UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

DISTRIBUTIONAL AND ASYMPTOTIC RESULTS OF CHAIN MAXIMA FROM INDEPENDENT RANDOM VECTORS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

ALEXIS ADRIÁN FUENTES PEZOA

PROFESOR GUÍA:
RAÚL GOUET BAÑARES

MIEMBROS DE LA COMISIÓN:
JOAQUÍN FONTBONA TORRES
F. JAVIER LÓPEZ LORENTE
DANIEL REMENIK ZISIS
GERARDO SANZ SAIZ

Este trabajo ha sido parcialmente financiado por Beca CONICYT-PFCHA/Doctorado Nacional/2013- 21130825 y Beca CMM ANID PIA AFB170001

SANTIAGO DE CHILE
2021

RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE
Doctor en Ciencias de la Ingeniería, Mención Modelación Matemática
POR: Alexis Fuentes Pezoa
FECHA: 21 ENERO 2021
PROFESOR GUÍA: Raúl Gouet

RESULTADOS DISTRIBUCIONALES Y ASINTOTICOS DE CHAIN MAXIMA PARA VECTORES ALEATORIOS INDEPENDIENTES

Esta tesis tiene como objetivo principal el estudio de la sucesión $(\mathcal{M}_n)$, de los llamados "chain-maxima" (máximos en cadena), bajo la hipótesis de observaciones independientes e idénticamente distribuidas (iid), con valores en $\mathbb{R}^{\mathrm{d}}$. Se trata de un nuevo tipo de máximo multidimensional, que se define recursivamente, a partir de un orden parcial en $\mathbb{R}^{\mathrm{d}}$. Los chain-maxima son usados como base para definir otros procesos como los chain-records $(\mathcal{R}_n)$, los cuales han sido previamente estudiados en [25] y [41].

Este trabajo se divide en tres partes. En la primera se dan las definiciones de los procesos de chain-maxima, chain-records y variables asociadas, para los cuales se obtienen resultados estructurales y asintóticos, en el marco general de datos iid. En la segunda y tercera parte se estudian, respectivamente, dos modelos particulares, a saber el de observaciones iid uniformes en $[0,1]^{\mathrm{d}}$ y el de observaciones iid uniformes en el simplex d-dimensional Δ^{d}. En el Capítulo 1 se presentan brevemente temas clásicos de récords unidimensionales así como una introducción a los récords multidimensionales, seguida de una discusión de la bibliografía. En el Capítulo 2 se dan las definiciones básicas y un estudio general de propiedades de los chain-maxima y chain-records, bajo supuestos probabilísticos razonables. Se exponen propiedades de la estructura Markoviana y resultados asintóticos, que pueden ser vistos como extensiones naturales aunque no triviales, de resultados unidimensionales similares, obtenidos en [40, 52, 53]. Asimismo, se presentan y se exploran brevemente las ideas de chain-maxima y chain-records asociados a órdenes estrictos y órdenes de conos. En el Capítulo 3 se estudia el modelo con datos iid uniformes en $[0,1]^{\mathrm{d}}$ y, entre otros, se derivan resultados para el proceso $(\mathcal{R}_n)$: independencia de sus componentes, probabilidades de transición, densidades marginales, representación como solución de una equación de diferencias, etc. Asimismo, se realiza un estudio exacto y asintótico de los "record-heights" $\mathcal{H}_n$ (probabilidad condicional de un chain-record). Mediante el uso de técnicas de análisis de polos, de Flajolet y Sedgewick [22], se obtiene una descripción fina de los momentos $\mathbb{E}(\mathcal{H}_n^k)$, así como de los momentos cruzados. Finalmente, con dichos resultados se establecen convergencias para los record-heights y para el proceso de conteo de chain-records. En el Capítulo 4 se considera el modelo uniforme en el d-simplex Δ^{d} y se estudian resultados similares a los del Capítulo 3, teniendo presente la dificultad adicional que implica la dependencia entre las componentes de las observaciones. En particular, el análisis de singularidades presenta una complejidad muy superior al del caso $[0,1]^{\mathrm{d}}$. Además se destaca el estudio asintótico de $(\mathcal{R}_n)$, mediante una representación como perpetuidad y la caracterización de la ley límite como probabilidad estacionaria de una cadena de Markov. Finalmente el Capítulo 5 trata temas en desarrollo, con avances parciales, especialmente enfocados en la convergencia en distribución de $(\mathcal{M}_n)$, tema de inesperada complejidad. Este problema se ha relacionado con los números de Delannoy, y con un modelo de tiempo continuo, denominado "board-breaking".

RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE
Doctor en Ciencias de la Ingeniería, Mención Modelación Matemática
POR: Alexis Fuentes Pezoa
FECHA: 21 ENERO 2021
PROFESOR GUÍA: Raúl Gouet

DISTRIBUTIONAL AND ASYMPTOTIC RESULTS OF CHAIN MAXIMA FROM INDEPENDENT RANDOM VECTORS

The main goal of this thesis is the study of the sequence $(\mathcal{M}_n)$, the so-called "chain-maxima", under the assumption of independent and identically distributed, $\mathbb{R}^{\mathrm{d}}$-valued observations. It is a new kind of multivariate maximum, which is defined recursively from a partial order on $\mathbb{R}^{\mathrm{d}}$. Chain-maxima are used as a base process for defining chain-records, which have been studied previously in [25] and [41].

This work is divided into three main parts. In the first, definitions of chain-maxima, chain-records and related variables are given. Structural and asymptotic results for these processes are obtained in the general framework of iid observations. In the second and third parts, two particular models are studied, namely iid observations uniformly distributed on en $[0,1]^{\mathrm{d}}$ and iid uniform observations on the d-dimensional simplex Δ^{d}, respectively. In Chapter 1 a few classical results of record theory are reviewed, followed by some elements of the multidimensional case and a bibliographical discussion. In Chapter 2 the definitions and a general study of chain-maxima and chain-records are presented, under reasonable probabilistic assumptions. Their Markovian structure is analyzed and several asymptotic results are given. Such results can be seen as natural, though nontrivial, extensions of one-dimensional results, first published in [40, 52, 53]. Moreover, the concepts of chain-maxima and chain-records, related to strict orders or cone-orders, are defined and briefly explored. Chapter 3 is devoted to study chain-maxima and chain-records of iid observations, uniformly distributed on $[0,1]^{\mathrm{d}}$. Results for $(\mathcal{R}_n)$ include, among others, the independence of the components, the derivation of transition functions and marginal densities, a distributional representation by means of a difference equation, etc. Furthermore, a detailed analysis of "record-heights" $\mathcal{H}_n$ (conditional probability of chain-records) is carried out. Using tools of singularity analysis, developed by Flajolet and Sedgewick [22], a precise asymptotic description of moments $\mathbb{E}(\mathcal{H}_n^k)$ is given, and the same is done with mixed moments. From these results, convergence theorems for record heights and for the counting process of records are established. The model with uniform iid observations in the d-simplex is presented in Chapter 4. Results similar to those of Chapter 3 are obtained, under the extra technical difficulty implied by the dependence of the components of observations. In particular, the singularity analysis becomes significantly more complicated than in $[0,1]^{\mathrm{d}}$. Also, the chapter contains an asymptotic study of $(\mathcal{R}_n)$ using a perpetuity representation, which allows to characterize the distribution of $\lim \mathcal{R}_n$ as a stationary probability of a Markov chain. Finally, Chapter 5 is devoted to topics in progress, for future research, especially focused on the weak convergence of $(\mathcal{M}_n)$, which turned out to be a problem of unexpected complexity. This problem has been connected with objects such as Delannoy's numbers and a continuous-time model of fragmentation, named "board-breaking" model.

This thesis is dedicated to my mother, Patricia Pezoa, and especially to the memory of my maternal grandparents Elvia Román and Gustavo Pezoa. Also, I dedicate this work to our lovely pet Daysy, her left recently us in a really sadly and painful way. I am profoundly sorry that all them not with us any longer.

Acknowledgments

I deeply thank my advisor, Professor Raúl Gouet, for his invaluable support and patience, especially during a difficult period in the middle of my doctorate studies. Professor Raúl is not only a clever mathematician and academic, but also a great human being. His help and advice were essential to carry out my work and take it to a good end. I am much indebted to him.

I also thank Professors Javier López and Gerardo Sanz, from the University of Zaragoza, for their encouragement and support, so important in my work, and for accepting to be members of my doctorate committee. I hope to collaborate with them in the future. I thank as well Professors Joaquin Fontbona and Daniel Remenik, from the Departamento de Ingeniería Matemática, Universidad de Chile (DIM) for their time and for kindly accepting to be members of my doctorate committee.

I am very grateful for the crucial financial support provided by CONICYT, through the PFCHA/Doctorado Nacional/2013- 21130825 grant and by the Center of Mathematical Modeling CMM, through PROYECTO BASAL PFB03.

I want to thank my maternal grandparents, Elvia and Gustavo, for giving me the basic principles, the real sense of family unity and for teaching me to live honestly without sacrificing moral values. To my mother Patricia, who showed me that the effort manages to overcome any adversity.

I thank my alma mater, our dear Engineering School, for its training and for providing life tools. I thank the DIM for the rigor, for the knowledge and for giving me the best years of my life. To my inspiring professors who left a mark on me as a student: Martín Matamala, Manuel del Pino, Patricio Felmer, Servet Martínez, Alejandro Maass and Jaime San Martín. The students of this department are really lucky to be trained and encouraged by academics of such quality and prestige, a fact that distinguishes the DIM in our country and abroad. I am also grateful to the DIM's administrative staff, for helping me in one way or another: Karen, Natacha, Eterin, Georgina, Oscar, Luis, Carlos, Gladys, María Cecilia, Silvia and Maria Inés.

Finally to my childhood friends, for the good times, and to friends of life for helping me in many ways. To my friends during the university years: Alejandra, Cinthia, Daniela, Martita, Llarela, Viviana, Mónica, Joce, Diana and especially Yenisei, whom I hope to continue seeing. Also to Andrés, José Luis, Rodrigo, Ignacio, Erwin, Víctor, Carlos, Pablo Z. Finally, to my soccer partners: César, Miguel, Pablo, Víctor, Cristóbal and Luis Alejandro.

Contents

List of Figures

Glossary of Notations and Abbreviations

Operations and Symbols

$:=$	equal by definition
$\stackrel{\mathcal{D}}{=}$	equal in distribution
$\|\cdot\|$	euclidean norm
$\lfloor\cdot\rfloor$	greatest integer less than
$x \to \bar{x}$	x converges to $\bar{x}$
$\nearrow$	increasing sequence
$\searrow$	decreasing sequence
$a.s.$	almost sure
$\xrightarrow{\mathcal{D}}$	weak convergence or convergence in distribution
$\xrightarrow{\mathbb{P}}$	convergence in probability
$\xrightarrow{L_2}$	convergence in L_2
$\mathbb{P}(\cdot)$	probability measure
$\mathbb{E}(\cdot)$	expectation
$\mathrm{Var}(\cdot)$	variance
$\mathrm{Cov}(\cdot)$	covariance
$\mathrm{Res}_{x_0} f$	residue of f in x_0
$\liminf$	lower limit
$\limsup$	upper limit
$\sim$	asymptotic equivalence
$O(\cdot), o(\cdot)$	Standard O, o notation
H_n	n-th harmonic number
$N(0,1)$	standard normal distribution

Sets and Spaces

$\emptyset$	empty set
$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^{\mathrm{d}}$	d-dimensional real space
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers
Ω	sample space
$\mathcal{F}$	sigma algebra of events
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$L^p(a, b; \mathbb{R})$	real valued Lebesgue p-integrable functions over $[a, b]$

Functions and abbreviations

$1_S(\cdot)$	indicator function of S
$\mathrm{d}_S(\cdot,\cdot)$ or $\mathrm{d}(\cdot;S)$	distance function
a.c.	absolutely continuous
a.s.	almost surely
df	distribution function
iid	independent and identically distributed
i.o.	infinitely often
mgf	moment generating function
pdf	probability density function
rhs	right-hand side
rv	random variable
sde	stochastic difference equation
CLT	central limit theorem
DCT	dominated convergence theorem
EVT	extreme value theory
LLN	law of large numbers
MCT	monotone convergence theorem
SLLN	strong law of large numbers
WLLN	weak law of large numbers

Mappings

$f: X \to Y$	Single-valued map from X to Y
$\mathrm{dom}A$	domain of operator A
A^{-1}	Inverse of an operator A

Chapter 1

Introduction

A record, as it is understood in Extreme Value Theory (EVT), is an extraordinary value of a variable, which surpasses all of its kind. Probably everyone has heard and become interested on records, in some context. For instance, in sports a great challenge for the elite athletes is to break the current best mark. In finance there are many phenomena of this type as well, such as record values of stocks or commodities. In climatology, especially when related to global warming, record temperatures or record rainfall are of great public interest.

Records are rare events and we can formulate, from a stochastic modeling point of view, some questions such as how often they appear or how long do we have to wait until the next record and what its value will be, etc. From a statistical perspective, we can wonder if a given model fits the athletic or climatological record data or if it can be used to predict future records and prevent disasters.

The mathematical theory of records, based on the assumption of real-valued, independent and identically distributed (iid) observations, is very well developed. There is an abundant literature on distributional and asymptotic aspects of records and related objects, as can be seen, for example, in [1], [4], [44]. More recent results on records from discrete or general discontinuous distributions, can be found in [30] and [32].

Records are part of EVT, which deals with the behavior of extremes (maxima, minima, records) of random processes. This theory has become increasingly popular in recent years, due to many applications in fields such as finance and insurance; see, for example, [20]. EVT has been developed mainly under the assumption of iid observations and its most famous result is, by far, the so called Gnedenko's trinity theorem. This result is a kind of Central Limit Theorem for the sequence of partial maxima where the Gaussian limit is replaced by three possible limit laws, dependent of the distribution tails of the observations. See [14] and [49].

The classical theory of extremes, for real-valued, iid random variables, has been greatly generalized and is now well established for stationary mixing sequences and other processes. Also, the multivariate theory of extremes for iid random vectors, began to emerge in the late 70's and is also fairly complete in terms of the possible limit laws and domains of attraction.

On the other hand, multivariate (componentwise) records did not have a parallel development because, unfortunately for asymptotics, there are, almost surely, a finite number of records in the whole sequence of iid (vector-valued) observations. However, see [16] and [28] for an interesting result about multivariate records approaching a curve, conditional on there being many records.

Other definitions of multivariate records, which are natural and amenable to asymptotic analysis, have been proposed in the literature. For instance, chain-records and Pareto-records, studied in some detail in [25] and [41]. This thesis is devoted to the analysis of chain-records and related processes.

1.1 One-dimensional records

Let (X_n) be a sequence of real-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By convention we declare X_1 to be a record and, for $n \geq 2$, X_n is said to be a (upper) record if it is greater than all previous observations, that is, if $X_n > M_{n-1}$ (or $M_n > M_{n-1}$), where $M_n = \max\{X_1, \ldots, X_n\}$. Clearly, records correspond to observations where the sequence of partial maxima (M_n) jumps.

The number of records, say N_n, among the first n observations can be written in terms of record indicators as $N_n = \sum_{\text{i}=1}^n I_{\text{i}}$, where $I_1 = 1$ and $I_n = 1_{\{X_n > M_{n-1}\}}$, for $n \geq 2$. Record times L_n are the jump times of the sequence (M_n). That is, $L_1 = 1$ by convention and $L_{n+1} := \min\{k > L_n : X_k > X_{L_n}\}$, for $n \geq 1$. Observe that N_n is equal to the number of elements of the set $\{k \geq 1 : L_k \leq n\}$.

Finally, record values R_n are defined as the values of partial maxima at jump times, that is, $R_n = X_{L_n} = M_{L_n}, n \geq 1$. The theory of records has focused on studying the sequences defined above. We recall below a few well-known results about one-dimensional records; the interested reader can consult [4] and references therein, for proofs and additional information.

If the observations X_n are iid, with common continuous underlying distribution F, then it is clear that record times L_n and indicators I_n are distribution free. For instance, a famous result discovered by A. Renyi states that indicators I_n are independent Bernoulli variables, with $\mathbb{E}(I_n) = 1/n$ and so, one can immediately see that $\mathbb{E}(N_n) = \log n + O(1)$ and that $Var(N_n) = \log n + O(1)$. Furthermore, the probability generating function of N_n is given by

$$\mathbb{E}(s^{N_n}) = \prod_{j=1}^{n} \Big(\frac{s-1}{j} + 1\Big) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{s^k}{n!},$$

where the $\left[{n \atop k}\right]$'s are the Stirling numbers of the first kind. From the above it follows that

$$\mathbb{P}(N_n = k) = \frac{1}{n!} \begin{bmatrix} n \\ k \end{bmatrix},$$

for $k \geq 1, n \geq 1$. Asymptotic results for N_n also follow at once from Renyi's result. One can easily prove a strong law of large number and asymptotic normality. Namely, $N_n / \log n \to 1$

almost surely and $(N_n - \log n)/\sqrt{\log n}$ converges weakly to the standard Gaussian distribution.

The sequence (L_n) of record times L_n (which are closely related to record counts N_n because $N_n < k \Leftrightarrow L_k > n$) is easily seen to be a Markov chain. More precisely, $\mathbb{P}(L_2 = k) = \frac{1}{k(k-1)}$, for $k \geq 2$, and

$$\mathbb{P}(L_n = k | L_{n-1} = j) = \frac{j}{k(k-1)},$$

for $k > j \geq n-1 \geq 2$. The formula above reveals that the waiting times for records can be very long, since $\mathbb{E}(L_2) = \infty$. Also, it can be shown that the random variables $K_n := \lceil L_n / L_{n-1} \rceil, n \geq 2$, are iid with $\mathbb{P}(K_n = k) = \frac{1}{k(k-1)}, k \geq 2$. This fact gives additional information on how fast record times grow because, from the Borel-Cantelli lemma, we have that $\mathbb{P}(K_n > n \text{ i.o.}) = 1$ but $\mathbb{P}(K_n > n(\log n)^2 \text{ i.o.}) = 0$.

Concerning the record values it is worth mentioning that the sequence (R_n) behaves as the arrival times of a non-homogeneous Poisson process on the line, with intensity measure given by $H(\mathrm{d}x) = F(\mathrm{d}x)/(1 - F(x))$; see [53]. This implies that (R_n) is a Markov chain with transition measure

$$\mathbb{P}(R_n \in A | R_{n-1} = x) = H(A) = \int_A \frac{F(\mathrm{d}x)}{1 - F(x)}.$$

In particular, for exponential observations (X_n), we get a homogeneous Poisson process and this means that R_n is distributed a sum of iid exponential random variables. This representation yields asymptotic results for R_n at once.

In all results above, the continuity of F is crucial. If F is allowed to have discontinuities, the situation gets considerably more complicated. For example, record indicators are no longer independent and their expectations depend on F. This means that asymptotic results for N_n are not easily obtained and more sophisticated tools are required. Vervaat [57] was the first to obtain asymptotic normality for N_n, assuming iid geometrically distributed observations; see also [5]. Later, martingale techniques were used in [29] and [30] to obtain strong and weak convergence results for N_n, assuming iid observations, distributed according to a variety of discrete models.

With respect to record values, the situation also changes significantly when F is discontinuous. In the general case, the sequence (R_n) is characterized as superposition of two independent point processes: a non-homogeneous Poisson process (the continuous component) and a Bernoulli process (the discrete component) on the atoms of F, with probabilities given by the so-called hazard rates. This process is known in the literature as Shorrock's process and was used in [32] to obtain asymptotic normality for N_n, when the iid observations have a general (possibly discontinuous) distribution F.

Finally, we can consider studying records from dependent and/or non-identically distributed observations. It could be expected, for example, that in the case of stationary mixing case, the fairly complete asymptotic theory for normalized maxima had a corresponding theory for records, but this is not the case. For independent, non-identically distributed observations, some interesting asymptotic results are related to the so-called Nevzorov's power model and observations with trend. For recent developments in this topic, see [34].

1.2 Multidimensional record

Assume that observations are random elements $\mathbf{X}_n = (X_n^{(1)}, \ldots, X_n^{(\mathrm{d})})$ taking values in $\mathbb{R}^{\mathrm{d}}, \mathrm{d} \geq 2$. The classical multivariate EVT is based on the definition of component-wise maxima $\mathbf{M}_n, n \geq 1$, where $\mathbf{M}_n = (M_n^{(1)}, \ldots, M_n^{(\mathrm{d})})$, with $M_n^{(j)} = \max\{X_1^{(j)}, \ldots, X_n^{(j)}\}$, for $j = 1, \ldots, \mathrm{d}$. The asymptotic theory of multivariate extremes is connected with interesting concepts, such as max-infinite divisibility and max-stable distributions; for more information, see [49].

Although the component wise definition of maxima seems natural and is useful in applications, a corresponding (natural) notion of multivariate record is not obvious. There are several plausible definitions to consider, some of them being more amenable to asymptotic study than others. The first and most common is that of strong record, defined as follows: $\mathbf{X}_n$ is a strong record observation if $X_n^{(j)} \geq M_{n-1}^{(j)}$, for $1 \leq j \leq \mathrm{d}$, with at least one strict inequality. This definition can be written more compactly in terms of the dominance relation, as follows: let $\mathbf{x}_{\mathrm{i}} = (x_{\mathrm{i}}^{(1)}, \ldots, x_{\mathrm{i}}^{(\mathrm{d})}) \in \mathbb{R}^{\mathrm{d}}, \mathrm{i} = 1, 2$, then $\mathbf{x}_1 \succ \mathbf{x}_2$ if $\mathbf{x}_1 \neq \mathbf{x}_2$ and $x_1^{(j)} \geq x_2^{(j)}, j = 1, \ldots, \mathrm{d}$. So, $\mathbf{X}_n$ is a strong record if $\mathbf{X}_n \succ \mathbf{M}_{n-1}$.

The problem with strong records is their scarceness. For example, if the observations are iid vectors with independent components, their total number is finite. In general, the number of records in a region of $\mathbb{R}^{\mathrm{d}}$ depends on the hazard measure of the region; see [27].

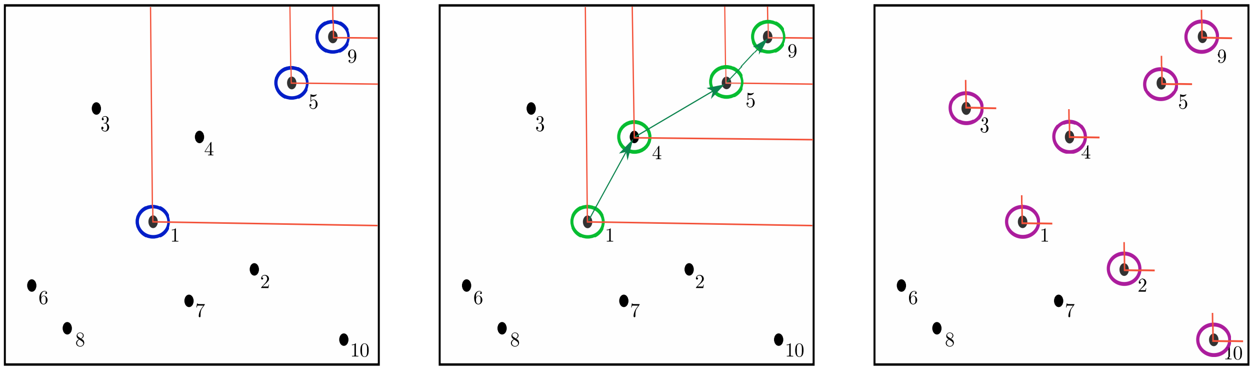
The concept of chain-record was introduced by Gnedin in [25], as a natural and tractable alternative to strong records. It turns out that their behavior is close to that of one-dimensional records. An observation $\mathbf{X}_n$ is a chain-record if it dominates, in the sense of $\succ$, the current chain-record. As in other types of record, the first observation $\mathbf{X}_1$ is conventionally a chain-record; rigorous definitions are given in Chapter 2.

For completeness, we mention here the interesting notion of Pareto record, which we do not investigate in this thesis. An observation $\mathbf{X}_n$ is a Pareto record if $\mathbf{X}_{\mathrm{i}} \not\succ \mathbf{X}_n$, for $\mathrm{i} = 1, \ldots, n-1$. In other words, a Pareto record is not dominated by any of the previous observations.

It is easy to see that Pareto records are more abundant than chain records, which in turn are more abundant than strong records. See Figure 1.1 for an illustrative example.

1.3 Review of the literature on multidimensional records

In the late 80's Goldie and Resnick [27] develop a theory of records for iid sequences of random elements, with values in a partially ordered set. In particular, they analyze several notions of multivariate records and they show, among other results, that the total number of records in a region A is finite if and only if its hazard measure $H(A)$ is finite. The same authors [28] focus in iid $\mathbb{R}^2$-valued random vectors, and study the behavior of record sequences in a fixed rectangle A, conditionally on there being n records in A, as $n \to \infty$.



(a) Strong records. (b) Chain records. (c) Pareto records.

Figure 1.1: Strong, chain and weak records from ten point in the unit square. Records are highlighted using colored circles.

They show, under suitable conditions, that the random set of record points converges in probability to a deterministic parameterized curve, which solves a variational problem. See also [16], for further results and a connection of this phenomenon with the problem of the longest increasing subsequence.

Later, Gnedin [24] studies the asymptotic behavior of the probability that $\mathbf{X}_n$ is a strong record, for iid multivariate Gaussian observations, showing that their total number is finite, unless the correlations between components are $+1$.

In [25] Gnedin introduces chain records and studies mainly the associated counting process. Using random partitions techniques, he shows that the number of chain records up to time n is asymptotically Gaussian, as $n \to \infty$. He also establishes a strong law of large numbers.

Pareto and chain records are extensively studied in [41]. The authors derive expressions for the mean and variance of the number of records, for iid observations uniformly distributed in the hypercube $[0,1]^{\mathrm{d}}$ and the d-simplex, from which they obtain the uniform convergence to the Gaussian distribution.

The most recent contributions in the domain of strong and Pareto records, up to the author's knowledge, can be found in [10], [18], [21]. In particular, the distribution of the total (finite) number of records and the distribution of the last record, from iid observation with independent components, is investigated in [21]. Another type of multivariate record, known as north-east records, is studied in [7].

1.4 Outline of the thesis

Chapter 1 contains a brief introduction to the mathematical theory of record observations. Chapter 2 is devoted to the general theory of chain-maxima ($\mathcal{M}_n$) and chain-records ($\mathcal{R}_n$). The dominance relation is defined, followed by the introduction of the new concept of chain-maxima, in Definition 2.1.2. The study of chain-maxima and related processes, is the main goal of this thesis. Chain-records are then defined from $(\mathcal{M}_n)$ in a (equivalent) way that differs from Gnedin's original definition but we believe more natural. Chain-record times are also defined and they are shown to be stopping times for the natural filtration of the data. Then, the Markovian property of chain-maxima, of inter-record times and of chain-records are established. It is also shown that, as in the classical one-dimensional case, inter-record times are conditionally distributed as geometric random variables. A useful formula for the expectation $\mathbb{E}(g(\mathcal{M}_n))$ is given, which allows among other applications, to obtain a recursion for the distribution function $\mathbf{F}_n(\mathbf{x})$ of $\mathcal{M}_n$. The idea of terminal atom of the distribution of $\mathbf{X}$ (a generic observation) is introduced, in order to initiate the study of the counting process $(\mathcal{N}_n)$ of chain-records. Conditions are given to ensure that the total number of chain-records, from an iid sequence of random vectors, is infinite, almost surely. Once the question of infiniteness of chain-records is settled, the asymptotic study is initiated. A series of limiting results are exhibited and a martingale is defined to derive a strong law of large numbers for $\mathcal{N}_n$. A result is proven about the logarithmic closeness of inter-record times Δ_n and $\bar{\mathbf{F}}(\mathcal{R}_n)$. This allows to generalize some well-known results that Holmes and Strawderman [40] obtained for one-dimensional records. The chapter also contains results about the point process of chain-record values, the particular but important case of observations with independent components, the definition of cone-chain-records and strict chain-records. Some of these topics are only briefly studied and a good deal remains to be done.

The uniform model on the hypercube $[0,1]^{\mathrm{d}}$ is studied en Chapter 3. Some results of Chapter 1 are restated for this particular model. Record heights $\mathcal{H}_n$ are introduced and studied and a detailed analysis of moments is carried out. First, we prove that moments solve a recursion and that the general form of such solution is a so-called alternating sum, also known as Euler transform. Then we use some very nice tools from complex analysis, such as the Nörlund-Rice representation and the singularity analysis of Flajolet and Sedgewick. The asymptotic study turns out to be laborious, because of the technical necessity of constructing a sequence of growing circles, avoiding the poles of a complex function, which is the meromorphic extension of the coefficients in the alternating sum related to the moments. From the computation of residues, we are able to give very precises convergence rates for the moments and, as a by-product, to obtain weak convergence for record heights. Mixed moments are also studied using the same tools and results are applied to the convergence of the counting process $(\mathcal{N}_n)$.

Chapter 4 is devoted to the uniform model on the d-simplex Δ^{d} (nonnegative coordinates adding to less than or equal to 1), i.e. the observations $\mathbf{X}_n$ are iid uniform on Δ^{d} and this implies that components of $\mathbf{X}_n$ are dependent random variables, unlike the case of $[0,1]^{\mathrm{d}}$. The Markovian nature of chain-records is explored and recursions are obtained for the distribution and the density of $\mathcal{R}_n$. Then, a representation in distribution for $(\mathcal{R}_n)$, as a stochastic recurrence is obtained. The solution of the recurrence is found and convergence of $\mathcal{R}_n$ to a limit $\mathcal{R}_\infty$, is established. Then, in order to characterize the distribution of $\mathcal{R}_\infty$, we obtain a perpetuity representation of $\mathcal{R}_\infty$. From the above analysis and using a result of Grincevičius [37], we conclude that the distribution of $\mathcal{R}_\infty$ is absolutely continuous or singular and continuous; it can't be neither discrete nor degenerate. Finally, for $\mathrm{d} = 2$ we are able to prove that $\mathcal{R}_\infty$ has a Dirichlet distribution, by showing that the law of $\mathcal{R}_\infty$ is the invariant distribution of a certain Markov chain. The study of record-heights is carried out following the same strategy of the previous chapter but the calculations get very messy. Nonetheless, the analysis of residues leads to precise rates for the moments and, as a by-product, to weak convergence for record heights and the counting process of chain-records. There remain many unanswered questions about this model and its variants, for future research.

Chapter 5 is about work in progress. It contains many more questions than answers. It was decided to include this chapter in the thesis because it reflects many hours of discussion and calculations about ideas that could prove interesting in the future. A serious effort was made to obtain a limiting distributions for $\mathcal{M}_n$, similar to those existing in classical EVT, for usual maxima. We fell short of our goal but managed to obtain partial answers revealing an intriguing behavior. When considering the recurrence satisfied by the density of $\mathcal{M}_n$, we found a structure of certain coefficients that was close to Delannoy's numbers. We also dedicated some effort to the analysis of a continuous-time fragmentation model, inspired from a work by Brennan and Durrett. The model was called the board-breaking model and can be considered a continuous time analog of chain-maxima, mimicking the one-dimensional stick-breaking process of Brennan and Durrett. We were interested in the asymptotic behavior of a tagged fragment (rectangle), whose dimensions (and consequently its area) obviously tend to zero. Knowing that the area can be normalized so that a limiting distribution exists, we asked if it is possible to find limiting distributions for the sides. It turns out that the solution of this problem is linked to the convergence of two dependent renewal processes. The problem is also linked to semi-stable Markov processes.

Chapter 2

General theory of chain-maxima and chain-records

In this chapter, we introduce the concepts of chain-maxima and chain-records. We formulate the probabilistic hypotheses to be used throughout this work and obtain distributional and asymptotic results.

2.1 Preliminaries

We introduce below the dominance relations between vectors in $\mathbb{R}^{\mathrm{d}}$, which allow to define different notions of record more compactly.

Definition 2.1.1 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathrm{d}}$, with respective components $x^{(j)}, y^{(j)}, j = 1, \ldots, \mathrm{d}$.

