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MONOCHROMATIC PARTITIONS IN RANDOM GRAPHS

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En 1991 Erdős, Gyárfás y Pyber conjeturaron que para todo r-coloreo de un grafo completo K_n este puede ser particionado en a lo más $r-1$ árboles monocromáticos. Paralelamente Gyárfás y Lehel conjeturaron un resultado similar para un tipo diferente de grafos. Ellos propusieron que para todo r-coloreo de las aristas de un grafo bipartito completo este puede ser cubierto por a lo más $2r-2$ árboles monocromáticos.

En 2017 Bal y DeBiasio fueron los primeros en estudiar este problema para grafos aleatorios $G(n,p) \in \mathcal{G}_{n,p}$ y conjeturaron que si $p \gg \left(\frac{rlog(n)}{n}\right)^{1/r}$ entonces $G(n,p)$ puede ser particionado, casi seguramente, por a lo más r árboles monocromáticos. En esta memoria proponemos una version de este problema para grafos bipartitos aleatorios en el caso $r=2$. Conjeturamos que para la misma cota de p podemos, casi seguramente, particionar $G \in \mathcal{B}_{n,n,p}$ por a lo más 3 árboles monocromáticos. Además, probamos esta conjetura aproximadamente.

Finalmente, damos un ejemplo de un 2-coloreo de las aristas tal que, casi seguramente, necesitamos al menos tres árboles monocromáticos para particionar el grafo, y por lo tanto, nuestra conjetura sería mejor posible.

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In 1991 Erdős, Gyárfás and Pyber conjectured that every r-coloured complete graph K_n can be partitioned by at most $r-1$ monochromatic trees. Following this line, Gyárfás and Lehel conjectured a similar result for a different host graph. They proposed that every r-coloured complete bipartite graph can be covered by at most $2r-2$ monochromatic trees.

In 2017 Bal and DeBiasio were the first to study this problem for random graphs $G(n,p) \in \mathcal{G}_{n,p}$ and they conjectured that if $p \gg \left(\frac{r log(n)}{n}\right)^{1/r}$ then $G(n,p)$ could be a.a.s. partitioned by at most r monochromatic trees. We propose a version of this problem for random bipartite graphs in the case $r = 2$. We conjecture that for the same p we can a.a.s. partition $G \in \mathcal{B}_{n,n,p}$, the random bipartite graph, by at most 3 monochromatic trees. Also, we prove this conjecture approximately.

Finally we give an example of a 2-colouring of the edges such that a.a.s. we need at least three monochromatic trees to partition the graph, and therefore, our conjecture would be best possible.

Why are numbers beautiful?
It's like asking why is Beethoven's Ninth Symphony beautiful.
If you don't see why, someone can't tell you.
I know numbers are beautiful.
If they aren't beautiful, nothing is.

Paul Erdős

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Contents

Introduction **1**

1 Known Results **4**
1.1 Complete Graphs 4
1.2 Random Graphs 6
1.3 Bipartite Graphs 8
1.4 Random Bipartite Graphs 9

2 Random Graphs **11**
2.1 General Notation 11
2.2 Random Graphs 12
2.3 Random Bipartite Graphs 13
2.4 Chernoff Bound 14

3 Preliminaries **16**
3.1 Preliminary Lemma 16
3.2 Lower Bound 19

4 Proof of the Main Theorem **21**

Conclusion **26**

Bibliography **28**

Introduction

Ramsey theory dates from the end of the twenties and was named after the British mathematician Frank Ramsey. Ramsey theory is a branch of mathematics which deals with finding monochromatic structures in large complete graphs. The Ramsey number has a fundamental role in this theory and is defined as follows. The Ramsey number $R(G)$ of a graph G is the smallest number n for which any 2-coloured complete graph on n vertices contains either a red or a blue copy of G. In 1967 Gerencsér and Gyárfás [13] determined the Ramsey number of paths with given lengths and proved that for any 2-colouring of the edges of a complete graph K_n on n vertices there are two monochromatic paths partitioning all the vertices of the graph. This was the beginning of countless papers and articles that have opened up a very important line of research for graph theory and combinatorics in recent years. As in [13], there are some Ramsey numbers that have been determined by partitioning coloured graphs into monochromatic subgraphs [19, 37], and conversely, Ramsey numbers are a very useful tool to partition graphs. (For more information about Ramsey theory see [14])

In 1977, Gyárfás [15] proposed that for $r > 1$ and every r-colouring of the edges of K_n, there is a cover of the vertex set with $r-1$ monochromatic trees. This conjecture, if true, is best possible because if we consider a complete graph whose vertices are the points of an affine plane of order $r-1$ and then we colour the edge pq with colour i if the line through p and q is in the i-th parallel class, we need at least $r-1$ trees to cover the graph. This conjecture is a particular case of his formulation of Ryser's conjecture [24], which proposes that the maximal cardinality of a matching of an r-partite hypergraph H is at most $r-1$ times the minimal cardinality of a cover of H. Ryser's conjecture is believed to be very difficult, since it was raised in 1971 and so far only cases $r = 2, 3$ have been solved [28, 1], while the case $r \geq 4$ is still open. Therefore, it is natural to think that Gyárfás's formulation is difficult too, as it remains open for $r \geq 6$. Also, it is worth mentioning that if we wanted to cover the graph with r monochromatic trees (instead of $r-1$) then it would be very easy to prove. This is because if we pick any vertex of K_n and we consider all its neighbourhoods in the r different colours, each of these neighbourhoods, together with the vertex itself induces a monochromatic connected component and as the graph is complete we cover all vertices.

In 1991, Erdős, Gyárfás and Pyber [10] conjectured that every r-coloured K_n can not only be covered by $r-1$ but even partitioned by at most $r-1$ monochromatic trees. This conjecture, if true, is best possible because of the affine-plane example from the previous paragraph. The conjecture is a strengthening of Gyárfás's formulation of Ryser's conjecture where the trees are only required to form a cover. It is easier to cover than to partition a graph by monochromatic trees, because in a cover the trees are not required to be disjoint. That is why generally the number of monochromatic trees required to partition an edge-coloured graph is larger than the number of monochromatic trees needed to cover it. In 1996, it was proved by Haxell and Kohayakawa [23] that every r-coloured K_n it can be partitioned by r monochromatic trees if n is big enough. This bound on n was later improved by Bal and DeBiasio [3].

In parallel, Gyárfás and Lehel [19] conjectured a similar result for the case where the graph is a complete bipartite graph. They proposed that for every r-colouring of a complete bipartite graph the vertex set can be covered by at most $2r-2$ monochromatic trees and this was proved for the cases $r = 2, 3, 4, 5$ in [8]. Also this conjecture, if true, is best possible.

Bal and DeBiasio were the first to study this problem for random graphs in 2017. We say a graph is a random graph $G(n, p)$ if it is an element of the probability space generated by fixing n vertices and letting each edge occur with probability p. Bal and DeBiasio [3] found a lower bound for p such that for every r-coloring of $G(n, p)$ there exists, with high probability, a cover of at most r^2 monochromatic trees. Moreover, they found an upper bound for p such that, asymptotically almost surely, there does not exist a cover with a bounded in r number of trees. With these two bounds they gave a range for p in which it is expected, with high probability, to find a bounded number of monochromatic trees that cover a graph $G(n, p)$. While the latter results were all for covers, there are also results for partitions. They conjectured that if $p \gg \left(\frac{rlog(n)}{n}\right)^{1/r}$ then $G(n, p)$ could be a.a.s. partitioned by at most r monochromatic trees. Kohayakawa, Mota and Schacht [27] proved the conjecture for the case $r = 2$. Furthermore, they show a construction due to Ebsen, Mota and Schnitzer in [27] which disproves the conjecture for $r \geq 3$.

Kohayakawa, Mota and Schacht conjectured a new lower bound for p slightly larger than the one of Bal and DeBiasio and for the same number of monochromatic trees. This bound for p was recently refuted by Bucić, Korándi and Sudakov [6]. They found different bounds for the quantity of trees required to cover a random graph throughout the probability range, specifically they showed that $G(n, p)$ can be covered by r monochromatic trees only if p is exponentially larger than conjectured.

