

Quantum particles that behave as free classical particles

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The existence of non-vanishing Bohm potentials, in the Madelung–Bohm version of the Schrödinger equation, allows for the construction of particular solutions for states of quantum particles interacting with non-trivial external potentials whose propagation is equivalent to the one for classical free particles.

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I. INTRODUCTION

In 1979, Berry and Balazs [1] showed that a quantum free wavepacket can show unexpected accelerating characteristics. In this work we address the opposite question. That is, if an interacting particle, that satisfies the Schrödinger equation with a potential V , can still behave as if it were a free particle. In this work we establish the conditions which make this behavior possible and present numerous examples.

We prove that such possibility indeed exists in the framework of non-relativistic quantum mechanics and its related to the existence of the so-called Bohm potential. In other words, there are quantum solutions, for families of external potentials V , in which the particle behaves with the same propagation as a free classical particle. This is only possible for a non-vanishing Bohm potential, which in turn implies that amplitude of the wavefunction is not constant. We focus in one-dimensional systems, although our results can be generalized to higher dimensions [2–4], or to relativistic regime [5] following the ideas presented here.

By free classical particle, we understand any particle of mass m satisfying the free Hamilton–Jacobi (HJ) equation

$$\frac{1}{2m}(S')^2 + \dot{S} = 0. \quad (1)$$

for an action $S = S(x, t)$, where $' \equiv \partial_x$, and $\dot{} \equiv \partial_t$. This action may be considered as the phase of a solution to the Schrödinger equation. Therefore, we are looking for wavefunctions with a phase satisfying Eq. (1), and with an amplitude that allow us to solve Schrödinger equation for a given potential V .

Let us consider the wavefunction $\psi = \psi(x, t)$ of a one-dimensional Schrödinger equation (and its complex con-

jugate) for a real potential $V(x, t)$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi - i\hbar\dot{\psi} = 0. \quad (2)$$

The wavefunction may be written in terms of a polar decomposition as $\psi = A \exp(iS/\hbar)$, where the amplitude $A(x, t)$ and the phase $S(x, t)$ are real functions. Thereby, the Schrödinger equations become as [6, 7]

$$\frac{1}{2m}(S')^2 + V_B + V + \dot{S} = 0, \quad (3)$$

$$\frac{1}{m}(A^2 S')' + (A^2)\dot{} = 0. \quad (4)$$

where the Bohm potential is given by

$$V_B \equiv -\frac{\hbar^2}{2m} \frac{A''}{A}. \quad (5)$$

The first equation (3) is the quantum Hamilton–Jacobi (QHJ) equation for the (external) potential V . The quantum modification consists in the addition of the Bohm potential to the classical HJ equation. The second equation (4) is the continuity (probability conservation) equation.

In order to answer if a quantum interacting particle can behave as a free classical particle, we need to enforce the condition that Bohm potential cancels out any contribution of the external potential

$$V_B + V = 0. \quad (6)$$

allowing the phase, from Eq. (3), to fulfill the HJ equation (1). Above condition implies that the external potential determines completely the dynamics of the amplitude A , through the Bohm potential. This also must be consistent with the continuity equation (4).

The continuity equation (4) is identically solved by defining the arbitrary potential function $f = f(x, t)$, such that $A^2 = f'$, and $A^2 S' = -m\dot{f}$. For a one-dimensional system, once the free particle action S is found by solving HJ equation (1), $f = f(x, t)$ can be determined by the relation

$$f' S' + m\dot{f} = 0. \quad (7)$$

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This equation states that f depends on x and t through one variable only. On the other hand, the amplitude of the wavefunction is found to be (with $f' > 0$)

$$A = \sqrt{f'}. \quad (8)$$

The exact form of the amplitude A (or function f) is found by solving Eq. (6). In this way, a quantum particle in the presence of a potential V propagates as a free particle, provided Eq. (6) is solvable.

II. SEPARABLE ACTION FOR FREE CLASSICAL PARTICLE

Let us study the simplest case for a classical free particle, in which the spatial and temporal dependence are separated. The phase (action) is given by

$$S(x, t) = kx - \frac{k^2}{2m}t. \quad (9)$$

This action is a solution of (1) for any constant k . Eq. (7) allows us to find that f depends on x and t through one variable z only. In this case, we obtain that it has the form $f(x, t) = f(z) = f(x - kt/m)$, and thus, by Eq (8), we obtain that the amplitude depends on the same variable z

$$A(x, t) = A(z) = \sqrt{\frac{df}{dz}}, \quad (10)$$

where z is defined as

$$z \equiv x - \frac{k}{m}t. \quad (11)$$

A quantum particle interacting with an external potential $V(x, t) = V(z)$, behaves as a free particle with phase (9), if the amplitude fulfills Eq. (6), in the form

$$V(z) = \frac{\hbar^2}{2mA(z)} \frac{d^2 A(z)}{dz^2}. \quad (12)$$

For this case, all considered external potentials V must depend exactly on z variable, and therefore, they are not static. Several cases are described below.

Constant force. Consider a constant force $F = -V'$, with potential $V(z) = -Fz$. Thus, Eq. (12) produces an amplitude given in terms of Airy functions

$$A(x, t) = \text{Ai} \left(- \left(\frac{2mF}{\hbar^2} \right)^{1/3} z \right). \quad (13)$$

This Airy wavepacket propagates as a free classical particle under a constant force.

Moving potential trap. An attractive potential with the form $V(z) = -\gamma \delta(z)$ is used to manipulate particles [8, 9]. Here γ is a constant, and δ is the Dirac delta function. The amplitude solution of Eq. (12) becomes

$$A(x, t) = \frac{m\gamma\beta}{\hbar^2} z \text{sgn}(z) - \beta, \quad (14)$$

for an arbitrary constant β , and where sgn is the sign function.