1. $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{y} \prec \mathbf{x}$ if $\mathbf{x} \neq \mathbf{y}$ and $x^{(j)} \geq y^{(j)}$, for $j = 1, \ldots, \mathrm{d}$ ($\mathbf{x}$ *dominates* $\mathbf{y}$).
2. $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \preceq \mathbf{x}$ if $x^{(j)} \geq y^{(j)}$, for $j = 1, \ldots, \mathrm{d}$.
3. $\mathbf{x} \succ_s \mathbf{y}$ or $\mathbf{y} \prec_s \mathbf{x}$ if $x^{(j)} > y^{(j)}$, for $j = 1, \ldots, \mathrm{d}$ ($\mathbf{x}$ *strictly dominates* $\mathbf{y}$).

Let $(\mathbf{X}_n)$ be a sequence of iid $\mathbb{R}^{\mathrm{d}}$-valued random vectors, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathrm{d} \geq 1$. We begin by defining the sequence $(\mathcal{M}_n)$ of chain-maxima, using the dominance relation of Definition 2.1.1. The notation 1_A stands for the indicator function of event A.

Definition 2.1.2 The sequence $(\mathcal{M}_n)$ of chain-maxima associated to $(\mathbf{X}_n)$ is defined by $\mathcal{M}_1 = \mathbf{X}_1$ and

$$\mathcal{M}_n = \mathbf{X}_n 1_{\{\mathbf{X}_n \succ \mathcal{M}_{n-1}\}} + \mathcal{M}_{n-1} 1_{\{\mathbf{X}_n \not\succ \mathcal{M}_{n-1}\}},\ n \geq 2. \tag{2.1.1}$$

Remark 2.1.1 Note that chain-maxima $(\mathcal{M}_n)$ bear some resemblance with componentwise maxima $(\mathbf{M}_n)$, such as being monotone, that is, $\mathcal{M}_{n-1} \preceq \mathcal{M}_n$, for $n \geq 2$. A major difference is that chain-maxima are observed vectors, in the sense that $\{\mathcal{M}_n : n \geq 1\} \subset \{\mathbf{X}_n : n \geq 1\}$, while componentwise maxima are not.

It is also clear that, while $M_n^{(j)}$ depends only on $X_1^{(j)}, \ldots, X_n^{(j)}$, $\mathcal{M}_n^{(j)}$ (the j-th component of $\mathcal{M}_n$) depends on all coordinates of $\mathbf{X}_1, \ldots, \mathbf{X}_n$, not just the j-th, a fact which complicates the analysis of chain-maxima. Another distinctive feature is their sensitivity to the labeling of observations. That is, $\mathcal{M}_n$ is not permutation invariant because it depends on the order in which the observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are considered ($\mathrm{d} > 1$).

Observe also that for $\mathrm{d} = 1$, $\mathcal{M}_n = M_n = \max\{X_1, \ldots X_n\}$. In fact, equation (2.1.1) is a d-dimensional extension of the well-known formula $M_n = X_n 1_{\{X_n > M_{n-1}\}} + M_{n-1} 1_{\{X_n \not> M_{n-1}\}}$, for real random variables X_n. Of course, relation (2.1.1) does not hold for componentwise maxima ($\mathbf{M}_n$).

Chain-records are defined below using the concept of chain-maxima. We believe that this definition is simpler and more natural than Gnedin's [25], which requires the previous introduction of record times. For completeness and comparison, we also define strong records.

Definition 2.1.3 By convention $\mathbf{X}_1$ is a chain-record and, for $n \geq 2$, $\mathbf{X}_n$ is a chain-record if $\mathcal{M}_n \succ \mathcal{M}_{n-1}$.

Definition 2.1.4 By convention $\mathbf{X}_1$ is a strong record and, for $n \geq 2$, $\mathbf{X}_n$ is a strong record if $\mathbf{X}_n \succ_s \mathbf{M}_{n-1}$ or, equivalently, if $\mathbf{X}_n \succ_s \mathbf{X}_\mathrm{i}$, for $\mathrm{i} = 1, \ldots, n-1$.

Associated to chain records, we define below the sequences of chain-record times, chain-inter-record times, chain- record-values, etc. For simplicity, since we only deal with chain-extremes in this and subsequent chapters, we drop the prefix "chain" from these objects. So, chain-records become records, etc.

Definition 2.1.5 (i) Record times $(\mathcal{T}_n)$ are defined by $\mathcal{T}_1 = 1$ and

$$\mathcal{T}_n = \min\{j > \mathcal{T}_{n-1} : \mathcal{M}_j \succ \mathcal{M}_{j-1}\}, \ n \geq 2, \tag{2.1.2}$$

where $\mathcal{T}_m = \infty$, for all $m \geq n$, if $\mathcal{M}_j \not\succ \mathcal{M}_{j-1}$, for $j > \mathcal{T}_{n-1}$.

(ii) Inter record times (Δ_n) are defined by $\Delta_n = \mathcal{T}_{n+1} - \mathcal{T}_n$, if $\mathcal{T}_{n+1} < \infty$; $\Delta_n = \infty$ if $\mathcal{T}_{n+1} = \infty, \mathcal{T}_n < \infty$ and $\Delta_n = \diamond$ otherwise, where $\diamond$ is a cemetery value.

(i) Record values (vectors) $(\mathcal{R}_n)$ are defined by $\mathcal{R}_n = \mathcal{M}_{\mathcal{T}_n}$, if $\mathcal{T}_n < \infty$, and $\mathcal{R}_n = \diamond$ otherwise.

(iii) The number of records among the first n observations is defined by

$$\mathcal{N}_n = \#\{k : \mathcal{T}_k \leq n\} = \sum_{\mathrm{i}=1}^{n} I_\mathrm{i},$$

where $\#$ denotes cardinality, $I_1 = 1$ and $I_n = 1_{\{\mathcal{M}_n \succ \mathcal{M}_{n-1}\}} = 1_{\{\mathbf{X}_n \text{ is a record}\}}$, $n \geq 2$, is the indicator of $\mathbf{X}_n$ being a record.

Note, from Definitions 2.1.3, 2.1.5, that $\mathcal{M}_j \succ \mathcal{M}_{j-1}$ if and only if $\mathbf{X}_j \succ \mathcal{M}_{j-1}$. Also, $\mathcal{M}_{\mathcal{T}_n} \succ \mathcal{M}_{\mathcal{T}_n - 1}$, so that $\mathbf{X}_{\mathcal{T}_n} \succ \mathcal{M}_{\mathcal{T}_n - 1}$ and, from equation (2.1.1), we have $\mathcal{M}_{\mathcal{T}_n} = \mathbf{X}_{\mathcal{T}_n}$.

Remark 2.1.2 The previous definitions correspond to upper chain extremes. It is also possible to define minima and lower records, simply by inverting the dominance relation. For example, minima, denoted μ_m, are obtained from the recurrence

$$\mu_n = \mathbf{X}_n 1_{\{\mathbf{X}_n \prec \mu_{n-1}\}} + \mu_{n-1} 1_{\{\mathbf{X}_n \not\prec \mu_{n-1}\}}. \tag{2.1.3}$$

Other variables such as lower record values, lower record times, etc. are defined similarly.

We begin the presentation of results for the processes defined above. Of course, we require some notation and distributional hypotheses on the sequence $(\mathbf{X}_n)$, which are stated below.

Let $\mathcal{C}$ be any collection of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\sigma(\mathcal{C})$ denote the sub-σ-algebra of $\mathcal{F}$ generated by $\mathcal{C}$. Let $\mathcal{F}_n = \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be the σ-algebra generated by the first n observations. A filtration is an increasing family of σ-algebras. We have the following generalization of the corresponding one-dimensional result.

Proposition 2.1.1 Record times $\mathcal{T}_k, k \geq 1$, are stopping times for the filtration $(\mathcal{F}_n)$.

Proof. Recall that $\mathcal{T}$ is a stopping time for $(\mathcal{F}_n)$ if $\{\mathcal{T} = n\} \in \mathcal{F}_n$, for all $n \geq 1$. To prove the result we proceed inductively. First note that $\mathcal{T}_1$ is obviously a stopping time. Now, let $k > 1$ and suppose that $\mathcal{T}_2, \ldots, \mathcal{T}_{k-1}$ are stopping times. Then

$$\begin{aligned} \{\mathcal{T}_k = n\} &= \bigcap_{\mathcal{T}_{k-1} < j < n} \{\mathbf{X}_j \not\succ \mathbf{X}_{\mathcal{T}_{k-1}}, \mathbf{X}_n \succ \mathbf{X}_{\mathcal{T}_{k-1}}\} \\ &= \bigcup_{m<n} \bigcap_{m<j<n} \{\mathbf{X}_j \not\succ \mathbf{X}_m, \mathbf{X}_n \succ \mathbf{X}_m, \mathcal{T}_{k-1} = m\} \in \mathcal{F}_n, \end{aligned}$$

which proves that $\mathcal{T}_k$ is a stopping time for $(\mathcal{F}_n)$. □

2.2 The Markov property

We now consider the Markovian nature of maxima, records and record times, which is a well-known characteristic in dimension $\mathrm{d} = 1$. Henceforth we assume that $(\mathbf{X}_n)$ is an iid sequence, with common distribution function $\mathbf{F}$, that is, $\mathbf{F}(\mathbf{x}) = \mathbb{P}(\mathbf{X}_1 \preceq \mathbf{x}), \mathbf{x} \in \mathbb{R}^{\mathrm{d}}$. Also, let $\bar{\mathbf{F}}(\mathbf{x}) = \mathbb{P}(\mathbf{X}_1 \succ \mathbf{x})$ and $\mathbf{F}(A) = \int_A \mathbf{F}(\mathrm{d}\mathbf{x})$, for any Borel subset A of $\mathbb{R}^{\mathrm{d}}$. Note that the components $X_n^{(j)}, j = 1, \ldots, \mathrm{d}$, need not be independent.

Proposition 2.2.1 The sequence of maxima $(\mathcal{M}_n)$ is a Markov chain with transition measure

$$\mathbb{P}(\mathcal{M}_{n+1} \preceq \mathbf{x} | \mathcal{M}_n) = \mathbf{F}((\mathcal{M}_n, \mathbf{x}]) + (1 - \bar{\mathbf{F}}(\mathcal{M}_n)) 1_{\{\mathcal{M}_n \preceq \mathbf{x}\}}, \tag{2.2.1}$$

where $(\mathcal{M}_n, \mathbf{x}] = \{\mathbf{y} \in \mathbb{R}^{\mathrm{d}} : \mathcal{M}_n \prec \mathbf{y} \preceq \mathbf{x}\}$.

Proof. The Markovian property is direct from (2.1.1). For simplicity, let $\mathbb{P}(\cdot \mid \mathcal{M}_n) := \mathbb{P}_n(\cdot)$, then

$$\begin{aligned} \mathbb{P}_n(\mathcal{M}_{n+1} \preceq \mathbf{x}) &= \mathbb{P}_n(\mathcal{M}_{n+1} \preceq \mathbf{x}, \mathbf{X}_{n+1} \succ \mathcal{M}_n) + \mathbb{P}_n(\mathcal{M}_{n+1} \preceq \mathbf{x}, \mathbf{X}_{n+1} \not\succ \mathcal{M}_n) \\ &= \mathbb{P}_n(\mathcal{M}_n \prec \mathbf{X}_{n+1} \preceq \mathbf{x}) + \mathbb{P}_n(\mathbf{X}_{n+1} \not\succ \mathcal{M}_n) 1_{\{\mathcal{M}_n \preceq x\}}. \end{aligned}$$

So,(2.2.1) follows noting that $\mathbf{X}_{n+1}$ is independent of $\mathcal{M}_n$. □

Remark 2.2.1 Note that the inter record times Δ_n are just the holding times of the Markov chain $(\mathcal{M}_n)$.

The following results show that the behavior of records, in the case $\mathrm{d} > 1$, is very close to that of usual records ($\mathrm{d} = 1$); see [53].

Proposition 2.2.2 For $n \geq 1, k \geq 1$,

$$\mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x}, \Delta_k = n \mid \mathcal{R}_k) = (1 - \bar{\mathbf{F}}(\mathcal{R}_k))^{n-1}\mathbf{F}((\mathcal{R}_k, \mathbf{x}])1_{\{\mathcal{R}_k \neq \diamond\}} \tag{2.2.2}$$

and

$$\mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x} \mid \mathcal{R}_k) = \frac{\mathbf{F}((\mathcal{R}_k, \mathbf{x}])}{\bar{\mathbf{F}}(\mathcal{R}_k)}1_{\{\mathcal{R}_k \neq \diamond\}}. \tag{2.2.3}$$

Proof. For (2.2.2) note that, if the inter record time after $\mathcal{R}_k$ is n, then $n-1$ iid observations do not dominate $\mathcal{R}_k$, with (conditional) probability $(1 - \bar{\mathbf{F}}(\mathcal{R}_k))^{n-1}$. Then the next observation must dominate $\mathcal{R}_k$ and stay $\preceq \mathbf{x}$. Such event has probability $\mathbf{F}((\mathcal{R}_k, \mathbf{x}])$. For (2.2.3) it suffices to sum over n in the first equation. Finally, the cemetery state is absorbing, that is, $\mathbb{P}(\mathcal{R}_{k+1} = \diamond \mid \mathcal{R}_k = \diamond) = 1$. □

Corollary 2.2.1 $(\mathcal{R}_{k+1}, \Delta_k)_{k\geq 1}$ and $(\mathcal{R}_k)_{k\geq 1}$ are Markov chains with transition probabilities given by (2.2.2) and (2.2.3), respectively.

Proof. Let $\mathcal{A}_k = \sigma\{(\mathcal{R}_{\mathrm{i}+1}, \Delta_{\mathrm{i}}), 1 \leq \mathrm{i} \leq k\}$, $k \geq 1$, then

$$\mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x}, \Delta_k = n \mid \mathcal{A}_{k-1}) = \mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x}, \Delta_k = n \mid \mathcal{R}_k),$$

which shows that $(\mathcal{R}_{k+1}, \Delta_k)_{k\geq 1}$ is a Markov chain. The Markovian property of $(\mathcal{R}_k)$ follows from the formula above, after adding over n. □

Proposition 2.2.3 Let $\mathcal{G} = \sigma\{\mathcal{R}_k, k \geq 1\}$. Then the inter record times Δ_n are independent and geometrically distributed, conditionally on $\mathcal{G}$, provided that $\mathbb{P}(\mathcal{R}_k = \diamond) = 0$ for all k, with

$$\mathbb{P}(\Delta_k = n|\mathcal{G}) = \mathbb{P}(\Delta_k = n|\mathcal{R}_k) = (1 - \bar{\mathbf{F}}(\mathcal{R}_k))^{n-1}\bar{\mathbf{F}}(\mathcal{R}_k)1_{\{\mathcal{R}_k \neq \diamond\}},\ k \geq 1, n \geq 1. \tag{2.2.4}$$

Proof. Same argument as in Proposition 2.2.2. □

We now obtain a recurrence for the distribution and the expected value of a function of $\mathcal{M}_n$.

Lemma 2.2.1 Let $(\mathcal{M}_n)$ be the sequence of maxima and let $g : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ measurable. Then

$$\mathbb{E}(g(\mathcal{M}_{n+1})) = \int g(\mathbf{y})\mathbf{F}_n^-(\mathbf{y})\mathbf{F}(\mathrm{d}\mathbf{y}) + \int g(\mathbf{y})(1 - \bar{\mathbf{F}}(\mathbf{y}))\mathbf{F}_n(\mathrm{d}\mathbf{y}), \tag{2.2.5}$$

where $\mathbf{F}_n(\mathbf{x}) = \mathbb{P}(\mathcal{M}_n \preceq \mathbf{x})$ and $\mathbf{F}_n^-(\mathbf{x}) = \mathbb{P}(\mathcal{M}_n \prec \mathbf{x}), n \geq 1$. In particular, with $g(\mathbf{y}) = 1_{\{\mathbf{y}\preceq\mathbf{x}\}}$, for $\mathbf{x} \in \mathbb{R}^{\mathrm{d}}$, we have

$$\mathbf{F}_{n+1}(\mathbf{x}) = \int_{\mathbf{y}\preceq\mathbf{x}} \mathbf{F}_n^-(\mathbf{y})\mathbf{F}(\mathrm{d}\mathbf{y}) + \int_{\mathbf{y}\preceq\mathbf{x}} (1-\bar{\mathbf{F}}(\mathbf{y}))\mathbf{F}_n(\mathrm{d}\mathbf{y}),\ n \geq 1. \tag{2.2.6}$$

Proof. From (2.1.1) we have

$$g(\mathcal{M}_{n+1}) = g(\mathbf{X}_{n+1})1_{\{\mathbf{X}_{n+1}\succ\mathcal{M}_n\}} + g(\mathcal{M}_n)1_{\{\mathbf{X}_{n+1}\not\succ\mathcal{M}_n\}}.$$

Taking conditional expectations we get

$$\mathbb{E}(g(\mathcal{M}_{n+1})|\mathcal{M}_n) = \int g(\mathbf{x})1_{\{\mathbf{x}\succ\mathcal{M}_n\}}\mathbf{F}(\mathrm{d}\mathbf{x}) + g(\mathcal{M}_n)(1-\bar{\mathbf{F}}(\mathcal{M}_n)). \tag{2.2.7}$$

Then, taking expectations in (2.2.7), formula (2.2.5) follows. □

Remark 2.2.2 Observe that, as expected, with $\mathbf{x} = (x^{(1)}, \dots, x^{(\mathrm{d})})$,

$$\begin{aligned}\lim_{\substack{x^{(j)}\to\infty\\ j=1,\dots,\mathrm{d}}} \mathbf{F}_{n+1}(\mathbf{x}) &= \int \mathbf{F}_n^-(\mathbf{y})\mathbf{F}(\mathrm{d}\mathbf{y}) + \int (1-\bar{\mathbf{F}}(\mathbf{y}))\mathbf{F}_n(\mathrm{d}\mathbf{y}) \\ &= \mathbb{P}(\mathcal{M}_n \prec \mathbf{X}_{n+1}) + \mathbb{P}(\mathbf{X}_{n+1} \not\succ \mathcal{M}_n) = 1.\end{aligned}$$

Proposition 2.2.4 Let $\mathbf{F}$ be absolutely continuous (with respect to the Lebesgue measure on $\mathbb{R}^{\mathrm{d}}$), with density $\mathbf{f}$. Then $\mathbf{F}_n$, given in equation (2.2.6), is also absolutely continuous, with density $\mathbf{f}_n$ satisfying

$$\mathbf{f}_{n+1}(\mathbf{x}) = \mathbf{F}_n(\mathbf{x})\mathbf{f}(\mathbf{x}) + (1-\bar{\mathbf{F}}(\mathbf{x}))\mathbf{f}_n(\mathbf{x}). \tag{2.2.8}$$

Proof. The result follows from induction and equation (2.2.6). Indeed, assuming that $\mathbf{F}_1, \dots, \mathbf{F}_n$ are a.c., with densities $\mathbf{f}_1, \dots, \mathbf{f}_n$, $\mathbf{F}_{n+1}$ is obtained by integrating formula (2.2.8). □

Example 2.2.1 We solve recurrence (2.2.6) for $\mathrm{d} = 1$ (with $\mathbf{F}$ is replaced by F). Noting that $1 - \bar{F}(y) = F(y)$, we have

$$F_{n+1}(x) = \int_{-\infty}^{x} F_n^-(y)F(\mathrm{d}y) + \int_{-\infty}^{x} F(y)F_n(\mathrm{d}y),$$

with solution $F_n(x) = F^n(x)$, obtained by induction and the integration by parts formula; see [51].

2.3 Asymptotic results

2.3.1 On the total number of records

The counting process of records $(\mathcal{N}_n)$ is central in our study of chain extremes. As mentioned before, the problem with strong records is their finiteness, if the observations $(\mathbf{X}_n)$ are iid with independent components; see, for example, [24] and [28]. Chain records are supposed to be more convenient than strong records, in terms of asymptotic analysis, because they do not have such problem. But in fact, $\mathcal{N}_\infty$ can also be finite, even if $\mathrm{d} = 1$. For example, if $\mathrm{d} = 1$ and the distribution F has a terminal atom, then $\mathcal{N}_\infty < \infty$ a.s.

Note first that, for one dimensional observations, with general distribution function F, $\mathcal{N}_\infty = \infty$ a.s. if and only if there is no terminal atom, which is also equivalent to saying that the right end-point of F, defined by $\omega_F = \sup\{x \in \mathbb{R} : F(x) < 1\}$, is not an atom. In fact, if ω_F is an atom, then $\mathbb{P}(\mathcal{N}_\infty = \infty) = 0$. For $\mathrm{d} > 1$ the situation is quite different.

Definition 2.3.1 Let $\mathbf{X}$ have distribution $\mathbf{F}$. Then $\mathbf{x} \in \mathbb{R}^{\mathrm{d}}$ is a terminal atom of $\mathbf{X}$ (or $\mathbf{F}$), if $\mathbb{P}(\mathbf{X} = \mathbf{x}) > 0$ and $\mathbb{P}(\mathbf{X} \succ \mathbf{x}) = 0$.

If $\mathbf{F}$ has a terminal atom then it may happen that $0 < \mathbb{P}(\mathcal{N}_\infty = \infty) < 1$. Also, even is $\mathbf{F}$ has no atoms we may have $\mathcal{N}_\infty < \infty$, with positive probability. For example, let $\mathbf{F}$ be the uniform probability on the segment $\{(x, y) \in [0, 1]^2 : x + y = 1\} \subset \mathbb{R}^2$, then the only (chain) record is the first observation.

We consider the following assumption, ensuring the existence of infinitely many records in the whole sequence $(\mathbf{X}_n)$.

Assumption 1 There exists a Borel subset A of $\mathbb{R}^{\mathrm{d}}$ such that $\mathbf{F}(A) = 1$ and $\bar{\mathbf{F}}(\mathbf{x}) > 0$, for all $\mathbf{x} \in A$.

For example, if $\mathrm{d} = 1$ and F is continuous, with support $[a, b]$, then A can be chosen as $[a, b[$.

Proposition 2.3.1 $\mathcal{N}_\infty = \infty$ a.s. under Assumption 1.

Proof. The proof consists in showing inductively that $\mathcal{R}_k \neq \diamond$, for all $k \geq 1$, which is clearly equivalent to $\mathcal{N}_\infty = \infty$. So let $k \geq 1$, such that $\mathbb{P}(\mathcal{R}_k \neq \diamond) = 1$, then $\mathbb{P}(\mathcal{R}_k \in A) = 1$, where A is the set from Assumption 1. Hence $\bar{\mathbf{F}}(\mathcal{R}_k) > 0$ a.s. and so, from (2.2.4), $\mathbb{P}(\Delta_k < \infty) = 1$, that is, $\mathbb{P}(\mathcal{R}_{k+1} \neq \diamond) = 1$ and this yields the conclusion by induction, since $\mathcal{R}_1 = \mathbf{X}_1$. □

In order to avoid unnecessary complexity, we require the sequence $(\mathbf{X}_n)$ to have an infinite number of records a.s., that is, $\mathbb{P}(\mathcal{N}_\infty = \infty) = 1$. Hereafter we work with distributions $\mathbf{F}$ satisfying Assumption 1.

2.3.2 Limiting results for maxima and records

We begin the study of asymptotic properties of the sequences $(\mathcal{M}_n)$ and $(\mathcal{R}_n)$. Limits are written without subscript (save if necessary to avoid confusion), and are understood as the appropriate index increases to ∞.

Proposition 2.3.2 The sequences of maxima $(\mathcal{M}_n)$ and records $(\mathcal{R}_n)$ are increasing, in the sense that, for all $k \geq 1$, $\mathcal{M}_k \preceq \mathcal{M}_{k+1}$ and $\mathcal{R}_k \prec \mathcal{R}_{k+1}$. Also, both sequences converge a.s. to the same, possibly random, limit in $\bar{\mathbb{R}}^{\mathrm{d}} = [-\infty, \infty]^{\mathrm{d}}$. Furthermore, $\bar{\mathbf{F}}(\mathcal{M}_n)$ and $\bar{\mathbf{F}}(\mathcal{R}_n)$ converge a.s. to the same limit in $[0, 1]$.

Proof. Monotonicity is direct from the definitions. Also, because records are a subsequence of maxima, the limits, if they exist, must be the same, a.s. Since the one-dimensional component sequences of $(\mathcal{M}_n)$ and $(\mathcal{R}_n)$ are increasing, they necessarily converge or diverge to ∞. On the other hand, since $\bar{\mathbf{F}}(\mathcal{M}_n), \bar{\mathbf{F}}(\mathcal{R}_n)$ are nonnegative decreasing sequences, they converge to the same limit. □

Proposition 2.3.3 Let $\mathcal{R}_\infty = \lim \mathcal{R}_n$. Then $\mathbb{P}(\mathcal{R}_\infty \preceq \mathbf{x}) = 0$, for any $\mathbf{x} \in \mathbb{R}^{\mathrm{d}}$, such that $\bar{\mathbf{F}}(\mathbf{x}) > 0$.

Proof. Clearly, $\mathcal{R}_\infty \preceq \mathbf{x}$ implies $\mathcal{R}_k \preceq \mathbf{x}$, for all $k \geq 1$. It also implies that $\mathbf{X}_n \not\succ \mathbf{x}$, for all $n \geq 1$. But $\mathbb{P}(\mathbf{X}_n \not\succ \mathbf{x}, n \geq 1) = 0$ because the $\mathbf{X}_n$ are iid and $\mathbb{P}(\mathbf{X}_1 \succ \mathbf{x}) = \bar{\mathbf{F}}(\mathbf{x}) > 0$. □

Corollary 2.3.1 $\bar{\mathbf{F}}(\mathcal{R}_\infty) = 0$ a.s.

Proof. Let $\mathbf{F}_\infty(\mathbf{x}) = \mathbb{P}(\mathcal{R}_\infty \preceq \mathbf{x})$, for $\mathbf{x} \in \mathbb{R}^{\mathrm{d}}$, then

$$\mathbb{P}(\bar{\mathbf{F}}(\mathcal{R}_\infty) > 0) = \int_{\{\mathbf{x}:\bar{\mathbf{F}}(\mathbf{x})>0\}} \mathbf{F}_\infty(\mathrm{d}\mathbf{x}) \leq \int_{\{\mathbf{x}:\mathbf{F}_\infty(\mathbf{x})=0\}} \mathbf{F}_\infty(\mathrm{d}\mathbf{x}) = 0,$$

where the inequality follows from Proposition 2.3.3. □

Corollary 2.3.2 Let $A_{\mathbf{F}}$ be the set of atoms of $\mathbf{F}$ then $\mathbb{P}(\mathcal{R}_\infty \in A_{\mathbf{F}}) = 0$.

Proof. Let $\mathbf{x} \in A_{\mathbf{F}}$ then, by Assumption 1, $\bar{\mathbf{F}}(\mathbf{x}) > 0$. Further, from Proposition 2.3.3 we have $\mathbb{P}(\mathcal{R}_\infty = \mathbf{x}) \leq \mathbb{P}(\mathcal{R}_\infty \preceq \mathbf{x}) = 0$, which yields the conclusion because $A_{\mathbf{F}}$ is countable. □

Lemma 2.3.1 Let A be the set specified in Assumption 1 and let $(\mathbf{r}_n)$ be a strictly increasing sequence in A, with $\lim \mathbf{r}_n = \mathbf{r}_\infty \in \bar{\mathbb{R}}^{\mathrm{d}}_+$, such that $\bar{\mathbf{F}}(\mathbf{r}_\infty) + \mathbf{F}(\{\mathbf{r}_\infty\}) = 0$. Then $\lim \bar{\mathbf{F}}(\mathbf{r}_n) = 0$.

Proof. Let $\mathbf{X}$ have distribution $\mathbf{F}$ and note that $\bigcap_{n\geq 1}\{\mathbf{X} \succ \mathbf{r}_n\} = \{\mathbf{X} \succ \mathbf{r}_\infty\} \cup \{\mathbf{X} = \mathbf{r}_\infty\}$. Then $\lim \bar{\mathbf{F}}(\mathbf{r}_n) = \lim \mathbb{P}(\mathbf{X} \succ \mathbf{r}_n) = \bar{\mathbf{F}}(\mathbf{r}_\infty) + \mathbf{F}(\{\mathbf{r}_\infty\}) = 0$. □

Corollary 2.3.3 $\lim \bar{\mathbf{F}}(\mathcal{R}_n) = 0$. a.s.

Proof. By Proposition 2.3.2 and Corollaries 2.3.1, 2.3.2, we have that $(\mathcal{R}_n)$ a.s. satisfies the hypotheses of the sequence $(\mathbf{r}_n)$ in Lemma 2.3.1., so the conclusion follows. □

Remark 2.3.1 Note, from Corollary 2.3.1, that $\mathcal{R}_\infty \notin A$ a.s. and that, as shown in Example 2.5.1, $\mathcal{R}_\infty$ is not necessarily a point of continuity of $\mathbf{F}$.

2.3.3 A martingale and laws of large numbers

Proposition 2.3.4 The sequence $(S_n)_{n\geq 1}$, defined by

$$S_n = \sum_{k=1}^{n} (\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k) - 1), \; n \geq 1, \tag{2.3.1}$$

is a square-integrable $(\mathcal{A}_k)$-martingale, with $\mathcal{A}_k = \sigma\{(\mathcal{R}_{\mathrm{i}+1}, \Delta_{\mathrm{i}}), 1 \leq \mathrm{i} \leq k\}$, $k \geq 1$.

Proof. Note first that, conditionally on $\mathcal{A}_{k-1}$, Δ_k is geometric (starting at 1), with parameter $\bar{\mathbf{F}}(\mathcal{R}_k)$, and that $\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k)$ is $\mathcal{A}_k$-measurable. Also, by (2.2.4), $\mathbb{E}(\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k) | \mathcal{A}_{k-1}) = 1$. Finally, for square integrability observe, from the formula of the variance of a geometric random variable, that

$$\mathbb{E}((\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k) - 1)^2 | \mathcal{A}_{k-1}) = 1 - \bar{\mathbf{F}}(\mathcal{R}_k) \leq 1,$$

and the conclusion follows. □

From well-known results for square integrable martingales, we have the following law of large numbers.

Proposition 2.3.5

$$\frac{1}{n} \sum_{k=1}^{n} \Delta_k \bar{\mathbf{F}}(\mathcal{R}_k) \to 1 \text{ a.s.} \tag{2.3.2}$$

Proof. Let S_n be as defined in (2.3.1) and let $\langle S_n \rangle = \sum_{k=1}^{n} \mathbb{E}((\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k) - 1)^2 | \mathcal{A}_{k-1}) = \sum_{k=1}^{n} (1 - \bar{\mathbf{F}}(\mathcal{R}_k))$. By Corollary 2.3.3, we have $\langle S_n \rangle / n \to 1$ a.s. Also, recalling that $S_n / \langle S_n \rangle \to 0$ a.s. (this is the well known convergence of a martingale S_n, normalized by its associated increasing process $\langle S_n \rangle$, related to Doob's decomposition of the submartingale S_n^2; see Proposition VII-2-4 in [43] or [38]), the conclusion follows. □

Recall that $I_n = \mathbb{1}_{\{\mathcal{M}_n \succ \mathcal{M}_{n-1}\}} = \mathbb{1}_{\{\mathbf{X}_n \succ \mathcal{M}_{n-1}\}}$ is the indicator that $\mathbf{X}_n$ is a (chain) record and $\mathcal{N}_n = \sum_{\mathrm{i}=1}^{n} I_{\mathrm{i}}$ is the number of records up to the n-th observation. Some simple identities and relations involving $\mathcal{N}_n$ are shown below.

Lemma 2.3.2 For $n, k \geq 1$ and $g : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ measurable, we have

(i) $\mathbb{P}(\mathcal{N}_n < k) = \mathbb{P}(\mathcal{T}_k > n)$,

(ii) $\mathbb{E}(I_n) = \mathbb{E}(\bar{\mathbf{F}}(\mathcal{M}_{n-1}))$,

(iii) $\mathcal{T}_{\mathcal{N}_n} \leq n < \mathcal{T}_{\mathcal{N}_n+1}$, $\mathcal{N}_{\mathcal{T}_n} = n$,

(iv) $\mathcal{R}_{\mathcal{N}_n} = \mathcal{M}_n$, $\mathcal{M}_{\mathcal{T}_n} = \mathcal{R}_n$,

(v) $\sum_{k=1}^{n} g(\mathcal{M}_k) = \sum_{k=1}^{\mathcal{N}_n} \Delta_k g(\mathcal{R}_k) - (\mathcal{T}_{\mathcal{N}_n+1} - n) g(\mathcal{R}_{\mathcal{N}_n})$.

