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Implicit constitutive relations for describing the response of visco-elastic bodies

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ABSTRACT

A constitutive relation is proposed for viscoelastic bodies that is a generalization of the classic Kelvin–Voigt model, wherein the left Cauchy–Green tensor, the symmetric part of the velocity gradient, and the Cauchy stress tensor are implicitly related. The model developed includes several models that are being used in the literature to describe the elastic and viscoelastic response of bodies. In this paper, we study special homogeneous deformations of a slab within the context of the implicit viscoelastic model.

1. Introduction

The material moduli of many polymeric materials depends on pressure (see [3–7]) as do the properties of geological and composite materials, especially soils (see [8,9]; see also the discussion in [10] concerning the dependence of the material properties of solids on pressure). Also, it is well known that many viscoelastic solids exhibit shear-thinning. Thus, it would seem reasonable to consider the response characteristics of shear-thinning viscoelastic solids whose properties are pressure dependent, and this is the main purpose of this study. Unfortunately, none of the experimental literature cited determine the pressure dependence of materials that can also shear thin and hence are not relevant to the thrust of this study, namely the response of compressible viscoelastic bodies that are also capable of shear-thinning. The experimental data determine the effect of pressure on the relaxation modulus and glass transition. Thus, even though experimental results pertinent to our study are not available now, we believe that they will become available in the near future as understanding the response of such materials is important. One of the difficulties that has proved to be an impediment with regard to experiments that can describe materials whose properties depend on the mean value of the stress and the shear rate is that data reduction would require implicit models that can be used to interpret the data (see [10–12] for reasons why implicit models are necessary). As classical models are incapable of describing such behaviour, we generalize a popular viscoelastic solid model, namely the Kelvin–Voigt model so that its properties are pressure dependent and the model is also capable of describing shear-thinning, by developing an implicit constitutive relation.

Kelvin [13], Voigt [14] and Boltzmann [15] developed early models to describe the viscoelastic response of solids. Boltzmann developed an integral model to describe the linear response of viscoelastic solids while Kelvin and Voigt independently developed the eponymous differential viscoelastic solid model. Nearly a hundred years later, these efforts were greatly generalized by the development of integral models by Green and Rivlin [16] and Green et al. [17] to describe the finite deformation of viscoelastic solids, but such models are not amenable to use as the solution of specific initial–boundary value problems lead to very cumbersome nonlinear integral equations that rarely allow simple analytical solutions and also lead to very difficult computational problems within the context of numerical solutions. Also, in view of the stress being given by non-linear integral representation, there is an inherent non-uniqueness in determining the material moduli that appear in the constitutive relation since many functions can lead to the same value for the integral. The models developed by Kelvin and Voigt, Boltzmann, Green and Rivlin and others are constitutive relations wherein the stress is expressed explicitly in terms of appropriate kinematical quantities.

A detailed treatment of constitutive relations for the viscoelastic response of bodies as well as extensive literature can be found in Lockett [18] and Truesdell and Noll [19]. In this paper, we shall discuss a generalization of the differential type models developed by Kelvin and Voigt. We shall consider constitutive relations wherein the stress, and the kinematical quantities are implicitly related,[1] and as we are interested in the finite deformation of isotropic viscoelastic solids we shall assume that the Cauchy stress $\mathbf{T}$, the Cauchy–Green tensor $\mathbf{B}$ and

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[1] An early implicit model to describe the viscoelastic response of a fluid is due to Burgers [1]. The even earlier model due to Maxwell [2] is not an implicit constitutive relation as the symmetric part of the velocity gradient can be expressed explicitly in terms of the stress and a time derivative of the stress.

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the symmetric part of the velocity gradient $\mathbf{D}$ are the main variables with regard to the modelling. This constitutive relation includes as a special case isotropic elastic bodies defined through implicit constitutive relations (see Rajagopal [11,20,21]) and fluids defined by algebraic implicit constitutive relations, namely constitutive relations between the stress and the symmetric part of the velocity gradient (see Rajagopal [11,12]). Rajagopal and Srinivasa [22,23] have provided a thermodynamic framework for implicit constitutive relations for elastic bodies. Also, Rajagopal and Srinivasa [24] have recently developed a rate-type model, within a thermodynamic framework, to describe the response of thermo-viscoelastic bodies. A multi-network approach is adopted to describe the constitution of the body and attention is restricted to small deformations, though such an approach can be generalized to take into account large deformations. The model developed in this paper does not appeal to thermodynamics, and unlike the model put into place by Rajagopal and Srinivasa [24] it is an algebraic model in the sense that the model is an algebraic expression of the Cauchy stress, the left Cauchy–Green tensor and the symmetric part of the velocity gradient, no rates of any of these quantities appear in the constitutive relation. Unlike the case of the analysis by Rajagopal and Srinivasa [24] there is no restrictions concerning the smallness of the deformation when boundary value problems are considered.

We introduce a generalization of the classical Kelvin–Voigt model with four material moduli, one of which is a constant and the others depending on three of the invariants, one of them being the mean value of the stress $\mathrm{tr}\,\mathbf{T}$ (thus allowing us to take into account the material moduli depending on the 'mechanical pressure'[2]), the second being $\mathrm{tr}\,(\mathbf{D}^2)$ (allowing one to allow for shear thinning and shear thickening) and the third being the $\det \mathbf{F}$ which allows us to take into account compressibility of the body. Using such a reduced model we consider the homogeneous deformations of a slab which is subject to a special form for the stress (shear stress superposed on a normal stress). In addition to assuming a special structure for the stress field, we also assume a specific expression for the deformation. The general governing equations, even for such a special case are much too complicated to be solved exactly. On making further simplifying assumptions for the constitutive model, we first obtain an exact analytical solution to the problem. Later, we solve the full nonlinear governing equations numerically.

Since experimental results are not available for the class of initial–boundary value problems we are considering, we are unable to corroborate our work against such experimental data. However, since such models could be useful for describing the response of a variety of viscoelastic solids, especially polymeric and geological matter, we feel that such experimental data will be eventually available. While the study is a parametric study as it stands, we feel the model being considered, a model for viscoelastic solid body that is compressible and exhibits shear thinning/shear thickening, normal stress differences in shear and pressure dependent material moduli, would be useful.

The organization of the paper is as follows. In the next section we provide the basic kinematical definitions and record the balance laws. In Section 3, we introduce a generalization of the Kelvin–Voigt constitutive relation. The homogeneous deformation of a slab is introduced and an exact solution is established in Section 4. We also solve the fully nonlinear problem numerically in that section. Some concluding remarks are made in Section 5.

2. Basic equations

A point X in a body $\mathscr{B}$ occupies the position $\mathbf{X} = \boldsymbol{\kappa}_r(X)$ in the reference configuration $\boldsymbol{\kappa}_r(\mathscr{B})$. In the current configuration the position of the point is denoted $\mathbf{x}$, and it is assumed that there exists a one-to-one mapping χ such that $\mathbf{x} = \chi(\mathbf{X}, t)$. The current configuration is denoted $\boldsymbol{\kappa}_t(\mathscr{B})$. The deformation gradient, the left Cauchy–Green tensor, and the symmetric part of the velocity gradient are defined through

$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{D} = \frac{1}{2}\left[\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}}^{\mathrm{T}}\right], \tag{1}$$

The Cauchy stress tensor is denoted by $\mathbf{T}$ and it satisfies the equations of motion

$$\rho \ddot{\mathbf{x}} = \mathrm{div}\,\mathbf{T} + \rho \mathbf{b}, \tag{2}$$

where ρ is the density of the body and $\mathbf{b}$ represents the body forces in the current configuration, and where we use the notation $[\ \dot{}\ ]$ for the material time derivative. More details on the kinematics of continua can be found in [26].

We use the round brackets ( ) to denote the arguments of a function.

3. Implicit constitutive relation

We are interested in a special sub-class of the viscoelastic materials described by the implicit constitutive relation:

$$\mathfrak{F}(\mathbf{T}, \mathbf{B}, \mathbf{D}) = \mathbf{0}, \tag{3}$$

namely

$$\mathbf{T} + \varphi \mathbf{I} - \alpha \mathbf{B} - \mu \mathbf{D} = \mathbf{0}, \tag{4}$$

where α is a constant and we assume that

$$\mu = \mu(I_1, I_2) = \mu_0 e^{\delta I_1}[1+2\beta I_2]^n, \quad \varphi = \varphi(I_1, \det \mathbf{F}) = [\gamma I_1 + \lambda][\det \mathbf{F}]^m, \tag{5}$$

where μ_0, δ, β, n, γ, λ and m are constants[3] and

$$I_1 = \mathrm{tr}\,\mathbf{T}, \quad I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{D}^2). \tag{6}$$

The above model allows us to describe a material whose response depends on the 'mechanical pressure' and that is capable of 'shear thinning/shear thickening'.

