $j\neq i$, $F_j(\tau)=(p/F_i(\tau))^{\frac{1}{n-1}}$. It follows that, since σ and V_i are independent,

$$
\begin{aligned}
\mathbb{P}(\sigma_T=i|V_i>\tau) &= \frac{1}{n}\sum_{j=1}^{n}\mathbb{P}(\sigma_T=i|V_i>\tau,\sigma_j=i) \\
&\geq \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{p}{F_i(\tau)}\right)^{\frac{j}{n-1}} \\
&\geq \frac{1}{n}\sum_{j=0}^{n-1} p^{\frac{j}{n-1}} = \frac{1}{n}\frac{1-p^{\frac{n}{n-1}}}{1-p^{\frac{1}{n-1}}}.
\end{aligned}
$$

Now we note that the left hand side does not depend on n: we can add some dummy variables $(V_{n+1}, V_{n+2},\ldots\equiv 0)$ and the probability does not change. Therefore, taking limit on $n\longrightarrow\infty$ we get

$$
\mathbb{P}(\sigma_T=i|V_i>\tau)\geq\frac{1-p}{-\ln p}.
$$

□

Step 3: optimization and proof of Proposition 2.1 The previous parts imply the following result:

Lemma 2.3

$$
\mathbb{P}(V_{\sigma_T}>t)\geq\min\left\{1-p,\frac{1-p}{-\ln p}\right\}\cdot\mathbb{P}(\max_{i\in[n]}\{V_i\}>t)
$$

Proof The inequality stems from 2.1, Lemma 2.2 and the inequality $\sum_{i\in[n]}\mathbb{P}(V_i>t)\geq\mathbb{P}(\max_{i\in[n]}\{V_i\}>t)$. □

We now turn to optimize the choice of $\alpha \equiv p \in (0, 1)$. The optimum is obtained by taking $p = 1/e$, which gives Proposition 2.1.

3 Reduction to deterministic blind strategies

This section reduces the analysis of blind strategies to a simpler class of strategies, and constitutes a preliminary step towards proving Theorem 1.1.

Recall that a blind strategy is determined by a nonincreasing function $\alpha : [0, 1] \to [0, 1]$ which is turned into an algorithm as follows: given an instance $F_1, \ldots, F_n$ of continuous distributions, we independently draw $u_1, \ldots, u_n$ from a uniform distribution on $[0, 1]$ and find thresholds $\tau_1, \ldots, \tau_n$ such that

$$\mathbb{P}(\max_{i \in [n]}\{V_i\} \leq \tau_i) = \alpha(u_{[i]}),$$

where $u_{[i]}$ is the ith order statistic of $u_1, \ldots, u_n$. Then the algorithm for the gambler is to apply $TTA(\tau_1, \ldots, \tau_n)$. Therefore, to analyze blind strategies, we would have to take into account randomness from $u_1, \ldots, u_n$ and $V_1, \ldots, V_n$.

Consider a deterministic version of a blind strategy defined as follows.

Definition 3.1 Consider a nonincreasing function $\alpha : [0, 1] \to [0, 1]$. The deterministic blind strategy given by α, upon the instance $F_1, \ldots, F_n$, is the strategy that applies $TTA(\tau_1, \ldots, \tau_n)$, where the thresholds $\tau_1, \ldots, \tau_n$ are defined by the following condition:

$$\forall j \in [n], \quad \mathbb{P}(\max_{i \in [n]}\{V_i\} \leq \tau_j) = \alpha\left(\frac{j}{n}\right).$$

Given a nonincreasing function $\alpha : [0, 1] \to [0, 1]$, the difference between its corresponding deterministic blind strategy and blind strategy is that $u_{[i]} = i/n$ for the deterministic case, instead of being drawn from a uniform distribution between 0 and 1. The following lemma shows that in order to analyze the performance of blind strategies, it is sufficient to study the performance of deterministic blind strategies in the limit as n grows to infinity.

Lemma 3.1 *For any nonincreasing function* $\alpha : [0, 1] \to [0, 1]$,

$$\inf_{F_1,\ldots,F_n} \frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}(\max_{i \in [n]}\{V_i\})} \geq \lim_{n\to\infty} \inf_{F_1,\ldots,F_n} \frac{\mathbb{E}(V_{\sigma_{T_d}})}{\mathbb{E}(\max_{i \in [n]}\{V_i\})},$$

where T (respectively T_d) are the stopping times given by the blind strategy (respectively deterministic blind strategy) defined by α.

Proof Consider a nonincreasing function $\alpha : [0, 1] \to [0, 1]$, an instance of size n given by $F_1, \ldots, F_n$ and $m \geq 1$. Consider the new instance of size $n + m$ given by $F_1, \ldots, F_n, F_{n+1}, \ldots, F_{n+m}$, where $F_{n+i} = \mathbb{1}_{[0,\infty)}$ for $i = 1, \ldots, m$, i.e. : now there are m deterministic random variables equal to zero. Denoting by

T_m the stopping time given by the deterministic blind strategy α upon the instance $F_1, \ldots, F_n, F_{n+1}, \ldots, F_{n+m}$, we claim that

$$\lim_{m\to\infty} \mathbb{E}(V_{\sigma_{T_m}}) = \mathbb{E}(V_{\sigma_T}).$$

Indeed, recalling the definition of blind strategies in Sect. 1.2, it is easy to see that a deterministic blind strategy applied to instance $F_1, \ldots, F_{n+m}$ is approximately a blind strategy applied to instance $F_1, \ldots, F_n$, where the random variables $u_1, \ldots, u_n$ are drawn from $U(\{\frac{1}{n+m}, \ldots, 1\})$, rather than from $U(0, 1)$. Thus, by taking the limit as $m \to \infty$, the lemma follows. $\square$

4 Beating $1 - \frac{1}{e}$

In this section, we prove the following proposition, that is a weaker version of Theorem 1.1 (the constant is 0.665 instead of 0.669):

Proposition 4.1 *There exists a nonincreasing function* $\alpha : [0, 1] \to [0, 1]$ *such that*

$$\mathbb{E}(V_{\sigma_T}) \geq 0.665 \cdot \mathbb{E}(\max_{i\in[n]}\{V_i\}),$$

where T *is the stopping time of the blind strategy* α.

We first present this result because it already beats significantly the best known constant in the literature, and it is simpler to prove than Theorem 1.1. As in Sect. 2, we proceed in three steps.

Step 1: Decomposition

Lemma 4.1 *Given an instance* $F_1, F_2, \ldots, F_n$ *and nonincreasing thresholds* $\infty = \tau_0 \geq \tau_1 \geq \cdots \geq \tau_n \geq \tau_{n+1} = -\infty$, *it holds that, for* $j \in [n+1]$ *and* $t \in [\tau_j, \tau_{j-1})$,

$$\mathbb{P}(V_{\sigma_T} > t) = \mathbb{P}(T \leq j-1) + \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=j}^{n} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{n} \right),$$

where T *is the stopping time given by* $TTA(\tau_1, \ldots, \tau_n)$.

Proof Let $j \in [n+1]$ and $t \in [\tau_j, \tau_{j-1})$. Then, since thresholds are nonincreasing,

$$\mathbb{P}(V_{\sigma_T} > t) = \mathbb{P}(T \leq j-1) + \mathbb{P}(V_{\sigma_T} > t, T \geq j).$$

Therefore, we must simply analyze the second term.

By partitioning among all possible values of σ_T, and then of T, we notice that:

$$\begin{aligned}\mathbb{P}(V_{\sigma_T} > t, T \geq j) &= \sum_{i\in[n]} \mathbb{P}(V_i > t, \sigma_T = i, T \geq j)\\ &= \sum_{i\in[n]} \sum_{k=j}^{n} \mathbb{P}(V_i > t, \sigma_k = i, T = k).\end{aligned}$$

Notice that, since $t \in [\tau_j, \tau_{j-1})$, we have that for all $k \geq j$, if $V_i > t$ and $\sigma_k = i$, then $T \leq k$. Therefore,

$$\mathbb{P}(V_i > t, \sigma_k = i, T = k) = \mathbb{P}(V_i > t, \sigma_k = i, T \geq k).$$

Define $\Sigma_{-i}(k) := \{\sigma,$ ordered subset of $[n]\backslash\{i\}$ with size $k\}$. Then, by partitioning over all possible realizations of $\sigma_1, \ldots, \sigma_{k-1}$, and using the independence of $V_1, \ldots, V_n, \sigma$, we have that

$$\begin{aligned}&\mathbb{P}(V_i > t, \sigma_k = i, T \geq k)\\ &\quad= \frac{1}{|\Sigma_{-i}(k-1)|} \sum_{\sigma\in\Sigma_{-i}(k-1)} \mathbb{P}(V_i > t, \sigma_k = i, V_{\sigma_1} \leq \tau_1, \ldots, V_{\sigma_{k-1}} \leq \tau_{k-1})\\ &\quad= \frac{1}{|\Sigma_{-i}(k-1)|} \sum_{\sigma\in\Sigma_{-i}(k-1)} \mathbb{P}(V_i > t)\mathbb{P}(\sigma_k = i, V_{\sigma_1} \leq \tau_1, \ldots, V_{\sigma_{k-1}} \leq \tau_{k-1})\\ &\quad= \mathbb{P}(V_i > t)\mathbb{P}(\sigma_k = i, T \geq k).\end{aligned}$$

Using all previous equalities, and the fact that σ is uniform, we conclude that

$$\begin{aligned}\mathbb{P}(V_{\sigma_T} > t, T \geq j) &= \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=j}^{n} \mathbb{P}(\sigma_k = i, T \geq k) \right)\\ &= \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=j}^{n} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{n} \right).\end{aligned}$$

□

Step 2: Lower bounds and weak Schur convexity In this section, we give lower bounds on the terms that appear in Lemma 4.1, namely $\mathbb{P}(T \leq k)$ and $\mathbb{P}(T \geq k|\sigma_k = i)$, $i, k \in [n]$. We start with the following simple inequality:

Lemma 4.2 *For all* $i, k \in [n]$,

$$\mathbb{P}(T \geq k|\sigma_k = i) \geq \mathbb{P}(T > k). \tag{4.1}$$

Proof Inspired by the proof given by Esfandiari et al. [13], let $i, k \in [n]$. Conditioning on the value of $\sigma^{-1}(i)$,

$$\mathbb{P}(T > k) = \frac{1}{n} \sum_{l \in [n]} \mathbb{P}(T > k | \sigma_l = i). \tag{4.2}$$