The main problem studied in this thesis is a variant of the above described problem for random bipartite graphs. We are interested in finding the minimum number m of disjoint monochromatic trees such that we can partition, with high probability, any r-edge coloured random bipartite graph into at most m monochromatic trees, particularly, we will focus on the case $r = 2$. We conjecture that if $p \gg \left(\frac{log(n)}{n}\right)^{1/2}$ then a.a.s. every 2-edge coloured bipartite random graph can be partitioned into at most 3 monochromatic trees. We found a 2-colouring of the edges such that the graph cannot be partitioned into less than three monochromatic trees. This proves that our conjecture is best possible. Also, we prove that for every 2-colouring of the graph, all but at most $O(\frac{1}{p})$ vertices can be partitioned by three monochromatic trees. Our main proof will be inspired by Kohayakawa, Mota and

Schacht's [27] result for general random graphs.

The work is organized as follows. In the first chapter we will give a survey of known results in order to present the history and the motivation behind our problem. In second chapter we give basic concepts and notation necessary for the understanding of the problem and some preliminary results required for our main proof, principally, random graph tools. Chapter 3 will present the proof of a lemma which consists of the random bipartite version of a lemma for usual random graphs. In addition, in Chapter 3 we will present the example that provides the lower bound on the number of trees required to partition the graph. Finally, Chapter 4 is devoted to the proof of the main result.

Chapter 1

Known Results

In this chapter we will present a survey of known results which are the motivation of our problem. We first define given an r-edge coloured graph G the tree cover number $tc_r(G)$ as the minimum number f of monochromatic trees such that for every r-colouring of the edges of G its vertex set can be covered by the vertices of f monochromatic trees. Also, we write $\alpha(G)$ for the maximum size of an independent set in G. We will use standard notation, for further information see Section 2.1 of Chapter 2.

1.1 Complete Graphs

There is a famous conjecture which is due to Ryser and appeared in his student Henderson's thesis [24]. This conjecture presents the relationship between the maximal cardinality of a matching of a given r-partite hypergraph H and the minimal cardinality of a covering of H. Nevertheless, we will state Ryser's conjecture in the following form which was noted by Gyárfás [15] to be an equivalent formulation.

Conjecture 1.1.1 (Ryser [24], Formulation by Gyárfás [15]) Let $r \geq 2$. For all graphs G, we have $tc_r(G) \leq (r-1)\alpha(G)$.

If true, this conjecture is best possible when $r-1$ is a prime power. For $r = 2$ it is equivalent to Kőnig's theorem[1] and it was proved for the case $r = 3$ by Aharoni [1]. The case $r \geq 4$ is still open. Slightly more is known for the case $\alpha(G) = 1$, where the conjecture has been proved for $r \leq 5$ (see [12, 20]). We present this case of Conjecture 1.1.1 separately, since it is more relevant for our work.

Conjecture 1.1.2 (Gyárfás [15]) Let $r \geq 2$. For all complete graphs $G = K_n$, we have $tc_r(K_n) \leq (r-1)$.

Let us note that in Conjecture 1.1.2 the trees are only required to form a cover. In 1991,

[1]This is easy to see for the original formulation of Ryser's conjecture, which we will not show here.

Erdős, Gyárfás and Pyber [10] gave a strengthening of Conjecture 1.1.2 where the trees form a partition of the vertex set and the graph is a complete graph. We define $tp_r(G)$ as the minimum number f of monochromatic trees such that for every r-colouring of the edges of G its vertex set can be partitioned by the vertices of f monochromatic trees.

Conjecture 1.1.3 (Erdős, Gyárfás, Pyber [10]) For all $r \geq 2$, we have $tp_r(K_n) = r - 1$.

For the case $r = 2$ this conjecture is equivalent to the fact that either a graph or its complement is connected which was proved by Erdős and Rado [18]. Nagy and Szentmiklóssy proved that $tp_3(K_{\mathbb{N}}) = 2$ (see [10]), where $K_{\mathbb{N}}$ is the complete graph with countably many vertices and a path is an infinite one-way sequence of distinct vertices such that each pair of consecutive vertices is connected by an edge.

On the other hand, for finite complete graphs, Erdős, Gyárfás and Pyber [10] showed that $tp_3(K_n) = 2$ and Haxell and Kohayakawa [23] showed in 1996 that the tree partition number of K_n is at most r if n is large enough. The bound on n presented by Haxell and Kohayakawa was improved by Bal and DeBiasio in 2017.

Theorem 1.1.4 (Bal, DeBiasio [3]) Let $r \geq 2$. If $n \geq 3r^2r!log(r)$, then $tp_r(K_n) \leq r$.

If we require the components to be cycles, Lehel conjectured in 1979 that the vertex set of any 2-edge coloured complete graph K_n can be partitioned into two cycles of distinct colours. Gyárfás [16] proved in 1983 that this is true if we allow the cycles to intersect in at most one vertex. Many years later, in 1998, Łuczak, Rödl and Szemerédi [35] succeeded in proving that Lehel's conjecture is true for large n and Allen [2] gave in 2008 a simpler proof for a still large but smaller n. Finally, in 2010, Bessy and Thomassé [4] found a short inductive proof that works for all n.

Theorem 1.1.5 (Bessy, Thomassé [4]) The vertex set of any 2-edge coloured complete graphs K_n can be partitioned into two cycles of distinct colours.

For the multicolor case, Erdős, Gyárfás and Pyber [10] conjectured that $cp_r(G) = r$, where $cp_r(G)$ is the cycle partition number defined similarly as the tree partition number tp_r.

Erdős, Gyárfás and Pyber [10] also showed that there is a constant $c > 0$ such that $cp_r(K_n) \leq cr^2log(r)$. And in 2006, Gyárfás, Ruszinkó, Sárközy and Szemerédi [21] found the best known bound for the cycle partition number, namely $cp_r(K_n) \leq 100rlog(r)$.

In 2011, Gyárfás, Sárközy and Szemerédi [20] showed that Erdős, Gyárfás and Pyber's conjecture is asymptotically true for $r = 3$, that is, apart from $o(n)$ vertices, the vertex set of any 3-coloured K_n can be partitioned into 3 monochromatic cycles.

Finally, Pokrovskiy [36] found a counterexample for this conjecture in the case $r \geq 3$. He gave a 3-colouring of a complete graph such that for every three disjoint monochromatic cycles there is a vertex not covered by any of these cycles.

If we required the components to be paths, Gyárfás [17] conjectured that $pp_r(G) = r$, where the path partition number $pp_r(G)$ is the obvious analog to tp_r.

Conjecture 1.1.6 (Gyárfás, [17]) The vertices of every r-edge coloured complete graph can be covered by r vertex-disjoint monochromatic paths.

Conjecture 1.1.6 is true for case $r = 2$ as was proved in [16]. Gyárfás also showed in [17] that the path partition number is bounded from above by a function on r. A countably infinite version of this conjecture is known to be true for all r and it was proved by Rado [38].

Later, in 2012, Pokrovskiy [36] settled the path version for $r = 3$ and showed that the vertex set of any 3-coloured complete graph K_n can be covered by three monochromatic paths of different colours.

There are many variations of this problem such as considering different host graphs, as random graphs or bipartite graphs which will be presented in Section 1.2 and 1.3 respectively. Also, considering different conditions for the graph such as large minimum degree [40], [34], a complete graph with few edges missing or small independence number [39]. Another variant of the problem is to consider covering the graph by other monochromatic structures such as graphs with bounded degree.

1.2 Random Graphs

In 2017 Bal and DeBiasio [3] raised the problem of covering random graphs with monochromatic trees. They proved that the number of trees needed becomes bounded when p is somewhere between $\left(\frac{rlog(n)}{n}\right)^{1/r}$ and $\left(\frac{rlog(n)}{n}\right)^{1/(r+1)}$.

Theorem 1.2.1 (Bal, DeBiasio [3]) Let r be a positive integer and let $G \in \mathcal{G}_{n,p}$.

(i) If $p \ll \left(\frac{rlog(n)}{n}\right)^{1/r}$, then a.a.s. $tc_r(G) \to \infty$.

(ii) If $p \gg \left(\frac{rlog(n)}{n}\right)^{1/(r+1)}$, then a.a.s. $tc_r(G) \leq r^2$.