In Table 1 we present the values of the different constants to be used in our calculations in the later sections.

Table 1
Material constants for the model (5).

μ_0	[Pa s]	6.9×10^7, 6.24×10^9
δ	1/[Pa]	8×10^{-9}
β	[s^2]	1, 10
n		−0.5, 0, 0.5
γ		0, 9.52, 95.2, 952, 9520, 47600
λ	[Pa]	0, 9.52, 95.2, 95200, 1.05×10^5, 9.52×10^6
m		0, 1, 5, 10

An important restriction that the model (4), (5) must satisfy is that when $\mathbf{F} \equiv \mathbf{I}$, for all time, then the stress has to be zero, that is, $\mathbf{T} \equiv \mathbf{0}$ and the condition is satisfied if

$$\alpha = \lambda, \quad \text{and} \quad \gamma \neq \frac{1}{3}. \tag{7}$$

That $\mathbf{F} \equiv \mathbf{Q}$, where $\mathbf{Q}$ is an orthogonal transformation, when $\mathbf{T} \equiv \mathbf{0}$, is far from easy to show.

[2] The terminology 'pressure' has been used to signify a variety of different physical quantities (see Rajagopal [25] for a discussion of the same).

[3] It is not at all surprising that one needs eight constants to capture the response of a viscoelastic solids whose viscosity depends on the pressure and shear-rate, whose elastic response depends on the pressure, and the material is compressible, given a popular model for an isotropic viscoelastic fluid model developed by Oldroyd requires six constants (see [27]), popular models due to Ogden to describe the isotropic elastic solid behaviour requires six and eight constants (see [28]), and a linearized orthotropic material requires nine constants to describe its response (see [29]). The point is, if one needs to describe a variety of response characteristics one needs as many constants as necessary to capture the response.

4. Homogeneous deformations and stresses of a slab

Let us study the response of a viscoelastic slab described in the reference configuration through

$$-\frac{L_i}{2} \leq X_i \leq \frac{L_i}{2}, \quad i = 1,2,3. \tag{8}$$

This slab is subjected to a stress tensor field of the form

$$\mathbf{T} = \sigma_2(t)\mathbf{e}_2 \otimes \mathbf{e}_2 + \tau(t)[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1], \tag{9}$$

which represents time dependent extension/compression in the 2-direction and shear in the 1-2 plane.

In this problem we assume that there is no body force and that $|\ddot{\mathbf{u}}|$ can be neglected, such that the equation of motion is approximately satisfied for the above homogeneous distribution of stresses.

Let us assume that the slab deforms under the influence of the above stress field that is given by

$$x = \aleph(t)X + \kappa(t)Y, \quad y = \wp(t)Y, \quad z = \ell(t)Z, \tag{10}$$

where in this and in some of the subsequent problems we use the notation x, y, z for x_i, $i = 1,2,3$ and X, Y, Z for X_i, $i = 1,2,3$ respectively. In general, the assumptions (9) and (10) might not be compatible, however for the problem under consideration, it is.

The deformation gradient and the left Cauchy–Green tensors are

$$\mathbf{F} = \aleph(t)\mathbf{e}_1 \otimes \mathbf{E}_1 + \kappa(t)\mathbf{e}_1 \otimes \mathbf{E}_2 + \wp(t)\mathbf{e}_2 \otimes \mathbf{E}_2 + \ell(t)\mathbf{e}_3 \otimes \mathbf{E}_3, \tag{11}$$

$$\begin{aligned}\mathbf{B} = {} & [\aleph(t)^2 + \kappa(t)^2]\mathbf{e}_1 \otimes \mathbf{e}_1 + \kappa(t)\wp(t)[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1] \\ & + \wp(t)^2\mathbf{e}_2 \otimes \mathbf{e}_2 + \ell(t)^2\mathbf{e}_3 \otimes \mathbf{e}_3,\end{aligned} \tag{12}$$

whereas

$$\mathbf{D} = \frac{\dot{\aleph}}{\aleph}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{2}\left[\frac{\dot{\kappa}}{\wp} - \frac{\kappa\dot{\aleph}}{\wp\aleph}\right][\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1] + \frac{\dot{\wp}}{\wp}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{\dot{\ell}}{\ell}\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{13}$$

where we have used the relations $\frac{\partial X}{\partial x} = \frac{1}{\aleph}$, $\frac{\partial X}{\partial y} = -\frac{\kappa}{\aleph\wp}$, $\frac{\partial Y}{\partial y} = \frac{1}{\wp}$ and $\frac{\partial Z}{\partial z} = \frac{1}{\ell}$

From (11), (12), (13) and (6) we have

$$I_1 = \sigma_2(t), \quad \det \mathbf{F} = \ell(t)\wp(t)\aleph(t), \tag{14}$$

and

$$I_2 = \frac{1}{2}\left\{\left[\frac{\dot{\aleph}}{\aleph}\right]^2 + \frac{[\aleph\dot{\kappa} - \kappa\dot{\aleph}]^2}{2\wp^2\aleph^2} + \left[\frac{\dot{\wp}}{\wp}\right]^2 + \left[\frac{\dot{\ell}}{\ell}\right]^2\right\}. \tag{15}$$

Using (9), (11), (12) and (13) in (4) we obtain the four ordinary differential equations:

$$\lambda[\aleph^2 + \kappa^2] + \mu\frac{\dot{\aleph}}{\aleph} = \varphi, \tag{16}$$

$$\lambda\wp^2 + \mu\frac{\dot{\wp}}{\wp} = \sigma_2 + \varphi, \tag{17}$$

$$\lambda\ell^2 + \mu\frac{\dot{\ell}}{\ell} = \varphi, \tag{18}$$

$$\lambda\kappa\wp + \frac{\mu}{2}\left[\frac{\dot{\kappa}}{\wp} - \frac{\kappa\dot{\aleph}}{\wp\aleph}\right] = \tau, \tag{19}$$

where φ and μ are given in (5) in terms of the invariants I_1, I_2 and $\det \mathbf{F}$ that are defined in (14), (15), i.e.,

$$\mu = \mu(t) = \mu(\sigma_2(t), \aleph(t), \dot{\aleph}(t), \wp(t), \dot{\wp}(t), \ell(t), \dot{\ell}(t), \kappa(t), \dot{\kappa}(t)),$$

$$\varphi = \varphi(t) = \varphi(\sigma_2(t), \aleph(t), \wp(t), \ell(t)).$$

The Eqs. (16)–(19) must be solved to find the functions $\aleph(t)$, $\kappa(t)$, $\wp(t)$ and $\ell(t)$ for given $\tau(t)$ and $\sigma_2(t)$. The initial conditions are

$$\aleph(0) = 1, \quad \kappa(0) = 0, \quad \wp(0) = 1, \quad \ell(0) = 1. \tag{20}$$

4.1. Exact solutions for the case $n = 0$ and $m = 0$

In the special case $n = 0$ and $m = 0$ from (5) we have

$$\mu = \mu_0 e^{\delta I_1}, \quad \varphi = \gamma(I_1) + \lambda. \tag{21}$$

From the above results we deduce that $\mu = \mu(I_1) = \mu(\sigma_2(t))$ and $\varphi = \varphi(I_1) = \varphi(\sigma_2(t))$.

