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VORTICES IN THE ANISOTROPIC GINZBURG-LANDAU EQUATION FOR THIN NEMATIC LIQUID CRYSTAL CELLS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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VORTICES IN THE ANISOTROPIC GINZBURG-LANDAU EQUATION FOR THIN NEMATIC LIQUID CRYSTAL CELLS

In this thesis, a version of the anisotropic Ginzburg-Landau equation is studied as a model of defects in thin cells of nematic liquid crystals.

This thesis is composed of 6 chapters and 2 appendices. The first 3 chapters are introductory and serve to present the problem to be solved. The first chapter is focused on presenting the nematic liquid crystals. In the next two chapters, we introduce the vortices of the Ginzburg-Landau model in the context of liquid crystals with or without anisotropy. In addition some results that will be useful in the construction of anisotropic vortex-type solutions are stated.

Chapter 4 is dedicated to constructing vortices of positive and negative degree of the Ginzburg-Landau anisotropic equation, based on a perturbative approach through an application of Banach's Fixed Point Theorem and the invertibility of the linearized operator around the symmetric vortices in the suitable spaces. In addition the Fourier decomposition of the linear approximation of the negative degree perturbed vortex, for these solutions, energy expansion and stability, are also studied.

The chapter 5 is dedicated to the finite element method, of the linear approximation of the anisotropic vortex of negative degree, as well as a quantitative estimation of the quadratic coefficient in its energy expansion. With these, the complete bifurcation diagram of the energy of the anisotropic vortices is deduced for both negative and positive degrees.

In chapter 6, from the extension to $\mathbb{R}^2$ of the system of differential equations for the dominant mode of the Fourier series decomposition of the linear approximation perturbed solution found in chapter 4, we built a linear approximation of the negative anisotropic vortex in the entire plane.

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VÓRTICES DE LA ECUACIÓN ANISOTRÓPICA DE GINZBURG-LANDAU EN CELDAS DELGADAS DE CRISTALES LÍQUIDOS NEMÁTICOS

En esta tesis se estudia una versión de la ecuación anisotrópica de Ginzburg-Landau como un modelo de defectos en celdas delgadas de cristales líquidos nemáticos.

Esta tesis está compuesta de 6 capítulos y 2 apéndices. Los primeros 3 capítulos son introductorios y sirven para presentar el problema que se quiere resolver. El primer capítulo está enfocado en presentar los cristales líquidos nemáticos. En los siguientes 2 capítulos, introducimos los vórtices del modelo de Ginzburg-Landau en el contexto de cristales líquidos con o sin anisotropía. Además de resultados que nos serán útiles en la construcción de soluciones tipo vórtice anisotrópicos.

El capítulo 4 está dedicado a construir vórtices de grado positivo y negativo de la ecuación anisotrópica de Ginzburg-Landau, basándose en un enfoque perturbativo a través de una aplicación del Teorema de Punto Fijo de Banach y la invertibilidad del operador linealizado en torno a los vórtices simétricos en los espacios adecuados. También se estudia su expansión de energía y estabilidad de estas soluciones obtenidas. Además de su descomposición de Fourier de la aproximación lineal del vórtice de grado negativo perturbado.

El capítulo 5, está dedicado a hacer simulaciones a través del método de elementos finitos, de la aproximación lineal del vórtice anisotrópico de grado negativo, así como una estimación cuantitativa del coeficiente cuadrático en su expansión de energía. Con esto se deduce el diagrama de bifurcación completo de la energía de los vórtices anisotrópicos tanto de grado negativo y positivo.

En el capítulo 6, a partir de la extensión a $\mathbb{R}^2$ del sistema de ecuaciones diferenciales para el modo dominante de la descomposición en serie de Fourier de la aproximación lineal de la perturbación de la solución encontrada en el capítulo 4, se construye una aproximación lineal del vórtice anisotrópico de grado negativo en todo el plano.

A mi madre, Graciela Araya,
y a mi padre, José Luis Gómez,
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Chapter 1

Preliminar Concepts

1.1 Liquid Crystals

Liquid Crystals, as the name suggests, are phases which combine properties of ordered matter (like solid crystal) and disordered matter (like an isotropic liquid). Therefore, liquid crystals are materials that have local position and orientation correlation, [14, 24], which allows them to flow and also present crystal-like properties.

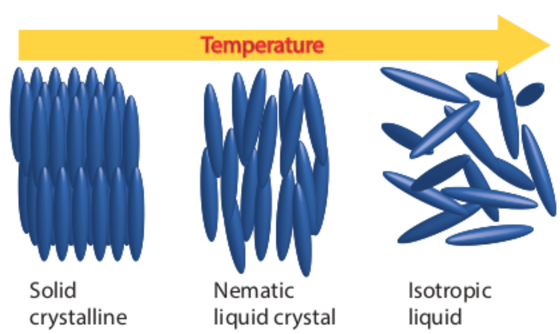

Figure 1.1: Different states of matter are represented according to the increase in temperature. When the temperature is low enough, the molecules are positionally arranged like a solid crystal. When the temperature is high, the order is lost and it becomes an isotropic liquid. In the middle, the liquid crystal state presents directional order, but not positional.

Liquid crystals have been a great source of interest since their discovery due to their optical properties and their use in technological applications, the best known being the Liquid Crystal Display (LCD).

Whereas there are many types of liquid crystals, we will focus in the so-called nematic liquid crystals, where the molecules that make them up, are like rod-shaped elongated. The preferred direction may vary throughout the medium and is called a director. The orientation of the director is represented by a unit vector $\vec{n}(\vec{r}, t)$, that describes the average molecule position in the liquid crystal at position $\vec{r}$ and a time $t > 0$.

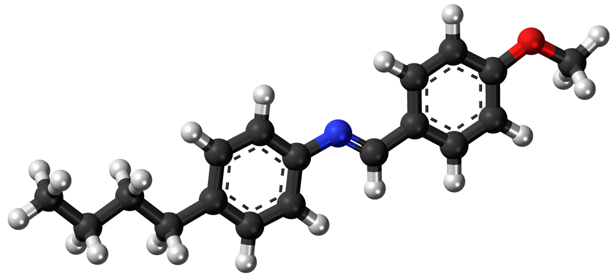

Figure 1.2: Molecular structure of MBBA (N-(4-Methoxybenzylidene)-4-butylaniline). A typical nematic liquid crystal. [3]

The Oseen-Frank theory regards $\vec{n}$ as a vector field. However, due to statistical head-to-tail symmetry of the constituent, this vector has the symmetry $\vec{n} = -\vec{n}$, or in other words «it does not have an arrowhead». Hence $\vec{n}$ can be understood as a line field, or equivalently, as a map from Ω to the set of all line through the origin. The set of such lines forms the real proyective plane $\mathbb{R}P^2$.

1.2 The Frank-Oseen model

Liquid crystal are a highly dissipative medium whose dynamic is characterized by minimizing their elastic energy. In nematic materials, there are three principal distinct director axis deformations: splay, twist and bend (figure 1.3). Each of these deformations has its own elastic constant, giving rise to the elastic energy of Frank-Oseen:

$$F_{\mathrm{d}} = \int_V \frac{K_1}{2}(\nabla \cdot \vec{\boldsymbol{n}})^2 + \frac{K_2}{2}(\vec{\boldsymbol{n}} \cdot (\nabla \times \vec{\boldsymbol{n}}))^2 + \frac{K_3}{2}\|\vec{\boldsymbol{n}} \times (\nabla \times \vec{\boldsymbol{n}})\|^2 \,\mathrm{d}\boldsymbol{x}$$

where K_1 corresponds to the splay deformation, K_2 to the twist deformation, K_3 to the bend one, and V corresponds a certain volume sample where the average of the molecules of the director vector is taken. These constants are usually of the order of 10^{-6} dyne, for example for MBBA their values are 5.8×10^{-7}, 3.4×10^{-7}, and 7×10^{-7} dyne, for K_1, K_2, and K_3, respectively.

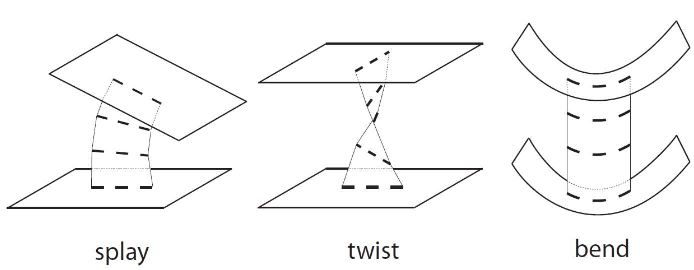

Figure 1.3: Principal elastic deformations on a nematic liquid crystal.

If the sample of liquid crystal is subject to electric or magnetic fields, for example, if we have the electric field is given by $\vec{E}$, then

$$F_{\mathrm{e}} = -\int_V \frac{\varepsilon_a}{2} (\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{n}})^2 \, \mathrm{d}\boldsymbol{x},$$

where $\varepsilon_a = \varepsilon_{\|} - \varepsilon_{\perp}$ is the anisotropic dielectric constant that accounts for nonlinear response of the electric field, with $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ the dielectric susceptibility for low-frequency electric fields parallel and orthogonal, respectively, to the molecular director.

Chapter 2

Ginzburg-Landau equations

2.1 Pitchfork Bifurcation

Bifurcations are qualitative changes on the properties of a system as a control parameter is changed [42]. One example of our interest, is the case when a solution changes its stability and two new stable solutions appear, it is called a **Supercritical Pitchfork Bifurcation**, and it is common in system with reflection symmetry.

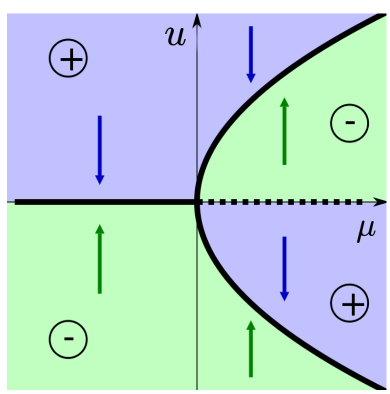

Figure 2.1: Supercritical Pitchfork Bifurcation [2]

The simplest system that presents this bifurcation

$$\partial_t u = \mu\, u - u^3 \tag{2.1}$$

here we recognize the bifurcation parameter μ, the bifurcation that occurs at $\mu = 0$. Besides, the equation (2.1) presents the reflection symmetry $u \mapsto -u$.

This system has three steady states: $u_0 = 0$ and $u_\pm = \pm\sqrt{\mu}$. For $\mu < 0$, there is one stable equilibrium at $u_0 = 0$. For $\mu > 0$, there is an unstable equilibrium at $u_0 = 0$, and two stable equilibria at $u_\pm = \pm\sqrt{\mu}$.

2.1.1 Degenerated Pitchfork Bifurcation

Now, we try to understand the Pitchfork Bifurcation in a two dimension variable, instead of just one. The system must have rotational invariance instead of reflection symmetry. For example, we can imagine this situation with an elastic rod subject to gravitational pull, fixed in its base. If the rod is short, it can stay straight, but if the rod is too long it will inevitable bend in any direction, or, in cylindrical coordinates, in any angle between 0 and 2π, as is shown in Figure 2.2.

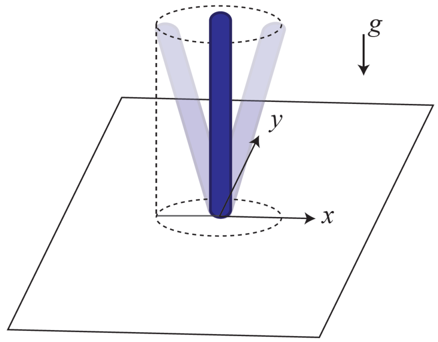

Figure 2.2: Illustration of rod can bend in any direction. Extracted from [48].

To describe a system which accounts the above situation, we need to introduce a complex parameter $A \in \mathbb{C}$, in this way $\mathbb{Re}(A)$ and $\mathbb{Im}(A)$, respectively, represent the deviation of the rod in the x-axis, and in the perpendicular y-axis. We will call this parameter $A = A(\vec{r}, t)$, where $\vec{r}$ denotes the spatial variable and t the time variable, order parameter, a concept that was first introduced by Ginzburg and Landau in the context of phase transitions [44]. Generally, this order parameter is a combination of the relevant fields in the system in a way that important changes in the system, can be easily visualized as changes in the order parameter.

The amplitude equation that describes this type of bifurcation, is the following normal form

$$\partial_t A = \mu A - |A|^2 A \tag{2.2}$$

and it is called a **Degenerated Pitchfork Bifurcation.**

We can consider this type of bifurcation in more complex systems, for example considering an extended system. If we work, first in the one-dimensional case, we can consider the real parameter $A = u(x, t)$, satisfying

$$\partial_t u = \mu u - u^3 + \partial_{xx} u \tag{2.3}$$

which has a structure similar to the equation (2.2), since it also presents the symmetry $u \mapsto -u$. In this equation (2.3), there are two symmetrical homogeneous solutions, when $\mu > 0$, $u = \sqrt{\mu}$ and $u = -\sqrt{\mu}$, these solutions are both stable and have the same energy. Besides, owing to initial conditions or fluctuations, two solutions can exist in different locations simultaneously, these different locations are called domains. When this happens the two solutions are connected in a smooth way through the Kink solution, $u = \sqrt{\mu}\tanh(x\sqrt{\mu/2})$.

Furthermore, owing to the symmetry of the system, the analogous Antikink also exists $u = -\sqrt{\mu}\tanh(x\sqrt{\mu/2})$.

Now, considering the bifurcation in an extended system in the two-dimensional case, we have that the easiest equation that describes this, is the Ginzburg Landau-Equation with real coefficients

$$\partial_t A = \mu A - |A|^2 A + \Delta A \tag{2.4}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is understood as the usual Laplacian in 2 coordinates.

The system above presents a Degenerated Pitchfork Bifurcation when $\mu = 0$, which means that the homogeneous state $A = \sqrt{\mu}\,\mathrm{e}^{\mathrm{i}\phi_0}$ can take any value of ϕ_0, when $\mu > 0$. And just like in 1-D case, the system can take different directions in different zones. If there are two different directions (values of ϕ_0) in the system, they are connected through a wall solution $A = \mathrm{e}^{\mathrm{i}\phi_0}\sqrt{\mu}\tanh(x\sqrt{\mu/2})$, which is very similar to a kink, but extended, and therefore it is known as an extended defect, since there is a continuum of points where the amplitude becomes zero.

It is important note, that the equation (2.4) can be rewritten in the form $\partial_t A = -\frac{\delta\mathcal{E}}{\delta\overline{A}}$, where the free energy is

$$\mathcal{E}(A) = \int \left(|\nabla A|^2 + \frac{1}{2}(\mu - |A|^2)^2\right) \mathrm{d}S$$

Namely, the Ginzburg-Landau equation (2.4) is simply a gradient flow of the free energy. Moreover, $\mathcal{E}$ is a Lyapunov functional:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int \left(\frac{\delta\mathcal{E}}{\delta A}\partial_t A + \frac{\delta\mathcal{E}}{\delta\overline{A}}\partial_t\overline{A}\right) \mathrm{d}S = -2\int \frac{\delta\mathcal{E}}{\delta A}\frac{\delta\mathcal{E}}{\delta\overline{A}}\,\mathrm{d}S \;\leq\; 0$$

and the minimal energy solution corresponds to the homogeneous state $A = \sqrt{\mu}\,\mathrm{e}^{\mathrm{i}\phi_0}$.

2.2 Vortex solutions of Ginzburg-Landau Equation

The complex Ginzburg-Landau equation appears in different systems such as fluids, superfluids, superconductors, granular matter and liquid crystals, to mention a few [42]. The main properties of the complex Ginzburg-Landau equation are reported in the review [7]. In 2-D this equation describes any stationary degenerate supercritical bifurcation [17], which appears in the most system that presents vortices, understanding them as structures with zero amplitude and a phase discontinuity at their center.

The stationary equation is written simply as

$$\Delta A + (\mu - |A|^2)\,A = 0, \tag{2.5}$$

where (in the case of the entire plane) $A : \mathbb{R}^2 \to \mathbb{C}$, we add the following boundary condition:

$$|A| \to \sqrt{\mu} \quad \text{as} \quad |x| \to \infty, \tag{2.6}$$

The most studied solution of this equation is the Dissipative Vortex Solution: a rotationally symmetric solution, where the amplitude is axisymmetric and the phase grows continuously around the vortex: $A_m = R_m(r)e^{i(m\theta+\theta_0)}$, where (r, θ) are the polar coordinates in $\mathbb{R}^2$, with the origin in the vortex position and θ_0 is a continuous parameter that accounts for the phase invariance of the amplitude and shows explicitly the position of the phase discontinuity. Solutions A_m are called m-vortices. We note that the boundary condition (2.6) allows one to introduce $A = \text{deg}A_m$, the degree of A as the winding number at ∞ (vorticity), considered as a vector field in $\mathbb{R}^2$:

$$\text{deg}A = \frac{1}{2\pi}\int_{|x|=R} \text{d}(\arg A), \qquad R \gg 1$$

The function $R_m(r)$, satisfies the ordinary differential equation

$$\begin{cases} \dfrac{\partial^2 R}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial R}{\partial r} - \dfrac{m^2 R}{r^2} + \mu R - R^3 = 0, \qquad r \in (0, +\infty) \\ \\ R(0) = 0, \quad R(+\infty) = \sqrt{\mu} \end{cases} \tag{2.7}$$

The vanishing of the real amplitude at the origin, is necessary to eliminate the divergence due to the phase singularity, and the condition at $+\infty$ is compatible with (2.6).

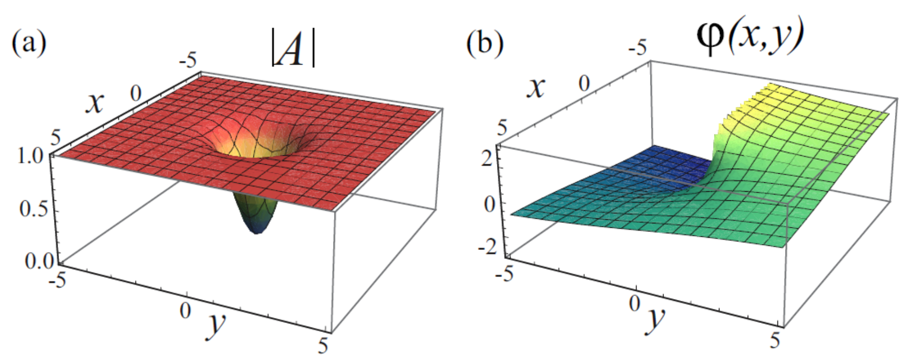

Figure 2.3: Vortex solution with charge $m = +1$ of Ginzburg-Landau equation (2.5) (here $\mu = 1$). Structure of the magnitude (a) and phase of the positive vortex (b) [48]

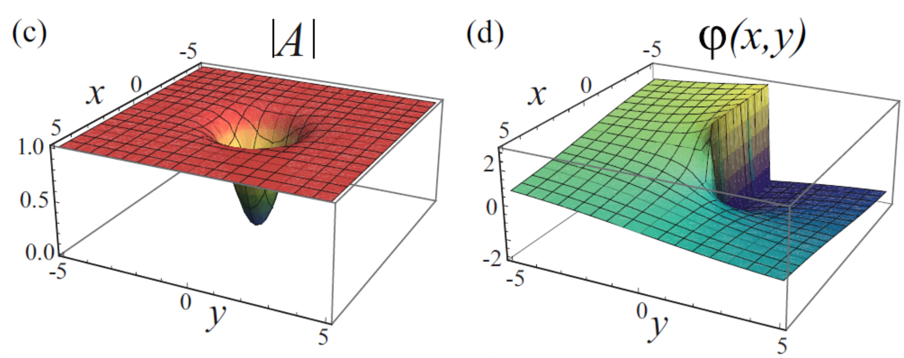

Figure 2.4: Vortex solution with charge $m = -1$ of Ginzburg-Landau equation (2.5) (here $\mu = 1$). Structure of the magnitude (c) and phase of the positive vortex (d) [48]

The solutions of (2.7) are not explicit, nevertheless we can find their asymptotic behavior close to infinity and near the origin (see for instance, [15, 30])

$$R(r) \approx \begin{cases} \alpha_m \, r^{|m|} + \ldots, & r \to 0 \\ \sqrt{\mu} - \frac{m^2}{2} r^{-2} + \ldots & r \to \infty \end{cases} \tag{2.8}$$

where α_m is a positive constant that depends on μ. A good approximation developed by Pismen in [42] for $\mu = 1$ and vorticity $m = \pm 1$ in the form of a Padé approximation for the square of the amplitude magnitude is

$$R(r) \approx \sqrt{\frac{0.34r^2 + 0.07r^4}{1 + 0.42r^2 + 0.07r^4}}$$

For different values of μ this approximation has to be scaled as $R_\mu(r) = \sqrt{\mu} R_{\mu=1}(\sqrt{\mu} r)$.

The existence of other non-rotationally symmetric solutions is still an open problem, see for example, [40, 20].

2.2.1 Symmetries of the Ginzburg-Landau equation

Lemma 2.2.1. *The Ginzburg-Landau equation* (2.4) *is invariant under the following symmetries:*

- *Spatial translation transformation:*

$$\forall\, h \in \mathbb{R}^2, \quad T_h : A(\vec{r}, t) \mapsto A(\vec{r} + h, t)$$

- *Coordinates rotation and reflection transformation:*

$$\forall\, R \in O(2), \quad T_R : A(\vec{r}, t) \mapsto A(R\,\vec{r}, t).$$

- *Gauge transformations or phase invariance:*

$$\forall\, e^{i\phi_0} \in U(1), \quad T_{\phi_0} : A(\vec{r}, t) \mapsto e^{i\phi_0} A(\vec{r}, t).$$

- *Charge transformations:* $A(\vec{r}, t) \mapsto \bar{A}(\vec{r}, t)$.

We note that the **symmetry group**, G_{sym}, of equation (2.5), i.e. the maximal group of transformations, g, of A, such that, is A is a solution to (2.4) , then so is gA, is

$$G_{sym} = \mathbb{R}^2 \times O(2) \times U(1) \times \text{Charge}$$

By the *symmetry group* G_ψ of a *solution* ψ, we understand the largest subgroup of G_{sym} which leaves ψ fixed, i.e $G_\psi = \{g \in G_{sym} \,|\, g\psi = \psi\}$. Then the part of G_{sym} broken by ψ is G_{sym}/G_ψ. It is also considered (one parameter) subgroup $H \subset G_{sym}$ *preserved* (or *broken*) by ψ meaning by this that $h\psi = \psi \;\forall\, h \in H$.

As an example, the subgroup of translations, $\mathbb{R}^2$, is preserved iff ψ is independent of x. This happens only if $\deg\psi = 0$ and the solution ψ in this case is $\psi = \mathrm{e}^{\mathrm{i}\alpha}, \alpha \in \mathbb{R}$. This solution preserves also the subgroup of rotations but breaks the gauge and Charge subgroups.

Another class of solutions, more of our interest, are the rotationally symmetric solutions, m-vortices,

$$A_m = R_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2$. The symmetry group of A_m, $m \neq 0$, is

$$\Gamma \times U(1)^{-m}O(2)$$

where Γ is the discrete subgroup of $O(2)$ of rotations by the angles $\dfrac{2\pi k}{m}$, $k \in \mathbb{Z}$, and

$$U(1)^m O(2) = \{\mathrm{e}^{\mathrm{i}\varphi m} r(\varphi) \mid \varphi \in [0, 2\pi]\},$$

where $r(\varphi)\psi(x) = \psi(R(\varphi)^{-1}x)$ with $R(\varphi)$, the rotation by the angle φ. Thus A_m breaks the translations subgroup, $\mathbb{R}^2$, the rotation subgroup $O(2)/\Gamma$ and the charge subgroup.

2.3 Energy

In order to simplify calculations, without loss of generality, we can take $\mu = 1$. Thus, we will consider the equation (2.5):

$$\Delta A + (1 - |A|^2)\, A = 0, \quad A : \mathbb{R}^2 \to \mathbb{C} \qquad \text{Repeat eq. (2.5)}$$

with the boundary condition (2.6):

$$|A| \to 1 \quad \text{as} \quad |x| \to \infty, \qquad \text{Repeat eq. (2.6)}$$

Therefore, the free energy associated to (2.5) it is:

$$\mathcal{E}(A) = \frac{1}{2}\int \left(|\nabla A|^2 + \frac{1}{2}(1 - |A|^2)^2 \right) \mathrm{d}S \tag{2.9}$$

replacing the vortex solution $A = R_v(r)\mathrm{e}^{\mathrm{i}m\theta}$, where R_v corresponds to the profile of the vortex:

$$\mathcal{E}(A) = \frac{1}{2}\int (\partial_r R_v)^2 + \frac{m^2 R_v^2}{r^2} + \frac{1}{2}(1 - R_v^2)^2\,\mathrm{d}S$$

separating the energy terms, we can write the energy $\mathcal{E}$ as:

$$\mathcal{E}(A) = \underbrace{\pi \int_0^\infty \left((\partial_r R_v)^2 + \frac{m^2 R_v^2}{r^2} \right) r\,\mathrm{d}r}_{\mathcal{E}_1} + \underbrace{\frac{\pi}{2}\int_0^\infty (1 - R_v^2)^2\, r\,\mathrm{d}r}_{\mathcal{E}_2}$$

the term $\mathcal{E}_1$ is divergent in an infinite domain, but the term $\mathcal{E}_2$ can be solved analytically using the equation (2.7) satisfied by vortex profile R_v, and integrating by parts:

$$\mathcal{E}_2 = \pi \int_0^\infty (1 - R_v^2) R_v R_v' \, r^2 \, \mathrm{d}r = -\pi \int_0^\infty \left(r R_v' \frac{\mathrm{d}(r R_v')}{\mathrm{d}r} - m^2 R_v R_v' \right) \mathrm{d}r = \frac{\pi m^2}{2}$$

Now, to calculate $\mathcal{E}_1$ we need to introduce a cut-off at a distance L (see Theorem 2.3.1), the divergent term depends on a numerical constant a_0 giving by the specific shape of the vortex-core solution:

$$\mathcal{E}_1 \approx \pi \, m^2 \ln \left(\frac{L}{a_0} \right), \tag{2.10}$$

it follows that, for $A = R_v(r) \mathrm{e}^{im\theta}$ (the vortex solution):

$$\mathcal{E}(A) \approx \pi \, m^2 \ln \left(\frac{L\sqrt{\mathrm{e}}}{a_0} \right) \tag{2.11}$$

Therefore, both vortices are indistinguishable from the point of view of their vorticity magnitude.

In order to justify the use of the cut-off function in $\mathcal{E}_1$, we have the following theorem [39]
Theorem 2.3.1. *Let ψ be a C^1 vector field on $\mathbb{R}^2$ such that $|\psi| \to 1$ as $|x| \to \infty$.*
If deg(ψ) $\neq 0$, then $\mathcal{E}[\psi] = \infty$.

Proof. If we write $\psi = f \, \mathrm{e}^{\mathrm{i}\varphi}$ with $f = |\psi|$ and $\varphi = \arg \psi$, then $|\nabla \psi|^2 = |\nabla f|^2 + f^2 |\nabla \varphi|^2$, and hence

$$\int |\nabla \psi|^2 \geq \int f^2 |\nabla \varphi|^2$$

Moreover, by the condition at $+\infty$ on $f = |\psi|$, there is R such that $f \geq \frac{1}{\sqrt{2}}$ for all $|x| \geq R$. Thus

$$\int |\nabla \psi|^2 \geq \frac{1}{2} \int_{|x| \geq R} |\nabla \varphi|^2 \tag{2.12}$$

In addition, the relation $\displaystyle\int_{|x|=r} \mathrm{d}\varphi = 2\pi \deg \psi$ implies that

$$2\pi |\deg \psi| \leq r \int_0^{2\pi} |\nabla \varphi| \, \mathrm{d}\theta \leq r \left(2\pi \int_0^{2\pi} |\nabla \varphi|^2 \, \mathrm{d}\theta \right)^{1/2}$$

by the Cauchy-Schwartz inequality. This implies that $\displaystyle\int_0^{2\pi} |\nabla \varphi|^2 \, \mathrm{d}\theta \geq \frac{2\pi (\deg \psi)^2}{r^2}$ that together with (2.12) yields

$$\int |\nabla \psi|^2 \geq \frac{1}{2} \int_{|x| \geq R} |\nabla \varphi|^2 = \frac{1}{2} \int_0^{2\pi} \int_R^\infty |\nabla \varphi|^2 \, r \, \mathrm{d}r \, \mathrm{d}\theta \geq \pi (\deg \psi)^2 \int_R^\infty \frac{1}{r^2} r \, \mathrm{d}r = \infty.$$

■

Thus if we want to use energy arguments for vortices, the proof above shows us how to modify $\mathcal{E}$ as follows:

$$\mathcal{E}_{\text{ren}}[\psi] = \frac{1}{2}\int \left(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + \frac{1}{2}(1-|\psi|^2)^2 \right) \mathrm{d}S \tag{2.13}$$

where $r = |x|$ and $\chi \in C^\infty(\mathbb{R}^2)$ such that

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \geq 2 \\ 0 & \text{for } |x| \leq 1 \end{cases}$$

This modification (2.13) of the energy functional, is called the renormalized Ginzburg-Landau energy functional [39, 45].

2.4 Linearized operator of Ginzburg-Landau equation around vortex solution

2.4.1 Kernel of Linearized operator around vortex solution in the plane

Definition 2.4.1 (**Gâteaux-derivative**). *[4] Let W, V normed vector spaces, Ω an open subset of W, $w \in \Omega$ and $g : \Omega \subset W \to V$ a function. We say that g is Gâteaux-differentiable at w, if there exists $A \in \mathcal{L}(W, V)$ such that:*

$$(\forall\, z \in W) \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} g(w+\lambda z)\bigg|_{\lambda=0} = \lim_{\lambda\to 0^+} \frac{g(w+\lambda z)-g(w)}{\lambda} = A[z]$$

The map A is unique determined, called the G-differential or Gâteaux-derivative of g at w and denoted by $A = g'_G(w)$.

Given an equation

$$\mathcal{M}(u) = 0 \tag{2.14}$$

and its solution u_0, the **linearization** of this equation around u_0 is the equation

$$D_G\mathcal{M}(u_0)(w) = 0$$

where $D_G\mathcal{M}(u_0)$ is the Gâteaux-derivative, and the operator $D_G\mathcal{M}(u_0) : X \to Y$ is the **linearized operator** associated to (2.14).