Notice that for all $l = k+1, \ldots, n$,

$$\mathbb{P}(T > k | \sigma_l = i) = \mathbb{P}(T > k | \sigma_{k+1} = i) \leq \mathbb{P}(T \geq k | \sigma_k = i).$$

Define

$$\Sigma_{-i}(k) := \{\sigma, \text{ ordered subset of } [n] \backslash \{i\} \text{ with size } k\}.$$

Then, for $l \in [k]$, since the thresholds are nonincreasing,

$$\begin{aligned} \mathbb{P}(T > k | \sigma_l = i) &= \frac{\mathbb{P}(V_i \leq \tau_l)}{|\Sigma_{-i}(k-1)|} \sum_{\sigma \in \Sigma_{-i}(k-1)} \prod_{j \in [k-1]} F_{\sigma_j}(\tau_{j + \mathbb{1}_{j \geq l}}) \\ &\leq \frac{1}{|\Sigma_{-i}(k-1)|} \sum_{\sigma \in \Sigma_{-i}(k-1)} \prod_{j \in [k-1]} F_{\sigma_j}(\tau_j) \\ &= \mathbb{P}(T \geq k | \sigma_k = i). \end{aligned}$$

Plugging both inequalities back into Eq. (4.2) give the result. □

We can now bound from below $\mathbb{P}(V_{\sigma_T} > t)$ by a quantity that depends only on the cumulative distribution of T and on α.

Proposition 4.2 *Let $\alpha : [0, 1] \to [0, 1]$ be nonincreasing, and let T be the deterministic blind strategy stopping time. For every instance $F_1, \ldots, F_n$ and $t > 0$,*

$$\mathbb{P}(V_{\sigma_T} > t) \geq \min_{j \in [n+1]} \left\{ \frac{\mathbb{P}(T \leq j-1)}{1 - \alpha(\frac{j}{n})} + \left(\frac{1}{n} \sum_{k=j}^{n} \mathbb{P}(T > k) \right) \right\} \mathbb{P}(\max_{i \in [n]}\{V_i\} > t),$$

where $\alpha(\frac{n+1}{n}) = 0$.

Proof Note that for all $j \in [n+1]$ and $t \in [\tau_j, \tau_{j-1})$, $1 - \alpha(j/n) = \mathbb{P}(\max_{i \in [n]}\{V_i\} > \tau_j) \geq \mathbb{P}(\max_{i \in [n]}\{V_i\} > t)$. Plugging this inequality and inequalities $\sum_{i \in [n]} \mathbb{P}(V_i > t) \geq \mathbb{P}(\max_{i \in [n]}\{V_i\} > t)$ and (4.1) into Lemma 4.1 yield the result. □

To achieve a lower bound from this point, we need to have a lower and an upper bound of the distribution of T in terms of $\{\alpha(j/n)\}_{j \in [n+1]}$. It turns out that $\mathbb{P}(T \leq \cdot)$ is maximized and minimized in opposite instances: the maximum is achieved when there is only one non-zero variable, while the minimum is achieved when all distributions are equal. This behavior is closely related to Schur-convexity.

Recall that in the proof of Lemma 2.2 we solved the following optimization problem: $\min \left\{ \phi(y);\ s.t. \sum_{j\in[n-1]} y_j = \beta \right\}$. This problem was solved by noticing that ϕ is Schur-convex. This time we consider the problem

$$\begin{cases} \text{opt } \mathbb{P}(T > k; F_1, \ldots, F_n) \\ s.t. \prod_{i\in[n]} F_i = F \text{ and } F_i \text{ is a distribution.} \end{cases}$$

where "opt" is a symbol in {min, max}. This problem is harder since it involves optimizing over functions rather than real numbers. Trying to apply Schur-convexity theory again, one could see $\mathbb{P}(T > k; F_1, \ldots, F_n)$ as a function of the distributions evaluated at each threshold, that is, as a function of the vector $(F_1(\tau_1), \ldots, F_1(\tau_n), \ldots, F_n(\tau_1), \ldots, F_n(\tau_n))$. Unfortunately, the corresponding domain is not symmetric and the constraint on the product being constant results in n different constraints. Both difficulties are not addressed in the Schur-convexity literature. However, we are only interested in the value of the optimization problem. Therefore, weaker properties than Schur-convexity can solve our problem.

The following lemma formalizes the idea that $\mathbb{P}(T > k; F_1, \ldots, F_n)$, as a function of $F_1, \ldots, F_n$, increases upon concentration and decreases upon homogenization.

Lemma 4.3 *Fix* $\alpha_1, \ldots, \alpha_n \in [0, 1]$. *For every instance* $F_1, \ldots, F_n$*, consider* $\tau_1, \ldots, \tau_n$ *the sequence of thresholds such that*

$$\mathbb{P}(\max_{i\in[n]}\{V_i\} \leq \tau_i) = \alpha_i.$$

Denote by T *the stopping time of* $TTA(\tau_1, \ldots, \tau_n)$. *For all* $k \in [n]$*, we have*

$$\frac{1}{n}\sum_{j\in[k]}(1-\alpha_j) \leq \mathbb{P}(T \leq k) \leq 1 - \left(\prod_{j=1}^{k}\alpha_j\right)^{\frac{1}{n}}.$$

Proof Despite that $\mathbb{P}(T > k; F_1, \ldots, F_n)$ is not always monotone along the curve $\lambda \in [0, 1] \mapsto (F_1F_2^{\lambda}, F_2^{1-\lambda}, \ldots, F_n)$, a property that would be satisfied by a log-Schur-convex function if $F_1, \ldots, F_n$ were numbers, there is a step by step way to go from $(F_1, F_2, \ldots, F_n)$ to $(F_1F_2 \ldots F_n, \mathbb{1}_{\mathbb{R}_+}, \ldots, \mathbb{1}_{\mathbb{R}_+})$ that exhibits a monotonic behavior, while maintaining the product. This property is enough to deduce the upper bound. For the lower bound, there is a step by step monotonic way to go from $(F_1, F_2, \ldots, F_n)$ to $(\sqrt[n]{F_1F_2 \ldots F_n}, \ldots, \sqrt[n]{F_1F_2 \ldots F_n})$.

The proof consists in describing this step by step process. Specifically, we highlight the role of F_1 and F_2 in $\mathbb{P}(T > k)$ in order to compare the instance $F_1, \ldots, F_n$ with the instances given by $F_1F_2, \mathbb{1}_{\mathbb{R}_+}, F_3, \ldots, F_n$ and $\sqrt{F_1F_2}, \sqrt{F_1F_2}, F_3, \ldots, F_n$. Then, we use the symmetry of the random order σ and a step by step iteration to derive the claimed inequalities.

We start by introducing a decomposition in three cases:

1. $\sigma^{-1}(1) \leq k \veebar \sigma^{-1}(2) \leq k$, i.e. : only one of the variables V_1 and V_2 shows before time k.

2. $\sigma^{-1}(1) \leq k \wedge \sigma^{-1}(2) \leq k$, i.e. : both V_1 and V_2 show before time k.
3. $\sigma^{-1}(1) > k \wedge \sigma^{-1}(2) > k$, i.e. : neither V_1 nor V_2 shows before time k.

To express this formally, define

$$\begin{aligned}\Sigma(k) &:= \{\sigma,\ \text{ordered subset of } [n] \text{ with size } k\}\\ \Sigma_{-1,-2}(k) &:= \{\sigma,\ \text{ordered subset of } [n]\backslash\{1,2\} \text{ with size } k\}.\end{aligned}$$

For $\sigma \in \Sigma(k)$, we have that either

1. $\exists p \in [k], \exists i \in \{1,2\}$ s.t. $\quad \sigma_p = i$ and $(\sigma_j)_{j\in[k]\backslash\{p\}} \in \Sigma_{-1,-2}(k-1)$.
2. $\exists p < q \in [k]$ s.t. $\quad \{\sigma_p, \sigma_q\} = \{1,2\}$ and $(\sigma_j)_{j\in[k]\backslash\{p,q\}} \in \Sigma_{-1,-2}(k-2)$.
3. $\sigma \in \Sigma_{-1,-2}(k)$.

In what follows, we write $\mathbb{P}(T > k;\, F_1, \ldots, F_n)$ for the probability that T is strictly larger than k, given that the instance is $F_1, \ldots, F_n$. Using the previous decomposition, we can derive the following identity:

$$\begin{aligned}\mathbb{P}(T > k;\, F_1, \ldots, F_n) &= \frac{1}{|\Sigma(k)|} \sum_{\sigma\in\Sigma(k)} \prod_{i\in[k]} F_{\sigma_i}(\tau_i)\\ &= \frac{(n-k)!}{n!} \Bigg(\sum_{\sigma\in\Sigma_{-1,-2}(k)} \prod_{i\in[k]} F_{\sigma_i}(\tau_i)\\ &\quad + \sum_{\substack{\sigma\in\Sigma_{-1,-2}(k-1)\\ p\in[k]}} \prod_{i=1}^{p-1} F_{\sigma_i}(\tau_i) \left[F_1(\tau_p) + F_2(\tau_p)\right] \prod_{i=p}^{k-1} F_{\sigma_i}(\tau_{i+1})\\ &\quad + \sum_{\substack{\sigma\in\Sigma_{-1,-2}(k-2)\\ p<q\in[k]}} \prod_{i=1}^{p-1} F_{\sigma_i}(\tau_i) \big[F_1(\tau_p)F_2(\tau_q)\\ &\quad + F_2(\tau_p)F_1(\tau_q)\big] \prod_{i=p}^{q-1} F_{\sigma_i}(\tau_{i+1}) \prod_{i=q}^{k-2} F_{\sigma_i}(\tau_{i+2}) \Bigg).\end{aligned}$$