Corollary 2.3.4

$$\frac{\mathcal{N}_n}{\sum_{k=1}^{n} \bar{\mathbf{F}}(\mathcal{M}_k)} \to 1 \text{ a.s.} \tag{2.3.3}$$

Proof. Consider (v) of Lemma 2.3.2, with $g = \bar{\mathbf{F}}$. Then, the result follows from the strong LLN of Proposition 2.3.5, if we show that $(\mathcal{T}_{\mathcal{N}_n+1} - n)\bar{\mathbf{F}}(\mathcal{R}_{\mathcal{N}_n})/\mathcal{N}_n \to 0$ a.s. But, since $\mathcal{N}_n \to \infty$ and $(\mathcal{T}_{\mathcal{N}_n+1} - n) \leq \Delta_{\mathcal{N}_n}$, it suffices to show that $\Delta_n\bar{\mathbf{F}}(\mathcal{R}_n)/n \to 0$ a.s.

From the proof of Proposition 2.3.4, we have $\mathbb{E}((\Delta_k\bar{\mathbf{F}}(\mathcal{R}_k))^2|\mathcal{A}_{k-1}) = 2 - \bar{\mathbf{F}}(\mathcal{R}_k)$ and so, $\mathbb{E}((\Delta_k\bar{\mathbf{F}}(\mathcal{R}_k))^2) \leq 2$. Finally, for any $\varepsilon > 0$, we have

$$\mathbb{P}\Big(\frac{\Delta_n\bar{\mathbf{F}}(\mathcal{R}_n)}{n} > \varepsilon\Big) \leq \frac{\mathbb{E}(\Delta_n\bar{\mathbf{F}}(\mathcal{R}_n))^2}{\varepsilon^2 n^2} \leq \frac{2}{\varepsilon^2 n^2},$$

and the conclusion follows from the Borel-Cantelli lemma. □

A natural martingale related to $\mathcal{N}_n$ is shown below.

Proposition 2.3.6 Let $\mathcal{F}_k = \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_k), k \geq 1$. Then

$$T_n = \mathcal{N}_n - \sum_{k=1}^{n} \bar{\mathbf{F}}(\mathcal{M}_{k-1}),\ n \geq 1, \tag{2.3.4}$$

is a $(\mathcal{F}_k)$-martingale, with bounded increments, where $\bar{\mathbf{F}}(\mathcal{M}_0) = 1$, conventionally.

Proof. The result is obtained by noting that $\mathbb{E}(I_k|\mathcal{F}_{k-1}) = \mathbb{P}(\mathbf{X}_k \succ \mathcal{M}_{k-1}|\mathcal{F}_{k-1}) = \bar{\mathbf{F}}(\mathcal{M}_{k-1})$. □

Remark 2.3.2 The LLN of Corollary 2.3.4 can also be obtained from Proposition 2.3.6 above. Note that $\langle T_n \rangle := \sum_{k=1}^n \mathbb{E}((I_k - \bar{\mathbf{F}}(\mathcal{M}_{k-1}))^2|\mathcal{F}_{k-1}) = \sum_{k=1}^n (1 - \bar{\mathbf{F}}(\mathcal{M}_{k-1}))\bar{\mathbf{F}}(\mathcal{M}_{k-1})$, then (2.3.3) follows from $T_n/\langle T_n \rangle \to 0$ a.s.

The following result is a generalization of theorem 3 of [52], to chain records from a general distribution $\mathbf{F}$, under Assumption 1. In this version we only consider the upper bound of the limsup, which is enough to derive asymptotic results for the inter record times Δ_k.

Theorem 2.3.1

$$\limsup \left| \frac{\log(\Delta_n\bar{\mathbf{F}}(\mathcal{R}_n))}{\log n} \right| \leq 1, \text{ a.s.} \tag{2.3.5}$$

Proof. Recall from Proposition 2.2.3 that, conditionally on $\mathcal{G} = \sigma\{\mathcal{R}_k, k \geq 1\}$, the random variables Δ_k are independent, geometrically distributed, with $\mathbb{P}(\Delta_k > n|\mathcal{G}) = (1 - \bar{\mathbf{F}}(\mathcal{R}_k))^n$.

Let $Z_k = \log(\Delta_k\bar{\mathbf{F}}(\mathcal{R}_k))/\log k, k \geq 2$, and the events $C = \{\bar{\mathbf{F}}(\mathcal{R}_k) \to 0\}$ and $L = \{\limsup_k |Z_k| \leq 1\}$. Then, by Corollary 2.3.3, $\mathbb{P}(C) = 1$. We have to prove that $\mathbb{P}(L) = 1$, which is equivalent to $\mathbb{P}(L \cap C) = 1$ but, in order to take full advantage of the distributional properties of the Δ_k, we must deal with $\mathbb{P}(L \cap C|\mathcal{G}) = \mathbb{P}(L|\mathcal{G})1_C$.

Observe that L is equivalent to $\{|Z_k| > 1 + \varepsilon,$ f.o.$\}$, for any $\varepsilon > 0$, where f.o. stands for "finitely often". By Borel-Cantelli lemma it suffices to show that

$$\sum_{k=1}^{\infty} \mathbb{P}(|Z_k| > 1 + \varepsilon|\mathcal{G})1_C < \infty \tag{2.3.6}$$

to obtain $\mathbb{P}(L|\mathcal{G})1_C = 1$ a.s., which, after taking expectation, yields the conclusion. In the following expressions the convergence $\bar{\mathbf{F}}(\mathcal{R}_k) \to 0$ holds but, for simplicity, we omit the indicator 1_C. Note that

$$\mathbb{P}(|Z_k| > 1+\varepsilon|\mathcal{G}) \leq \mathbb{P}(Z_k > 1+\varepsilon|\mathcal{G}) + \mathbb{P}(Z_k \leq -(1+\varepsilon)|\mathcal{G}) \tag{2.3.7}$$

Also, from Proposition 2.2.3, we have

$$\mathbb{P}(Z_k > 1+\varepsilon|\mathcal{G}) = (1-\bar{\mathbf{F}}(\mathcal{R}_k))^{\left\lfloor \frac{k^{1+\varepsilon}}{\bar{\mathbf{F}}(\mathcal{R}_k)} \right\rfloor}. \tag{2.3.8}$$

and

$$\mathbb{P}(Z_k \leq -(1+\varepsilon)|\mathcal{G}) = 1-(1-\bar{\mathbf{F}}(\mathcal{R}_k))^{\left\lfloor \frac{k^{-(1+\varepsilon)}}{\bar{\mathbf{F}}(\mathcal{R}_k)} \right\rfloor}. \tag{2.3.9}$$

Furthermore, because $\bar{\mathbf{F}}(\mathcal{R}_k) \to 0$ a.s., it holds

$$(1-\bar{\mathbf{F}}(\mathcal{R}_k))^{\frac{1}{\bar{\mathbf{F}}(\mathcal{R}_k)}} \to \mathrm{e}^{-1},$$

a.s. Hence, for sufficiently large k, there exist $a, b \in (0,1)$, such that $a < (1-\bar{\mathbf{F}}(\mathcal{R}_k))^{\frac{1}{\bar{\mathbf{F}}(\mathcal{R}_k)}} < b$ and so, from (2.3.8) and (2.3.9), we have

$$\mathbb{P}(Z_k > 1+\varepsilon|\mathcal{G}) \leq b^{\bar{\mathbf{F}}(\mathcal{R}_k)\left\lfloor \frac{k^{1+\varepsilon}}{\bar{\mathbf{F}}(\mathcal{R}_k)} \right\rfloor} \leq b^{k^{1+\varepsilon}},$$

and

$$\mathbb{P}(Z_k \leq -(1+\varepsilon)|\mathcal{G}) \leq 1 - a^{k^{-(1+\varepsilon)}} = O\left(\frac{1}{k^{1+\varepsilon}}\right),$$

for sufficiently large k. Hence, from (2.3.7) we see that (2.3.6) holds, and the conclusion is obtained. □

Remark 2.3.3 The result of Theorem 2.3.1 shows that the sequences (Δ_k) and $(\bar{\mathbf{F}}(\mathcal{R}_k))$ are close in a "logarithmic" sense. Note also that we make no assumption about the speed of convergence of $\bar{\mathbf{F}}(\mathcal{R}_k)$ to 0. In [52] (lemma 2, theorem 3) it is required that $\sum_{k=1}^{\infty} \bar{\mathbf{F}}(\mathcal{R}_k) < \infty$.

Corollary 2.3.5 Let $c \in (0,1)$, then

$$\frac{\log \bar{\mathbf{F}}(\mathcal{R}_k)}{k} \to \log c \text{ a.s.} \quad \Longleftrightarrow \quad \frac{\log \Delta_k}{k} \to -\log c \text{ a.s.} \tag{2.3.10}$$

Proof. Observe that

$$\left|\left|\frac{\log \Delta_k}{k} + \log c\right| - \left|\frac{\log \bar{\mathbf{F}}(\mathcal{R}_k)}{k} - \log c\right|\right| \leq \left|\frac{\log(\Delta_k \bar{\mathbf{F}}(\mathcal{R}_k))}{\log k}\right| \frac{\log k}{k}.$$

Then, from Theorem 2.3.1, we see that either convergence in (2.3.10) implies the other. □

The result above is well-known for one dimensional records, from a continuous parent distribution F, and was obtained by Holmes and Strawderman [40]. In such case it can be shown that $-\log \bar{F}(\mathcal{R}_k)$ behaves as a sum of iid exponential random variables, so that $\bar{F}(\mathcal{R}_k)^{\frac{1}{k}} \to \mathrm{e}^{-1}$, by the SLLN, and thus $\frac{\log \Delta_k}{k} \to 1$ a.s.

Another result than can be derived from Theorem 2.3.1 is the following CLT for $\log \Delta_k$.

Corollary 2.3.6 Let $a \in \mathbb{R}$, then

$$\frac{\log \bar{\mathbf{F}}(\mathcal{R}_k) + ak}{\sqrt{k}} \xrightarrow{\mathcal{D}} N(0, \sigma^2) \quad \Longleftrightarrow \quad \frac{\log \Delta_k - ak}{\sqrt{k}} \xrightarrow{\mathcal{D}} N(0, \sigma^2).$$

Proof. The result follows from the identity

$$\frac{\log \Delta_k - ak}{\sqrt{k}} = \frac{\log \Delta_k + \log \bar{\mathbf{F}}(\mathcal{R}_k)}{\sqrt{k}} - \frac{\log \bar{\mathbf{F}}(\mathcal{R}_k) + ak}{\sqrt{k}}$$

and Theorem 2.3.1. □

2.4 On the point process of chain record values

The behavior of chain record values is well understood. We know from Proposition 2.2.2 that $(\mathcal{R}_k)$ is a Markov chain with initial probability $\mathbf{F}$ and transition probabilities given by (2.2.3). We have the following

Proposition 2.4.1 Suppose the iid observations $\mathbf{X}_n$ have common probability density function $\mathbf{f}$. Then the joint density of the first k records $\mathcal{R}_1, \ldots, \mathcal{R}_k$ is given by

$$\mathbf{f}_{\mathcal{R}_1,\ldots,\mathcal{R}_k}(\mathbf{r}_1, \ldots, \mathbf{r}_k) = \mathbf{f}(\mathbf{r}_1)\frac{\mathbf{f}(\mathbf{r}_2)}{\bar{\mathbf{F}}(\mathbf{r}_1)} \cdots \frac{\mathbf{f}(\mathbf{r}_k)}{\bar{\mathbf{F}}(\mathbf{r}_{k-1})}, \quad \mathbf{r}_1 \prec \ldots \prec \mathbf{r}_k. \tag{2.4.1}$$

Proof. Density (2.4.1) follows from the densities $\mathbf{f}_{\mathcal{R}_{k+1}|\mathcal{R}_k}(\mathbf{r}_{k+1}|\mathbf{r}_k) = \mathbf{f}(\mathbf{r}_{k+1})/\bar{\mathbf{F}}(\mathbf{r}_k)$, obtained from (2.2.3). □

It is interesting to describe the set of record values as a point process in $\mathbb{R}^{\mathrm{d}}$. The idea comes from a result found in [53] (theorem 1), where standard records ($\mathrm{d} = 1$) are characterized as arrival times of a point process on the line, with independent increments. This process is the superposition of two independent point processes: a non-homogeneous Poisson process, with intensity $-\log(1 - F_c(x))$ and a Bernoulli process on the atoms of F, where F_c is the continuous part of F.

For $\mathrm{d} > 1$ it makes little sense to consider the $\mathcal{R}_k$ values as arrival times and even the concept of independent increments does not have an obvious meaning. Nonetheless, it may be rewarding to describe records using the language of point processes, as presented, for example, in [42]. For simplicity, we restrict attention to continuous distributions $\mathbf{F}$ on $\mathbb{R}^{\mathrm{d}}$.

A point process in $\mathbb{R}^{\mathrm{d}}$ can be described as a countable set of randomly distributed points (Z_k) on $\mathbb{R}^{\mathrm{d}}$. It can also be seen as an integer-valued, atomic random measure M, with $M(A) = \#\{k : Z_k \in A\}$, for any Borel set $A \subseteq \mathbb{R}^{\mathrm{d}}$, where $\#$ indicates cardinality.

Definition 2.4.1 The multivariate counting process of records $\{\xi(\mathbf{r}), \mathbf{r} \in \mathbb{R}^{\mathrm{d}}\}$ is defined by

$$\xi(\mathbf{r}) = \#\{k \geq 1 : \mathcal{R}_k \preceq \mathbf{r}\}. \tag{2.4.2}$$

The marginal counting processes $\xi^{(j)}(r)$ of $\xi(\mathbf{r})$, for $r \in \mathbb{R}$ and $j = 1, \ldots, \mathrm{d}$, are defined by

$$\xi^{(j)}(r) = \#\{k \geq 1 : \mathcal{R}_k^{(j)} \leq r\}. \tag{2.4.3}$$

Lemma 2.4.1 Let $\mathbf{r} = (r^{(1)}, \ldots, r^{(\mathrm{d})}), \tau_j = \min\{k : \mathcal{R}_k^{(j)} > r^{(j)}\}$, for $j = 1, \ldots, \mathrm{d}$, $\tau = \min\{k : \mathcal{R}_k \succ \mathbf{r}\}$ and $\tilde{\tau} = \min\{k : \mathcal{R}_k \not\preceq \mathbf{r}\}$. Then

$$\xi(\mathbf{r}) = \min\{\xi^{(j)}(r^{(j)}) : 1 \leq j \leq \mathrm{d}\}, \tag{2.4.4}$$

$$\tau = \max_{1 \leq j \leq \mathrm{d}} \tau_j \leq \infty \quad \text{and} \quad \tilde{\tau} = \min_{1 \leq j \leq \mathrm{d}} \tau_j \leq \infty. \tag{2.4.5}$$

Proof. The conclusions follows from the definitions. □

2.4.1 Independent continuous components

We consider below the analysis of maxima and records under the additional assumption that the iid observations $\mathbf{X}_n$ have independent continuous components. This hypothesis makes some calculations simpler and allows to illustrate the results of Corollaries 2.3.5 and 2.3.6. We investigate also the eventual independence of components of maxima and records.

Assumption 2 The observation vectors $\mathbf{X}_n$ have independent components $X_n^{(j)}, j = 1, \ldots, \mathrm{d}$ with respective continuous marginal distributions $F^{(j)}, j = 1, \ldots, \mathrm{d}$.

Note that Assumption 2 implies $\mathcal{N}_\infty = \infty$ a.s. See the discussion in Section 2.3.1.

Remark 2.4.1 It is intuitively clear that the components $\mathcal{M}_n^{(j)}$ of $\mathcal{M}_n$ are not independent, even under Assumption 2. For example, if $\mathrm{d} = 2$ and the $\mathbf{X}_n$ are uniform in $[0, 1]^2$, then the joint density of $\mathcal{M}_2^{(1)}, \mathcal{M}_2^{(2)}$ is $f_2(x, y) = (x + y)1_{[0,1]^2}(x, y)$, showing that $\mathcal{M}_2^{(1)}$ and $\mathcal{M}_2^{(2)}$ are dependent; see Chapter 3. However, as shown in the next proposition, the components $\mathcal{R}_n^{(j)}$ of records $\mathcal{R}_n$ are independent.

Proposition 2.4.2 Let $(\mathcal{R}_n)$ be the sequence of records, with $\mathcal{R}_n = (\mathcal{R}_n^{(1)}, \ldots, \mathcal{R}_n^{(\mathrm{d})})$. Then, under Assumption 2, the sequences $(\mathcal{R}_n^{(j)}), j = 1, \ldots, \mathrm{d}$, are independent Markov chains on $\mathbb{R}$, with initial state $\mathcal{R}_n^{(j)} = X_1^{(j)}$ and transition probabilities

$$\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \leq x^{(j)} | \mathcal{R}_k^{(j)}) = \frac{F^{(j)}(x^{(j)}) - F^{(j)}(\mathcal{R}_k^{(j)})}{1 - F^{(j)}(\mathcal{R}_k^{(j)})} 1_{\{\mathcal{R}_k^{(j)} \leq x^{(j)}\}}. \tag{2.4.6}$$

Proof. Observe that, by Assumption 2, $\bar{\mathbf{F}}(\mathbf{x}) = \prod_{j=1}^{\mathrm{d}} \bar{F}^{(j)}(x^{(j)}) = \prod_{j=1}^{\mathrm{d}}(1 - F^{(j)}(x^{(j)}))$ (continuity of $\mathbf{F}$ is crucial). From Proposition 2.2.2,

$$\mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x} | \mathcal{R}_k) = \frac{\prod_{j=1}^{\mathrm{d}}(F^{(j)}(x^{(j)}) - F^{(j)}(\mathcal{R}_k^{(j)}))}{\prod_{j=1}^{\mathrm{d}}(1 - F^{(j)}(\mathcal{R}_k^{(j)}))} 1_{\{\mathcal{R}_k \preceq \mathbf{x}\}}.$$

Then

$$\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k) = \lim_{x^{(l)} \to \infty, l \neq j} \mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x} | \mathcal{R}_k) = \frac{F^{(j)}(x^{(j)}) - F^{(j)}(\mathcal{R}_k^{(j)})}{1 - F^{(j)}(\mathcal{R}_k^{(j)})} 1_{\{\mathcal{R}_k^{(j)} \le x^{(j)}\}}$$

and so, $\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k^{(j)}) = \mathbb{E}(\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k) | \mathcal{R}_k^{(j)})$ yields (2.4.6). Finally, for the independence of the Markov chains $(\mathcal{R}_n^{(j)}), j = 1, \ldots, \mathrm{d}$, we argue inductively and consider, for simplicity, only two coordinates. Suppose that $\mathcal{R}_k^{(j)}, \mathcal{R}_k^{(l)}$ are independent, then

$$\begin{aligned}
\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)}, \mathcal{R}_{k+1}^{(j)} \le x^{(l)}) &= \mathbb{E}(\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)}, \mathcal{R}_{k+1}^{(l)} \le x^{(l)} | \mathcal{R}_k)) \\
&= \mathbb{E}(\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k^{(j)}) \mathbb{P}(\mathcal{R}_{k+1}^{(l)} \le x^{(l)} | \mathcal{R}_k^{(l)})) \\
&= \mathbb{E}(\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k^{(j)})) \mathbb{E}(\mathbb{P}(\mathcal{R}_{k+1}^{(l)} \le x^{(l)} | \mathcal{R}_k^{(l)})) \\
&= \mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)}) \mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(l)}).
\end{aligned}$$

Hence, $\mathcal{R}_{k+1}^{(j)}, \mathcal{R}_{k+1}^{(l)}$ are independent. □

Remark 2.4.2 The result of Proposition 2.4.2 above says that, under the assumption of independence and continuity of components of $\mathbf{X}_n$, the marginal processes of chain records $(\mathcal{R}_n^{(j)}), j = 1, \ldots, \mathrm{d}$ behave as independent ordinary record processes, from continuous distributions $F^{(j)}, j = 1, \ldots, \mathrm{d}$. One may be tempted to conclude that chain records behave exactly as d-dimensional strong records. Such conclusion is erroneous because the sequence of chain records $(\mathbf{X}_{\mathcal{T}_k})$ is distributed as the Markov chain $(\mathcal{R}_k)$, with transitions (2.2.3), whereas the sequence of strong record values is not; see Definition 2.1.4.

Corollary 2.4.1 Under Assumption 2, the point process ξ of records, in Definition 2.4.1, is distributed as the minimum of d independent, non-homogeneous Poisson processes, with respective intensities $-\log \bar{F}^{(j)}, j = 1, \ldots, \mathrm{d}$.

Proof. From theorem 1 in [53], it follows that the marginal record value process $(\mathcal{R}_k^{(j)})$ is a non-homogeneous Poisson processes, with intensity $-\log \bar{F}^{(j)}$. The conclusion is derived from Lemma 2.4.1. □

Proposition 2.4.3 Under Assumption 2, the sequence $(-\log \bar{\mathbf{F}}(\mathcal{R}_k))_k$ is distributed as $(\sum_{j=1}^{\mathrm{d}} \sum_{\mathrm{i}=1}^{k} Y_{\mathrm{i}j})_k$, where the $Y_{\mathrm{i}j}$ are iid, exponentially distributed random variables, with parameter 1.

Proof. $(-\log \bar{F}^{(j)}(\mathcal{R}_k^{(j)}))_k$ is distributed as the sequence of records, from the exponential distribution with parameter 1. From theorem 1 in [53] we know that exponential records are distributed as arrival times of the homogeneous Poisson process, with parameter 1, hence, as partial sums $(\sum_{\mathrm{i}=1}^{k} Y_{\mathrm{i}j})$ of iid random variables $Y_{\mathrm{i}j}$, exponentially distributed, with parameter 1. Then, since $\bar{\mathbf{F}}(\mathbf{x}) = \prod_{j=1}^{\mathrm{d}} \bar{F}^{(j)}(x^{(j)})$, we conclude that $(-\log \bar{\mathbf{F}}(\mathcal{R}_k))$ is distributed as $(\sum_{j=1}^{\mathrm{d}} \sum_{\mathrm{i}=1}^{k} Y_{\mathrm{i}j})$. □

Corollary 2.4.2 Under Assumption 2,

$$\frac{\log \Delta_k}{k} \to \mathrm{d} \quad \text{a.s.} \quad \text{and} \quad \frac{\log \Delta_k - k\mathrm{d}}{\sqrt{k}} \xrightarrow{\mathcal{D}} N(0, \mathrm{d}). \tag{2.4.7}$$

PROOF. The first convergence follows from Corollary 2.3.5, Proposition 2.4.3 and the Strong Law of Large Numbers. The second follows from Corollary 2.3.6, Proposition 2.4.3 and the Central Limit Theorem. □

Proposition 2.4.4 Under Assumption 2, the number of records among the first n observations $\mathcal{N}_n$, the k-th record time $\mathcal{T}_k$ and the k-th inter-record time Δ_k, are distribution free. That is, their distributions do not depend on $\mathbf{F}$.

PROOF. $\mathbf{X}_n = (X_n^{(1)}, \ldots, X_n^{(\mathrm{d})})$ is a record if and only if $(F^{(1)}(X_n^{(1)}), \ldots, F^{(\mathrm{d})}(X_n^{(\mathrm{d})}))$ is a record. Therefore, $\mathcal{N}_n, \mathcal{T}_k, \Delta_k$, defined either on $(\mathbf{X}_n)$ or $(F^{(1)}(X_n^{(1)}), \ldots, F^{(\mathrm{d})}(X_n^{(\mathrm{d})}))$, are the same. Finally note that $F^{(j)}(X_n^{(j)}), j = 1, \ldots, \mathrm{d}$, are iid, uniform in $[0, 1]$. □

For illustration, we compute below the probability distribution of $\mathcal{T}_2$, for $\mathrm{d} = 2$.

Example 2.4.1 Let $\mathrm{d} = 2$ and $\mathbf{F}$ with independent uniform marginals in $[0, 1]$. Then, conditionally on $\mathbf{X}_1 = (x, y)$, $\mathcal{T}_2 - 1$ is geometrically distributed (starting at 1), with success parameter $(1-x)(1-y)$. Hence $\mathbb{P}(\mathcal{T}_2 = k|\mathbf{X}_1 = (x, y)) = (1-(1-x)(1-y))^{k-2}(1-x)(1-y)$, and, from the binomial formula,

$$\begin{aligned}\mathbb{P}(\mathcal{T}_2 = k) &= \int_0^1 \int_0^1 (1 - uv)^{k-2} uv \mathrm{d}u \mathrm{d}v \\ &= \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(j+2)^2}, \quad k \geq 2.\end{aligned}$$

Note that $\mathbb{E}(\mathcal{T}_2|\mathbf{X}_1 = (x, y)) = ((1 - x)(1 - y))^{-1}$, so $\mathbb{E}(\mathcal{T}_2) = \infty$.

Remark 2.4.3 The behavior of the number of chain records $\mathcal{N}_n$, which is a distribution-free variable in the sense of Proposition 2.4.4, is studied in Chapter 3, in the context of observations $\mathbf{X}_n$ uniformly distributed in $[0, 1]^{\mathrm{d}}$.

Proposition 2.4.5 Under Assumption 2, the sequences $(\mathcal{R}_k^{(j)}), j = 1, \ldots, \mathrm{d}$, are independent, $\mathbb{R}$-valued and increasing Markov chains, such that $\mathcal{R}_k^{(j)} \to \omega_{F^{(j)}}$ a.s., where we denote as $\omega_{F^{(j)}} = \sup\{x \in \mathbb{R} : F^{(j)}(x) < 1\}$ the right-end point of distribution $F^{(j)}$.

PROOF. For $\mathrm{d} = 1$, the convergence of record values to the right-end point of the distribution, is a well known result of record theory. □

Remark 2.4.4 Asymptotic laws for $\mathcal{R}_k$ can also be obtained. Each marginal process $(\mathcal{R}_k^{(j)})$, with appropriate centering and scaling sequences, can be shown to converge to a limiting distribution, which depends on the domain of attraction for maxima of $F^{(j)}$; see [49]. On the other hand, as stated in Proposition 2.3.2, records and (chain) maxima have the same limits, therefore, under Assumption 2, $\mathcal{M}_k^{(j)} \to \omega_{F^{(j)}}$ a.s. It would also be interesting to investigate limiting distributions for $\mathcal{M}_k$.

2.5 Cone induced order and strict dominance

In what follows we briefly consider extensions of previous results. The first idea is to replace the dominance relation $\preceq$ by the partial order induced by a cone in $\mathbb{R}^{\mathrm{d}}$. Then we explore a new definition of chain extreme based on strict dominance.

2.5.1 Cone records

For completeness we recall some well-known concepts of convex geometry. A subset $\mathcal{K}$ of $\mathbb{R}^{\mathrm{d}}$ is a cone if $\mathbf{x} \in \mathcal{K}$ implies $\lambda\mathbf{x} \in \mathcal{K}$, for all $\lambda > 0$. When $\mathcal{K}$ is a closed and convex set, $\mathcal{K}$ is said to be a closed convex cone. A closed convex cone $\mathcal{K}$ is pointed if it contains no line or, equivalently, $\mathcal{K}$ is not pointed if there exists $\mathbf{x} \in \mathcal{K}$, $\mathbf{x} \neq 0$, such that $-\mathbf{x} \in \mathcal{K}$. A cone $\mathcal{K}$ is solid if it contains d linearly independent vectors. Finally, a closed, convex, pointed and solid cone is called a proper cone.

It is easy to verify that a proper cone induces a partial order in $\mathbb{R}^{\mathrm{d}}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathrm{d}}$ and a proper cone $\mathcal{K}$ we have $\mathbf{x} \preceq_{\mathcal{K}} \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \mathcal{K}$. For example, the dominance relation of Definition 2.1.1 is a cone-induced order with $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^{\mathrm{d}} : x^{(j)} \geq 0, 1 \leq j \leq \mathrm{d}\}$, the positive orthant.

Maxima and records can now be defined with respect to any order $\preceq_{\mathcal{K}}$ related to a proper cone $\mathcal{K}$. If, additionally, $\mathcal{K}$ is assumed to be contained in the positive orthant of $\mathbb{R}^{\mathrm{d}}$, then it is clear that $\mathbf{x} \preceq_{\mathcal{K}} \mathbf{y}$ implies $\mathbf{x} \preceq \mathbf{y}$. However, this does not mean that if $\mathbf{X}_n$ is a record for $\preceq_{\mathcal{K}}$ then it is also a record for $\preceq$ and so, the sequence of records for $\preceq_{\mathcal{K}}$ is not a subsequence of records for $\preceq$. Asymptotic results such as in Propositions 2.3.2 and 2.3.3 can be developed in this new setting.

It may also be worth exploring records related to proper cones containing the positive orthant such as, for example, $\mathcal{K} = \{\mathbf{x} = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : 10x^{(1)} + x^{(2)} \geq 0, 10x^{(2)} + x^{(1)} \geq 0\}$. However, the asymptotic analysis is not simple because monotone sequences with respect to $\preceq_{\mathcal{K}}$ can behave oddly. We shall not develop this topic further in this thesis.

2.5.2 Strict records

Here we consider a new definition of chain maxima and records based on the strict dominance relation $\succ_s$; see Definition 2.1.1. The advantage of this definition is to allow a rather simple analysis in the case of observations with independent components, having possibly discontinuous marginal distributions. We explore aspects of this new type or multivariate record, following the structure of the previous sections, omitting unnecessary details.

The relation $\succ_s$ is nothing but strict inequality over all components and clearly implies $\succ$. Using $\succ_s$ we define strict chain maxima and records, which should not be confused with strong maxima and records. As done before, the word *chain* is implicit in our definitions.

Definition 2.5.1 The sequence $(\mathcal{M}_n^s)_{n\geq 1}$, of strict maxima from observations $(\mathbf{X}_n)$, is defined by $\mathcal{M}_1^s = \mathbf{X}_1$ and, for $n \geq 2$,

$$\mathcal{M}_n^s = \mathbf{X}_n 1_{\{\mathbf{X}_n \succ_s \mathcal{M}_{n-1}^s\}} + \mathcal{M}_{n-1}^s 1_{\{\mathbf{X}_n \not\succ_s \mathcal{M}_{n-1}^s\}}. \tag{2.5.1}$$

Observe that $(\mathcal{M}_n^s)$ and $(\mathcal{M}_n)$, obtained from the same sequence of iid observations $(\mathbf{X}_n)$, coincide a.s if $\mathbf{F}$ is continuous. Also, since $\succ_s$ is stronger than $\succ$, the sequence of strict records $(\mathcal{R}_k^s)$, defined below, is a subsequence of records $(\mathcal{R}_k)$. Moreover, this means that the total numbers of strict records in the whole sequence $(\mathbf{X}_n)$ could be finite while records are infinite. Finally note that, for $\mathrm{d} = 1$, all definitions (strong, chain and strict chain) coincide, regardless of the continuity of the distribution.

Definition 2.5.2 By convention $\mathbf{X}_1$ is a strict record and, for $n \geq 2$, $\mathbf{X}_n$ is a strict record if $\mathcal{M}_n^s \succ_s \mathcal{M}_{n-1}^s$.

The sequences of strict record times, inter record times, record values, record indicators and the counting process of strict records can be defined exactly as done for records, in Definition 2.1.5, by replacing $\succ$ by $\succ_s$. The notation for these random variables is the same, except for the exponent s which stands for strict.

We consider the question of finiteness of $\mathcal{N}_\infty^s$. Of course, if $\mathcal{N}_\infty < \infty$ then $\mathcal{N}_\infty^s < \infty$ because $(\mathcal{R}_k^s)$ is a subsequence of $(\mathcal{R}_k)$. But, is it true that $\mathcal{N}_\infty = \infty$ implies $\mathcal{N}_\infty^s = \infty$? The following trivial counterexample shows that this implication does not hold. The distribution $\mathbf{F}$ in the counterexample is, of course, discontinuous.

Example 2.5.1 For $\mathrm{d} = 2$ let $\mathbf{X} = (X^{(1)}, 0)$, where $X^{(1)}$ is uniformly distributed on $[0, 1]$. Then, the bivariate distribution $\mathbf{F}$ of $\mathbf{X}$ is discontinuous along the positive x axis. The sequence of records $(\mathcal{R}_k)$, from iid observations distributed as $\mathbf{X}$, consists of pairs $(R_k, 0)$, where the R_k are standard records from the uniform distribution, and so $\mathcal{N}_\infty = \infty$, while there is just one strict record.