Coulomb potential for a moving charge. Let us assume a potential with the form $V(z) = \alpha/z$, for a moving charge with constant non-relativistic velocity (α is a constant). This corresponds to the non-relativistic expression for the Liénard–Wiechert four-potential [10]. In this case, Eq. (12) gives an amplitude in terms of Bessel functions K_1

$$A(x, t) = \frac{\sqrt{2m\alpha z}}{\hbar} K_1 \left(\frac{2\sqrt{2m\alpha z}}{\hbar} \right). \quad (15)$$

Solutions in terms of Bessel functions I_1 are also possible. Thus, this Coulomb potential produces Bessel wavepackets that allows the particles propagate freely.

Electromagnetic wave. A particle interacting with an electromagnetic wave (with wavenumber κ and frequency $\kappa k/m$) experience a potential of the form $V(z) = \gamma \cos(\kappa z)$ (with constant γ). In this case, Eq. (12) becomes a Mathieu equation

$$\frac{d^2 A}{dz^2} - \frac{2m\gamma}{\hbar^2} \cos(\kappa z) A = 0. \quad (16)$$

Explicit solutions are written in terms of the recurrence relations [11, 12]. In this form, Mathieu beam wavepackets support quantum solutions that propagate in a free classical fashion.

Harmonic oscillator. For a shifted harmonic oscillator $V(z) = m\omega^2 z^2/2$ [13], with frequency ω , Eq. (12) has a solution Parabolic cylinder functions [12]

$$A(x, t) = D_{-\frac{1}{2}} \left(\sqrt{\frac{2m\omega}{\hbar}} z \right). \quad (17)$$

Pöschl–Teller potential. Consider the moving potential $V(z) = -\gamma \text{sech}^2 z$, with constant γ . The amplitude solution of (12) is written in terms of a Legendre polynomial P and a Legendre function of the second kind Q as

$$A(x, t) = a_1 P_n(\tanh z) + a_2 Q_n(\tanh z) \quad (18)$$

with arbitrary a_1 and a_2 , and $n = (\sqrt{1 + 8m\gamma/\hbar^2} - 1)/2$.

III. NON-SEPARABLE ACTION FOR FREE CLASSICAL PARTICLE

Another very well-known solution for the classical HJ (1) for classical free particles is

$$S(x, t) = \frac{m x^2}{2t}. \quad (19)$$

This action is a non-separable function of space and time.

In this case, Eq. (7) allow us to find that any function with the functionality $f(x, t) = f(x/t)$ solves the continuity equation. Therefore, amplitude is given by

$$A(x, t) = \frac{1}{\sqrt{t}} \mathcal{A}(y), \quad (20)$$

where $\mathcal{A} = \sqrt{df/dy}$, and where we have introduced the variable

$$y \equiv \frac{x}{t}. \quad (21)$$

In this case, any external potential with the form

$$V(x, t) = \frac{1}{t^2} \mathcal{V}(y), \quad (22)$$

allow us to re-write Eq. (6) as

$$\mathcal{V}(y) = \frac{\hbar^2}{2m\mathcal{A}(y)} \frac{d^2 \mathcal{A}(y)}{dy^2}. \quad (23)$$

Potentials with the exact space and time dependence of form (22), have been shown to produce exact Feynman propagators [14]. Below we study some of them in our context.

Time decreasing force. For a force decreasing in time with the form $F(t) = F_0/t^3$, a potential $\mathcal{V}(y) = -F_0 y$ can be used. In this case, Eq. (23) produces Airy solutions, and amplitude (20) is

$$A(x, t) = \frac{1}{\sqrt{t}} \text{Ai} \left(- \left(\frac{2mF_0}{\hbar^2} \right)^{1/3} y \right). \quad (24)$$

Thus, for such forces, the quantum system is solved exactly, and the particle propagates as it were free.

Harmonic oscillator. For a time-decreasing frequency in the form $\omega = \omega_0/t^2$ [14] (with constant ω_0), then a harmonic oscillator with potential $\mathcal{V}(y) = m\omega_0^2 y^2/2$ can

be solved exactly. Using Eq. (23), amplitudes are given in terms of Parabolic Cylinder functions [12]

$$A(x, t) = \frac{1}{\sqrt{t}} D_{-\frac{1}{2}} \left(\sqrt{\frac{2m\omega_0}{\hbar}} y \right). \quad (25)$$

Coulomb-like potentials. Consider a potential with the form $V(x, t) = Z(t)/x$. When Z decrease in time as $Z(t) = Z_0/t$ [14], then $\mathcal{V} = Z_0/y$, and there exist solutions using our approach. The amplitude of the wavefunction is again given in terms of Bessel functions K_1

$$A(x, t) = \frac{\sqrt{2mZ_0 y}}{\hbar\sqrt{t}} K_1 \left(\frac{2\sqrt{2mZ_0 y}}{\hbar} \right). \quad (26)$$

IV. DISCUSSION

With the above several examples and calculations we have shown that is possible for interacting quantum particles to propagate as a free classical particle for a wide range of known potentials. This is only achieved because the Bohm potential of the wavefunction cancels out the external potential. By doing this, the external potential completely determines the amplitude of the wavepackets, as it can be seen in Eqs. (12) and (23).

It is remarkable that this fact occurs for the large family of potentials treated here. We think that these kind of solutions have remained largely unexplored and they can bring new insights in the propagation of quantum particles, as the quantum characteristics remain confined to the amplitude, while the phase is associated to the action of a free classical particle.

Any solution fulfilling condition (6) can now be interpreted as a wavepacket that modified its own probability density in order to propagate as if it were free. The implications of this behavior are not difficult to be envisaged as very interesting.

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