To simplify the notation, let us define

$$A(F_1, F_2) := \sum_{\substack{\sigma\in\Sigma_{-1,-2}(k-1)\\ p\in[k]}} \left[F_1(\tau_p) + F_2(\tau_p)\right] \prod_{i\in[k-1]} F_{\sigma_i}(\tau_{i+\mathbb{1}_{i\geq p}}),$$

$$B(F_1, F_2) := \sum_{\substack{\sigma \in \Sigma_{-1,-2}(k-2) \\ p<q \in [k]}} \left[F_1(\tau_p)F_2(\tau_q) + F_2(\tau_p)F_1(\tau_q)\right] \prod_{i \in [k-2]} F_{\sigma_i}(\tau_{i+\mathbb{1}_{i \geq p}+\mathbb{1}_{i \geq q}}),$$

$$C := \sum_{\sigma \in \Sigma_{-1,-2}(k)} \prod_{i \in [k]} F_{\sigma_i}(\tau_i).$$

Then,

$$\mathbb{P}(T > k; F_1, \ldots, F_n) = \frac{(n-k)!}{n!} \left[A(F_1, F_2) + B(F_1, F_2) + C\right].$$

The next step in the proof is to show that

$$A(\sqrt{F_1 F_2}, \sqrt{F_1 F_2}) \leq A(F_1, F_2) \leq A(F_1 F_2, \mathbb{1}_{\mathbb{R}_+})$$

and

$$B(\sqrt{F_1 F_2}, \sqrt{F_1 F_2}) \leq B(F_1, F_2) \leq B(F_1 F_2, \mathbb{1}_{\mathbb{R}_+}).$$

This is a consequence of the two following points:

1. For all $p \in [k]$,

$$2\sqrt{F_1(\tau_p)F_2(\tau_p)} \leq F_1(\tau_p) + F_2(\tau_p) \leq 1 + F_1(\tau_p)F_2(\tau_p).$$

2. For all $p < q \in [k]$,

$$\begin{aligned} 2\sqrt{F_1(\tau_p)F_2(\tau_p)F_1(\tau_q)F_2(\tau_q)} &\leq F_1(\tau_p)F_2(\tau_q) + F_2(\tau_p)F_1(\tau_q) \\ &\leq F_1(\tau_p)F_2(\tau_p) + F_2(\tau_q)F_1(\tau_q). \end{aligned}$$

Thus, we have proven that

$$\begin{aligned} &\mathbb{P}(T > k; \sqrt{F_1 F_2}, \sqrt{F_1 F_2}, F_3, \ldots, F_n) \leq \mathbb{P}(T > k; F_1, \ldots, F_n) \\ &\quad \leq \mathbb{P}(T > k; F_1 F_2, \mathbb{1}_{\mathbb{R}_+}, \ldots, F_n). \end{aligned}$$

The lower bound of Lemma 4.3 is derived by applying the inequality n times and noticing that

$$\mathbb{P}\left(T \leq k; \prod_{i \in [n]} F_i, \mathbb{1}_{\mathbb{R}_+}, \ldots, \mathbb{1}_{\mathbb{R}_+}\right) = \frac{1}{n} \sum_{j \in [k]} (1 - \alpha_j).$$

The upper bound of Lemma 4.3 follows from applying the inequality infinitely many times and noticing that

$$\mathbb{P}\left(T \leq k; \prod_{i\in[n]} F_i^{\frac{1}{n}}, \ldots, \prod_{i\in[n]} F_i^{\frac{1}{n}}\right) = 1 - \prod_{j\in[k]} \alpha_j^{\frac{1}{n}}.$$

□

Step 3: Optimization and proof of Theorem 4.1 The previous analysis gives a lower bound on $\mathbb{P}(V_{\sigma_T} > t)$ that only depends on α:

Theorem 4.4 *Let $\alpha : [0,1] \to [0,1]$ be nonincreasing, and let T be the deterministic blind strategy stopping time. For every instance $F_1, \ldots, F_n$ and $t > 0$,*

$$\mathbb{P}(V_{\sigma_T} > t) \geq \min_{j\in[n+1]} \left\{ f_j(\alpha) \right\} \cdot \mathbb{P}(\max_{i\in[n]}\{V_i\} > t),$$

where, for all $j \in [n+1]$, taking $\alpha(\frac{n+1}{n}) = 0$,

$$f_j(\alpha) = \sum_{k=1}^{j-1} \frac{1-\alpha\left(\frac{k}{n}\right)}{n\left(1-\alpha\left(\frac{j}{n}\right)\right)} + \frac{1}{n}\sum_{k=j}^{n}\left(\prod_{l=1}^{k}\alpha\left(\frac{l}{n}\right)\right)^{\frac{1}{n}}.$$

Proof This is a direct consequence of Lemma 4.1 and the lower bounds of Lemmas 4.2 and 4.3, as well as the inequality $\sum_{i\in[n]} \mathbb{P}(V_i > t) \geq \mathbb{P}(\max_{i\in[n]}\{V_i\} > t)$. □

Thus, for every n, we get a lower bound on the performance of a deterministic blind strategy α, that only depends on $\alpha(\frac{1}{n}), \ldots, \alpha(\frac{n}{n})$. As we explained before, we only care about the performance of this strategy when n tends to $+\infty$. Assume that α is continuous. A standard Riemann sum analysis shows that

$$\begin{aligned} &\lim_{n\to\infty} \min_{j\in[n+1]} \{f_j(\alpha)\} \\ &= \min\left\{ \int_0^1 1-\alpha(y)dy\,,\ \inf_{x\in[0,1]} \int_0^x \frac{1-\alpha(y)}{1-\alpha(x)}dy + \int_x^1 e^{\int_0^y \ln\alpha(w)dw}dy \right\}. \end{aligned} \tag{4.3}$$

Thus, in order to prove Theorem 1.1, we would like to find a blind strategy α maximizing the latter expression. As this is a nontrivial optimal control problem, we aim at finding a function α such that the above expression is larger than 0.665.

Remark Consider α being constant equal to $1/e$. Then the above quantity is equal to $1-1/e$. Thus, we recover the one-threshold result in Proposition 2.1. Furthermore, if for instance we take $\alpha(x) = 0.53 - 0.38x$ the guarantee of the strategy (given by expression (4.3)) is greater than 0.657. This gives an explicit α that beats significantly $1-1/e$.

To maximize over expression (4.3), we resort to a numerical approximation. Note that if α is such that $\alpha(1) = 0$ and $x \mapsto \int_0^x \frac{1-\alpha(y)}{1-\alpha(x)} dy + \int_x^1 e^{\int_0^y \ln \alpha(w) dw} dy$ is a constant, then this constant is a lower bound for the infimum in (4.3).

Consequently, we solve the following integro-differential equation:

$$\begin{cases} \frac{d}{dx}\left(\int\limits_0^x \frac{1-\alpha(y)}{1-\alpha(x)} dy + \int\limits_x^1 e^{\int\limits_0^y \ln \alpha(w) dw} dy \right) = 0; \quad x \in (0,1) \\ \alpha(1) = 0. \end{cases}$$

To this end we consider a change of variables leading to the following second order ODE:

$$\begin{cases} (u'(x))^2 K(x,u) - u''(x)u(x) = 0; \quad x \in (0,1) \\ u'(1) = 1 \\ u(0) = 0, \end{cases}$$

where

$$u(x) := \int\limits_0^x 1 - \alpha(x) dx$$

$$K(x,u) := 1 - \exp\left(\int\limits_0^x \ln(1 - u'(t)) dt \right).$$

We approximately solved this equation by taking an initial guess u_0 and defining u_{n+1} as the solution to $(u'(x))^2 K(x, u_n) - u''(x)u(x) = 0$. To be more precise, the initial guess u_0 was the result of maximizing over α $\min_{j \in [n+1]}\{f_j(\alpha)\}$, given in Theorem 4.4, for $n = 23$. Then, we iterated the process eleven times and obtained an α with $\alpha(1) = 0$ and such that the function $x \mapsto \int_0^x \frac{1-\alpha(y)}{1-\alpha(x)} dy + \int_x^1 \exp\left(\int_0^y \ln \alpha(w) dw\right) dy$ varies between 0.6653 and 0.6720. The code to achieve this result is available at https://github.com/rasa200/prophet-secretary-through-blind-strategies.git. Even if we did not find an exact solution for the ODE, its performance is given by computing (4.3). This gives the claimed factor of 0.665.

5 Improved analysis and proof of Theorem 1.1

In this section, we present how to get the factor 0.669 in Theorem 1.1. As done in Sect. 4, we analyze the performance of the corresponding deterministic blind strategy with an instance of size n and we only care about the performance guarantee of α as n grows to ∞. The difference is that in this section we restrict ourselves to use blind strategies that take only m possible values, which allows us to derive different inequalities. Specifically, we consider $\alpha = \alpha_{\alpha_1,\ldots,\alpha_m}$ given by

$$\alpha_{\alpha_1,\ldots,\alpha_m}(x) = \sum_{j\in[m]} \alpha_j \mathbb{1}_{\left[\frac{j-1}{m},\frac{j}{m}\right)}(x),$$

in other words, piece-wise constant functions. The general method is similar and proceeds in three steps.

Step 1: Decomposition In the same spirit as in Lemma 4.1, we have the following lemma, in which the notation $\lceil x \rceil$ stands for the upper integer part of x.

Lemma 5.1 *Given an instance $F_1, F_2, \ldots, F_{mN}$ and $\infty = \tau_0 \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_m \geq \tau_{m+1} = -\infty$ a sequence of nonincreasing thresholds it holds that, for $j \in [m+1]$ and $t \in [\tau_j, \tau_{j-1})$,*

$$\mathbb{P}(V_{\sigma_T} > t) = \mathbb{P}(T \leq (j-1)N) + \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=(j-1)N+1}^{Nm} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{Nm} \right),$$

where T is the stopping time given by $TTA(\tau_1 = \tau_1, \ldots, \tau_i = \tau_{\lceil i/N \rceil}, \ldots, \tau_{mN} = \tau_m)$.