In the particular case that $n = 0$ and $m = 0$ (18) becomes

$$\lambda\ell^2 + \mu_0 e^{\delta\sigma_2}\frac{\dot{\ell}}{\ell} = \gamma\sigma_2 + \lambda, \tag{22}$$

which is an equation of the Bernoulli type. Let us assume that $L(t) = [\ell(t)]^{-2}$ then the above equation becomes

$$\dot{L}(t) = -2\frac{[\gamma\sigma_2(t) + \lambda]}{\mu_0}e^{-\delta\sigma_2(t)}L(t) + \frac{2\lambda}{\mu_0}e^{-\delta\sigma_2(t)}. \tag{23}$$

In the special case $\sigma_2(t) = 0$ the solution of the above equation is

$$L(t) = \tilde{A}e^{-\frac{2\lambda t}{\mu_0}} + 1, \tag{24}$$

where $\tilde{A}$ is a constant. In the more general case where $\sigma_2(t) \neq 0$ the solution of (22) for $\ell(t)$ when $n = 0$, $m = 0$ and of (17) for $\wp(t) > 0$ are:

$$\ell(t) = \frac{\exp\left(\int_0^t \frac{\varphi(\sigma_2(\zeta))}{\mu(\sigma_2(\zeta))}\,\mathrm{d}\zeta\right)}{\left[1 + 2\lambda\int_0^t \frac{\exp\left(2\int_0^\xi \frac{\varphi(\sigma_2(\eta))}{\mu(\sigma_2(\eta))}\,\mathrm{d}\eta\right)}{\mu(\sigma_2(\xi))}\,\mathrm{d}\xi\right]^{1/2}}, \tag{25}$$

$$\wp(t) = \frac{\exp\left(\int_0^t \frac{\varphi(\sigma_2(\zeta))+\sigma_2(\zeta)}{\mu(\sigma_2(\zeta))}\,\mathrm{d}\zeta\right)}{\left[1 + 2\lambda\int_0^t \frac{\exp\left(2\int_0^\xi \frac{\varphi(\sigma_2(\eta))+\sigma_2(\eta)}{\mu(\sigma_2(\eta))}\,\mathrm{d}\eta\right)}{\mu(\sigma_2(\xi))}\,\mathrm{d}\xi\right]^{1/2}}. \tag{26}$$

The above solutions are studied for the following cases:

$$\text{(A)}\ \sigma_2(t) = \frac{\sigma_{2_o}}{t_o}t[1 - \mathcal{H}(t - t_o)] + \sigma_{2_o}\mathcal{H}(t - t_o), \quad \tau(t) = 0, \tag{27}$$

$$\text{(B)}\ \sigma_2(t) = 0, \quad \tau(t) = \frac{\tau_o}{t_o}t[1 - \mathcal{H}(t - t_o)] + \tau_o\mathcal{H}(t - t_o), \tag{28}$$

where $\mathcal{H}(t)$ is the Heaviside step function, and σ_{2_o}, $\tau_o > 0$ and $t_o > 0$ are constants (for some of the cases to be studied in Section 4.2 we also consider the case $\sigma_2(t) = \frac{\sigma_{2_o}}{t_o}t[1 - \mathcal{H}(t - t_o)] + \sigma_{2_o}\mathcal{H}(t - t_o)$ and $\tau(t) = \frac{\tau_o}{t_o}t[1 - \mathcal{H}(t - t_o)] + \tau_o\mathcal{H}(t - t_o)$ simultaneously). The behaviour of $\ell(t)$, $\wp(t)$, $\kappa(t)$ and $\aleph(t)$ when $t \to \infty$ is presented now (some details of the calculations are omitted for the sake of brevity):

Case (A): In the case $\sigma_2(t) = \frac{\sigma_{2_o}}{t_o}t[1 - \mathcal{H}(t - t_o)] + \sigma_{2_o}\mathcal{H}(t - t_o)$, if $t \gg t_o$ we have $\sigma_2 = \sigma_{2_o}$ that is constant. Then from (23) we obtain

$$\dot{L} = \tilde{A}\left[\frac{2\lambda e^{-\delta\sigma_{2_o}}}{\hat{\mu}_0\tilde{A}} - L\right], \tag{29}$$

where

$$\tilde{A} = 2\frac{[\gamma\sigma_{2_o} + \lambda]}{\hat{\mu}_0}e^{-\delta\sigma_{2_o}}, \tag{30}$$

where $\hat{\mu}_0 = \mu_0 e^{\delta\gamma\sigma_{2_o}}$. Therefore for L we have

$$\lim_{t\to\infty} L(t) = \frac{\lambda}{\gamma\sigma_{2_o} + \lambda}, \tag{31}$$

then

$$\lim_{t\to\infty} \ell(t) = \left[\frac{\gamma\sigma_{2_o} + \lambda}{\lambda}\right]^{1/2}. \tag{32}$$

With regard to $\wp(t)$ in the case $t \gg t_o$ Eq. (17) becomes

$$\lambda\wp^2 + \hat{\mu}_0\frac{\dot{\wp}}{\wp} = [1 + \gamma]\sigma_{2_o} + \lambda. \tag{33}$$

Taking $p = \wp^{-2}$ we obtain

$$\dot{p} = -\frac{2\{[1 + \gamma]\sigma_{2_o} + \lambda\}}{\hat{\mu}_0}p + \frac{2\lambda}{\hat{\mu}_0}. \tag{34}$$

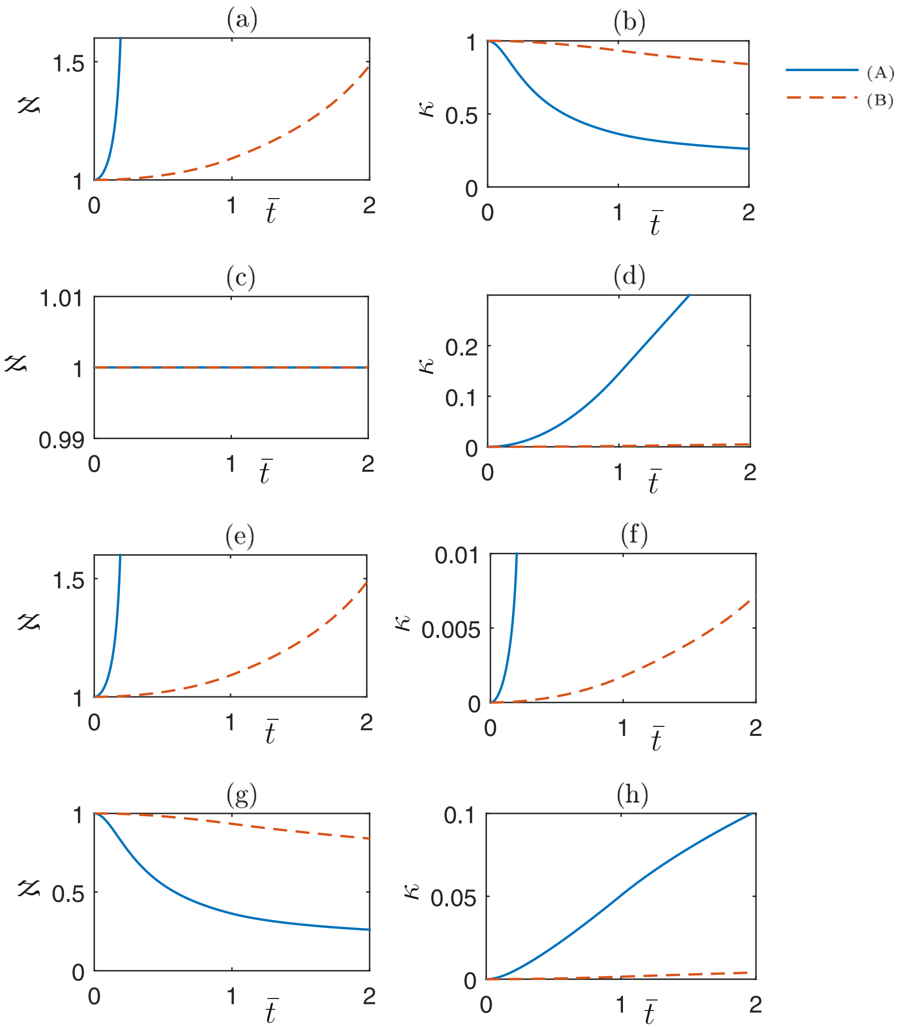

Fig. 1. Comparison of the behaviour of the slab considering two possible values for the constant μ_0 in (5), namely (A) $\mu_0 = 6.9\times10^7$, (B) $\mu_0 = 6.24\times10^9$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$.

Following the same steps as before we have

$$\lim_{t\to\infty} \wp(t) = \left\{ \frac{[1+\gamma]\sigma_{2_o} + \lambda}{\lambda} \right\}^{1/2}. \tag{35}$$

In this case $\kappa(t) = 0$ is a solution of Eq. (19) that satisfies the initial condition $\kappa(0) = 0$. Taking $M = \aleph^{-2}$, from Eq. (16) we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{2\lambda}{\mu} - \frac{2\varphi}{\mu} M, \tag{36}$$

and we obtain

$$\aleph(t) = \frac{\exp\left(\int_0^t \frac{2\varphi(\zeta)}{\mu(\zeta)}\,\mathrm{d}\zeta\right)}{\left[1 + 2\lambda \int_0^t \frac{\exp\left(\int_0^\eta \frac{2\varphi(\eta)}{\mu(\eta)}\,\mathrm{d}\eta\right)}{\mu(\zeta)}\,\mathrm{d}\zeta\right]^{1/2}}. \tag{37}$$

Case (B): In the case $\sigma_2(t) = 0$, $\tau(t) = \frac{\tau_o}{t_o} t[1 - \mathcal{H}(t - t_o)] + \tau_o \mathcal{H}(t - t_o)$, from (22) we obtain

$$\lambda \ell^2(t) + \mu_0 \frac{\dot{\ell}(t)}{\ell(t)} = \lambda, \tag{38}$$

which has three solutions $\ell(t) = 1$ and $\ell(t) = \frac{\pm 1}{\sqrt{1+\exp(2[C - t\lambda/\mu_0])}}$, where C is a constant, and since $\ell(t) > 0$ only the + solution is valid. From the initial condition $\ell(0) = 1$ only the solution $\ell(t) = 1$ is possible, because we can observe that it is not possible to find C such that $\ell(0) = 1$ for the solution $\ell(t) = \frac{1}{\sqrt{1+\exp(2[C - t\lambda/\mu_0])}}$.