In our case, $\mathcal{M}(u) = \Delta u + u(1-|u|^2)$ and therefore

$$D_G\mathcal{M}(u_0)(w) = \Delta w + (1-|u_0|^2)w - 2(u_0 \cdot w)\, u_0$$

where, by definition,

$$u \cdot w := \frac{\bar{u}\, w + u\, \bar{w}}{2} = \mathbb{R}\mathrm{e}(u\, \bar{w})$$

is the usual scalar product in $\mathbb{C}$.

When $u_0 = U_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ is the symmetric m-vortex solution, we will denote the corresponding linearized operator by $\mathbb{L}_m$. In order to study $\mathbb{L}_m$, it happens to be more convenient to write any complex valued function w as

$$w := (\alpha + \mathrm{i}\,\beta)\,\mathrm{e}^{\mathrm{i}m\theta},$$

where α and β are real valued functions. Because of this decomposition, it is natural to define the conjugate linearized operator by

$$\mathcal{L}_m := \mathrm{e}^{-\mathrm{i}m\theta}\mathbb{L}_m \mathrm{e}^{\mathrm{i}m\theta} \tag{2.15}$$

If follows from a simple computation that

$$\mathcal{L}_m(\alpha + \mathrm{i}\,\beta) = \left(\mathfrak{L}_1^{(m)}\alpha - \frac{2m}{r^2}\partial_\theta\beta\right) + \mathrm{i}\left(\mathfrak{L}_2^{(m)}\beta + \frac{2m}{r^2}\partial_\theta\alpha\right) \tag{2.16}$$

where

$$\mathfrak{L}_1^{(m)}\alpha = \Delta\alpha + \left(-\frac{m^2}{r^2} + 1 - 3\,U_m^2(r)\right)\alpha \tag{2.17}$$

$$\mathfrak{L}_2^{(m)}\beta = \Delta\beta + \left(-\frac{m^2}{r^2} + 1 - U_m^2(r)\right)\beta \tag{2.18}$$

Linearized operator when $|m| = 1$

Theorem 2.4.2. *[39] Let u_0 be a solution to the equation $\mathcal{M}(u_0) = 0$ breaking an one parameter subgroup $g(s) \in G_{sym}$ (the symmetry group of this equation). Let T be the generator of $g(s)$. Then Tu_0 solves the linearized equation $D_G\mathcal{M}(u_0)w = 0$,*

Applying this result to our case and observing that the generators of translations, rotations and gauge transformations are ∇_x, $x_1\partial_{x_2} - x_2\partial_{x_1} = \partial_\theta$ and i, respectively. Thus, we have the following result:

Lemma 2.4.3 (Kernel of $\mathbb{L}_{\pm 1}$ Part I)**.** *[39, 41] If we denote by $u_0^\pm = U(r)\mathrm{e}^{\pm\mathrm{i}\theta}$ the vortex solution of degree $|m| = 1$, the following linearly independent Jacobi Fields are in the Kernel of $\mathbb{L}_{\pm 1}$, given by the invariance of the equation* (2.5) *under rotation and translations*

1. *The rotational invariance yields the solution*

$$\mathrm{i}\,u_0^\pm = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\theta}\,U(r)$$

2. *The translational invariance along x_1 and x_2 direction leads to the solutions*

$$\frac{\partial u_0^\pm}{\partial x_1} = \mathrm{e}^{\pm\mathrm{i}\,\theta}\left[U'(r)\cos\theta \mp \mathrm{i}\,\frac{U(r)}{r}\sin\theta\right]$$

$$\frac{\partial u_0^\pm}{\partial x_2} = \mathrm{e}^{\pm\mathrm{i}\,\theta}\left[U'(r)\sin\theta \pm i\,\frac{U(r)}{r}\cos\theta\right]$$

Theorem 2.4.4 (Kernel of $\mathbb{L}_{\pm 1}$ Part II).
[41] All solutions of $\mathbb{L}_{\pm 1} w = 0$ which are defined on all $\mathbb{C}$ and are bounded in L^∞-norm, are linear combinations of

$$\frac{\partial u_0^\pm}{\partial x_1}, \quad \frac{\partial u_0^\pm}{\partial x_2}, \quad \mathrm{i}\, u_0^\pm$$

Therefore, if we denote by

$$\mathcal{Z} = span_{\mathbb{C}} \left\{ \frac{\partial u_0^\pm}{\partial x_1}, \frac{\partial u_0^\pm}{\partial x_2}, \mathrm{i}\, u_0^\pm \right\}$$

We have that

$$\ker(\mathbb{L}_{\pm 1}) \cap L^\infty(\mathbb{R}^2, \mathbb{C}) \supset \mathcal{Z}$$

where $L^\infty(\mathbb{R}^2, \mathbb{C})$ is the set of functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ such that $\|\psi\|_\infty := \sup_{z \in \mathbb{R}^2} |\psi(z)| < \infty$.

Remark 2.4.5. *An observation we should make, is that $\mathcal{Z}$ is not contained in $L^2(\mathbb{R}^2; \mathbb{C})$.*

2.4.2 Linearized operator on a bounded domain

In this part, we consider the linearized operator $\mathbb{L}_{\pm 1}$ defined for functions in the bounded domain $B_R = B(0, R)$ with $R > 0$, we will denote this operator by L, where $u_0 = U(r)\mathrm{e}^{\pm \mathrm{i}\theta}$ is the vortex solution of the Ginzburg-Landau equation in the ball B_R:

$$\begin{cases} \Delta u + u(1 - |u|^2) = 0, & \text{in } B_R \\ u = \mathrm{e}^{\pm \mathrm{i}\theta} & \text{on } \partial B_R \end{cases} \tag{2.19}$$

and $U(r) = U_R(r)$ satisfies the following ordinary differential equation:

$$\begin{cases} \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} - \dfrac{U}{r^2} + U(1 - U^2) = 0, & r \in (0, R) \\ \\ U(0) = 0, \quad U(R) = 1, \quad U \geq 0 \end{cases} \tag{2.20}$$

The energy functional associated to equation (2.19) is given by

$$E(v) = \int_{B_R} \frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1 - |v|^2)^2$$

and Q_0 the bilinear form given by the (formal) second variation of E around u_0,

$$Q_0(w) := \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} E(u_0 + \lambda w)\bigg|_{\lambda = 0} = \int_{B_R} |\nabla w|^2 - \int_{B_R} (1 - |u_0|^2)|w|^2 + 2\int_{B_R} (u_0 \cdot w)^2 \tag{2.21}$$

We observe that $Q_0(w) = \langle -Lw, w \rangle$, where here and in what follows

$$\langle w_1, w_2 \rangle := \mathbb{R}\mathrm{e} \int_{B_R} w_1 \overline{w_2}$$

Since L is self-adjoint, then all the eigenvalues of L must be real. Hence the eigenvalue becomes

$$Lw = \lambda w, \quad \lambda \in \mathbb{R}, \qquad w \in H_0^1(B_R;\mathbb{C}) \cap H^2(B_R;\mathbb{C}). \tag{2.22}$$

Lieb and Loss [33] proved that the first eigenvalue on (2.22) is non-negative. Later Mironescu [36] showed that

Theorem 2.4.6. *The first eigenvalue λ_1 of $-L$ is positive.*

Besides, he showed that

Theorem 2.4.7. *The symmetric vortex solution u_0 is stable, in the sense that the quadratic form Q_0 associated to E is positive definite.*
Moreover, $\forall\, w \in H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C})$ we have that

$$Q_0(w) \geq \lambda_1 \|w\|^2_{L^2(B_R)} \tag{2.23}$$

where λ_1 is the first eigenvalue of the linearized operator L around u_0.

As a consequence of theorem 2.4.7, we have the following property about L as operator from $H^2(B_R,\mathbb{C}) \cap H_0^1(B_R,\mathbb{C})$ onto $L^2(B_R,\mathbb{C})$:

Corollary 2.4.7.1 (The invertibility of the linearized operator L).
The linearized operator $L : H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C}) \to L^2(B_R;\mathbb{C})$ around u_0 defined by

$$Lv = \Delta v + v(1 - |u_0|^2) - 2(u_0 \cdot v)u_0$$

is an isomorphism, that is, $L^{-1} \in \mathcal{L}(L^2(B_R;\mathbb{C}), H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C}))$ is such that

$$\|L^{-1}(w)\|_{H^2(B_R)} \leq \|L^{-1}\|\|w\|_{L^2(B_R)}$$

where $\|L^{-1}\| := \|L^{-1}\|_{\mathcal{L}(L^2(B_R),H^2(B_R))} = \sup_{w\neq 0} \dfrac{\|L^{-1}(w)\|_{H^2(B_R)}}{\|w\|_{L^2(B_R)}} < \infty.$

Proof. Is a direct application of Lax-Milgram Theorem (2.4.8) to $Lv = f$, with

$$a(v,w) = \int_\Omega (-Lv \cdot w)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad l(w) = -\int_\Omega (f \cdot w)\,\mathrm{d}x\,\mathrm{d}y \tag{2.24}$$

clearly a, l are continuous, bilinear and linear respectively. The coercivity of a, it follows from (2.23). ■

Theorem 2.4.8 (Lax-Milgram). *[12] Let H a Hilbert space equipped with a scalar product $\langle v,w\rangle \in \mathbb{R}$, and let $|v| = \langle v,v\rangle^{1/2}$ his norm.*
Assume that $a : H \times H \to \mathbb{R}$ is a continuous, coercive bilinear form. Then, given any $l \in H^*$, there exists a unique element $u \in H$ such that

$$a(v,w) = \langle l,w\rangle \qquad \forall\, v \in H$$

Chapter 3

Anisotropic Ginzburg Landau Equation

3.1 Experimental Setup

The nematic liquid crystal cell is composed of a thin nematic liquid crystal film sandwiched between two glass plates, one of them has a photoconductive slab, usually these plates have a surface of about 1-5 cm^2, while the separation between the plates is $5-50\mu m$. Thus a large surface is available to observe different patterns. The two glass plates have been chemically treated to provide an homeotropic anchoring to the molecules, that is, the alignment direction of liquid crystal molecules close to the confining layers are perpendicular.

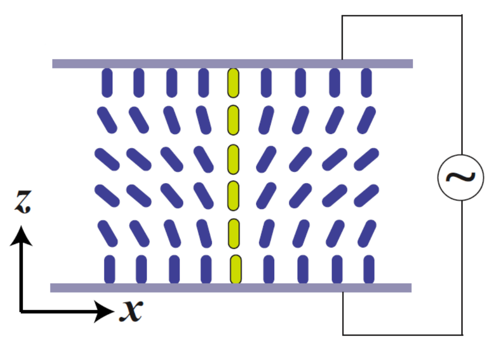

Figure 3.1: Schematic representation of the system under study, the rods describe the orientation of the director and the gray rods (green rods) stand for the vortex position. Adapted from [16]

If the material has negative dielectric constant ε_a, when a voltage is applied to the plates, the molecules will tend to align perpendicular to the electric field in order to minimize the interaction energy. This electric forces opposes the elasticity, therefore, for low voltages the samples remains in equilibrium, but if the voltages is increased above a threshold, a transitions occurs at a voltage known as **Freedericksz Voltage** [21] and a Schlieren texture appears as in figure (3.2).

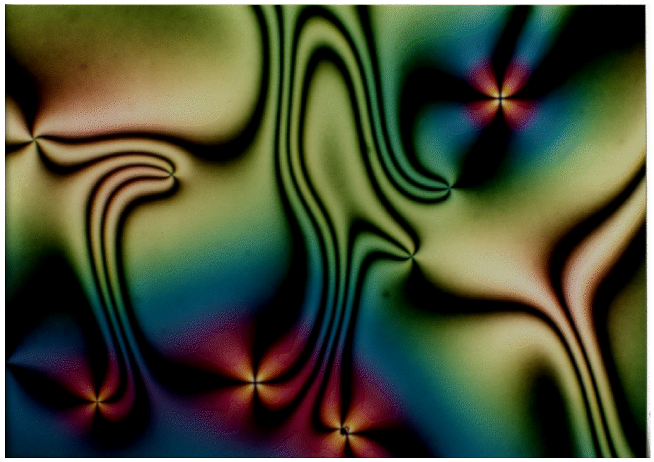

Figure 3.2: Schlieren Texture of a Nematic Film [1]

This transition is a Degenerated Pitchfork Bifurcation, where the molecules leave the vertical axis in a angle that depends on the magnitude of the voltage, but there is a cone of possible equilibrium positions for the molecules.

3.2 Amplitude Equation Derivation

The description of the nematic liquid crystal inside the cell is given completely by the molecular director $\vec{\boldsymbol{n}}(x, y)$, which corresponds to the molecular order in the position (x, y).
We have to minimize the free energy, it is the sum between the Frank-Oseen elastic energy and the energy due to the interaction of the liquid crystal with the electric field produced by the applied voltage.

$$F = \frac{1}{2}\int_V (K_1(\nabla\cdot\vec{\boldsymbol{n}})^2 + K_2(\vec{\boldsymbol{n}}\cdot(\nabla\times\vec{\boldsymbol{n}}))^2 + K_3\|\vec{\boldsymbol{n}}\times(\nabla\times\vec{\boldsymbol{n}})\|^2 - \varepsilon_a(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{n}})^2)\,\mathrm{d}\boldsymbol{x}$$

Before we arrive at the dynamic equation, note that the free energy has the form

$$W(\vec{\boldsymbol{n}}) = \int_V w(\vec{\boldsymbol{n}}, \nabla\vec{\boldsymbol{n}})\,\mathrm{d}\boldsymbol{x}.$$

Therefore minimizing the free energy, with the additional constraint $\|\vec{\boldsymbol{n}}\| = 1$, the Euler-Lagrange equation for $\vec{\boldsymbol{n}}$ is:

$$\gamma\,\partial_t\boldsymbol{n} = -\frac{\delta W}{\delta\boldsymbol{n}} \quad \text{s.a. } \|\boldsymbol{n}\| = 1 \quad \Longleftrightarrow \quad \gamma\,\partial_t\boldsymbol{n} = -\frac{\delta W}{\delta\boldsymbol{n}} + \boldsymbol{n}\left(\boldsymbol{n}\cdot\frac{\delta W}{\delta\boldsymbol{n}}\right) \tag{3.1}$$

where γ is the relaxation time, and W is the free energy.

Besides $\dfrac{\delta W}{\delta\boldsymbol{n}}$ is the gradient derivative of W, which is define given a variation $\mathbf{d}(\boldsymbol{x})$, it has

$$\left.\frac{\mathrm{d}W[\boldsymbol{n}+\varepsilon\,\mathbf{d}]}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} =: \int \frac{\delta W[\boldsymbol{n}]}{\delta\boldsymbol{n}(\boldsymbol{x})}\mathbf{d}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

Its known that the gradient derivative is given by the Euler-Lagrange equations:

$$\frac{\delta W}{\delta \boldsymbol{n}} = -\operatorname{div}\left(\frac{\partial W}{\partial \nabla \boldsymbol{n}}(\boldsymbol{n}, \nabla \boldsymbol{n})\right) + \frac{\partial W}{\partial \boldsymbol{n}} = \left[-\frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial n_{\mathrm{i},j}}\right) + \frac{\partial W}{\partial n_{\mathrm{i}}}\right]\mathbf{e}_{\mathbf{i}}$$

where we have used the notation $\nabla \boldsymbol{n} = (\boldsymbol{n}_{\mathrm{i},j})\,\mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\boldsymbol{j}} := \left(\dfrac{\partial n_{\mathrm{i}}}{\partial x_j}\right)\mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\boldsymbol{j}}$.

Computing the terms of the gradient derivative, as in [24, page 107] and replacing in (3.1), we find that $\vec{\boldsymbol{n}}$ satisfies the following nonlinear partial differential equation

$$\begin{aligned}\gamma\frac{\mathrm{d}\vec{\boldsymbol{n}}}{\mathrm{d}t} &= K_3\left[\Delta\vec{\boldsymbol{n}} - \vec{\boldsymbol{n}}\,(\vec{\boldsymbol{n}}\cdot\Delta\vec{\boldsymbol{n}})\right] + (K_3 - K_1)[\vec{\boldsymbol{n}}(\vec{\boldsymbol{n}}\cdot\nabla)(\nabla\cdot\vec{\boldsymbol{n}}) - \nabla(\nabla\cdot\vec{\boldsymbol{n}})] \\ &\quad + (K_2 - K_3)[2(\vec{\boldsymbol{n}}\cdot\nabla\times\vec{\boldsymbol{n}})(\vec{\boldsymbol{n}}(\vec{\boldsymbol{n}}\cdot\nabla\times\vec{\boldsymbol{n}}) - \nabla\times\vec{\boldsymbol{n}}) + \vec{\boldsymbol{n}}\times\nabla(\vec{\boldsymbol{n}}\cdot\nabla\times\vec{\boldsymbol{n}})] \\ &\quad - \varepsilon_a(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}})(\vec{\boldsymbol{n}}(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}})\end{aligned} \tag{3.2}$$

where γ is the rotational viscosity of liquid crystal, $\varepsilon_a < 0$ is the anisotropic dielectric constant that accounts for nonlinear response to electric fields, $\{K_1, K_2, K_3\}$ are the elastic constants. Under uniform illumination, the electric field is given by $\vec{E} = (V/\mathrm{d})\hat{z} = E_z\hat{z}$, where E_z is the root mean square amplitude of the electric field, V is the applied voltage, and d is the thickness of the liquid crystal cell.

3.2.1 Linear Analysis

We must first find the threshold voltage where the homeotropic position destabilizes due to the effect of the electric field. For this we note that, in the absence of electric field, a trivial equilibrium of (3.2) is $\vec{\boldsymbol{n}} = \hat{z}$, which is compatible with the homeotropic anchoring $\vec{\boldsymbol{n}}(z=0) = \vec{\boldsymbol{n}}(z=\mathrm{d}) = \hat{z}$.

Let us find the point where this equilibrium position is destabilized due to the effect of the electric field, for this, let be $\vec{\boldsymbol{n}} = \left(u,\ v,\ 1 - \dfrac{u^2+v^2}{2}\right)$ where $\{u(z,t), v(z,t)\}$ are small perturbations.
Replacing in (3.2) and retaining only linear terms, we get

$$\begin{aligned}\gamma\,\dot{u} &= K_3\partial_{zz}u - \varepsilon_a E^2 u \\ \gamma\,\dot{v} &= K_3\partial_{zz}v - \varepsilon_a E^2 v\end{aligned}$$

Now we take an ansatz consistent with the homeotropic boundary conditions, $u = v = 0$ in $z = 0$ and $z = \mathrm{d}$, and find when it destabilizes. Thus we take a perturbation of the form $u = v = \mathrm{e}^{\sigma t}\sin(kz)$ with $k = \pi m/\mathrm{d}$, and obtain

$$\gamma\,\sigma_m = -K_3 k_n^2 - E^2\varepsilon_a,$$

this growth relation implies that the perturbation $\mathrm{e}^{\sigma t}\sin(kz)$ destabilizes when

$$\gamma\,\sigma_m = -K_3 k_,^2 - E^2\varepsilon_a > 0$$

Thus, solving for the electric field, we get $E^2 = \dfrac{-K_3 k_m^2}{\varepsilon_a} = \dfrac{-K_3 \pi^2 m^2}{\mathrm{d}^2 \varepsilon_a}$, therefore the minimum value for electric field that causes movement is for $m = 1$:

$$E_c = \sqrt{\frac{-K_3\,\pi^2}{\varepsilon_a\,\mathrm{d}^2}},$$

Therefore, if a voltage $V = E\,\mathrm{d}$ is applied to the cell we obtain the critical **Fréedericksz Voltage** [21] :

$$V_{FT} = \sqrt{\frac{-K_3\,\pi^2}{\varepsilon_a}}$$

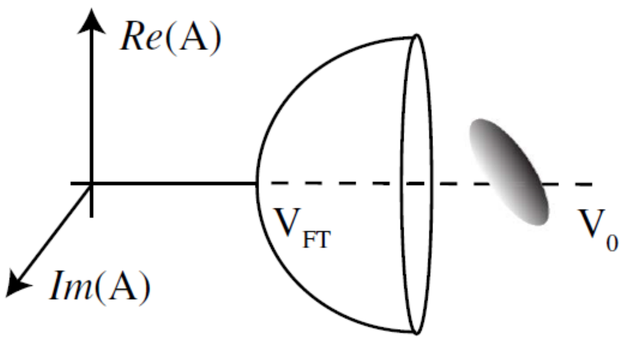

Figure 3.3: **Schematic representation of Fréedericksz Bifurcation.** The horizontal axis represents the voltage applied to the liquid crystal cell. The transverse plane stands for the projection of the director in the horizontal plane of the sample [8].

3.2.2 Weakly Nonlinear Analysis

Close to the bifurcation voltage, we can assume the deviation from the homeotropic state $\hat{\boldsymbol{n}} = (0, 0, 1)$ is small, that is

$$\vec{\boldsymbol{n}} = \left(n_1,\, n_2,\, 1 - \frac{n_1^2 + n_2^2}{2}\right)$$

replacing in (3.2) and under uniform electric field $\vec{\boldsymbol{E}} = E\hat{z}$, we get:

$$\begin{aligned}\gamma \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} &= K_3 \begin{pmatrix} \Delta n_1 + n_1((\partial_z n_1)^2 + (\partial_z n_2)^2) \\ \Delta n_2 + n_2((\partial_z n_1)^2 + (\partial_z n_2)^2) \end{pmatrix} \\ &- (K_3 - K_1) \begin{pmatrix} n_1\,\partial_{zz}(n_1^2 + n_2^2)/2 + \partial_{xx} n_1 + \partial_{xy} n_2 \\ n_2\,\partial_{zz}(n_1^2 + n_2^2)/2 + \partial_{xy} n_1 + \partial_{yy} n_2 \end{pmatrix} \\ &+ (K_2 - K_3) \begin{pmatrix} -\partial_{xy} n_2 + \partial_{yy} n_1 \\ \partial_{xx} n_2 - \partial_{xy} n_1 \end{pmatrix} - \varepsilon_a \begin{pmatrix} n_1 E^2 (1 - n_1^2 - n_2^2) \\ n_2 E^2 (1 - n_1^2 - n_2^2) \end{pmatrix} \end{aligned} \tag{3.3}$$

If we consider $n_1 = X \sin(k_c z) + W_1$, $n_2 = Y \sin(k_c z) + W_2$ which describes the amplitude of the first critical mode ($k_c = \pi/\mathrm{d}$), and $\vec{W} = (W_1, W_2)$, which stands for higher order

corrections.
Using $k = k_c$ to simplify notation, and replacing in (3.3), we get

$$\begin{aligned}\begin{pmatrix}\gamma\dot{X}\sin(kz)\\ \gamma\dot{Y}\sin(kz)\end{pmatrix} = &\begin{pmatrix}(K_3\partial_{zz}-\varepsilon_a E^2)W_1\\ (K_3\partial_{zz}-\varepsilon_a E^2)W_2\end{pmatrix} + K_3\sin(kz)\begin{bmatrix}\Delta X - k^2X + k^2X(X^2+Y^2)\cos^2(kz)\\ \Delta Y - k^2Y + k^2Y(X^2+Y^2)\cos^2(kz)\end{bmatrix}\\ &-(K_3-K_1)\sin(kz)\begin{bmatrix}Xk^2(X^2+Y^2)(\cos^2(kz)-\sin^2(kz)) + \partial_{xx}X + \partial_{xy}Y\\ Yk^2(X^2+Y^2)(\cos^2(kz)-\sin^2(kz)) + \partial_{xy}X + \partial_{yy}Y\end{bmatrix}\\ &+(K_2-K_3)\sin(kz)\begin{bmatrix}\partial_{yy}X - \partial_{xy}Y\\ \partial_{xx}Y - \partial_{xy}X\end{bmatrix} - \varepsilon_a\sin(kz)E^2\begin{bmatrix}X(1-(X^2+Y^2)\sin^2(kz))\\ Y(1-(X^2+Y^2)\sin^2(kz))\end{bmatrix}\end{aligned}$$

Denoting by $\boldsymbol{\mathcal{L}}$ the linear operator acting on W:

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix}K_3\partial_{zz}-\varepsilon_a E^2 & 0\\ 0 & K_3\partial_{zz}-\varepsilon_a E^2\end{pmatrix}$$

we have that this system can be written as $\boldsymbol{\mathcal{L}}\vec{\boldsymbol{W}} = \vec{\boldsymbol{b}}$.

Theorem 3.2.1 (**Fredholm's Alternative**). *[22] If we consider the linear problem*

$$\mathcal{L}w = b \tag{3.4}$$

where $\mathcal{L}$ is a linear operator and w is the unknown variable. Fredholm's alternative states that for a given internal product $\langle\cdot|\cdot\rangle$ the linear problem (3.4) *has solution if and only if*

$$\langle b|\psi\rangle = 0, \quad \forall\,\psi \in \ker(\mathcal{L}^\dagger).$$

By introducing the inner product $\langle \boldsymbol{f}|\boldsymbol{g}\rangle = \int_0^{\mathrm{d}} \boldsymbol{f}\cdot\boldsymbol{g}\,\mathrm{d}z$, the operator $\boldsymbol{\mathcal{L}}$ is self-adjoint and its kernel is $\ker(\boldsymbol{\mathcal{L}}^\dagger) = \{(\sin(kz),0),(0,\sin(kz))\}$.

Therefore using Fredholm's alternative we obtain

$$\begin{aligned}\gamma\,\dot{X} =&(-K_3k^2-\varepsilon_a E^2)X + \left[3/4(K_3k^2+\varepsilon_a E^2) - K_1k^2/2\right]X(X^2+Y^2)\\ &+(K_1\partial_{xx}X + K_2\partial_{yy}X + (K_1-K_2)\partial_{xy}Y)\\ \gamma\,\dot{Y} =&(-K_3k^2-\varepsilon_a E^2)Y + \left[3/4(K_3k^2+\varepsilon_a E^2) - K_1k^2/2\right]Y(X^2+Y^2)\\ &+(K_1\partial_{yy}Y + K_2\partial_{xx}Y + (K_1-K_2)\partial_{xy}X)\end{aligned}$$

Rewriting this system with the complex parameter $A = X + \mathrm{i}Y$, and defining $\partial_\eta = \partial_x + \mathrm{i}\partial_y$, we get

$$\gamma\,\partial_t A = \mu A - a\,A|A|^2 + \frac{K_1+K_2}{2}\Delta A + \frac{K_1-K_2}{2}\partial_{\eta\eta}\overline{A} \tag{3.5}$$

where $\mu = -K_3(\pi/\mathrm{d})^2 - \varepsilon_a E^2$ is the bifurcation parameter, $a = -3/4(K_3k^2+\varepsilon_a E^2) + K_1k^2/2 \simeq K_1\,k^2/2$ is a parameter of order 1 that accounts for the nonlinear response. Finally reescaling the parameter A:

$$A(\vec{\rho},t) \to \frac{\gamma}{\sqrt{a}}A\left(\frac{2}{K_1+K_2}\vec{r},t\right)$$

we obtain

$$\partial_t A = \mu A - A|A|^2 + \Delta A + \delta\partial_{\eta\eta}\overline{A} \tag{3.6}$$

with $\delta \neq 0$ and where $\delta = (K_1 - K_2)(K_1 + K_2)$ is the anisotropic parameter, stands for the anisotropy elasticity of the system. This equation is called the ***Anisotropic Ginzburg-Landau equation***.

Similar equations were derived before: using the method of homogenization fro nematic liquid crystals near the Fréedericksz bifurcation [21], and for modeling self-organization in an array of microtubules via molecular motors [31].

If $\delta = 0$ this is the well known **Ginzburg-Landau equation** with real coefficients.

3.3 Anisotropic Ginzburg Landau Equation

In this section, we will review the know results for our anisotropic Ginzburg Landau equation, in particular of the vortex-type solutions, mostly taken from the article [16] and the master thesis [48]. We are considering the following **anisotropic Ginzburg-Landau equation**

$$\partial_t A = \mu A - A|A|^2 + \Delta A + \delta \partial_{\eta\eta} \overline{A} \tag{3.7}$$

This equation has lost the independent rotational symmetries $A \to A\,\mathrm{e}^{\mathrm{i}\theta}$ and $z \to z\mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in (0, 2\pi)$, retaining only the joint symmetry $A(z) \to \mathrm{e}^{-\mathrm{i}\theta} A(z\mathrm{e}^{\mathrm{i}\theta})$. Note that equation (3.7) can be rewritten in the form

$$\partial_t A = -\frac{\delta \mathcal{E}}{\delta \bar{A}},$$

where the free energy is:

$$\mathcal{E}(A, \delta) = \frac{1}{2} \int_\Omega |\nabla A|^2 + \frac{1}{2}(\mu - |A|^2)^2 + \delta \operatorname{Re}\{(\partial_\eta \bar{A})^2\} \,\mathrm{d}S \tag{3.8}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain[1].
The trivial equilibria that minimize the free energy are $|A|^2 = \mu$. However, this equation has nontrivial inhomogeneous equilibria.

Using the notation $\mathcal{R}_{\varphi_0}$ for a rotation by a angle φ_0 of $\mathbb{R}^2$ about the origin, a short calculation [16] shows that when $\varphi_0 = \pi/2$:

$$\mathcal{E}(A, \delta) = \mathcal{E}(A \circ \mathcal{R}_{\varphi_0}, -\delta) = \mathcal{E}(\mathcal{R}_{\varphi_0} A, -\delta).$$

Moreover, $\mathcal{E}$ has a fourfold symmetry in the sense that:

$$\mathcal{E}(A, \delta) = \mathcal{E}[\mathcal{R}_{m\pi/2} A \circ \mathcal{R}_{k\pi/2}, (-1)^{m+k} \delta]. \tag{3.9}$$

This formula relates different equations and energies when $m + k$ is odd, and at the same time it shows that energy and bifurcations diagrams have to be even symmetric with respect to $\delta = 0$. Functionals with fourfold symmetries appear, for example in the so called d-wave Ginzburg-Landau equation (see [28, 32]).