Proof This is a special instance of Lemma 4.1. Indeed, define $\tau'_0, \tau'_1, \ldots, \tau'_{mN+1}$ by $\tau'_l := \tau_{\lceil l/N \rceil}$, for $l \in \{0, 1, \ldots, mN+1\}$. Notice that

$$\bigcup_{l\in[mN+1]} [\tau'_l, \tau'_{l-1}) = \bigcup_{j\in[mN+1]} [\tau'_{(j-1)N+1}, \tau'_{(j-1)N}).$$

Taking $j \in [m+1]$ and $l = (j-1)N + 1$, by Lemma 4.1 we have that, for all $t \in [\tau'_l = \tau_j, \tau'_{l-1} = \tau_{j-1})$,

$$\begin{aligned}\mathbb{P}(V_{\sigma_T} > t) &= \mathbb{P}(T \leq l-1) + \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=l}^{Nm} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{Nm} \right) \\ &= \mathbb{P}(T \leq (j-1)N) + \sum_{i\in[n]} \mathbb{P}(V_i > t) \left(\sum_{k=(j-1)N+1}^{Nm} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{Nm} \right),\end{aligned}$$

which is the desired decomposition. □

Step 2: Lower bounds and weak Schur convexity The main novelty is to sharpen the lower bound of Lemma 4.2:

Lemma 5.2 *Given $V_1, \ldots, V_n$ independent random variables and $\tau_1 \geq \cdots \geq \tau_n$ a sequence of nonincreasing thresholds, we denote by T the stopping time of $TTA(\tau_1, \ldots, \tau_n)$. Then, for all $i, k \in [n]$,*

$$\mathbb{P}(T \geq k | \sigma_k = i) \geq \frac{\mathbb{P}(T > k)}{1 - \frac{k}{n} + \frac{1}{n} \sum_{l\in[k]} \mathbb{P}(V_i \leq \tau_l)}.$$

Proof Inspired by the proof given by Esfandiari et al. [13], fix $i, k \in [n]$. Conditioning on the value of $\sigma^{-1}(i)$,

$$\mathbb{P}(T > k) = \frac{1}{n} \sum_{l \in [n]} \mathbb{P}(T > k | \sigma_l = i). \tag{5.1}$$

Note that for all $l = k+1, \ldots, n$,

$$\mathbb{P}(T > k | \sigma_l = i) = \mathbb{P}(T > k | \sigma_{k+1} = i) \leq \mathbb{P}(T \geq k | \sigma_k = i).$$

Define

$$\Sigma_{-i}(k) := \{\sigma, \text{ ordered subset of } [n] \backslash \{i\} \text{ with size } k\}.$$

Then, given $l \in [k]$, since the thresholds are nonincreasing,

$$\begin{aligned}
\mathbb{P}(T > k | \sigma_l = i) &= \frac{\mathbb{P}(V_i \leq \tau_l)}{|\Sigma_{-i}(k-1)|} \sum_{\sigma \in \Sigma_{-i}(k-1)} \prod_{j \in [k-1]} F_{\sigma_j}(\tau_{j+\mathbb{1}_{j \geq l}}) \\
&\leq \frac{\mathbb{P}(V_i \leq \tau_l)}{|\Sigma_{-i}(k-1)|} \sum_{\sigma \in \Sigma_{-i}(k-1)} \prod_{j \in [k-1]} F_{\sigma_j}(\tau_j) \\
&= \mathbb{P}(V_i \leq \tau_l) \mathbb{P}(T \geq k | \sigma_k = i).
\end{aligned}$$

Plugging both inequalities back into Eq. (5.1) we get the result. □

Recall that in the proof of Proposition 4.1, we used the inequality $\sum_{i \in [n]} \mathbb{P}(V_i > t) \geq \mathbb{P}(\max_{i \in [n]}\{V_i\} > t)$. We will now use the following more sophisticated inequality:

Lemma 5.3 *Let $V_1, \ldots, V_n$ be independent random variables and $\tau_1 \geq \max\{\tau_2, \tau_3, \ldots, \tau_n\}$ be a sequence of thresholds. Then, for any $t < \tau_1$ and $k \leq \frac{n}{2}$,*

$$\sum_{i \in [n]} \frac{\mathbb{P}(V_i > t)}{1 - \frac{1}{n} \sum_{l \in [k]} \mathbb{P}(V_i > \tau_l)} \geq \frac{\mathbb{P}(\max_{i \in [n]}\{V_i\} > t)}{1 - \frac{k}{n} \mathbb{P}(\max_{i \in [n]}\{V_i\} > \tau_1)}.$$

Proof Define $\lambda := \frac{k}{n} \in [0, \frac{1}{2}]$ and notice that

$$\begin{aligned}
\sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{1 - \frac{1}{n}\sum_{l\in[k]} \mathbb{P}(V_i > \tau_l)} &= \sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{\frac{n-k}{n} + \frac{1}{n}\sum_{l\in[k]} \mathbb{P}(V_i \le \tau_l)} \\
&\ge \sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{1 - \lambda + \lambda\mathbb{P}(V_i \le \tau_1)} \\
&= \sum_{i\in[n]} \frac{1 - F_i(t)}{1 - \lambda + \lambda F_i(\tau_1)} \\
&=: C(t; \lambda, F_1, \ldots, F_n).
\end{aligned}$$

Therefore, it is sufficient to prove that

$$\frac{1 - F_1(t)}{1 - \lambda + \lambda F_1(\tau_1)} + \frac{1 - F_2(t)}{1 - \lambda + \lambda F_2(\tau_1)} \ge \frac{1 - F_1(t)F_2(t)}{1 - \lambda + \lambda F_1(\tau_1)F_2(\tau_1)}, \tag{5.2}$$

since we can iterate this argument n times to deduce the result. To prove this inequality, define the following variables

$$\begin{aligned}
&\beta := F_1(\tau_1)F_2(\tau_1), \quad \gamma := F_1(t)F_2(t), \\
&x := F_1(\tau_1), \quad y := F_1(t).
\end{aligned}$$

Note that if $\beta = 0$, the inequality is trivial, so we can assume that $\beta > 0$. Consider the following optimization problem.

$$(P)\begin{cases}
\min_{x,y}\ f_\lambda(x, y) := \frac{1-y}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{y}}{1-\lambda+\lambda\frac{\beta}{x}} \\
s.t. \qquad \beta \le x \le 1 \\
\qquad\quad\ \beta \le \frac{\beta}{x} \le 1 \\
\qquad\quad\ \gamma \le y \le x \\
\qquad\quad\ \gamma \le \frac{\gamma}{y} \le \frac{\beta}{x}.
\end{cases}$$

Notice that the function f_λ defined by

$$f_\lambda(x, y) = \frac{1}{1 - \lambda + \lambda x} - \left(\frac{1}{1 - \lambda + \lambda x}\right) y + \frac{1}{1 - \lambda + \lambda\frac{\beta}{x}} - \left(\frac{\gamma}{1 - \lambda + \lambda\frac{\beta}{x}}\right)\frac{1}{y},$$

is concave in y. Rearranging the inequalities, the problem reduces to

$$(P)\begin{cases}
\min_{x}\ f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}} \\
s.t. \qquad \beta \le x \le 1
\end{cases}$$

Then, all we have to show is that, if $\lambda \in [0, \frac{1}{2}]$, $x = 1$ is the minimum, since this would imply inequality (5.2). The following are three sufficient conditions for $x = 1$ to be the minimum:

1. $f_\lambda(\beta) \geq f_\lambda(1)$.
2. $x = 1$ is local minimum.
3. There exists at most one critical point in the interval $[\beta, 1]$.

All these conditions are true for $\lambda \in [0, 1/2]$, but since the formal computations are long they are in Appendix A. This implies inequality (5.2). Then, the lemma is proved iterating this inequality n times.

□

Step 3: Optimization and proof of Theorem 1.1 The previous analysis gives a new lower bound on $\mathbb{P}(V_{\sigma_T} > t)$ that only depends on α. An auxiliary function to express this lower bound is useful. Let us define for $m \geq 1$ and $p \in (0, 1)$ the function $g_{m,p} : [m] \to \mathbb{R}_+$ by

$$g_{m,p}(k) = \begin{cases} \frac{1}{1-\frac{k}{m}(1-p)}; & k \leq \frac{m}{2} \\ \frac{2}{1+p}; & k > \frac{m}{2}. \end{cases}$$

Notice that it is a nondecreasing function in k that is always greater than 1.

We must bound from below $\mathbb{P}(V_{\sigma_T} > t)$ by a quantity that depends only on α. In light of the decomposition of Lemma 5.1, we first handle the case where $j \in \{2, \ldots, m\}$, which is more complicated than the extreme cases $j = 1$ and $j = m+1$. The notation $\lfloor x \rfloor$ stands for the lower integer part of x.

Proposition 5.1 *Given an instance $F_1, F_2, \ldots, F_{mN}$ and $\infty = \tau_0 \geq \tau_1 \geq \tau_2 \geq \ldots \geq \tau_m \geq \tau_{m+1} = -\infty$ a sequence of nonincreasing thresholds it holds that, for $j \in \{2, \ldots, m\}$ and $t \in [\tau_j, \tau_{j-1})$,*

$$\mathbb{P}(V_{\sigma_T} > t) \geq \sum_{k \in [j-1]} \frac{1-\alpha_k}{m} + \left[\sum_{k=(j-1)N+1}^{Nm} \mathbb{P}(T > k) \frac{g_{Nm,\alpha_1}(k)}{Nm} \right] \mathbb{P}(\max_{i \in [n]}\{V_i\} > t),$$

where T is the stopping time given by $TTA(\tau_1 = \tau_1, \ldots, \tau_i = \tau_{\lfloor i/N \rfloor + 1}, \ldots, \tau_{mN} = \tau_{m+1})$.

Proof From Lemma 5.1, we have that

$$\mathbb{P}(V_{\sigma_T} > t) = \mathbb{P}(T \leq (j-1)N) + \sum_{i \in [n]} \mathbb{P}(V_i > t) \sum_{k=(j-1)N+1}^{Nm} \frac{\mathbb{P}(T \geq k | \sigma_k = i)}{Nm}.$$

We will bound each of the terms in this decomposition.

For the first term, using Lemma 4.3 we get that

$$\mathbb{P}(T \le (j-1)N) \ge \sum_{k\in[(j-1)N]} \frac{1-\alpha(k/Nm)}{Nm} = \sum_{k\in[j-1]} \frac{1-\alpha_k}{m},$$

which is the desired bound.