In the case of $\wp(t)$ the structure of the Eq. (17) is the same as (38) and the solutions are similar as the ones just discussed for $\ell(t)$, therefore $\wp(t) = 1$ is the only valid solution. Taking into account the above and recalling the definition $\hat{\mu} = \mu_0 e^{\lambda\delta}$, for $t > t_0$ we have $\tau = \tau_o$, and we proceed to obtain asymptotic solutions for $\aleph$ and κ rewriting (16) and (19) as

$$\frac{\mathrm{d}\aleph}{\mathrm{d}t} = \frac{\lambda}{\hat{\mu}} \left\{1 - [\aleph^2 + \kappa^2]\right\} \aleph, \quad \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \frac{1}{\hat{\mu}} \left\{2\tau_0 - \lambda[\aleph^2 + \kappa^2]\kappa\right\}. \tag{39}$$

Considering the classical approach, the study of the equilibrium points is translated into the following system $\aleph^2 + \kappa^2 = 1$, $2\tau_0 - \lambda\kappa = 0$. Then, the following equilibrium points are obtained

$$\kappa_e = \frac{2\tau_0}{\lambda}, \quad \aleph_e = \sqrt{1 - \left[\frac{2\tau_0}{\lambda}\right]^2}. \tag{40}$$

Writing the system field as $F(\aleph, \kappa) = \begin{pmatrix} \frac{\lambda}{\hat{\mu}} \left\{1 - [\aleph^2 + \kappa^2]\right\} \aleph \\ \frac{1}{\hat{\mu}} \left\{2\tau_0 - \lambda[\aleph^2 + \kappa^2]\kappa\right\} \end{pmatrix}$, linearizing around the equilibrium point $(\aleph_e, \kappa_e)$ we obtain $F(\aleph, \kappa) \approx -\frac{2\lambda}{\hat{\mu}} \begin{pmatrix} \aleph_e^2 & \aleph_e\kappa_e \\ \aleph_e\kappa_e & \kappa - \frac{1}{2} \end{pmatrix} \begin{pmatrix} [\aleph - \aleph_e] \\ [\kappa - \kappa_e], \end{pmatrix}$ and the eigenvalues

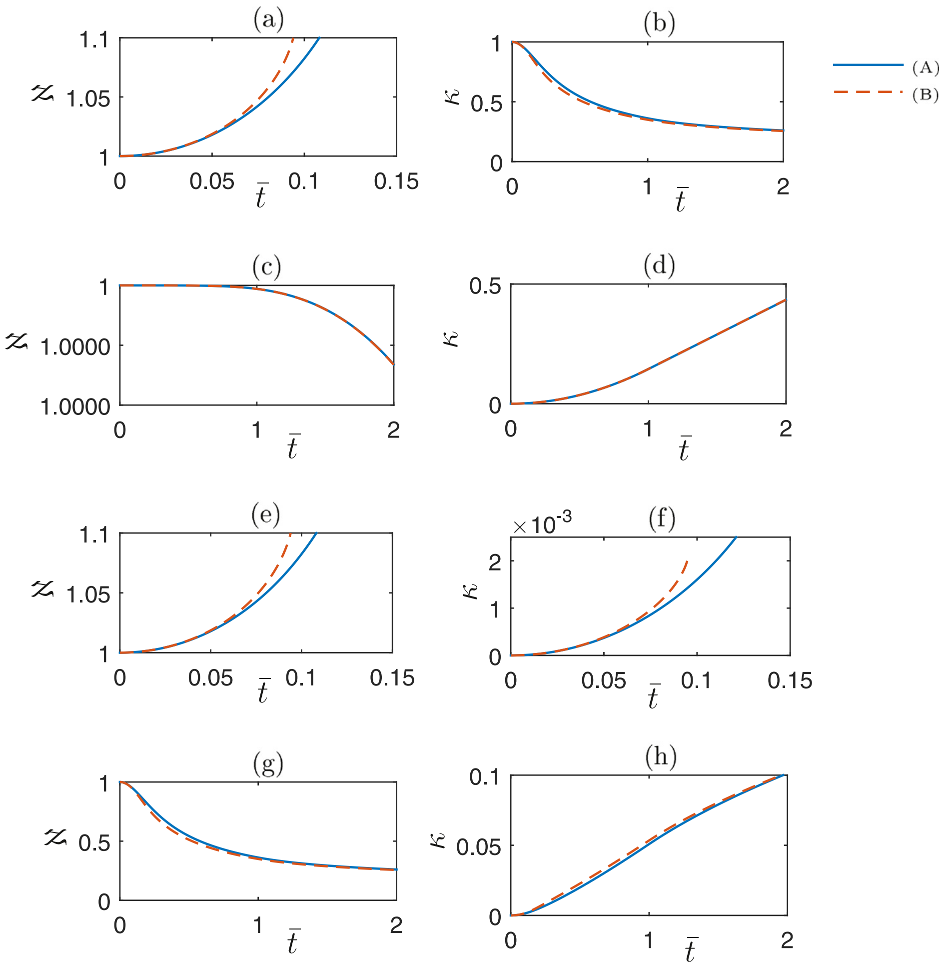

Fig. 2. Comparison of the behaviour of the slab for two possible values for the constant β in (5), namely (A) $\beta = 1$, (B) $\beta = 10$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$.

of the matrix associated with the above system are

$$\iota_1 = -\frac{\lambda}{2\hat{\mu}}\left\{1 + [\kappa_e - 1]\kappa_e - \sqrt{9 - 4\kappa_e[1+\kappa_e][3 + \kappa_e\{3\kappa_e - 5\}]}\right\},$$
$$\iota_2 = -\frac{\lambda}{2\hat{\mu}}\left\{1 + [\kappa_e - 1]\kappa_e + \sqrt{9 - 4\kappa_e[1+\kappa_e][3 + \kappa_e\{3\kappa_e - 5\}]}\right\}.$$

If $\Re(\iota_i) < 0$ then $\aleph \to \aleph_e$, and $\kappa \to \kappa_e$.

Case (B) when $\lambda = 0$: In the case $\sigma_2(t) = 0$ and $\lambda = 0$ from (16) and (19) we obtain

$$\aleph(t) = \exp\left(\int_0^t \frac{\varphi(\zeta)}{\mu(\zeta)}\, d\zeta\right), \tag{41}$$

$$\kappa(t) = \exp\left(\int_0^t \frac{\varphi(\zeta)}{\mu(\zeta)}\, \mathrm{d}\zeta\right)\int_0^t \frac{2\tau(\zeta)\wp(\zeta)}{\mu(\zeta)}\exp\left(-\int_0^\zeta \frac{\varphi(\eta)}{\mu(\eta)}\, d\eta\right) \mathrm{d}\zeta. \tag{42}$$