Observe that $\mathbf{F}$, in the example above, satisfies Assumption 1, which is known to be sufficient for $\mathcal{N}_\infty = \infty$. We consider in Assumption 3, a similar condition for strict records, based in a modified version of $\bar{\mathbf{F}}$, denoted $\bar{\mathbf{F}}^s$. We state first the strict version of Proposition 2.2.2, followed by corollaries and other related results. Proofs are omitted when the same arguments of classical chain records apply and a reference is made to the corresponding non-strict result.

Proposition 2.5.1 For $n \geq 1, k \geq 1$,

$$\mathbb{P}(\mathcal{R}_{k+1}^s \preceq \mathbf{x}, \Delta_k^s = n | \mathcal{R}_k^s) = (1 - \bar{\mathbf{F}}^s(\mathcal{R}_k^s))^{n-1} \mathbf{F}((\mathcal{R}_k^s, \mathbf{x}]^s) 1_{\{\mathcal{R}_k^s \neq \diamond\}} \tag{2.5.2}$$

and

$$\mathbb{P}(\mathcal{R}_{k+1}^s \preceq_s \mathbf{x} | \mathcal{R}_k^s) = \frac{\mathbf{F}((\mathcal{R}_k^s, \mathbf{x}]^s)}{\bar{\mathbf{F}}^s(\mathcal{R}_k^s)} 1_{\{\mathcal{R}_k^s \neq \diamond\}}, \tag{2.5.3}$$

where $\bar{\mathbf{F}}^s(\mathbf{x}) := \mathbb{P}(\mathbf{X}_n \succ_s \mathbf{x})$ and $(\mathcal{R}_n^s, \mathbf{x}]^s = \{\mathbf{y} \in \mathbb{R}^\mathrm{d} : \mathcal{R}_n^s \prec_s \mathbf{y} \preceq \mathbf{x}\}$.

Proof. See Proposition 2.2.2. □

Corollary 2.5.1 $(\mathcal{R}^s_{k+1}, \Delta^s_k)_{k\geq 1}$ and $(\mathcal{R}^s_k)_{k\geq 1}$ are Markov chains with transition probabilities given by (2.5.2) and (2.5.3), respectively.

PROOF. See Corollary 2.2.1. □

Proposition 2.5.2 Conditionally on $\mathcal{G}^s = \sigma\{\mathcal{R}^s_k, k \geq 1\}$, the strict inter record times Δ^s_n are independent and geometrically distributed, provided that $\mathbb{P}(\mathcal{R}^s_k = \diamond) = 0$, for all k, with

$$\mathbb{P}(\Delta^s_k = n|\mathcal{G}^s) = \mathbb{P}(\Delta^s_k = n|\mathcal{R}^s_k) = (1 - \bar{\mathbf{F}}^s(\mathcal{R}^s_k))^{n-1}\bar{\mathbf{F}}^s(\mathcal{R}^s_k)1_{\{\mathcal{R}^s_k\neq\diamond\}}, \tag{2.5.4}$$

for $k \geq 1, n \geq 1$.

PROOF. See Proposition 2.2.3 □

Assumption 3 There exists a Borel subset A of $\mathbb{R}^{\mathrm{d}}$, such that $\mathbf{F}(A) = 1$ and $\bar{\mathbf{F}}^s(\mathbf{x}) > 0$, for all $\mathbf{x} \in A$.

Observe that Assumption 3 implies Assumption 1, because $\bar{\mathbf{F}}^s(\mathbf{x}) \leq \bar{\mathbf{F}}(\mathbf{x})$. In the remaining results of this section, $\mathbf{F}$ satisfies Assumption 3.

Proposition 2.5.3 $\mathcal{N}^s_\infty = \infty$ a.s.

PROOF. See Proposition 2.3.1. □

Proposition 2.5.4 The sequences of strict maxima $(\mathcal{M}^s_n)$ and strict records $(\mathcal{R}^s_n)$ are increasing, in the sense that, for all $k \geq 1$, $\mathcal{M}^s_k \preceq \mathcal{M}^s_{k+1}$ and $\mathcal{R}^s_k \prec_s \mathcal{R}^s_{k+1}$. Also, both sequences converge a.s. to the same, possibly random limit in $\bar{\mathbb{R}}^{\mathrm{d}} = [-\infty, \infty]^{\mathrm{d}}$. Furthermore, $\bar{\mathbf{F}}^s(\mathcal{M}^s_n)$ and $\bar{\mathbf{F}}^s(\mathcal{R}^s_n)$ converge a.s. to the same limit in $[0, 1]$.

PROOF. See Proposition 2.3.2. □

Proposition 2.5.5 Let $\mathcal{R}^s_\infty = \lim_n \mathcal{R}^s_n$. Then $\mathbb{P}(\mathcal{R}^s_\infty \preceq \mathbf{x}) = 0$, for any $\mathbf{x} \in \mathbb{R}^{\mathrm{d}}$, such that $\bar{\mathbf{F}}(\mathbf{x}) > 0$.

PROOF. See Proposition 2.3.3. □

Remark 2.5.1 Recalling that $\mathcal{R}^s_\infty = \mathcal{R}_\infty$, since $(\mathcal{R}^s_n)$ is subsequence of $(\mathcal{R}_n)$, the result of Proposition 2.5.5 does not look informative. Of course, it also holds under the stronger condition $\bar{\mathbf{F}}^s(\mathbf{x}) > 0$. The same comment applies to the analogues of Corollaries 2.3.1 and 2.3.2, since it trivially holds that $\bar{\mathbf{F}}^s(\mathcal{R}^s_\infty) = 0$ and $\mathbb{P}(\mathcal{R}^s_\infty \in A_{\mathbf{F}}) = 0$

Corollary 2.5.2 $\lim_n \bar{\mathbf{F}}^s(\mathcal{R}^s_n) = 0$. a.s.

PROOF. From Corollary 2.3.3, and because $(\mathcal{R}^s_n)$ is a subsequence of $(\mathcal{R}_n)$,

$$\lim_n \bar{\mathbf{F}}^s(\mathcal{R}^s_n) = \lim_n \bar{\mathbf{F}}^s(\mathcal{R}_n) \leq \lim_n \bar{\mathbf{F}}(\mathcal{R}_n) = 0.$$

□

We end this section about strict records, with the analogue of Proposition 2.3.1 and the corresponding corollaries.

Proposition 2.5.6

$$\limsup_{n\to\infty}\left|\frac{\log(\Delta_n^s \bar{\mathbf{F}}^s(\mathcal{R}_n^s))}{\log n}\right| \leq 1, \text{ a.s.} \tag{2.5.5}$$

Proof. See Proposition 2.3.1. □

Corollary 2.5.3 Let $c \in (0,1)$, then

$$\frac{\log \bar{\mathbf{F}}^s(\mathcal{R}_k^s)}{k} \to \log c \text{ a.s.} \quad \text{if and only if} \quad \frac{\log \Delta_k^s}{k} \to -\log c \text{ a.s.}$$

Proof. See Corollary 2.3.5. □

Corollary 2.5.4 Let $a \in \mathbb{R}$, then

$$\frac{\log \bar{\mathbf{F}}^s(\mathcal{R}_k^s) + ak}{\sqrt{k}} \overset{\mathcal{D}}{\to} N(0,\sigma^2) \quad \text{if and only if} \quad \frac{\log \Delta_k^s - ak}{\sqrt{k}} \overset{\mathcal{D}}{\to} N(0,\sigma^2).$$

Proof. See Corollary 2.3.6. □

2.5.3 Discrete observations with independent components

We study here strict maxima and record under the assumption that the iid observations $\mathbf{X}_n$ have independent discrete components. We use strict maxima instead of non-strict, because only the strict dominance guarantees independence of the marginal record processes. We focus on $\mathbb{Z}_+^{\mathrm{d}}$-valued observations $\mathbf{X}_n$, for simplicity.

Assumption 4 The observation vectors $\mathbf{X}_n$ are iid and have independent components $X_n^{(j)}, j = 1, \ldots, \mathrm{d}$, with respective discrete marginal distributions $F^{(j)}$. The support of $F^{(j)}$ is contained in $\mathbb{Z}_+$ and $\omega_{F^{(j)}} = +\infty$, $j = 1, \ldots, \mathrm{d}$.

It is easy to see that, under Assumption 4, $\mathcal{N}_\infty = \infty$ a.s. In the following results, Assumption 4 holds.

Proposition 2.5.7 Let $(\mathcal{R}_n^s)$ be the process of strict records, with $\mathcal{R}_n^s = (\mathcal{R}_n^{s(1)}, \ldots, \mathcal{R}_n^{s(\mathrm{d})})$. Then the sequences $(\mathcal{R}_n^{s(j)}), j = 1, \ldots, \mathrm{d}$, are independent Markov chains on $\mathbb{Z}_+$, with transition probabilities

$$\mathbb{P}(\mathcal{R}_{k+1}^{s(j)} \leq x^{(j)} | \mathcal{R}_k^{s(j)}) = \frac{F^{(j)}(x^{(j)}) - F^{(j)}(\mathcal{R}_k^{s(j)})}{1 - F^{(j)}(\mathcal{R}_k^{s(j)})} 1_{\{\mathcal{R}_k^{s(j)} \leq x^{(j)}\}}. \tag{2.5.6}$$

Proof. From Assumption 4, $\bar{\mathbf{F}}^s(\mathbf{x}) = \prod_{j=1}^{\mathrm{d}} \bar{F}^{(j)}(x^{(j)}) = \prod_{j=1}^{\mathrm{d}}(1 - F^{(j)}(x^{(j)}))$. From Proposition 2.5.1,

$$\mathbb{P}(\mathcal{R}_{k+1}^s \preceq \mathbf{x} | \mathcal{R}_k^s) = \frac{\prod_{j=1}^{\mathrm{d}}(F^{(j)}(x^{(j)}) - F^{(j)}(\mathcal{R}_k^{s(j)}))}{\prod_{j=1}^{\mathrm{d}}(1 - F^{(j)}(\mathcal{R}_k^{s(j)}))} 1_{\{\mathcal{R}_k^s \preceq \mathbf{x}\}}.$$

The rest of the proof is like that of Proposition 2.4.2. □

Remark 2.5.2 It is easy to see that, in general, the non-strict dominance $\succ$ does not yield a multiplicative decomposition of $\bar{\mathbf{F}}$. Indeed, for d = 2 we have

$$\begin{aligned}\bar{\mathbf{F}}(x^{(1)}, x^{(2)}) &= \mathbb{P}(X^{(1)} > x^{(1)}, X^{(2)} \geq x^{(2)}) + \mathbb{P}(X^{(1)} \geq x^{(1)}, X^{(2)} > x^{(2)}) \\ &\quad - \mathbb{P}(X^{(1)} > x^{(1)}, X^{(2)} > x^{(2)}) \\ &= \bar{F}^{(1)}(x^{(1)})\bar{F}^{(2)}(x^{(2)-}) + \bar{F}^{(1)}(x^{(1)^-})\bar{F}^{(2)}(x^{(2)}) - \bar{F}^{(1)}(x^{(1)})\bar{F}^{(2)}(x^{(2)}) \\ &= \bar{F}^{(1)}(x^{(1)^-})\bar{F}^{(2)}(x^{(2)-}) - F^{(1)}(\{x^{(1)}\})F^{(2)}(\{x^{(2)}\}),\end{aligned}$$

where $\bar{F}^{(j)}(x^{(j)-}) = \mathbb{P}(X^{(j)} \geq x^{(j)})$.

Corollary 2.5.5 The point process ξ of records of Definition 2.4.1 is distributed as the minimum of d independent Bernoulli processes on $\mathbb{Z}_+$, with respective probabilities $h_k^{(j)} := \mathbb{P}(X^{(j)} = k)/\mathbb{P}(X^{(j)} \geq k)$.

Proof. From theorem 1 in [53], it follows that the marginal record value process $(\mathcal{R}_k^{(j)})$ is a Bernoulli processes with probabilities equal to the hazard rates $h_k^{(j)}$ defined above. The conclusion follows from Lemma 2.4.1. □

The geometric distribution

Lemma 2.5.1 Let (X_n) be a sequence of iid geometric random variables, with parameter $p \in (0, 1)$, starting at 1. That is, $\mathbb{P}(X_n = k) = (1 - p)^{k-1}p, k = 1, 2, \ldots$. Let (R_k) be the sequence of record value from (X_n), then (R_k) is distributed as $(\sum_{\mathrm{i}=1}^{k} Y_{\mathrm{i}})$, where the Y_{i} are iid geometric, with parameter p.

Proof. The result is a direct consequence of Shorrock's theorem 1 in [53], which states that the point process of record values, for discrete random variables, is a Bernoulli process, with probabilities given by the hazard rates h_k. In the case of the geometric distribution, $h_k = p$, for all $k \geq 1$. □

Proposition 2.5.8 Let $\mathbf{X}_n$ have geometric marginals of parameters $p_j \in (0, 1), j = 1, \ldots, \mathrm{d}$. Then the sequence $(-\log \bar{\mathbf{F}}^s(\mathcal{R}_k^s))$ is distributed as the sequence $(-\sum_{j=1}^{\mathrm{d}} \log(1-p_j) \sum_{\mathrm{i}=1}^{k} Y_{\mathrm{i}j})$, where the random variables $Y_{\mathrm{i}j}, \mathrm{i} \geq 1, j = 1, \ldots, \mathrm{d}$, are iid geometric, with parameter p_j.

Proof. Observe that $\bar{F}^{(j)}(k) = (1-p_j)^k, k \geq 1, j = 1, \ldots, \mathrm{d}$. Then $\bar{\mathbf{F}}^s(\mathbf{k}) = \prod_{j=1}^{\mathrm{d}} \bar{F}^{(j)}(k^{(j)}) = \prod_{j=1}^{\mathrm{d}}(1-p_j)^{k^{(j)}}$, and $\log \bar{\mathbf{F}}^s(\mathcal{R}_k^s) = \sum_{j=1}^{\mathrm{d}} \log(1-p_j)\mathcal{R}_k^{s(j)}$. From Lemma 2.5.1, the conclusion follows. □

Corollary 2.5.6 Under the hypotheses of Proposition 2.5.8,

$$\frac{\log \Delta_k^s}{k} \to \mu \text{ a.s.} \quad \text{and} \qquad \frac{\log \Delta_k^s - k\mu}{\sqrt{k}} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \tag{2.5.7}$$

where $\mu = -\sum_{j=1}^{\mathrm{d}} \frac{\log(1-p_j)}{p_j}$ and $\sigma^2 = \sum_{j=1}^{\mathrm{d}}(1-p_j)\left(\frac{\log(1-p_j)}{p_j}\right)^2$.

Proof. By Proposition 2.5.8 and the SLLN applied to the variables $Z_{\mathrm{i}} = \sum_{j=1}^{\mathrm{d}} \log(1-p_j)Y_{\mathrm{i}j}$, we have $\frac{-\log \bar{\mathbf{F}}^s(\mathcal{R}_k^s)}{k} \to \mu$ a.s. Then, the first convergence in (2.5.7) follows from Corollary 2.5.3. The second convergence in (2.5.7) follows from the CLT and Corollary 2.5.4, noting that $Var(Z_{\mathrm{i}}) = \sigma^2$. □

Observe that, from Lemma 2.5.1, we obtain a limiting distribution for record values of the geometric model.

Proposition 2.5.9 Under the hypotheses of Proposition 2.5.8,

$$\frac{\mathcal{R}_k^{s(j)} - k/p_j}{\sqrt{k}} \xrightarrow{\mathcal{D}} N(0, \tau_j^2), \tag{2.5.8}$$

where $\tau_j^2 = (1-p_j)^2/p_j^2$. Also

$$\frac{\mathcal{R}_k^s - k\mathbf{u}}{\sqrt{k}} \xrightarrow{\mathcal{D}} MN(\mathbf{0}, \Sigma), \tag{2.5.9}$$

where $\mathbf{u} = (1/p_1, \ldots, 1/p_{\mathrm{d}})$, Σ is the diagonal matrix with elements τ_j^2 and $MN(\mathbf{0}, \Sigma)$ denotes the multivariate centered normal distribution, with covariance matrix Σ.

Proof. The convergence of each marginal record process follows from Lemma 2.5.1 and the CLT. The multivariate limit follows from Proposition 2.5.7. □

Remark 2.5.3 In Proposition 2.5.9 we show that records from the geometric model have a limiting distribution, although the sequence of maxima does not. This is in contrast with continuous distributions; see [49].

Chapter 3

The uniform model on the hypercube

In this chapter we study chain maxima and records from observations $\mathbf{X}_n$, distributed according to the $\mathbf{U}[0,1]^{\mathrm{d}}$ model, that is, the components $X_n^{(j)}$ are iid uniformly distributed in $[0,1]$. This means that all results in Section 2.4.1 are valid here. We analyze this particular model because this is the case considered in [25], and also because the analysis has a combinatorial aspect that we consider interesting. As in Chapter 2, we omit the word *chain* from objects, such as maxima or records, if no confusion arises. We consider Assumption 5 below to hold in this chapter, unless stated otherwise. Observe that, under this assumption, $\mathcal{N}_\infty = \infty$ a.s. (see Section 2.4.1).

Assumption 5 Along this chapter, the observation vectors $\mathbf{X}_n$ are iid, uniformly distributed on $[0,1]^{\mathrm{d}}$. This is model $\mathbf{U}[0,1]^{\mathrm{d}}$.

3.1 Records

We restate some results of Section 2.4.1 in the context of Assumption 5.

Proposition 3.1.1 Let $(\mathcal{R}_n) = (\mathcal{R}_n^{(1)}, \ldots, \mathcal{R}_n^{(\mathrm{d})})$. Then the processes $(\mathcal{R}_n^{(j)})$, $j = 1, \ldots, \mathrm{d}$, are independent and identically distributed Markov chains on $[0,1]$, with initial states $\mathcal{R}_1^{(j)} = X_1^{(j)}$ and transition probabilities

$$\mathbb{P}(\mathcal{R}_{k+1}^{(j)} \le x^{(j)} | \mathcal{R}_k^{(j)}) = \frac{x^{(j)} - \mathcal{R}_k^{(j)}}{1 - \mathcal{R}_k^{(j)}} 1_{\{\mathcal{R}_k^{(j)} \le x^{(j)}\}}, \ x^{(j)} \in [0,1]. \tag{3.1.1}$$

Proof. Direct from Proposition 2.4.2. □

Proposition 3.1.2 $\mathcal{R}_k^{(j)}$ has density function given by

$$g_k(x) = \frac{(-\log(1-x))^{k-1}}{(k-1)!} 1_{[0,1]}(x), \quad k \ge 1. \tag{3.1.2}$$

That is, $-\log(1-\mathcal{R}_k^{(j)})$ has $\Gamma(1,k)$ distribution.

Proof. $-\log(1-X_k^{(j)})$ has exponential distribution so, from Proposition 2.4.3, $-\log(1-\mathcal{R}_k^{(j)})$ is distributed as the k-th arrival time, of a homogeneous Poisson process of parameter 1. □

Remark 3.1.1 The result of Proposition 3.1.2 can also be derived from a recurrence related to the Markovian property of $\mathcal{R}_k^{(j)}$. If $G_k(x)$ denotes the distribution function of $\mathcal{R}_k^{(j)}$, then, from (3.1.1) we have

$$G_{k+1}(x) = \int_0^x \frac{x-s}{1-s} G_k(\mathrm{d}s),$$

which yields the distribution, with density g_k in (3.1.2). It is also interesting to see that $\frac{\log(1-\mathcal{R}_k^{(j)})+k}{\sqrt{k}}$ is asymptotically $N(0,1)$.

Proposition 3.1.3 Let (U_n) be a sequence of iid random variables, uniformly distributed on $[0,1]$ ($U[0,1]$ for short), and let (V_n) be defined by the recursion

$$V_{n+1} = V_n + (1-V_n)U_{n+1},\ n \geq 1, \tag{3.1.3}$$

with $V_1 = U_1$. Then $(V_n) \stackrel{\mathcal{D}}{=} (\mathcal{R}_n^{(j)})$, $j = 1, \ldots, \mathrm{d}$.

Proof. It suffices to show that (V_n) is a Markov chain, with initial state distributed as $X_1^{(1)}$ and transitions given by (3.1.1). Clearly V_1 and $X_1^{(1)}$ are both $U[0,1]$. Furthermore, (V_n) is also clearly Markovian, with

$$\mathbb{P}(V_{n+1} \leq v | V_n) = \mathbb{P}\left(U_{n+1} \leq \frac{v-V_n}{1-V_n}\Big| V_n\right) = \frac{v-V_n}{1-V_n} 1_{\{V_n \leq v\}},\ v \in [0,1]. \tag{3.1.4}$$

□

We consider recurrence (3.1.3) as a stochastic difference equation (sde) and investigate its solutions. Note that it can be equivalently written as

$$V_{n+1} = V_n(1-U_{n+1}) + U_{n+1}, \tag{3.1.5}$$

which has the structure of a sde studied in [58], namely $Y_n = A_n Y_{n-1} + B_n$. In our case we have $A_n = 1 - U_n$ and $B_n = U_n$, with (U_n) iid $U[0,1]$. Such recursions appear in a variety of models in finance, chemistry and biology, and are also related to random walks in random environment. For more information and references, see [58].

Lemma 3.1.1 Let (U_n) be an iid sequence of $U[0,1]$ random variables. Then, the sde in (3.1.5), with initial condition $V_1 = U_1$, has (strong) solution

$$\sum_{\mathrm{i}=1}^{n} U_\mathrm{i} \prod_{j=\mathrm{i}+1}^{n} (1-U_j) = 1 - \prod_{\mathrm{i}=1}^{n}(1-U_\mathrm{i}),\ n \geq 1. \tag{3.1.6}$$

Proof. By a strong solution of (3.1.5) we mean a sequence, which solves the recurrence a.s. The first formula in (3.1.6) is obtained by iterating (3.1.5). The second follows from the recurrence $1 - V_{n+1} = (1 - V_n)(1 - U_{n+1})$, equivalent to (3.1.3). As usual, sums (products) over an empty set of indexes are given the value 0 (1). □

Remark 3.1.2 From Lemma 3.1.1 we have a view of how fast records $\mathcal{R}_k^{(j)}$ converge to 1. Observe also that $\sum_{\mathrm{i}=1}^{n} U_{\mathrm{i}} \prod_{j=\mathrm{i}+1}^{n} (1 - U_j)$ and $\sum_{\mathrm{i}=1}^{n} U_{\mathrm{i}} \prod_{j=1}^{\mathrm{i}-1} (1 - U_j)$ are equally distributed sequences but the second is not a strong solution although it can be seen as weak solution or solution in distribution. There exists abundant literature on the behavior of sums of the form $\sum_{\mathrm{i}=1}^{\infty} U_{\mathrm{i}} \prod_{j=1}^{\mathrm{i}-1} (1 - U_j)$, which are commonly known as perpetuities, and are related to iterative schemes such as (3.1.5). The interested reader can consult [17, 26, 39].

3.2 Record heights

In this section we study the so-called record heights, as defined in [25]. We start by obtaining their moments and then derive weak convergence, with a suitable normalization. The asymptotic results are obtained using tools from the theory of singularity analysis, developed mainly by Ph. Flajolet.

Definition 3.2.1 Let the n-th record height be defined by

$$\mathcal{H}_n = \bar{\mathbf{F}}(\mathcal{M}_n), \ n \geq 1.$$

Note that $\bar{\mathbf{F}}(\mathcal{M}_n) = \prod_{j=1}^{\mathrm{d}} (1 - \mathcal{M}_n^{(j)})$, for observations from the $\mathbf{U}([0,1]^{\mathrm{d}})$ model and that $\mathcal{H}_n$ is the probability of $\mathbf{X}_{n+1}$ being a record, conditional on the past observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$. In the following proposition we exhibit a recurrence for the moments of $\mathcal{H}_n$.

Proposition 3.2.1 Let $\mu_n^{(k)} = \mathbb{E}(\mathcal{H}_n^k)$, for $k \geq 0, n \geq 1$ integers. Then the following recursion holds

$$\mu_{n+1}^{(k)} = \mu_n^{(k)} - \left(1 - \mu_1^{(k)}\right) \mu_n^{(k+1)}, \tag{3.2.1}$$

where $\mu_1^{(k)} = \frac{1}{(k+1)^{\mathrm{d}}}$.

PROOF. From equation (2.2.5) in Lemma 2.2.1, with $g(\mathbf{y}) = \bar{\mathbf{F}}(\mathbf{y})^k$, we have

$$
\begin{aligned}
\mu_{n+1}^{(k)} &= \mathbb{E}\left(\int_{[0,1]^{\mathrm{d}} \cap \{\mathbf{y} \succ \mathcal{M}_n\}} \prod_{j=1}^{\mathrm{d}} (1 - y^{(j)})^k \mathrm{d}\mathbf{y} \right) + \mathbb{E}\left(\prod_{j=1}^{\mathrm{d}} (1 - \mathcal{M}_n^{(j)})^k (1 - \mathcal{H}_n) \right) \\
&= \mathbb{E}\left(\prod_{j=1}^{\mathrm{d}} \int_{\mathcal{M}_n^{(j)}}^{1} (1 - y^{(j)})^k \mathrm{d}y^{(j)} \right) + \mu_n^{(k)} - \mu_n^{(k+1)} \\
&= \mathbb{E}\left(\prod_{j=1}^{\mathrm{d}} \frac{(1 - \mathcal{M}_n^{(j)})^{k+1}}{k+1} \right) + \mu_n^{(k)} - \mu_n^{(k+1)} \\
&= \frac{\mu_n^{(k+1)}}{(k+1)^{\mathrm{d}}} + \mu_n^{(k)} - \mu_n^{(k+1)}.
\end{aligned}
$$

□

Proposition 3.2.2 The solution of the recurrence (3.2.1) has the form

$$
\mu_{n+1}^{(k)} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mu_1^{(k+j)} \prod_{\mathrm{i}=0}^{j-1} \left(1 - \mu_1^{(k+\mathrm{i})}\right), \ n \geq 0. \tag{3.2.2}
$$

PROOF. Formula (3.2.2) is obtained by iterating (3.2.1) and can be checked by direct substitution. Indeed, let $b_j^{(k)} = \mu_1^{(k+j)} \prod_{\mathrm{i}=0}^{j-1} (1 - \mu_1^{(k+\mathrm{i})})$ and observe that $b_{j+1}^{(k)} = b_j^{(k+1)}(1 - \mu_1^{(k)})$. Then

$$
\begin{aligned}
\mu_{n+1}^{(k)} - \mu_n^{(k)} &= \sum_{j=1}^{n-1} \left[\binom{n}{j} - \binom{n-1}{j} \right] (-1)^j b_j^{(k)} + (-1)^n b_n^{(k)} \\
&= \sum_{j=1}^{n-1} \binom{n-1}{j-1} (-1)^j b_j^{(k)} + (-1)^n b_n^{(k)} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j+1} b_{j+1}^{(k)} \\
&= -(1 - \mu_1^{(k)}) \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j+1} b_{j+1}^{(k)} \\
&= -\left(1 - \mu_1^{(k)}\right) \mu_n^{(k+1)}.
\end{aligned}
$$

□

In one and two dimensions we have closed-form expressions for $\mu_n^{(k)}$.

Lemma 3.2.1 (i) For $\mathrm{d} = 1$,

$$
\mu_n^{(k)} = \binom{n+k}{k}^{-1}. \tag{3.2.3}
$$

(ii) For $\mathrm{d} = 2$,

$$
\mu_n^{(k)} = \frac{1}{k+1} \binom{n+k}{k}^{-1}. \tag{3.2.4}
$$

PROOF. It suffices to iterate the recursion (3.2.1). □

Remark 3.2.1 The result for d = 1 is well-known and corresponds to the k-th moment of the minimum of n iid $U[0,1]$ random variables, which can be calculated as $n\int_0^1 x^k(1-x)^{n-1}\mathrm{d}x$.

3.2.1 Asymptotic analysis of $\mu_n^{(k)}$

We investigate the asymptotic behavior of moments $\mu_n^{(k)}$, as $n \to \infty$. This information will be used to establish convergence in distribution of record heights. We start with the special cases d = 1, 2, which are relatively easy, since we have the explicit formulas. We use the symbol $\sim$ to denote asymptotic equivalence of two sequences: $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.

Lemma 3.2.2 (i) For d=1

$$\mu_n^{(k)} \sim k!n^{-k}. \tag{3.2.5}$$

(ii) For d=2

$$\mu_n^{(k)} \sim \frac{k!}{k+1}n^{-k}. \tag{3.2.6}$$

PROOF. From Stirling's approximation $n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$, we have

$$\binom{n+k}{k}^{-1} \sim \frac{k!\sqrt{2\pi n}n^n\mathrm{e}^{-n}}{\sqrt{2\pi(n+k)}(n+k)^{n+k}\mathrm{e}^{-(n+k)}} \sim k!n^{-k}.$$

The results then follows from Lemma 3.2.1. □

Euler transform and the Nörlund-Rice representation

Unlike the simple asymptotic results for $\mu_n^{(k)}$ in dimensions d = 1, 2, the situation for d > 2 is quite different. We have not been able to find a closed-form expression for $\mu_n^{(k)}$, such as those in Lemma 3.2.1 and so we must deal directly with (3.2.2), which is a so-called alternating sum. This type of expression appears frequently in the analysis of algorithms and data structures. It is of some interest here to recall that a sum of the form

$$s_n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j a_j \tag{3.2.7}$$

defines the alternating binomial transform of the sequence (a_n), also called Euler transform in [22]. This is an involutive linear transform on sequences of real numbers. In fact, it is just the iterated forward-difference operator.

The asymptotic analysis of sums such as (3.2.7) was considered in the pioneering paper of Flajolet and Sedgewick [22], where the authors state that the problem is delicate. The difficulty, which prevents from using elementary techniques, is the phenomenon of *exponential* *cancellation*, which is often coupled with small periodic oscillations. The starting point of this technique is the Nörlund-Rice integral representation, which allows to write (3.2.7) as a complex integral of an analytical continuation $a(z)$ of the function $n \to a_n$, in the sense that $a(z)$ is analytic in a suitable domain $D \subset \mathbb{C}$ and interpolates a_n, that is $a(n) = a_n$, for all $n \geq 0$. For completeness, we state the following result; for a proof, see lemma 1 in [22].

Lemma 3.2.3 Let $a(z)$ be an analytic continuation of the sequence a_n in a domain containing the interval $[0, \infty)$. Then

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j a_j = \frac{(-1)^n n!}{2\mathrm{i}\pi} \int_{\mathcal{C}} \frac{a(z)}{z(z-1)\dots(z-n)} \mathrm{d}z, \tag{3.2.8}$$

where $\mathcal{C}$ is a positively oriented and closed curve, contained in the domain of analyticity of $a(s)$, which encircles the poles $0, 1, \dots, n$ and no others.

When the analytic continuation $a(z)$ of a_n is a rational function, the Nörlund Rice representation can be given in terms of residues, as indicated in theorem 1 of [22]. More generally, if $a(z)$ is meromorphic on $\mathbb{C}$, we have the following.

Theorem 3.2.1 If $a(z)$ is meromorphic in $\mathbb{C}$ and analytic in $[0, \infty) \cup \Omega$, where $\Omega = \bigcup_{j \geq 1} \gamma_j$ and the γ_j are concentric circles, positively oriented, with radii tending to infinity and if, additionally, $a(z)$ has at most polynomial growth on Ω then, for n large enough, then

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j a_j = -(-1)^n n! \sum_{z \in \mathcal{P}} \mathrm{Res} \left[\frac{a(z)}{z(z-1)\dots(z-n)} \right], \tag{3.2.9}$$

where $\mathcal{P}$ is the set of poles of $\frac{a(z)}{z(z-1)\dots(z-n)}$, not belonging to $[0, \infty)$.

Proof. See theorem 2 in [22]. □

Proposition 3.2.3 Let $b_j^{(k)} = \mu_1^{(k+j)} \prod_{\mathrm{i}=0}^{j-1} (1 - \mu_1^{(k+\mathrm{i})}), j \geq 0$, be the sequence in the alternating sum (3.2.2) and let

$$\mathcal{P}_{\mathrm{d}} = \bigcup_{l=1}^{\mathrm{d}} \{r_l - k - 1, r_l - k - 2, \dots\} \setminus \{-k-1, -k-2, \dots\},$$

where $r_l := \mathrm{e}^{2\mathrm{i}\pi l/\mathrm{d}}, l = 1, \dots, \mathrm{d}$, are the d-th roots of unity. Let $D = \mathbb{C} \setminus \mathcal{P}_{\mathrm{d}}$ and $B_k = \prod_{n \geq k+1} \left(1 - \frac{1}{n^{\mathrm{d}}}\right)$. Then

$$\varphi(z) = \frac{B_k}{(k+1+z)^{\mathrm{d}}} \prod_{n \geq k+1} \frac{(n+z)^{\mathrm{d}}}{(n+z)^{\mathrm{d}} - 1}, \ z \in D, \tag{3.2.10}$$

is the analytic continuation of the sequence $(b_j^{(k)})_{j \geq 0}$ to the domain D. Furthermore, the singularities of φ are isolated poles, elements of the countable set $\mathcal{P}_{\mathrm{d}}$, while the integers $-k-2, -k-3, \dots$ are the zeros of φ.