They also conjectured that $tc_r(G(n,p)) \leq r$ when $p \gg \left(\frac{rlog(n)}{n}\right)^{1/r}$, let us notice that where we write $G(n,p)$ to say that the graph G is drawn from $\mathcal{G}_{n,p}$. This conjecture was proved by Kohayakawa, Mota and Schacht [27] for the case $r = 2$, as follows.

Theorem 1.2.2 (Kohayakawa, Mota, Schacht [27]) If $p \gg \left(\frac{log(n)}{n}\right)^{1/2}$, then a.a.s. $tp_2(G(n,p)) \leq 2$.

The condition on p is best possible. This is because if $p < (1-\varepsilon)\left(\frac{2log(n)}{n}\right)^{1/2}$ for some $\varepsilon > 0$ then a.a.s. $G(n,p)$ has diameter at least three (see [5], Chapter 10), and hence, there are two non-adjacent vertices u and v with disjoint neighbourhoods. Colouring all the edges incident to u and v with red and all others edges blue we produce a colouring that requires at least three monochromatic trees to cover $V(G)$, and therefore, to partition the vertex set.

In [27] the authors also present an example due to Ebsen, Mota and Schnitzer showing that $tc_r(G(n,p)) \geq r+1$ for $p \ll \left(\frac{rlog(n)}{n}\right)^{1/(r+1)}$ for $r \geq 3$ which disproves the conjecture from [3]. For simplicity they only presented the case $r = 3$ since the adjustments for $r > 3$ are rather straightforward. This example is mostly based on the fact that by the choice of p a.a.s. G has four independent (mutually non-adjacent) vertices having no common neighbourhood. This allows us to build a 3-colouring of the edges where we cannot cover these four vertices by three monochromatic trees. This lead the authors of [27] to ask whether r trees are enough to cover $G(n,p)$ when p is slightly larger than $\left(\frac{rlog(n)}{n}\right)^{1/(r+1)}$.

This question was recently solved by Bucić, Korándi and Sudakov [6]. They showed that the conjecture is wrong and obtained a good understanding of the behaviour of $tc_r(G(n,p)$ throughout the probability range. First, they proved that $tc_r(G(n,p))$ only becomes equal to r when p is much larger than initially conjectured.

Theorem 1.2.3 (Bucić, Korándi, Sudakov [6]) Let r be a positive integer and $G \in \mathcal{G}_{n,p}$. Then there are constants c, C such that the following hold.

(i) If $p < \left(\frac{clog(n)}{n}\right)^{\sqrt{r}/2^{r-2}}$, then a.a.s. $tc_r(G) > r$.

(ii) If $p > \left(\frac{Clog(n)}{n}\right)^{1/2^r}$, then a.a.s. $tc_r(G) \leq r$.

It is easy to see that $tc_r(G) \geq r$ as long as $\alpha(G) \geq r$, this is because we can take an independent set S with $|S| = r$ and for each vertex we paint its edges with a different colour and then we will need at least r trees to cover the graph. Therefore, Theorem 1.2.3 (ii) implies that $tc_r(G(n,p)) = r$ for all larger values of p, as long as $\alpha(G(n,p)) \geq r$. Moreover, the authors of [6] proved that near the threshold $tc_r(G)$ ceases to be linear in r and is of order $\Theta(r^2)$.

Theorem 1.2.4 (Bucić, Korándi, Sudakov [6]) Let r be a positive integer, let $\mathrm{d} > 1$ be a constant and let $G \in \mathcal{G}_{n,p}$. There are constants c, C such that if $\left(\frac{Clog(n)}{n}\right)^{1/r} < p < \left(\frac{clog(n)}{n}\right)^{1/\mathrm{d}(r+1)}$ then a.a.s. $tc_r(G) = \Theta(r^2)$.

Furthermore, the same authors gave a bound for the number of monochromatic trees needed to partition G when p is between the ranges of the last two theorems. The following theorem establishes a slightly better bound and a connection between the last two theorems.

Theorem 1.2.5 (Bucić, Korándi, Sudakov [6]) Let $k > r \geq 2$ be integers. Then there are constants c, C such that for $G \in \mathcal{G}_{n,p}$ if $\left(\frac{Clog(n)}{n}\right)^{1/k} < p < \left(\frac{clog(n)}{n}\right)^{1/(k+1)}$ then a.a.s. $\frac{r^2}{20log(k)} \leq tc_r(G) \leq \frac{12r^2log(r)}{log(k)}$.

To prove these theorems the authors of [6] found a strong relation between the minimum number of vertices required to cover the edge set of an hypergraph and the number of trees required to cover a random graph $G(n,p)$, this relationship is also what connects this problem with Ryser's conjecture. With these theorems we are able to say that Bucić, Korándi and Sudakov essentially solved the problem posed by Bal and DeBiasio. Now, it only remains to improve the bounds in Theorem 1.2.5. This was recently done by Kohayakawa, Mendonça, Mota and Schülke [26] for the case $r = 3$.

On the other hand, there is a cycle version of this problem for random graphs which was first studied by Korándi, Mousset, Nenadov, Škorić and Sudakov [30]. They showed that, with high probability, $G \in \mathcal{G}_{n,p}$ can be covered by $O(r^8log(r))$, not necessarily disjoint, monochromatic cycles. This bound was recently improved by Lang and Lo [31] with the additional condition that the cycles form a partition of the vertex set, i.e., that the graph can be partitioned, with high probability, by an even smaller number of vertex disjoint monochromatic cycles. The authors of [31] showed that a.a.s. $cp_r(G) \leq 1000r^4log(r)$.

Theorem 1.2.1 shows that if $p = o((rlog(n)/n)^{1/r})$ the number of components needed to cover the graph is unbounded, in particular, then the number of cycles needed to cover the graph is unbounded. Moreover, Korándi, Lang, Letzter and Pokrovskiy [29] recently found that for $\varepsilon > 0$ and r sufficiently large an r-edge coloured graph on n vertices with minimum degree $(1-\varepsilon)n$ which cannot be partitioned in fewer than $O(\varepsilon^2 r^2)$ cycles. This motivated Lang and Lo to conjecture that for $p = O((rlog(n)/n)^{1/r})$ a.a.s. $cp_r(G(n,p)) = o(r^2)$.

Finally, for the path partition case there is no literature but it is easy to see that every theorem that applies to cycles also applies to paths.

1.3 Bipartite Graphs

Gyárfás and Lehel ([15] and [33]) proposed a version of Conjecture 1.1.2 for complete bipartite graphs. A complete bipartite graph G with vertex classes on n and m vertices will be referred to as $K_{n,m}$.

Conjecture 1.3.1 (Gyárfás, [15]) For $r > 1$ and every r-colouring of the edges of $K_{n,m}$ the vertex set of $K_{n,m}$ can be covered by the vertices of at most $2r-2$ monochromatic trees.

This conjecture was proved in [8] by Chen, Fujita, Gyárfás, Lehel and Tóth for the cases $r = 2, 3, 4, 5$. Also, it is demonstrated in [8] that if the conjecture is true it is best possible. Finally, it was shown in [8] that if we allow the number of trees to be $2r-1$, then Conjecture 1.3.1 holds.

This is because for any edge uv of the graph $K_{n,m}$ we take the monochromatic double star consisting of all the edges incident with u or v and having the same colour as uv. In the other colours we consider the monochromatic stars with centers u and v, of which there are at most $2(r-1)$. This gives at most $2r-1$ monochromatic trees covering the vertices of $K_{n,m}$.

In 2012, Pokrovskiy [36] suggested the following conjecture for path partitions of the $K_{n,n}$.

Conjecture 1.3.2 (Pokrovskiy, [36]) For $r > 0$ and for every r-coloured complete bipartite graph $K_{n,n}$ there is a vertex partition into $2r-1$ monochromatic paths.

Pokrovskiy showed his conjecture for $r = 2$. This conjecture would be optimal, since there exists an r colouring of the edges of the complete bipartite $K_{n,n}$ such that the graph cannot be partitioned into $2r-2$ monochromatic paths. This is because we can partition X into $X_1, \ldots, X_r$ and Y into $Y_1, \ldots Y_r$ such that $|X_\mathrm{i}| = 10^\mathrm{i} + \mathrm{i}$ and $|Y_\mathrm{i}| = 10^\mathrm{i} + r - \mathrm{i}$, and then we colour the edges between X_i and Y_j in colour $\mathrm{i} + j \pmod r$. It is possible to prove that this coloured graph cannot be partitioned into $2r-2$ monochromatic paths.