4.2. The fully nonlinear problem

The original system of nonlinear ordinary differential Eqs. (16)–(19) is solved numerically. Taking into account that the function μ contains the first derivatives of the different unknown functions, in order to use the standard methods for solving ordinary differential equations with initial conditions, we take the derivative of (16)–(19) in time, obtaining respectively[4]:

$$2\lambda[\aleph\dot{\aleph} + \kappa\dot{\kappa}] + \mu\left[\frac{\ddot{\aleph}}{\aleph} - \frac{\dot{\aleph}^2}{\aleph^2}\right] + \left\{\frac{\partial\mu}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2 + \frac{\partial\mu}{\partial I_2}\left[\frac{\partial I_2}{\partial\aleph}\dot{\aleph} + \frac{\partial I_2}{\partial\dot{\aleph}}\ddot{\aleph}\right.\right.$$
$$\left.\left. + \frac{\partial I_2}{\partial\wp}\dot{\wp} + \frac{\partial I_2}{\partial\dot{\wp}}\ddot{\wp} + \frac{\partial I_2}{\partial\ell}\dot{\ell} + \frac{\partial I_2}{\partial\dot{\ell}}\ddot{\ell} + \frac{\partial I_2}{\partial\kappa}\dot{\kappa} + \frac{\partial I_2}{\partial\dot{\kappa}}\ddot{\kappa}\right]\right\}\frac{\dot{\aleph}}{\aleph} = \frac{\partial\varphi}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2$$
$$+ \frac{\partial\varphi}{\partial \det\mathbf{F}}\left[\frac{\partial\det\mathbf{F}}{\partial\ell}\dot{\ell} + \frac{\partial\det\mathbf{F}}{\partial\wp}\dot{\wp} + \frac{\partial\det\mathbf{F}}{\partial\aleph}\dot{\aleph}\right], \tag{43}$$

$$2\lambda\wp\dot{\wp} + \mu\left[\frac{\ddot{\wp}}{\wp} - \frac{\dot{\wp}^2}{\wp^2}\right] + \left\{\frac{\partial\mu}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2 + \frac{\partial\mu}{\partial I_2}\left[\frac{\partial I_2}{\partial\aleph}\dot{\aleph} + \frac{\partial I_2}{\partial\dot{\aleph}}\ddot{\aleph}\right.\right.$$
$$\left.\left. + \frac{\partial I_2}{\partial\wp}\dot{\wp} + \frac{\partial I_2}{\partial\dot{\wp}}\ddot{\wp} + \frac{\partial I_2}{\partial\ell}\dot{\ell} + \frac{\partial I_2}{\partial\dot{\ell}}\ddot{\ell} + \frac{\partial I_2}{\partial\kappa}\dot{\kappa} + \frac{\partial I_2}{\partial\dot{\kappa}}\ddot{\kappa}\right]\right\}\frac{\dot{\wp}}{\wp}$$
$$= \dot{\sigma}_2 + \frac{\partial\varphi}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2$$
$$+ \frac{\partial\varphi}{\partial \det\mathbf{F}}\left[\frac{\partial\det\mathbf{F}}{\partial\ell}\dot{\ell} + \frac{\partial\det\mathbf{F}}{\partial\wp}\dot{\wp} + \frac{\partial\det\mathbf{F}}{\partial\aleph}\dot{\aleph}\right], \tag{44}$$

$$2\lambda\ell\dot{\ell} + \mu\left[\frac{\ddot{\ell}}{\ell} - \frac{\dot{\ell}^2}{\ell^2}\right]$$

[4] By taking the time derivative of (16)–(19) we have increased the order of the equation and hence need to supply additional initial conditions to obtain a determinate set of equations. We discuss how we augment the initial conditions later.

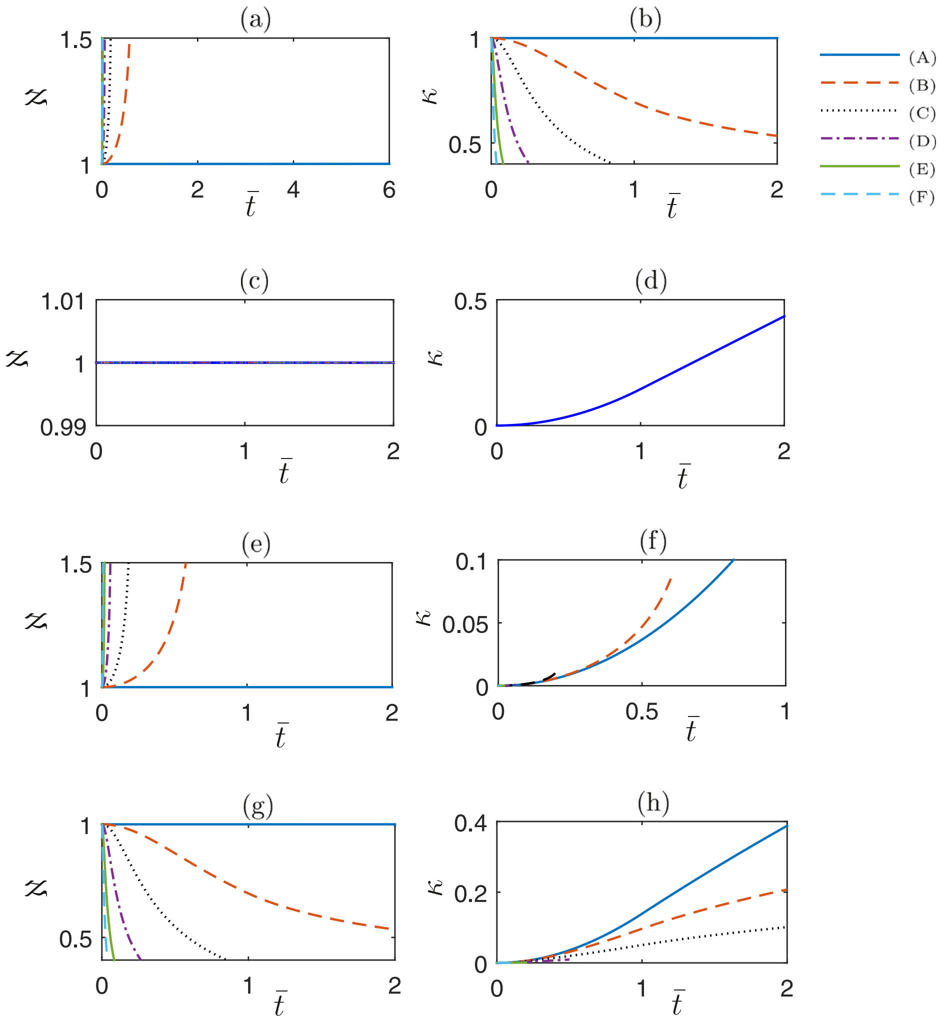

Fig. 3. Comparison of the behaviour of the slab for six possible values for the constant γ in (5), namely (A) $\gamma = 0$, (B) $\gamma = 9.52$, (C) $\gamma = 95.2$, (D) $\gamma = 952$, (E) $\gamma = 9520$, (F) $\gamma = 4.76 \times 10^4$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$.

$$+\left\{\frac{\partial\mu}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2+\frac{\partial\mu}{\partial I_2}\left[\frac{\partial I_2}{\partial\aleph}\dot{\aleph}+\frac{\partial I_2}{\partial\dot{\aleph}}\ddot{\aleph}+\frac{\partial I_2}{\partial\wp}\dot{\wp}+\frac{\partial I_2}{\partial\dot{\wp}}\ddot{\wp}+\frac{\partial I_2}{\partial\ell}\dot{\ell}+\frac{\partial I_2}{\partial\dot{\ell}}\ddot{\ell}+\frac{\partial I_2}{\partial\kappa}\dot{\kappa}+\frac{\partial I_2}{\partial\dot{\kappa}}\ddot{\kappa}\right]\right\}\frac{\dot{\ell}}{\ell}=\frac{\partial\varphi}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2+\frac{\partial\varphi}{\partial\det\mathbf{F}}\left[\frac{\partial\det\mathbf{F}}{\partial\ell}\dot{\ell}+\frac{\partial\det\mathbf{F}}{\partial\wp}\dot{\wp}+\frac{\partial\det\mathbf{F}}{\partial\aleph}\dot{\aleph}\right], \tag{45}$$

and

$$\lambda[\dot{\kappa}\wp+\kappa\dot{\wp}]+\frac{\mu}{2}\left\{\frac{\ddot{\kappa}}{\wp}-\frac{\kappa\ddot{\aleph}}{\wp\aleph}-\frac{1}{\wp^2\aleph^2}[\aleph\dot{\kappa}-\kappa\dot{\aleph}][\aleph\dot{\wp}+\wp\dot{\aleph}]\right\}+\left\{\frac{\partial\mu}{\partial I_1}\frac{\partial I_1}{\partial\sigma_2}\dot{\sigma}_2+\frac{\partial\mu}{\partial I_2}\left[\frac{\partial I_2}{\partial\aleph}\dot{\aleph}+\frac{\partial I_2}{\partial\dot{\aleph}}\ddot{\aleph}+\frac{\partial I_2}{\partial\wp}\dot{\wp}+\frac{\partial I_2}{\partial\dot{\wp}}\ddot{\wp}+\frac{\partial I_2}{\partial\ell}\dot{\ell}+\frac{\partial I_2}{\partial\dot{\ell}}\ddot{\ell}+\frac{\partial I_2}{\partial\kappa}\dot{\kappa}+\frac{\partial I_2}{\partial\dot{\kappa}}\ddot{\kappa}\right]\right\}\times\frac{1}{2}\left[\frac{\dot{\kappa}}{\wp}-\frac{\kappa\dot{\aleph}}{\wp\aleph}\right]=\dot{\tau}, \tag{46}$$