PROOF. We check first that φ is analytic in D. To that end we observe that

$$\prod_{n\geq k+1} \frac{(n+z)^{\mathrm{d}}}{(n+z)^{\mathrm{d}}-1} = \lim_{m\to\infty} \prod_{n=k+1}^{m} \frac{(n+z)^{\mathrm{d}}}{(n+z)^{\mathrm{d}}-1}$$

exists and is analytic, if the series $\sum_{n\geq k+1} \frac{1}{|(n+z)^{\mathrm{d}}-1|}$ converges locally uniformly. Let V be a disk contained in D. Then, for any $z \in V$ and sufficiently large n, we have $|z+n| > n-C > 1$, where C is a positive constant, and so, $|(n+z)^{\mathrm{d}}-1| > (n-C)^{\mathrm{d}}-1$, and we conclude that φ is analytic in D.

Moreover, it is apparent from formula (3.2.10), that the only potential singularities of φ are poles, given by $-n+r_l, n \geq k+1, l = 1,\ldots,\mathrm{d}$. This set of potential poles include the integers $-k, -k-1, -k-2,\ldots$ but, as we see below, only $-k$ and $-k-1$ are actual poles, the rest being zeros of φ, due to cancellation. Indeed, for $m \geq k+2$,

$$(m+z)^{\mathrm{d}}-1 = (m-1+z)\left[(m+z)^{\mathrm{d}-1}+\cdots+(m+z)+1\right]$$

and so, $(m-1+z)$ in the denominator of the m-th term of the product, also appears in the numerator of the $(n-1)$-term and they cancel out. To conclude, we verify that the function φ interpolates the sequence $b_j^{(k)}$. Observe that, for $j \geq 0$,

$$\begin{aligned}\varphi(j) &= \frac{B_k}{(k+1+j)^{\mathrm{d}}} \prod_{n\geq k+1} \frac{(n+j)^{\mathrm{d}}}{(n+j)^{\mathrm{d}}-1} \\ &= \frac{1}{(k+1+j)^{\mathrm{d}}} \prod_{n\geq k+1} \frac{n^{\mathrm{d}}-1}{n^{\mathrm{d}}} \prod_{n\geq k+j+1} \frac{n^{\mathrm{d}}}{n^{\mathrm{d}}-1} \\ &= b_j^{(k)}.\end{aligned}$$

□

We state some technical lemmas related to the problem of bounding $|\varphi(z)|$ on circles, as required in Theorem 3.2.1.

Lemma 3.2.4 Let $m \in \mathbb{Z}$, such that $m > k+2$ and let

$$A_m = \{n : n \geq k+1, |n-m| > 1\}.$$

Then

$$S_m := \sum_{n\in A_m} \frac{1}{(n-m)^2-1} \leq \frac{3}{2}. \tag{3.2.11}$$

PROOF. Observe that $A_m = \{k+1,\ldots,m-2\} \cup \{m+2,\ldots\}$, so

$$\begin{aligned}S_m &= \sum_{n=k+1}^{m-2} \frac{1}{(n-m)^2-1} + \sum_{n\geq m+2} \frac{1}{(n-m)^2-1} \\ &= \sum_{n=2}^{m-k-1} \frac{1}{n^2-1} + \sum_{n\geq 2} \frac{1}{n^2-1} \leq 2\sum_{n\geq 2} \frac{1}{n^2-1} = \frac{3}{2}.\end{aligned}$$

□

Lemma 3.2.5 Let $m \in \mathbb{Z}$, such that $m > k+2$, $\varepsilon \in (0,1)$ and let

$$A_{m,\varepsilon} = \{n : n \geq k+1, |n-m-\varepsilon| > 1\}.$$

Then

$$S_{m,\varepsilon} := \sum_{n \in A_{m,\varepsilon}} \frac{1}{(n-m-\varepsilon)^2 - 1} \leq \frac{3}{2} + \frac{1}{(1+\varepsilon)^2 - 1} + \frac{1}{(2-\varepsilon)^2 - 1}.$$

Proof. Notice that $A_{m,\varepsilon} = \{k+1, \ldots, m-1\} \cup \{m+2, \ldots\}$, so

$$\begin{aligned} S_{m,\varepsilon} &= \sum_{n=k+1}^{m-1} \frac{1}{(n-m-\varepsilon)^2 - 1} + \sum_{n \geq m+2} \frac{1}{(n-m-\varepsilon)^2 - 1} \\ &= \sum_{n=1}^{m-k-1} \frac{1}{(n+\varepsilon)^2 - 1} + \sum_{n \geq 2} \frac{1}{(n-\varepsilon)^2 - 1} \\ &\leq \frac{1}{(1+\varepsilon)^2 - 1} + \sum_{n=2}^{m-k-1} \frac{1}{n^2 - 1} + \frac{1}{(2-\varepsilon)^2 - 1} + \sum_{n \geq 2} \frac{1}{n^2 - 1} \end{aligned}$$

and the conclusion follows from Lemma 3.2.4. □

Lemma 3.2.6 Let

$$R_{m,\delta} := \{z \in \mathbb{C} : -m-\delta < \Re(z) < -m\},$$

for $m \geq k+1$ and $\delta \in (0,1)$. Then there exists δ, such that $R_{m,\delta} \cap \mathcal{P}_{\mathrm{d}} = \emptyset$.

Proof. The assertion is a simple consequence of the fact that poles of φ are d-th roots of unity, around the negative integers $-m \leq -k-1$. A graphical analysis yields that δ can be chosen equal to (or less than) the distance from the line $\Re(z) = -m$ and the nearest pole, with real part less than $-m$. See Figure 3.1. □

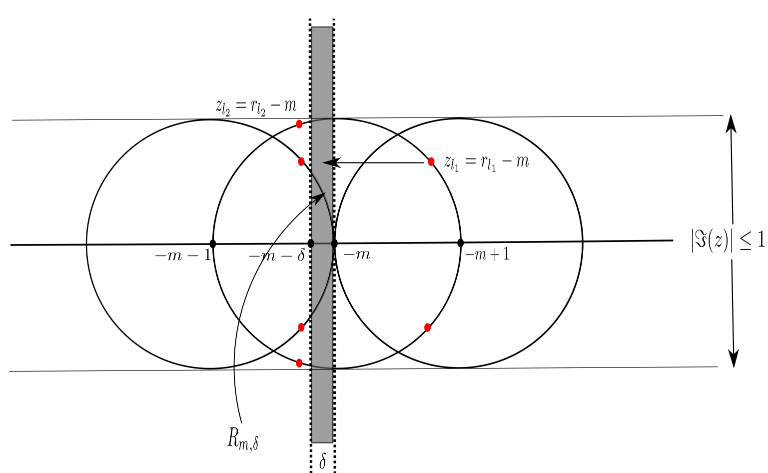


Figure 3.1: The nearest poles to the band $R_{m,\delta}$, are associated to $l_1 = 1$ and $l_2 = \lfloor \mathrm{d}/4 \rfloor + 1$.

Proposition 3.2.4 For $m \geq k+2$, let $\gamma_m = \{z \in \mathbb{C} : |z| = m + \varepsilon\}$ be the circle centered at 0, with radius $m + \varepsilon$, where $\varepsilon < \delta/2$ and δ satisfies the condition of Lemma 3.2.6. Then there exists M_0, such that $|\varphi(z)|$ is bounded on the set $\Omega := \bigcup_{m \geq M_0} \gamma_m$, where φ is defined in (3.2.10).

Proof. For $m \geq k+2$, let $A_{m,\varepsilon}$ be as defined in Lemma 3.2.5. Recalling that the infinite product defining φ converges absolutely, then, for any $z \in D$, we have

$$\begin{aligned} |\varphi(z)| =& \frac{B_k}{|k+1+z|^{\mathrm{d}}} \prod_{n \geq k+1} \frac{|n+z|^{\mathrm{d}}}{|(n+z)^{\mathrm{d}} - 1|} \\ =& \frac{B_k}{|k+1+z|^{\mathrm{d}}} \prod_{n \notin A_{m,\varepsilon}} \frac{|n+z|^{\mathrm{d}}}{|(n+z)^{\mathrm{d}} - 1|} \prod_{n \in A_{m,\varepsilon}} \frac{|n+z|^{\mathrm{d}}}{|(n+z)^{\mathrm{d}} - 1|}. \end{aligned}$$

Let $z \in \gamma_m$. Then, from the definition of $A_{m,\varepsilon}$, $|n+z| \geq |n - |z|| = |n - m - \varepsilon| > 1$. So, $|(n+z)^{\mathrm{d}} - 1| > |n+z|^{\mathrm{d}} - 1$, which implies

$$\prod_{n \in A_{m,\varepsilon}} \frac{|n+z|^{\mathrm{d}}}{|(n+z)^{\mathrm{d}} - 1|} \leq \prod_{n \in A_{m,\varepsilon}} \frac{|n+z|^{\mathrm{d}}}{|n+z|^{\mathrm{d}} - 1} \leq \prod_{n \in A_{m,\varepsilon}} \left(1 + \frac{1}{|n-m-\varepsilon|^{\mathrm{d}} - 1}\right). \tag{3.2.12}$$

Observe that, from the elementary inequality $\log(1+x) \leq x, x > -1$, the rightmost term of (3.2.12) can be bounded by

$$\begin{aligned} \exp\left(\sum_{n \in A_{m,\varepsilon}} \frac{1}{|n-m-\varepsilon|^{\mathrm{d}} - 1}\right) &\leq \exp\left(\sum_{n \in A_{m,\varepsilon}} \frac{1}{(n-m-\varepsilon)^2 - 1}\right) \\ &\leq \exp\left(\frac{3}{2} + \frac{1}{(1+\varepsilon)^2 - 1} + \frac{1}{(2-\varepsilon)^2 - 1}\right), \end{aligned}$$

by Lemma 3.2.5. Furthermore, for $z \in \gamma_m$, we have

$$\frac{B_k}{|k+1+z|^{\mathrm{d}}} \leq |k+1-m-\varepsilon|^{-\mathrm{d}} \leq (1+\varepsilon)^{-\mathrm{d}}.$$

Finally, we consider the remaining term of $|\varphi(z)|$, for $z \in \gamma_m$, that is,

$$\prod_{n \notin A_{m,\varepsilon}} \frac{|n+z|^{\mathrm{d}}}{|(n+z)^{\mathrm{d}} - 1|} = \left|\frac{(m+z)^{\mathrm{d}}}{(m+z)^{\mathrm{d}} - 1}\right| \left|\frac{(m+1+z)^{\mathrm{d}}}{(m+1+z)^{\mathrm{d}} - 1}\right|, \tag{3.2.13}$$

provided that $z+m$ or $z+m+1$ are not d-th roots of unity. This is so if m is chosen such that $\gamma_m \cap \{z : |\Im(z)| \leq 1\} \subset R_{m,\delta}$, where $R_{m,\delta}$ is defined in Lemma 3.2.6. It is easy to see that such value of m satisfies $\sqrt{(m+\varepsilon)^2 - 1} > m$, which yields $m > (1-\varepsilon^2)/(2\varepsilon)$. So it suffices to take $M_0 > 1/(2\varepsilon)$.

We proceed to check that the function in (3.2.13) is bounded in γ_m, for $m \geq M_0$. Observe that

$$g_m(z) := \left|\frac{(m+z)^{\mathrm{d}}}{(m+z)^{\mathrm{d}} - 1}\right| = \left|1 + \frac{1}{(m+z)^{\mathrm{d}} - 1}\right| \leq 1 + \frac{1}{|(m+z)^{\mathrm{d}} - 1|},$$

hence,

$$\sup_{z\in\gamma_m} g_m(z) \le 1 + \frac{1}{\inf_{z\in\gamma_m} |(m+z)^{\mathrm{d}} - 1|}.$$

Note that $|(m+z)^{\mathrm{d}} - 1| = \prod_{l=0}^{\mathrm{d}-1} |z - z_l|$, where z_l is such that $(m+z_l)^{\mathrm{d}} = 1$. Hence, $\inf_{z\in\gamma_m} |(m+z)^{\mathrm{d}} - 1| \ge \prod_{l=0}^{\mathrm{d}-1} \inf_{z\in\gamma_m} |z - z_l|$. Finally observe, from elementary geometric arguments (see Figure 3.2), that

$$\inf_{z\in\gamma_m} |z - z_l| = |m + \varepsilon - |z_l||. \tag{3.2.14}$$

If $|z_l| \le m + \varepsilon$, then $|z_l| \le \sqrt{m^2+1}$ and

$$\inf_{z\in\gamma_m} |z - z_l| = m + \varepsilon - |z_l| \ge m + \varepsilon - \sqrt{m^2+1} = \frac{2m\varepsilon + \varepsilon^2 - 1}{m + \varepsilon + \sqrt{m^2+1}} \to \varepsilon,$$

as $m \to \infty$. Then, for sufficiently large m, $\inf_{z\in\gamma_m} |z - z_l| \ge \varepsilon/2$. On the other hand, if $|z_l| > m + \varepsilon$, then $|z_l| \ge \sqrt{(m+\delta)^2 + 1 - \delta^2}$ and

$$\inf_{z\in\gamma_m} |z - z_l| = |z_l| - (m+\varepsilon) \ge \sqrt{(m+\delta)^2 + 1 - \delta^2} - (m+\varepsilon) = \frac{2m(\delta-\varepsilon) + \varepsilon^2 - 1}{m + \varepsilon + \sqrt{m^2 + 2\delta m + 1}} \to \delta - \varepsilon,$$

as $m \to \infty$.

Then, for sufficiently large m, we have $\inf_{z\in\gamma_m} |z - z_l| \ge \delta - 2\varepsilon$. From the two cases considered above, we conclude that $\sup_{z\in\gamma_m} g_m(z) \le 1 + C^{\mathrm{d}}$, where $C = \max\{2/\varepsilon, 1/(\delta - 2\varepsilon)\}$. The same argument is valid for g_{m+1} and we see that the function in (3.2.13) is bounded on γ_m by a constant independent of m and so, is bounded on Ω. Last, we conclude from the bounds above that $|\varphi|$ is bounded on Ω. □

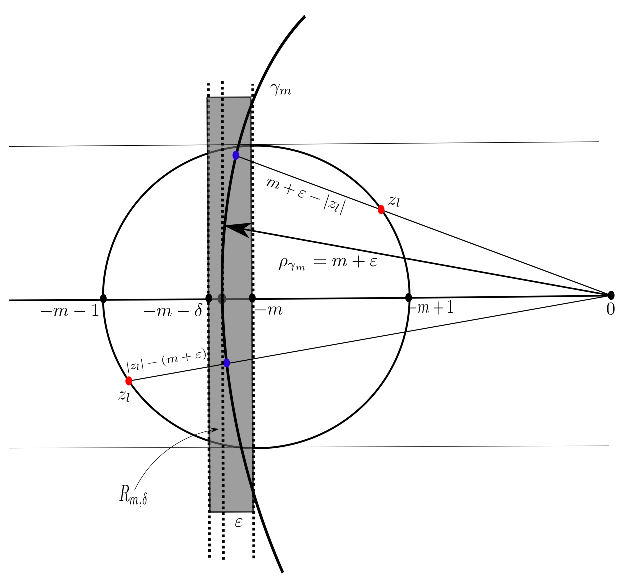


Figure 3.2: Geometric point of view in (3.2.14).

Theorem 3.2.2 For sufficiently large n,

$$\mu_{n+1}^{(k)} = -(-1)^n \sum_{s\in\mathcal{P}_{\mathrm{d}}} \operatorname*{Res}_{z=s} \left[\frac{n!\varphi(z)}{z(z-1)\dots(z-n)} \right], \tag{3.2.15}$$

where for a ψ function $\operatorname*{Res}_{z=s}\psi(z)$ denotes the residue of $\psi(z)$ at $z=s$, while the function φ and set $\mathcal{P}_{\mathrm{d}}$ are defined in Proposition 3.2.3.

Proof. The result follows from Propositions 3.2.3, 3.2.4 and Theorem 3.2.1. □

Remark 3.2.2 The asymptotic analysis based on the Nörlund-Rice integral representation has been widely used in combinatorics and theoretical computer science. However, up to our knowledge, this is the first application of such techniques in Extreme Value Theory.

Computation of residues

We proceed to determine the residues in the right-hand side of (3.2.15).

Proposition 3.2.5 Let $\psi(z)=\frac{n!\varphi(z)}{z(z-1)\ldots(z-n)}$, then

$$\operatorname*{Res}_{z=-k}\psi(z)=\frac{(-1)^{n+1}}{k\mathrm{d}}\binom{n+k}{k}^{-1}\prod_{j=2}^{k}\Big(\frac{j^{\mathrm{d}}}{j^{\mathrm{d}}-1}\Big), \tag{3.2.16}$$

$$\operatorname*{Res}_{z=-k-1}\psi(z)=\frac{(-1)^{n+1}}{(n+k+1)\mathrm{d}}\binom{n+k}{k}^{-1}\prod_{j=2}^{k}\Big(\frac{j^{\mathrm{d}}}{j^{\mathrm{d}}-1}\Big), \tag{3.2.17}$$

$$\operatorname*{Res}_{z=-m+r}\psi(z)=\frac{n!}{\prod_{j=m}^{m+n}(r-j)}\frac{B_k r}{\mathrm{d}(1-m+k+r)^{\mathrm{d}}}\prod_{\substack{j\geq k+1\\ j\neq m}}\frac{(j-m+r)^{\mathrm{d}}}{(j-m+r)^{\mathrm{d}}-1}, \tag{3.2.18}$$

where r is a d-th root of 1, such that $r\notin\{\pm1,\mathrm{e}^{\pm\mathrm{i}\pi/3},\mathrm{e}^{\pm2\mathrm{i}\pi/3}\}$ and $m\geq k+1$.

If d is a multiple of 6, then the poles $\mathrm{e}^{\pm\mathrm{i}\pi/3},\mathrm{e}^{\pm2\mathrm{i}\pi/3}$ are of order 2. In such case the residues are given by

$$\operatorname*{Res}_{z=-m+r}\psi(z)=\frac{n!}{\prod_{j=m}^{m+n}(r-j)}\frac{B_k r(r+1)}{\mathrm{d}^2(1-m+k+r)^{\mathrm{d}}}\prod_{\substack{j\geq k+1\\ j\neq m,m+1}}\frac{(j-m+r)^{\mathrm{d}}}{(j-m+r)^{\mathrm{d}}-1}, \tag{3.2.19}$$

for $r=\mathrm{e}^{\pm\mathrm{i}\pi/3}$, and

$$\operatorname*{Res}_{z=-m+r}\psi(z)=\frac{n!}{\prod_{j=m}^{m+n}(r-j)}\frac{B_k r(r-1)}{\mathrm{d}^2(1-m+k+r)^{\mathrm{d}}}\prod_{\substack{j\geq k+1\\ j\neq m,m+1}}\frac{(j-m+r)^{\mathrm{d}}}{(j-m+r)^{\mathrm{d}}-1}, \tag{3.2.20}$$

for $r=\mathrm{e}^{\pm2\mathrm{i}\pi/3}$.

Proof. We compute residues by taking limits, as shown below; details are given for the pole

$-k$. From (3.2.10) we have

$$\begin{aligned}\operatorname*{Res}_{z=-k}\psi(z) &= \lim_{s\to -k}(s+k)\frac{n!}{s(s-1)\cdots(s-n)}\frac{B_k}{(k+s+1)^{\mathrm{d}}}\prod_{j\geq k+1}\frac{(j+s)^{\mathrm{d}}}{(j+s)^{\mathrm{d}}-1}\\ &= \lim_{s\to -k}\frac{B_k n!}{s(s-1)\cdots(s-n)}\frac{(s+k)}{(k+s+1)^{\mathrm{d}}-1}\prod_{j\geq k+2}\frac{(j+s)^{\mathrm{d}}}{(j+s)^{\mathrm{d}}-1}\\ &= \frac{(-1)^{n+1}}{k\mathrm{d}}\binom{n+k}{k}^{-1}\prod_{j=2}^{k}\Big(\frac{j^{\mathrm{d}}}{j^{\mathrm{d}}-1}\Big).\end{aligned}$$

□

We present below the asymptotic behavior of residues.

Lemma 3.2.7 As $n\to\infty$,

$$\operatorname*{Res}_{z=-k}\psi(z)\sim(-1)^{n+1}\frac{(k-1)!}{\mathrm{d}}\Big(\prod_{j=2}^{k}\frac{j^{\mathrm{d}}}{j^{\mathrm{d}}-1}\Big)n^{-k} \tag{3.2.21}$$

$$\operatorname*{Res}_{z=-k-1}\psi(z)\sim(-1)^{n}\frac{k!}{\mathrm{d}}\Big(\prod_{j=2}^{k}\frac{j^{\mathrm{d}}}{j^{\mathrm{d}}-1}\Big)n^{-(k+1)} \tag{3.2.22}$$

Proof. From Stirling's approximation, equations (3.2.16) and (3.2.17) yield (3.2.21) and (3.2.22) respectively. □

We turn our attention to the complex poles, which are distributed in unit circles around the negative integers, starting from $-k-1$. This means a denumerable number of residues to be taken into account, so we study the asymptotic order (as $n\to\infty$) of the series

$$\sum_{l=1}^{\mathrm{d}-1}\sum_{m\geq k+1}\operatorname*{Res}_{z=-m+r_l}\psi(z).$$

For technical reasons, we analyze separately the cases $m=k+1$ and $m\geq k+2$.

Lemma 3.2.8 Let $m\geq k+2$ and let r be a d-th root of unity, such that $r\neq\pm1$. Then

$$\Big|\operatorname*{Res}_{z=-m+r}\psi(z)\Big|\leq C(r)\frac{1}{n+1}\binom{n+m-1}{m-2}^{-1}, \tag{3.2.23}$$

for sufficiently large n, where $C(r)$ is a positive constant depending on r but not on n nor m.

Proof. We proceed to bound the terms in the rhs of (3.2.18) (the same idea applies to (3.2.19) and (3.2.20)). Note first that $|x-r|>|x-1|$, for all $x\neq0$, since $|r|=1$ and $\Re(r)<1$, so $|x-r|^2=x^2+1-2x\Re(r)>(x-1)^2$. Therefore

$$\prod_{j=m}^{m+n}\frac{1}{|j-r|}\leq\prod_{j=m}^{m+n}\frac{1}{|j-1|}=\frac{(m-2)!}{(m+n-1)!}.$$

Also, it is easy to see that, for $m \geq k+1$, $|1-m+k+r| \geq \min\{1, |1-r|\}$.

Additionally, for the infinite product in (3.2.18), we have

$$\prod_{\substack{j\geq k+1\\ j\neq m}} \left|\frac{(j-m+r)^{\mathrm{d}}}{(j-m+r)^{\mathrm{d}}-1}\right| = \prod_{j\in A_m} \left|\frac{(j-m+r)^{\mathrm{d}}}{(j-m+r)^{\mathrm{d}}-1}\right| \prod_{\mathrm{i}\in\{-1,1,2\}} \left|\frac{(r-\mathrm{i})^{\mathrm{d}}}{(r-\mathrm{i})^{\mathrm{d}}-1}\right|,$$

where $A_m = \{k+1,\dots,m-2\}\cup\{m+3,\dots\}$. Now we proceed as in the proof of Proposition 3.2.4.

From the definition of A_m, $|j-m+r|>1$ and so, $|(j-m+r)^{\mathrm{d}}-1| > |j-m+r|^{\mathrm{d}}-1$, which implies

$$\prod_{j\in A_m} \frac{|j-m+r|^{\mathrm{d}}}{|(j-m+r)^{\mathrm{d}}-1|} \leq \prod_{j\in A_m} \frac{|j-m+r|^{\mathrm{d}}}{|j-m+r|^{\mathrm{d}}-1} \leq \prod_{j\in A_m}\left(1+\frac{1}{(j-m-1)^{\mathrm{d}}-1}\right). \quad (3.2.24)$$

Finally, from the elementary inequality $\log(1+x)\leq x$, the rightmost term of (3.2.24) is bounded above by

$$\begin{aligned}
\exp\left(\sum_{j\in A_m}\frac{1}{(j-m-1)^{\mathrm{d}}-1}\right) &\leq \exp\left(\sum_{j\in A_m}\frac{1}{(j-m-1)^2-1}\right)\\
&= \exp\left(\sum_{n=3}^{m-k}\frac{1}{n^2-1}+\sum_{n\geq 2}\frac{1}{n^2-1}\right)\\
&\leq \exp\left(2\sum_{n\geq 2}\frac{1}{n^2-1}\right)\\
&= \mathrm{e}^{\frac{3}{2}}.
\end{aligned}$$

Finally, collecting the bounds above, we get

$$\left|\operatorname*{Res}_{z=-m+r}\psi(z)\right| \leq C(r)\frac{n!(m-2)!}{(n+m-1)!} = C(r)\frac{1}{n+1}\binom{n+m-1}{m-2}^{-1}.$$

□

Corollary 3.2.1 As $n\to\infty$,

$$Z_n := \left|\sum_{l=1}^{\mathrm{d}-1}\sum_{m\geq k+2}\operatorname*{Res}_{z=-m+r_l}\psi(z)\right| \leq Cn^{-k-1}, \quad (3.2.25)$$

where C is a positive constant.

Proof. From Lemma 3.2.8,

$$\begin{aligned}
Z_n &\leq \frac{1}{n+1}\sum_{l=1}^{\mathrm{d}-1}C(r_l)\sum_{m\geq k+2}\binom{m+n-1}{m-2}^{-1}\\
&= \frac{n+k+1}{n(n+1)}\binom{n+k+1}{k}^{-1}\sum_{l=1}^{\mathrm{d}-1}C(r_l) \leq Cn^{-k-1},
\end{aligned}$$

as $n\to\infty$. □

Lemma 3.2.9 Let r be a d-th root of 1, such that $r \neq \pm 1$. Then, as $n \to \infty$,

$$\left| \operatorname*{Res}_{z=-k-1+r} \psi(z) \right| \leq C(r) n^{-k-1} |n^r| \leq C(r) n^{-k-1+\cos(2\pi/\mathrm{d})}, \tag{3.2.26}$$

where $C(r)$ is a positive constant.

Proof. We consider the formula (3.2.18), with $m = k+1$ and bound its modulus as in the proof of Lemma 3.2.8, except for the bound of the product, which now is approximated asymptotically as follows

$$\prod_{j=k+1}^{k+1+n} \frac{1}{|j-r|} = \frac{|\Gamma(k+1-r)|}{|\Gamma(n+k+2-r)|} \sim \frac{|\Gamma(k+1-r)|}{(n-1)!} n^{r-k-2}.$$

The result is obtained by collecting constants. Finally, observe that, for $l = 1, \ldots, \mathrm{d}-1$, $|n^{r_l}| = n^{\cos(2\pi l/\mathrm{d})} \leq n^{\cos(2\pi/\mathrm{d})} = o(n)$. □

Corollary 3.2.2 For $k \geq 1$,

$$n^k \mu_n^{(k)} = \frac{(k-1)!}{\mathrm{d}} \prod_{j=2}^{k} \left(1 - \frac{1}{j^{\mathrm{d}}}\right)^{-1} + O(n^{\cos(2\pi/\mathrm{d})-1}), \tag{3.2.27}$$

as $n \to \infty$.

Proof. The result follows from the formula in Theorem 3.2.2 and collecting the asymptotic results of Lemma 3.2.7, Corollary 3.2.1 and Lemma 3.2.9. □

3.2.2 Weak convergence of record heights

We now state a result of weak convergence for the sequence $\mathcal{H}_n$. The proof is based on the Frechet-Shohat theorem about convergence of moments. According to this well-known result, if a sequence (Y_n) of random variables has convergent moments $\mathbb{E}(Y_n^k) \to M_k < \infty$, then (M_k) is the sequence of moments of a random variable Y and $Y_n \xrightarrow{\mathcal{D}} Y$ if the distribution of Y is determined by its moments. The latter property is obtained from Carleman's criterion.

Theorem 3.2.3 Let $(\mathcal{H}_n)$ be the sequence of record heights of Definition 3.2.1. Then

$$n\mathcal{H}_n \xrightarrow{\mathcal{D}} \mathcal{H},$$

where $\mathcal{H}$ is a random variable, with moments given by

$$\nu_k^{(\mathrm{d})} := \frac{(k-1)!}{\mathrm{d}} \prod_{j=2}^{k} \left(1 - \frac{1}{j^{\mathrm{d}}}\right)^{-1},$$

for $k \geq 1$. In the case $\mathrm{d} = 2$, $\mathcal{H} \stackrel{\mathcal{D}}{=} UV$, where U is $U[0,1]$, V is exponential with mean 1 and U, V are independent.

Proof. From Corollary 3.2.2 and noting that $\cos(2\pi/\mathrm{d}) < 1$, it holds that $\mathbb{E}(n\mathcal{H}_n)^k = n^k\mu_n^{(k)} \to \nu_k^{(\mathrm{d})}$. We have convergence of moments and so, the result follows if we show that the sequence $(\nu_k^{(\mathrm{d})})$ determines a distribution. We check Carleman's condition, namely that $\sum_{k\geq 1}(\nu_{2k}^{(\mathrm{d})})^{-\frac{1}{2k}} = \infty$, see [2]. Note that $\nu_k^{(\mathrm{d})}$ is decreasing in d, so

$$\nu_k^{(\mathrm{d})} \leq \nu_k^{(2)} = \frac{\Gamma(k)}{2}\frac{2\Gamma(k+1)^2}{\Gamma(k)\Gamma(k+2)} = \frac{k!}{k+1}.$$

Hence, $(\nu_k^{(\mathrm{d})})^{-\frac{1}{k}} \geq (\nu_k^{(2)})^{-\frac{1}{k}} \sim \frac{\mathrm{e}}{k}$, and the corresponding series diverges. For $\mathrm{d} = 2$ we have $\mathbb{E}(U^k) = \frac{1}{k+1}$, $\mathbb{E}(V^k) = k!$, so $\nu_k^{(2)} = \mathbb{E}((UV)^k)$. □

For $\mathrm{d} = 2$, $\mathcal{H}$ can be seen as having, conditionally on U, exponential density, with expectation U. For $\mathrm{d} > 2$, it is not clear if $\mathcal{H}$ can be represented as a product UV, with U, V independent, V unit exponential and U concentrated on $[0,1]$. To that end, it would suffice to prove that $m_k := \frac{1}{k\mathrm{d}}\prod_{j=2}^{k}\left(1-\frac{1}{j^{\mathrm{d}}}\right)^{-1}$, $k \geq 1$, is the sequence of moments of a random variable, with values in $[0,1]$. A necessary and sufficient condition for such property is that

$$\sum_{\mathrm{i}=0}^{r}\binom{r}{\mathrm{i}}(-1)^{\mathrm{i}}m_{n+\mathrm{i}} \geq 0,$$

for all $r, n \geq 1$; see E18.6 in [60]. Numerical evidence obtained with Maple®, for some values of $\mathrm{d} > 2$, leads to conjecture that it holds. We do not pursue this investigation further.

3.3 Mixed moments of $\mathcal{M}_n$

In this section we study the expectations of the random variables $\prod_{j=1}^{\mathrm{d}}(1-\mathcal{M}_n^{(j)})^{k_j}$, which we call mixed moments of $\mathcal{M}_n$, with $k_1,\ldots,k_{\mathrm{d}}$ nonnegative integers. The idea is to gain insight on the asymptotic behavior of $\mathcal{M}_n$.