[1] The energy of vortex solutions diverges in unbounded domain, (see Theorem 2.3.1)

3.3.1 Fourfold symmetry

In fact, if we consider a function f defined in $\Omega = B(0, L)$ with Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} f_n(r)\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{where } z = r\mathrm{e}^{\mathrm{i}\theta}$$

and $f(z)$ has the form

$$f(z) = \sum_{n=-\infty}^{\infty} f_{4n\pm1}(r)\mathrm{e}^{\mathrm{i}(4n\pm1)\theta}$$

that it, only modes indexed by $4n \pm 1$ are present, it can be proven (see propos. A.0.2) that

$$\mu A - A|A|^2 + \nabla_\perp^2 A + \delta\partial_{\eta\eta}\overline{A}$$

has an expansion in Fourier, where only modes indexed by $4n \pm 1$ appear.

Besides, the presence of anisotropy also breaks the symmetry between the vortices with positive and negative vorticity.

Figure 3.4 shows vortices with positive and negative topological charge found in the anisotropic Ginzburg-Landau equation (3.7) . For the vortex with charge +1, the modulus remains rotationally invariant, while for the −1 vortex the rotational invariance around the core is broken by the fourfold symmetry.

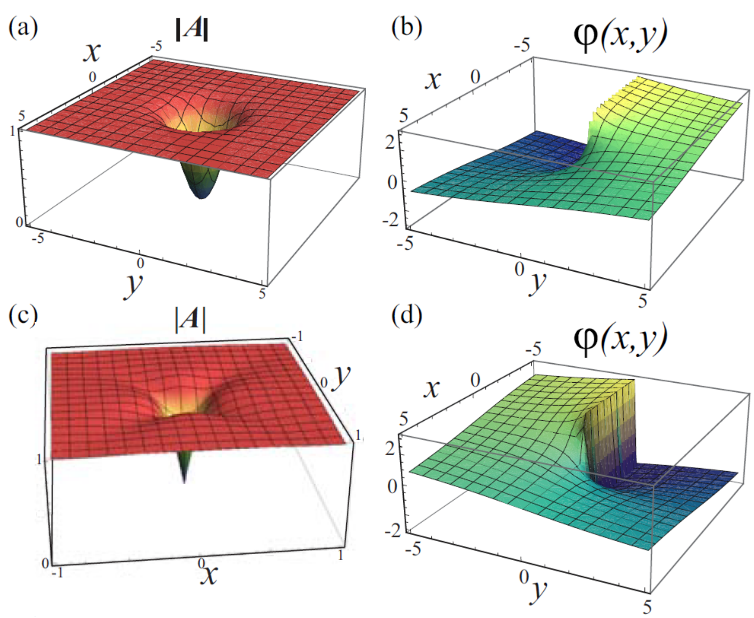

Figure 3.4: Vortex solution of anisotropic Ginzburg-Landau equation (3.7) with $\mu = 1$. Structure of the magnitude (a) and phase of the positive vortex (b). Structure of the magnitude (c) and phase of the positive vortex (d) [48]

Moreover, this is also seen experimentally, as shown the following figure

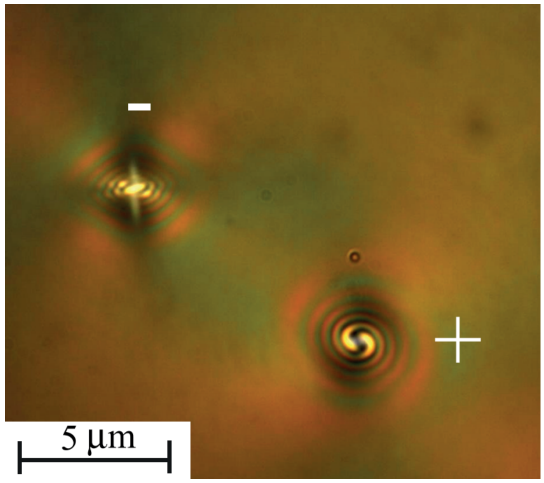

Figure 3.5: Snapshot of vortices of opposite charges observed in a nematic liquid crystal cell within circular crossed polarizers (CCP). Vortex of positive (negative) charge has circular (square) shape [51].

3.3.2 Positive Vortex Solution in Anisotropic Ginzburg-Landau

By introducing the ansatz $A_m(r,\theta,\{\theta_0\}) = R(r)\mathrm{e}^{\mathrm{i}(m\theta+\varphi_0)}$ in the anisotropic Ginzburg-Landau equation (3.7), we obtain:

$$0 = \mathrm{e}^{\mathrm{i}(m\theta+\theta_0)}\left(\mu R - R^3 + (1+\delta\,\mathrm{e}^{\mathrm{i}(2-2m)\theta}\mathrm{e}^{-2\mathrm{i}\theta_0})\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{m^2\,R}{r^2}\right]\right)$$

which simplifies when $m = +1$:

$$0 = \mu R - R^3 + (1+\delta\mathrm{e}^{-2\mathrm{i}\theta_0})\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{R}{r^2}\right]. \tag{3.10}$$

Taking the imaginary part implies

$$0 = \delta\sin 2\theta_0\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{R}{r^2}\right]. \tag{3.11}$$

the only possible way to obtain a nontrivial solution is to consider the phase parameter $\sin\theta_0 = 0$, which gives

$$\theta_0 = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}.$$

Besides, taking the real part of (3.10), we get an equation for the amplitude:

$$0 = \mu R - R^3 + (1+\delta\cos 2\theta_0)\left[\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{R}{r^2}\right]$$

where the isotropic solution $R = R_v$ is recovered by the factor scaling $\sqrt{1+\delta\cos\theta_0}$:

$$R(r) = R_v\left(\frac{r}{\sqrt{1+\delta\cos 2\theta_0}}\right)$$

Therefore, the positive anisotropic vortex solution is

$$A = R_v\left(\frac{r}{\sqrt{1\pm\delta}}\right)\mathrm{e}^{\mathrm{i}\left(\theta+\frac{\pi}{4}\mp\frac{\pi}{4}+n\pi\right)}, \qquad n = 0, 1 \tag{3.12}$$

with R_v the magnitude of the vortex solution of the Ginzburg-Landau equation with real coefficients.

In consequence, the anisotropic vortex solution corresponds to a scaling of the positive symmetric vortex solution, with a finite number of possible phase jumps.

In order to study the stability properties of the vortex solution with $+1$ degree, we can study the properties of the anisotropic free energy $\mathcal{E}(A,\delta)$. Using the vortex solution $A = R_v(r/\sqrt{1\pm\delta})\mathrm{e}^{\mathrm{i}(\theta+\theta_0)}$, where $+$ stands for $\theta_0 \in \{0,\pi\}$ and $-$ for $\theta_0 \in \{\pi/2, 3\pi/2\}$, and taking $\Omega = B(0,L)$, we obtain

$$\mathcal{E}(A,\delta) = \pi \int_0^L \left\{ (\partial_r R_v)^2 + \frac{R_v^2}{r^2} + \frac{1}{2}(1-R_v^2)^2 + \delta\cos(2\theta_0)\left(\partial_r R_v + \frac{R_v}{r}\right)^2 \right\} r\,\mathrm{d}r$$

changing variables $\rho = r/\sqrt{1\pm\delta}$, we get:

$$\mathcal{E}(A,\delta) = \pi \int_0^{L/\sqrt{1\pm\delta}} \left\{ (\partial_\rho R_v(\rho))^2 + \frac{R_v^2(\rho)}{\rho^2} + \frac{(1\pm\delta)(1-R_v^2(\rho))^2}{2} \pm \delta\left(\partial_\rho R_v(\rho) + \frac{R_v(\rho)}{\rho}\right)^2 \right\} \rho\,\mathrm{d}\rho,$$

after straightforward calculations (see [48]), we derive the energy of the vortex with positive topological charge:

$$\mathcal{E}(A,\delta) \approx \pi \ln\left(\frac{L}{a_0\sqrt{1\pm\delta}}\right) + \frac{\pi(1\pm\delta)}{2} \pm \pi\delta\left(\ln\left(\frac{L}{a_0\sqrt{1\pm\delta}}\right) + 1\right) \tag{3.13}$$

This expression shows that the scaling that makes the core smaller is the one with less energy and, therefore, preferred by the system.

For the negative vortex solution in anisotropic Ginzburg-Landau, in the next chapter we will construct a negative anisotropic solution for δ small in a ball, and in the chapter 6, we approximate a negative anisotropic vortex solution in the plane $\mathbb{R}^2$.

Chapter 4

Anisotropic Vortices: Existence, Stability and Energy

In the present chapter, using properties of the linearized operator of the Ginzburg-Landau equation around the negative symmetric vortex solution.

We will construct a solution similar to a negative vortex in a perturbative approach to the anisotropic Ginzburg-Landau equation, rewriting this perturbation as a fixed point of certain operator $\mathcal{F}$.

On the other hand, we study the energy expansion and certain stability for these anisotropic vortex solutions (negative and positive). Finally, we obtain a system of differential equations from the Fourier decomposition of the linear approximation of the perturbation of the negative symmetric vortex-like solution, in order to characterize the vortex core structure of negative anisotropic vortex solution, as well as, to get a quantitative study of the energy expansion around anisotropic negative vortex solution.

4.1 Construction of the solution

Let $R > 0$ be a positive number, and let us consider $B_R = B(0, R)$, the ball in $\mathbb{C}$ with radius R and center at the origin.
We will work with the anisotropic Ginzburg-Landau Equation in $\Omega = B_R$ with negative degree boundary condition,

$$\begin{cases} \Delta u + u(1 - |u|^2) + \delta B u = 0, & \text{in } B_R, \\ u = \mathrm{e}^{-\mathrm{i}\theta} & \text{on } \partial B_R. \end{cases} \tag{4.1}$$

here $u : \overline{B_R} \to \mathbb{C}$, where we denote $Bu := \partial_{\eta\eta}\bar{u}$ as an anisotropic operator and $\partial_\eta =: \partial_x + \mathrm{i}\partial_y$. This equation can be considered as a perturbation of the Ginzburg-Landau equation with negative degree at ∂B_R:

$$\begin{cases} \Delta u + u(1 - |u|^2) = 0, & \text{in } B_R, \\ u = \mathrm{e}^{-\mathrm{i}\theta} & \text{on } \partial B_R. \end{cases} \tag{4.2}$$

Let $u_0 = U(r)\mathrm{e}^{-\mathrm{i}\theta}$ the symmetric vortex solution of degree -1 of (4.2) and we consider a small perturbation of u_0 of the form $u = u_0 + \delta v$ solution of (4.1) with $\delta \neq 0$, where $v \in X$, with X a functional space to be determined, replacing this perturbation u in (4.2) yields:

$$\Delta u_0 \ + \ \delta\Delta v \ + (1-|u_0+\delta v|^2)(u_0+\delta v) + \ \delta B(u_0+\delta v) = 0$$

we can rewrite this as

$$\Delta u_0 + u_0(1-|u_0|^2) + \delta\,[\Delta v + v(1-|u_0|^2) - 2(u_0\cdot v)u_0 + Bu_0] + \delta^2\,[Bv - u_0|v|^2 - v(u_0\cdot v) - \delta v|v|^2] = 0 \quad (4.3)$$

since u_0 is solution of (4.2), then $\Delta u_0 + u_0(1-|u_0|^2) = 0$, and replacing this in (4.3), we can simplify by $\delta > 0$ and we get

$$[\Delta v + v(1-|u_0|^2) - 2(u_0\cdot v)u_0 + Bu_0] + \delta\,[Bv - u_0|v|^2 - v(u_0\cdot v) - \delta v|v|^2] = 0 \qquad (4.4)$$

we can rewrite (4.4) in a more convenient way

$$\underbrace{\Delta v + v(1-|u_0|^2) - 2(u_0\cdot v)u_0}_{Lv} = -Bu_0 + \underbrace{\delta\,[\,u_0|v|^2 + v(u_0\cdot v) + \delta v|v|^2 - Bv\,]}_{\mathcal{N}_\delta(v)} \qquad (4.5)$$

where L is the linearized operator of the equation (4.2) around the solution u_0, and $\mathcal{N}_\delta(v)$ is the nonlinear term of (4.5) given below by (4.8). Besides by the boundary condition in (4.1) we must have $v = 0$ at ∂B_R and also by Corollary 2.4.7.1, L is an isomorphism from $H^2(B_R;\mathbb{C})\cap H_0^1(B_R;\mathbb{C})$ onto $L^2(B_R;\mathbb{C})$. Hence, denoting L^{-1} as the inverse of L, we have to consider the functional space

$$X = H^2(B_R;\mathbb{C})\cap H_0^1(B_R;\mathbb{C}) \qquad \text{equipped with the norm} \qquad \|\cdot\|_X = \|\cdot\|_{H^2(B_R)} \qquad (4.6)$$

where it is known that $(X, \|\cdot\|_X)$ is a Banach space. Therefore (4.5) is equivalent to

$$v = L^{-1}(-Bu_0 + \mathcal{N}_\delta(v)) \qquad (4.7)$$

where

$$\mathcal{N}_\delta(v) = \delta\,[\,u_0|v|^2 + v(u_0\cdot v) + \delta v|v|^2 - Bv\,] \qquad (4.8)$$

Thus, if we define

$$\mathcal{F} : \mathcal{B}_K \subset X \longrightarrow X \quad \text{ by } \quad \mathcal{F}(w) = L^{-1}(-Bu_0 + \mathcal{N}_\delta(w))$$

where $\mathcal{B}_K = \{w \in X : \|w\|_{H^2} \le K\}$ is the closed ball in X with radius K and center at $\mathbf{0}$. We note that (4.7) can be expressed as a fixed point of the operator $\mathcal{F}$, and therefore if we found K such that: $\mathcal{F}(\mathcal{B}_K) \subset \mathcal{B}_K$ and $\mathcal{F}$ is a contraction in $\mathcal{B}_K$, then by the Banach Fixed Point Theorem, we found a unique $v \in \mathcal{B}_K$ such that $v = \mathcal{F}(v) = L^{-1}(-Bu_0 + \mathcal{N}_\delta(v))$.

Let's find K such that the hypotheses just mentioned are fulfilled:

(1) $\boldsymbol{\mathcal{F}(\mathcal{B}_K) \subset \mathcal{B}_K}$: Let $w \in \mathcal{B}_K$, then

$$\begin{aligned}\|\mathcal{F}(w)\|_X &= \|L^{-1}(-Bu_0 + \mathcal{N}_\delta(w))\|_X \le \|L^{-1}\|\|-Bu_0 + \mathcal{N}_\delta(w)\|_{L^2(B_R)} \\ &\le \|L^{-1}\|\|-Bu_0\|_{L^2(B_R)} + \|L^{-1}\|\|\mathcal{N}_\delta(w)\|_{L^2(B_R)}\end{aligned} \qquad (4.9)$$

We estimate the nonlinear term:

$$\|\mathcal{N}_\delta(w)\|_{L^2} \le |\delta|\Big[\underbrace{\|\,u_0|w|^2\,\|_{L^2}}_{\le\ \|w\|_{L^4}^2} + \underbrace{\|w(u_0\cdot w)\|_{L^2}}_{\le\ \|w\|_{L^4}^2} + |\delta|\underbrace{\|\,w|w|^2\,\|_{L^2}}_{\le\ \|w\|_{L^6}^3} + \underbrace{\|\,Bw\,\|_{L^2}}_{\lesssim\ \|w\|_{H^2}}\Big]$$

where we have used that $|u_0| \leq 1$ and Cauchy-Schwartz inequality. Using that the following Sobolev embedding is continuous

$$H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C}) \hookrightarrow H_0^1(B_R;\mathbb{C}) \hookrightarrow L^p(B_R;\mathbb{C}), \quad \forall\, 1 \leq p < \infty,$$

we have that, there are constants $C_1, C_2, C_3 > 0$ independent of w, such that

$$\|\mathcal{N}_\delta(w)\|_{L^2} \leq |\delta| \Big[C_1 \|w\|_{H^2(B_R)} + C_2 \|w\|^2_{H^2(B_R)} + C_3 |\delta| \|w\|^3_{H^2(B_R)} \Big]. \tag{4.10}$$

Since $\|w\|_{H^2(B_R)} \leq K$, from (4.9) and (4.10) we obtain that $\|\mathcal{F}(w)\|_{H^2} \leq K$ holds if and only if

$$\varphi_K(\delta) = \|L^{-1}\| \, \|B\, u_0\|_{L^2} - K + |\delta| \, \|L^{-1}\| \Big[C_1 K + C_2 K^2 + C_3 |\delta| K^3 \Big] \leq 0. \tag{4.11}$$

Then, taking $K = 2 \|L^{-1}\| \, \|B\, u_0\|_{L^2}$, we get $\varphi_K(0) = -\|L^{-1}\| \, \|B\, u_0\|_{L^2} < 0$, thus by the continuity of the function $\varphi_K(\cdot)$ around $\delta = 0$, exists $\delta_K > 0$, such that $\forall\, 0 < |\delta| < \delta_K$, $\varphi_K(\delta) \approx \varphi_K(0) < 0$.

Before continuing with the proof, the following lemmas will be useful for us to simplify the arguments to prove that $\mathcal{F}$ is contraction in $\mathcal{B}_K$:

Lemma 4.1.1 (Gâteaux-derivative of $\mathcal{N}_\delta$).
The nonlinear map $\mathcal{N}_\delta : H^2(B_R;\mathbb{C}) \to L^2(B_R;\mathbb{C})$ given by

$$\mathcal{N}_\delta(v) = \delta \left[u_0 |v|^2 + v(u_0 \cdot v) + \delta v |v|^2 - Bv \right], \tag{4.12}$$

is Gateaux-differentiable at w, $\forall\, w \in H^2(B_R;\mathbb{C})$ with $(\mathcal{N}_\delta)'_G(w) \in \mathcal{L}(H^2(B_R;\mathbb{C}), L^2(B_R;\mathbb{C}))$ given by

$$(\mathcal{N}_\delta)'_G(w)(z) = \delta \Big[2(w \cdot z) u_0 + (u_0 \cdot z) w + (u_0 \cdot w) z + 2\delta\, (w \cdot z) w + \delta z |w|^2 - Bz \Big], \tag{4.13}$$

and moreover, we have the following estimate

$$\|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R), L^2(B_R))} \leq C(\delta, \|w\|_{H^2}), \tag{4.14}$$

where

$$C(\delta, \|w\|_{H^2}) = |\delta| \Big[\widetilde{C}_1 + \widetilde{C}_2 \|w\|_{H^2} + \widetilde{C}_3 |\delta| \, \|w\|^2_{H^2} \Big], \tag{4.15}$$

and the constants $\widetilde{C}_1, \widetilde{C}_2$ and $\widetilde{C}_3$ are positive.

Proof. Given $w \in H^2(B_R;\mathbb{C})$ fixed, we compute $A[z] = \left.\dfrac{\mathrm{d}\mathcal{N}_\delta(w + \lambda z)}{\mathrm{d}\lambda}\right|_{\lambda=0}$ with $z \in H^2(B_R;\mathbb{C})$:

$$\begin{aligned} A[z] =& \left.\frac{\mathrm{d}\mathcal{N}_\delta(w + \lambda z)}{\mathrm{d}\lambda}\right|_{\lambda=0} \\ =& \left.\frac{\mathrm{d}}{\mathrm{d}\lambda} \Big(\delta \Big[u_0 |w + \lambda z|^2 + (w + \lambda z)(u_0 \cdot (w + \lambda z)) + \delta (w + \lambda z) |w + \lambda z|^2 - B(w + \lambda z) \Big] \Big) \right|_{\lambda=0} \\ =& \delta \Big[2(w \cdot z) u_0 + (u_0 \cdot z) w + (u_0 \cdot w) z + 2\delta\, (w \cdot z) w + \delta z |w|^2 - Bz \Big], \end{aligned}$$

It is clear that $A[\bullet]$ is linear, now let's see that it is a bounded operator from $H^2(B_R;\mathbb{C})$ to $L^2(B_R;\mathbb{C})$:

$$\begin{aligned} &\|A[z]\|_{L^2}\\ &\leq |\delta|\Big[2\|(w\cdot z)u_0\|_{L^2}+\|(u_0\cdot z)w\|_{L^2}+\|(u_0\cdot w)z\|_{L^2}+2|\delta|\,\|(w\cdot z)w\|_{L^2}+|\delta|\|z|w|^2\|_{L^2}+\|Bz\|_{L^2}\Big]\\ &\leq\ |\delta|\,\Big[\,2\|w\|_{L^4}\|z\|_{L^4}+2\|w\|_{L^4}\|z\|_{L^4}+2|\delta|\,\|w\|^2_{L^8}\|z\|_{L^4}+|\delta|\|w\|^2_{L^8}\|z\|_{L^4}+\|Bz\|_{L^2}\,\Big].\end{aligned}$$

Using that $H^2(B_R,\mathbb{C})\hookrightarrow L^p(B_R,\mathbb{C})$ with $1\leq p<\infty$ is an continuous embedding, then there are constants $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3>0$ independent of w such that

$$\begin{aligned} &\leq\ |\delta|\,\Big[\widetilde{C}_2\|w\|_{H^2}\|z\|_{H^2}+\widetilde{C}_3\,|\delta|\,\|w\|^2_{H^2}\|z\|_{H^2}+\widetilde{C}_1\|z\|_{H^2}\Big]\\ &=\ |\delta|\,\|z\|_{H^2}\,\Big[\,\widetilde{C}_1+\widetilde{C}_2\,\|w\|_{H^2}+\widetilde{C}_3\,|\delta|\,\|w\|^2_{H^2}\Big]=C(\delta,\|w\|_{H^2})\|z\|_{H^2},\end{aligned}$$

it follows that $A\in\mathcal{L}(H^2(B_R;\mathbb{C}),L^2(B_R;\mathbb{C}))$ and by definition 2.4.1 of Gâteaux-differentiability, we have that the nonlinear map $\mathcal{N}_\delta$, defined by (4.12), is Gâteaux-differentiable at w with $(\mathcal{N}_\delta)'_G(w)=A$ satisfying the estimate (4.14)-(4.15).

■

Notation: For $w_1,w_2\in\Omega$ let $[w_1,w_2]$ denote the segment $\{\lambda\,w_1+(1-\lambda)\,w_2\,|\,\lambda\in[0,1]\}$.
Lemma 4.1.2 (**Mean Value Theorem**). *[4] Let $(W,\|\cdot\|_W)$, $(V,\|\cdot\|_V)$ normed vector spaces, Ω an open set of W and $g:\Omega\subset W\to V$ a function. Let $w_1,w_2\in\Omega$ such that $[w_1,w_2]\subset\Omega$ and we suppose the Gateaux derivative of g in the point w, $g'_G(w)$, exists $\forall\,w\in[w_1,w_2]$, therefore*

$$\|g(w_1)-g(w_2)\|_V\ \leq\ \sup_{w\in[w_1,w_2]}\|g'_G(w)\|_{\mathcal{L}(W,V)}\ \ \|w_1-w_2\|_W.$$

(2) $\boldsymbol{\mathcal{F}}$ **is contraction in** $\boldsymbol{\mathcal{B}_K}$: Let $w_1,w_2\in\mathcal{B}_K$:

$$\begin{aligned}\|\mathcal{F}(w_1)-\mathcal{F}(w_2)\|_{H^2(B_R)} &=\ \|L^{-1}\|\ \ \ \|\mathcal{N}_\delta(w_1)-\mathcal{N}_\delta(w_2)\|_{L^2(B_R)}\\ &\leq\ \|L^{-1}\|\ \sup_{w\in[w_1,w_2]}\|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R),L^2(B_R))}\ \|w_1-w_2\|_{H^2(B_R)},\end{aligned}$$

where in the inequality we used the Mean Value Theorem, with $(\mathcal{N}_\delta)'_G(w)(\cdot)$ the Gateaux derivative of $\mathcal{N}_\delta$ in the point $w\in[w_1,w_2]\subset\mathcal{B}_K$ (since $w_1,w_2\in\mathcal{B}_K$ and $\mathcal{B}_K$ is convex).
It follows that for $\mathcal{F}$ to be a contraction in $\mathcal{B}_K$, we should prove that:

$$\|L^{-1}\|\sup_{w\in[w_1,w_2]}\|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R),L^2(B_R))}<1.$$

From estimate (4.14)-(4.15), and since $w\in\mathcal{B}_K=\{w\in X:\|w\|_{H^2}\leq K\}$, it follows that

$$\|L^{-1}\|\sup_{w\in[w_1,w_2]}\|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R),L^2(B_R))}\leq|\delta|\,\|L^{-1}\|\,\Big[\,\widetilde{C}_1+\widetilde{C}_2\,K+\widetilde{C}_3\,|\delta|\,K^2\Big].$$

Thus, the right side is less than 1, if and only if

$$\eta_K(\delta)=|\delta|\,\|L^{-1}\|\,\Big[\,\widetilde{C}_1K+\widetilde{C}_2\,K^2+\widetilde{C}_3\,|\delta|\,K^3\Big]-K<0.. \tag{4.16}$$

Using the same argument as before, we noticed that $\eta_K(0)=-K<0$, thus by the continuity of the function $\eta_K(\cdot)$ around $\delta=0$, exists $\widetilde{\delta}_K>0$, such that $\forall\,0<|\delta|<\widetilde{\delta}_K,\ \ \eta_K(\delta)\approx\eta_K(0)<0$.

Finally, taking $\delta^* = \min(\delta_K, \widetilde{\delta}_K)$, such that (4.11) and (4.16) remains true for all $0 < |\delta| < \delta^*$. Therefore, we find K such that hypothesis of Banach Fixed-Point Theorem to the operator $\mathcal{F}$ are fullfilled, it follows $\exists!\, v \in \mathcal{B}_K$ such that $v = \mathcal{F}(v)$.

We conclude the following result:

Theorem 4.1.3 (**Existence of negative anisotropic vortex solution**).
Let $\delta \neq 0$, for the anisotropic Ginzburg-Landau with negative vorticity at ∂B_R

$$\begin{cases} \Delta u + u(1-|u|^2) + \delta B u = 0, & \text{in } B_R \\ u = \mathrm{e}^{-\mathrm{i}\theta} & \text{on } \partial B_R \end{cases} \tag{4.1}$$

exists $\delta^* > 0$ such that for all $0 < |\delta| < \delta^*$, the equation (4.1) has a solution of the form $u = u_0^- + \delta v$ with $u_0^- = U(r)\mathrm{e}^{-\mathrm{i}\theta}$ the symmetric negative vortex-like solution of

$$\begin{cases} \Delta u + u(1-|u|^2) = 0, & \text{in } B_R \\ u = \mathrm{e}^{-\mathrm{i}\theta} & \text{on } \partial B_R \end{cases} \tag{4.2}$$

and where $v \in H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C})$ satisfies

$$\begin{cases} Lv = -Bu_0^- + \mathcal{N}_\delta(v) & \text{in } B_R \\ v = 0 & \text{on } \partial B_R \end{cases} \tag{4.17}$$

where L is the linearized operator of the equation (4.2) around u_0^-, and $\mathcal{N}_\delta(v)$ is the nonlinear term given by

$$\mathcal{N}_\delta(v) = \delta\,[\, u_0^-|v|^2 + v(u_0^- \cdot v) + \delta v|v|^2 - Bv\,],$$

Additionally it has $\|v\|_{H^2(B_R)} \leq K$, with $K = 2\,\|L^{-1}\|\,\|B\,u_0^-\|_{L^2}$.

Analogously, taking in all this construction $u_0^+ = U(r)\mathrm{e}^{\mathrm{i}\theta}$ the positive vortex-like solution, we obtain the next version of the previous theorem:

Theorem 4.1.4 (**Existence of positive anisotropic vortex solution**).
Let $\delta \neq 0$, for the anisotropic Ginzburg-Landau with positive vorticity at ∂B_R

$$\begin{cases} \Delta u + u(1-|u|^2) + \delta B u = 0, & \text{in } B_R \\ u = \mathrm{e}^{\mathrm{i}\theta} & \text{on } \partial B_R \end{cases} \tag{4.18}$$

exists $\delta^* > 0$ such that for all $0 < |\delta| < \delta^*$, the equation (4.18) has a solution of the form $u = u_0^+ + \delta v$ with $u_0^+ = U(r)\mathrm{e}^{\mathrm{i}\theta}$ the symmetric positive vortex-like solution of

$$\begin{cases} \Delta u + u(1-|u|^2) = 0, & \text{in } B_R \\ u = \mathrm{e}^{\mathrm{i}\theta} & \text{on } \partial B_R \end{cases} \tag{4.19}$$

and where $v \in H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C})$ satisfies

$$\begin{cases} Lv = -Bu_0^+ + \mathcal{N}_\delta(v) & \text{in } B_R \\ v = 0 & \text{on } \partial B_R \end{cases} \tag{4.20}$$

where L is the linearized operator of the equation (4.2) around u_0^+, and $\mathcal{N}_\delta(v)$ is the nonlinear term given by

$$\mathcal{N}_\delta(v) = \delta\,[\, u_0^+|v|^2 + v(u_0^+ \cdot v) + \delta v|v|^2 - Bv\,],$$

Additionally it has $\|v\|_{H^2(B_R)} \leq K$, with $K = 2\,\|L^{-1}\|\,\|B\,u_0^+\|_{L^2}$.
Moreover, we have the same result for $u_0^+ = \mathrm{e}^{\mathrm{i}(\theta+\theta_0)}$ with $\theta_0 \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ (in order to coincide with the positive anisotropic vortex in the plane) with the corresponding slight modifications.

Remark 4.1.5. *Note, we have explicit solution for the positive anisotropic vortex solution, given by the formula in* (3.12).