where from (14)$_1$ we have

$$\frac{\partial I_1}{\partial\sigma_2}=1, \tag{47}$$

and from (15) we obtain

$$\frac{\partial I_2}{\partial\aleph}=\frac{\dot{\aleph}}{2\wp^2\aleph^3}\{\kappa\dot{\kappa}\aleph-[\kappa^2+2\wp^2]\dot{\aleph}\}, \tag{48}$$

$$\frac{\partial I_2}{\partial\dot{\aleph}}=\frac{1}{2\wp^2\aleph^2}[\kappa^2\dot{\aleph}+2\wp^2\dot{\aleph}-\kappa\dot{\kappa}\aleph], \tag{49}$$

$$\frac{\partial I_2}{\partial\wp}=-\frac{1}{2\wp^3}\left\{2\dot{\wp}^2+\frac{1}{\aleph^2}[\dot{\kappa}\aleph-\kappa\dot{\aleph}]^2\right\}, \tag{50}$$

$$\frac{\partial I_2}{\partial\dot{\wp}}=\frac{\dot{\wp}}{\wp^2},\quad \frac{\partial I_2}{\partial\ell}=-\frac{\dot{\ell}}{\ell^2},\quad \frac{\partial I_2}{\partial\dot{\ell}}=\frac{\dot{\ell}^2}{\ell^2}, \tag{51}$$

$$\frac{\partial I_2}{\partial\kappa}=\frac{\dot{\aleph}}{2\wp^2\aleph^2}[\kappa\dot{\aleph}-\dot{\kappa}\aleph],\quad \frac{\partial I_2}{\partial\dot{\kappa}}=\frac{\dot{\kappa}\aleph-\kappa\dot{\aleph}^2}{2\wp^2\aleph}, \tag{52}$$

and finally from (14)$_2$

$$\frac{\partial\det\mathbf{F}}{\partial\ell}=\wp\aleph,\quad \frac{\partial\det\mathbf{F}}{\partial\wp}=\ell\aleph,\quad \frac{\partial\det\mathbf{F}}{\partial\aleph}=\ell\wp. \tag{53}$$

The system of Eqs. (43)–(46) can be rewritten as

$$\ddot{\aleph}(t)=\mathcal{F}_1(\aleph(t),\dot{\aleph}(t),\wp(t),\dot{\wp}(t),\ell(t),\dot{\ell}(t),\kappa(t),\dot{\kappa}(t),\sigma_2(t),\dot{\sigma}_2(t),\dot{\tau}(t)), \tag{54}$$

$$\ddot{\wp}(t)=\mathcal{F}_2(\aleph(t),\dot{\aleph}(t),\wp(t),\dot{\wp}(t),\ell(t),\dot{\ell}(t),\kappa(t),\dot{\kappa}(t),\sigma_2(t),\dot{\sigma}_2(t),\dot{\tau}(t)), \tag{55}$$

$$\ddot{\ell}(t)=\mathcal{F}_3(\aleph(t),\dot{\aleph}(t),\wp(t),\dot{\wp}(t),\ell(t),\dot{\ell}(t),\kappa(t),\dot{\kappa}(t),\sigma_2(t),\dot{\sigma}_2(t),\dot{\tau}(t)), \tag{56}$$

$$\ddot{\kappa}(t)=\mathcal{F}_4(\aleph(t),\dot{\aleph}(t),\wp(t),\ell(t),\dot{\ell}(t),\dot{\wp}(t),\kappa(t),\dot{\kappa}(t),\sigma_2(t),\dot{\sigma}_2(t),\dot{\tau}(t)), \tag{57}$$

where the functions $\mathcal{F}_i$, $i = 1, 2, 3, 4$ are not shown explicitly for brevity. The above system of nonlinear ordinary differential equations can be solved using standard methods that have been developed for handling such equations.

Apart from the initial conditions (see (20)) $\aleph(0) = 1$, $\kappa(0) = 0$, $\wp(0) = 1$ and $\ell(0) = 1$, we need conditions for $\dot{\aleph}(0)$, $\dot{\kappa}(0)$, $\dot{\wp}(0)$ and $\dot{\ell}(0)$, which can be obtained by replacing (20) in the original system of

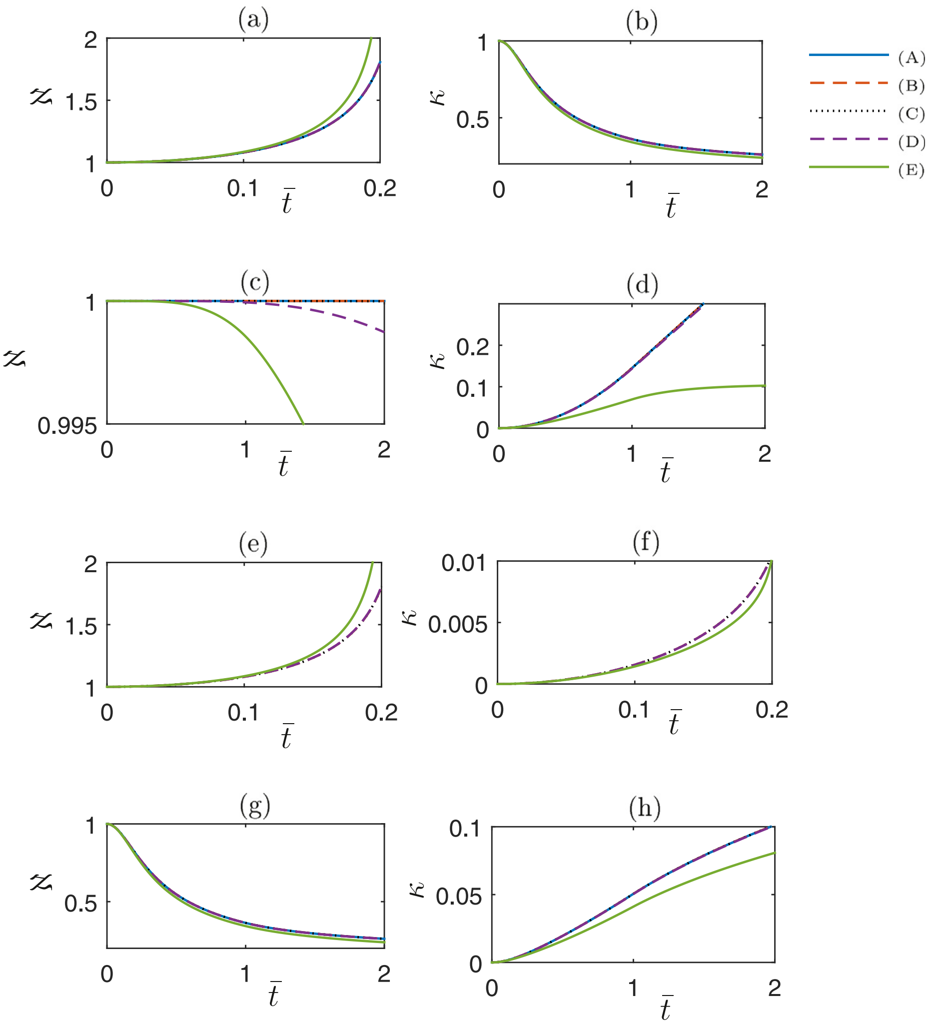

Fig. 4. Comparison of the behaviour of the slab for five possible values for the constant λ in (5), namely (A) $\lambda = 0$, (B) $\lambda = 9.52$, (C) $\lambda = 952$, (D) $\lambda = 9.52\times 10^4$, (E) $\lambda = 9.52\times 10^6$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$.

equations[5] (16)–(19) thereby obtaining

$$\lambda + \mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0))\dot{\aleph}(0) = \varphi(\sigma_2(0),1,1,1), \tag{58}$$

$$\lambda + \mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0))\dot{\wp}(0) = \varphi(\sigma_2(0),1,1,1)+\sigma_2(0), \tag{59}$$

$$\lambda + \mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0))\dot{\ell}(0) = \varphi(\sigma_2(0),1,1,1), \tag{60}$$

$$\mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0))\frac{\dot{\kappa}(0)}{2} = \tau(0), \tag{61}$$

which is a system of equations whose solutions are $\dot{\aleph}(0)$, $\dot{\kappa}(0)$, $\dot{\wp}(0)$ and $\dot{\ell}(0)$. From (58) and (60) we obtain

$$\mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0))[\dot{\ell}(0)-\dot{\aleph}(0)] = 0, \tag{62}$$

and from (5) we have that $\mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\ell}(0),0,\dot{\kappa}(0)) \neq 0$ therefore

$$\dot{\ell}(0) = \dot{\aleph}(0). \tag{63}$$

Taking the difference between (59) and (58) we have

$$\mu(\sigma_2(0),1,\dot{\aleph}(0),1,\dot{\wp}(0),1,\dot{\aleph}(0),0,\dot{\kappa}(0))[\dot{\wp}(0)-\dot{\aleph}(0)] = \sigma_2(0). \tag{64}$$

[5] Such a procedure of evaluating the equation at $t = 0$ and obtaining a condition can be adopted if the problem under consideration has sufficiently smooth solutions. We assume such to be the case.