Proposition 3.3.1 Let $\mu_n^{k_1,\ldots,k_{\mathrm{d}}} = \mathbb{E}\Big(\prod_{j=1}^{\mathrm{d}}(1-\mathcal{M}_n^{(j)})^{k_{\mathrm{i}}}\Big)$, with $k_1,\ldots,k_{\mathrm{d}}$ nonnegative integers. Then the following recursion holds

$$\mu_{n+1}^{k_1,\ldots,k_{\mathrm{d}}} = \mu_n^{k_1,\ldots,k_{\mathrm{d}}} - \left(1-\prod_{\mathrm{i}=1}^{\mathrm{d}}(k_{\mathrm{i}}+1)^{-1}\right)\mu_n^{k_1+1,\ldots,k_{\mathrm{d}}+1},\ n \geq 1, \tag{3.3.1}$$

with $\mu_1^{k_1,\ldots,k_{\mathrm{d}}} = \prod_{\mathrm{i}=1}^{\mathrm{d}}(k_{\mathrm{i}}+1)^{-1}$.

Proof. We proceed as in the proof of Proposition 3.2.1. From equation (2.2.5) in Lemma

2.2.1, recalling that $\mathcal{H}_n = \prod_{j=1}^{\mathrm{d}}(1-\mathcal{M}_n^{(j)})$, we have

$$\begin{aligned}
\mu_{n+1}^{k_1,\ldots,k_{\mathrm{d}}} &= \mathbb{E}\left(\int_{[0,1]^{\mathrm{d}}\cap\{\mathbf{x}\succ\mathcal{M}_n\}}\prod_{j=1}^{\mathrm{d}}(1-x^{(j)})^{k_j}\mathrm{d}\mathbf{x}\right)+\mathbb{E}\left(\prod_{j=1}^{\mathrm{d}}(1-\mathcal{M}_n^{(j)})^{k_j}(1-\mathcal{H}_n)\right)\\
&= \mathbb{E}\left(\prod_{j=1}^{\mathrm{d}}\int_{\mathcal{M}_n^{(j)}}^{1}(1-x^{(j)})^{k_j}\mathrm{d}x^{(j)}\right)+\mu_n^{k_1,\ldots,k_{\mathrm{d}}}-\mu_n^{k_1+1,\ldots,k_{\mathrm{d}}+1}\\
&= \mathbb{E}\left(\prod_{j=1}^{\mathrm{d}}\frac{(1-\mathcal{M}_n^{(j)})^{k_j+1}}{k_j+1}\right)+\mu_n^{k_1,\ldots,k_{\mathrm{d}}}-\mu_n^{k_1+1,\ldots,k_{\mathrm{d}}+1}\\
&= \mu_n^{k_1+1,\ldots,k_{\mathrm{d}}+1}\prod_{\mathrm{i}=1}^{\mathrm{d}}(k_{\mathrm{i}}+1)^{-1}+\mu_n^{k_1,\ldots,k_{\mathrm{d}}}-\mu_n^{k_1+1,\ldots,k_{\mathrm{d}}+1}.
\end{aligned}$$

□

For simplicity, we only carry out detailed calculations for the particular case d = 2. Letting $k_1 = k, k_2 = l$ in (3.3.1), we get

$$\mu_{n+1}^{k,l} = \mu_n^{k,l} - \left(1-\frac{1}{(k+1)(l+1)}\right)\mu_n^{k+1,l+1},\ n\geq 1. \tag{3.3.2}$$

Proposition 3.3.2 The solution of the recurrence (3.3.2) has the form

$$\begin{aligned}
\mu_{n+1}^{k,l} &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j\mu_1^{k+1,j+1}\prod_{\mathrm{i}=0}^{j-1}(1-\mu_1^{k+\mathrm{i},l+\mathrm{i}})\\
&= \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(k+j+1)(l+j+1)}\prod_{\mathrm{i}=1}^{j}\left(1-\frac{1}{(k+\mathrm{i})(l+\mathrm{i})}\right),\ n\geq 0.
\end{aligned} \tag{3.3.3}$$

Proof. Similar to the proof of Proposition 3.2.2. □

3.3.1 Asymptotic analysis of $\mu_n^{k,l}$

Note that (3.3.3) is the Euler's transform (see (3.2.7)) of the sequence

$$b_j^{k,l} := \frac{1}{(k+j+1)(l+j+1)}\prod_{\mathrm{i}=1}^{j}\left(1-\frac{1}{(k+\mathrm{i})(l+\mathrm{i})}\right),\ j\geq 0. \tag{3.3.4}$$

We proceed, as before, by checking the conditions of Theorem 3.2.1 in order to obtain an asymptotic expression for $\mu_n^{k,l}$.

Proposition 3.3.3 Let $\mathcal{P} := \{z_m^+, z_m^- : m\geq 1\}$, where

$$z_m^{\pm} = -m - \frac{k+l}{2} \pm r_{kl}, \tag{3.3.5}$$

with $r_{kl} = \frac{1}{2}\sqrt{(k-l)^2+4}$. Then

$$\varphi(z) = \frac{B_{k,l}}{(k+1+z)(l+1+z)-1} \prod_{n\geq 2} \frac{(k+n+z)(l+n+z)}{(k+n+z)(l+n+z)-1}, \; z \in D, \tag{3.3.6}$$

is the analytic continuation of the sequence $b_j^{k,l}$ in (3.3.4) to the domain $D = \mathbb{C} \setminus \mathcal{P}$, where $B_{k,l} = \prod_{n\geq 1}\left(1 - \frac{1}{(k+n)(l+n)}\right)$. Furthermore, the singularities of φ are isolated poles, elements of the countable set $\mathcal{P}$.

Proof. Clearly, the poles of φ are the roots of the quadratic equations $(k+n+z)(l+n+z) = 1$, for $n \geq 1$, and these roots (real and irrational if $k \neq l$) are precisely the elements of $\mathcal{P}$, shown in (3.3.5). We check next that φ is analytic on D. To that end, observe that

$$\prod_{n\geq 2} \frac{(k+n+z)(l+n+z)}{(k+n+z)(l+n+z)-1} = \lim_{m\to\infty} \prod_{n=2}^{m} \frac{(k+n+z)(l+n+z)}{(k+n+z)(l+n+z)-1}$$

exists and is analytic if the series

$$\sum_{n\geq 1} \frac{1}{|(k+n+z)(l+n+z)-1|} \tag{3.3.7}$$

converges locally uniformly. Let V be a disk contained in D then, for any $z \in V$ and n large enough, $|z+n+k| > n+k-C > 1$, where C is a positive constant. Hence, (3.3.7) converges locally uniformly and we conclude that φ is analytic in D. Finally, it can be easily verified that φ interpolates the sequence $b_j^{k,l}$. □

Proposition 3.3.4 Let $\gamma_m = \{z \in \mathbb{C} : |z| = m\}$, for $m \geq 1$. Then $|\varphi(z)|$ is bounded on $\Omega = \bigcup_{m\geq M_0} \gamma_m$, for some $M_0 > 0$, where φ is defined in (3.3.6).

Proof. We follow the proof of Proposition 3.2.4, recalling that $k \neq l$ and, without loss of generality, we assume that $k < l$. For convenience we also assume that $m \geq l+3$.

Let

$$B_m = \{n \geq 2 : |k+n-m||l+n-m| > 1\}, \; m \geq 1,$$

then

$$\begin{aligned} |\varphi(z)| = {} & \frac{B_{k,l}}{|(k+1+z)(l+1+z)-1|} \prod_{n\notin B_m} \frac{|k+n+z||l+n+z|}{|(k+n+z)(l+n+z)-1|} \\ & \times \prod_{n\in B_m} \frac{|k+n+z||l+n+z|}{|(k+n+z)(l+n+z)-1|}. \end{aligned} \tag{3.3.8}$$

For $z \in \gamma_m$ and from the definition of B_m we have

$$|k+n+z||l+n+z| \geq |k+n-|z|||l+n-|z|| = |k+n-m||l+n-m| > 1.$$

Hence,

$$\prod_{n\in B_m} \frac{|k+n+z||l+n+z|}{|(k+n+z)(l+n+z)-1|} \leq \prod_{n\in B_m} \frac{|k+n-m||l+n-m|}{|k+n-m||l+n-m|-1}. \tag{3.3.9}$$

The infinite product on the right of the display above is bounded if and only if the series

$$S_m := \sum_{n\in B_m} \frac{1}{|k+n-m||l+n-m|-1} \tag{3.3.10}$$

is bounded (by a constant not depending on m). To show that such is the case, we take a closer look at B_m, noting that $n \notin B_m$ if either $|k+n-m||l+n-m| = 0$ or $|k+n-m||l+n-m| = 1$. The first equation has solutions $n = m-k$ and $n = m-l$, while the second has no solution, if $l \neq k+2$. But, if $l = k+2$ then the second equation has the unique solution $n = m-l+1$.

So, let us assume first that $l \neq k+2$, in which case

$$B_m = \{2, \ldots, m-l-1\} \cup \{m-l+1, \ldots, m-k-1\} \cup \{m-k+1, \ldots\}$$

and

$$\begin{aligned} S_m = &\sum_{n=2}^{m-l-1} \frac{1}{(m-k-n)(m-l-n)-1} + \sum_{n=m-l+1}^{m-k-1} \frac{1}{(m-k-n)(l+n-m)-1} \\ &+ \sum_{n\geq m-k+1} \frac{1}{(k+n-m)(l+n-m)-1}. \end{aligned} \tag{3.3.11}$$

Moreover,

$$\sum_{n=2}^{m-l-1} \frac{1}{(m-k-n)(m-l-n)-1} \leq \frac{1}{l-k} + \sum_{n=2}^{m-l-2} \frac{1}{(m-l-n)^2-1} < \frac{1}{l-k} + \sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Also, the second sum is finite and independent of m, since

$$\sum_{n=m-l+1}^{m-k-1} \frac{1}{(m-k-n)(l+n-m)-1} = \sum_{\mathrm{i}=1}^{l-k-1} \frac{1}{\mathrm{i}(l-k-\mathrm{i})-1}.$$

Finally,

$$\begin{aligned} \sum_{n\geq m-k+1} \frac{1}{(k+n-m)(l+n-m)-1} &\leq \frac{1}{l-k} + \sum_{n\geq m-k+2} \frac{1}{(k+n-m)^2-1} \\ &\leq \frac{1}{l-k} + \sum_{n\geq 2} \frac{1}{n^2-1}. \end{aligned}$$

From the computations above we see that the product over B_m in (3.3.8) is bounded by a constant independent of m, if $l \neq k+2$. If $l = k+2$

$$B_m = \{2, \ldots, m-l-1\} \cup \{m-l+3, \ldots\}$$

and

$$\begin{aligned} S_m = &\sum_{n=2}^{m-l-1} \frac{1}{(m-l-n+2)(m-l-n)-1} + \sum_{n\geq m-l+3} \frac{1}{(l-2+n-m)(l+n-m)-1} \\ \leq &\frac{1}{2} + \sum_{n=2}^{m-l-2} \frac{1}{(m-l-n)^2-1} + \frac{1}{2} + \sum_{n\geq m-l+4} \frac{1}{(l-2+n-m)^2-1} \\ < &1 + 2\sum_{n\geq 2} \frac{1}{n^2-1}. \end{aligned}$$

Again we see that the product over B_m in (3.3.8) is bounded by a constant independent of m, if $l = k+2$. We now consider the product over $n \notin B_m$, assuming first that $l \neq k+2$, which is equal to

$$\frac{|m+z||l-k+m+z|}{|(m+z)(l-k+m+z)-1|}\frac{|k-l+m+z||m+z|}{|(k-l+m+z)(m+z)-1|}. \tag{3.3.12}$$

Note that the first term in (3.3.12) can be written and bounded as

$$\left|1+\frac{1}{(m+z)(l-k+m+z)-1}\right| \leq 1+\frac{1}{|(m+z)(l-k+m+z)-1|},$$

so we must bound $|(m+z)(l-k+m+z)-1|$ from below, for $z \in \gamma_m$. But

$$|(m+z)(l-k+m+z)-1| = |z-z_m^+-k||z-z_m^--k| \geq ||z_m^++k|-m|||z_m^-+k|-m|.$$

If m is large enough, $|z_m^\pm + k| = -k - z_m^\pm$, which yields $||z_m^\pm+k|-m| = |\frac{l-k}{2} \mp r_{kl}|$. So

$$|(m+z)(l-k+m+z)-1| \geq \left|\frac{l-k}{2}+r_{kl}\right|\left|\frac{l-k}{2}-r_{kl}\right| = 1$$

and the first term in (3.3.12) is bounded by 2. The analysis of the second term is similar and is thus omitted. There remains to check the case $l = k+2$. The product over $n \notin B_m$ has three terms and is given by

$$\frac{|m-2+z||m+z|}{|(m-2+z)(m+z)-1|}\frac{|m-1+z||m+1+z|}{|(m-1+z)(m+1+z)-1|}\frac{|m+z||m+2+z|}{|(m+z)(m+2+z)-1|}. \tag{3.3.13}$$

We proceed as in the previous case, noting that the first term is written as

$$\left|1+\frac{1}{(m+z)(m-2+z)-1}\right| \leq 1+\frac{1}{|(m+z)(m-2+z)-1|},$$

so we bound $|(m+z)(m-2+z)-1|$, for $z \in \gamma_m$. We have, for m large enough,

$$\begin{aligned}|(m+z)(m-2+z)-1| &= |r_+-m-z||r_--m-z| \\ &\geq ||r_+-m|-m|||r_--m|-m| \\ &= |r_+r_-| = 1,\end{aligned}$$

where $r_\pm = 1 \pm \sqrt{2}$. Hence, the first term of (3.3.13) is bounded by 2. The analysis of the remaining two terms is similar and is omitted.

Last, we have to bound the first term of (3.3.8), namely $\frac{B_{k,l}}{|(k+1+z)(l+1+z)-1|}$. It is easy to see that, for m large enough, we have $|(k+1+z)(l+1+z)-1| \geq (m-c)^2$, for some positive constant c, and the conclusion follows. □

Theorem 3.3.1 For sufficiently large n,

$$\mu_{n+1}^{k,l} = -(-1)^n \sum_{s\in\mathcal{P}} \operatorname*{Res}_{z=s}\left[\frac{n!\varphi(z)}{z(z-1)\dots(z-n)}\right], \tag{3.3.14}$$

where $\operatorname*{Res}_{z=s}\psi(z)$ denotes the residue of a function $\psi(z)$ at $z = s$, and φ, $\mathcal{P}$ are defined in Proposition 3.3.3.

Proof. The result follows from Propositions 3.3.3, 3.3.4 and Theorem 3.2.1. □

Residues

We evaluate the residues in the right-hand side of (3.3.14).

Proposition 3.3.5 Let $\psi(z) = \frac{n!\varphi(z)}{z(z-1)\ldots(z-n)}$ and

$$\varphi_m(z) = \frac{B_{k,l}}{(k+1+z)(l+1+z)} \prod_{n\geq 1, n\neq m} \frac{(k+n+z)(l+n+z)}{(k+n+z)(l+n+z)-1}, \; z \in D, m \geq 1.$$

Then

$$\operatorname*{Res}_{z=z_m^\pm} \psi(z) = \pm \frac{n!}{(z_m^\pm)_{n+1}} \frac{\varphi_m(z_m^\pm)}{2r_{kl}}, \tag{3.3.15}$$

where $(s)_{n+1} = s(s-1)\cdots(s-n)$ is the falling factorial.

For the asymptotic behavior of residues we have

Lemma 3.3.1 As $n \to \infty$, for $m \geq 1$,

$$\operatorname*{Res}_{z=z_m^\pm} \psi(z) \sim \pm(-1)^{n+1} n^{z_m^\pm} \Gamma(-z_m^\pm) \frac{\varphi_m(z_m^\pm)}{2r_{kl}}. \tag{3.3.16}$$

Proof. The result follows from Stirling's approximation, $n!/(s)_{n+1} \sim (-1)^{n+1}\Gamma(-s)n^s$. □

Lemma 3.3.2 For $m \geq 2$ and $n \geq 1$,

$$\frac{n!}{|(z_m^\pm)_{n+1}|} \leq \binom{m+n+a}{n}^{-1},$$

where $a = \lfloor c \rfloor$ and $c = \frac{k+l}{2} \mp r_{kl} > 0$.

Proof. Note that

$$\frac{n!}{(z_m^+)_{n+1}} = (-1)^{n-1} \frac{n!}{\prod_{i=0}^{n}(i+m+c)} = (-1)^{n-1} \frac{n!}{\prod_{i=m}^{n+m}(i+c)}.$$

Also,

$$\prod_{i=m}^{n+m}(i+c) \geq \prod_{i=m}^{n+m}(i+a) = n!(m+a)\binom{m+n+a}{n},$$

and the conclusion follows. □

Lemma 3.3.3 The sequence $|\varphi_m(z_m^\pm)|, m \geq 1$, is bounded.

Proof. From the definition of φ_m in Proposition 3.3.5, we have

$$\begin{aligned}\left|\prod_{n\geq 1, n\neq m} \frac{(k+n+z_m^{\pm})(l+n+z_m^{\pm})}{(k+n+z_m^{\pm})(l+n+z_m^{\pm})-1}\right| &= \prod_{n\geq 1, n\neq m}\left|1+\frac{1}{(k+n+z_m^{\pm})(l+n+z_m^{\pm})-1}\right| \\ &\leq \prod_{n\geq 1, n\neq m}\left(1+\frac{1}{|(k+n+z_m^{\pm})(l+n+z_m^{\pm})-1|}\right) \\ &= \prod_{n\geq 1, n\neq m}\left(1+\frac{1}{|(z_m^{\pm}-z_n^{+})(z_m^{\pm}-z_n^{-})|}\right) \\ &= \prod_{n\geq 1, n\neq m}\left(1+\frac{1}{|(n-m)(n-m\pm 2r_{kl})|}\right) \\ &\leq \exp\left(\sum_{n\geq 1, n\neq m}\frac{1}{|n-m||n-m\pm 2r_{kl}|}\right) \leq C,\end{aligned}$$

where C is a constant independent of m. Finally, the conclusion is reached, noting that $B_{k,l}\leq 1$ and that $|k+1+z_m^{\pm}||l+1+z_m^{\pm}|\to\infty$. □

Corollary 3.3.1 As $n\to\infty$,

$$\left|\sum_{m\geq 2}\operatorname*{Res}_{z=z_m^{\pm}}\psi(z)\right| \leq Cn^{-(a+2)}, \tag{3.3.17}$$

where C is a positive constant and $a>0$ is defined in Lemma 3.3.2.

Proof. From Lemmas 3.3.2 and 3.3.3 we get

$$\left|\sum_{m\geq 2}\operatorname*{Res}_{z=z_m^{\pm}}\psi(z)\right| \leq C_1\sum_{m\geq 2}\binom{m+n+a}{n}^{-1} \leq C_2\binom{n+2+a}{n}^{-1} \leq Cn^{-a-2}.$$

□

Corollary 3.3.2 As $n\to\infty$,

$$n^{-z_1^+}\mu_n^{k,l} = \Gamma(-z_1^+)\frac{\varphi_1(z_1^+)}{2r_{kl}} + o(1). \tag{3.3.18}$$

Proof. Note from (3.3.16) that $n^{-z_1^+}\operatorname*{Res}_{z=z_1^-}\psi(z) = O(n^{z_1^- - z_1^+}) = O(n^{-2r_{kl}}) = o(1)$. Also, from (3.3.17),

$$n^{-z_1^+}\left|\sum_{m\geq 2}\operatorname*{Res}_{z=z_m^{\pm}}\psi(z)\right| \leq Cn^{-a-2-z_1^+} = o(1),$$

because $a+2+z_1^+ = 1+\lfloor\frac{k+l}{2}\mp r_{kl}\rfloor - \frac{k+l}{2} - r_{kl} > 0$. The result follows from (3.3.16) . □

Theorem 3.3.2 As $n \to \infty$,

$$n^{\alpha}(1 - \mathcal{M}_n^{(j)}) \to 0, \; j = 1, 2,$$

in L_k, if $\alpha < \alpha_k^* := 1/k - \sqrt{k^2+4}/(2k) + 1/2$. In particular, convergence holds in L_1, for all $\alpha < \alpha_1^* := (3 - \sqrt{5})/2 \approx 0.38$. Furthermore, the sequence $(n^{\alpha_1^*}(1 - \mathcal{M}_n^{(j)}))_n$ is tight.

Proof. From (3.3.18), with $l = 0$, we have

$$\begin{aligned}
\mathbb{E}(n^{\alpha}(1 - \mathcal{M}_n^{(j)}))^k &= n^{k\alpha}\mu_n^{k,0} \\
&= Cn^{k\alpha - 1 - k/2 + r_{k0}} + o(1) \\
&= Cn^{(\alpha - 1/2)k + \sqrt{k^2+4}/2 - 1} + o(1) \to 0,
\end{aligned}$$

if $\alpha < \alpha_k^*$. For $l = 0, k = 1$, $n^{\alpha}\mathbb{E}(1 - \mathcal{M}_n^{(j)}) \sim Cn^{\alpha + (\sqrt{5}-3)/2} \to 0$ if $\alpha < \alpha_1^*$. Tightness is a consequence of Markov's inequality. $\square$

Remark 3.3.1 Note that, for $\alpha \in (\alpha_1^*, 1/2)$, we have $n^{2\alpha}\mathbb{E}(1 - \mathcal{M}_n^{(1)})(1 - \mathcal{M}_n^{(2)}) \to 0$, but $n^{\alpha}\mathbb{E}(1 - \mathcal{M}_n^{(j)}) \to \infty$, for $j = 1, 2$.

3.4 On the counting process of records

This section is dedicated to the study of the total number of chain-records among the first n observations. We recall that this variable was defined in Chapter 2, as

$$\mathcal{N}_n = \sum_{\mathrm{i}=1}^{n} I_{\mathrm{i}},$$

where $I_n = 1_{\{\mathcal{M}_n \succ \mathcal{M}_{n-1}\}} = 1_{\{\mathbf{X}_n \text{ is a record}\}}$ is the indicator of $\mathbf{X}_n$ being a record. Observe that $\mathcal{N}_n$ also represent the number of jumps of maxima $\mathcal{M}_k$ up to n.

Our objective is to find a sequence of real numbers (a_n) such that $\frac{\mathcal{N}_n}{a_n} \to 1$ as $n \to \infty$, in some sense, such a.s., in probability or in L_p. We only consider the particular case $\mathrm{d} = 2$ and start with some preliminaries results.

Lemma 3.4.1 $\mathbb{E}(\mathcal{N}_n) = \frac{1}{2}H_n \sim \frac{1}{2}\log n$, where $H_n = \sum_{j=1}^{n} 1/j$ is the n-th harmonic number.

Proof. Note, from Definition 3.2.1, that $\mathbb{E}(I_n|\mathcal{F}_{n-1}) = \mathcal{H}_{n-1}$ for $n \geq 1$, where $\mathcal{F}_n = \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Then by 3.2.4, we have

$$\mathbb{E}(I_n) = \mathbb{E}(\mathcal{H}_{n-1}) = \mu_{n-1}^{(1)} = \frac{1}{2n}.$$

Note that the expectation above is also in (ii) of Lemma 2.3.2. The result then follows. $\square$

Lemma 3.4.2 It holds

$$\begin{aligned}\sum_{\mathrm{i}=1}^{n} \mathrm{Var}(I_{\mathrm{i}}) &= O(\log n),\\ \sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(I_{\mathrm{i}})\mathbb{E}(I_j) &= \frac{1}{8}\log^2 n + O(\log n),\end{aligned} \tag{3.4.1}$$

as $n \to \infty$.

PROOF. First, note that $\mathrm{Var}(I_{\mathrm{i}}) = \mathbb{E}\big(I_{\mathrm{i}} - \frac{1}{2\mathrm{i}}\big)^2 = \frac{1}{2\mathrm{i}}\big(1 - \frac{1}{2\mathrm{i}}\big)$, then

$$\sum_{I=1}^{n} \frac{1}{2\mathrm{i}}\Big(1 - \frac{1}{2\mathrm{i}}\Big) = \frac{1}{2}H_n - \frac{1}{4}\sum_{\mathrm{i}=1}^{n}\frac{1}{\mathrm{i}^2} = O(\log n).$$

Also,

$$\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(I_{\mathrm{i}})\mathbb{E}(I_j) = \sum_{\mathrm{i}=1}^{n-1}\sum_{j=\mathrm{i}+1}^{n}\frac{1}{4\mathrm{i}j} = \frac{1}{4}H_nH_{n-1} - \frac{1}{4}\sum_{\mathrm{i}=1}^{n-1}\tfrac{H_{\mathrm{i}}}{\mathrm{i}}.$$

Finally, from [[3], Eq. (3.62)] we have $\sum_{j=1}^{n}\frac{H_j}{j} = \frac{1}{2}(H_n^2 + H_n^{(2)})$, where $H_n^{(s)} = \sum_{j=1}^{n}\frac{1}{j^s}$, $s \in \mathbb{C}$, is the generalized harmonic number of order s (see [35]). Then

$$\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(I_{\mathrm{i}})\mathbb{E}(I_j) = \frac{1}{8}\log^2 n + O(\log n).$$

□

Remark 3.4.1 Lemma 3.4.2 suggests the study the L_2 norm of $\frac{2}{\log n}\mathcal{N}_n - 1$. To that end, we could proceed, for example, as in [36], hoping that $\sum\limits_{1\le \mathrm{i}<j\le n}\mathbb{E}(I_{\mathrm{i}}I_j)$ contributes with the term $\frac{1}{8}\log^2 n$, to cancel the dominant term in the last expression but, unfortunately, we do not have a manageable expression for $\mathbb{E}(I_{\mathrm{i}}I_j)$.

3.4.1 Convergence in L_2

To avoid the inconvenience mentioned in Remark 3.4.1, we can apply Corollary 2.3.4, which states, essentially, that $\mathcal{N}_n$ and $\sum\limits_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}}$ are asymptotically equivalent. So, in what follows we concentrate on the latter sum.

Lemma 3.4.3 We have

$$\begin{aligned}\sum_{\mathrm{i}=1}^{n}\mathbb{E}(\mathcal{H}_{\mathrm{i}}) &\sim \frac{1}{2}\log n,\\ \mathbb{E}(\mathcal{N}_n) &\sim \frac{1}{2}\log n,\\ \sum_{\mathrm{i}=1}^{n}\mathrm{Var}(\mathcal{H}_{\mathrm{i}}) &= O(1),\\ \sum_{1\le \mathrm{i}<j\le n}\mathbb{E}(\mathcal{H}_{\mathrm{i}})\mathbb{E}(\mathcal{H}_j) &= \frac{1}{8}H_n^2 + O(\log n).\end{aligned} \tag{3.4.2}$$

Proof. The first expression in (3.4.2) follows from $\mathbb{E}(\mathcal{H}_{\mathrm{i}}) = \mu_{\mathrm{i}}^{(1)} = \frac{1}{\mathrm{i}+1}$, due to (3.2.4). The second follows from $\mathbb{E}(I_{\mathrm{i}}\,|\,\mathcal{F}_{\mathrm{i}-1}) = \mathcal{H}_{\mathrm{i}-1}$, which implies $\mathbb{E}(I_{\mathrm{i}}) = \mathbb{E}(\mathcal{H}_{\mathrm{i}-1})$, so $\mathbb{E}(\mathcal{N}_n) = \frac{1}{2}H_n \sim \frac{1}{2}\log n$. Moreover, since

$$\mathrm{Var}(\mathcal{H}_{\mathrm{i}}) = \mu_{\mathrm{i}}^{(2)} - \left(\mu_{\mathrm{i}}^{(1)}\right)^2 = \frac{2}{3}\frac{1}{(\mathrm{i}+2)(\mathrm{i}+1)} - \frac{1}{4}\frac{1}{(\mathrm{i}+1)^2},$$

the third assertion is proved. Finally, for the last one, using [[3], Eq. (3.62)], we obtain

$$\begin{aligned}\sum_{1\le \mathrm{i}<j\le n}\mathbb{E}(\mathcal{H}_{\mathrm{i}})\mathbb{E}(\mathcal{H}_j) &= \frac{1}{4}\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}}\frac{1}{(\mathrm{i}+1)(\mathrm{i}+k+1)}\\ &= \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\mathrm{i}+1}(H_{n+1} - H_{\mathrm{i}+1})\\ &= \frac{1}{4}(H_n + \frac{1}{n+1})(H_n - 1) - \frac{1}{4}\left[\frac{1}{2}(H_n^2 + H_n^{(2)}) - 1\right]\\ &= \frac{1}{8}H_n^2 - \frac{1}{4}H_n\Big(1 - \frac{1}{n+1}\Big) - \frac{1}{4(n+1)} - \frac{1}{8}H_n^{(2)} + \frac{1}{4}\\ &= \frac{1}{8}H_n^2 + O(\log n).\end{aligned}$$

□

Proposition 3.4.1 Let $\nu_k^{(l)}(n) = \mathbb{E}(\mathcal{H}_n\mathcal{H}_{n+k}^l)$, for integers $l \ge 0, k \ge 1, n \ge 1$, then the following recurrence holds

$$\nu_k^{(l)}(n) = \nu_{k-1}^{(l)}(n) - a_{l+1}\nu_{k-1}^{(l+1)}(n), \text{ where } a_l = 1 - \frac{1}{l^2}. \tag{3.4.3}$$

Proof. Let $g(x,y) = (1-x)^l(1-y)^l$, for $x, y \in [0,1]$, so $g(\mathcal{M}_{n+k}) = \mathcal{H}_{n+k}^l$. Let also

$S = \{(x,y) \in [0,1] : (x,y) \succ \mathcal{M}_{n+k-1}\}$, then, from equation (2.2.7) in Lemma 2.2.1, we have

$$\begin{aligned}
\mathbb{E}(\mathcal{H}^l_{n+k}|\mathcal{M}_{n+k-1}) &= \int_S (1-x)^l(1-y)^l \mathrm{d}x\mathrm{d}y + \mathcal{H}^l_{n+k-1}(1-\mathcal{H}_{n+k-1}) \\
&= \int_{\mathcal{M}^{(1)}_{n+k-1}}^1 (1-x)^l \mathrm{d}x \int_{\mathcal{M}^{(2)}_{n+k-1}}^1 (1-y)^l \mathrm{d}y + \mathcal{H}^l_{n+k-1} - \mathcal{H}^{l+1}_{n+k-1} \\
&= \frac{\mathcal{H}^{l+1}_{n+k-1}}{(l+1)^2} + \mathcal{H}^l_{n+k-1} - \mathcal{H}^{l+1}_{n+k-1} \\
&= \mathcal{H}^l_{n+k-1} - a_{l+1}\mathcal{H}^{l+1}_{n+k-1}.
\end{aligned}$$

Finally, multiplying by $\mathcal{H}_n$ and taking expectation, the result is obtained. □

Remark 3.4.2 Note that recurrence (3.4.3), in any dimension d, has the same form, with a_l given by $a_l = 1 - 1/l^{\mathrm{d}}$. Also, by checking some boundary conditions we recover some known cases, such as

$$\begin{aligned}
\nu^{(0)}_k(n) &= \mathbb{E}(\mathcal{H}_n) = \frac{1}{2}\binom{n+1}{1}^{-1} = \frac{1}{2(n+1)}, \\
\nu^{(l)}_0(n) &= \mathbb{E}(\mathcal{H}^l_n) = \frac{1}{l+2}\binom{n+l+1}{l+1}^{-1}.
\end{aligned}$$

In addition, after comparing (3.4.3) to (3.2.1), we realize that both recurrences are structurally identical.

Proposition 3.4.2 For $n, k \geq 1$

$$\mathbb{E}(\mathcal{H}_n\mathcal{H}_{n+k}) = \frac{1}{2}\sum_{j=0}^{k-1}\binom{n-1}{j}(-1)^j\left[\tfrac{j+2}{(j+1)(j+3)}\binom{n+j+2}{j+2}^{-1} - \tfrac{j+3}{(j+2)(j+4)}\binom{n+j+3}{j+3}^{-1}\right] \tag{3.4.4}$$

Proof. The solution of (3.4.3) has the form

$$\nu^{(l)}_k(n) = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j b_j(l)\nu^{(l+j)}_1(n).$$

From (3.4.3) and Remark 3.4.2, we have

$$\nu^{(l+j)}_1(n) = \nu^{(l+j)}_0(n) - a_{l+j+1}\nu^{(l+j+1)}_0(n), \quad \nu^{(k)}_0(n) = \frac{1}{k+2}\binom{n+k+1}{k+1}^{-1} = \mu^{(k+1)}_n,$$

so we can rewrite $\nu^{(l)}_k(n)$ as

$$\nu^{(l)}_k(n) = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j b_j(l)\mu^{(l+j+1)}_n - \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j b_j(l)a_{l+j+1}\mu^{(l+j+2)}_n.$$

Also, noting that $b_j(l)a_{l+j+1} = b_{j+1}(l)$, and letting $l = 1$, we obtain

$$\mathbb{E}(\mathcal{H}_n\mathcal{H}_{n+k}) = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j b_j(1)\mu_n^{(j+2)} - \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j b_{j+1}(1)\mu_n^{(j+3)},$$

Finally, (3.4.4) follows after replacing $b_j(1)$ by its value $\prod_{\mathrm{i}=1}^{j} a_{1+\mathrm{i}} = \frac{1}{2}\frac{j+2}{j+1}$ and by using formula (3.2.4) for $\mu_n^{(k)}$. □

We recall here the definitions of the binomial and the Euler transform (also known as alternating binomial transform) and show a couple of elementary properties. The Euler transform was introduced in Section 3.2.1, in the context of singularity analysis; see (3.2.7).