Moreover, we have as corollary the following extension of the previous theorem to a particular boundary condition

Theorem 4.1.6 (Vortex solution with disturbed boundary condition at ∂B_R). *Let $\varepsilon > 0$, we consider the following anisotropic Ginzburg-Landau equation*

$$\begin{cases} \Delta u + u(1-|u|^2) + \delta B u = 0, & \text{in } B_R \\ \qquad\qquad u = \mathrm{e}^{\pm \mathrm{i}\theta} + \varepsilon\, g(\theta) & \text{on } \partial B_R \end{cases} \tag{4.21}$$

where $g : \partial B_R \to \mathbb{C}$ is a continuous function. If we consider h the C-harmonic function in the closed disk B_R with boundary values $g(\theta)$, that is

$$\begin{cases} \Delta h = 0 & \text{in } B_R \\ \ \ h = g & \text{on } \partial B_R \end{cases} \tag{4.22}$$

we have the existence of $\delta^*(\epsilon)$, such that for all $0 < |\delta| \leq \delta^*$ exists solution u_δ of (4.21) *of the form $u_\delta = u_0^\pm + \delta v$ where $u_0^\pm = U(r)\mathrm{e}^{\pm \mathrm{i}\theta}$ is the symmetric vortex solution and $v \in C(B_R;\mathbb{C})$ solve the following nonlinear elliptic system*

$$\begin{cases} Lv = -Bu_0^\pm + \mathcal{N}_\delta(v) & \text{in } B_R \\ \ \ v = \varepsilon\, g & \text{on } \partial B_R \end{cases} \tag{4.23}$$

where L is the linearized operator of the equation (4.2) *around $u_0^\pm$, and $\mathcal{N}_\delta(v)$ is the nonlinear term, given respectively by*

$$Lv = \Delta v + v(1-|u_0^\pm|^2) - 2(u_0^\pm \cdot v)u_0^\pm,$$

$$\mathcal{N}_\delta(v) = \delta\,[\, u_0^\pm |v|^2 + v(u_0^\pm \cdot v) + \delta v|v|^2 - Bv\,],$$

Additionally it has $\|v\|_{H^2(B_R)} \leq K$, here $K = M\|L^{-1}\|(\|B\, u_0^\pm\|_{L^2} + \varepsilon\,\|L\|\|h\|_{H^2})$ with M some real constant such that $M > 1$.

Proof. Analogously to the case (4.5), if we impose a solution of the form $u = u_0^\pm + \delta v$ to (4.21), with $v \in H^2(B_R;\mathbb{C})$ and $u_0^\pm$ the symmetric vortex solution, we obtain

$$\begin{cases} Lv = -Bu_0^\pm + \mathcal{N}_\delta(v) & \text{in } B_R \\ \ \ v = \varepsilon\, g & \text{on } \partial B_R \end{cases} \tag{4.24}$$

however, unlike the proof of theorem 4.1.3, the perturbation $v \notin H_0^1(B_R;\mathbb{C})$, but we can consider $\widetilde{v} = v - \varepsilon\, h$, then we have that $\widetilde{v}$ solves the following system:

$$\begin{cases} L\widetilde{v} = \widetilde{f} + \mathcal{N}_\delta(\widetilde{v} + \varepsilon\, h) & \text{in } B_R \\ \ \ \widetilde{v} = 0 & \text{on } \partial B_R \end{cases} \tag{4.25}$$

where $\widetilde{f} = -Bu_0^\pm + \varepsilon\, Lh$. Therefore, using the same ideas of the proof in theorem 4.1.3, we can rewrite (4.25) as a fixed point of the following operator

$$\widetilde{\mathcal{F}} : \mathcal{B}_K \subset X \longrightarrow X \quad \text{by} \quad \widetilde{\mathcal{F}}(w) = L^{-1}(\widetilde{f} + \mathcal{N}_\delta(w + \varepsilon\, h))$$

where $\mathcal{B}_K = \{w \in X : \|w\|_{H^2} \leq K\}$ is the closed ball in X with radius K and center at $\mathbf{0}$. Here $X = H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C})$ equipped with the norm $\|\cdot\|_X = \|\cdot\|_{H^2(B_R)}$. Let's find K such that the hypotheses of Banach Fixed Point Theorem are fulfilled:

(1) $\widetilde{\boldsymbol{\mathcal{F}}}\boldsymbol{(\mathcal{B}_K)} \subset \boldsymbol{\mathcal{B}_K}$: Let $w \in \mathcal{B}_K$, then

$$\begin{aligned}\|\widetilde{\mathcal{F}}(w)\|_X &= \|L^{-1}(\widetilde{f} + \mathcal{N}_\delta(w + \varepsilon\, h))\|_X \le \|L^{-1}\|\|\widetilde{f} + \mathcal{N}_\delta(w + \varepsilon\, h)\|_{L^2(B_R)} \\ &\le \|L^{-1}\|\|\widetilde{f}\|_{L^2(B_R)} + \|L^{-1}\|\|\mathcal{N}_\delta(w)\|_{L^2(B_R)} \end{aligned} \tag{4.26}$$

Using the (4.10), we have that there are constants $C_1, C_2, C_3 > 0$ independent of w, such that

$$\|\mathcal{N}_\delta(w+\varepsilon\, h)\|_{L^2} \le |\delta| \Big[C_1 \, \|w+\varepsilon\, h\|_{H^2(B_R)} + C_2 \, \|w+\varepsilon\, h\|^2_{H^2(B_R)} + C_3 \, |\delta| \|w+\varepsilon\, h\|^3_{H^2(B_R)} \Big] \tag{4.27}$$

Since $\|w\|_{H^2(B_R)} \le K$, from (4.26) and (4.27) we obtain that $\|\widetilde{\mathcal{F}}(w)\|_{H^2} \le K$ holds if and only if

$$\xi_K(\delta) = \|L^{-1}\| \, \|\widetilde{f}\|_{L^2} - K + |\delta| \, \|L^{-1}\| \Big[C_1(K+\varepsilon\|h\|_X) + C_2(K+\varepsilon\|h\|_X)^2 + C_3|\delta|(K+\varepsilon\|h\|_X)^3 \Big] \le 0 \tag{4.28}$$

Then, taking $K = M \, \|L^{-1}\| \, \|\widetilde{f}\|_{L^2}$ with M any constant such that $M > 1$, we get $\xi_K(0) = -(M-1)\|L^{-1}\| \, \|\widetilde{f}\|_{L^2} < 0$, thus by the continuity of the function $\xi_K(\cdot)$ around $\delta = 0$, exists $\delta_K > 0$, such that $\forall\, 0 < |\delta| < \delta_K, \quad \xi_K(\delta) \approx \xi_K(0) < 0$.

(2) $\widetilde{\boldsymbol{\mathcal{F}}}$ **is contraction in** $\boldsymbol{\mathcal{B}_K}$: Let $w_1, w_2 \in \mathcal{B}_K$:

$$\begin{aligned}\|\widetilde{\mathcal{F}}(w_1) - \widetilde{\mathcal{F}}(w_2)\|_{H^2(B_R)} &= \|L^{-1}\| \quad \|\mathcal{N}_\delta(w_1 + \varepsilon\, h) - \mathcal{N}_\delta(w_2 + \varepsilon\, h)\|_{L^2(B_R)} \\ &\le \|L^{-1}\| \sup_{w \in [\widetilde{w}_1, \widetilde{w}_2]} \|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R), L^2(B_R))} \, \|w_1 - w_2\|_{H^2(B_R)}\end{aligned}$$

where in the inequality we used the Mean Value Theorem, with $(\mathcal{N}_\delta)'_G(w)(\cdot)$ the Gateaux derivative of $\mathcal{N}_\delta$ in the point $w \in [\widetilde{w}_1, \widetilde{w}_2] \subset \mathcal{B}_K$, here $\widetilde{w}_1 = w_1 + \varepsilon\, h$ and $\widetilde{w}_2 = w_2 + \varepsilon\, h$.
It follows that for $\widetilde{\mathcal{F}}$ to be a contraction in $\mathcal{B}_K$, we should prove that:

$$\|L^{-1}\| \sup_{w \in [\widetilde{w}_1, \widetilde{w}_2]} \|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R), L^2(B_R))} < 1$$

First, we note that for $w \in [\widetilde{w}_1, \widetilde{w}_2]$, $\exists\, \lambda_w \in [0,1]$ such that

$$w = \lambda_w \widetilde{w}_1 + (1-\lambda_w)\widetilde{w}_2 = \lambda_w w_1 + (1-\lambda_w) w_2 + \varepsilon\, h$$

Thus, since $w_1, w_2 \in \mathcal{B}_K = \{w \in X : \|w\|_{H^2} \le K\}$ we obtain

$$\|w\|_{H^2} \le \lambda_w \|w_1\|_{H^2} + (1-\lambda_w)\|w_2\|_{H^2} + \varepsilon \|h\|_{H^2} \le K + \varepsilon \|h\|_{H^2}$$

therefore, from the estimate (4.14)-(4.15), it follows that

$$\|L^{-1}\| \sup_{w \in [\widetilde{w}_1, \widetilde{w}_2]} \|(\mathcal{N}_\delta)'_G(w)\|_{\mathcal{L}(H^2(B_R), L^2(B_R))} \le |\delta| \, \|L^{-1}\| \, \Big[\widetilde{C}_1 + \widetilde{C}_2 \, (K + \varepsilon\, h) + \widetilde{C}_3 \, |\delta| \, (K + \varepsilon\, h)^2 \Big]$$

Hence the right side is less than 1, if

$$\nu_K(\delta) = |\delta| \, \|L^{-1}\| \, \Big[\widetilde{C}_1(K + \varepsilon \, \|h\|_{H^2}) + \widetilde{C}_2 \, (K + \varepsilon \, \|h\|_{H^2})^2 + \widetilde{C}_3 \, |\delta| \, (K + \varepsilon \, \|h\|)^3 \Big] - K < 0 \tag{4.29}$$

Using the same argument as before, we noticed that $\nu_K(0) = -(K + \varepsilon \, \|h\|_{H^2}) < 0$, thus by the continuity of the function $\nu_K(\cdot)$ around $\delta = 0$, exists $\widetilde{\delta}_K > 0$, such that $\forall\, 0 < |\delta| < \widetilde{\delta}_K$, $\nu_K(\delta) \approx \nu_K(0) < 0$.

Finally, taking $\delta^* = \min(\delta_K, \widetilde{\delta}_K)$, such that (4.28) and (4.29) remains true for all $0 < |\delta| \le \delta^*$. Therefore, we find K such that hypothesis of Banach Fixed-Point Theorem to the operator $\widetilde{\mathcal{F}}$ are fullfilled, it follows $\exists!\, \widetilde{v} \in \mathcal{B}_K$ such that $\widetilde{v} = \widetilde{\mathcal{F}}(\widetilde{v})$.

■

4.2 Anisotropic energy

Let $u_\delta^\pm = u_0^\pm + \delta v : B_R \to \mathbb{C}$ be the solution of:

$$\begin{cases} \Delta u + u(1-|u|^2) + \delta\, \partial_{\eta\eta}\bar{u} = 0, & \text{in } B_R, \\ u = \mathrm{e}^{\pm \mathrm{i}\theta} & \text{on } \partial B_R. \end{cases} \tag{4.30}$$

given in the previous subsection. We denote by $E_\delta(\cdot)$ the energy functional associated to equation (4.30), given by

$$E_\delta(w) = \int_{B_R} \frac{1}{2}|\nabla w|^2 + \frac{1}{4}(1-|w|^2)^2 + \frac{\delta}{2}\,\mathbb{R}\mathrm{e}\{(\partial_\eta \bar{w})^2\}\,\mathrm{d}x\,\mathrm{d}y \tag{4.31}$$

4.2.1 Stability

Lemma 4.2.1 (Continuity of v). *The perturbation $v \in H^2(B_R,\mathbb{C}) \cap H_0^1(B_R,\mathbb{C})$ from theorems 4.1.3 and 4.1.4, satisfies $v \in C(B_R,\mathbb{C})$.*

Proof. Is a direct consequence of the Sobolev's Embedding Theorem

$$W^{j+m,p}(\Omega) \hookrightarrow C_B^j(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain sufficiently smooth, $j \geq 0$ and $m \geq 1$ are integers, $p \in [1,\infty)$ and $mp > n$ (see [5, page 85]) ■

Corollary 4.2.1.1. *Under the assumptions as theorem 4.1.3 and 4.1.4, the solution $u_\delta^\pm = u_0^\pm + \delta\, v$ is stable if the parameter δ is sufficiently small, in the sense that, the following associated quadratic form associated to $E_\delta(\cdot)$*

$$Q_\delta(w) = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} E_\delta(u_\delta^\pm + \lambda w)\Big|_{\lambda=0} = \int_{B_R} \Big(|\nabla w|^2 - (1-|u_\delta^\pm|^2)|w|^2 + 2(u_\delta^\pm \cdot w)^2 + \delta\, \mathbb{R}\mathrm{e}\left\{(\partial_\eta \overline{w})^2\right\}\Big)\mathrm{d}S \tag{4.32}$$

for $w \in V_0 = H_0^1(B_R;\mathbb{C})$, is positive definite; that is, $Q_\delta(w) > 0$ for $w \in V_0$, $\|w\|_{L^2(B_R)} \neq 0$.

Proof. Using that $u_\delta^\pm = u_0^\pm + v_\delta$, where $v_\delta = \delta v$, we have

$$Q_\delta(w) = \int_{B_R} \Big(|\nabla w|^2 - (1-|u_0^\pm|^2)|w|^2 + 2(u_0^\pm \cdot w)^2 + \delta\, \mathbb{R}\mathrm{e}\left\{(\partial_\eta \overline{w})^2\right\}\Big)\mathrm{d}S + \mathcal{O}(\|v_\delta\|_{L^\infty} + \|v_\delta\|_{L^\infty}^2)\|w\|_{L^2}^2$$

Note we have the inequality $\delta \int_{B_R} \mathbb{R}\mathrm{e}(\partial_\eta \overline{w})^2\,\mathrm{d}S \geq -|\delta| \int_{B_R} |\nabla w|^2\,\mathrm{d}S$ for all $\delta \neq 0$, then we get

$$\begin{aligned}
&\geq \int_{B_R} \Big((1-|\delta|)|\nabla w|^2 - (1-|u_0^\pm|^2)|w|^2 + 2(u_0^\pm \cdot w)^2\Big)\,\mathrm{d}S + \mathcal{O}(\|v_\delta\|_{L^\infty} + \|v_\delta\|_{L^\infty}^2)\|w\|_{L^2}^2 \\
&= (1-|\delta|)Q_0(w) + \delta \int_{B_R} \Big(-(1-|u_0|^2)|w|^2 + 2(u_0 \cdot w)^2\Big)\,\mathrm{d}S + \mathcal{O}(\|v_\delta\|_{L^\infty} + \|v_\delta\|_{L^\infty}^2)\|w\|_{L^2}^2 \\
&\geq (1-|\delta|)Q_0(w) - |\delta| \sup_{B_R}(1-|u_0|^2)\|w\|_{L^2}^2 + \mathcal{O}(\|v_\delta\|_{L^\infty} + \|v_\delta\|_{L^\infty}^2)\|w\|_{L^2}^2 \\
&= (1-|\delta|)Q_0(w) - |\delta| \sup_{B_R}(1-|u_0|^2)\|w\|_{L^2}^2 + \mathcal{O}(|\delta|\|v\|_{L^\infty} + |\delta|^2\|v\|_{L^\infty}^2)\|w\|_{L^2}^2
\end{aligned}$$

where in last term we use $\|v_\delta\|_{L^2} = |\delta|\|v\|_{L^\infty}$, therefore if we use that $Q_0(w)$ is definite positive in V_0, moreover we have that $Q_0(w) \geq \lambda_1 \|w\|^2_{L^2}$, where $\lambda_1 > 0$ is the first eigenvalue of the linearized operator L, and also, by the lemma 4.2.1 $\|v\|_{L^\infty(B_R)} < \infty$, we obtain the following, for δ sufficiently small:

$$Q_\delta(w) \geq \left[\,(1-|\delta|)\,\lambda_1 - |\delta| \sup_{B_R}(1-|u_0|^2) + \mathcal{O}(|\delta|)\,\right] \|w\|^2_{L^2(B_R)} \tag{4.33}$$

Finally for $w \in V_0$, $\|w\|_{L^2(B_R)} \neq 0$ and $\delta \neq 0$ sufficiently small, we have that $Q_\delta(w) > 0$. ∎

4.2.2 Anisotropic energy expansion of anisotropic vortex solution

Lemma 4.2.2. *For any $R > 0$ and for all $\delta \in (0, \delta^*)$ with δ^*, given by the Theorem* (4.1.3) *or* (4.1.4)*, the following expansion holds*

$$E_\delta(u_0 + \delta v) = E_\delta(u_0) + \delta^2 \left[\int_{B_R} \mathrm{Re}(\partial_\eta \bar{u_0}\, \partial_\eta \bar{v})\, \mathrm{d}S + \frac{1}{2}\, Q_0(v) \right] + \mathcal{O}(\delta^3) \tag{4.34}$$

where

$$\mathcal{O}(\delta^3) = \frac{\delta^3}{2} \left[\int_{B_R} \mathrm{Re}(\partial_\eta \bar{v})^2 + \int_{B_R} (u_0 \cdot v)|v|^2 \right] + \frac{\delta^4}{4} \int_{B_R} |v|^4 \tag{4.35}$$

and $Q_0(v) = \langle -Lv, v\rangle$ is the quadratic form associated to $E_0(\cdot)$, here $\langle w_1, w_2 \rangle = \mathrm{Re} \int_{B_R} w_1 \overline{w_2}$.

Proof. By the definition (4.31), we have

$$\begin{aligned}
E_\delta(u_0 + \delta v) &= \int_{B_R} \frac{1}{2}|\nabla(u_0 + \delta v)|^2 + \frac{1}{4}(1 - |u_0 + \delta v|^2)^2 + \frac{\delta}{2}\,\mathrm{Re}\{(\partial_\eta(\bar{u_0} + \delta \bar{v}))^2\}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{B_R} \frac{1}{2}|\nabla u_0|^2 + \frac{1}{4}(1 - |u_0|^2)^2 + \frac{\delta}{2}\,\mathrm{Re}\{(\partial_\eta \bar{u_0})^2\} + \delta \int_{B_R} \left[(\nabla u_0 \cdot \nabla v) - (1 - |u_0|^2)(u_0 \cdot v)\right] \\
&\quad + \delta^2 \int_{B_R} \mathrm{Re}(\partial_\eta \bar{u_0}\, \partial_\eta \bar{v}) + \frac{\delta^2}{2} \int_{B_R} \left[|\nabla v|^2 + 2(u_0 \cdot v)^2 - (1 - |u_0|^2)|v|^2\right] \\
&\quad + \frac{\delta^3}{2} \int_{B_R} \mathrm{Re}(\partial_\eta \bar{v})^2 + \frac{\delta^3}{2} \int_{B_R} (u_0 \cdot v)|v|^2 + \frac{\delta^4}{4} \int_{B_R} |v|^4
\end{aligned} \tag{4.36}$$

integrating by parts the second integral, we can rewrite (4.36) as

$$\begin{aligned}
E_\delta(u_0 + \delta v) =& E_\delta(u_0) + \delta \int_{B_R} \mathrm{Re}\left[(-\Delta u_0 - (1 - |u_0|^2))\, \bar{v}\right] + \delta^2 \int_{B_R} \mathrm{Re}(\partial_\eta \bar{u_0}\, \partial_\eta \bar{v}) \\
&+ \frac{\delta^2}{2} \int_{B_R} \left[|\nabla v|^2 + 2(u_0 \cdot v)^2 - (1 - |u_0|^2)|v|^2\right] + \mathcal{O}(\delta^3)
\end{aligned} \tag{4.37}$$

where $\mathcal{O}(\delta^3)$ is given explicitly in (4.35). Since $Q_0(v) = \int_{B_R} \left[|\nabla v|^2 + 2(u_0 \cdot v)^2 - (1 - |u_0|^2)|v|^2\right]$ (see (2.21)) and u_0 is solution of (4.2), that is, $-\Delta u_0 - (1 - |u_0|^2) = 0$, we obtain the expansion (4.34). ∎

Energy comparison between vortices at order delta

Now, we compare the energy expansion between the anisotropic vortex solutions of degrees +1 and −1, at order $\mathcal{O}(\delta)$:

$$E_\delta(u_0 + \delta v) = E_\delta(u_0) + \mathcal{O}(\delta^2) \qquad\qquad \text{Repeat eq. (4.34) at order } \mathcal{O}(\delta)$$

where $u_0 = U(r)\mathrm{e}^{\pm\mathrm{i}\,(\theta+\theta_0)}$ is the symmetric vortex solution of (4.2). Noting that

$$E_\delta(u_0) = E_0(u_0) + \frac{\delta}{2}\int_{B_R} \mathbb{R}\mathrm{e}\{(\partial_\eta \bar{u}_0)^2\},$$

and $E_0(u_0) =: E_0$ is the isotropic energy of the symmetric vortex solution, which is an independent expression of δ and of the magnitude of degree of u_0. Therefore, just analyze the term with δ:

- **Vortex of positive degree:** $u_0^+ = U(r)\mathrm{e}^{\mathrm{i}(\theta+\theta_0^\pm)}$, where $\theta_0^+ = \{0, \pi\}$, $\theta_0^- = \{\pi/2, 3\pi/2\}$, then

  $$\mathbb{R}\mathrm{e}\{(\partial_\eta \bar{u}_0^+)^2\} = \underbrace{\cos(2\theta_0)}_{\pm 1}\left(\partial_r U + \frac{U}{r}\right)^2,$$

  thus

  $$\therefore \qquad E_\delta(u_0^+ + \delta v) = E_0 \pm \delta\,\pi \int_0^R \left(\partial_r U + \frac{U}{r}\right)^2 r\,\mathrm{d}r + \mathcal{O}(\delta^2) \tag{4.38}$$

- **Vortex of negative degree:** $u_0^- = U(r)\mathrm{e}^{-\mathrm{i}(\theta+\theta_0)}$, where $\theta_0 \in \mathbb{R}$, then

  $$\mathbb{R}\mathrm{e}\{(\partial_\eta \bar{u}_0^-)^2\} = \cos(4\theta + 2\theta_0)\left(\partial_r U - \frac{U}{r}\right)^2,$$

  it follows that

  $$\int_{B_R} \mathbb{R}\mathrm{e}\{(\partial_\eta \bar{u}_0^-)^2\} = \int_0^{2\pi} \cos(4\theta + 2\theta_0)\,\mathrm{d}\theta \int_0^R \left(\partial_r U - \frac{U}{r}\right)^2 r\,\mathrm{d}r = 0,$$

  $$\therefore \qquad E_\delta(u_0^- + \delta v) = E_0 + \mathcal{O}(\delta^2) \tag{4.39}$$

Thus, we can mathematically justify the quantitative behaviour of $E_\delta(\cdot)$ for the different degrees of anisotropic vortex solutions using (4.38) and (4.39) at order $\mathcal{O}(\delta)$ for δ small enough (in particular with $|\delta| < \delta^*$).

Energy comparison between vortices at order delta square

As was previously stated in (4.38), we have that the energy at $u^+ = u_0^+ + \delta v$ depends linearly on δ at order $\mathcal{O}(\delta)$, and δ is small enough, so the linear dependence on δ dominates over the quadratic δ^2 dependence. In contrast, (4.39) shows that the energy at $u^- = u_0^- + \delta v$ doesn't depend linearly on δ, but only depends quadratically on δ.

Thus, we seek to analyze the energy expansion $E_\delta(u_0^- + \delta v)$ at order $\mathcal{O}(\delta^2)$. Using (4.39), we note that

$$\therefore \qquad E_\delta(u_0^- + \delta v) = E_0 + G(v, u_0)\,\delta^2 \tag{4.40}$$

where

$$G(v,u_0)=\int\limits_{B_R}\mathbb{R}\mathrm{e}(\partial_\eta\bar{u}_0\,\partial_\eta\bar{v})\,\mathrm{d}S\;+\;\frac{1}{2}\,Q_0(v) \tag{4.41}$$

with u_0 the negative symmetric vortex solution of degree -1 and $v\in H^2(B_R,\mathbb{C})\cap H_0^1(B_R,\mathbb{C})$ satisfies the nonlinear elliptic partial differential equation (4.17), that is, $Lv=-Bu_0+\mathcal{N}_\delta(v)$. Therefore, it seeks to analyze the sign of $G(v,u_0)$.

4.2.3 Linear approximation of the perturbation

In order to analyze the sign of $G(v,u_0)$ and due to the fact that $\mathcal{N}_\delta(v)=\mathcal{O}(\delta)$, we can consider the linear approximation of v, that is, $Lv=-Bu_0$ with $v\in H^2(B_R,\mathbb{C})\cap H_0^1(B_R,\mathbb{C})$. Therefore, we can derive the following lemma:

Lemma 4.2.3 (Integration by parts for the linear approximation of v).
For the linear approximation of v, that is, $Lv=-Bu_0$, with L the linearized operator of the equation (4.2) around u_0 and we had denoted by $Bw:=\partial_{\eta\eta}\overline{w}$ as the anisotropic operator with $\partial_\eta=:\partial_x+\mathrm{i}\,\partial_y$. We have the identity:

$$\int\limits_{B_R}\mathbb{R}\mathrm{e}(\partial_\eta\bar{u}_0\,\partial_\eta\bar{v})\,\mathrm{d}S=-Q_0(v) \tag{4.42}$$

where $Q_0(v)=\langle -Lv,v\rangle$ is the quadratic form associated to $E_0(\cdot)$, here $\langle w_1,w_2\rangle=\mathbb{R}\mathrm{e}\int\limits_{B_R}w_1\overline{w_2}$.

Proof. Let $v\in H^2(B_R,\mathbb{C})\cap H_0^1(B_R,\mathbb{C})$ with $Lv=-Bu_0$. First, we note that

$$Q_0(v)=\mathbb{R}\mathrm{e}\int\limits_{B_R}(-Lv\,\overline{v})\,\mathrm{d}S=\mathbb{R}\mathrm{e}\int\limits_{B_R}(Bu_0\,\overline{v})\,\mathrm{d}S=\mathbb{R}\mathrm{e}\int\limits_{B_R}(\partial_\eta(\partial_\eta\overline{u}_0)\,\overline{v})\,\mathrm{d}S \tag{4.43}$$

and if we denote by $\phi=\partial_\eta\overline{u}_0$, then

$$\int\limits_{B_R}\partial_\eta(\phi)\,\overline{v}=\int\limits_{B_R}\partial_x\phi\,\overline{v}+\mathrm{i}\int\limits_{B_R}\partial_y\phi\,\overline{v} \tag{4.44}$$

$$=\int\limits_{\partial B_R}\phi\,\overline{v}\,n_x+\mathrm{i}\int\limits_{\partial B_R}\phi\,\overline{v}\,n_y-\int\limits_{B_R}\phi\,\partial_x\overline{v}-\mathrm{i}\int\limits_{B_R}\phi\,\partial_y\overline{v} \tag{4.45}$$

$$=-\int\limits_{B_R}\phi\,\partial_\eta\overline{v} \tag{4.46}$$

where in (4.45) we integrate by parts with n_x and n_y are the normal derivatives to ∂B_R in the directions x and y respectively, and in (4.46) we use that $v\in H_0^1(B_R,\mathbb{C})$.
Finally, we conclude (4.42) using (4.43) and the integration by parts before:

$$Q_0(v)=\mathbb{R}\mathrm{e}\int\limits_{B_R}(\partial_\eta(\partial_\eta\overline{u}_0)\,\overline{v})\,\mathrm{d}S=\mathbb{R}\mathrm{e}\left(-\int\limits_{B_R}\partial_\eta\overline{u}_0\,\partial_\eta\overline{v}\,\mathrm{d}S\right)=-\int\limits_{B_R}\mathbb{R}\mathrm{e}(\partial_\eta\overline{u}_0\,\partial_\eta\overline{v})\,\mathrm{d}S$$

■

Thus, in the linear approximation of v, using this lemma, we have that

$$G(v,u_0) = \int\limits_{B_R} \mathbb{R}\mathrm{e}(\partial_\eta \bar{u_0}\, \partial_\eta \bar{v})\,\mathrm{d}S \,+\, \frac{1}{2}\, Q_0(v) = -\frac{1}{2}Q_0(v) < 0$$

where in the last inequality, we have used Theorem 2.4.7, due the fact that $v \in H^2(B_R,\mathbb{C}) \cap H_0^1(B_R,\mathbb{C})$. We conclude, the sign of $G(v,u_0)$ is negative.

In order to compare the energies between u^+ and u^-, it only remains to analyze quantitatively the value $G(v,u_0)$ for u_0 the negative symmetric vortex of degree -1, For this purpose, we note that, in the linear approximation of v, that is, $Lv = -Bu_0$, we can compute the right hand side, considering $u_0 = U(r)\mathrm{e}^{-\mathrm{i}\theta}$:

$$-\,Bu_0 = -\partial_\eta^2(U(r)\mathrm{e}^{-\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}3\theta}\cdot \underbrace{-\left[\partial_r^2 U(r) - 3\frac{\partial_r U(r)}{r} + 3\frac{U(r)}{r^2}\right]}_{c_0(r)}, \tag{4.47}$$

therefore, $v = L^{-1}(c_0(r))\mathrm{e}^{3\mathrm{i}\theta}$, in other words, the linear approximation of v, gives the following form for solution u:

$$u = U(r)\,\mathrm{e}^{-\mathrm{i}\theta} + \delta\, U_1(r)\,\mathrm{e}^{\mathrm{i}3\theta}$$

Therefore, if we consider the form $v = V(r)\mathrm{e}^{3\mathrm{i}\theta}$, by (4.42) we have that

$$\begin{aligned} G(v,u_0) =& \frac{1}{2}\int\limits_{B_R} \mathbb{R}\mathrm{e}(\partial_\eta \bar{u_0}\partial_\eta \bar{v})\,\mathrm{d}S = \frac{1}{2}\int\limits_{B_R} \mathbb{R}\mathrm{e}\left(\underbrace{\mathrm{e}^{2\mathrm{i}\theta}\left(U'(r) - \frac{U(r)}{r}\right)}_{\partial_\eta \bar{u_0}}\underbrace{\mathrm{e}^{-2\mathrm{i}\theta}\left(V'(r) + \frac{3}{r}V(r)\right)}_{\partial_\eta \bar{v}}\right)\mathrm{d}S \\ =& \frac{1}{2}\int\limits_{B_R} \mathbb{R}\mathrm{e}\left(U' - \frac{U}{r}\right)\left(V' + \frac{3}{r}V\right)\, r\,\mathrm{d}r\,\mathrm{d}\theta = \pi\int\limits_0^R \left(U'(r) - \frac{U(r)}{r}\right)\left(V'(r) + \frac{3}{r}V(r)\right)\, r\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned}$$

Thus, an approximation for the value $G(v,u_0)$ we can reduce to compute the value of

$$G(v,u_0) = \pi\int\limits_0^R \left(U'(r) - \frac{U(r)}{r}\right)\left(V'(r) + \frac{3}{r}V(r)\right)\, r\,\mathrm{d}r\,\mathrm{d}\theta \tag{4.48}$$

where $U(r)$ is the vortex profile of the symmetric vortex solution, which satisfies the following ordinary differential equation

$$\frac{\partial^2 U(r)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r)}{\partial r} - \frac{U(r)}{r^2} + U(r)(1 - U^2(r)) = 0, \quad r \in (0,R), \qquad U(0) = 0, U(R) = 1, \tag{4.49}$$

and $V(r)$ is the vortex profile of $v = V(r)\mathrm{e}^{3\mathrm{i}\theta}$ solution of $Lv = -Bu_0$ en B_R.
For the purpose of computing numerically the value of $G(v,u_0)$ in (4.48), we can replace $v = V(r)\mathrm{e}^{3\mathrm{i}\theta}$ in $Lv = -Bu_0$,

$$\frac{\partial^2 V(r)}{\partial r^2} + \frac{1}{r}\frac{\partial V(r)}{\partial r} - \frac{9}{r^2}V(r) + V(r)(1 - U^2(r)) - (1 + \mathrm{e}^{-8\mathrm{i}\theta})U^2 V = c_0(r)$$

We note the vortex profile $V(r)$ does not decouple from θ, therefore we need a series Fourier decomposition.