If we require the components to be cycles, Erdős, Gyárfás and Pyber [10] raised the question whether the cycle partition number for $K_{n,m}$ is also independent of n. Haxell [22] proved in 1997 that the cycle partition number of an r-edge coloured $K_{n,m}$ is $O((rlogr)^2)$. This bound was improved in 2017 by Lang and Stein [32] to $O(r^2)$. Also, we note that if the requirement that the cycles be disjoint is dropped, then, we can cover any r-edge coloured $K_{n,m}$ by $O(r^2)$ monochromatic cycles, which was proved in 1991 by Erdős, Gyárfás and Pyber [10].

Finally, in 2018, Bürger and Pitz [7] showed an infinite version of Conjecture 1.3.2.

1.4 Random Bipartite Graphs

We are interested in finding the number of monochromatic trees required to partition a random bipartite graph. The random bipartite graph $B(n, m, p)$ is a bipartite graph whose edges occur with some probability p independently of each other (The graph $B(n, m, p)$ will be defined properly in the next chapter).

Inspired by the results of [27] and taking into account the results from the previous section we suggest an analogue of Conjecture 1.3.1. for the random case.

Conjecture 1.4.1 If $p \gg \left(\frac{log(n)}{n}\right)^{1/2}$, then for every 2-colouring of the edges of a random bipartite graph $B(n, n, p)$ a.a.s. $tp_2(B(n, n, p)) \leq 3$.

We prove in this thesis that for every 2-colouring of the edges of the random balanced bipartite graph we can cover all but at most $O(\frac{1}{p})$ vertices by three disjoint monochromatic trees.

Theorem 1.4.2 Let $G = B(n,n,p) \in \mathcal{B}_{n,n,p}$ with $p(n) \gg \left(\frac{log(n)}{n}\right)^{\frac{1}{2}}$ then a.a.s. for every 2-colouring of the edge set of G, all but at most $O(\frac{1}{p})$ vertices can be covered by at most three vertex disjoint monochromatic trees.

Also, we prove that there exists a 2-colouring of the edges where we need at least three monochromatic trees to partition the random balanced bipartite graph, and consequently, if Conjecture 1.4.1 is true, it is best possible. The following theorem establishes a lower bound for the tree partition number of a random balanced bipartite graph.

Theorem 1.4.3 For $1 - \frac{1}{n} \gg p(n) \gg \left(\frac{log(n)}{n}\right)^{\frac{1}{2}}$ a.a.s. $G \in \mathcal{B}_{n,n,p}$ has a 2-colouring of the edges such that G cannot be covered by two monochromatic trees.

This theorem allows us to say that the if the condition to form a partition is dropped we still need at least three monochromatic trees to cover the graph and, therefore, the tree cover number will have the same lower bound. Also, for general r, if we raise the bound of p to $p \gg \left(\frac{log(n)}{n}\right)^{1/2^r}$ as in Theorem 1.2.3, and following Conjecture 1.3.1, it is natural to think that we would need $2r - 1$ trees to cover the graph with high probability.

In addition, it would be interesting to study the case where components are cycles or paths. For the path partition, keeping the bound $p \gg \left(\frac{log(n)}{n}\right)^{1/2^r}$, due to Conjecture 1.3.3, we think that the path partition number for the random bipartite graph could be $2r - 1$ a.a.s. as in the previous case. For the cycle partition number is harder to say as there is a big difference between the random graph bound which is $O(r^4 log(r))$ and the bound for deterministic bipartite graph which is $O(r^2)$. Also, it is worth noting that none of these bounds are linear, so our understanding of cp_r is not as good as our understanding of tp_r or pp_r.

Finally, it would be interesting to vary the range of p so we could understand the behavior of this property throughout the probability range as in [6].

Chapter 2

Random Graphs

2.1 General Notation

We begin with some standard graph theory notation. Given a graph G we let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Given $U, W \subseteq V(G)$, we write $\mathrm{e}_G(U, W)$ for the number of edges $uv \in E(G)$ such that $u \in U$ and $v \in W$. Also, we write $\mathrm{e}_G(U)$ for $\mathrm{e}(U, U)$. Given $v \in V(G)$ we write $N_G(v)$ for the neighbourhood of v, that is, the set of vertices that form an edge with v and we denote by $\mathrm{d}_G(v) = |N_G(v)|$ the degree of v. Similarly, given $S \subseteq V(G)$ we write $N_G(v, S)$ for the neighbours of v in the set S and $\mathrm{d}_G(v, S)$ for the number of neighbours that v has in S. If the underlying graph is clear we omit the subscript. We write $\delta(G) = \min_{v \in V(G)} \mathrm{d}(v)$ for the minimum degree of G. Given a set $U \subseteq V(G)$ we denote by $G[U]$ the graph induced by the vertices of U.

An r-edge colouring of the graph G is a function $c : E(G) \to [r]$ which assigns to each edge one of the r colours in $[r] = \{1, \cdots, r\}$. We denote by G_i the graph induced by the i-coloured edges. We write $N_i(v)$ for the set of vertices that v sees in colour i and set $\mathrm{d}_i(v) = |N_i(v)|$. Also, given $U, W \subseteq V(G)$ we write $\mathrm{e}_i(U, W)$ for the number of edges of colour i between U and W. We say a subgraph C of G is monochromatic if there exists an $i \in [r]$ such that C is subgraph of G_i. Given a set $S \subseteq V(G)$ we write $N^\cap(S) = \bigcap_{v \in S} N(v)$ and $N^\cup(S) = \bigcup_{v \in S} N(v)$. Similarly, given two sets $L, S \subseteq V(G)$ we write $N^\cap(S, L) = \bigcap_{v \in S} N(v, L)$. Given an r-edge coloured graph G, we define the tree partition (cover) number $tp_r(G)$ ($tc_r(G)$) as the minimum number f of monochromatic trees such that for every r-colouring of the edges of G its vertex set might be partitioned (covered) by the vertices of f monochromatic trees. (We remark that it is equivalent to ask for monochromatic connected subgraphs or for monochromatic trees, because every connected graph has a spanning tree). Analogously we write $pp_r(G)$ and $cp_r(G)$ ($pc_r(G), cc_r(G)$) when we require the elements of the partition (cover) to be paths or cycles.

We write $a_n \gg b_n$ to mean that given a_n we can choose b_n small enough so that b_n satisfies all of necessary conditions throughout the proof. More precisely, we say $a_n \gg b_n$ if there exists a constant $C > 0$ such that $a_n \geq C \cdot b_n$. Analogously we write $a_n \ll b_n$. In order to simplify the presentation, we will not determine these constants. We say $c = a \pm b$ if$a - b \leq c \leq a + b$. We will ignore floors and ceilings when they are not crucial for the calculation. Logarithms are assumed to be base e.

2.2 Random Graphs

In this section we will present some basic and standard notation and definitions following [9]. Let n be a positive number and $p = p(n) \in [0,1]$ a function, and set $[V] = \{v_1, \cdots, v_n\}$. We define for every potential edge $e = v_{\mathrm{i}} v_j \in [V]^2$ with i $\neq j$ its own probability space $\Omega_e = \{0_e, 1_e\}$, choosing $\mathbb{P}_e(1_e) = p$ and $\mathbb{P}_e(0_e) = 1 - p$. Therefore, we define the probability space $\mathcal{G}_{n,p}$ as the following product space

$$\Omega = \prod_{e \in [V]^2} \Omega_e,$$

where the probability measure $\mathbb{P}$ on Ω is the product measure of all the measures $\mathbb{P}_e$ and an element in Ω is a map w assigning to every $e \in [V]^2$ either 1_e or 0_e. We identify w with a graph G on V whose edge set is

$$E(G) = \{e : w(e) = 1_e\}$$

and we call G a random graph on V with probability p. We call any set of graphs on V an event in $\mathcal{G}_{n,p}$. In particular, for every $e \in [V]^2$ we call A_e the event that e is an edge of G. These events occur with probability p and are independent.