Definition 3.4.1 Let $(a_k)_k$ denote the sequence of real numbers $a_k, k \geq 0$.

(i) The sequence $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k, n \geq 0$, is called the *binomial transform* of $(a_k)_k$.

(ii) The sequence $c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k, n \geq 0$, is called the *Euler transform* of $(a_k)_k$.

Lemma 3.4.4 Let $(a_k)_k$ be a sequence and $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k, n \geq 0$. Then $b_{n+1} - b_n = \sum_{k=0}^{n} \binom{n}{k} a_{k+1}, n \geq 0$. Also, let $c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, then $c_{n+1} - c_n = -\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+1}, n \geq 0$.

Proof. Direct computation; see [54]. □

Lemma 3.4.5 For $k, \mathrm{i} \geq 1$, let

$$p_1(\mathrm{i}, k) = \frac{1}{4} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+1} \binom{\mathrm{i}+j+2}{j+2}^{-1}, \quad p_2(\mathrm{i}, k) = \frac{1}{4} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+3} \binom{\mathrm{i}+j+2}{j+2}^{-1}, \tag{3.4.5}$$

$$p_3(\mathrm{i}, k) = \frac{1}{4} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+2} \binom{\mathrm{i}+j+3}{j+3}^{-1}, \quad p_4(\mathrm{i}, k) = \frac{1}{4} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+4} \binom{\mathrm{i}+j+3}{j+3}^{-1}. \tag{3.4.6}$$

Then

$$p_3(\mathrm{i}, k) = p_1(\mathrm{i}, k) - p_1(\mathrm{i}, k+1), \quad p_4(\mathrm{i}, k) = p_2(\mathrm{i}, k) - p_2(\mathrm{i}, k+1) \tag{3.4.7}$$

and

$$\mathbb{E}(\mathcal{H}_\mathrm{i}\mathcal{H}_{\mathrm{i}+k}) = [p_1 - p_3 + p_2 - p_4](\mathrm{i}, k). \tag{3.4.8}$$

Proof. For (3.4.7) observe that $(p_1(\mathrm{i}, k))_k$ is the Euler transform of $(a_j)_j$, where $a_j = \frac{1}{4(j+1)} \binom{\mathrm{i}+j+2}{j+2}^{-1}$ and that $(p_3(\mathrm{i}, k))_k$ is the Euler transform of $(a_{j+1})_j$. The conclusion is then derived from Lemma 3.4.4. The argument for the identity involving p_2, p_4 is analogous. Formula (3.4.8) follows from using partial fraction decomposition in (3.4.4) □

Lemma 3.4.6 $p_2(\mathrm{i}, k)$ and $p_4(\mathrm{i}, k)$ are positive for every $\mathrm{i}, k \geq 1$.

Proof. Observe that

$$\begin{aligned}
p_2(\mathrm{i},k) &= \frac{1}{4}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\mathbb{E}(\mathcal{H}_{\mathrm{i}}^{j+2}) \\
&= \frac{1}{4}\mathbb{E}\left[\mathcal{H}_{\mathrm{i}}^2\sum_{j=0}^{k-1}\binom{k-1}{j}(-\mathcal{H}_{\mathrm{i}})^j\right] \\
&= \frac{1}{4}\mathbb{E}\big(\mathcal{H}_{\mathrm{i}}^2(1-\mathcal{H}_{\mathrm{i}})^{k-1}\big) > 0.
\end{aligned}$$

Analogously, it can be shown that $p_4(\mathrm{i},k) = \frac{1}{4}\mathbb{E}\big(\mathcal{H}_{\mathrm{i}}^3(1-\mathcal{H}_{\mathrm{i}})^{k-1}\big) > 0$. □

Lemma 3.4.7 Let $p_1(\mathrm{i},k)$ be as in (3.4.5). Then

$$p_1(\mathrm{i},k) = \frac{1}{4}\frac{2\mathrm{i}+k+1}{(\mathrm{i}+1)(\mathrm{i}+k)(\mathrm{i}+k+1)}$$

and

$$\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_1(\mathrm{i},k) = \frac{1}{8}H_n^2 + O(1). \tag{3.4.9}$$

Proof. Consider the identity $\binom{k-1}{j}\frac{1}{j+1} = \frac{1}{k(k+1)}\binom{k+1}{j+2}(j+2)$. Then, from (3.4.5), we get

$$p_1(\mathrm{i},k) = \frac{1}{4k(k+1)}\left[\sum_{j=1}^{k+1}\binom{k+1}{j}(-1)^j j\binom{\mathrm{i}+j}{j}^{-1} + \frac{k+1}{\mathrm{i}+1}\right].$$

Sury and Purkait in [47], provide a variety of results about sums involving the reciprocal of binomial coefficients. Among them, we apply the second formula of their lemma 1.1, to obtain

$$p_1(\mathrm{i},k) = \frac{1}{4k(k+1)}\left[-\frac{(k+1)\mathrm{i}}{(k+\mathrm{i}+1)(k+1)} + \frac{k+1}{\mathrm{i}+1}\right] = \frac{1}{4}\frac{2\mathrm{i}+k+1}{(\mathrm{i}+1)(\mathrm{i}+k)(\mathrm{i}+k+1)}.$$

For the double sum we have

$$\begin{aligned}
\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_1(\mathrm{i},k) &= \sum_{l=2}^{n}\sum_{k=1}^{l-1} p_1(l-k,k) \\
&= \sum_{l=2}^{n}\sum_{k=1}^{l-1}\frac{1}{4}\frac{2l-k+1}{l(l+1)(l-k+1)} \\
&= \sum_{l=2}^{n}\sum_{k=1}^{l-1}\frac{1}{4}\left[\frac{1}{(l+1)(l-k+1)} + \frac{1}{l(l+1)}\right] \\
&= \sum_{l=2}^{n}\sum_{k=1}^{l-1}\frac{1}{4}\left[\frac{H_l-1}{l+1} + \frac{l-1}{l(l+1)}\right] \\
&= \frac{1}{4}\sum_{l=1}^{n-1}\frac{H_l}{l+2}.
\end{aligned}$$

Sums involving harmonic numbers have been intensively studied and many results are available. From formula (1.32) in [13], we have

$$\sum_{l=1}^{n-1} \frac{H_l}{l+2} = \frac{1}{2}\Big(H_{n+1}^2 - H_{n+1}^{(2)}\Big) + \frac{1}{n+1} - 1$$

and so, (3.4.9) follows. □

Lemma 3.4.8 Let $p_3(\mathrm{i}, k)$ be as in (3.4.6). Then

$$\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_3(\mathrm{i}, k) = O(1). \tag{3.4.10}$$

Proof. By Lemma 3.4.5, we have $p_3(\mathrm{i}, k) = p_1(\mathrm{i}, k) - p_1(\mathrm{i}, k+1)$, so by the telescoping property and the expression for $p_1(\mathrm{i}, k)$ in Lemma 3.4.7, we get

$$\begin{aligned}\sum_{k=1}^{n-\mathrm{i}} p_3(\mathrm{i}, k) &= p_1(\mathrm{i}, 1) - p_1(\mathrm{i}, n-\mathrm{i}+1)\\ &= \frac{1}{2}\left(\frac{1}{\mathrm{i}+1} - \frac{1}{\mathrm{i}+2}\right) - \frac{1}{4}\left(\frac{1}{(n+1)(n+2)} + \frac{1}{(\mathrm{i}+1)(n+2)}\right).\end{aligned}$$

Then

$$\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_3(\mathrm{i}, k) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1}\right) - \frac{1}{4}\left(\frac{n-1}{(n+1)(n+2)} + \frac{H_n - 1}{n+2}\right) = O(1).$$

□

Lemma 3.4.9 Let $p_2(\mathrm{i}, k)$ be as in (3.4.5). Then

$$\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_2(\mathrm{i}, k) \le \frac{H_n}{8}. \tag{3.4.11}$$

Proof. Let

$$p_0(\mathrm{i}, k) = \frac{1}{4}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j+2}\binom{\mathrm{i}+j+1}{j+1}^{-1} = \frac{1}{4}\mathbb{E}\big(\mathcal{H}_\mathrm{i}(1-\mathcal{H}_\mathrm{i})^{k-1}\big) > 0$$

and observe that the sequence $(p_0(\mathrm{i}, k))_k$ is the Euler transform of $\left(\frac{1}{4(j+2)}\binom{\mathrm{i}+j+1}{j+1}^{-1}\right)_j$ and $(p_2(\mathrm{i}, k))_k$ is the Euler transform of $\left(\frac{1}{4(j+3)}\binom{\mathrm{i}+j+2}{j+2}^{-1}\right)_j$. Then, by Lemma 3.4.4, we have $p_2(\mathrm{i}, k) = p_0(\mathrm{i}, k) - p_0(\mathrm{i}, k+1) > 0$. So,

$$\sum_{k=1}^{n-\mathrm{i}} p_2(\mathrm{i}, k) = p_0(\mathrm{i}, 1) - p_0(\mathrm{i}, n-\mathrm{i}+1) \le p_0(\mathrm{i}, 1) = \frac{1}{8(\mathrm{i}+1)}$$

and the conclusion follows. □

Lemma 3.4.10
$$\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i}, \mathcal{H}_j) = o(\log^2 n). \tag{3.4.12}$$

Proof. We have
$$\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i}, \mathcal{H}_j) = \sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i}\mathcal{H}_j) - \sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i})\mathbb{E}(\mathcal{H}_j).$$
From (3.4.2), $\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i})\mathbb{E}(\mathcal{H}_j) = \frac{1}{8}H_n^2 + O(\log n)$. Also, from Lemmas 3.4.5, 3.4.7 and 3.4.9, we get
$$\begin{aligned}\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i},\mathcal{H}_j) &= \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}}(p_1 - p_3)(\mathrm{i}, k) + \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}}(p_2 - p_4)(\mathrm{i}, k)\\ &= \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_1(\mathrm{i}, k+1) + \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_2(\mathrm{i}, k+1)\\ &= \frac{1}{8}H_n^2 + O(\log n).\end{aligned}$$
Hence, $\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i}, \mathcal{H}_j)$ is at most $O(\log n)$ and the conclusion follows. □

Now we state a law of large numbers for the sum of record heights, which is the main result of this section, and a law of large numbers for $\mathcal{N}_n$ as corollary.

Theorem 3.4.1 The record heights $\mathcal{H}_\mathrm{i}$, for the uniform model on $[0,1]^2$ satisfy the following law of large numbers,
$$\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n} \mathcal{H}_\mathrm{i} \xrightarrow{L_2} \frac{1}{2}. \tag{3.4.13}$$
Furthermore
$$\frac{\mathcal{N}_n}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{2}. \tag{3.4.14}$$

Proof. Note that
$$\mathbb{E}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i} - \frac{1}{2}\right)^2 = \mathrm{Var}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i}\right) + \left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathbb{E}(\mathcal{H}_\mathrm{i}) - \frac{1}{2}\right)^2.$$
The second term in the rhs of the decomposition above goes to 0 because of (3.4.2). Furthermore,
$$\mathrm{Var}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i}\right) = \frac{1}{\log^2 n}\sum_{\mathrm{i}=1}^{n}\mathrm{Var}(\mathcal{H}_\mathrm{i}) + \frac{2}{\log^2 n}\sum_{1\le \mathrm{i}<j\le n}\mathrm{Cov}(\mathcal{H}_\mathrm{i}, \mathcal{H}_j) \to 0,$$
because of (3.4.2) and (3.4.12), hence (3.4.13) holds. For (3.4.14), we have from (2.3.3) that
$$\frac{\mathcal{N}_n}{\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i}} \to 1,$$
a.s. Then, (3.4.14) follows at once from (3.4.13). □

We conclude our asymptotic analysis of $\mathcal{N}_n$ with a central limit theorem for the martingale obtained by compensating $\mathcal{N}_n$. Recall that $\mathbb{E}(I_n|\mathcal{F}_{n-1}) = \mathcal{H}_{n-1}$, where $\mathcal{F}_n$ is the σ-algebra generated by $\mathbf{X}_1, \dots, \mathbf{X}_n$. Then, $\sum_{k=1}^{n-1} \mathcal{H}_k$ is the predictable compensator of $\mathcal{N}_n$ and so, $\mathcal{N}_n - \sum_{k=1}^{n-1} \mathcal{H}_k$ is a martingale with bounded increments. We can apply a central limit theorem for martingales, such as the following

Theorem 3.4.2 Let (ξ_n) be a sequence of square-integrable random variables, adapted to the filtration $(\mathcal{F}_n)$, such that $\mathbb{E}(\xi_n \,|\, \mathcal{F}_{n-1}) = 0$, for all $n \geq 1$. Let (b_n) be a sequence of real numbers such that $b_n \nearrow \infty$ and suppose that the following conditions hold

$$(1) \quad \frac{1}{b_n^2} \sum_{i=1}^{n} \mathbb{E}(\xi_i^2 \,|\, \mathcal{F}_{i-1}) \xrightarrow{\mathbb{P}} \sigma^2 > 0,$$

$$(2) \quad \frac{1}{b_n^2} \sum_{i=1}^{n} \mathbb{E}(\xi_i^2 1_{\{|\xi_i| > \varepsilon b_n\}} \,|\, \mathcal{F}_{i-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{for all } \varepsilon > 0.$$

Then $\frac{1}{b_n} \sum\limits_{i=1}^{n} \xi_i \xrightarrow{\mathcal{D}} N(0, \sigma^2)$.

Theorem 3.4.3

$$\frac{\mathcal{N}_n - \sum_{i=1}^{n} \mathcal{H}_{i-1}}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} N(0, \tfrac{1}{2}) \tag{3.4.15}$$

Proof. Condition (1) of Theorem 3.4.2 is satisfied with $b_n^2 = \log n$ and $\sigma^2 = 1/2$, since $\mathbb{E}(\xi_i^2|\mathcal{F}_{i-1}) = \mathcal{H}_{i-1}(1 - \mathcal{H}_{i-1})$, with $\mathcal{H}_0 := 1$. Moreover, condition (2) is trivially satisfied since the martingale has bounded increments. $\square$

Chapter 4

The uniform model on the d-simplex

In this chapter we study chain maxima and records from observations $\mathbf{X}_n$, distributed according to the uniform model on the d-simplex. That is, $(\mathbf{X}_n)$ is an iid sequence of random vectors, uniformly distributed on

$$\Delta^{\mathrm{d}} = \{\mathbf{x} = (x^{(1)}, \ldots, x^{(\mathrm{d})}) \in \mathbb{R}_+^{\mathrm{d}} : \sum_{\mathrm{i}=1}^{\mathrm{d}} x^{(\mathrm{i})} \le 1\}.$$

Observe that the components $X_n^{(j)}$ of $\mathbf{X}_n$ are not independent and that $\mathcal{N}_\infty = \infty$ a.s., since Assumption 3 is satisfied with A equal to the interior of Δ^{d}. We consider the assumption below to hold in this chapter, unless stated otherwise.

Assumption 6 The observation vectors $\mathbf{X}_n$ are iid, uniformly distributed in Δ^{d}. This is the $\mathbf{U}(\Delta^{\mathrm{d}})$ model.

By definition, $\mathbf{X}_n$ has density function given by

$$\mathbf{f}(\mathbf{x}) = \mathrm{d}!\mathbb{1}_{\Delta^{\mathrm{d}}}(\mathbf{x}).$$

Also, for $\mathbf{x} \in \Delta^{\mathrm{d}}$,

$$\bar{\mathbf{F}}(\mathbf{x}) = \Big(1 - \sum_{j=1}^{\mathrm{d}} x^{(j)}\Big)^{\mathrm{d}}. \tag{4.0.1}$$

As commented above, the components $X_n^{(j)}$ are dependent, identically distributed random variables, with common Beta$(1, \mathrm{d})$ density, given by $f(x) = \mathrm{d}(1-x)^{\mathrm{d}-1}, x \in [0, 1]$.

4.1 Records

In this section we focus on the study of $\mathcal{R}_n = (\mathcal{R}_n^{(1)}, \ldots, \mathcal{R}_n^{(\mathrm{d})})$. We start by deriving a recurrence for the density function of $\mathcal{R}_n$ and then establish a perpetuity representation which, in the case $\mathrm{d}=2$, allows us to study the asymptotic behavior and identify an associated limit in distribution.

Proposition 4.1.1 The process $(\mathcal{R}_k)$ is a Markov chain with transition probabilities

$$\mathbb{P}(\mathcal{R}_{k+1} \preceq \mathbf{x} | \mathcal{R}_k) = \frac{\mathrm{d}! \prod_{j=1}^{\mathrm{d}} (x^{(j)} - \mathcal{R}_k^{(j)})}{\left(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} \mathcal{R}_k^{(\mathrm{i})}\right)^{\mathrm{d}}} 1_{\{\mathcal{R}_k \preceq \mathbf{x}\}}, \ \mathbf{x} \in \Delta^{\mathrm{d}}. \tag{4.1.1}$$

In addition, $\mathcal{R}_k$ has probability density function g_k satisfying the recurrence

$$g_{k+1}(\mathbf{x}) = 1_{\Delta^{\mathrm{d}}}(\mathbf{x}) \int_{[\mathbf{0},\mathbf{x}]} \frac{\mathrm{d}! g_k(\mathbf{u})}{\left(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} u^{(\mathrm{i})}\right)^{\mathrm{d}}} \mathrm{d}\mathbf{u}, \ k \geq 1. \tag{4.1.2}$$

where $g_1(\mathbf{u}) = \mathbf{f}(\mathbf{u}) = \mathrm{d}! 1_{\Delta^{\mathrm{d}}}(\mathbf{u})$ and $[\mathbf{0}, \mathbf{x}] = \{\mathbf{y} \in \mathbb{R}^{\mathrm{d}} : \mathbf{0} \preceq \mathbf{y} \preceq \mathbf{x}\}$.

Proof. Formula (4.1.1) follows directly from (2.2.3). Also, if G_k denotes the distribution function of $\mathcal{R}_k$ then, taking expectation in (4.1.1), we have

$$G_{k+1}(\mathbf{x}) = \int_{[\mathbf{0},\mathbf{x}]} \frac{\mathrm{d}! \prod_{j=1}^{\mathrm{d}} (x^{(j)} - u^{(j)})}{\left(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} u^{(\mathrm{i})}\right)^{\mathrm{d}}} G_k(\mathrm{d}\mathbf{u}). \tag{4.1.3}$$

It can be shown, by induction, that the density g_k exists for any k. Then (4.1.2) is obtained by differentiating in (4.1.3). □

Remark 4.1.1 For $\mathrm{d}=2$, we can compute a few terms of the recurrence in (4.1.2):

$$\begin{aligned} g_1(x,y) &= 2 1_{\Delta^2}(x,y), \\ g_2(x,y) &= 4 \log\left(\frac{(1-x)(1-y)}{1-x-y}\right) 1_{\Delta^2}(x,y), \\ g_3(x,y) &= 8 \left[\log\left(\frac{1-x-y}{(1-x)(1-y)}\right) - \mathrm{Li}_2\left(\frac{xy-x-y-1}{1-x-y}\right)\right] 1_{\Delta^2}(x,y), \end{aligned}$$

where $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} \mathrm{d}t$, $z \in \mathbb{C}$, is Spence's function, also known as dilogarithm. We observe that higher order terms in the sequence involve intractable expressions. So, given the complexity of the sequence g_k, we have to consider a different strategy to analyze the asymptotic behavior of chain records in Δ^{d}.

Conditionally on $\mathcal{R}_k$, the random variable $\mathcal{R}_{k+1}$ is uniformly distributed on a region Δ_k, which is the d-simplex Δ^{d}, after appropriate translation and change of scale, namely, Δ_k is given by

$$\Delta_k = \mathcal{R}_k + \left(1 - \sum_{j=1}^{\mathrm{d}} R_k^{(j)}\right) \Delta^{\mathrm{d}}.$$

In other words, if $\mathbf{X}$ is $\mathbf{U}(\Delta^{\mathrm{d}})$, then

$$\mathcal{R}_{k+1} \stackrel{\mathcal{D}}{=} \mathcal{R}_k + (1 - \sum_{j=1}^{\mathrm{d}} R_k^{(j)})\mathbf{X}, \tag{4.1.4}$$

conditionally on $\mathcal{R}_k$, and the density of $\mathcal{R}_{k+1}$, conditional on $\mathcal{R}_k$, is given by

$$f_{\mathcal{R}_{k+1}|\mathcal{R}_k}(\mathbf{x}) = \frac{\mathrm{d}!}{1-\sum_{j=1}^{\mathrm{d}} R_k^{(j)}} 1_{\Delta^{\mathrm{d}}}\Big(\frac{\mathbf{x}-\mathcal{R}_k}{1-\sum_{j=1}^{\mathrm{d}} R_k^{(j)}}\Big).$$

Moreover, for $\mathbf{x} \in \Delta^{\mathrm{d}}$,

$$\mathbb{P}(\mathcal{R}_{k+1} \succ \mathbf{x}|\mathcal{R}_k) = \bar{\mathbf{F}}\Big(\frac{\mathbf{x}-\mathcal{R}_k}{1-\sum_{j=1}^{\mathrm{d}} R_k^{(j)}}\Big) = \Big(\frac{1-\sum_{j=1}^{\mathrm{d}} x^{(j)}}{1-\sum_{j=1}^{\mathrm{d}} R_k^{(j)}}\Big)^{\mathrm{d}}.$$

Also, setting all coordinates to 0, except one, we get the conditional marginals, for $j = 1, \ldots, \mathrm{d}$, as

$$\mathbb{P}(\mathcal{R}_{k+1}^{(j)} > x|\mathcal{R}_k) = \Big(\frac{1-x}{1-\sum_{j=1}^{\mathrm{d}} R_k^{(j)}}\Big)^{\mathrm{d}}.$$

Thanks to (4.1.4), we have the following analog of Proposition 3.1.3.

Proposition 4.1.2 Let $(\mathbf{U}_n)$ be a sequence of iid $\mathbf{U}(\Delta^{\mathrm{d}})$ random vectors and let $(\mathbf{V}_n)$ be defined by $\mathbf{V}_1 = \mathbf{U}_1$ and

$$\mathbf{V}_{n+1} = \mathbf{V}_n + \Big(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} V_n^{(\mathrm{i})}\Big)\mathbf{U}_{n+1}, \tag{4.1.5}$$

for $n \geq 1$. Then $(\mathbf{V}_n) \stackrel{\mathcal{D}}{=} (\mathcal{R}_n)$.

Proof. $(\mathbf{V}_n)$ is clearly a Markov chain, with initial state distributed as $\mathbf{U}_1$ and transitions given by (4.1.1). Hence, the conclusion follows. □

Our objective is to investigate (4.1.5) as a stochastic difference equation. In particular we are interested in convergence and characterization of the distributions of its solutions. A solution is meant in the strong sense.

Lemma 4.1.1 Recurrence (4.1.5) has solution

$$\mathbf{V}_n = \sum_{\mathrm{i}=1}^{n} \mathbf{U}_{\mathrm{i}} \prod_{k=1}^{\mathrm{i}-1} \Big(1 - \sum_{j=1}^{\mathrm{d}} U_k^{(j)}\Big), \tag{4.1.6}$$

where $(\mathbf{U}_n)$ is a sequence of iid $\mathbf{U}(\Delta^{\mathrm{d}})$ vectors.

Proof. We proceed by induction. The case $n = 1$ is evident. Assume that (4.1.6) holds for

$n > 1$, then, adding by coordinates in (4.1.6), we have

$$\begin{aligned}\sum_{l=1}^{\mathrm{d}} V_n^{(l)} &= \sum_{\mathrm{i}=1}^{n}\sum_{l=1}^{\mathrm{d}} U_{\mathrm{i}}^{(l)} \prod_{k=1}^{\mathrm{i}-1}(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}) \\ &= \sum_{\mathrm{i}=1}^{n}\left(1-\left(1-\sum_{l=1}^{\mathrm{d}} U_{\mathrm{i}}^{(l)}\right)\right)\prod_{k=1}^{\mathrm{i}-1}(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}) \\ &= \sum_{\mathrm{i}=1}^{n}\left(\prod_{k=1}^{\mathrm{i}-1}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right)-\prod_{k=1}^{\mathrm{i}}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right)\right) \\ &= 1-\prod_{k=1}^{n}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right).\end{aligned}$$

Now, replacing in (4.1.5) the expression obtained above for $\sum_{l=1}^{\mathrm{d}} V_n^{(l)}$, we get

$$\begin{aligned}\mathbf{V}_{n+1} &= \mathbf{V}_n + \prod_{k=1}^{n}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right)\mathbf{U}_{n+1} \\ &= \sum_{\mathrm{i}=1}^{n+1}\mathbf{U}_{\mathrm{i}}\prod_{k=1}^{\mathrm{i}-1}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right).\end{aligned}$$

□

4.1.1 Convergence of $\mathcal{R}_n$

Proposition 4.1.3 The sequence of records $(\mathcal{R}_n)$ converges a.s. to a random variable $\mathcal{R}_\infty$, satisfying

$$\mathcal{R}_\infty \overset{\mathcal{D}}{=} \sum_{\mathrm{i}=1}^{\infty}\mathbf{U}_{\mathrm{i}}\prod_{k=1}^{\mathrm{i}-1}\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right), \tag{4.1.7}$$

where $(\mathbf{U}_n)$ is a sequence of iid $\mathbf{U}(\Delta^{\mathrm{d}})$ vectors.

Proof. The a.s. convergence of $(\mathcal{R}_n)$ is immediate because the marginal sequences $(\mathcal{R}_n^{(j)})$, $j = 1, \ldots, \mathrm{d}$, are increasing and bounded. The series representation (4.1.7) follows from Proposition 4.1.2. Moreover, it can be shown that, independently of the relationship between records and the random vectors $\mathbf{V}_n$, the series in the rhs of (4.1.7) is well defined. To that end we can apply the root test for convergence by components. Note that, for $j = 1, \ldots, \mathrm{d}$,

$$-\log\left(U_n^{(j)}\prod_{k=1}^{n-1}(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)})\right)^{1/n} = -\frac{\log U_n^{(j)}}{n} - \frac{1}{n}\sum_{k=1}^{n-1}\log\left(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\right). \tag{4.1.8}$$

From (4.0.1) we have $\mathbb{P}(U_n^{(j)} > u) = (1-u)^{\mathrm{d}}$, and so $\mathbb{P}(-\log U_n^{(j)} > n\varepsilon) = 1-(1-\mathrm{e}^{-n\varepsilon})^{\mathrm{d}} \leq \mathrm{d}\mathrm{e}^{-n\varepsilon}$, for $\varepsilon > 0$ and sufficiently large n. Hence, by the Borel-Cantelli lemma, we conclude that $-\frac{\log U_n^{(j)}}{n} \to 0$ a.s.

For the second term in the rhs of (4.1.8), we have a.s. convergence to the expectation $\mathbb{E}\Big(-\log\Big(1-\sum_{j=1}^{\mathrm{d}} U_k^{(j)}\Big)\Big) < \infty$, thanks to the SLLN. □

Remark 4.1.2 Observe that $S_\infty := \sum_{j=1}^{\mathrm{d}} \mathcal{R}_\infty^{(j)} = 1$ a.s. This follows directly from the rhs of (4.1.7), by summing through coordinates. Also, $S_\infty = 1$ implies that $\mathcal{R}_\infty$ does not have a density with respect to the Lebesgue measure on $\mathbb{R}_+^{\mathrm{d}}$, and also that $\mathbb{E}(\mathcal{R}_\infty^{(j)}) = 1/\mathrm{d}, j = 1, \ldots, \mathrm{d}$, since the coordinates $\mathcal{R}_\infty^{(j)}$ are equally distributed.

4.1.2 Perpetuity representation of $\mathcal{R}_\infty$

We are interested in the distribution of $\mathcal{R}_\infty$ but the series representation (4.1.7) does not look simple to analyze. That is why we turn our attention to the so-called perpetuity representation of $\mathcal{R}_\infty$. To that end we use some ideas related to perpetuities, that we take from [39].

Let (R_n) be a sequence of real random variables, satisfying the recursion

$$R_n = Q_n + M_n R_{n-1}, \tag{4.1.9}$$

for $n \geq 1$, with arbitrary R_0, where (Q_n, M_n) are iid copies of random pairs (Q, M), independent of R_{n-1}. Then, iterating (4.1.9) we get

$$\begin{aligned} R_n &= Q_n + M_n Q_{n-1} + M_n M_{n-1} Q_{n-2} + \cdots + M_n \cdots M_2 Q_1 + M_n \cdots M_1 R_0 \\ &= R_0 \prod_{j=1}^{n} M_j + \sum_{k=1}^{n} Q_k \prod_{j=k+1}^{n} M_j. \end{aligned} \tag{4.1.10}$$

Now, renumbering the (Q_n, M_n) pairs in the opposite direction, from (4.1.10) we obtain

$$\tilde{R}_n := R_0 \prod_{j=1}^{n} M_j + \sum_{k=1}^{n} Q_k \prod_{j=1}^{k-1} M_j. \tag{4.1.11}$$

Note that, because the random pairs (Q_n, M_n) are iid, we have that $\tilde{R}_n \stackrel{\mathcal{D}}{=} R_n$ but it is not true that $\tilde{R}_n = R_n$ and, in general, it is not even true that $(\tilde{R}_n) \stackrel{\mathcal{D}}{=} (R_n)$. However weak this relation might seem, it can be useful, for example, when the sequence (R_n) converges in distribution and the partial sums defining $\tilde{R}_n$, converge. For further information, see [39] and references therein.

We will use the same trick of renumbering iid random variables in recurrences related to records, in order to obtain equations in distribution involving $\mathcal{R}_\infty$.

Let us recall the expression (4.1.6), written for simplicity as

$$\begin{aligned} \mathbf{V}_n &= \sum_{\mathrm{i}=1}^{n} \mathbf{U}_{\mathrm{i}} \prod_{k=1}^{\mathrm{i}-1} \delta(\mathbf{U}_k) \\ &= \mathbf{U}_1 + \mathbf{U}_2 \delta(\mathbf{U}_1) + \cdots + \mathbf{U}_n \delta(\mathbf{U}_1) \cdots \delta(\mathbf{U}_{n-1}), \end{aligned}$$

where $\delta(\mathbf{U}_k) = 1 - \sum_{j=1}^{\mathrm{d}} U_k^{(j)}$. Then, by circularly permuting the vectors $\mathbf{U}_k$, in such a way that $\mathbf{U}_1$ becomes $\mathbf{U}_n$, $\mathbf{U}_2$ becomes $\mathbf{U}_1$ and so on, we get the following equation in distribution

$$\begin{aligned}\mathbf{V}_n &\overset{\mathcal{D}}{=} \mathbf{U}_n + \delta(\mathbf{U}_n)(\mathbf{U}_1 + \cdots + \mathbf{U}_{n-1}\delta(\mathbf{U}_1)\cdots\delta(\mathbf{U}_{n-2})) \\ &= \mathbf{U}_n + \delta(\mathbf{U}_n)\sum_{\mathrm{i}=1}^{n-1}\mathbf{U}_{\mathrm{i}}\prod_{k=1}^{\mathrm{i}-1}\delta(\mathbf{U}_k) \\ &= \mathbf{U}_n + \delta(\mathbf{U}_n)\mathbf{V}_{n-1}.\end{aligned} \tag{4.1.12}$$

Remark 4.1.3 Note that, since $\mathbf{V}_n$ depends only on $\mathbf{U}_1, \ldots, \mathbf{U}_n$ (see Lemma 4.1.1), it holds that $\mathbf{V}_{n-1}$ is independent of $\mathbf{U}_n$ and $\delta(\mathbf{U}_n)$, because the $\mathbf{U}_n$ are iid.