4.2.4 Fourier Series decomposition

Since $v \in C(B_R, \mathbb{C}) \subset L^2(B_R, \mathbb{C})$, we can decompose v in Fourier series using complex exponential

$$v(z) = \sum_{m\in\mathbb{Z}} v_m(r)\, \mathrm{e}^{\mathrm{i}m\theta} \tag{4.50}$$

where $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ and the functions $v_m : [0, R] \longrightarrow \mathbb{C}$ are continuous function complex valued. Since we are considering v as a perturbation of the negative symmetric vortex solution, and due to invariance of the anisotropic Ginzburg-Landau equation in the subspace W^- (see proposition A.0.2) defined as

$$W^- = \left\{ u = \sum_{k\in\mathbb{Z}} a_{4k-1}(r)\mathrm{e}^{\mathrm{i}(4k-1)\theta} \text{ on } B_R, a_{4k-1}(r) \in \mathbb{C}, \forall\, r \in [0, R] \right\},$$

we must consider only modes indexed by $4m - 1$ in (4.50), that is

$$v(z) = \sum_{m\in\mathbb{Z}} v_{4m-1}(r)\, \mathrm{e}^{\mathrm{i}(4m-1)\theta} = \left(A_0(r) + \sum_{m\geq 1} \left(A_m(r)\, \mathrm{e}^{4\mathrm{i}m\theta} + A_{-m}(r)\mathrm{e}^{-4\mathrm{i}m\theta} \right) \right) \mathrm{e}^{-\mathrm{i}\theta}, \tag{4.51}$$

where $A_0(r) := v_{-1}(r)$, $A_m(r) := v_{4m-1}(r)$ and $A_{-m} := v_{-4m-1}(r)$ for $m \geq 1$, then if we replace this in $Lv = -Bu_0$ and using the computation in (4.47), that is $-Bu_0 = c_o(r)\mathrm{e}^{3\mathrm{i}\theta}$ we get

$$L(\mathrm{e}^{-\mathrm{i}\theta} A_0(r)) + \sum_{m\geq 1} \left(L\mathrm{e}^{-\mathrm{i}\theta}(A_m(r)\, \mathrm{e}^{4\mathrm{i}m\theta}) + L\mathrm{e}^{-\mathrm{i}\theta}(A_{-m}(r)\mathrm{e}^{-4\mathrm{i}m\theta}) \right) = c_0(r)\mathrm{e}^{3\mathrm{i}\theta} \tag{4.52}$$

using $\mathcal{L}_{-1} = \mathrm{e}^{\mathrm{i}\theta} L\, \mathrm{e}^{-\mathrm{i}\theta}$ the conjugate linearized operator around $u_0 = U(r)\mathrm{e}^{-\mathrm{i}\theta}$ (see (2.15)), in view that u has the form $u = (U(r) + \delta\, U_1(r)\, \mathrm{e}^{\mathrm{i}4\theta})\mathrm{e}^{-\mathrm{i}\theta}$, we only consider modes m that $|m| = 1$, that is

$$\mathcal{L}_{-1}(\alpha(r)\mathrm{e}^{-4\mathrm{i}\theta}) + \mathcal{L}_{-1}(\beta(r)\mathrm{e}^{4\mathrm{i}\theta}) = c_o(r)\mathrm{e}^{4\mathrm{i}\theta} \tag{4.53}$$

where $\alpha = A_{-1}, \beta = A_1$ are complex valued functions, i.e. $v = (\alpha\, \mathrm{e}^{-4\mathrm{i}\theta} + \beta\, \mathrm{e}^{4\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta}$.

Therefore, it is more convenient to use the Fourier decomposition in trigonometric form, that is,

$$v = (w_r + \mathrm{i}w_\mathrm{i})\, \mathrm{e}^{-\mathrm{i}\theta}$$

with $w_r = w_r(r, \theta)$, $w_\mathrm{i} = w_\mathrm{i}(r, \theta)$ real-valued functions, then by (2.16), we get

$$\mathcal{L}_{-1}(w_r + \mathrm{i}w_r) = \Delta w_r + (1 - 3U^2)w_r - \frac{1}{r^2}w_r + \frac{2}{r^2}\partial_\theta w_\mathrm{i} + \mathrm{i}\left(\Delta w_\mathrm{i} + (1 - U^2)w_\mathrm{i} - \frac{1}{r^2}w_\mathrm{i} - \frac{2}{r^2}\partial_\theta w_r \right) \tag{4.54}$$

thus replacing this in (4.53), we get

$$\begin{aligned} \Delta w_r + (1 - 3U^2)w_r - \frac{1}{r^2}w_r + \frac{2}{r^2}\partial_\theta w_\mathrm{i} &= c_0(r)\cos(4\theta) \\ \Delta w_\mathrm{i} + (1 - U^2)w_\mathrm{i} - \frac{1}{r^2}w_\mathrm{i} - \frac{2}{r^2}\partial_\theta w_r &= c_0(r)\sin(4\theta) \end{aligned} \tag{4.55}$$

and using the Fourier decomposition, that is,

$$w_r(r, \theta) = x(r)\cos(4\theta) + y(r)\sin(4\theta), \qquad w_\mathrm{i}(r, \theta) = w(r)\cos(4\theta) + z(r)\sin(4\theta),$$

(4.55) becomes

$$\left\{\begin{array}{rcll} x''(r)+\dfrac{1}{r}x'(r)-\dfrac{17}{r^2}x(r)+(1-3U^2(r))x(r)+\dfrac{8}{r^2}z(r) & = & c_0(r), & r\in(0,R)\\[2ex] y''(r)+\dfrac{1}{r}y'(r)-\dfrac{17}{r^2}y(r)+(1-3U^2(r))y(r)-\dfrac{8}{r^2}w(r) & = & 0, & r\in(0,R)\\[2ex] w''(r)+\dfrac{1}{r}w'(r)-\dfrac{17}{r^2}w(r)+(1-U^2(r))w(r)-\dfrac{8}{r^2}y(r) & = & 0, & r\in(0,R)\\[2ex] z''(r)+\dfrac{1}{r}z'(r)-\dfrac{17}{r^2}z(r)+(1-U^2(r))z(r)+\dfrac{8}{r^2}x(r) & = & c_0(r), & r\in(0,R) \end{array}\right. \tag{4.56}$$

In particular, since we looking for particular solutions of the system, we can take $y = w \equiv 0$, then the system reduces to

$$\left\{\begin{array}{rcl} x''(r)+\dfrac{1}{r}x'(r)-\dfrac{17}{r^2}x(r)+(1-3\,U^2(r))x(r)+\dfrac{8}{r^2}z(r) & = & c_0(r),\\[2ex] z''(r)+\dfrac{1}{r}z'(r)-\dfrac{17}{r^2}z(r)+(1-U^2(r))z(r)+\dfrac{8}{r^2}x(r) & = & c_0(r), \end{array}\right. \quad r\in(0,R) \tag{4.57}$$

with the boundary condition $x(R) = z(R) = 0$ (or equivalently $v = 0$ on ∂B_R), where

$$c_o(r) = -\left[U''(r) - 3\frac{U'(r)}{r} + 3\frac{U(r)}{r^2}\right].$$

In the following chapters, we will try to solve this system, from different points of view, both numerically and analytically when $R \to +\infty$.

Chapter 5

Numerical Results

In this chapter, we give a numerical analysis of the linear approximation of v, that is, $Lv = -Bu_0$, in order to obtain a quantitative description of the energy expansion of $E_\delta(u_0^- + \delta v)$ when u_0^- is the negative symmetric solution of degree -1 of equation (4.2). Therefore, through a weak formulation of $Lv = -Bu_0^-$ and using the Padé approximation of the profile $U(r)$, we use the finite element method to give a complete diagram bifurcation of anisotropic energy between anisotropic vortex solutions of positive and negative degree.

5.1 Finite element method

5.1.1 Introduction

The finite element method is a numerical method used to solve linear and nonlinear partial differential equations with specific boundary conditions. It allows by a discretization of the domain, to transform a system of partial differential equations into an algebraic system. The process is structured as follows:

- The continuous domain on which the function is defined gets divided in finite elements trough a process named tessellation. Hence, the mesh is automatically built using Delaunay-Voronoi algorithm. Each tassel is triangular. If we identify each triangle as T_k, then the finite element approximation of the domain $\Omega \to$ f.e.m. $\to \Omega_h = \bigcup_{k=1}^{n_t} T_k$ where n_t is the number of triangles.
- The finite element space is defined, it usually consists of a Hilbert space of polynomial functions defined on each element of the mesh and affine in x, y. These functions build the canonical basis of the Hilbert space and are continuous, piecewise equal to 1 on one vertex and 0 on all others. Naming then ϕ_k, and indicating with $\mathcal{T}_h = \{T_k\}_{k=1,\dots,n_t}$ the family of triangles in the mesh, we can define the space as:

$$V_h(\mathcal{T}_h, \mathbb{P}_2) = \left\{ w(x,y) \,|\, w(x,y) = \sum_{k=1}^{M} w_k\, \phi_k(x,y),\ w_k \in \mathbb{R} \right\} \tag{5.1}$$

  Here M is the dimension of V_h, ie. the number of vertices. $\mathbb{P}_2$ indicates that the basis functions are continuous piecewise quadratic. Finally, the sets of coefficients w_k are called «degrees of freedom of w», and effectively contain all the information about the projection of the analytic function we are interested in, onto the finite dimensional Hilbert space.

- The algebraic system is set depending on the particular differential equation solved.

5.1.2 Finite element method for the linear approximation of anisotropic vortex solution

With this in mind, we proceed to compute a numerical solution through finite element method for the following problem:
For $\Omega = B(0, R)$ the ball of center 0 and radius R in $\mathbb{R}^2$, find $v : \Omega \to \mathbb{C}$ such that

$$\begin{cases} Lv = -Bu_0^- & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

where L is the linearized operator of the equation (4.2) around u_0 the negative symmetric solution of degree -1 of equation (4.2).

This problem is implemented with *FreeFem++* [29], in the following steps

- **First step: Construction of the domain Ω.** In *FreeFem++* the domain is assumed to described by its boundary that is on the left side of the boundary which is implicitly oriented by the parametrization.

```
real R=5.; //domain radius
border domega ( t = 0.0, 2.0 * pi ) { x =R* cos(t); y = R*sin(t); label=1;}
int n=200;
mesh Th=buildmesh(domega(n));
plot(Th,wait=1,ps="dominio.eps", cmm="domain_meshing");
```

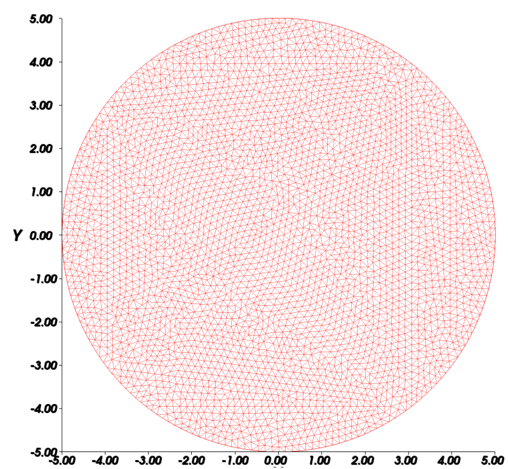

Figure 5.1: Mesh of Ω with $R = 5$ and $n = 200$ triangles

- **Second step: Solve the linear approximation.** The basic variational formulation of (5.2) is:Find $w \in H^2(\Omega,\mathbb{C}) \cap H_0^1(\Omega,\mathbb{C})$, such that for all $w \in H^2(\Omega,\mathbb{C}) \cap H_0^1(\Omega,\mathbb{C})$ we have

$$a(v,w) = l(w) \tag{5.3}$$

where $a(v,w) = \displaystyle\int_\Omega Lv \cdot w \,\mathrm{d}x\,\mathrm{d}y$ and $l(w) = -\displaystyle\int_\Omega Bu_0^- \cdot w \,\mathrm{d}x\,\mathrm{d}y$.
To discretize (5.3), let $\mathcal{T}_h$ the regular uniform triangulation of Ω with triangles of maximum size $h < 1$, let $V_h = \{v_h \in C^0(\overline{\Omega}) : \; v_h|_T \in \mathbb{P}_2(T), \forall T \in \mathcal{T}_h; \; v_h = 0 \text{ on } \partial\Omega\}$ denote a finite-dimensional subspace of $H^2(\Omega,\mathbb{C}) \cap H_0^1(\Omega,\mathbb{C})$ where $\mathbb{P}_2$ is the set of polynomials of $\mathbb{R}^2$ of degree ≤ 2. Thus the discretize weak formulation of (5.3) is:

$$\text{Find } v_h \in V_h \,:\, a(v_h,w_h) - l(w) = 0 \qquad \forall\, w_h \in V_h \tag{5.4}$$

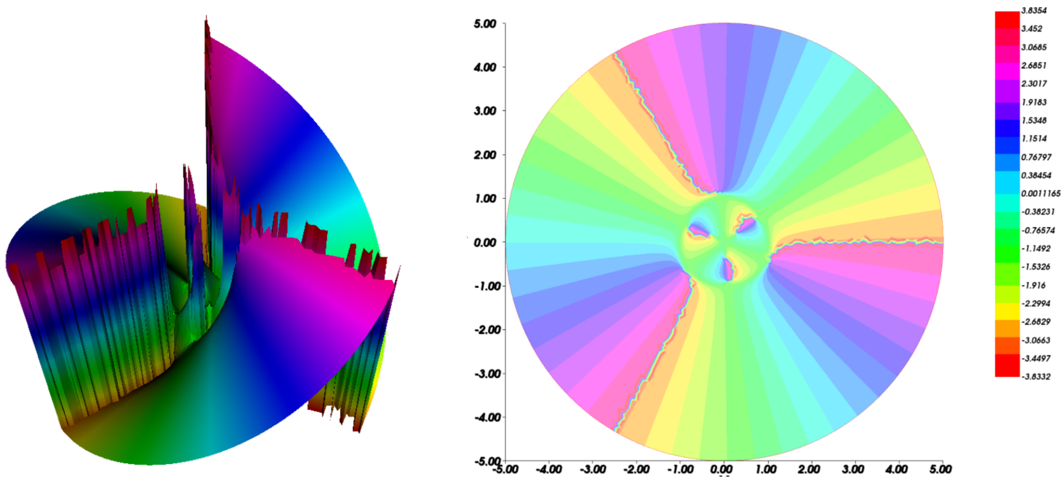

Figure 5.2: Phase of perturbation v with $R = 5$.
(Left) 3D representation (Right) 2D representation.

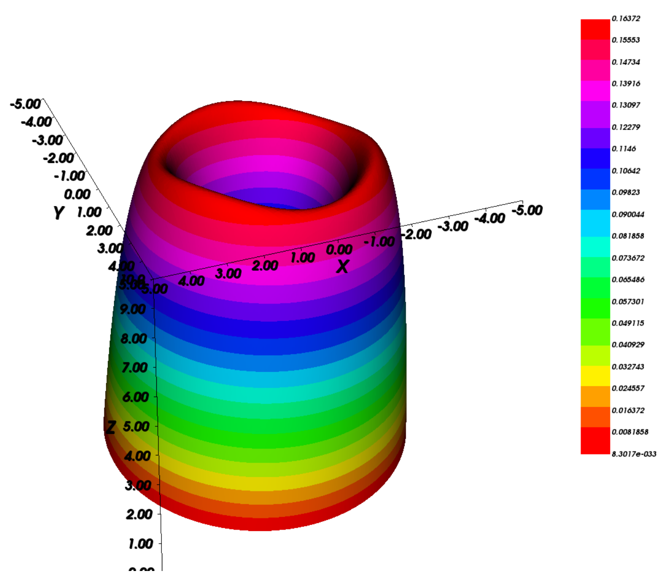

Figure 5.3: Modulus magnitude perturbation v. The z-scale has been adjusted, be guided by the values in color bar.

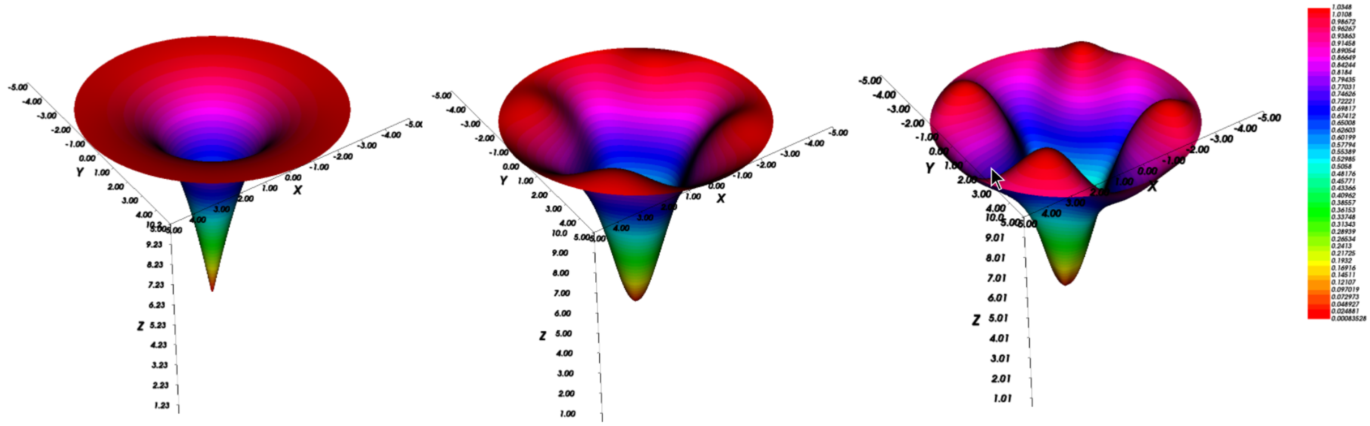

Figure 5.4: The negative anisotropic vortex solution $u = u_0^- + \delta v$,
with (Left) $\delta = 0$, (Center) $\delta = 0.2$ and (Right) $\delta = 0.5$.

Figure 5.2 illustrates the phase perturbation of the negative anisotropic vortex solution, which has a degree +3, induced by the right hand-side of the linear approximation $-Bu_0^- = U_1(r)\mathrm{e}^{\mathrm{i}3\theta}$ where $U_1 = U_1(r)$ is a real-valued function depending only of $r \in [0, R]$.

Figure 5.3 shows that, the modulus of v satisfies the boundary condition $v = 0$ on ∂B_R, and also, shows that $v = 0$ in a neighborhood of origin, but you have to be careful that the number of triangles of the domain mesh, as discussed in Remark 5.1.1. See also [6, page 186].
Finally, the fourfold symmetry of the negative anisotropic vortex solution, is shown in the numerical simulation through its modulus magnitude, Figure 5.4. This fourfold structure being clearer, when the anisotropy constant is greater.

Remark 5.1.1. *If we write the linearized operator in polar coordinates, we see that it is singular at the origin. And that singularity can be transferred to the numerical solution by the finite element method, if we increase the number of triangles in the domain mesh, as shown in the following Figure:*

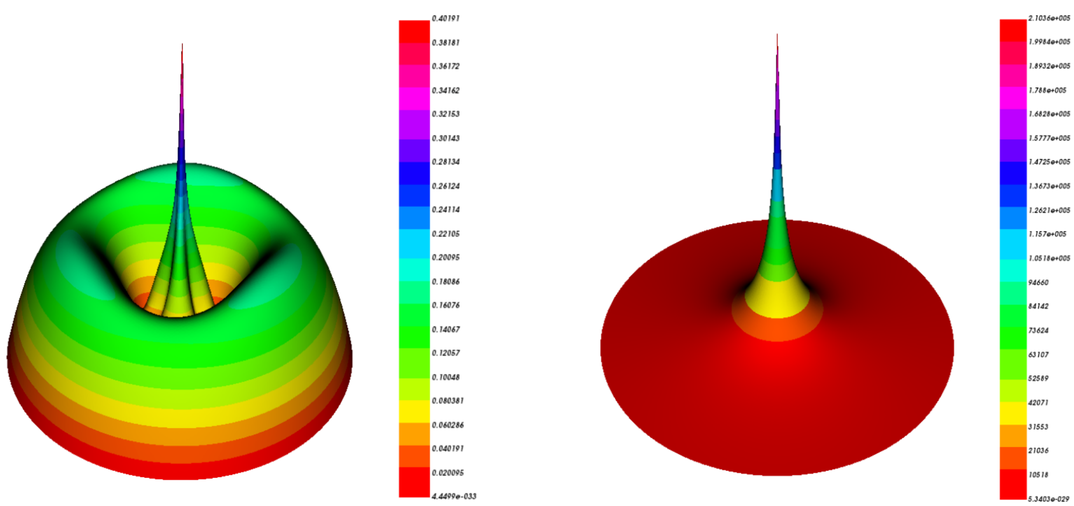

Figure 5.5: Representation v_h is not regular. Modulus magnitude of v on B_R with $R = 5$.
(Left) $n = 370$ triangles, (Right) $n = 400$ triangles

5.1.3 Quadratic coefficient of negative anisotropic vortex solution

In order to analyze the value of $G(v, u_0)$, using the lemma 4.2.3, we have that

$$G(v, u_0) = -\frac{1}{2}Q_0(v) = -\frac{1}{2}\langle -Lv, v\rangle = \frac{1}{2}\langle Bu_0, v\rangle = \frac{1}{2}\int_{B_R} \mathbb{R}\mathrm{e}(Bu_0\,\overline{v})\,\mathrm{d}x\,\mathrm{d}y$$

we compute this expression for different solutions v of (5.2) depending for different values of the domain radius R and particular triangulations of Ω, the numerical results obtained by the finite element method are shown below by the table 5.1 and by the figure 5.6.

R	n	$G(v, u_0; R)$	$\widetilde{G}(v, u_0; R)$	|Error|
3	150	-1.43212	-0.07721	1.35491
4	80	-2.95169	-2.89304	0.05865
5	250	-4.56483	-5.07717	0.51234
10	300	-11.0234	-11.861702	0.838302
15	700	-15.2255	-15.83039	0.60489
20	700	-18.2545	-18.64622	0.39172
50	700	-27.9452	-27.61488	0.33032
100	700	-35.225	-34.399405	0.825595
150	600	-38.1544	-38.36809	0.21369

Table 5.1: Numerical values of $G(v, u_0)$ as a function of domain radius R. The number of triangles n used in the mesh of B_R of the finite element method are also included. Also, is included the values of $\widetilde{G}(v, u_0; R)$ (the curve fitting of values of $G(v, u_0; R)$) and its fitting error.

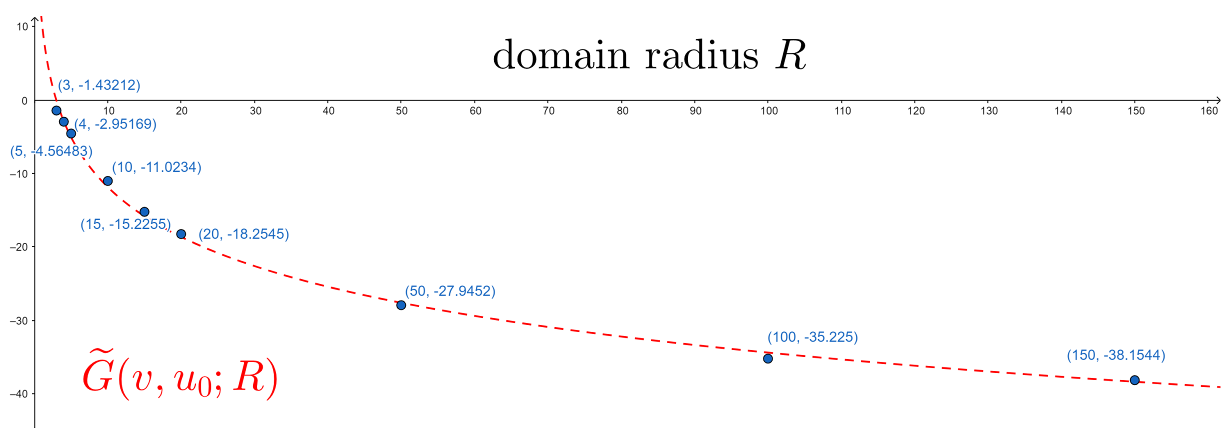

Figure 5.6: Numerical calculations (circles) and logarithmic curve fitting $\widetilde{G}(v, u_0; R)$ (dashed red line) for the quadratic coefficient $G(v, u_0)$ with $u_0 = U(r)\mathrm{e}^{-\mathrm{i}\theta}$ as a function of domain radius R.

Furthermore, due to the divergence behaviour of the energy in the symmetric vortex solutions, we can compute a curve logarithmic fitting, giving the following

$$\widetilde{G}(v, u_0; R) = -9.788 \cdot ln(R) + 10.676 \tag{5.5}$$

Note that in Table 5.6, for each fixed domain radius R, a particular number n of triangles is used for domain meshing. This is not so arbitrary, but is due to the singularity described in Remark 5.1.1. So this process was carried out in a heuristic way, and can be seen in more detail in Appendix B.

5.1.4 Numerical calculation of anisotropic Energy

In this part, using the numerical curve fitting for $G(v, u_0; R)$ obtained in before part, we plot the anisotropic energy of the vortex solutions as a function of the anisotropy parameter δ. For this, we use the analysis of the energy expansion obtained in the previous chapter:

$$E_\delta(u_0^+ + \delta v) = E_0 \pm \delta\, \pi \int_0^R \left(\partial_r U + \frac{U}{r}\right)^2 r\, \mathrm{d}r + \mathcal{O}(\delta^2) \qquad \text{Repeat eq. (4.38)}$$

$$E_\delta(u_0^- + \delta v) = E_0 + G(v, u_0; R)\, \delta^2 + \mathcal{O}(\delta^3) \qquad \text{Repeat eq. (4.40)}$$

where E_0 is the isotropic energy of the isotropic vortex solution $u_0^\pm$, given by

$$E_0 \approx \pi \ln\left(\frac{R\sqrt{\mathrm{e}}}{a_0}\right), \tag{5.6}$$

where a_0 is a constant related to the vortex core of isotropic vortex solution.

Computation of isotropic vortex energy

In the liquid crystal context, we can think that the domain $\Omega \subseteq \mathbb{R}^2$ is a surface of about $1 - 5cm^2$ and the diameter of vortex core is $1.2\mu m$ [8]. Therefore, since for the numerical simulations we use the Padé approximation for the vortex profile of symmetric vortex solution, we have that the numerical value for radius of vortex core is $a_0 \approx 1.126$ [42], then rescaling with the liquid crystal context, we must consider $R_{min} \approx 5641$ and $R_{max} \approx 12615$. Thus, we have

$$E_0(R_{min}) \approx 28.33 \qquad \text{and} \qquad E_0(R_{max}) \approx 30.86 \tag{5.7}$$

Bifurcation Diagram

For $R_{max} = 12615$, we have the following diagram of the anisotropic energy as function of delta for anisotropic vortices of degree -1 and $+1$ (Figures 5.7 and 5.8).

For $R_{min} = 5641$, we have the following diagram of the anisotropic energy as function of delta for anisotropic vortices of degree -1 and $+1$ (Figures 5.9 and 5.10).

We observe from the figures 5.7 and 5.9, that the energy of negative anisotropic vortex solution is greater than the energy of positive vortex solution $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^-)} + \delta v$, which has less energy than $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^+)} + \delta v$, for $\delta > 0$ small (and viceversa for $\delta < 0$). This shows the validity range of the Theorems 4.1.3 and 4.1.4, also shows that these solutions are only stable in the sense of the Corollary 4.2.1.1, for small delta.

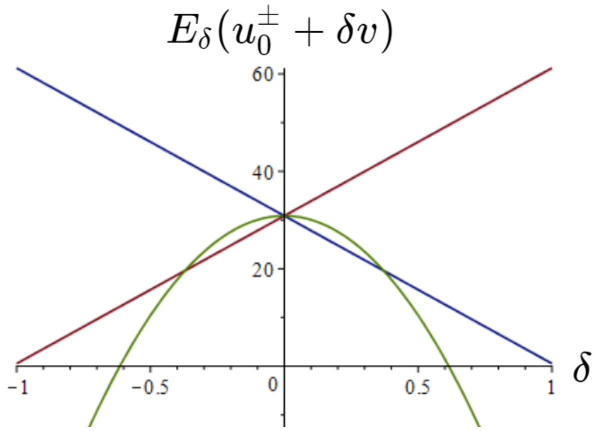

Figure 5.7: Anisotropic energy E_δ of the negative anisotropic vortex $u = u_0^- + \delta v$ (green) and the positive anisotropic vortex $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^\pm)} + \delta v$ where $\theta^+ \in \{0, \pi\}$ (red) and $\theta^- \in \{\pi/3, 3\pi/3\}$ (blue). Diagram made in Maple.