For the second term, since α is nonincreasing, the corresponding thresholds are nonincreasing too. Then, we can use both Lemmas 5.2 and 5.3. First, for every $i, k \in [Nm]$,

$$\mathbb{P}(T \ge k|\sigma_k = i) \ge \frac{\mathbb{P}(T > k)}{1 - \frac{1}{Nm}\sum_{l\in[k]} \mathbb{P}(V_i > \tau_l)}.$$

Then, interchanging the order of the sums,

$$\begin{aligned}\sum_{i\in[n]} \mathbb{P}(V_i > t) &\sum_{k=(j-1)N+1}^{Nm} \frac{\mathbb{P}(T \ge k|\sigma_k = i)}{Nm}\\ &\ge \sum_{k=(j-1)N+1}^{Nm} \mathbb{P}(T > k) \sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{Nm - \sum_{l\in[k]} \mathbb{P}(V_i > \tau_l)}.\end{aligned}$$

Now, by Lemma 5.3, for $k \le Nm/2$,

$$\begin{aligned}\sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{Nm - \sum_{l\in[k]} \mathbb{P}(V_i > \tau_l)} &\ge \frac{\mathbb{P}(\max_{i\in[n]}\{V_i\} > t)}{Nm - k\mathbb{P}(\max_{i\in[n]}\{V_i\} > \tau_1)}\\ &= \frac{g_{Nm,\alpha_1}(k)}{Nm}\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right),\end{aligned}$$

and for $k > Nm/2$,

$$\begin{aligned}\sum_{i\in[n]} \frac{\mathbb{P}(V_i > t)}{Nm - \sum_{l\in[k]} \mathbb{P}(V_i > \tau_l)} &\ge \frac{\mathbb{P}(\max_{i\in[n]}\{V_i\} > t)}{Nm - \frac{Nm}{2}\mathbb{P}(\max_{i\in[n]}\{V_i\} > \tau_1)}\\ &= \frac{g_{Nm,\alpha_1}(k)}{Nm}\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right).\end{aligned}$$

All in all, we have proven the following bound:

$$\begin{aligned}\sum_{i\in[n]} \mathbb{P}(V_i > t) &\sum_{k>(j-1)N}^{Nm} \frac{\mathbb{P}(T \ge k|\sigma_k = i)}{Nm}\\ &\ge \left[\sum_{k=(j-1)N+1}^{Nm} \mathbb{P}(T > k)\frac{g_{Nm,\alpha_1}(k)}{Nm}\right]\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right),\end{aligned}$$

which is the desired bound for the second term, and so we have proven the proposition. □

The following lemma has the same flavor as Theorem 4.4 and allows us to deduce a lower bound for the performance of blind strategies taking only m values.

Lemma 5.4 *Let $\alpha = \alpha_{\alpha_1,\dots,\alpha_m}$ be a nonincreasing function where $\alpha_m > 0$, and let T be the blind strategy stopping time. Then, for every instance $F_1,\dots,F_n$ and $t > 0$,*

$$\mathbb{P}(V_{\sigma_T} > t) \geq \min_{j\in[m+1]}\{f_j(\alpha_1,\dots,\alpha_m)\}\cdot\mathbb{P}(\max_{i\in[n]}\{V_i\} > t),$$

with

$$f_j(\alpha_1,\dots,\alpha_m) = \begin{cases} \sum_{k=1}^{m}\left(\prod_{l\in[k-1]}\alpha_l\right)^{\frac{1}{m}}\left(\frac{1-\alpha_k^{\frac{1}{m}}}{-\ln\alpha_k}\right); & j=1\\ \sum_{k\in[j-1]}\frac{1-\alpha_k}{m(1-\alpha_j)}+\sum_{k=j}^{m}\left(\prod_{l\in[k-1]}\alpha_l\right)^{\frac{1}{m}} g_{m,\alpha_1}(k-1)\left(\frac{1-\alpha_k^{\frac{1}{m}}}{-\ln\alpha_k}\right); & j\in\{2,\dots,m\}\\ \sum_{k\in[m]}\frac{1-\alpha_k}{m}; & j=m+1. \end{cases}$$

Proof *Case* $j = m+1$ (i.e. : $t\in[0,\tau_m)$)

By the decomposition of Lemma 5.1 and using Lemma 4.3, we have that

$$\mathbb{P}(V_{\sigma_T} > t) = \mathbb{P}(T \leq mN) \geq \sum_{k\in[mN]}\frac{1-\alpha(k/Nm)}{Nm} = \sum_{k\in[m]}\frac{1-\alpha_k}{m},$$

which concludes the case $j = m+1$.

Case $j\in\{2,\dots,m\}$

By Proposition 5.1, we have that

$$\mathbb{P}(V_{\sigma_T} > t) \geq \sum_{k\in[j-1]}\frac{1-\alpha_k}{m} + \left[\sum_{k=(j-1)N+1}^{Nm}\mathbb{P}(T > k)\frac{g_{Nm,\alpha_1}(k)}{Nm}\right] \mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right).$$

Notice that, since $t\in[\tau_j,\tau_{j-1})$, we have that $1-\alpha_j \geq \mathbb{P}(\max_{i\in[n]}\{V_i\} > t)$. Then,

$$\sum_{k\in[j-1]}\frac{1-\alpha_k}{m} \geq \sum_{k\in[j-1]}\frac{1-\alpha_k}{m(1-\alpha_j)}\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right).$$

On the other hand,

$$\begin{aligned}\sum_{k=(j-1)N+1}^{Nm} \mathbb{P}(T > k)\frac{g_{Nm,\alpha_1}(k)}{Nm} &= \sum_{l=j}^{m}\sum_{k=1}^{N}\mathbb{P}(T > (l-1)N+k)\frac{g_{Nm,\alpha_1}((l-1)N+k)}{Nm}\\ &\geq \sum_{l=j}^{m}\left(\prod_{l'=1}^{l-1}\alpha_{l'}\right)^{\frac{1}{m}}\sum_{k=1}^{N}\left(\alpha_l^{\frac{1}{Nm}}\right)^k\frac{g_{Nm,\alpha_1}((l-1)N)}{Nm}\\ &= \sum_{l=j}^{m}\left(\prod_{l'=1}^{l-1}\alpha_{l'}\right)^{\frac{1}{m}} g_{m,\alpha_1}(l-1)\frac{\alpha_l^{\frac{1}{Nm}}}{Nm}\frac{1-\alpha_l^{\frac{1}{m}}}{1-\alpha_l^{\frac{1}{Nm}}}\\ &\xrightarrow[N\to\infty]{} \sum_{k=j}^{m}\left(\prod_{l\in[k-1]}\alpha_l\right)^{\frac{1}{m}} g_{m,\alpha_1}(k-1)\left(\frac{1-\alpha_k^{\frac{1}{m}}}{-\ln\alpha_k}\right).\end{aligned}$$

Putting these two inequalities together, we can conclude the case $j \in \{2, \ldots, m\}$.

Case $j = 1$ (i.e. : $t \in [\tau_1, \infty)$)
We first decompose according to Lemma 5.1, then use Lemma 4.2 and finally we reuse the previous computation replacing $j = 1$ and noticing that $g_{m,p}(\cdot) \geq 1$.

$$\begin{aligned}\mathbb{P}(V_{\sigma_T} > t) &= \frac{1}{n}\sum_{i\in[n]}\mathbb{P}(V_i > t)\sum_{k=1}^{Nm}\mathbb{P}(T \geq k|\sigma_k = i)\\ &\geq \left[\frac{1}{n}\sum_{k=1}^{Nm}\mathbb{P}(T > k)\right]\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right)\\ &\geq \left[\sum_{k=1}^{m}\left(\prod_{l\in[k-1]}\alpha_l\right)^{\frac{1}{m}}\left(\frac{1-\alpha_k^{\frac{1}{m}}}{-\ln\alpha_k}\right)\right]\mathbb{P}\left(\max_{i\in[n]}\{V_i\} > t\right),\end{aligned}$$

where the last inequality is only valid in the limit as $N \to \infty$. □

With the previous lemma we can easily establish the improved guarantee. Indeed, take the right-hand-side of the expression in Lemma 5.4 and optimize over the choice of $\alpha_1, \ldots, \alpha_m$. We do this optimization numerically and find a particular collection $\alpha_1, \ldots, \alpha_m$ such that the guarantee evaluates to 0.669, as stated in the following corollary. We must note however that there might be other choices leading to slightly improved guarantees.

Corollary 5.5 *There exists* $1 \geq \alpha_1 \geq \ldots \geq \alpha_m \geq 0$ *such that*

$$\frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}(\max_{i\in[n]}\{V_i\})} \geq 0.66975,$$

where T is the stopping time corresponding to the blind strategy $\alpha = \alpha_{\alpha_1,\ldots,\alpha_m}$.

In particular, taking $m = 30$ was enough to derive this result. The code to achieve this result is available at https://github.com/rasa200/prophet-secretary-through-blind-strategies.git

6 A 0.675 upper bound for blind strategies: proof of Theorem 1.2

In order to prove Theorem 1.2, we consider two instances and show that no blind strategy can guarantee better than 0.675 for both instances.

The first instance consists simply of a single random variable which is nearly deterministic, given by $V_1 \sim U(1-\varepsilon, 1+\varepsilon)$. The second instance has n i.i.d. random variables defined by (and we take $n \to \infty$):

$$V_i \sim \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ U(0,\varepsilon) & \text{w.p. } 1-\varepsilon. \end{cases}$$

Combining these two instances one can show the following result.

Lemma 6.1 *Let T be the stopping time corresponding to the blind strategy given by α. Then,*

$$\sup_{\alpha} \inf_{n;F_1,\ldots,F_n} \frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}(\max\limits_{i\in[n]}\{V_i\})} \le \sup_{\alpha} \min \left\{ 1 - \int\limits_0^1 \alpha(s)ds \quad , \quad \int\limits_0^1 e^{\int\limits_0^s \ln \alpha(w)dw} ds \right\}.$$

With this result we need to compute the quantity on the right-hand-side of the previous lemma to obtain an upper bound on the performance guarantee of any blind strategy. This is done using optimal control theory. The basic procedure consists first in proving that the supremum right-hand-side in Lemma 6.1 is attained, then we have Mayer's optimal control problem for which the necessary optimality conditions can be expressed as an integro-differential equation. We conclude by solving this equation numerically, and thus have the following result.