In the context of random graphs, any graph invariant may be interpreted as a non-negative random variable in $\mathcal{G}_{n,p}$, that is, a function

$$X : \mathcal{G}_{n,p} \to [0, \infty).$$

Then the expected value of X will be

$$\mathbb{E}(X) = \sum_{G \in \mathcal{G}_{n,p}} \mathbb{P}(\{G\}) X(G).$$

Computing the expected value of a random variable X may be a simple and effective way to establish the existence of a graph G with some desired property $\mathcal{P}$. This is due to

Markov's inequality which is used in the proof of some of the preliminary results. We present this inequality for completeness.

Lemma 2.2.1 (Markov's inequality) Let $X \geq 0$ be a random variable on $\mathcal{G}_{n,p}$ and let $a > 0$ be a constant. Then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

In addition, we shall work with asymptotic properties. This means that given $p = p(n)$ and a graph property $\mathcal{P}$ we ask how the probability $\mathbb{P}(G \in \mathcal{P})$ behaves for $G \in \mathcal{G}_{n,p}$ as $n \to \infty$. If this probability tends to 1, that is, if

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{P}) = 1,$$

then we say that given $G \in \mathcal{G}_{n,p}$ asymptotically almost surely (a.a.s.) $G \in \mathcal{P}$.

Finally, let us make a general frame of the thresholds for some properties when p varies along n as shown in [9]. For edge probabilities p lying below n^{-2}, a random graph $G \in \mathcal{G}_{n,p}$ almost surely has no edges at all. As p grows from about $\sqrt{n}n^{-2}$ onward, $G \in \mathcal{G}_{n,p}$ almost surely has a component with more than two vertices, these components grow into trees and around n^{-1} a cycle appears. Then, as p continues to grow around $(logn)n^{-1}$ the graph becomes connected, and hardly later, at $p = (1+\varepsilon)(logn)n^{-1}$, the graph almost surely has a Hamilton cycle.

2.3 Random Bipartite Graphs

In this section we will present the definitions required to work with random bipartite graphs, we will follow the formulations given in Section 2.2. Given numbers $n, m \in \mathbb{N}$ and $p(n,m) \in [0,1]$, we fix X, Y such that $|X| = n,\ |Y| = m$m and $V = X \cup Y$ is the vertex set of our random bipartite graph $\mathcal{B}_{n,m,p}$. We define for every potential edge $e \in X \times Y$ its probability space $\widehat{\Omega}_e = \{0_e, 1_e\}$ where $\mathbb{P}_e(1_e) = p$ and $\mathbb{P}_e(0_e) = 1 - p$. Then, we define the probability space $\mathcal{B}_{n,m,p}$ as follows

$$\widehat{\Omega} = \prod_{e \in X \times Y} \widehat{\Omega}_e\ ,$$

with probability measure being the product of the measures $\mathbb{P}_e$. Let us note that the only difference between this space and the space $\mathcal{G}_{n,p}$ given in Section 2.2 is that here we only allow edges between X and Y.

Therefore, each element in $\widehat{\Omega}$ is an assignment $\widehat{w}$ of 1_e or 0_e for every possible edge $e \in X \times Y$. As we did before, we identify each assignment with a graph G that we call a random bipartite graph on V with probability p whose edge set is

$$E(G) = \{e \in X \times Y : \widehat{w}(e) = 1_e\}.$$

2.4 Chernoff Bound

The Chernoff bound is useful as part of a technique to obtain exponentially decreasing bounds on tail probabilities. It is a sharper bound than Markov's inequality. We will use the following version of the Chernoff bound (see Corollary 21.7 in [11]).

Theorem 2.4.1 (Chernoff) Let X be a binomially distributed random variable. Then for $1 > \alpha > 0$

$$\mathbb{P}\big(X \leq (1-\alpha)\mathbb{E}(X)\big) \leq exp\left(-\alpha^2 \frac{\mathbb{E}(X)}{2}\right), \text{ and} \tag{2.1}$$

$$\mathbb{P}\big(X \geq (1+\alpha)\mathbb{E}(X)\big) \leq exp\left(-\alpha^2 \frac{\mathbb{E}(X)}{3}\right). \tag{2.2}$$

The following results in this section are corollaries of the Chernoff bound version from [25].

Corollary 2.4.2 If $X \in Bin(n,p)$ and $\varepsilon > 0$, then

$$\mathbb{P}\big(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)\big) \leq 2exp\big(-\phi(\varepsilon)\mathbb{E}(X)\big), \tag{2.3}$$

where $\phi(x) = (1+x)log(1+x) - x$ ($\phi(x) = \infty$ for $x < -1$). In particular, if $\varepsilon \leq \frac{3}{2}$,

$$\mathbb{P}\big(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)\big) \leq 2exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}(X)\right). \tag{2.4}$$

Corollary 2.4.3 If $X \in Bin(n,p)$, $\lambda = np$ and $c > 1$, then

$$\mathbb{P}(X \geq x) \leq 2exp\left(-c'x\right), \qquad x \geq c\lambda, \tag{2.5}$$

where $c' = log(c) - 1 + \frac{1}{c} > 0$. In particular,

$$\mathbb{P}(X \geq x) \leq 2exp\left(-x\right). \qquad if \;\; x \geq 7\lambda. \tag{2.6}$$

Chapter 3

Preliminaries

3.1 Preliminary Lemma

To prove Theorem 1.2.2, Kohayakawa, Mota and Schacht used the following lemma on random graphs.

Lemma 3.1.1 If $p \gg \left(\frac{log(n)}{n}\right)^{1/2}$, then for every $\varepsilon > 0$ a.a.s. $G \in \mathcal{G}_{n,p}$ satisfies the following properties.

(i) Every vertex $v \in V(G)$ has degree $\mathrm{d}_G(v) = (1 \pm \varepsilon)pn$ and every pair of distinct vertices $u, w \in V(G)$ has $|N_G(u) \cap N_G(w)| = (1 \pm \varepsilon)p^2 n$ joint neighbours.

(ii) For every vertex $v \in V(G)$ and all disjoint subsets $U \subseteq V$ and $W \subseteq N_G(v)$ with $|U| \geq 100/p$ and $|W| \geq pn/100$ we have $\mathrm{e}_G(U, W) > p|U||W|/2$.

(iii) For every vertex $v \in V(G)$ and $J \subseteq N_G(v)$ with $|J| \geq pn/100$, we have that all but at most $100/p$ vertices $x \in V(G) \setminus J$ satisfy $|N_G(x) \cap J| > p^2 n/100$.

(iv) Every subgraph $H \subseteq G$ with minimum degree $\delta(H) \geq (1/2 + \varepsilon)pn$ is connected.

We now state a bipartite version of Lemma 3.1.1 and give its proof. Most of the proof is similar to the proof of the previous lemma but we will give all the details because these properties are fundamental to our argument.

Lemma 3.1.2 If $p = p(n) \gg \left(\frac{log(n)}{n}\right)^{\frac{1}{2}}$ then for every $\varepsilon > 0$ and $G = G(X,Y) \in \mathcal{B}_{n,n,p}$ a.a.s G satisfies the following properties:

(i) For every $v \in X$, we have $d(v) = (1 \pm \varepsilon)pn$ and for every $u, w \in X$ we have $|N(u) \cap N(w)| = (1 \pm \varepsilon)p^2 n$.

(ii) For every $v \in X$ and all disjoint subsets $U \subseteq X$ and $W \subseteq N(v)$ with $|U| \geq \frac{100}{p}$, and $|W| \geq \frac{pn}{100}$ we have $e(U,W) > \frac{p}{2}|U||W|$.

(iii) For every $v \in X$ and $J \subseteq N(v)$ with $|J| \geq \frac{pn}{100}$ we have that all but at most $\frac{100}{p}$ vertices $x \in X$ satisfy $|N(x) \cap J| > \frac{p^2 n}{200}$.

(iv) Every $H \subseteq G$ with $\delta(H) \geq (\frac{1}{2} + \varepsilon)pn$ is connected.

Remark Let us notice that we may exchange the roles of X and Y.

Proof. This proof is mainly a consequence of the binomial concentration and the Chernoff bound.