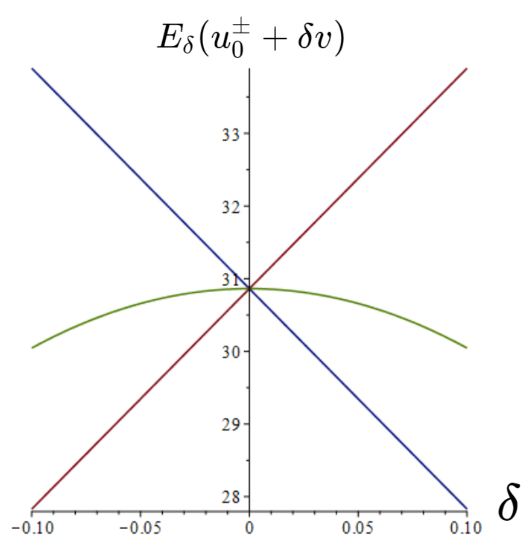

Figure 5.8: Detail of the previous picture, with $\delta \in (-0.1, 0.1)$. Anisotropic energy E_δ of the negative anisotropic vortex $u = u_0^- + \delta v$ (green) and the positive anisotropic vortex $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^\pm)} + \delta v$ where $\theta^+ \in \{0, \pi\}$ (red) and $\theta^- \in \{\pi/3, 3\pi/2\}$ (blue). Diagram made in Maple.

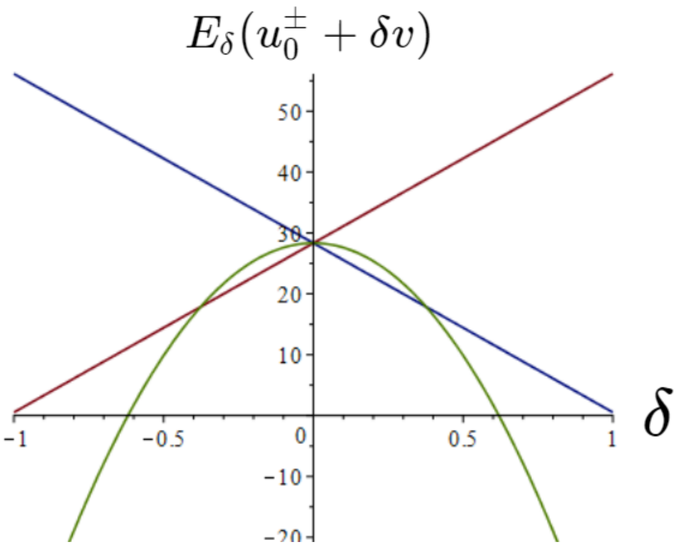

Figure 5.9: Anisotropic energy E_δ of the negative anisotropic vortex $u = u_0^- + \delta v$ (green) and the positive anisotropic vortex $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^\pm)} + \delta v$ where $\theta^+ \in \{0, \pi\}$ (red) and $\theta^- \in \{\pi/3, 3\pi/3\}$ (blue). Diagram made in Maple.

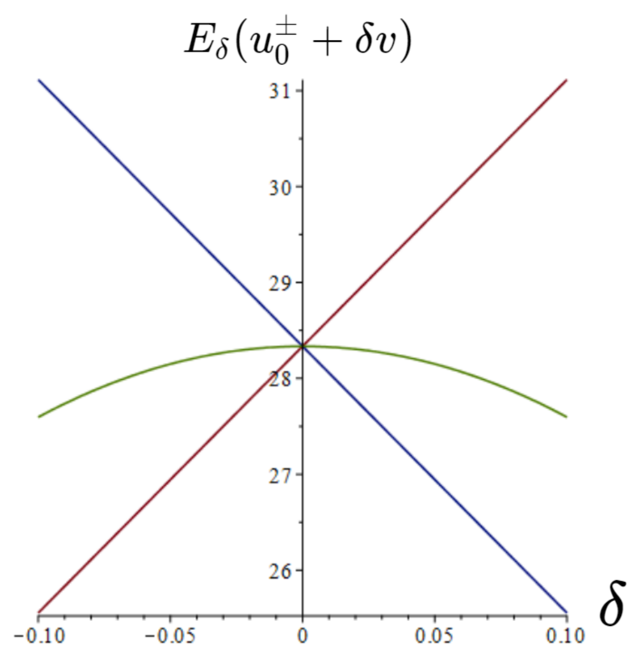

Figure 5.10: Detail of the previous picture, with $\delta \in (-0.1, 0.1)$. Anisotropic energy E_δ of the negative anisotropic vortex $u = u_0^- + \delta v$ (green) and the positive anisotropic vortex $u = u_0^+ \mathrm{e}^{\mathrm{i}(\theta+\theta^\pm)} + \delta v$ where $\theta^+ \in \{0, \pi\}$ (red) and $\theta^- \in \{\pi/3, 3\pi/2\}$ (blue). Diagram made in Maple.

Chapter 6

Analysis for an O.D.E. system from negative anisotropic vortex

In this chapter, in order to approximate a negative anisotropic vortex solution in the whole plane, we consider the system of differential equations for the dominant mode of the Fourier series decomposition of the linear approximation $Lv = -Bu_0^{-1}$ with $v = 0$ on ∂B_R:

$$(EQ)\begin{cases} x'' + \frac{1}{r}x' - \frac{17}{r^2}x + (1 - 3\,U^2(r))x + \frac{8}{r^2}z &= c(r), \\ \\ z'' + \frac{1}{r}z' - \frac{17}{r^2}z + (1 - U^2(r))z + \frac{8}{r^2}x &= c(r), \end{cases} \qquad r \in (0, +\infty) \qquad \text{Repeat eq. (6.1)}$$

We want to construct bounded solutions. We first study the homogeneous system, constructing a base of solutions with certain asymptotic behaviors at zero and at infinity , using a fixed-point argument from the succession of its Picard iterates. This idea comes from the book of Pacard and Riviere [41, Chapter 3], and the recent articles of Anne Beaulieu [9, 10].

Besides, we obtain the no existence of bounded global solutions in $\mathbb{R}^2$, to the homogeneous system, which implies (in case of existence) the uniqueness of bounded solutions in case they exist. Moreover, we connect behaviours of the solutions of the base at 0 to $+\infty$.

Finally, rewriting the extended system of differential equations as a first order system, and with the help of the method of variation of parameters, we can construct a bounded solution to the inhomogeneous system.

6.1 Introduction

We study the existence of bounded solutions of the following system, which arises taking $R \to \infty$ in the system (4.57):

$$\begin{cases} x'' + \dfrac{1}{r}x' - \dfrac{17}{r^2}x + (1-3\,U^2)x + \dfrac{8}{r^2}z & = & c\,(r), \\ \\ z'' + \dfrac{1}{r}z' - \dfrac{17}{r^2}z + (1-U^2)z + \dfrac{8}{r^2}x & = & c\,(r), \end{cases} \qquad r \in (0,+\infty) \tag{6.1}$$

where the functions $x = x(r)$ and $z = z(r)$ are real-valued.
Here $U = U(r)$ is the unique solution of the differential equation

$$\begin{cases} U'' + \dfrac{1}{r}\,U' - \dfrac{1}{r^2}\,U + U(1-U^2) = 0, \qquad r > 0 \\ \\ U(0) = 0, \qquad \lim\limits_{r\to\infty} U(r) = 1 \end{cases} \tag{6.2}$$

and $c(r) = -\left[U''(r) - 3\dfrac{U'(r)}{r} + 3\dfrac{U(r)}{r^2}\right]$.

Theorem 6.1.1. *[15, 30] There exists a unique, non-constant solution of* (6.2)*. This solution U is strictly increasing and $0 < U < 1$. Furthermore,*

$$U(r) = 1 - \frac{1}{2r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

for large r, and there exists some constant $\kappa > 0$ such that

$$U(r) = \kappa\, r - \frac{\kappa}{8}r^3 + \left(\frac{\kappa^3}{24} + \frac{\kappa}{192}\right) r^5 + \mathcal{O}(r^7)$$

for r close to 0.

Corollary 6.1.1.1. *In consequence, the function $c(r)$ defined before, has the following asymptotic expansions:*

$$c\,(r) = -\kappa\left(\frac{\kappa^3}{3} + \frac{1}{24}\right) r^3 + \mathcal{O}(r^5) \quad \text{as} \;\; r \to 0^+$$

$$c\,(r) = -\frac{3}{r^2} + \frac{15}{2\,r^4} + \mathcal{O}\left(\frac{1}{r^6}\right) \quad \text{as} \;\; r \to +\infty$$

Remark 6.1.2. *Since the system* (6.1) *comes from extending to* $(0,+\infty)$ *the system* (4.57)*, we have that*

$$v = (x(r)\cos(4\theta) + \mathrm{i}\, z(r)\, \sin(4\theta))\, \mathrm{e}^{-\mathrm{i}\,\theta} \tag{6.3}$$

is a solution of the system

$$\mathbb{L}_{-1} v = c(r)\, \mathrm{e}^{3\mathrm{i}\theta} \tag{6.4}$$

where $\mathbb{L}_{-1}$ is the linearized operator of the Ginzburg-Landau equation around $u_0^- = U(r)\mathrm{e}^{-\mathrm{i}\theta}$.

Remark 6.1.3. *In terms of $\mathcal{L}_{-1}$, the conjugate linearized operator of $\mathbb{L}_{-1}$, we have that*

$$w := x(r)\cos(4\theta) + \mathrm{i}\, z(r)\, \sin(4\theta) = \left(\frac{x+z}{2}\right) \mathrm{e}^{4\mathrm{i}\theta} + \left(\frac{x-z}{2}\right) \mathrm{e}^{-4\mathrm{i}\theta} \tag{6.5}$$

is a solution of the system

$$\mathcal{L}_{-1} w = c(r)\, \mathrm{e}^{4\mathrm{i}\theta} \tag{6.6}$$

6.2 Asymptotic behavior of solutions of the homogeneous system

In this section, we study the asymptotic behavior of solutions of the homogeneous version of the system defined in (6.1):

$$\begin{cases} x'' + \frac{1}{r}x' - \frac{17}{r^2}x + (1-3\,U^2)x + \frac{8}{r^2}z &= 0 \\ \\ z'' + \frac{1}{r}z' - \frac{17}{r^2}z + (1-U^2)z + \frac{8}{r^2}x &= 0 \end{cases} \tag{6.7}$$

These ordinary differential equations are second order and the functions x and z are real valued, hence the space of solutions of (6.7) is a 4-dimensional real vector space.
Letting $a = (x+z)/2$, $b = (x-z)/2$, we consider the system for (a, b)

$$\begin{cases} a'' + \frac{1}{r}a' - \frac{9}{r^2}a + (1-2\,U^2)\,a - U^2\,b &= 0 \\ \\ b'' + \frac{1}{r}b' - \frac{25}{r^2}b + (1-2\,U^2)\,b - U^2\,a &= 0 \end{cases} \tag{6.8}$$

We will give a complete description of two solution bases for the system (6.7), one base being defined near 0, and another base being defined near $+\infty$.

6.2.1 The possible behaviours at zero

Theorem 6.2.1. *We have a base of four solutions (a, b) of (6.8), with the following behaviors at 0.*

1. *There exist 2 linearly independent solutions that are bounded near at 0,*

$$(a_1(r), b_1(r)) \sim_0 (\mathcal{O}(r^9), r^5), \qquad (a_3(r), b_3(r)) \sim_0 (r^3, \mathcal{O}(r^7))$$

2. *There exist 2 linearly independent solutions that blow up at 0,*

$$(a_2(r), b_2(r)) \sim_0 (\mathcal{O}(r^2\theta(r)), r^{-5}), \qquad (a_4(r), b_4(r)) \sim_0 (r^{-3}, \mathcal{O}(r^2\widetilde{\theta}(r)))$$

where

$$\theta(r) = \frac{-r + r^{-3}}{4} \quad \textit{and} \quad \widetilde{\theta}(r) = \frac{-r^3 + r^{-1}}{4}$$

First we explain the idea of the proof. We can rewrite the system (6.8) as

$$\begin{cases} a'' + \frac{1}{r}a' - \frac{9}{r^2}a &= U^2\,b - (1-2\,U^2)\,a \\ \\ b'' + \frac{1}{r}b' - \frac{25}{r^2}b &= U^2\,a - (1-2\,U^2)\,b \end{cases} \tag{6.9}$$

We use a constructive method, similar to the proof of the Banach fixed point Theorem. We define a fixed point problem of the form $(a, b) = T(a, b)$. For this purpose, we explain the reduction of order method:

Remark 6.2.2. ***(Reduction of order method for non-homogeneous linear second-order equation)*** *Given the general non-homogeneous second order equation*

$$y'' + c_1(r)\, y' + c_2(r)\, y = g(r) \tag{6.10}$$

and a single solution $\varphi(r)$ of the homogeneous equation $[g(r) = 0]$, then we can try a solution of (6.10) *in the form $y(r) = \varphi(r)v(r)$, where $v(r)$ is an arbitrary function.*
If we replace this ansatz in (6.10)*, we get the following first-order equation (reduction of order of second-order equation) for v':*

$$\frac{\mathrm{d}}{\mathrm{d}r}(\mu(r)v'(r)) = \varphi(r)g(r)\mathrm{e}^{\int c_1(r)\,\mathrm{d}r} \tag{6.11}$$

where the integrating factor is $\mu(r) = \varphi(r)^2\, \mathrm{e}^{\int c_1(r)\,\mathrm{d}r}$.
After integrating the last equation, $v'(r)$ is found, containing one constant of integration. Then, integrate $v'(r)$ to find the full solution of the equation (6.10)*, exhibiting two constants of integration as it should:*

$$y(r) = \varphi(r)\, v(r)$$

In our case, we work with the non-homogeneous linear second-order system (6.8) of two variables $a = a(r)$ and $b = b(r)$, where we apply the remark 6.2.2 to each equation of the system.
Note that the solutions of the homogeneous to the first equation of the system are $\phi_1(r) = r^3$ and $\phi_2(r) = r^{-3}$, and for the second equation are $\phi_1(r) = r^5$ and $\phi_2(r) = r^{-5}$.
Thus, if we are looking for solutions a and b that are bounded at 0, we must use the respective homogeneous solutions that are bounded at 0. Hence, $a = r^3 v_1(r)$ and $b = r^5 v_2(r)$, with the following equations (note that $\mathrm{e}^{\int \frac{1}{r}\,\mathrm{d}r} = r$):

$$\frac{\mathrm{d}}{\mathrm{d}r}((r^3)^2 r\, v_1'(r)) = r^4[U^2\, b - (1 - 2\,U^2)\, a], \quad \frac{\mathrm{d}}{\mathrm{d}r}((r^5)^2 r\, v_2'(r)) = r^6[U^2\, b - (1 - 2\,U^2)\, a]$$

After integrating between 0 and t, and after integrating between 0 and r, we have

$$\begin{cases} v_1(r) = v_1(0) + \displaystyle\int_0^r t^{-7} \int_0^t s^4\, [U^2(s)\, b(s) - (1 - 2\,U^2(s))\, a(s)]\, \mathrm{d}s\, \mathrm{d}t, \\ v_2(r) = v_2(0) + \displaystyle\int_0^r t^{-11} \int_0^t s^6\, [U^2(s)\, a(s) - (1 - 2\,U^2(s))\, b(s)]\, \mathrm{d}s\, \mathrm{d}t \end{cases}$$

If we denote $\alpha := v_1(0)$ and $\beta := v_2(0)$, and we replace the previous expressions in $a = r^3 v_1(r)$ and $b = r^5 v_2(r)$, we have the following integral system:

$$\begin{cases} a = \alpha\, r^3 + r^3 \displaystyle\int_0^r t^{-7} \int_0^t s^4\, [U^2(s)\, b(s) - (1 - 2\,U^2(s))\, a(s)]\, \mathrm{d}s\, \mathrm{d}t, \\ \\ b = \beta\, r^5 + r^5 \displaystyle\int_0^r t^{-11} \int_0^t s^6\, [U^2(s)\, a(s) - (1 - 2\,U^2(s))\, b(s)]\, \mathrm{d}s\, \mathrm{d}t \end{cases} \tag{6.12}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are parameters.

Analogously, if we are looking for solutions a and b that blow up at 0, we must use the respective homogeneous solutions that blow up at 0. Hence, $a = r^{-3} v_1(r)$ and $b = r^{-5} v_2(r)$ and following the same procedure as in the bounded case, we have the following integral system:

$$
\begin{cases}
a = \alpha\, r^{-3} + r^{-3} \displaystyle\int_0^r t^5 \int_0^t s^{-2} \left[U^2(s)\, b(s) - (1 - 2\,U^2(s))\, a(s)\right] \mathrm{d}s\, \mathrm{d}t, \\[2ex]
b = \beta\, r^{-5} + r^{-5} \displaystyle\int_0^r t^9 \int_0^t s^{-4} \left[U^2(s)\, a(s) - (1 - 2\,U^2(s))\, b(s)\right] \mathrm{d}s\, \mathrm{d}t
\end{cases}
\tag{6.13}
$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are parameters.

The solution (a_1, b_1)

Proposition 6.2.3. *[41, 9, 10] There exists a solution (a_1, b_1) of (6.8) such that, there exists some real number R and C verifying*

$$
\forall\, r \leq R, \qquad |a_1(r)\, r^{-2}| + |b_1(r) - r^5| \leq C\, r^7 \tag{6.14}
$$

$$
\forall\, r < R, \qquad |a_1'(r)\, r^{-2}| + |b_1'(r) - 5\, r^4| \leq C\, r^6 \tag{6.15}
$$

Proof. Let us consider the integral system (6.12) with $(\alpha, \beta) = (1, 0)$, then we consider (a_1, b_1) solution of the following integral system:

$$
\begin{cases}
a = r^3 + r^3 \displaystyle\int_0^r t^{-7} \int_0^t s^4 \left[U^2(s)\, b(s) - (1 - 2\,U^2(s))\, a(s)\right] \mathrm{d}s\, \mathrm{d}t, \\[2ex]
b = r^5 \displaystyle\int_0^r t^{-11} \int_0^t s^6 \left[U^2(s)\, a(s) - (1 - 2\,U^2(s))\, b(s)\right] \mathrm{d}s\, \mathrm{d}t
\end{cases}
\tag{6.16}
$$

Let us denote by $T_1(a, b)$ the right-hand side of (6.16), we define the maps $\eta_1(r) = r^7$ and $\eta_2(r) = r^5$, and we define two sequences

$$
\begin{cases}
\alpha_0 = 0, \quad \beta_0 = \eta_2 \\
(\alpha_{k+1}, \beta_{k+1}) = T_1(\alpha_k, \beta_k)
\end{cases}
\tag{6.17}
$$

We prove that for all $0 < r < 1$, we have

$$
|\alpha_{k+1} - \alpha_k|(r) \leq C\, \eta_1(r)\, r^2 \left(\|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])}\right) \tag{6.18}
$$

$$
|\beta_{k+1} - \beta_k|(r) \leq C\, \eta_2(r)\, r^2 \left(\|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])}\right) \tag{6.19}
$$

and

$$
|\alpha_1 - \alpha_0|(r) \leq C\, r^2\, \eta_1(r), \quad |\beta_1 - \beta_0|(r) \leq C\, r^2\, \eta_2(r) \tag{6.20}
$$

then it follows

$$
\begin{aligned}
&\|\eta_1^{-1}(\alpha_{k+1} - \alpha_k)\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_{k+1} - \beta_k)\|_{L^\infty([0,r])} \\
&\leq (Cr)^{2k} (\|\eta_1^{-1}(\alpha_1 - \alpha_0)\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_1 - \beta_0)\|_{L^\infty([0,r])})
\end{aligned}
$$

Thus, if we choose R such that $CR < 1$, we can define,

$$
\text{for all } 0 < r < R, \qquad a_1(r) = \alpha_0 + \sum_{k=0}^{\infty} (\alpha_k - \alpha_{k-1}), \qquad b_1(r) = \beta_0 + \sum_{k=0}^{\infty} (\beta_k - \beta_{k-1}) \tag{6.21}
$$

Then, we have $(a_1, b_1) = T_1(a_1, b_1)$ and the continuity of $(a_1(r), b_1(r))$ for $r \in (0, R]$ follows from the continuity of (α_k, β_k) for all k (this will be proved below, together with the estimates (6.18), (6.19) and (6.20)) and from the uniform convergence of the sums in $(0, R]$. The extension of this solution in $(0, +\infty)$ follows from the ODE theory.

The continuity of (α_k, β_k) for all k follows by induction, this is, for $k \geq 1$ assuming that $\alpha_k - \alpha_{k-1}$ and $\beta_k - \beta_{k-1}$ are continuous in $(0, R]$, and the use of the estimates (6.18), (6.19), it gives us the continuity of $\alpha_{k+1} - \alpha_k$ and $\beta_{k+1} - \beta_k$ in $(0, R]$.

Then, it only remains to prove the estimates (6.18) and (6.19), and also the estimates in (6.20). To prove the first, we note that for $0 < t \leq r$, we have

$$\int_0^t s^4 \left[U^2(s) \, |\beta_k - \beta_{k-1}| + |1 - 2\,U^2(s)| \, |\alpha_k - \alpha_{k-1}| \right] \mathrm{d}s$$

$$\leq M \, \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \int_0^t s^4 s^2 s^5 \, \mathrm{d}s + M \, \|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} \int_0^t s^4 s^7 \, \mathrm{d}s$$

$$= M \left(\|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \right) \int_0^t s^{11} \, \mathrm{d}s$$

where we have used the estimates $U^2(s) \leq M s^2$ and $|1 - 2\,U^2(s)| \leq M$. Thus,

$$|\alpha_{k+1} - \alpha_k| \leq M \left(\|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \right) r^3 \int_0^r t^{-7} \int_0^t s^{11} \, \mathrm{d}s \, \mathrm{d}t$$

Then, the desired estimate (6.18) remains to the estimation for all $0 < r < 1$,

$$r^3 \int_0^r t^{-7} \int_0^t s^{11} \, \mathrm{d}s \, \mathrm{d}t = \frac{r^9}{6 \cdot 12} \leq C r^9 = C r^2 \, \eta_1(r)$$

Thus, we have (6.18) and also the estimate of $|\alpha_1 - \alpha_0|$.
Analogously for the second estimate (6.19), we have for $0 < t \leq r$

$$\int_0^t s^6 \left[U^2(s) \, |\alpha_k - \alpha_{k-1}| + |1 - 2\,U^2(s)| \, |\beta_k - \beta_{k-1}| \right] \mathrm{d}s$$

$$\leq M \, \|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} \int_0^t s^6 s^2 s^7 \, \mathrm{d}s + M \, \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \int_0^t s^6 s^5 \, \mathrm{d}s$$

$$= M \, \|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} \int_0^t s^{15} \, \mathrm{d}s + M \, \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \int_0^t s^{11} \, \mathrm{d}s$$

where we have used the estimates $U^2(s) \leq M s^2$ and $|1 - 2\,U^2(s)| \leq M$. Thus,

$$|\alpha_{k+1} - \alpha_k| \leq M \, \|\eta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} \, r^5 \int_0^r t^{-11} \int_0^t s^{15} \, \mathrm{d}s \, \mathrm{d}t$$
$$+ M \, \|\eta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} \, r^5 \int_0^r t^{-11} \int_0^t s^{11} \, \mathrm{d}s \, \mathrm{d}t$$

Then, the desired estimate (6.19) remains to the estimates for all $0 < r < 1$,

$$r^5 \int_0^r t^{-11} \int_0^t s^{15} \, \mathrm{d}s \, \mathrm{d}t \leq C r^{11} = C r^2 \, r^9 \leq C r^2 \, \eta_2(s)$$

$$r^5 \int_0^r t^{-11} \int_0^t s^{11} \, \mathrm{d}s \, \mathrm{d}t = \frac{r^7}{24} \leq C r^2 \, r^5 = C r^2 \, \eta_2(s)$$

Thus, we have (6.19) and also the estimate of $|\beta_1 - \beta_0|$.
Therefore, this proves the the existence of (a_1, b_1). ∎

Similarly, following the same procedure, the following behaviors are observed for the rest of the base solutions:

The solution (a_2, b_2)

Proposition 6.2.4. *[41, 9, 10] There exists a solution* (a_2, b_2) *of* (6.8) *such that, there exists some real number* R *and* C *verifying*

$$\forall\, r \le R, \qquad |a_2(r)| \le C(r^2\theta(r) + r^3), \quad |b_2(r) - r^{-5}| \le C\, r^3 \tag{6.22}$$

$$\forall\, r < R, \qquad |a_2'(r)| \le C(r\theta(r) + r^2), \quad |b_2'(r) + 5\, r^{-6}| \le C\, r^{-6} \tag{6.23}$$

The solution (a_3, b_3)

Proposition 6.2.5. *[41, 9, 10] There exists a solution* (a_3, b_3) *of* (6.8) *such that, there exists some real number* R *and* C *verifying*

$$\forall\, r \le R, \qquad |a_3(r) - r^3| \le C\, r^5, \quad |b_3(r)| \le C\, r^7 \tag{6.24}$$

$$\forall\, r < R, \qquad |a_3'(r) - 3r^2| \le C\, r^4, \quad |b_3'(r)| \le C\, r^6 \tag{6.25}$$

The solution (a_4, b_4)

Proposition 6.2.6. *[41, 9, 10] There exists a solution* (a_4, b_4) *of* (6.8) *such that, there exists some real number* R *and* C *verifying*

$$\forall\, r \le R, \qquad |a_4(r) - r^{-3}| \le C\, r^{-1}, \quad |b_4(r)| \le C(r^2\widetilde{\theta} + r^5) \tag{6.26}$$

$$\forall\, r < R, \qquad |a_4'(r) + 3\, r^{-4}| \le C\, r^{-2}, \quad |b_4'(r) - 5\, r^4| \le C(r\widetilde{\theta} + r^4) \tag{6.27}$$

6.2.2 The possible behaviours at infinity

In this subsection, we use the system (6.7) and we construct a base of four solutions (x_j, z_j) $j = 1, \dots, 4$ characterized by their behaviors at $+\infty$.
Theorem 6.2.7. *[41, 9, 10] We have a base of four solutions* (x, z) *of* (6.7)*, such that*

$$\begin{aligned}
x_1(r) &= J_4^+(1 + \mathcal{O}(r^{-2})) && \textit{and} && z_1(r) = J_4^+\mathcal{O}(r^{-2}) \\
x_3(r) &= \mathcal{O}(r^2) && \textit{and} && z_3(r) = r^4(1 + \mathcal{O}(r^{-2})) \\
x_2(r) &= J_4^-(1 + \mathcal{O}(r^{-2})) && \textit{and} && z_2(r) = J_4^-\mathcal{O}(r^{-2}) \\
x_4(r) &= \mathcal{O}(r^{-6}) && \textit{and} && z_4(r) = r^{-4}(1 + \mathcal{O}(r^{-2}))
\end{aligned} \tag{6.28}$$

where $J_4^{\pm}$ *is defined in lemma 6.2.8.*

We can rewrite the system (6.7) as

$$\left\{\begin{array}{rcl} x'' + \frac{1}{r}x' - \frac{14}{r^2}x - 2\,x &=& -\frac{8}{r^2}z - 3\left(1 - U^2 - \frac{1}{r^2}\right)x \\ \\ z'' + \frac{1}{r}z' - \frac{16}{r^2}z &=& -\frac{8}{r^2}x - \left(1 - U^2 - \frac{1}{r^2}\right)z \end{array}\right. \tag{6.29}$$

We use the same idea as in the bounded case. For some given $R_0 > 0$, we define a fixed point problem of the form $(x, y) = T(x, y)$ for (x, y) defined on $[R_0, +\infty)$.
For this purpose, we will need the following lemma, a result concerning the asymptotic behavior of some Bessel functions at $+\infty$, and we will also use as a homogeneous solution of first equation in (6.29),
Lemma 6.2.8. *[41] For all $n \geq 0$, and all $\eta > 0$, there exist $R_0 > 0$, $C_\eta > c_\eta > 0$ and a function J_n^- solution of*

$$\frac{\mathrm{d}^2 J}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}J}{\mathrm{d}r} - \frac{1}{r^2}(n^2 - 2)J - 2J = 0, \qquad \textit{in} \ \ (R_0, +\infty) \tag{6.30}$$

such that

$$c_\eta\, r^{-1/2-\eta}\, \mathrm{e}^{-\sqrt{2}r} \leq J_n^- \leq C_\eta\, r^{-1/2+\eta}\, \mathrm{e}^{-\sqrt{2}r}, \qquad \forall\, r \geq R_0 \tag{6.31}$$

For all $n \geq 0$, there exist R_0, $C > c > 0$ and a function J_n^+, solution of (6.30) *in* $(R_0, +\infty)$ *such that*

$$c\, r^{-1/2}\, \mathrm{e}^{\sqrt{2}r} \leq J_n^+ \leq C\, r^{-1/2}\, \mathrm{e}^{\sqrt{2}r}, \qquad \forall\, r \geq R_0 \tag{6.32}$$

Proof. For a complete reference about Bessel functions, in particular, about asymptotic expansion of modified Bessel functions, see [49, pages 202-203]. Also, for a alternative proof of this lemma, see [41, pages 56-57]. ■

The idea is the same as in the construction of base of solutions near $r = 0$, we apply the lemma 6.2.2 over each equation of the system, if we apply the lemma 6.2.8 to the first equation, then we get that the homogeneous solutions for the first equation of (6.29) are $\phi_1(r) = J_4^+$ and $\phi_2(r) = J_4^-$, and for the second equation are $\phi_1(r) = r^{-4}$ and $\phi_2(r) = r^4$.