Therefore, in order to find $\dot{\aleph}(0)$, $\dot{\kappa}(0)$ and $\dot{\wp}(0)$ we need in general to solve numerically the nonlinear algebraic system of Eqs. (58), (61) and (64).

In the special case that $\tau(0) = 0$ and $\sigma_2(0) = 0$ from (61) and (64) we have that

$$\dot{\kappa}(0) = 0, \quad \dot{\wp}(0) = \dot{\aleph}(0), \tag{65}$$

and in this case we need to find, for example, $\dot{\aleph}(0)$ from (58), which becomes

$$\lambda + \mu(0,1,\dot{\aleph}(0),1,\dot{\aleph}(0),1,\dot{\aleph}(0),0,0)\dot{\aleph}(0) = \varphi(0,1,1,1). \tag{66}$$

For the special case $\tau(0) = 0$ and $\sigma_2(0) = 0$ from (5) and (66) we obtain $\mu_0[1+2\beta I_2]^n\dot{\chi}(0) = 0$, where $I_2 = \frac{3}{2}\dot{\aleph}(0)^2$, but the above equation is satisfied if $\dot{\aleph}(0) = 0$ and thus we conclude $\dot{\wp}(0) = 0$ and $\dot{\ell}(0) = 0$.

We display some numerical results for (54)–(57) in the case the loads $\sigma_2(t)$ and $\tau(t)$ are given alternatively as in (27) and (28). The values taken for σ_{2_o} and τ_o are $\pm 10^6$ Pa and 10^6 Pa, respectively, and for t_o we assume that $t_o = 10$ s. In these plots the dimensionless time $\bar{t}$ is defined as $\bar{t} = t/t_o$.

In Fig. 1 results are presented for the functions $\aleph(t)$ and $\kappa(t)$ for the two possible values for the constant μ_0 from Table 1, for some combinations of external loads taking into consideration (27) and (28). In Fig. 1(a) and (b) we have results when we consider only the effect of $\sigma_2(t)$ (see (27)) in tension (case (a)) and compression (case (b)).

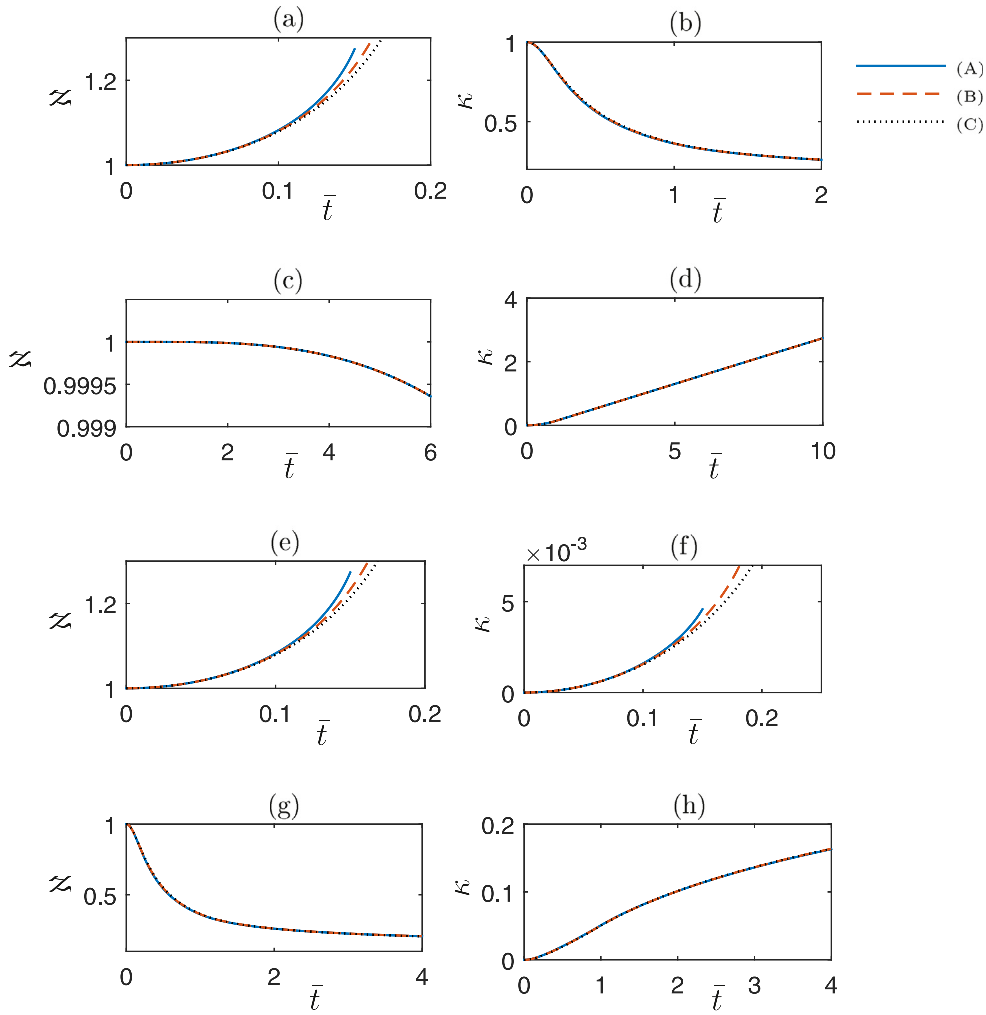

Fig. 5. Comparison of the behaviour of the slab considering three possible values for the constant n in (5), namely (A) $n = -0.5$, (B) $n = 0$, (C) $n = 0.5$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$.

In Figs. 1(c) and (d) we show results for the case $\sigma_2(t) = 0$ and $\tau(t)$ given through (28), in the case of (c) we portray the behaviour of $\aleph(t)$ and in (d) of $\kappa(t)$. In Figs. 1(e), (f) and (g), (h) we present results when both external loads $\sigma_2(t)$ and $\tau(t)$ are present, i.e., when $\sigma_2(t) = \frac{\sigma_{2_o}}{t_o} t[1 - \mathcal{H}(t - t_o)] + \sigma_{2_o}\mathcal{H}(t - t_o)$ and $\tau(t) = \frac{\tau_o}{t_o} t[1 - \mathcal{H}(t - t_o)] + \tau_o\mathcal{H}(t - t_o)$. From these results we can see that for a larger value of μ_0 we have a body that is stiffer. In cases (a) and (b) the shear $\kappa(t)$ was not affected when only normal loads $\sigma_2(t)$ are applied, and so results are not shown for that function. In cases (c) and (d) where we only have shear stress $\tau(t)$, the function $\aleph(t)$ almost does not change in time. In all the cases presented in these plots the behaviour of the functions $\wp(t)$ and $\ell(t)$ are very similar to $\aleph(t)$ and for the sake of brevity such results are not shown here. To obtain the results portrayed in this figure we assumed that $\gamma = 95.2$, $\lambda = 952$, $\beta = 1$, $n = 0$ and $m = 1$.

In Fig. 2 we have displayed the results with regard to the behaviour of the slab when we consider two possible values for β. As in the previous case we show results for $\aleph(t)$ and $\kappa(t)$ for different combinations of external loads $\sigma_2(t)$, $\tau(t)$ (see (27) and (28)). For the results presented in this figure we assumed that $\gamma = 95.2$, $\mu_0 = 6.9 \times 10^7$, $\lambda = 952$, $n = -0.5$ and $m = 1$. From the different plots it is possible to observe that the body becomes less stiffer when β is increased.