Corollary 6.2 *Let T be the stopping time corresponding to the blind strategy given by α. Then,*

$$\sup_{\alpha} \inf_{n;F_1,\ldots,F_n} \frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}(\max\limits_{i\in[n]}\{V_i\})} \le 0.675.$$

Proof of Lemma 6.1 The first instance is $V_1 = U(1-\varepsilon, 1+\varepsilon)$, with $\varepsilon > 0$. Notice that, $\tau = 1-\varepsilon+\alpha(u)2\varepsilon$, where $u \sim U(0,1)$ and, by direct computation, we have that

$$\begin{aligned}\mathbb{E}(V_{\sigma_T}) &= \int_0^1 \mathbb{E}(V_{\sigma_T}|u=s)ds\\ &= \int_0^1 \left(1+\varepsilon\left(\alpha(s)-\frac{1}{2}\right)\right)(1-\alpha(s))ds\\ &\xrightarrow[\varepsilon\to 0]{} \int_0^1 (1-\alpha(s))ds.\end{aligned}$$

The second instance has n i.i.d. random variables defined by:

$$V_i \sim \begin{cases} \frac{1}{\varepsilon} & w.p.\ \varepsilon \\ U(0,\varepsilon) & w.p.\ 1-\varepsilon. \end{cases}$$

Moreover, for ε small enough,

$$\mathbb{P}\left(\max_{i\in[n]}\{V_i\}\le t\right) = \begin{cases} 0; & t<0 \\ \left(\frac{1-\varepsilon}{\varepsilon}\right)^n t^n; & 0\le t<\varepsilon \\ (1-\varepsilon)^n; & \varepsilon\le t<\frac{1}{\varepsilon} \\ 1; & \frac{1}{\varepsilon}\le t \end{cases}$$

Notice that we can assume $\alpha(x) < 1$, for $x > 0$, since there is no gain in rejecting all instances. Then, $\tau_i = \sqrt[n]{\alpha(u_{[i]})}\frac{\varepsilon}{1-\varepsilon}$ and we have that $\mathbb{P}(V_i \le \tau_i) = \sqrt[n]{\alpha(u_{[i]})}$. By direct computation,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}(V_{\sigma_T}|u) &= 1+\sqrt[n]{\alpha(u_{[1]})}+\sqrt[n]{\alpha(u_{[1]})\alpha(u_{[2]})}+\cdots+\sqrt[n]{\alpha(u_{[1]})\ldots\alpha(u_{[n-1]})}\\ &= \sum_{i=0}^{n-1}\prod_{j=1}^{i}\sqrt[n]{\alpha(u_{[j]})}.\end{aligned}$$

In addition, we have $\mathbb{E}(\max_{i\in[n]}\{V_i\}) = \frac{1}{\varepsilon}(1-(1-\varepsilon)^n) \xrightarrow[\varepsilon\to 0]{} n$. Then,

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}\left(\max_{i\in[n]}\{V_i\}\right)} = \lim_{n\to\infty}\mathbb{E}_u\left[\frac{1}{n}\sum_{i=0}^{n-1}\prod_{j=1}^{i}\sqrt[n]{\alpha(u_{[j]})}\right] = \int_0^1 e^{\int_0^s \ln\alpha(w)dw}ds.$$

$\square$

Sketch of Proof of Corollary 6.2 We first show that the supremum given by Lemma 6.1 is attained at a certain α^*. To this end we note that, without loss of generality, we can consider the supremum over nonincreasing functions α, by a simple exchange of mass argument. Then, we note that the set of nonincreasing functions from [0, 1] to itself is compact for the $||\cdot||_\infty$ and the functional being optimized is continuous for that metric. Then we deduce the existence of α^*, and furthermore it satisfies

$$1-\int_0^1 \alpha^*(s)ds = \int_0^1 \exp\left[\int_0^s \ln\alpha^*(w)dw\right]ds.$$

Therefore, α^* is the solution of the following optimal control problem:

$$(P)\begin{cases} \min\limits_{\alpha} \;\; -x_1(1) = -\int\limits_0^1 1-\alpha(t)dt \\ s.t.: \dot{x}(t) = f(x(t),\alpha(t)) = \begin{pmatrix} 1-\alpha(t) \\ \exp[x_3(t)] \\ \ln\alpha(t) \end{pmatrix} \\ \qquad x(0) = (0,0,0)' \\ \qquad (t,x(t)) \in [0,1]\times\mathbb{R}^3 \\ \qquad \alpha(t)\in[0,1] \\ \qquad x_1(1) = x_2(1). \end{cases}$$

This is generally called a Mayer problem and the necessary optimality conditions (Pontryagin maximum principle) leads to identify α^* with $\alpha_{K,\bar{t}}$ defined by

$$\alpha_{K,\bar{t}}(t) = \begin{cases} 1; & 0\le t<\bar{t} \\ \beta_{K,\bar{t}}(t); & \bar{t}\le t\le 1, \end{cases}$$

where $K\in[0,3]$ and $\bar{t}\in[0,1/3]$ and $\beta_{K,\bar{t}}$ is the solution of the following integro-differential equation

$$\begin{cases} \dot{\beta}(t) = -K\exp\left[\int\limits_{\bar{t}}^{t} \ln\beta(s)ds\right] \; t\in(\bar{t},1) \\ \beta(1)=0. \end{cases} \tag{6.1}$$

To solve numerically this equation, consider the change of variables $g(t) = \int_{\bar{t}}^t \ln\beta(s)ds$, so that the equation becomes the second order ODE

$$\begin{cases} e^{\dot{g}(t)}\ddot{g}(t) = -Ke^{g(t)}\;; \quad t\in(\bar{t},1) \\ g(\bar{t}) = 0 \\ \dot{g}(\bar{t}) = \ln\beta(\bar{t}). \end{cases}$$

Because $\exp(\cdot)$ is continuous and locally Lipschitz, this is a well-posed Cauchy problem with a unique local solution. The initial condition $\dot{g}(\bar{t}) = \ln\beta(\bar{t})$ turns out to be simply a replacement for $\dot{g}(1) = -\infty$ in the sense that we search for the solutions g such that $g(\bar{t}) = 0$ and exploits at time 1. This seemingly numerical difficulty is well treated using solvers for stiff ODE such as *ode15s* of *Matlab*.

Then, we numerically compute (6.1) to determine that

$$
\begin{aligned}
&\sup_{\alpha} \min \left\{ 1 - \int_0^1 \alpha(s) ds, \int_0^1 e^{\int_0^s \ln \alpha(w) dw} ds \right\} \\
&= \sup_{\substack{K \in [0,3] \\ \bar{t} \in [0,1/3]}} \min \left\{ 1 - \int_0^1 \alpha_{K,\bar{t}}(s) ds, \int_0^1 e^{\int_0^s \ln \alpha_{K,\bar{t}}(w) dw} ds \right\} \\
&\le 0.675.
\end{aligned}
$$

Finally we note that if (6.1) has no solution, this simply means that α^* does not corresponds to $\alpha_{K,\bar{t}}$ and thus it is not taken into account in the previous supremum. The code to achieve this result is available at https://github.com/rasa200/prophet-secretary-through-blind-strategies.git. □

7 A 0.732 upper bound for any strategies: proof of Theorem 1.3

We now obtain an upper bound on the performance of any algorithm for the prophet secretary problem. Surprisingly, our bound comes from analyzing the following simple instance. Take $a \in [0, 1]$ and consider $n + 1$ random variables whose values are distributed as:

$$
V_1, \ldots, V_n \sim \begin{cases} n & w.p.\ \frac{1}{n^2} \\ 0 & w.p.\ 1 - \frac{1}{n^2} \end{cases}
$$
$$
V_{n+1} \equiv a.
$$

Clearly any reasonable algorithm would always accept a value of n and never accept a value of 0. Therefore the only decision an algorithm has to make is that of whether accepting a value of a or not. The algorithm picks the value a if it was presented after some time $j_n^* \in [n+1]$, and from j_n^*, the expectation of the future is less than the value a. Let σ be the random permutation and T be the implied stopping time, then we have that for $i = 1, \ldots, j_n^* - 1$

$$
\mathbb{E}(V_{\sigma_T} | \sigma_i = n+1) = n \left[1 - \left(1 - \frac{1}{n^2} \right)^n \right],
$$

and for $i = j_n^*, \ldots, n+1$,

$$
\mathbb{E}(V_{\sigma_T} | \sigma_i = n+1) = n \left[1 - \left(1 - \frac{1}{n^2} \right)^{i-1} \right] + \left(1 - \frac{1}{n^2} \right)^{i-1} a.
$$

Therefore,

$$\mathbb{E}(V_{\sigma_T}) = \frac{j_n^* - 1}{n+1} n \left[1 - \left(1 - \frac{1}{n^2}\right)^n\right] + \frac{1}{n+1} \sum_{i=j_n^*}^{n+1} n \left[1 - \left(1 - \frac{1}{n^2}\right)^{i-1}\right] + \left(1 - \frac{1}{n^2}\right)^{i-1} a.$$

Consider $\lambda := \limsup_{n\to\infty} \frac{j_n^*}{n} \in [0, 1]$, then

$$\limsup_{n\to\infty} \mathbb{E}(V_{\sigma_T}) = \lambda + \int_\lambda^1 x + a\, dx = \lambda + \frac{1-\lambda^2}{2} + a(1-\lambda) \le 1 + \frac{a^2}{2},$$

where the last inequality comes from maximizing over $\lambda \in [0, 1]$.

On the other hand,

$$\mathbb{E}\left(\max_{i\in[n]}\{V_i\}\right) = n\left[1 - \left(1 - \frac{1}{n^2}\right)^n\right] + \left(1 - \frac{1}{n^2}\right)^n a \xrightarrow[n\to\infty]{} 1 + a.$$

We notice that choosing $a = \sqrt{3} - 1$ we get that

$$\limsup_{n\to\infty} \frac{\mathbb{E}(V_{\sigma_T})}{\mathbb{E}\left(\max_{i\in[n]}\{V_i\}\right)} \le \frac{1 + \frac{a^2}{2}}{1+a} = \sqrt{3} - 1 \approx 0.732.$$

8 Dealing with discontinuous distributions

In this section we explain how to use a *blind strategy* in instances where the distributions $F_1, \dots, F_n$ are not necessarily continuous. Recall that in the definition of blind strategies in Sect. 1.2, we need the existence of τ_i such that $\mathbb{P}(\max_{i\in[n]}\{V_i\} \le \tau_i) = \alpha(u_{[i]})$. So, what happens if such thresholds $\tau_1, \dots, \tau_n$ do not exist? For the purpose of studying the prophet inequality, the performance of a strategy defined over instances with continuous distributions is always extendable to discontinuous ones allowing stochastic tie breaking. In this case, we can explicitly define the strategy that α induces over discontinuous instances. The resulting strategy no longer depends on the distribution of the maximum only.