(i) By the fact that $d(v)$ follows a $Bin(n,p)$ distribution. We have

$$\mathbb{P}\big(\exists v \in X : d(v) < (1-\varepsilon)pn\big) = \sum_{v \in X} \mathbb{P}\big(d(v) < (1-\varepsilon)pn\big) \leq n \cdot e^{-\varepsilon^2 pn/2} \to 0.$$

The second inequality is due to (2.1) from Theorem 2.4.1 and the fact that $p^2 n \gg log(n)$.

Similarly we can prove that $\mathbb{P}(\exists v \in X : d(v) > (1+\varepsilon)pn)$ goes to zero using (2.2) instead of (2.1). Also, for $u, w \in X$ we have that $|N(u) \cap N(w)|$ follows a $Bin(n,p^2)$ distribution. Therefore we may apply Chernoff's inequality to obtain

$$\begin{aligned}
\mathbb{P}\big(\exists (u,v) \in X \times X :& |N(u) \cap N(v)| \notin (1 \pm \varepsilon)p^2 n\big) \\
&\leq \sum_{(u,v) \in X \times X} \mathbb{P}\big(|N(u) \cap N(v)| \notin (1 \pm \varepsilon)p^2 n\big) \\
&\leq \binom{n}{2}(e^{-\varepsilon^2 p^2 n/2} + e^{-\varepsilon^2 p^2 n/3}) \\
&\leq 2e^{2log(n)}e^{-\varepsilon^2 p^2 n/3} \\
&\leq \frac{2}{n} \to 0.
\end{aligned}$$

As $p^2 n \gg log(n)$ we obtain that $p^2 n \geq 18log(n)/2\varepsilon^2$.

(ii) This property follows from Chernoff's bound by the following argument. For disjoint subsets $U, W \subseteq V(G)$ we observe that $e(U,W)$ follows a $Bin(|U||W|, p)$ distribution and we easily note that $\mathbb{E}(e(U,W)) = p|U||W|$. Applying Theorem 2.4.1 with $\alpha = 1/2$ we have

$$\mathbb{P}(e(U,W) \leq \frac{1}{2}p|U||W|) \leq e^{\frac{-p}{8}|U||W|} .$$

Now, summing over all the possibilities for v, U and W we obtain

$$\begin{aligned}\mathbb{P}\big(\exists v, U, W : e(U,W) \leq \frac{1}{2}p|U||W|\big) &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} \binom{n}{u}\binom{n}{w} p^w e^{-puw/8} \\ &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} e^{u \cdot log(n)} \left(\frac{en}{w}\right)^w p^w e^{-puw/8} \\ &\leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} e^{u \cdot log(n) + 6w - puw/8} .\end{aligned}$$

The last two inequalities are due to the usual bounds on the binomial coefficient $\binom{n}{u} \leq n^u = \mathrm{e}^{u \cdot log(n)}$ and $\binom{n}{w} \leq \frac{n^w}{w!} \leq \left(\frac{\mathrm{e}n}{w}\right)^w$, and a simple calculation using the fact that $w \geq pn/100 \geq pn/\mathrm{e}^5$. Therefore we have

$$\mathbb{P}\big(\exists v, U, W : e(U,W) \leq \frac{1}{2}p|U||W|\big) \leq n \sum_{u \geq 100/p} \sum_{w \geq pn/100} e^{-w/4} \to 0 ,$$

since $u \geq 100/p$ we have $puw/16 - 6w \geq w/4$ and for $w \geq pn/100$ we get $puw/16 \geq up^2n/1600 \gg u \cdot log(n)$. Finally the last sum goes to zero since $\mathrm{e}^{-w/4} \leq \mathrm{e}^{-pn/400}$ which converges to zero because $p^2 n \gg log(n)$ and this concludes the proof.

(iii) Let us define

$$U = \{x \in X : |N(x) \cap J| \leq \frac{p^2 n}{200}\}.$$

If we assume that $|U| > \frac{100}{p}$ we infer from (ii) that a.a.s.

$$e(U,J) > \frac{p}{2}|U||J| \geq p|U|\frac{pn}{200} = \frac{p^2 n|U|}{200} ,$$

a contradiction to the definition of U.

(iv) Let U be a connected component of H and set $U_x = U \cap X$ and $U_y = U \cap Y$. We shall show that both $|U_x|, |U_y| > \frac{n}{2}$ which implies that any connected component has at least $n/2+1$ vertices in X. Then, if there is a second component it must have $n/2+1$

vertices in X, and consequently $|X| \geq n+2$ which is false. We conclude that there is only one connected component meaning H is connected.

First, we show that for every fixed $\delta > 0$ that a.a.s.

$$e(U) < |U_x||U_y|p + \delta n|U_y|p \qquad \text{and} \tag{3.1}$$
$$e(U) < |U_x||U_y|p + \delta n|U_x|p. \tag{3.2}$$

To see these two inequalities we define $\beta = \frac{\delta n}{|U_x|}$ for inequality (3.1) and $\beta = \frac{\delta n}{|U_y|}$ for inequality (3.2). Next, we consider the cases $\beta < \frac{3}{2}$, $\frac{3}{2} \leq \beta \leq 7$ and $\beta > 7$ and we use Corollary 2.4.2 for the first two cases and (2.6) from Corollary 2.4.3 for the third case to conclude.

On the other hand, we have that

$$e(U_x, U_y) = \sum_{u \in U_x} d(u, U_y) = \sum_{u \in U_y} d(u, U_x).$$

Therefore, using our condition on the minimum degree of H implies that a.a.s. we get

$$e(U) \geq |U_y|(\frac{1}{2} + \varepsilon)pn \qquad \text{and}$$
$$e(U) \geq |U_x|(\frac{1}{2} + \varepsilon)pn.$$

Taking $\delta = \varepsilon$ and combining these inequalities with (3.1) and (3.2) we get

$$|U_y|(\frac{1}{2} + \varepsilon)pn \leq e(U) < |U_x||U_y|p + \varepsilon n|U_y|p \qquad \text{and}$$
$$|U_x|(\frac{1}{2} + \varepsilon)pn \leq e(U) < |U_x||U_y|p + \varepsilon n|U_x|p.$$

This implies that $|U_x|, |U_y| > n/2$ which finishes the proof.

□

3.2 Lower Bound

In this section we show Theorem 1.4.3 by exhibiting an example of a 2-colouring of the random bipartite graph, which a.a.s. does not have a cover of the vertex set in two monochromatic trees. First we note that if we cannot cover the graph by two monochromatic trees we cannot partition it with two monochromatic trees either. Also we note that this theorem is not true for the case $p = 1$. In this case we would have a complete bipartite graph which can always be partitioned into two monochromatic trees (see Section 1.3).

Proof of Theorem 1.4.3. By our choice of p we can assume that G satisfies a.a.s. the properties of Lemma 3.1.2 for every $\varepsilon > 0$.

First, we pick randomly any vertex $r \in X$. By Lemma 3.1.2 (i) a.a.s. $\mathrm{d}(r) \leq (1+\varepsilon)pn < n$, thus, there exists a vertex $b \in Y$ such that $b \notin N(r)$. We define $\overline{N(b)} = X \setminus (N(b) \cup r)$ and $\overline{N(r)} = Y \setminus (N(r) \cup b)$. Since $p \ll 1 - \frac{1}{n}$ we know that a.a.s. $\overline{N(r)}$ is not empty, that is, this condition ensures that there are at least 2 vertices in Y that are not neighbours of r.

Secondly, we colour by red all the edges between r and $N(r)$ and the ones between $\overline{N(r)}$ and $N(b)$. Also, the edges between $N(r)$ and $\overline{N(b)}$ will be coloured red. Next, we colour by blue all the edges between b and $N(b)$, the edges between $N(r)$ and $N(b)$ and also the ones between $\overline{N(r)}$ and $\overline{N(b)}$.

With this colouring is not possible to cover the vertex set of the graph by two monochromatic trees. Indeed, that is, suppose that we can cover the graph by two monochromatic trees. Let us note that r only is incident to red edges and b only sees blue edges, and therefore, these two vertices must lie in different trees. Since there is no monochromatic path from $\overline{N(r)}$ to either r or b, $\overline{N(r)}$ is not covered either by the tree containing r or by the tree containing b which is a contradiction.

$\square$

Chapter 4

Proof of the Main Theorem

In this section we will provide a proof for Theorem 1.4.2 which is the main result of this thesis.