Using this, we are going to construct four solutions (x_j, z_j), $j = 1, \ldots, 4$ of (6.7). The strategy is almost the same for each solution. First, for some given $R_0 > 0$, we consider $E = BC(I, \mathbb{R})$ the Banach space of all bounded continuous real-valued functions on the interval $I = [R_0, \infty)$, endowed with the sup-norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \sup_{r \in I} |x(r)| \qquad \text{for } x \in BC(I, \mathbb{R})$$

a fixed point problem of the form

$$(x, z) = T(x, z) \tag{6.33}$$

defined for $(x, z) \in \mathcal{C}([R_0, +\infty)) \times \mathcal{C}([R_0, +\infty))$. Therefore for a given function ξ, we will prove the existence of a fixed point (x, z) verifying, for some $C > 0$, an estimate of the form

$$|x_j(r) - \xi(r)| + |z_j(r)| \leq C\, \xi(r)\, r^{-2} \qquad \text{if } \ j = 1, 2, \tag{6.34}$$

$$\text{or} \quad |x_j(r)| + |z_j(r) - \xi(r)| \leq C\, \xi(r)\, r^{-2} \qquad \text{if } \ j = 3, 4. \tag{6.35}$$

in this way we will obtain the following

$$(x_j(r), z_j(r)) = (\xi(r)\,(1 + \mathcal{O}(r^{-2})), \xi(r)\,\mathcal{O}(r^{-2})) \qquad \text{if } \; j = 1, 2 \tag{6.36}$$

$$(x_j(r), z_j(r)) = (\xi(r)\,\mathcal{O}(r^{-2}), \xi(r)\,(1 + \mathcal{O}(r^{-2})) \qquad \text{if } \; j = 3, 4. \tag{6.37}$$

We define by induction, for (x_1, z_1) and for (x_2, z_2)

$$(\alpha_0, \beta_0) = (\xi, 0) \quad \text{ and } \quad (\alpha_{k+1}, \beta_{k+1}) = T(\alpha_k, \beta_k). \tag{6.38}$$

For (x_3, z_3) and for (x_4, z_4), we exchange the role of x and z, that gives

$$(\alpha_0, \beta_0) = (0, \xi) \quad \text{ and } \quad (\alpha_{k+1}, \beta_{k+1}) = T(\alpha_k, \beta_k). \tag{6.39}$$

We denote $\eta : r \mapsto r$, then we prove that there exists $C > 0$ independent of R_0, such that for all $R_0 \geq 0$ and all $k \geq 0$,
for $j = 1, 2$

$$|(\alpha_{k+1} - \alpha_k)\,\xi^{-1}|(r) \leq \frac{C}{r^2}\left(\|(\alpha_k - \alpha_{k-1})\,\xi^{-1}\|_\infty + \|(\beta_k - \beta_{k-1})\,\xi^{-1}\eta^2\|_\infty\right) \tag{6.40}$$

and

$$r^2|(\beta_{k+1} - \beta_k)\,\xi^{-1}|(r) \leq \frac{C}{r^2}\left(\|(\alpha_k - \alpha_{k-1})\,\xi^{-1}\|_\infty + \|(\beta_k - \beta_{k-1})\,\xi^{-1}\eta^2\|_\infty\right) \tag{6.41}$$

for $j = 3, 4$

$$r^2|(\alpha_{k+1} - \alpha_k)\,\xi^{-1}|(r) \leq \frac{C}{r^2}\left(\|(\alpha_k - \alpha_{k-1})\,\xi^{-1}\eta^2\|_\infty + \|(\beta_k - \beta_{k-1})\,\xi^{-1}\|_\infty\right) \tag{6.42}$$

and

$$|(\beta_{k+1} - \beta_k)\,\xi^{-1}|(r) \leq \frac{C}{r^2}\left(\|(\alpha_k - \alpha_{k-1})\,\xi^{-1}\eta^2\|_\infty + \|(\beta_k - \beta_{k-1})\,\xi^{-1}\|_\infty\right) \tag{6.43}$$

Then, we define

$$x(r) = \alpha_0(r) + \sum_{k\geq 0}(\alpha_{k+1} - \alpha_k)(r) \quad \text{ and } \quad z(r) = \beta_0(r) + \sum_{k\geq 0}(\beta_{k+1} - \beta_k)(r) \tag{6.44}$$

Since C is independent of R_0, we choose $R_0 > 0$ such that $(CR_0^{-2}) < 1$, the sums $\xi^{-1}x(r)$ and $\xi^{-1}\eta^2 z(r)$ (or $\xi^{-1}\eta^2 x(r)$ and $\xi^{-1}z(r)$) converge (uniformly) in E.

We will need the following estimates
Lemma 6.2.9. *Let $\alpha \in \mathbb{R}$ and $\beta > 0$ be given. Then*

$$\int_t^{+\infty} s^\alpha\,\mathrm{e}^{-\beta s}\,\mathrm{d}s \leq \frac{2}{\beta}t^\alpha\,\mathrm{e}^{-\beta t} \qquad \forall\, t \geq \frac{2\alpha}{\beta} \tag{6.45}$$

and

$$\int_R^t s^\alpha\,\mathrm{e}^{\beta s}\,\mathrm{d}s \leq \frac{2}{\beta}t^\alpha\,\mathrm{e}^{\beta t} \qquad t \geq R \geq \frac{-2\alpha}{\beta} \tag{6.46}$$

The fastest blow-up at $+\infty$: the solution (x_1, z_1)

Proposition 6.2.10. *[41, 9, 10] There exists a solution (x_1, z_1) of (6.7), such that there exists C and $R_0 > 0$ such that for all $r > R_0$*

$$|x_1(r) - J_4^+(r)| + |z_1(r)| \leq C J_4^+(r)\, r^{-2} \tag{6.47}$$

$$|x_1'(r) - (J_4^+)'(r)| \leq C J_4^+(r)\, r^{-3}, \qquad |z_1'(r)| \leq C J_4^+(r)\, r^{-4} \tag{6.48}$$

Proof. Let $R_0 > 0$ be given. Let us consider the following fixed point problem $(x, z) = T(x, z)$

$$\begin{cases} x &= J_4^+ + J_4^+ \displaystyle\int_{+\infty}^r (J_4^+)^{-2} t^{-1} \int_{R_0}^t s J_4^+ \left(-\frac{8}{s^2} z - 3 \left(1 - U^2 - \frac{1}{s^2}\right) x \right) \mathrm{d}s\, \mathrm{d}t \\ z &= r^4 \displaystyle\int_{R_0}^r t^{-9} \int_{R_0}^t s^5 \left(-\frac{8}{s^2} x - \left(1 - U^2 - \frac{1}{s^2}\right) z \right) \mathrm{d}s\, \mathrm{d}t \end{cases} \tag{6.49}$$

with $x, z \in \mathcal{C}([R_0, +\infty), \mathbb{R})$.

Let $\xi = J_4^+$. We define (α_k, β_k) by

$$(\alpha_0, \beta_0) = (\xi, 0) \qquad \text{and} \qquad (\alpha_{k+1}, \beta_{k+1}) = T(\alpha_k, \beta_k) \qquad \text{Repeat eq (6.38)}$$

First, we prove that, for R_0 large enough, if

$$((\alpha_k - \alpha_{k-1})(J_4^+)^{-1}, (\beta_k - \beta_{k-1})(J_4^+)^{-1}\eta^2) \in E \times E \tag{6.50}$$

then

$$((\alpha_{k+1} - \alpha_k)(J_4^+)^{-1}, (\beta_{k+1} - \beta_k)(J_4^+)^{-1}\eta^2) \in E \times E \tag{6.51}$$

where we have denoted by $\eta : r \mapsto r$. Therefore, we suppose that (6.50) is true, to show that (6.50) is true. We begin noting the following

$$|(\alpha_{k+1} - \alpha_k)(r)| \leq J_4^+ \int_{+\infty}^r (J_4^+)^{-2} t^{-1} \int_{R_0}^t s\, J_4^+ \left(\frac{8}{s^2} \big|\beta_k - \beta_{k-1}\big|(s) + 3 \left|1 - U^2 - \frac{1}{s^2}\right| \big|\alpha_k - \alpha_{k-1}\big|(s) \right) \mathrm{d}s\, \mathrm{d}t \tag{6.52}$$

$$|(\beta_{k+1} - \beta_k)(r)| \leq r^4 \int_{R_0}^r t^{-9} \int_{R_0}^t s^5 \left(\frac{8}{s^2} \big|\alpha_k - \alpha_{k-1}\big|(s) + \left|1 - U^2 - \frac{1}{s^2}\right| \big|\beta_k - \beta_{k-1}\big|(s) \right) \mathrm{d}s\, \mathrm{d}t \tag{6.53}$$

To prove that $(\alpha_{k+1} - \alpha_k)(J_4^+)^{-1} \in E$, we write

$$\int_{R_0}^t s\, J_4^+ \left(\frac{8}{s^2} \big|\beta_k - \beta_{k-1}\big|(s) + 3 \left|1 - U^2 - \frac{1}{s^2}\right| \big|\alpha_k - \alpha_{k-1}\big|(s) \right) \mathrm{d}s \tag{6.54}$$

$$\leq \int_{R_0}^t s (J_4^+)^2 \frac{8}{s^2} s^{-2}\, \mathrm{d}s \|(J_4^+)^{-1} \eta^2 (\beta_k - \beta_{k-1})\|_\infty + \int_{R_0}^t s \frac{M}{s^4} (J_4^+)^2\, \mathrm{d}s \|(J_4^+)^{-1}(\alpha_k - \alpha_{k-1})\|_\infty$$

where we have used $\left|1 - U^2 - \dfrac{1}{s^2}\right| \leq \dfrac{M}{s^4}$ for some constant $M > 0$.
Using that $J_4^+(r) = \mathcal{O}(\mathrm{e}^{\sqrt{2}r}/\sqrt{r})$ (by (6.32)), also the inequalities (6.45) and (6.46), we get that

$$(J_4^+)^{-2} t^{-1} \int_{R_0}^t s\, J_4^+ \left(\frac{8}{s^2} (\beta_k - \beta_{k-1})(s) + 3 \left(1 - U^2 - \frac{1}{s^2}\right) (\alpha_k - \alpha_{k-1})(s) \right) \mathrm{d}s$$

is integrable on $[r,+\infty)$, when $R_0 \geq \frac{6}{2\sqrt{2}}$, then $(\alpha_{k+1}-\alpha_k)(J_4^+)^{-1}$ is a bounded function on $[R_0,+\infty)$. Besides, by the Lebesgue dominated convergence theorem [12], we obtain that $(\alpha_{k+1}-\alpha_k)$ is a continuous functions on $[R_0,+\infty)$. It follows that, $(\alpha_{k+1}-\alpha_k)(J_4^+)^{-1} \in E$.

$$\int_{R_0}^{t} s^5 \left(\frac{8}{s^2}\left|\alpha_k-\alpha_{k-1}\right|(s)+\left|1-U^2-\frac{1}{s^2}\right|\left|\beta_k-\beta_{k-1}\right|(s)\right) \mathrm{d}s$$
$$\leq \int_{R_0}^{t} s^5 J_4^+ \left(\frac{1}{s^2}\|(J_4^+)^{-1}(\alpha_k-\alpha_{k-1})\|_\infty+\frac{M}{s^4}s^{-2}\|(J_4^+)^{-1}\eta^2(\beta_k-\beta_{k-1})\|_\infty\right) \mathrm{d}s$$

We use (6.45) with $\alpha = 3$ and for $\alpha = 1$, and $R_0 \geq 0$, it follows that

$$t^{-9}\int_{R_0}^{t} s^5 \left(\frac{8}{s^2}(\alpha_k-\alpha_{k-1})(s)+\left(1-U^2-\frac{1}{s^2}\right)(\beta_k-\beta_{k-1})(s)\right) \mathrm{d}s$$

is integrable in $[R_0, r)$, therefore $(\beta_{k+1}-\beta_k)$ is a bounded function, and by the Lebesgue dominated convergence theorem, we obtain that $(\beta_{k+1}-\beta_k)$ is a continuous function. We can conclude (6.51).

Now, we prove (6.40), we estimate, in view of (6.45) and for $R_0 \geq \frac{8}{2\sqrt{2}}$

$$J_4^+\int_r^{+\infty}(J_+^4)^{-2}\frac{1}{t}\int_{R_0}^{t} s(J_4^+)^2 s^{-4}\,\mathrm{d}s\,\mathrm{d}s$$
$$\leq CJ_4^+\int_r^{+\infty}(J_+^4)^{-2}\frac{1}{t}\frac{2}{2\sqrt{2}}t^{-4}\mathrm{e}^{2\sqrt{2}t}\,\mathrm{d}t \leq CJ_4^+\int_r^{+\infty}\frac{2}{2\sqrt{2}}t^{-4}\,\mathrm{d}t \leq Cr^{-3}J_4^+ \tag{6.55}$$

This gives (6.40), with $\xi = J_4^+$, and this gives also

$$|\alpha_1-\alpha_0|(r) \leq Cr^{-3}J_4^+ \tag{6.56}$$

Analogously, we prove (6.41), for this, we estimate, for $R_0 \geq \frac{15\sqrt{2}}{2}$

$$r^4\int_{R_0}^{r} t^{-9}\int_{R_0}^{t} s^5\frac{J_4^+}{s^4}\,\mathrm{d}s\,\mathrm{d}t \leq r^4\int_{R_0}^{r} t^{-9}\left(\frac{2}{\sqrt{2}}t^{3/2}\mathrm{e}^{\sqrt{2}t}\right)\mathrm{d}t \leq r^4\int_{R_0}^{r}\frac{2}{\sqrt{2}}t^{-15/2}\mathrm{e}^{\sqrt{2}t}\,\mathrm{d}t \leq Cr^{-3}J_4^+$$

This gives (6.41) and also gives

$$|\beta_1-\beta_0|(r) \leq Cr^{-3}J_4^+ \tag{6.57}$$

By (6.40) and (6.41) give, for all $k \geq 1$ and $r > R_0$

$$\begin{aligned}&(J_4^+)^{-1}|\alpha_{k+1}-\alpha_k|(r)+(J_4^+)^{-1}r^2|\beta_{k+1}-\beta_k|(r)\\ &\leq (CR_0^{-2})^{k-1}(\|(J_4^+)^{-1}(\alpha_1-\alpha_0)\|_\infty+\|(J_4^+)^{-1}\eta^2(\beta_1-\beta_0)\|_\infty)\end{aligned} \tag{6.58}$$

Thus, defining x_1 and z_1 as follows:

$$x_1(r) = J_4^+(r)+\sum_{k\geq 0}(\alpha_{k+1}-\alpha_k)(r), \qquad z_1(r) = \sum_{k\geq 0}(\beta_{k+1}-\beta_k)(r) \tag{6.59}$$

We use the following characterization through the absolute convergence of Banach spaces

Lemma 6.2.11. *A normed space $(E, \|\cdot\|_E)$ is a Banach space if and only if, each absolutely convergent series in E converges in E, that is*

$$\sum_{k\geq 0}\|v_k\|_E < \infty \qquad \textit{implies that} \qquad \sum_{k\geq 0} v_k \quad \textit{converges in } E \tag{6.60}$$

In our case, we have the Banach Space $(E, \|\cdot\|_\infty)$, and by (6.51),(6.58) we have that

$$\forall\, k \geq 0, \quad (J_4^+)^{-1}(\alpha_{k+1}-\alpha_k) \in E, \quad (J_4^+)^{-1}\eta^2(\beta_{k+1}-\beta_k) \in E$$

$$\|(J_4^+)^{-1}(\alpha_{k+1}-\alpha_k)\|_\infty \leq (CR_0^{-2})^{k-1}(\|(J_4^+)^{-1}(\alpha_1-\alpha_0)\|_\infty + \|(J_4^+)^{-1}\eta^2(\beta_1-\beta_0)\|_\infty)$$
$$\|(J_4^+)^{-1}\eta^2(\beta_{k+1}-\beta_k)\|_\infty \leq (CR_0^{-2})^{k-1}(\|(J_4^+)^{-1}(\alpha_1-\alpha_0)\|_\infty + \|(J_4^+)^{-1}\eta^2(\beta_1-\beta_0)\|_\infty)$$

Since C is independent of R_0, we choose $R_0 > 0$ such that $(CR_0^{-2}) < 1$, the series

$$\sum_{k\geq 0}\|(J_4^+)^{-1}(\alpha_{k+1}-\alpha_k)\|_\infty, \quad \sum_{k\geq 0}\|(J_4^+)^{-1}\eta^2(\beta_{k+1}-\beta_k)\|_\infty$$

converge, then by lemma 6.2.11 we have that

$$\sum_{k\geq 0}(J_4^+)^{-1}(\alpha_{k+1}-\alpha_k), \qquad \sum_{k\geq 0}(J_4^+)^{-1}(\beta_{k+1}-\beta_k)$$

converges in E, therefore we have that x_1 and z_1 defined by (6.59) are continuous functions on $[R_0, +\infty)$. It is a direct calculation to verify that x_1 and z_1 are solutions of the integral system (6.49). In consequence, (x_1, z_1) is a solution of (6.7).

In order to prove the behavior at $+\infty$ for x_1, we note that

$$|x_1(r) - J_4^+| \leq \sum_{k\geq 0}|\alpha_{k+1}-\alpha_k|(r)$$
$$\leq |\alpha_1-\alpha_0|(r) + J_4^+\left[\sum_{k\geq 1}(Cr_0^{-2})^{k-1}(\|(J_4^+)^{-1}(\alpha_1-\alpha_0)\|_\infty + \|(J_4^+)^{-1}\eta^2(\beta_1-\beta_0)\|_\infty)\right]$$

and by (6.56),(6.57) we obtain the behavior at $+\infty$ for x_1. A similar proof gives the desired behavior of z_1 at $+\infty$. In consequence, (x_1, z_1) satisfies (6.47).

Now, we prove (6.48). For $(x_1'(r), z_1'(r))$, we note that

$$(\alpha_{k+1}'-\alpha_k')(r) = (J_4^+)'(J_4^+)^{-1}(\alpha_{k+1}-\alpha_k)(r)$$

$$+\frac{(J_4^{+1})^{-1}}{r}\int_{R_0}^r sJ_4^+\left(-\frac{8}{s^2}(\beta_k-\beta_{k-1}) - 3\left(1-U^2-\frac{1}{s^2}\right)(\alpha_k-\alpha_{k-1})\right)\,\mathrm{d}s.$$

Thus, using successively (6.46) and (6.58)

$$(J_4^+)^{-1}|\alpha'_{k+1}-\alpha'_k|(r) \le C(J_4^+)^{-1}|\alpha_{k+1}-\alpha_k|(r)$$

$$+C\,r^{-2}(\|(J_4^+)^{-1}\eta^2(\beta_k-\beta_{k-1})\|_\infty + \|(J_4^+)^{-1}(\alpha_k-\alpha_{k-1})\|_\infty) \tag{6.61}$$

$$\le C\,r^{-3}(CR_0^{-3})^{k-1}(\|(J_4^+)^{-1}(\alpha_1-\alpha_0)\|_\infty + \|(J_4^+)^{-1}(\beta_1-\beta_0)\|_\infty)$$

This gives the convergence in E of

$$\sum_{k\ge1}(J_4^+)^{-1}(\alpha'_{k+1}-\alpha'_k)(r)$$

Besides, we have

$$|\alpha'_1(r)-\alpha'_0(r)|(J_4^+)^{-1} \le (J_4^+)^{-1}|\alpha_1-\alpha_0|(r) + C\,r^{-4}\,J_4^+((J_4^+)')^{-1} \le C\,r^{-3}$$

and then we get the behaviour of x_1 at $+\infty$, $(J_4^+)^{-1}|x'_1(r)-(J_4^+)'| \le C\,r^{-3}$.

■

Similarly, following the same procedure, the following behaviors are observed for the rest of the base solutions:

The fastest decaying at $+\infty$: the solution (x_2, z_2)

Proposition 6.2.12. *[41, 9, 10] There exists a solution (x_2, z_2) of (6.7), such that there exists C and $R_0 > 0$ such that for all $r > R_0$*

$$|x_2(r)-J_4^-(r)| + |z_2(r)| \le C J_4^-(r)\,r^{-2} \tag{6.62}$$

$$|x'_2(r)-(J_4^-)'(r)| + |z'_2(r)| \le C J_4^-(r)\,r^{-3}. \tag{6.63}$$

The intermediate blowing up behavior at $+\infty$: the solution (x_3, z_3)

Proposition 6.2.13. *[41, 9, 10] There exists a solution (x_3, z_3) of (6.7), such that there exists C and $R_0 > 0$ such that for all $r > R_0$*

$$|x_3(r)| + |z_3(r)-r^4| \le Cr^2 \tag{6.64}$$

$$|x'_3(r)| + |z'_3(r)-4r^3| \le C\,r \tag{6.65}$$

The intermediate vanishing behavior at $+\infty$: the solution (x_4, z_4)

Proposition 6.2.14. *[41, 9, 10] There exists a solution (x_4, z_4) of (6.7), such that there exists C and $R_0 > 0$ such that for all $r > R_0$*

$$|x_4(r)| + |z_4(r)-r^{-4}| \le C\,r^{-6} \tag{6.66}$$

$$|x'_4(r)| + |z'_4(r)+4\,r^{-5}| \le Cr^{-7} \tag{6.67}$$

6.2.3 The homogeneous system does not admit globally bounded solutions

Theorem 6.2.15. *The homogeneous system* (6.7) *doesn't admit globally bounded solutions in* $[0,+\infty)$.

Proof. We use the quadratic form associated to linearized Ginzburg-Landau operator defined by:

$$Q(w) := -\frac{1}{4\pi}\int_{\mathbb{R}^2} \langle w, \mathcal{L}_{-1} w\rangle r \,\mathrm{d}r\,\mathrm{d}\theta,$$

which is well-defined for all $w \in H^1(\mathbb{R}^2;\mathbb{C})$. Here, $\langle f, g\rangle = \mathbb{R}\mathrm{e}(f\bar{g})$.
In our case, we consider w, defined by

$$w := a(r)\,\mathrm{e}^{4\mathrm{i}\theta} + b(r)\,\mathrm{e}^{-4\mathrm{i}\theta} \tag{6.68}$$

where a and b are real valued functions only depend on $r \in (0,+\infty)$. In this case, we simply have

$$Q(w) = \int_0^{+\infty}\left(\left|\frac{\mathrm{d}a}{\mathrm{d}r}\right|^2 + \left|\frac{\mathrm{d}b}{\mathrm{d}r}\right|^2 + \frac{25}{r^2}a^2 + \frac{9}{r^2}b^2\right) r\,\mathrm{d}r - \int_0^{+\infty}(1-U^2)(a^2+b^2)\,r\,\mathrm{d}r$$
$$+\frac{1}{2}\int_0^{+\infty} U^2(a+b)^2\,r\,\mathrm{d}r.$$

and for $\psi = \psi(r)$ complex-valued function, we have:

$$Q(\psi) = \int_0^{+\infty}\left(\left|\frac{\mathrm{d}\psi}{\mathrm{d}r}\right|^2 + \frac{1}{r^2}|\psi|^2\right) r\,\mathrm{d}r - \int_0^{+\infty}(1-U^2)|\psi|^2\,r\,\mathrm{d}r + \frac{1}{2}\int_0^{+\infty} U^2|\psi+\bar{\psi}|^2\,r\,\mathrm{d}r$$

Now, for the function $w := a(r)\,\mathrm{e}^{4\mathrm{i}\theta} + b(r)\,\mathrm{e}^{-4\mathrm{i}\theta}$, defined before, we define the auxiliary function

$$\widetilde{w} := \mathrm{i}\,(a^2+b^2)^{1/2}.$$

Using the Cauchy-Schwartz inequality we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}(a^2+b^2)^{1/2}\right)^2 \le \left|\frac{\mathrm{d}a}{\mathrm{d}r}\right|^2 + \left|\frac{\mathrm{d}b}{\mathrm{d}r}\right|^2$$

Thus

$$\int_0^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}r}(a^2+b^2)^{1/2}\right)^2 r\,\mathrm{d}r \le \int_0^{+\infty}\left|\frac{\mathrm{d}a}{\mathrm{d}r}\right|^2 + \left|\frac{\mathrm{d}b}{\mathrm{d}r}\right|^2 r\,\mathrm{d}r.$$

This inequality gives

$$Q(w) - Q(\widetilde{w}) \ge \int_0^{+\infty}\left(\frac{24}{r^2}a^2 + \frac{8}{r^2}b^2\right) r\,\mathrm{d}r + \frac{1}{2}\int_0^{+\infty}(1-U^2)(a+b)^2\,r\,\mathrm{d}r$$

This implies,

$$Q(\widetilde{w}) \le Q(w) \tag{6.69}$$

Let us assume that (x,z) is a globally bounded solution of the homogeneous system (6.7), by the remark 6.1.3, we have that

$$w = \left(\frac{x+z}{2}\right)\mathrm{e}^{4\mathrm{i}\theta} + \left(\frac{x-z}{2}\right)\mathrm{e}^{-4\mathrm{i}\theta}$$

is a solution of the homogeneous system in $\mathbb{R}^2$

$$\mathcal{L}_{-1}w = 0$$

If we define $a := (x+z)/2$ and $b := (x-z)/2$, we have that (a,b) is a globally solution of the homogeneous system (6.9).

Therefore, from theorems 6.2.1 and 6.2.7, if follows that that $w \in H^1(\mathbb{R}^2;\mathbb{C})$ and since $\mathcal{L}_{-1}w = 0$, we obtain $Q(w) = 0$.

On the other hand, a result of del Pino, Felmer and Kowalczyk [18], asserts that

$$Q(\widetilde{w}) \geq 0$$

This, together with (6.69), gives

$$0 \leq Q(\widetilde{w}) \leq Q(w) = 0$$

and thus $Q(\widetilde{w}) = Q(w) = 0$.

In particular, it follows directly from the expression of $Q(w) - Q(\widetilde{w})$ that $a = b = 0$, or equivalently, $x = z = 0$. ■

From the previous theorem, the following corollary can be deduced immediately

Corollary 6.2.15.1 (Uniqueness of globally bounded solutions of the inhomogeneous system).
If there is a globally bounded solution to the system (6.1)*, it must be unique.*

6.2.4 The fastest vanishing at zero is related with the fastest blow-up at infinity

The following theorem connects the fastest vanishing at 0 and to the exponentially blowing up behaviour at $+\infty$ and the fastest decaying at $+\infty$ to the fastest blowing-up behaviour at 0.

Theorem 6.2.16. *[41, 9, 10] We have the following connections between the base of solutions at 0, defined in Theorem 6.2.1 and the base of solutions at $+\infty$ defined in Theorem 6.2.7*

(i) Let (a_1, b_1) solution of 6.8 defined by $(a_1, b_1) \sim_0 (\mathcal{O}(r^9), r^5)$. Then (a_1, b_1) blows up exponentially at $+\infty$ like $(J_4^+, J_4^+)/2$.

(ii) Let (x_2, z_2) be the solution of 6.7 defined by $(x_2, z_2) \sim_{+\infty} (J_4^-, J_4^-)$ Then $(x_2, z_2) \sim_0 C(o(r^{-5}), r^{-5})$

Proof. (i) First, we prove that (a_1, b_1) blows up exponentially at $+\infty$. We define $x = a_1 + b_1$ and $z = a_1 - b_1$. Then by the Theorem 6.2.1, $a_1(r) = \mathcal{O}(r^9)$ and $b_1(r) = r^5$, since the behavior of r^5 dominates over the behaviour of r^9 near $r = 0$, then we have

$$x(r) = a_1(r) + b_1(r) \sim r^5 \qquad \text{and} \qquad z(r) = a_1(r) - b_1(r) \sim -r^5 \qquad \text{near} \quad r = 0.$$

Now, let us prove that

$$x > 0, \qquad \text{and} \qquad z < 0 \qquad \text{for} \qquad r > 0. \tag{6.70}$$

Observe that those inequalities are already true for r small, near 0. Then, we proceed to prove (6.70) by contradiction.Let us suppose that (6.70) is not true, then there would exists $R_0 > 0$ such that

$$x(r) > 0 \quad \forall\, r \in (0, R_0), \qquad x(R_0) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}r}(R_0) \leq 0\,, \tag{6.71}$$

and

$$z(r) < 0 \quad \forall\, r \in (0, R_0), \qquad z(R_0) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}r}(R_0) \geq 0\,, \tag{6.72}$$

Taking the first equation of the system (6.7) with the equation for U, (6.2), we get the following system for all $r > 0$,

$$\begin{cases} x'' + \dfrac{1}{r}x' - \dfrac{17}{r^2}x + \dfrac{8}{r^2}z - 2\,U^2 x & = \quad -(1-U^2)\,x \\ U'' + \dfrac{1}{r}U' - \dfrac{1}{r^2}U & = \quad -U(1-U^2) \end{cases} \tag{6.73}$$

Multiplying the first equation of (6.73) by $r\,U$ and the second equation of (6.73) by $-r\,x$, and noting that $\dfrac{\mathrm{d}^2\bullet}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\bullet}{\mathrm{d}r} = \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\,(r\,\bullet)$

$$\begin{cases} U\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\,x'\right) - \dfrac{17}{r}x\,U + \dfrac{8}{r}z\,U - 2\,r\,U^2\,x & = \quad -r\,U(1-U^2)\,x \\ -x\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\,U'\right) + \dfrac{x\,U}{r} & = \quad -r\,U(1-U^2)x \end{cases} \tag{6.74}$$

Adding both equations and integrating between 0 and r, we get

$$\int_0^r \left(U\,\frac{\mathrm{d}}{\mathrm{d}s}(s\,x') - x\,\frac{\mathrm{d}}{\mathrm{d}s}(s\,U')\right)\,\mathrm{d}s + \int_0^r \left(\frac{-16}{s}x\,U + \frac{8}{s}z\,U\right)\,\mathrm{d}s - 2\int_0^r s\,U^3\,x\,\mathrm{d}s = 0 \tag{6.75}$$

An integration by parts, gives us the following equality

$$\begin{aligned} \int_0^r \left(U\,\frac{\mathrm{d}}{\mathrm{d}s}(s\,x') - x\,\frac{\mathrm{d}}{\mathrm{d}s}(s\,U')\right)\,\mathrm{d}s &= [s\,x'U]|_0^r - \int_0^r U'\,s\,x'\,\mathrm{d}s \;+ [-x\,s\,U']|_0^r + \int_0^r U's\,x'\,\mathrm{d}s \\ &= [s\,x'U - s\,U'x]|_0^r \end{aligned}$$

replacing this equality in (6.75), we get

$$r[x'\,U - U'\,x] \,+ \int_0^r \left(\frac{-16}{s}x\,U + \frac{8}{s}z\,U\right)\,\mathrm{d}s - 2\int_0^r s\,U^3\,x\,\mathrm{d}s = 0 \tag{6.76}$$

then noting $\dfrac{-16}{s}x\,U + \dfrac{8}{s}z\,U \leq 0$ for all $0 < s \leq r < R_0$, we obtain

$$\forall\, r \in (0, R_0], \qquad \frac{\mathrm{d}x}{\mathrm{d}r}\,U - \frac{\mathrm{d}U}{\mathrm{d}r}\,x \,\geq \frac{2}{r}\int_0^r s\,U^3\,x\,\mathrm{d}s \tag{6.77}$$

evaluating (6.77) at $r = R_0$ and using the fact that $U > 0$, we obtain

$$0 \geq \frac{\mathrm{d}x}{\mathrm{d}r}(R_0)\,U(R_0) \geq \frac{2}{R_0}\int_0^{R_0} s\,U^3(s)x(s)\,\mathrm{d}s > 0$$

which is a contradiction. The second case can be treated similarly, combining the second equation of the system (6.7) with the equation for U, (6.2), we get

$$r[z'\,U - U'\,z] + \int_0^r \left(\frac{-16}{s} z\,U\,\mathrm{d}s + \frac{8}{s} x\,U \right) \mathrm{d}s = 0 \tag{6.78}$$

and consequently

$$\forall\, r \in (0, R_0], \qquad \frac{\mathrm{d}z}{\mathrm{d}r} U - \frac{\mathrm{d}U}{\mathrm{d}r} z \leq \frac{16}{r} \int_0^r \frac{1}{s} z\,U\,\mathrm{d}s \tag{6.79}$$

evaluating (6.79) at $r = R_0$ and using the fact that $U > 0$, we obtain

$$0 \leq \frac{\mathrm{d}z}{\mathrm{d}r}(R_0)\,U(R_0) \leq \frac{16}{R_0} \int_0^{R_0} \frac{z(s)\,U(s)}{s}\,\mathrm{d}s < 0$$

which is the another contradiction. Therefore the proof of (6.70) is complete.
Now that we have proved $x > 0$ and $z < 0$ for $r > 0$, we may use (6.76) together with the fact that $r[x'U - U'x] = r\,U^2 \dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{x}{U}\right)$ to obtain

$$\forall\, r > 0, \qquad r\,U^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{x}{U}\right) \geq 2 \int_0^r s\,U^3\,x\,\mathrm{d}s \tag{6.80}$$

and analogously

$$\forall\, r > 0, \qquad r\,U^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{z}{U}\right) \leq 16 \int_0^r \frac{zU}{s}\,\mathrm{d}s \tag{6.81}$$

thus if follows x is increasing and z decreasing for all $r > 0$.