The procedure to compute the tie breaking is quite natural:

1. Approximate the instance.
2. Study the strategy induced by α in the approximated instance.
3. Replicate what would happen in the original instance, allowing tie breaking.

Given a realization of uniform random variables $u_1, \dots, u_n$, assume that τ_i does not exist, in other words, for some $i \in [n]$, there is a $\tau \in \mathbb{R}$ such that

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\max_{i\in[n]}\{V_i\} \le \tau - \varepsilon\right) < \alpha(u_{[i]}) < \mathbb{P}\left(\max_{i\in[n]}\{V_i\} \le \tau\right).$$

The stochastic tie breaking consists in accepting the value τ with some probability, say p_i. This acceptance rate depends on the whole instance, not only on the distribution of the maximum, and on the identity of the revealed variable. To compute these acceptance rates we use the following procedure. For $\varepsilon > 0$, consider the following approximated instance

$$F_i^\varepsilon(t) = \begin{cases} F_i(t) \\ F_i(\tau-\varepsilon) + \frac{t-\tau+\varepsilon}{\varepsilon}\,(F(\tau) - F_i(\tau-\varepsilon)) \end{cases},$$

for $t \notin [\tau-\varepsilon, \tau]$ in the first case and $t \in [\tau-\varepsilon, \tau]$ in the second case. This instance has a continuous distribution of the maximum in $[\tau-\varepsilon, \tau]$ and we are able to find τ^ε, the corresponding threshold for the approximated instance, such that

$$\mathbb{P}\left(\max_{i\in[n]}\{V_i\} \le \tau^\varepsilon\right) = \alpha(u_{[i]}).$$

Then, we compute, for $j \in [n]$, $\beta_j := \lim_{\varepsilon\to 0} F_j^\varepsilon(\tau^\varepsilon)$. To finish, we define, for j such that $\mathbb{P}(V_j = \tau) > 0$,

$$p_i(j; F_1, \ldots, F_n) := \frac{F_j(\tau) - \beta_j}{\mathbb{P}(V_j = \tau)}$$

and $p_i = 0$ otherwise. This will induce that, faced with V_j at time i, the gambler accepts its realization with probability $1-\beta_j$. To be more precise, we use the following procedure.

Algorithm 2 Stochastic TTA

```
1: for i = 1, ..., n do
2:     if V_{σ_i} > τ_i then
3:         Take V_{σ_i}
4:     else if V_{σ_i} = τ_i then
5:         Take V_{σ_i} with probability p_i(σ_i; F_1, ..., F_n)
6:     end if
7: end for
```

With this procedure, all results extend to general instances.

Appendix A Missing proofs in Lemma 5.3

Lemma A.1 *Consider* $0 < \beta \le 1$, $0 \le \gamma \le \beta$ *and the optimization problem*

$$(P)\begin{cases} \min_x \; f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}} \\ s.t. \qquad \beta \le x \le 1 \end{cases}.$$

If $\lambda \in [0, 1/2]$, *then the value of* (P) *is* $f_\lambda(1)$.

In order to prove Lemma A.1, we will prove three conditions that will imply the result, namely:

1. $f_\lambda(\beta) \geq f_\lambda(1)$.
2. $x = 1$ is local minimum.
3. There exists at most one critical point in the interval $[\beta, 1]$.

Each of these conditions are formally stated in the next three lemmata.

Lemma A.2 *Consider* $0 < \beta \leq 1$, $0 \leq \gamma \leq \beta$ *and* $f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}}$. *If* $0 \leq \lambda < 1$, *then*

$$f_\lambda(\beta) \geq f_\lambda(1).$$

Proof By direct computation, we have that

$$\begin{aligned} & f_\lambda(\beta) \geq f_\lambda(1) \\ & \quad \Leftrightarrow \frac{1-\beta}{1-\lambda+\lambda\beta} + \frac{1-\gamma/\beta}{1-\lambda+\lambda} \geq 0 + \frac{1-\gamma}{1-\lambda+\lambda\beta} \\ & \quad \Leftrightarrow \frac{\beta-\gamma}{\beta} \geq \frac{\beta-\gamma}{1-\lambda+\lambda\beta} \\ & \quad \Leftrightarrow (\beta-\gamma)(1-\lambda)(1-\beta) \geq 0, \end{aligned}$$

which is true by assumption. □

Lemma A.3 *Consider* $0 < \beta \leq 1$, $0 \leq \gamma \leq \beta$ *and* $f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}}$. *If* $0 \leq \lambda \leq 1/2$, *then*

$$x = 1 \textit{ is local minimum of } f_\lambda(\cdot) \textit{ in } [\beta, 1].$$

Proof Since the domain is $[\beta, 1]$, it's sufficient to prove that

$$\frac{d}{dx} f_\lambda(1) \leq 0.$$

Since $f_\lambda(x) = \frac{1-x}{1-\lambda+\lambda x} + \frac{x-\gamma}{(1-\lambda)x+\lambda\beta}$, we have that

$$\frac{d}{dx} f_\lambda(x) = \frac{-1}{(1-\lambda+\lambda x)^2} + \frac{\lambda\beta + (1-\lambda)\gamma}{((1-\lambda)x+\lambda\beta)^2}.$$

Therefore,

$$\begin{aligned} \frac{d}{dx} f_\lambda(1) &= -1 + \frac{\lambda\beta + (1-\lambda)\gamma}{(1-\lambda+\lambda\beta)^2} \\ &= \frac{\lambda\beta + (1-\lambda)\gamma - (1-\lambda+\lambda\beta)^2}{(1-\lambda+\lambda\beta)^2} \\ &= -\frac{\lambda^2[\beta-1]^2 + \lambda[\gamma+\beta-2] + [1-\gamma]}{(1-\lambda+\lambda\beta)^2}. \end{aligned}$$

Then, $\partial_x f_\lambda(1) \leq 0$ if and only if

$$g_{\beta,\gamma}(\lambda) := \lambda^2[\beta-1]^2 + \lambda[\gamma+\beta-2] + [1-\gamma] \geq 0.$$

The function $g_{\beta,\gamma}(\cdot)$ is a convex quadratic function. Moreover, $g_{\beta,\gamma}(0) = 1-\gamma \geq 0$ and $g_{\beta,\gamma}(1) = (\beta-1)\beta \leq 0$. There are some corner cases where it is easy to conclude. Assume that $\gamma = 1$, then $\beta = 1$ and $g_{\beta,\gamma}(\cdot) \equiv 0$, therefore, we can assume that $\gamma < \beta$. Consider the case $\beta = 1$, in which $g_{\beta,\gamma}(\cdot)$ is a linear function satisfying $g_{\beta,\gamma}(0) = 1-\gamma \geq 0$ and $g_{\beta,\gamma}(1) = 0$. Therefore, we can assume $\beta < 1$, i.e. : $g_{\beta,\gamma}(\cdot)$ is a strictly convex quadratic function such that $g_{\beta,\gamma}(0) > 0$ and $g_{\beta,\gamma}(1) < 0$. Moreover, if $\beta = 0$, then $g_{\beta,\gamma}(\lambda) = (\lambda-1)^2$, so we can also assume that $\beta > 0$. Define

$$\lambda_m(\beta,\gamma) := \inf\{\lambda > 0 : g_{\beta,\gamma}(\lambda) \leq 0\},$$

the smallest root of the polynomial $g_{\beta,\gamma}(\cdot)$. We will prove that

$$\inf_{\substack{0<\beta<1 \\ 0\leq\gamma\leq\beta}} \lambda_m(\beta,\gamma) = \frac{1}{2}. \tag{A.1}$$

By solving the quadratic equation,

$$\lambda_m(\beta,\gamma) = \frac{2-\beta-\gamma-\sqrt{(2-\beta-\gamma)^2 - 4(1-\gamma)(1-\beta)^2}}{2(1-\beta)^2}.$$

Note that $\lambda_m(\beta,\gamma) > 0$, since $0 \leq \gamma \leq \beta < 1$.

We will first show that for all $0 \leq \gamma \leq \beta$, $\partial_\gamma \lambda_m(\beta,\gamma) \leq 0$, i.e. : $\lambda_m(\beta,\cdot)$ is decreasing, which allows us to consider only $\lambda_m(\beta,\beta)$ to prove (A.1). We will finish the proof by proving that $\inf_{0<\beta<1} \lambda_m(\beta,\beta) = 1/2$.

To see that $\partial_\gamma \lambda_m(\beta,\gamma) \leq 0$, we'll prove that

$$\partial_\gamma \lambda_m(\beta,0) \leq 0 \quad \text{and} \quad \forall 0 \leq \gamma \leq \beta \quad \partial_{\gamma,\gamma} \lambda_m(\beta,\gamma) \leq 0.$$

By direct computation,

$$\begin{aligned}
&\partial_\gamma \lambda_m(\beta,\gamma) \leq 0 \\
&\qquad \Leftrightarrow \partial_\gamma \left(\frac{2-\beta-\gamma-\sqrt{(2-\beta-\gamma)^2 - 4(1-\gamma)(1-\beta)^2}}{2(1-\beta)^2} \right) \leq 0 \\
&\qquad \Leftrightarrow -1 - \partial_\gamma \left(\sqrt{(2-\beta-\gamma)^2 - 4(1-\gamma)(1-\beta)^2} \right) \leq 0
\end{aligned}$$

$$\Leftrightarrow -1 - \frac{-2(2-\beta-\gamma)+4(1-\beta)^2}{2\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2}} \leq 0$$
$$\Leftrightarrow -1 + \frac{3\beta-\gamma-2\beta^2}{\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2}} \leq 0$$

Therefore,

$$\begin{aligned}
\partial_\gamma \lambda_m(\beta, 0) \leq 0 & \\
&\Leftrightarrow -1 + \frac{3\beta-2\beta^2}{\sqrt{(2-\beta)^2-4(1-\beta)^2}} \leq 0 \\
&\Leftrightarrow \frac{\beta(3-2\beta)}{\sqrt{\beta(4-3\beta)}} \leq 1 \\
&\Leftrightarrow \sqrt{\beta}(3-2\beta) \leq \sqrt{(4-3\beta)} \\
&\Leftrightarrow 9\beta - 12\beta^2 + 4\beta^3 \leq 4 - 3\beta \\
&\Leftrightarrow \beta(3-2\beta)^2 \leq 4 - 3\beta,
\end{aligned}$$

which is true for all $\beta \in (0, 1)$.