PROOF. Let us note that if $p_1 < p_2$ and $G(n, p_1)$ can be partitioned by a certain number of trees, then $G(n, p_2)$ will fulfill that too. Therefore, and as $(log(n)/n)^{1/2}$ goes to zero, we may assume during this proof that $p < \frac{1}{200}$ for large enough n.

Let $G \in \mathcal{B}_{n,n,p}$ and assume we are given a 2-colouring of the edges of G in colours red and blue. We define G_b as the graph induced by the blue edges and G_r as the graph induced by the red edges. Since $p \gg \left(\frac{log(n)}{n}\right)^{1/2}$ we may assume that G satisfies a.a.s the properties of Lemma 3.2.2. We set $\varepsilon = \frac{1}{100}$.

We first note that as $p < \frac{1}{200}$ we have that

$$\max\left\{(1+\varepsilon)p^2 n, \frac{100}{p}\right\} \leq \frac{pn}{100} \tag{4.1}$$

for large enough n.

Let us define $R = \{v \in V(G) : \mathrm{d}_r(v) > \frac{1}{3}\mathrm{d}(v)\}$ and $B = \{v \in V(G) : \mathrm{d}_b(v) > \frac{1}{3}\mathrm{d}(v)\}$. Clearly, $R \cup B = V(G)$ and the two sets are not necessarily disjoint.

If one of the sets R or B, say R, is empty, it follows from Lemma 3.1.2 (i) that for every $v \in V(G)$ we have that $\mathrm{d}_b(v) > \frac{2}{3}(1-\varepsilon)pn > (\frac{1}{2}+\varepsilon)pn$. Consequently by Lemma 3.1.2 (iv), the blue graph G_b is connected. Thus G has a monochromatic spanning tree and we are done. We may thus assume that R and B are both not empty. After possibly swapping colours, we may also assume that there exist $r \in X \cap R$ and $b \in Y \cap B$.

We shall build two monochromatic trees having r and b as their respective roots. To this end, we consider a preference function $\rho : V(G) \to \{red, blue\}$ which assigns each vertex v the color $\rho(v)$ if there exist "many" monochromatic paths to the roots with colour $\rho(v)$ (we shall explain below what "many" means). In this process of assigning $\rho(v)$ to each vertex it might happen that some vertices are needed to connect vertices in the tree with the other colour, and for this reason preferences are not definitive and we will show the definitive assignment in a second step.

Let us note that although we are going to assign colours to the vertices, in order to establish monochromatic paths we will consider the colour of the edges.

First, we set $\rho(r) = \textit{red}$ and $\rho(b) = \textit{blue}$. Next we consider $N_r(r) \subseteq Y$ and $N_b(b) \subseteq X$ and we set

$$\rho(v) = \begin{cases} \textit{red} & \textit{if} \quad v \in N_r(r) \\ \textit{blue} & \textit{if} \quad v \in N_b(b). \end{cases}$$

Since we have $r \in R$ and $b \in B$, it is not hard to see that their neighbourhoods have at least $\frac{pn}{100}$ vertices. Also, by (4.1), we have $\frac{pn}{100} \gg \frac{100}{p}$ for large enough n. Then, $N_r(r)$ and $N_b(b)$ are big enough to satisfy the conditions of Lemma 3.1.2 (ii) and using this property we deduce that a.a.s.

$$e(N_r(r), N_b(b)) \geq \frac{p}{2}|N_r(r)||N_b(b)|,$$

with at least half of these edges having the same colour. Without loss of generality, we may assume this colour is red, and hence we have

$$\mathrm{e}_r(N_r(r), N_b(b)) \geq \frac{p}{4}|N_r(r)||N_b(b)|. \tag{4.2}$$

We define

$$J_1 = \left\{ v \in N_b(b) : |N_r(v) \cap N_r(r)| > \frac{p^2 n}{25} \right\}.$$

Observe that each of the vertices from J_1 can be connected to r by a red path of length two and to b by a blue edge. Hence, the vertices from J_1 may serve either for connecting all their blue neighbours to the blue tree rooted in b or their red neighbours to the red tree rooted in r.

Next, we will present a claim and its proof.

Claim 4.1 There are a.a.s. at least $\frac{pn}{100}$ vertices $v \in J_1$.

Proof. First, we define

$$L = \left\{ v \in N_b(b) : |N_r(v) \cap N_r(r)| \leq \frac{p}{8}|N_r(r)| \right\},$$

$$M = \left\{ v \in N_b(b) : |N_r(v) \cap N_r(r)| \geq \frac{p}{8}|N_r(r)| \right\}.$$

Therefore, we have that

$$\mathrm{e}_r(L, N_r(r)) \leq \frac{p}{8}|N_r(r)||N_b(b)|.$$

Let us recall that we have (4.2). So, considering the other $\frac{p}{8}|N_r(r)||N_b(b)|$ red edges and since by Lemma 3.2.2 (i) we have that for every vertex $v \in V$, $|N(v) \cap N(r)| < (1+\varepsilon)p^2 n$, there are at least

$$\frac{\frac{p}{8}|N_r(r)||N_b(b)|}{(1+\varepsilon)p^2 n}$$

vertices in M. As $r \in R$ satisfies $\mathrm{d}_r(r) \geq \frac{1}{3}(1-\varepsilon)pn$ we have

$$\frac{\frac{p}{8}|N_r(r)||N_b(b)|}{(1+\varepsilon)p^2 n} \geq \frac{p(1-\varepsilon)pn|N_b(b)|}{24(1+\varepsilon)p^2 n} > \frac{1}{25}|N_b(b)| \geq \frac{pn}{100}.$$

The last two inequalities are due to the fact that $|N_b(b)| \geq \frac{1}{3}(1-\varepsilon)pn$ and a simple calculation replacing $\varepsilon = \frac{1}{100}$. Hence, we have at least $\frac{pn}{100}$ vertices in M wich means that there are at least $\frac{pn}{100}$ with more than $\frac{p}{8}|N_r(r)| > \frac{p^2 n}{25}$ red neighbours in $N_r(r)$ which concludes the proof.

□

Let us set

$$V_1 = \left\{ x \in Y \setminus (N_r(r) \cup \{b\}) : |N(x) \cap J_1| \geq \frac{p^2 n}{200} \right\}$$

and

$$K_1 = Y \setminus (N_r(r) \cup V_1 \cup \{b\}).$$

Note that for each $x \in V_1$ we have that $|N(x) \cap J_1| \geq \frac{p^2 n}{400}$ in at least one of the colours. For $x \in V_1$ we set,

$$\rho(x) = \begin{cases} red & if \quad |N_r(x) \cap J_1| \geq \frac{p^2 n}{400} \\ blue & otherwise. \end{cases}$$

Note that for every vertex $v \in V(G)$, to which a preference $\rho(v)$ has been assigned, there are at least $\frac{p^2 n}{400}$ paths of colour $\rho(v)$ from v to either b or r.

Because of Lemma 3.1.2 (iii) we know that a.a.s.

$$|K_1| \leq \frac{100}{p}. \tag{4.3}$$

Also, we define,

$$\begin{aligned} V_1^r =& \{v \in V_1 : \rho(v) = red\}, \quad and \\ V_1^b =& \{v \in V_1 : \rho(v) = blue\}. \end{aligned}$$

Up to this point we have assigned the colour ρ to every vertex in $V(G)$ except for $X \setminus (N_b(b) \cup \{r\})$ and K_1, and this assignment was made in a way that every vertex v has a monochromatic path of colour $\rho(v)$ to its respective root. In order to define the trees we will also need this paths to be disjoint, but we will find an issue with the vertices in J_1. This is because we assigned them all blue and we will need some of them to be red in order to connect vertices from V_1^r to r by a red path.