Now, we have prove that the behaviour of x and z in $+\infty$ are exponential. First, we will give the details for the behaviour of x. So, suppose that x has increasing polynomial behaviour. We know that (6.80) is valid for all $r > 0$, differentiating this inequality with respect to r, we have

$$U\,x'' + \frac{1}{r} U\,x' + \left(-\frac{U'}{r} - U'' - 2\,U^3 \right) x \geq 0$$

Using the fact that $-U'' - \dfrac{U'}{r} = -\dfrac{1}{r^2} U + U(1 - U^2)$ and $U > 0$ for all $r > 0$, we can get

$$x'' + \frac{1}{r} x' + \left(1 - \frac{1}{r^2} - 3\,U^2 \right) x \geq 0$$

and noting that near $r = \infty$, $\left(1 - \dfrac{1}{r^2} - 3\,U^2\right) = -2 + \dfrac{2}{r^2} + \mathcal{O}\left(\dfrac{1}{r^4}\right)$, then we can choose $r_0 > 0$ large enough such that $x'' + \dfrac{1}{r}\,x' - C\,x \geq 0$ for all $r > r_0$, where $C > 0$. So, we consider the following ordinary differential inequality:

$$\begin{cases} x'' + \dfrac{1}{r}\,x' - C\,x \geq 0, & \forall\, r > r_0 \\ x(r_0) > 0 \end{cases} \tag{6.82}$$

Now, we consider $\widetilde{x}(r) = A\,\mathrm{e}^{\lambda_+ r} + B\,\mathrm{e}^{\lambda_- r}$, where $\lambda_\pm = \dfrac{-1 \pm \sqrt{1 + 4Cr_0^2}}{2\,r_0}$ are the roots of the polynomial $p(\lambda) = \lambda^2 + \frac{\lambda}{r_0} - C$, therefore for $A, B \geq 0$ and $r > r_0$ we get

$$\widetilde{x}'' + \frac{1}{r}\,\widetilde{x}' - C\,\widetilde{x} \;\leq\; \widetilde{x}'' + \frac{1}{r_0}\,\widetilde{x}' - C\,\widetilde{x} = A\,\mathrm{e}^{\lambda_+ r} p(\lambda_+) + B\,\mathrm{e}^{\lambda_- r} p(\lambda_-) = 0$$

Therefore, $\widetilde{x}$ is a super solution of the differential inequality (6.82), thus for $w(r) := \widetilde{x}(r) - x(r)$ it follows

$$w'' + \frac{1}{r}\,w' - C\,w \leq 0, \qquad \text{for all } r > r_0 \tag{6.83}$$

Note that, we can choose $B > 0$ such that $w(r_0) = A\,\mathrm{e}^{\lambda_+ r_0} + B\,\mathrm{e}^{\lambda_- r_0} - x(r_0) > 0$, then we affirm that $w(r) \geq 0$ for all $r > r_0$, otherwise if exists $r^* > r_0$ such that $w(r^*) < 0$, due to the fact that $\lim_{r\to\infty} \frac{x(r)}{\mathrm{e}^{\lambda_+ r}} = 0$ (x have polynomial increasing behaviour), therefore exists $r_1 > r^*$ such that $w(r) > 0\ \forall\, r > r_1$, then, without loss of generality we can also suppose that w in r_* is a local minimum, thus exists $r^* > r_0$ such that $w(r^*) > 0$, $w'(r^*) = 0$ and $w''(r^*) \geq 0$, then it follows

$$w''(r^*) + \frac{1}{r}\,w'(r^*) - C\,w(r^*) > 0$$

contradicting (6.83). Thus $w(r) \geq 0$ for all $r \geq r_0$, this implies

$$\forall\, A > 0, \qquad w(r) = A\,\mathrm{e}^{\lambda_+ r} + B\,\mathrm{e}^{\lambda_- r} - x(r) \geq 0, \qquad \text{for all } \ r > r_0$$

Finally, taking $A \to 0^+$, we obtain $x(r) \leq B\,\mathrm{e}^{\lambda_- r}$ for all $r > r_0$ and since $\lambda_- < 0$, we have that $x(r)$ has exponential decreasing behavior for $r > r_0$, but this is not compatible with the fact that $x(r)$ has increasing behaviour for $r > 0$.
Then, x in $+\infty$, cannot have increasing polynomial behaviour. Analogously we have the same result for z. Finally, by the Theorem 6.2.7, we identify the behavior of (x, z) at $+\infty$. Then, x and $-z$ have an exponentially increasing behaviour at $+\infty$.

(*ii*) See the proof in [10, Theorem 1.2] ■

6.3 Bounded solutions of the inhomogeneous system

In this section, we study the existence of bounded global solutions in $(0, +\infty)$ of

$$\begin{cases} x'' + \frac{1}{r}x' - \frac{17}{r^2}x + (1 - 3\,U^2)x + \frac{8}{r^2}z &= c\,(r) \\ z'' + \frac{1}{r}z' - \frac{17}{r^2}z + (1 - U^2)z + \frac{8}{r^2}x &= c\,(r) \end{cases} \qquad \text{Repeat equation (6.1)}$$

using the results obtained for the homogeneous system.
For this purpose, we make the transformation $a = \frac{x+z}{2}$, $b = \frac{x-z}{2}$. Thus we get

$$\begin{cases} a'' + \frac{1}{r}a' - \frac{9}{r^2}a + (1 - 2\,U^2)\,a - U^2\,b &= c(r) \\ b'' + \frac{1}{r}b' - \frac{25}{r^2}b + (1 - 2\,U^2)\,b - U^2\,a &= 0 \end{cases} \tag{6.84}$$

we can rewrite (6.84) as the following first order ode:

$$\boldsymbol{x}'(r) = A(r)\,\boldsymbol{x}(r) \,+\, \boldsymbol{f}(r) \qquad \text{with} \qquad \boldsymbol{x}(r) = (a, r\,a', b, r\,b')^t \tag{6.85}$$

where

$$A(r) = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -r(1-2U^2)+\frac{9}{r} & 0 & r\,U^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ r\,U^2 & 0 & -r(1-2U^2)+\frac{25}{r} & 0 \end{pmatrix}, \qquad \boldsymbol{f}(r) = \begin{pmatrix} 0 \\ r\,c(r) \\ 0 \\ 0 \end{pmatrix}$$

First, using the results obtained in the theorems 6.2.1 and 6.2.7, we construct a fundamental solution of

$$\boldsymbol{X}'(r) = A(r)\boldsymbol{X}(r) \tag{6.86}$$

where $\boldsymbol{X}(r)$ is a function with values in M_4 (the set of 4×4 with real entries) whose columns form a basis space of solution of homogeneous version of equation (6.85), with the following asymptotic behavior at 0 and at $+\infty$

$$W(r) \;=\; \begin{pmatrix} a_1 & a_3 & a_2 & a_4 \\ r\,a_1' & r\,a_3' & r\,a_2' & r\,a_4' \\ b_1 & b_3 & b_2 & b_4 \\ r\,b_1' & r\,b_3' & r\,b_2' & r\,b_4' \end{pmatrix} \sim_0 \begin{pmatrix} \mathcal{O}(r^9) & \mathcal{O}(r^{-1}) & r^3 & r^{-3} \\ \mathcal{O}(r^9) & \mathcal{O}(r^{-1}) & 3r^3 & -3r^{-3} \\ r^5 & r^{-5} & \mathcal{O}(r^7) & \mathcal{O}(r) \\ 5r^5 & -5r^{-5} & \mathcal{O}(r^7) & \mathcal{O}(r) \end{pmatrix}$$

$$W(r) \;=\; \begin{pmatrix} x_1+z_1 & x_2+z_2 & x_3+z_3 & x_4+z_4 \\ r\,(x_1'+z_1') & r\,(x_2'+z_2') & r\,(x_3'+z_3') & r\,(x_4'+z_4') \\ x_1-z_1 & x_2-z_2 & x_3-z_3 & x_4-z_4 \\ r\,(x_1'-z_1') & r\,(x_2'-z_2') & r\,(x_3'-z_3') & r\,(x_4'-z_4') \end{pmatrix} \sim_\infty \begin{pmatrix} J_4^+ & J_4^- & r^4 & r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & 4r^4 & -4r^{-4} \\ J_4^+ & J_4^- & -r^4 & -r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & -4r^4 & 4r^{-4} \end{pmatrix}$$

Moreover, since $W(r)$ is a fundamental solution of (6.86), it follows by Liouville's formula

$$\det W(r) = \det W(r_0)\,\exp\int_{r_0}^{r} \operatorname{tr} A(s)\,\mathrm{d}s = \det W(r_0)$$

where $r, r_0 \in (0,+\infty)$ and we have used that $\operatorname{tr} A(s) = 0 \;\; \forall\, s\in(0,+\infty)$. Then using the behaviour of $W(r)$ at $+\infty$, when $r_0\to+\infty$, we obtain

$$\det W(r) = \det \begin{pmatrix} J_4^+ & J_4^- & r^4 & r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & 4r^4 & -4r^{-4} \\ J_4^+ & J_4^- & -r^4 & -r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & -4r^4 & 4r^{-4} \end{pmatrix} = \frac{1}{r_0}\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ r_0\sqrt{2} & -r_0\sqrt{2} & 4 & -4 \\ 1 & 1 & -1 & -1 \\ r_0\sqrt{2} & -r_0\sqrt{2} & -4 & 4 \end{pmatrix} = 16{\cdot}4\sqrt{2}$$

We have by the method of variation of parameters, that a particular solution of (6.85) is given by

$$\boldsymbol{x}(r) = W(r)\int_0^r W(s)^{-1}\boldsymbol{f}(s)\,\mathrm{d}s \tag{6.87}$$

We will show that this particular solution is bounded, analyzing its asymptotic behavior at 0 and at infinity, but first note that $W(s)^{-1}\boldsymbol{f}(s) = s\,c(s)\,\mathcal{C}_2(s)$, where we have named $\mathcal{C}_2$ the second column of $W(s)^{-1}$. We have at 0 and at $+\infty$

$$\mathcal{C}_2(s) \sim_0 \begin{pmatrix} \mathcal{O}(s^{-1}) \\ \mathcal{O}(s^9) \\ \mathcal{O}(s^{-3}) \\ \mathcal{O}(s^3) \end{pmatrix}, \qquad \mathcal{C}_2(s) \sim_{+\infty} \frac{1}{64\sqrt{2}} \begin{pmatrix} 16\,J_4^- \\ -16\,J_4^+ \\ 4\sqrt{2}\,s^{-4} \\ -4\sqrt{2}\,s^4 \end{pmatrix} \tag{6.88}$$

where $-64\sqrt{2}$ is the determinant of $W(s)$, and by corollary (6.1.1.1), we have at 0 and at $+\infty$

$$s\,c(s) \sim_0 -\kappa \left(\frac{\kappa^3}{3} + \frac{1}{24}\right) s^4, \qquad s\,c(s) \sim_{+\infty} -3\,s^{-1}$$

then we have at 0

$$\boldsymbol{x}(r) \sim_0 W(r) \int_0^r s\,c(s)\,\mathcal{C}_2(s)\,\mathrm{d}s = W(r) \int_0^r \begin{pmatrix} \mathcal{O}(s^3) \\ \mathcal{O}(s^{13}) \\ \mathcal{O}(s) \\ \mathcal{O}(s^7) \end{pmatrix} \mathrm{d}s = W(r) \begin{pmatrix} \mathcal{O}(r^4) \\ \mathcal{O}(r^{14}) \\ \mathcal{O}(r^2) \\ \mathcal{O}(r^8) \end{pmatrix} \tag{6.89}$$

$$= \begin{pmatrix} \mathcal{O}(r^9) & \mathcal{O}(r^{-1}) & r^3 & r^{-3} \\ \mathcal{O}(r^9) & \mathcal{O}(r^{-1}) & 3r^3 & -3r^{-3} \\ r^5 & r^{-5} & \mathcal{O}(r^7) & \mathcal{O}(r) \\ 5r^5 & -5r^{-5} & \mathcal{O}(r^7) & \mathcal{O}(r) \end{pmatrix} \begin{pmatrix} \mathcal{O}(r^4) \\ \mathcal{O}(r^{14}) \\ \mathcal{O}(r^2) \\ \mathcal{O}(r^8) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(r^5) \\ \mathcal{O}(r^5) \\ \mathcal{O}(r^9) \\ \mathcal{O}(r^9) \end{pmatrix} \tag{6.90}$$

and for the analysis of $\boldsymbol{X}$ at ∞, we only analyze the integrating from $R_0 > 0$ big enough such that we have the behavior asymptotic at $+\infty$ as stated in the Theorem 6.2.7:

$$\boldsymbol{x}(r) = W(r) \int_0^r s\,c(s)\,\mathcal{C}_2(s)\,\mathrm{d}s \sim_\infty W(r) \int_{R_0}^r -3\,s^{-1} \frac{1}{64\sqrt{2}} \begin{pmatrix} 16\,J_4^- \\ -16\,J_4^+ \\ \sqrt{2}\,s^{-4} \\ -4\sqrt{2}\,s^4 \end{pmatrix} \mathrm{d}s \sim_\infty W(r) \begin{pmatrix} \mathcal{O}(r^{-1}J_4^-) \\ \mathcal{O}(r^{-1}J_4^+) \\ \mathcal{O}(r^{-4}) \\ \mathcal{O}(r^4) \end{pmatrix}$$

$$= \begin{pmatrix} J_4^+ & J_4^- & r^4 & r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & 4r^4 & -4r^{-4} \\ J_4^+ & J_4^- & -r^4 & -r^{-4} \\ r\sqrt{2}\,J_4^+ & -r\sqrt{2}\,J_4^- & -4r^4 & 4r^{-4} \end{pmatrix} \begin{pmatrix} \mathcal{O}(r^{-1}J_4^-) \\ \mathcal{O}(r^{-1}J_4^+) \\ \mathcal{O}(r^{-4}) \\ \mathcal{O}(r^4) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1+r^{-2}) \\ \mathcal{O}(r^{-1}) \\ \mathcal{O}(-1+r^{-2}) \\ \mathcal{O}(r^{-1}) \end{pmatrix}$$

Then, we can conclude the following result

Theorem 6.3.1 (Linear approximation of negative anisotropic vortex solution in $\mathbb{R}^2$). *We have that $v = a(r)\,\mathrm{e}^{3\mathrm{i}\theta} + b(r)\,\mathrm{e}^{-5\mathrm{i}\theta}$ is the solution of the system*

$$\mathbb{L}_{-1} v = -B u_0^-$$

where $\mathbb{L}_{-1}$ is the linearized operator of the Ginzburg-Landau equation around $u_0^- = U(r)\,\mathrm{e}^{-\mathrm{i}\theta}$ defined in the plane $\mathbb{R}^2$. Besides, (r,θ) are the polar coordinates in $\mathbb{R}^2$, and (a,b) are the unique bounded solution of the system (6.84) with the following asymptotic behavior at 0 and $+\infty$

$$(a(r), b(r)) \sim_0 (\mathcal{O}(r^5), \mathcal{O}(r^9)), \qquad (a(r), b(r)) \sim_\infty (\mathcal{O}(1+r^{-2})), \mathcal{O}(-1+r^{-2}))$$

Moreover, we have that

$$u(r,\theta) = U(r)\,\mathrm{e}^{-\mathrm{i}\,\theta} + \delta\,a(r)\,\mathrm{e}^{3\mathrm{i}\theta} + \delta\,b(r)\,\mathrm{e}^{-5\mathrm{i}\theta} + \mathcal{O}(\delta^2) \tag{6.91}$$

is a linear approximation in δ, of the anisotropic negative vortex solution of the anisotropic Ginzburg-Landau equation defined in the plane $\mathbb{R}^2$

$$\Delta u + u(1-|u|^2) + \delta\,\partial_{\eta\eta}\bar{u} = 0, \quad u : \mathbb{R}^2 \to \mathbb{C} \tag{6.92}$$

where $\partial_\eta = \partial_x + \mathrm{i}\partial_y$.

Remark 6.3.2. *We note that, the form of solution* (6.91)*, is compatible with invariance of the anisotropic Ginzburg-Landau equation* (6.92) *in the subspace*

$$W^{-}(\mathbb{R}^2) = \left\{ u = \sum_{k\in\mathbb{Z}} a_{4k-1}(r)\mathrm{e}^{\mathrm{i}(4k-1)\theta} \text{ on } \mathbb{R}^2, a_{4k-1}(r) \in \mathbb{C}, \forall\, r \in [0,+\infty) \right\}$$

where we have extended the definition of invariant subspace A.0.1 to the plane $\mathbb{R}^2$.

Remark 6.3.3. *We can conjecture that*

$$(a(r), b(r)) \sim_\infty (\mathcal{O}(r^{-2}), \mathcal{O}(r^{-2})) \tag{6.93}$$

this due to the following work in progress:

Theorem 6.3.4. *[46, 47] Let* $u^{-}_{\delta,R} \in X_R := \{u \in H^1(B_R;\mathbb{C}) : u = \mathrm{e}^{-\mathrm{i}\theta} \text{ on } \partial B_R\}$ *the minimizer of the following problem*

$$\min_{u\in X_R} E_{\delta,R}(u)$$

where

$$E_{\delta,R}(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-|u|^2)^2 + \frac{\delta}{2}\,\mathbb{R}\mathrm{e}\{(\partial_\eta \bar{u})^2\}\,\mathrm{d}S$$

We have that, exists $C > 0$ *independent of* R*, such that for all* $R > 0$

$$\int_{\mathbb{R}^2} (1 - |u^{-}_{\delta,R}|^2)^2\,\mathrm{d}S \le C$$

Therefore, for $u^{-}_{\delta} = \lim_{R\to\infty} u^{-}_{\delta,R}$ *in* $C^1_{loc}(\mathbb{R}^2;\mathbb{C})$*, we get*

$$\int_{\mathbb{R}^2} (1 - |u^{-}_{\delta}|^2)^2\,\mathrm{d}S < \infty$$

Moreover, based on the arguments of Brezis, Merle and Riviere [13], we have

$$\lim_{|\boldsymbol{x}|\to\infty} |u^{-}_{\delta}(\boldsymbol{x})| = 1$$

whence the conjecture (6.93) *follows.*

Conclusion and future work

In this thesis we have shown:

(i) The existence, via perturbative approach, of anisotropic vortices of both positive and negative degree of the Ginzburg-Landau anisotropic equation. Furthermore, it was proven that they are stable solutions, because their associated quadratic form, is positive definite for small perturbations.

(ii) Using the finite element method and theory, the anisotropic energy diagram of anisotropic vortices of positive and negative degree, is shown qualitatively and quantitatively, in the context of nematic liquid crystals.

(iii) Is proposed an extension to the system of differential equations from the Fourier decomposition of the negative anisotropic vortex, found via perturbative approach, and thus, we have the existence of a linear approximation in δ of negative degree anisotropic vortex in the plane.

It is proposed in the future to work in the following directions:

1. Study the uniqueness of the vortex-type anisotropic solutions from the previous results. This can be done by studying numerically, and because the solution is unique in the closed ball $\mathcal{B}_K \subseteq H^2(B_R;\mathbb{C}) \cap H_0^1(B_R;\mathbb{C})$ with $K = 2\|Bu_0^{\pm}\|\|L^{-1}\|$.
2. Using the above results, investigate the dynamics of the vortex in the anisotropic case, in particular a mathematical investigation of the anisotropic vortex interaction law.
3. To study the ode system equations from extending to the plane, in a functional framework of Sobolev spaces with weights, in order to avoid explosive solutions at zero and at infinity.
4. The existence of forced anisotropic vortices, using the perturbative approach, developed in this thesis.
5. Using the previous results, investigate the interaction in a vortex lattice, under the Ginzburg-Landau anisotropic equation.

Appendix A

Invariant function subspace of the anisotropic Ginzburg-Landau equation

Definition A.0.1. *Let W^+, W^- be the function subspaces defined by*

$$W^+ = \left\{ u = \sum_{k\in\mathbb{Z}} a_{4k+1}(r)\mathrm{e}^{\mathrm{i}(4k+1)\theta} \ \text{on } B_R, a_{4k+1}(r) \in \mathbb{C}, \forall\, r \in [0,R] \right\}$$

$$W^- = \left\{ u = \sum_{k\in\mathbb{Z}} a_{4k-1}(r)\mathrm{e}^{\mathrm{i}(4k-1)\theta} \ \text{on } B_R, a_{4k-1}(r) \in \mathbb{C}, \forall\, r \in [0,R] \right\}$$

where B_R is the disk of radius R and centered at 0 in $\mathbb{R}^2$. It is clear that $W^\pm$ is a subspace of $H^1_{g_0}(B_R)$, where $g_0 = \mathrm{e}^{\pm\mathrm{i}\theta}$.

In this section, we justify the invariance of the subspaces $W^\pm$, where the invariant space means that if $u \in W^\pm$ is smooth on B_R, then $\Delta u + u\,(\mu - |u|^2) + \delta B u \in W^\pm$, here $Bw := \partial_{\eta\eta}\overline{w}$ with $\partial_\eta = \partial_x + \mathrm{i}\,\partial_y$.

We will only do the invariance of subspace W^+, while the case for subspace W^- is analogous.

The invariance proof of W^+ is obtained directly from the following proposition

Proposition A.0.2. *Assume $u \in W^+$ is smooth, then Δu, Bu and $(\mu - |u|^2)u$ are in W^+.*

Proof. Assume $u \in W$ is smooth on B. Then

$$u = \sum_{k\in\mathbb{Z}} a_{4k+1}(r)\mathrm{e}^{\mathrm{i}(4k+1)\theta} \tag{A.1}$$

$$a_{4k+1}(r) = r^{|4k+1|}\widetilde{a}_{4k+1}(r^2), \qquad \forall\, k \in \mathbb{Z} \tag{A.2}$$

where $\widetilde{a}_{4k+1}$ are smooth complex-valued functions. To prove proposition A.0.2, we need the following lemma.

Lemma A.0.3. *Fix $k, l, m \in \mathbb{Z}$. Assume*

$$\begin{aligned} u =& r^{|4k+1|}a(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta}, \\ v =& r^{|4l+1|}b(r^2)\mathrm{e}^{\mathrm{i}(4l+1)\theta}, \\ w =& r^{4m+1}c(r^2)\mathrm{e}^{\mathrm{i}(4m+1)\theta}, \end{aligned} \tag{A.3}$$

where a, b and c are smooth complex-valued functions. Then

$$\Delta u = r^{|4k+1|}a_0(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta}, \tag{A.4}$$

$$Bu = r^{|4k-1|}a_1(r^2)\mathrm{e}^{\mathrm{i}(4k-1)\theta} + r^{|4k+3|}a_2(r^2)\mathrm{e}^{\mathrm{i}(4k+3)\theta}, \tag{A.5}$$

$$u\bar{v}w = r^{|1+4(k-l+m)|}a_3(r^2)\mathrm{e}^{[1+4(k-l+m)]\theta}, \tag{A.6}$$

where a_j are smooth complex-valued functions.

By (A.1),(A.2) and A.0.3, we have

$$\Delta u = \Delta\left(\sum_{k\in\mathbb{Z}} r^{|4k+1|}\widetilde{a}_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta}\right) = \sum_{k\in\mathbb{Z}} r^{|4k+1|}\widetilde{a}^0_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta} \qquad \text{from (A.4)}$$

$$\begin{aligned} Bu =& B\left(\sum_{k\in\mathbb{Z}} r^{|4k+1|}\widetilde{a}_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta}\right) \\ =& \sum_{k\in\mathbb{Z}} r^{|4k-1|}\widetilde{a}^1_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k-1)\theta} + r^{|4k+3|}\widetilde{a}^2_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k+3)\theta} \qquad \text{from (A.5)} \end{aligned}$$

$$\begin{aligned} |u|^2u =& u\bar{u}u \\ =& \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [(r^{|4k+1|}\widetilde{a}_{4k+1}(r^2)\mathrm{e}^{\mathrm{i}(4k+1)\theta})\overline{(r^{|4l+1|}\widetilde{a}_{4l+1}(r^2)\mathrm{e}^{\mathrm{i}(4l+1)\theta})}(r^{|4m+1|}\widetilde{a}_{4m+1}(r^2)\mathrm{e}^{\mathrm{i}(4m+1)\theta})] \\ =& \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} r^{|1+4(k-l+m)|}\widetilde{a}^3_{k,l,m}(r^2)\mathrm{e}^{\mathrm{i}[1+4(k-l+m)]\theta} \end{aligned}$$

where $\widetilde{a}^j_{4k+1}$ and $\widetilde{a}^3_{k,l,m}$ are smooth complex-valued functions. Thus we obtain that Δu, Bu and $(\mu - |u|^2)u$ are in W^+. Therefore, we complete the proof of proposition A.0.2 ■

Proof. Now we prove the lemma A.0.3. Since $\Delta = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}$, we can check that

$$\Delta u = \begin{cases} [(16k+8)a'(r^2) + 4r^2a''(r^2)]r^{4k+1}\mathrm{e}^{\mathrm{i}(4k+1)\theta} & \text{if } 4k+1>0 \\ \left[-16a'(r^2) + 4r^2a''(r^2)\right]r^{-4k-1}\mathrm{e}^{\mathrm{i}(4k+1)\theta} & \text{if } 4k+1<0 \end{cases}$$

Hence, we obtain (A.4).
From (A.3), we have

$$u\bar{v}w = r^{|4k+1|+|4l+1|+|4m+1|}a(r^2)b(r^2)c(r^2)\mathrm{e}^{\mathrm{i}[1+4(k-l+m)\theta}. \tag{A.7}$$

We may check in different cases that $|4k+1| + |4l+1| + |4m+1| = |1+4(k-l+m)| + 2n$ for some integer n. Thus by (A.7), we obtain (A.6).
Now we want to prove (A.5). To prove this, we see that follows from the following lemma:

Lemma A.0.4. *Fix $k_0 \in \mathbb{Z}$ and $|k_0| \geq 1$. Assume $u = r^{|k_0|}a(r^2)\mathrm{e}^{\mathrm{i}k_0\theta}$, where a is a smooth complex-valued function. Then*

$$Bu = r^{|k_0-2|}h(r^2)\mathrm{e}^{\mathrm{i}(k_0-2)\theta} + r^{|k_0+2|}H(r^2)\mathrm{e}^{\mathrm{i}(k_0+2)\theta}$$

where h, H are smooth complex-valued functions.

■

Appendix B

Computation details of the quadratic coefficient in energy expansion

In this appendix, we attach the detail of the numerical calculation of the quadratic coefficient, for a different number of triangles.

n	$G(v, u_0; 3)$
50	-1.42607
80	-1.43028
100	-1.43126
150	**-1.43212**
160	-1.4322
170	-2.69327
180	-279.684
200	-1.790740

Table B.1: $R = 3$

n	$G(v, u_0; 4)$
50	-2.94084
80	**-2.95169**
90	-2.95269
100	-12.4559
110	-2.95395
150	-2.95498
170	-5.19764
180	0.481622
190	-2.95541
200	-3.48964
220	-3.3419

Table B.2: $R = 4$

n	$G(v, u_0; 5)$
50	-4.53966
70	-4.55619
100	-4.56162
150	-4.56387
160	-4.5641
170	-8.073
180	-8.2357
190	-4.56446
200	-4.56455
250	**-4.56483**

Table B.3: $R = 5$

n	$G(v, u_0; 10)$
50	-10.856
80	-10.9842
100	-11.0069
120	-11.0141
150	-11.0189
200	-11.0221
250	-11.0223
300	**-11.0234**
350	-11.0236
400	-13.465
500	-12.0818

Table B.4: $R = 10$

n	$G(v, u_0; 15)$
50	-14.8962
75	-15.0533
100	-15.1717
150	-15.208
200	-15.22
300	-30.1209
400	-21.1889
500	-17.7968
700	**-15.2255**

Table B.5: $R = 15$

n	$G(v, u_0; 20)$
50	-18.162
100	-18.1351
150	-18.2087
200	-18.2396
300	-18.2506
500	-23.1577
700	**-18.2545**

Table B.6: $R = 20$

n	$G(v, u_0; 50)$
100	-38.0204
150	-29.0261
200	-27.7348
250	-27.8612
300	-27.8787
400	-641.547
500	-272.866
700	**-27.9452**

Table B.7: $R = 50$

n	$G(v, u_0; 100)$
100	4057
200	-34.2108
300	-36.7717
400	57.6687
500	-3.33288
700	**-35.225**
800	-32.7028
1000	-35.2695

Table B.8: $R = 100$

n	$G(v, u_0; 150)$
100	-22.91
150	-14.3147
200	-521.225
180	2885.68
190	6.53451
210	-48.0069
220	-480.421
250	-41.3135
300	261.034
350	-13.8576
400	-82.146
450	313.27
500	-17334.4
600	**-38.1544**
700	-38.9488
800	-33.6625

Table B.9: $R = 150$

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