On the other hand,

$$\begin{aligned}
&\partial_{\gamma,\gamma} \lambda_m(\beta, \gamma) \leq 0 \\
&\quad \Leftrightarrow \partial_{\gamma,\gamma} \left(\frac{2-\beta-\gamma-\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2}}{2(1-\beta)^2} \right) \leq 0 \\
&\quad \Leftrightarrow \partial_\gamma \left(-1 + \frac{3\beta-\gamma-2\beta^2}{\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2}} \right) \leq 0 \\
&\quad \Leftrightarrow -\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2} + \frac{(3\beta-\gamma-2\beta^2)^2}{\sqrt{(2-\beta-\gamma)^2-4(1-\gamma)(1-\beta)^2}} \leq 0 \\
&\quad \Leftrightarrow (3\beta-\gamma-2\beta^2)^2 - (2-\beta-\gamma)^2 + 4(1-\gamma)(1-\beta)^2 \leq 0 \\
&\quad \Leftrightarrow 9\beta^2 + \gamma^2 + 4\beta^4 - 6\beta\gamma + 4\gamma\beta^2 - 12\beta^3 - 4 - \beta^2 - \gamma^2 \\
&\qquad + 4\beta - 2\beta\gamma + 4\gamma + 4(1-\gamma)(1-2\beta+\beta^2) \leq 0 \\
&\quad \Leftrightarrow 8\beta^2 + 4\beta^4 - 8\beta\gamma + 4\gamma\beta^2 \\
&\qquad - 12\beta^3 - 4 + 4\beta + 4\gamma + 4(1-\gamma)(1-2\beta+\beta^2) \leq 0 \\
&\quad \Leftrightarrow 8\beta^2 + 4\beta^4 - 8\beta\gamma + 4\gamma\beta^2 - 12\beta^3 - 4 + 4\beta + 4\gamma + 4 - 8\beta \\
&\qquad + 4\beta^2 - 4\gamma + 8\beta\gamma - 4\beta^2\gamma \leq 0 \\
&\quad \Leftrightarrow 12\beta^2 + 4\beta^4 - 12\beta^3 - 4\beta \leq 0 \\
&\quad \Leftrightarrow 3 - 3\beta + \beta^2 \leq \frac{1}{\beta},
\end{aligned}$$

which, again, is true for all $\beta \in (0, 1)$.

We have proved that for all $\beta \in (0, 1)$, $\lambda_m(\beta, \cdot)$ is decreasing. Therefore, we just need to prove that $\inf_{0<\beta<1} \lambda_m(\beta, \beta) = 1/2$, which is true because:

$$\lambda_m(\beta, \beta) = \frac{2 - 2\beta - \sqrt{(2-2\beta)^2 - 4(1-\beta)^3}}{2(1-\beta)^2} = \frac{1-\sqrt{\beta}}{1-\beta} = \frac{1}{1+\sqrt{\beta}} \geq \frac{1}{2}.$$

This implies that $g_{\beta,\gamma}(\lambda) \geq 0$, for all $\lambda \in [0, 1/2]$, which in turn implies that $x = 1$ is a local minimum of $f_\lambda(\cdot)$ in $[\beta, 1]$. □

Lemma A.4 *Consider* $0 < \beta \leq 1$, $0 \leq \gamma \leq \beta$ *and* $f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}}$. *If* $0 \leq \lambda \leq 1/2$*, then*

$$\left|\{x \in (\beta, 1) : \frac{d}{dx} f_\lambda(x) = 0\}\right| \leq 1.$$

Proof Notice that

$$\begin{aligned} f_\lambda(x) &= \frac{1-x}{1-\lambda+\lambda x} + \frac{x-\gamma}{(1-\lambda)x+\lambda\beta} \\ &= \frac{x^2[2\lambda-1] + x[2-2\lambda-\lambda\beta-\lambda\gamma] + [\lambda\beta - \gamma(1-\lambda)]}{x^2[\lambda(1-\lambda)] + x[(1-\lambda)^2 + \lambda^2\beta] + [\lambda\beta(1-\lambda)]}, \end{aligned}$$

which implies that $f_\lambda(\cdot)$ has at most two extreme points. We will prove that one of them is always negative when $\lambda \leq 1/2$, which will conclude the proof. To do this, we compute $\frac{d}{dx} f_\lambda(x)$ and notice that extreme points solve a quadratic equation. By analyzing the corresponding coefficient we will conclude that, if there is a real extreme point, then there must be a real negative extreme point.

By direct computation, notice that $\frac{d}{dx} f_\lambda(x)$ if and only if

$$\begin{aligned} &(2x[2\lambda-1]+[2-2\lambda-\lambda\beta-\lambda\gamma])(x^2[\lambda(1-\lambda)] + x[(1-\lambda)^2+\lambda^2\beta] + [\lambda\beta(1-\lambda)) \\ &\quad - (2x[\lambda(1-\lambda)] + [(1-\lambda)^2 + \lambda^2\beta])(x^2[2\lambda-1] + x[2-2\lambda-\lambda\beta-\lambda\gamma] \\ &\quad + [\lambda\beta - \gamma(1-\lambda)]) = 0. \end{aligned}$$

Define and compute relevant terms by

$$\begin{aligned} a &:= (2-2\lambda-\lambda\beta-\lambda\gamma)\lambda(1-\lambda) + ((1-\lambda)^2+\lambda^2\beta)2(2\lambda-1) \\ &\quad \cdots - (2\lambda-1)((1-\lambda)^2+\lambda^2\beta) - (2-2\lambda-\lambda\beta-\lambda\gamma)2\lambda(1-\lambda) \\ &= (2\lambda - \lambda^2(2+\beta+\gamma))(1-\lambda) + (1-2\lambda+\lambda^2+\lambda^2\beta)(4\lambda-2) \\ &\quad \cdots - (1-2\lambda+\lambda^2+\lambda^2\beta)(2\lambda-1) - (2\lambda-\lambda^2(2+\beta+\gamma))(2-2\lambda) \\ &= (2\lambda - \lambda^2(2+\beta+\gamma))(\lambda-1) + (1-2\lambda+\lambda^2+\lambda^2\beta)(2\lambda-1) \\ &= \lambda^3[\beta-\gamma] + \lambda^2[\gamma-1] + \lambda[2] + [-1], \\ b &:= (2-2\lambda-\lambda\beta-\lambda\gamma)((1-\lambda)^2+\lambda^2\beta) + 2(2\lambda-1)\lambda\beta(1-\lambda) \end{aligned}$$

$$
\begin{aligned}
&\cdots - ((1-\lambda)^2 + \lambda^2\beta)(2 - 2\lambda - \lambda\beta - \lambda\gamma) - 2\lambda(1-\lambda)(\lambda\beta - \gamma(1-\lambda))\\
&= 2(2\lambda - 1)\lambda\beta(1-\lambda) - 2\lambda(1-\lambda)(\lambda\beta - \gamma(1-\lambda))\\
&= 2\lambda(1-\lambda)^2(\gamma - \beta),\\
c &= (2 - 2\lambda - \lambda\beta - \lambda\gamma)\lambda\beta(1-\lambda) - ((1-\lambda)^2 + \lambda^2\beta)(\lambda\beta - \gamma(1-\lambda)).
\end{aligned}
$$

Then, we have that, if x is an extreme point of $f_\lambda(\cdot)$, then x is a solution to $ax^2+bx+c = 0$.

Notice that $b \leq 0$, since $\gamma \leq \beta$. Then, one of the extreme points has the same sign as a (when it is a real number). Moreover,

$$
\begin{aligned}
\forall \gamma \leq \beta \in [0,1] \quad a &= \lambda^3[\beta - \gamma] + \lambda^2[\gamma - 1] + \lambda[2] + [-1]\\
&= \gamma\lambda^2(1-\lambda) + \beta\lambda^3 - \lambda^2 + 2\lambda - 1 \leq 0\\
&\Leftrightarrow \forall \beta \in [0,1](\beta - 1)\lambda^2 + 2\lambda - 1 \leq 0\\
&\Leftrightarrow 2\lambda - 1 \leq 0.
\end{aligned}
$$

Then, for all $\lambda \in [0, \frac{1}{2}]$, $a \leq 0$, therefore one of the extreme points of $f_\lambda(x)$, when real, is negative. □

Proof of Lemma A.1 Recall the optimization problem (P) is given by

$$
(P) \begin{cases} \min_x \ f_\lambda(x) := \frac{1-x}{1-\lambda+\lambda x} + \frac{1-\frac{\gamma}{x}}{1-\lambda+\lambda\frac{\beta}{x}} \\ s.t. \qquad\qquad \beta \leq x \leq 1 \end{cases}.
$$

Since $f_\lambda(\cdot)$ is a continuous function, there exists $x^* \in [\beta, 1]$ such that $f_\lambda(x^*) = \min_{\beta \leq x \leq 1} f_\lambda(x)$. By Lemma A.2, we can consider that $x^* \in (\beta, 1]$.

Assume by contradiction that there is $y \in (\beta, 1)$ such that $f_\lambda(y) < f_\lambda(1)$. Since we would also have that $f_\lambda(x^*) < f_\lambda(\beta)$, we can conclude that there exists $x^* \in (\beta, 1)$ local minimum of $f_\lambda(\cdot)$. But, by Lemma A.3, $f_\lambda(\cdot)$ is decreasing close to 1, so there must exist $y^* \in (x^*, 1)$ which is a local maximum of $f_\lambda(\cdot)$. Then,

$$
\left|\{x \in (\beta, 1) : \frac{d}{dx} f_\lambda(x) = 0\}\right| \geq 2,
$$

which contradicts Lemma A.4. Therefore, the value of (P) is $f_\lambda(1)$. □

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