In Fig. 3 results are presented for $\aleph(t)$ and $\kappa(t)$ when the constant γ takes six different values as presented in Table 1. From the results presented in the plots (a) and (b) it is possible to see the body becomes less stiff when γ is increased (see also plots (e) and (g)). The shear $\kappa(t)$ is not affected by changes in γ when there is only shear stress $\tau(t)$ on the slab. For the cases when there is combination of normal stress $\sigma_2(t)$ and shear $\tau(t)$, the function $\kappa(t)$ is affected by changes in γ. To obtain the results presented in this figure we assumed that $\mu_0 = 6.9 \times 10^7$, $\lambda = 952$, $\beta = 1$, $n = 0$ and $m = 1$.

In Fig. 4 we portray the results for the deformation of the slab, considering five possible values for the constant λ from Table 1. From these results it is observed that there is not much difference in the behaviour of the slab for the cases where λ is equal to 0, 9.52, 952 and 9.54×10^4, and the only difference is when $\lambda = 9.52 \times 10^6$, where from plots (a) and (b) we note that the body deforms more (see also the plots (e) and (g)). To obtain the results presented in this figure we assumed that $\mu_0 = 6.9 \times 10^7$, $\gamma = 95.2$, $\beta = 1$, $n = 0$ and $m = 1$.

In Fig. 5 results are displayed for $\aleph(t)$ and $\kappa(t)$ for three values for n (see Table 1). From Fig. 5(a) (the slab in tension) we can see that the body becomes softer for decreasing values for n. From Fig. 5(b) it is not possible to see any difference in the behaviour of the slab in compression, and the same happens in the cases (c) and (d) where only shear stresses are applied on the body (something similar is observed in the results presented in Fig. 5(g) and (h)). To obtain the results presented in this figure it was assumed that $\mu_0 = 6.9 \times 10^7$, $\gamma = 95.2$, $\lambda = 952$, $\beta = 1$ and $m = 1$.

In Fig. 6 results are presented for 4 different values for the constant m that appear in (5) (see Table 1). In the plots (a) and (b) results are portrayed for the case of the function $\aleph(t)$ for tension (case (a), compare

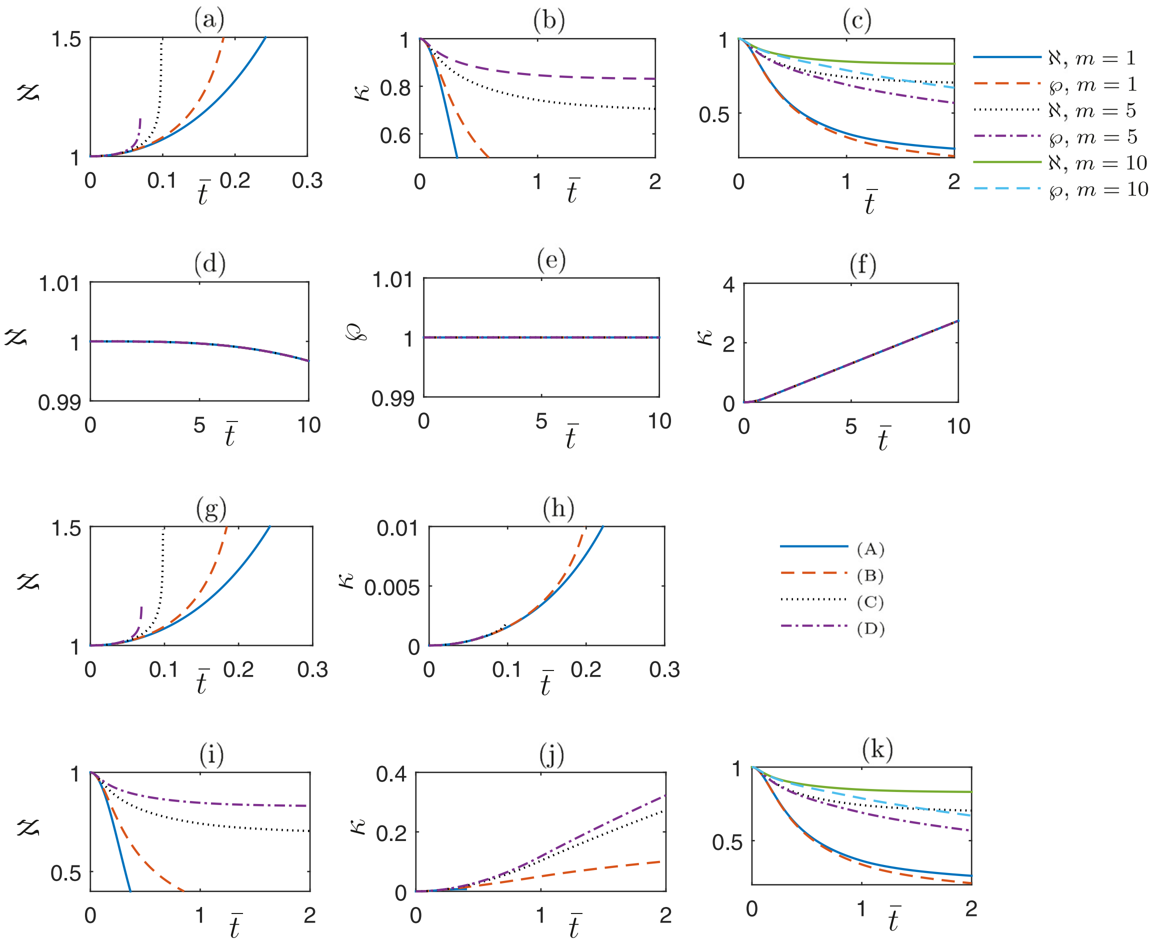

Fig. 6. Comparison of the behaviour of the slab for four possible values for the constant m in (5), namely (A) $m = 0$, (B) $m = 1$, (C) $m = 5$, (D) $m = 10$. (a) Case $\sigma_{2_o} = 10^6$, $\tau_o = 0$. (b) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (c) Case $\sigma_{2_o} = -10^6$, $\tau_o = 0$. (d) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (e) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (f) Case $\sigma_{2_o} = 0$, $\tau_o = 10^6$. (g) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (h) Case $\sigma_{2_o} = 10^6$, $\tau_o = 10^6$. (i) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (j) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. (k) Case $\sigma_{2_o} = -10^6$, $\tau_o = 10^6$. Legend for figures (a), (b), (d), (e), (f), (g), (h), (i) and (j) appears on the right of figure (h). Legend for figures (c) and (k) appears on the upper right side of the figure.

also with the case (g)), and compression (case (b)), where in both the cases considered there are no shear stresses. In Fig. 6(c) $\aleph(t)$ and $\wp(t)$ are displayed when there is only compression of the slab and no applied shear stress. From the plots presented in Fig. 6(d), (e) and (f) it is possible to glean that $\aleph(t)$, $\wp(t)$ and $\kappa(t)$ exhibit similar characteristics for the four different values for m (see also case (h) for $\kappa(t)$ for the slab in tension and shear). From the results presented in (a) (slab in tension) it is possible to see that the body becomes softer for larger values for m, however from (b) where the slab is in compression, the effect is the opposite, i.e., for larger values for m the body becomes stiffer. To obtain the results presented in this figure it was assumed that $\mu_0 = 6.9 \times 10^7$, $\gamma = 95.2$, $\lambda = 952$, $\beta = 1$, $n = 0$.

5. Conclusions

In this study, we have developed a model that is capable of describing the response of a compressible viscoelastic solid whose viscosity is dependent on the mean value of the stress and the shear rate and whose elastic response is also dependent on the pressure. The model is characterized by three material functions, a generalized viscosity $\mu = \mu_0[e^{\delta I_1}(1 + 2\beta I_2)^n]$, a generalized isotropic compressibility function $\varphi = [\gamma I_1 + \lambda](\det \mathbf{F})^m$ and an elastic modulus $\alpha = \lambda$, making a total of seven material constants. Four of the constants (μ_0, δ, β, and n) go towards describing the nature of the viscosity, namely its dependence on the mean value of the stress and the shear rate, and three constants (γ, λ, and m) go towards describing the dependence of the elastic response on the mean value of the stress and compressibility. When $n = m = \delta = \gamma = 0$, and $\lambda = 1$, the constitutive relation reduces to the classical Kelvin–Voigt constitutive relation. We find, not surprisingly, that the body becomes stiffer when μ_0 or β increase, other constants being held constant, as an increase in either μ_0 or β increases the viscosity.

CRediT authorship contribution statement

R. Bustamante: Formal analysis, Funding acquisition, Investigation, Methodology, Writing - original draft, Writing - review & editing. **K.R. Rajagopal:** Conceptualization, Formal analysis, Investigation, Writing - review & editing. **O. Orellana:** Investigation, Methodology, Validation. **R. Meneses:** Investigation, Methodology, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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