This problem has a simple solution. For every vertex v in J_1 we decide randomly and independently with probability $\frac{1}{2}$ whether we attach it to the red tree rooted in r or to the blue tree rooted in b and then we redefine $\rho(v)$ according to this result. Since every vertex in V_1 has at least $\frac{p^2 n}{400} \gg log(n)$ neighbours in J_1 and in its preferred colour, with high probability at least one of those neighbours will obtain that preferred colour in the random assignment. To see this, we consider $v \in V_1^r$ a vertex connected to J_1 by red, and bound the probability that every $u \in N_r(v, J_1)$ was assigned $\rho(u) = blue$ as follows.

$$\mathbb{P}\Big(\forall u \in N_r(v, J_1) : \quad \rho(u) = blue\Big) = \prod_{u \in N_r(v,J_1)} \mathbb{P}(\rho(u) = blue) \leq \prod_{\mathrm{i}=1}^{p^2 n/400} \mathbb{P}(\rho(u_\mathrm{i}) = blue).$$

The last inequality holds because $|N_r(v, J_1)| \geq \frac{p^2 n}{400}$. Thus, due to the fact that $\mathbb{P}(\rho(u) = blue) = \frac{1}{2}$ and that $\frac{p^2 n}{400} \gg log(n)$ we have

$$\mathbb{P}\big(\forall u \in N_r(v, J_1): \quad \rho(u) = blue\big) \leq \prod_{\text{i=1}}^{log(n)} \frac{1}{2} = \left(\frac{1}{2}\right)^{log(n)}$$

which tends to zero. We conclude that with positive probability every vertex $v \in V_1^r$ has at least one neighbour with assigned colour red.

Now we will assign ρ for the vertices in $X \setminus (N_b(b) \cup \{r\})$, so as to finish the construction of our trees and thus conclude the proof. In order to do that we will separate the proof in two cases, in the first case we will suppose that there exists a vertex $v \in X \setminus (N_b(b) \cup \{r\})$ such that $|N_b(v) \cap V_1^r| \geq \frac{pn}{100}$ or $|N_r(v) \cap V_1^b| \geq \frac{pn}{100}$. In the second case there will be no such vertex.

- Case 1: There exists a vertex $v \in X \setminus (N_b(b) \cup \{r\})$ such that $|N_b(v) \cap V_1^r| \geq \frac{pn}{100}$ or $|N_r(v) \cap V_1^b| \geq \frac{pn}{100}$. We first suppose that $|N_b(v) \cap V_1^r| \geq \frac{pn}{100}$. The case where $|N_r(v) \cap V_1^b| \geq \frac{pn}{100}$ is analogous. We define $\tilde{b} = v$ and

$$J_2 = N_b(\tilde{b}) \cap V_1^r.$$

We have that $|J_2| \geq \frac{pn}{100}$. The vertex $\tilde{b}$ will be a root for a third tree and it is important to note that the vertices from J_2 could be attached either to the red tree rooted in r or to the blue tree rooted in $\tilde{b}$. We define

$$V_2 = \left\{ x \in X \setminus (N_b(b) \cup \{r, \tilde{b}\}) : |N(x) \cap J_2| \geq \frac{p^2 n}{200} \right\}.$$

Then, for every $x \in V_2$, we have $|N(x) \cap J_2| \geq \frac{p^2 n}{400}$ in at least one of the colours. For $x \in V_2$ we set

$$\rho(x) = \begin{cases} red & if \quad |N_r(x) \cap J_2| > \frac{p^2 n}{400} \\ blue & otherwise. \end{cases}$$

We define

$$K_2 = X \setminus (N_b(b) \cup V_2 \cup \{r, \tilde{b}\}),$$

and as above we have $|K_2| < \frac{100}{p}$. At this point the only vertices u that have not been assigned $\rho(u)$ are the vertices from K_1 and K_2, which together are at most $\frac{200}{p}$. Finally, we will have the same issue we had with J_1 with J_2 and we can solve it in the same way.

Let us note that every vertex u but the vertices in $K_1 \cup K_2$ has been assigned a colour $\rho(u)$ such that u has a monochromatic path to one of the roots r, b or $\tilde{b}$. So now we can define the trees T_1, T_2 and T_3 such that

$$T_1 = \rho^{-1}(red)$$

$$T_2 = \{u \in \rho^{-1}(blue) : u \textit{ has a monochromatic path to } b\}$$
$$T_3 = \{u \in \rho^{-1}(blue) : u \textit{ has a monochromatic path to } \tilde{b}\}$$

We obtained three vertex disjoint monochromatic trees that partition all but at most $O(\frac{1}{p})$ vertices which concludes the proof of Theorem 1.4.2.

- Case 2: There is no vertex $v \in X \setminus (N_b(b) \cup \{r\})$ such that $|N_b(v) \cap V_1^r| \geq \frac{pn}{100}$ or $|N_r(v) \cap V_1^b| \geq \frac{pn}{100}$. In other words, every $v \in X \setminus (N_b(b) \cup \{r\})$ has at most $\frac{pn}{100}$ blue neighbours in V_1^r and at most $\frac{pn}{100}$ red neighbours in V_1^b. Let us note that each vertex $v \in X \setminus (N_b(b) \cup \{r\})$ satisfies

$$\mathrm{d}(v, V_1^r \cup V_1^b) = \mathrm{d}(v) - \mathrm{d}(v, K_1) - \mathrm{d}(v, N_r(r)) - 1.$$

Also, by (4.3) and given that property (i) in Lemma 3.1.2 implies that $\mathrm{d}(v, N_r(r)) \leq (1+\varepsilon)p^2 n$ and $\mathrm{d}(v) > (1-\varepsilon)pn$ we can see that:

$$\mathrm{d}(v, V_1^r \cup V_1^b) > (1-\varepsilon)pn - \frac{100}{p} - (1+\varepsilon)p^2 n - 1.$$

Moreover, by (4.1) we conclude that

$$\mathrm{d}(v, V_1^r \cup V_1^b) > \frac{99pn}{100} - \frac{pn}{100} - \frac{pn}{100} - \frac{pn}{100} = \frac{96pn}{100}.$$

Therefore, any $v \in X \setminus (N_b(b) \cup \{r\})$ has at least $\frac{48pn}{100}$ neighbours in one of the sets V_1^r or V_1^b. If most of these neighbours are in V_1^r then, as $|N_b(v) \cap V_1^r| \leq \frac{pn}{100}$, we have that

$$|N_r(v) \cap V_1^r| \geq \frac{47pn}{100}$$

and we define $\rho(v) = \mathit{red}$. If most of the neighbours are in V_1^b we proceed similarly and we obtain $|N_b(v) \cap V_1^b| \geq \frac{47pn}{100}$ so we define $\rho(v) = \mathit{blue}$. This means that we are able to connect all the vertices in $X \setminus (N_b(b) \cup \{r\})$ in red to V_1^r or in blue to V_1^b.

Now we can define the trees T_1 and T_2 such that

$$T_1 = \rho^{-1}(red)$$
$$T_2 = \rho^{-1}(blue)$$

This ends Case 2 and the only vertices that $\rho(v)$ has not been assigned to are the vertices in K_1 which size is at most $\frac{100}{p}$. This finishes the proof of Theorem 1.4.2.

□

Conclusion

As we have previously said, this work is the study of the monochromatic partition problem in the particular case where the host graph is a random bipartite graph. Moreover, this work can be seen as a continuation of the problem raised by Erdős, Gyárfás and Pyber [10] and a new variation of the case studied by Bal and DeBiasio [3].

As we have seen in Section 3.3, Theorem 1.4.3 establishes a lower bound for the tree partition number in the case $r = 2$ and $p \gg \left(\frac{log(n)}{n}\right)^{\frac{1}{2}}$. In this theorem we built and example of a 2-colouring of the edges of our graph where we need a.a.s. at least three monochromatic trees to cover, and therefore, to partition a random bipartite graph. Theorem 1.4.2 shows that for this same case for r and p we can partition a.a.s. all but at most $O\left(\frac{1}{p}\right)$ vertices by three monochromatic trees. This proof was inspired by Kohayakawa, Mota and Schacht's proof [27] for general random graphs. The key of the demonstration lies in the bound for p which assures us that we have enough edges to connect the vertices to some of the monochromatic trees. This last property was further studied in Lemma 3.1.2 of Section 3.1. where we presented the properties that are due to the chosen bound for p. This bound for p seems to be accurate since if we slightly decreases it we will not have these desired properties. Nevertheless, a possible extension of our presented work would be varying the range for p and trying to obtain better bounds for the tree partition number.

Another possible extension to this work would consist in studying this problem considering more colours. We believe it is possible to find a correct bound for p such that we can partition a.a.s. the random bipartite graph by $2r - 1$ monochromatic trees which would coincide with the conjecture bound for the deterministic case [36].

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