Proposition 4.1.4 Let $(\mathbf{U}_n), \mathbf{U}$ be iid $\mathbf{U}(\Delta^{\mathrm{d}})$ random vectors, independent of $(\mathcal{R}_n), \mathcal{R}_\infty$. Then, for $n \geq 1$,

$$\mathcal{R}_{n+1} \overset{\mathcal{D}}{=} \mathbf{U}_{n+1} + \delta(\mathbf{U}_{n+1})\mathcal{R}_n \tag{4.1.13}$$

and

$$\mathcal{R}_\infty \overset{\mathcal{D}}{=} \mathbf{U} + \delta(\mathbf{U})\mathcal{R}_\infty. \tag{4.1.14}$$

Proof. By Proposition (4.1.2) we know that $(\mathbf{V}_n) \overset{\mathcal{D}}{=} (\mathcal{R}_n)$. So, from (4.1.12) and Remark 4.1.3, we obtain (4.1.13). For (4.1.14) we have that $\mathcal{R}_n \to \mathcal{R}_\infty$ a.s, by Proposition 4.1.3, and that $\mathbf{U}_{n+1} \overset{\mathcal{D}}{=} \mathbf{U}$, so the rhs of (4.1.13) converges in distribution to $\mathbf{U} + \delta(\mathbf{U})\mathcal{R}_\infty$, and the conclusion follows. Observe that the hypothesis of independence between $\mathbf{U}_n, \mathbf{U}$ and the $\mathcal{R}_n, \mathcal{R}_\infty$ are crucial. □

Concerning the distribution of $\mathcal{R}_\infty$, we have the following result.

Proposition 4.1.5 The distribution of $\mathcal{R}_\infty^{(j)}$, for $j = 1, \ldots, \mathrm{d}$, is absolutely continuous or singular and continuous.

Proof. We quote, without proof, the following result, due to Grincevičius [37]; see also Vervaat [58], theorem 3.2. Let A, B, Y be real random variables such that $A \neq 0$ a.s. and Y is independent of (A, B). If

$$Y \overset{\mathcal{D}}{=} AY + B,$$

then the distribution of Y is either absolutely continuous, singular continuous or degenerate.

Observe that (4.1.14) is a vectorial recurrence, which yields for each coordinate $j = 1, \ldots, \mathrm{d}$, the following relation

$$\mathcal{R}_\infty^{(j)} \overset{\mathcal{D}}{=} U^{(j)} + \delta(\mathbf{U})\mathcal{R}_\infty^{(j)}. \tag{4.1.15}$$

We identify A with $\delta(\mathbf{U}) > 0$ a.s. and B with $U^{(j)}$, in Grincevičius' theorem, and the conclusion follows, if we prove that $\mathcal{R}_\infty^{(j)}$ is not degenerate. We know, from Remark 4.1.2, that $\mathbb{E}(\mathcal{R}_\infty^{(j)}) = 1/\mathrm{d}$, so, if $\mathcal{R}_\infty^{(j)}$ is degenerate, we must have $\mathbb{P}(\mathcal{R}_\infty^{(j)} = 1/\mathrm{d}) = 1$. But this is impossible since, by definition of chain-record, $\mathcal{R}_\infty^{(j)} \geq X_1^{(j)}$ a.s. and $\mathbb{P}(X_1^{(j)} > 1/\mathrm{d}) = (1 - 1/\mathrm{d})^{\mathrm{d}}$. So, $\mathbb{P}(\mathcal{R}_\infty^{(j)} > 1/\mathrm{d}) > 0$. □

We have obtained some valuable information about the distribution of $\mathcal{R}_\infty^{(j)}$ but still, far from being satisfactory. We consider below a rewriting of the recursion for $\mathcal{R}_\infty$, as a convex combination. Observe that (4.1.14) can be written as

$$\mathcal{R}_\infty \stackrel{\mathcal{D}}{=} (1-\delta(\mathbf{U}))\frac{\mathbf{U}}{1-\delta(\mathbf{U})} + \delta(\mathbf{U})\mathcal{R}_\infty. \tag{4.1.16}$$

Note that $1-\delta(\mathbf{U}) = \sum_{j=1}^{\mathrm{d}} U^{(j)} \in [0,1]$, so the coordinates of $\frac{\mathbf{U}}{1-\delta(\mathbf{U})}$ (and those of $\mathcal{R}_\infty$) add up to 1. Hence, the expression in the rhs of (4.1.16) is a convex combination of vectors $\frac{\mathbf{U}}{1-\delta(\mathbf{U})}$ and $\mathcal{R}_\infty$, both taking values in $D^{\mathrm{d}} = \{\mathbf{x} = (x^{(1)}, \ldots, x^{(\mathrm{d})}) \in \mathbb{R}_+^{\mathrm{d}} : \sum_{\mathrm{i}=1}^{\mathrm{d}} x^{(\mathrm{i})} = 1\} \subset \Delta^{\mathrm{d}}$. We have the following elementary result.

Lemma 4.1.2 Let $\mathbf{U}$ be a $\mathbf{U}(\Delta^{\mathrm{d}})$ random vector. Then $\frac{\mathbf{U}}{1-\delta(\mathbf{U})}$ is uniformly distributed in D^{d}. Furthermore, $1-\delta(\mathbf{U})$ and $\frac{\mathbf{U}}{1-\delta(\mathbf{U})}$ are independent.

Proof. Let ν be the Lebesgue's measure (or volume) on D^{d} and A a Borel subset of D^{d}, such that $\nu(A) > 0$. Let $\phi : \Delta^{\mathrm{d}} \to D^{\mathrm{d}}$ be the function defined by $\phi(u) = \frac{u}{1-\delta(u)}$. Then, the preimage $\phi^{-1}(A)$ is a cone in Δ^{d}, with tip at 0 (but not included), base A and height h, equal to the distance from 0 to D^{d}, that is, $h = 1/\sqrt{\mathrm{d}}$. Then the volume of $\phi^{-1}(A)$ (Lebesgue's measure on $\mathbb{R}_+^{\mathrm{d}}$) is $\frac{h}{\mathrm{d}}\nu(A)$ and so, $\mathbb{P}(\phi(\mathbf{U}) \in A) = \frac{(\mathrm{d}-1)!}{\sqrt{\mathrm{d}}}\nu(A)$, which means that $\phi(\mathbf{U})$ is uniformly distributed on D^{d}. Finally, noting that $\{1-\delta(\mathbf{U}) \le s\} = \{\mathbf{U} \in s\Delta^{\mathrm{d}}\}$, for $s \in (0,1]$, we have

$$\begin{aligned} p :&= \mathbb{P}(\phi(\mathbf{U}) \in A, 1-\delta(\mathbf{U}) \le s) \\ &= \mathbb{P}(\phi(\mathbf{U}) \in A, \mathbf{U} \in s\Delta^{\mathrm{d}}) \\ &= \mathbb{P}(\mathbf{U} \in \phi^{-1}(A) \cap (s\Delta^{\mathrm{d}})). \end{aligned}$$

Observe also that $\phi^{-1}(A) \cap (s\Delta^{\mathrm{d}})$ is a cone contained in $s\Delta^{\mathrm{d}}$, with tip at 0 (but not included), base sA and height $h = \frac{s}{\sqrt{\mathrm{d}}}$. So, its volume is

$$\frac{1}{\mathrm{d}}\frac{s}{\sqrt{\mathrm{d}}}\nu(sA) = \frac{s^{\mathrm{d}}}{\mathrm{d}\sqrt{\mathrm{d}}}\nu(A).$$

Hence,

$$p = \mathrm{d}!\frac{s^{\mathrm{d}}}{\mathrm{d}\sqrt{\mathrm{d}}}\nu(A) = \frac{(\mathrm{d}-1)!}{\sqrt{\mathrm{d}}}\nu(A)s^{\mathrm{d}} = \mathbb{P}(\phi(\mathbf{U}) \in A)\mathbb{P}(\mathbf{U} \in s\Delta^{\mathrm{d}})$$

and the claim of independence is thus proven. □

Remark 4.1.4 Note that $\phi(\mathbf{U})$ in the previous lemma has Dirichlet distribution, with parameters $\alpha_1 = \cdots = \alpha_{\mathrm{d}} = 1$, also known as flat Dirichlet distribution.

We now have the following representation for $\mathcal{R}_\infty$, as a random convex combination.

Proposition 4.1.6 Let $\mathbf{W}$ be a $\mathbf{U}(D^{\mathrm{d}})$ random vector (uniformly distributed on D^{d}) and Λ a random variable with density $f_\Lambda(\lambda) = \mathrm{d}\lambda^{\mathrm{d}-1}\mathbb{1}_{[0,1]}(\lambda)$, such that $\mathbf{W}, \Lambda$ and $\mathcal{R}_\infty$ are independent. Then

$$\mathcal{R}_\infty \stackrel{\mathcal{D}}{=} \Lambda\mathbf{W} + (1-\Lambda)\mathcal{R}_\infty. \tag{4.1.17}$$

PROOF. The result is a direct consequence of (4.1.16) and Lemma 4.1.2. The density of Λ is that of the sum of components of a $\mathbf{U}(\Delta^{\mathrm{d}})$ random vector. □

Remark 4.1.5 Recall that $(\mathcal{R}_n)$ is a Markov chain, with state space Δ^{d}, converging a.s. to $\mathcal{R}_\infty$, taking values in D^{d} and we may think that the law of $\mathcal{R}_\infty$ is a stationary distribution of the chain. Indeed, the law of $\mathcal{R}_\infty$ is stationary but the chain $(\mathcal{R}_n)$, starting with such initial distribution, never moves away from the initial point, because every point in D^{d} is terminal, in the sense that $\bar{\mathbf{F}}(\mathbf{x}) = 0$, for all $\mathbf{x} \in D^{\mathrm{d}}$. In fact, any distribution concentrated on D^{d} is stationary for the chain $(\mathcal{R}_n)$, just like the distribution of $\mathcal{R}_\infty$, and there are no other stationary distributions.

The result of Proposition 4.1.6 is a kind of invariance principle for the distribution of $\mathcal{R}_\infty$, by the transformation in the rhs of (4.1.17). The question is if such invariance characterizes the distribution of $\mathcal{R}_\infty$ or not. In order to explore this possibility, we define below a Markov chain on D^{d}, following the dynamics suggested by equation (4.1.16).

Definition 4.1.1 Let $(\mathbf{W}_n), (\Lambda_n)$ be independent iid copies of $\mathbf{W}$ and Λ respectively, with distributions as in Proposition 4.1.6. Let the sequence $(\mathcal{T}_n)_{n\geq 0}$ be defined as follows: $\mathcal{T}_0$ has a certain initial distribution on D^{d} and, for $n \geq 1$,

$$\mathcal{T}_n = \Lambda_n \mathbf{W}_n + (1-\Lambda_n)\mathcal{T}_{n-1}.$$

Proposition 4.1.7 Let A be a Borel subset of D^{d} and $s \in D^{\mathrm{d}}$. The sequence $(\mathcal{T}_n)$ from Definition 4.1.1 is a Markov chain with state space D^{d} and transition kernel K, given by

$$K(\mathbf{s}, A) = \mathbb{P}(\mathcal{T}_n \in A | \mathcal{T}_{n-1} = \mathbf{s}) = \mathbb{P}(\Lambda \mathbf{W} + (1-\Lambda)\mathbf{s} \in A).$$

PROOF. By construction, $(\mathcal{T}_n)$ is a Markov chain, with characteristics as described. □

Remark 4.1.6 Note that, from Propositions 4.1.6 and 4.1.7, the distribution of $\mathcal{R}_\infty$ is the invariant distribution of the chain $(\mathcal{T}_n)$.

For illustration we consider the case d = 2 in detail. In this situation Λ has density $\Lambda(\lambda) = 2\lambda \mathbb{1}_{[0,1]}(\lambda)$ and $\mathbf{W}$ is a $\mathbf{U}(D^2)$ random vector. Of course, since D^2 is one-dimensional, we can work with $\mathbf{W}$ by means of its projection W on $\Delta^1 = [0,1]$. Clearly, W is $U[0,1]$.

The transition kernel $K(s, \mathrm{d}y)$ of Proposition 4.1.7 is the distribution of $\Lambda W + (1-\Lambda)s$, given by

$$K(s, \mathrm{d}y) = 2\left(\tfrac{y}{s}\mathbb{1}_{[0,s]}(y) + \left(\tfrac{1-y}{1-s}\right)\mathbb{1}_{[s,1]}(y)\right)\mathrm{d}y, \quad s \in [0,1].$$

The formula above is obtained from the change of variable $(W,\Lambda) \to g(W,\Lambda) = ((W-s)\Lambda + s, \Lambda)$. We end this example by showing that $(\mathcal{T}_n)$ has invariant distribution Beta$(2,2)$. Indeed, the Beta$(2,2)$ density is given by $6s(1-s)\mathbb{1}_{[0,1]}(s)$ and we have

$$\begin{aligned}\int K(\mathrm{d}s, \mathrm{d}y) 6s(1-s)\mathrm{d}s &= 12y\mathrm{d}y \int_y^1 (1-s)\mathrm{d}s + 12(1-y)\mathrm{d}y \int_0^y s\mathrm{d}s \\ &= 6y(1-y)^2\mathrm{d}y + 6y^2(1-y)\mathrm{d}y \\ &= 6y(1-y)\mathrm{d}y.\end{aligned}$$

Hence, by Remark 4.1.6, we conclude that $\mathcal{R}_\infty$ has distribution Beta$(2, 2)$. We now turn our attention to the general case $\mathrm{d} > 2$, where calculations are more involved. In higher dimensions we consider the Dirichlet distribution as generalization of the Beta distribution. The Dirichlet distribution on D^{d}, with positive parameters $\alpha_1, \ldots, \alpha_{\mathrm{d}}$, is denoted $\mathrm{Dir}(\alpha_1, \ldots, \alpha_{\mathrm{d}})$ and has density (with respect to Lebesgue's measure on $\mathcal{R}^{\mathrm{d}-1}$) given by

$$\frac{\Gamma(\sum_{\mathrm{i}=1}^{\mathrm{d}} \alpha_{\mathrm{i}})}{\prod_{\mathrm{i}=1}^{\mathrm{d}} \Gamma(\alpha_{\mathrm{i}})} x_1^{\alpha_1} \cdots x_{\mathrm{d}-1}^{\alpha_{\mathrm{d}-1}} \Big(1 - \sum_{j=1}^{\mathrm{d}-1} x_j\Big)^{\alpha_{\mathrm{d}}}, \ (x_1, \ldots, x_{\mathrm{d}-1}) \in \Delta^{\mathrm{d}-1}.$$

For simplicity we can write $x_{\mathrm{d}} = 1 - \sum_{j=1}^{\mathrm{d}-1} x_j$. With such notation we have $(x_1, \ldots, x_{\mathrm{d}}) \in D^{\mathrm{d}}$. The $\mathbf{U}(D^{\mathrm{d}})$ model is a particular instance of the Dirichlet distribution, with parameters $\alpha_1 = \cdots = \alpha_{\mathrm{d}} = 1$, also known as flat Dirichlet distribution.

For fixed $\mathbf{s} \in D^{\mathrm{d}}$, we define $\mathbf{Y} = \Lambda\mathbf{W} + (1 - \Lambda)\mathbf{s} = \Lambda(\mathbf{W} - \mathbf{s}) + \mathbf{s}$ or, equivalently, $\mathbf{W} = \frac{\mathbf{Y} - s}{\Lambda} + \mathbf{s}$, which is a linear change of variable on D^{d}. In order to write the density of $\mathbf{Y}$ we must project on $\Delta^{\mathrm{d}-1}$, as we did for $\mathrm{d} = 2$. But, to keep the notation simple, we do not use a different symbol for the projections of $\mathbf{Y}, \mathbf{s}$ or $\mathbf{W}$ on $\Delta^{\mathrm{d}-1}$.

First we easily see that the density of $\mathbf{Y}$, conditional on Λ, denoted $f_{\mathbf{Y}|\Lambda}(\mathbf{y}|\lambda)$, is given by

$$f_{\mathbf{Y}|\Lambda}(\mathbf{y}|\lambda) = 1_{\Delta^{\mathrm{d}-1}}\left(\frac{\mathbf{y} - \mathbf{s}}{\lambda} + \mathbf{s}\right) \frac{(\mathrm{d} - 1)!}{\lambda^{\mathrm{d}-1}} 1_{\Delta^{\mathrm{d}-1}}(\mathbf{y}).$$

Then, the marginal density of $\mathbf{Y}$ is

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= 1_{\Delta^{\mathrm{d}-1}}(\mathbf{y}) \int_0^1 1_{\Delta^{\mathrm{d}-1}}\left(\frac{\mathbf{y} - \mathbf{s}}{\lambda} + \mathbf{s}\right) \frac{(\mathrm{d} - 1)!}{\lambda^{\mathrm{d}-1}} f_\Lambda(\lambda) \mathrm{d}\lambda \\ &= \mathrm{d}! 1_{\Delta^{\mathrm{d}-1}}(\mathbf{y}) \int_0^1 1_{\Delta^{\mathrm{d}-1}}\left(\frac{\mathbf{y} - \mathbf{s}}{\lambda} + \mathbf{s}\right) \mathrm{d}\lambda. \end{aligned} \tag{4.1.18}$$

We analyze the indicator within the integral above, as a function of λ. We have the following conditions:

$$\frac{y^{(j)} - s^{(j)}}{\lambda} + s^{(j)} \geq 0, \quad j = 1, \ldots, \mathrm{d} - 1,$$

equivalent to

$$\lambda \geq \max_{j=1,\ldots,\mathrm{d}-1} \Big(1 - \frac{y^{(j)}}{s^{(j)}}\Big), \tag{4.1.19}$$

and, additionally,

$$\sum_{j=1}^{\mathrm{d}-1} \Big(\frac{y^{(j)} - s^{(j)}}{\lambda} + s^{(j)}\Big) \leq 1. \tag{4.1.20}$$

We can consider vectors $\mathbf{x} \in \Delta^{\mathrm{d}-1}$ as projections from S^{d}, defining a d-th coordinate of $\mathbf{x}$ as $x^{(\mathrm{d})} = 1 - \sum_{j=1}^{\mathrm{d}-1} x^{(j)}$. With such notation, condition (4.1.20) is equivalent to

$$\frac{y^{(\mathrm{d})} - s^{(\mathrm{d})}}{\lambda} + s^{(\mathrm{d})} \geq 0.$$

So (4.1.19) and (4.1.20) can be summarized in the single condition

$$\lambda \geq M(\mathbf{y}, \mathbf{s}) := \max_{j=1,\ldots,\mathrm{d}} \Big(1 - \frac{y^{(j)}}{s^{(j)}}\Big), \tag{4.1.21}$$

which is necessary and sufficient for $\frac{\mathbf{y}-\mathbf{s}}{\lambda} + \mathbf{s}$ to be in $\Delta^{\mathrm{d}-1}$. From (4.1.18) and (4.1.21) we finally have

$$k(\mathbf{s}, \mathbf{y}) = \mathrm{d}!(1 - M(\mathbf{y}, \mathbf{s}))1_{\Delta^{\mathrm{d}-1}}(\mathbf{y}),$$

which is the density of the transition kernel $K(\mathbf{s}, \mathrm{d}\mathbf{y})$.

Based on the result of the case $\mathrm{d} = 2$, we would conjecture that the general invariant distribution for the above kernel is Dirichlet $\mathrm{Dir}(2, \ldots, 2)$, but have no evidence beyond the case $\mathrm{d} = 2$. Calculations for $\mathrm{d} = 3$ can be done with the help of a computer algebra system such as Maple®, but preliminary results do not seem to support the conjecture.

Another idea, that works well only for $\mathrm{d} = 2$, is to differentiate the invariance integral equation and solve the resulting ODE. Suppose g is the invariant density, then

$$g(y) = 2y \int_y^1 \frac{g(s)}{s} \mathrm{d}s + 2(1-y) \int_0^y \frac{g(s)}{1-s} \mathrm{d}s$$

and so,

$$g'(y) = 2 \int_y^1 \frac{g(s)}{s} \mathrm{d}s - 2 \int_0^y \frac{g(s)}{1-s} \mathrm{d}s,$$

which yields

$$g''(y) + \frac{2}{y(1-y)} g(y) = 0.$$

The ODE above has as general solution

$$f(y) = a_1 y(1-y) + a_2 y(1-y)\Big(\frac{1}{1-y} - 2\log(1-y) - \frac{1}{y} + 2\log(y)\Big),$$

with $a_1, a_2 \in \mathbb{R}$. But, since we are looking for a pdf with respect to the Lebesgue measure in $[0, 1]$, we choose $a_2 = 0$ and $a_1 = 6$. So the stationary distribution has density $6r(1-r)1_{[0,1]}(r)$, that is, $\mathrm{Beta}(2, 2)$.

4.2 Record heights

In this section we analyze record heights, as given in Definition 3.2.1, noting, from (4.0.1), that in the context of the $\mathbf{U}(\Delta^{\mathrm{d}})$ model, we have $\bar{\mathbf{F}}(\mathcal{M}_n) = \Big(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} \mathcal{M}_n^{(\mathrm{i})}\Big)^{\mathrm{d}}$, for $n \geq 1$. We proceed as in the previous chapter by first computing their moments. Then we study their asymptotic behavior and derive a weak convergence result, with suitable normalization.

Lemma 4.2.1 Let $\mathrm{d} \geq 1$, $l \geq 0$ and $a > 0$, then

$$\int_{a\Delta^{\mathrm{d}}} \Big(a - \sum_{\mathrm{i}=1}^{\mathrm{d}} u^{(\mathrm{i})}\Big)^{l} \mathrm{d}\mathbf{u} = a^{l+\mathrm{d}} \prod_{\mathrm{i}=1}^{\mathrm{d}} (l+\mathrm{i})^{-1},$$

where $a\Delta^{\mathrm{d}} := \{ax : x \in \Delta^{\mathrm{d}}\}$.

Proof. We proceed by induction on d, the base case $\mathrm{d} = 1$ is true since $\int\limits_{a\Delta^{\mathrm{d}}} (a-u)^{l}\mathrm{d}u = \frac{a^{l+1}}{l+1}$. Writing the corresponding integral and using the induction hypothesis

$$\begin{aligned}
\int_{a\Delta^{\mathrm{d}+1}} \Big(a - \sum_{\mathrm{i}=1}^{\mathrm{d}+1} u^{(\mathrm{i})}\Big)^{l} \mathrm{d}\mathbf{u} &= \int_0^a \bigg(\int_{(a-u)\Delta^{\mathrm{d}}} \Big(a - u - \sum_{\mathrm{i}=1}^{\mathrm{d}} u^{(\mathrm{i})}\Big)^{l} \mathrm{d}\mathbf{u}\bigg) \mathrm{d}u \\
&= \prod_{\mathrm{i}=1}^{\mathrm{d}} (l+\mathrm{i})^{-1} \int_0^a (a-u)^{l+\mathrm{d}} \mathrm{d}u \\
&= a^{l+\mathrm{d}+1} \prod_{\mathrm{i}=1}^{\mathrm{d}+1} (l+\mathrm{i})^{-1}.
\end{aligned}$$

□

Proposition 4.2.1 Let $\lambda_n^{(k)} = \mathbb{E}(\mathcal{H}_n^k)$, for k, n positive integers. Then the following recursion holds

$$\lambda_{n+1}^{(k)} = \lambda_n^{(k)} - \Big(1 - \lambda_1^{(k)}\Big) \lambda_n^{(k+1)}, \tag{4.2.1}$$

with $\lambda_1^{(k)} = \mathrm{d}! \prod\limits_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k + \mathrm{i})^{-1}$.

Proof. From (2.2.7), with $g(\mathbf{x}) = \bar{\mathbf{F}}(\mathbf{x})^k$, and taking expectation, we have

$$\lambda_{n+1}^{(k)} = \mathbb{E}\bigg(\int_{\Delta^{\mathrm{d}} \cap \{\mathbf{x} \succ \mathcal{M}_n\}} \mathrm{d}!\Big(1 - \sum_{j=1}^{\mathrm{d}} x^{(j)}\Big)^{\mathrm{d}k} \mathrm{d}\mathbf{x}\bigg) + \mathbb{E}\left(\mathcal{H}_n^k(1 - \mathcal{H}_n)\right). \tag{4.2.2}$$

Note that

$$\Delta^{\mathrm{d}} \cap \{\mathbf{x} \succ \mathcal{M}_n\} = \Big(1 - \sum_{\mathrm{i}=1}^{\mathrm{d}} \mathcal{M}_n^{(\mathrm{i})}\Big)\Delta^{\mathrm{d}} + \mathcal{M}_n = \mathcal{H}_n^{1/\mathrm{d}}\Delta^{\mathrm{d}} + \mathcal{M}_n.$$

Hence, after a change of variable and by Lemma 4.2.1, we have

$$\begin{aligned}
\int_{\Delta^{\mathrm{d}} \cap \{\mathbf{x} \succ \mathcal{M}_n\}} \mathrm{d}!\Big(1 - \sum_{j=1}^{\mathrm{d}} x^{(j)}\Big)^{\mathrm{d}k} \mathrm{d}\mathbf{x} &= \int_{\mathcal{H}_n^{1/\mathrm{d}}\Delta^{\mathrm{d}}} \mathrm{d}!\Big(\mathcal{H}_n^{1/\mathrm{d}} - \sum_{\mathrm{i}=1}^{\mathrm{d}} u^{(\mathrm{i})}\Big)^{\mathrm{d}k} \mathrm{d}\mathbf{u} \\
&= \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k + \mathrm{i})^{-1} \Big(\mathcal{H}_n^{1/\mathrm{d}}\Big)^{\mathrm{d}k+\mathrm{d}} \\
&= \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k + \mathrm{i})^{-1} \mathcal{H}_n^{k+1}.
\end{aligned}$$

Replacing in equation (4.2.2) yields

$$\lambda_{n+1}^{(k)} = \mathbb{E}\left(\mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k+\mathrm{i})^{-1} \mathcal{H}_n^{k+1}\right) + \lambda_n^{(k)} - \lambda_n^{(k+1)}$$
$$= \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k+\mathrm{i})^{-1} \lambda_n^{(k+1)} + \lambda_n^{(k)} - \lambda_n^{(k+1)}.$$

Finally, again by Lemma 4.2.1, we have $\lambda_1^{(k)} = \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}k+\mathrm{i})^{-1}$. □

Note that recurrence (4.2.1) has the same structure as (3.2.1).

Proposition 4.2.2 The solution of recurrence (4.2.1) has the form

$$\lambda_{n+1}^{(k)} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \lambda_1^{(k+j)} \prod_{\mathrm{i}=0}^{j-1} \left(1 - \lambda_1^{(k+\mathrm{i})}\right), \tag{4.2.3}$$

for $n \geq 0$.

Proof. Identical to the proof of Proposition 3.2.2. Formula (4.2.3) is derived from (4.2.1) by iteration. □

In one and two dimensions we have closed-form expressions for $\lambda_n^{(k)}$.

Lemma 4.2.2 (i) For $\mathrm{d} = 1$,

$$\lambda_n^{(k)} = \binom{n+k}{k}^{-1}. \tag{4.2.4}$$

(ii) For $\mathrm{d} = 2$,

$$\lambda_n^{(k)} = \frac{1}{2k+1} \binom{n+k}{k}^{-1}. \tag{4.2.5}$$

Proof. It suffices to iterate the recursion (4.2.1). □

4.2.1 Asymptotic analysis of $\lambda_n^{(k)}$

Observe, from (4.2.3), that the k-the moment of $\mathcal{H}_{n+1}$ is the Euler's transform (see (3.2.7)) of

$$b_j^{(k)} := \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{d}(k+j)+\mathrm{i})^{-1} \left[\prod_{\mathrm{i}=1}^{j} \left(1 - \mathrm{d}! \prod_{m=1}^{\mathrm{d}} (\mathrm{d}(k+\mathrm{i}-1)+m)^{-1}\right)\right], \tag{4.2.6}$$

for $j = 0, \ldots, n$. Therefore, for the asymptotic analysis of $\lambda_n^{(k)}$, we apply the same technique used for moments and mixed moments in Chapter 3. That is, we check the conditions of Theorem 3.2.1 in order to obtain an asymptotic expression for $\lambda_n^{(k)}$.

Definition 4.2.1 Let $p(z) = \prod_{i=1}^{d}(z+i) - d!$, for $z \in \mathbb{C}$, $\mathcal{R}(p) = \{\lambda_1, \dots, \lambda_d\}$ the set of roots of p and $\mathcal{R}_{\Im} = \{\lambda \in \mathcal{R}(p) : \Im(\lambda) \neq 0\}$. For $k \geq 1$, let

$$\mathcal{P}_{\Delta^d} = \bigcup_{\lambda \in \mathcal{R}_{\Im}} \{\tfrac{\lambda}{d} - k, \tfrac{\lambda}{d} - k - 1, \dots\} \cup \{-k, -k-1\}. \tag{4.2.7}$$

Lemma 4.2.3 Let $q(z) = p(z) + d! = \prod_{i=1}^{d}(z+i)$, for $z \in \mathbb{C}$.

(i) If d is odd, 0 is the only real root of p. If d is even then p has two real roots, given by 0 and $-d-1$.

(ii) The derivative q' of $q = \prod_{i=1}^{d}(z+i)$ has $d-1$ real roots (critical points of q), denoted r_j, such that $r_j \in (-1-j, -j)$, for $j = 1, \dots, d-1$. In addition, $r_1 \leq -1 - 1/d$.

Proof. For (i) note that, clearly, p has no positive root and that 0 is a root, for any d. It is also clear that $-d-1$ is a root, if d is even. Let us check that no other real roots exist.

Note that the roots of q are $-d, \cdots, -1$ and, since $q(0) = d!$, it is clear that $q'(-1) > 0$, so q is increasing in $(-1, 0]$. Hence, the only solution of $q(z) = d!$ (i.e. $p(z) = 0$) in $[-1, 0]$ is 0. Moreover, the sign of $q'(-j)$, for $j = 1, \dots, d$, alternates, which means that, if d is odd, $q'(-d) > 0$ while, if d is even, $q'(-d) < 0$. Thus, for odd d, q is increasing on $(-\infty, -d)$ so there is no solution of $q(z) = d!$ in $(-\infty, -d]$; for even d, q is decreasing in $(-\infty, -d)$, and $-d-1$ is the only solution of $q(z) = d!$ in $(-\infty, -d]$.

Finally, we show that $q(z) = d!$ has no solution in $(-d, -1)$, by proving that $|q(z)| < d!$ in that interval. Let $z = -k - \xi$, with $k \in \{1, \dots, d-1\}$, $0 < \xi < 1$. We consider the case $k = 1$ and $k > 1$ separately: For $k = 1$ we have

$$|q(z)| = \xi(1-\xi)\prod_{i=3}^{d}(i - 1 - \xi) \leq \prod_{i=3}^{d} i < d!,$$

and, for $k > 1$,

$$\begin{aligned}
|q(z)| &\leq \prod_{i=1}^{d}(|-k+i| + 1) \\
&= \prod_{i=1}^{k}(k - i + 1) \prod_{i=k+1}^{d}(i - k + 1) \\
&= k!2 \cdots (d - k + 1) \\
&< k!(k+1) \cdots d = d!.
\end{aligned}$$

For (ii) we can consider q as a real polynomial and it follows, from Rolle's theorem, that the roots of q' are real numbers r_j, with multiplicity one and locations $r_j \in (-j-1, -j)$, for $j = 1, \dots, d-1$. Moreover, the derivative of q is given by

$$q'(z) = q(z)\sum_{i=1}^{d}\frac{1}{z+i}.$$

Let $\eta \in (0,1)$, then $q(-1-\eta) < 0$ and so, $q'(-1-\eta) > 0$ if

$$\sum_{\mathrm{i}=1}^{\mathrm{d}} \frac{1}{-1-\eta+\mathrm{i}} = -\frac{1}{\eta} + \sum_{j=1}^{\mathrm{d}-1} \frac{1}{j-\eta} \leq -\frac{1}{\eta} + \frac{\mathrm{d}-1}{1-\eta} < 0.$$

That is, if $\eta < 1/\mathrm{d}$. Then, for the critical point r_1 we have $r_1 \leq -1 - 1/\mathrm{d}$. □

From the above lemma it follows that $\mathcal{R}_{\mathfrak{S}} = \{\lambda \in \mathcal{R}(p) : \lambda \notin \{0, -\mathrm{d}-1\}\}$.

Proposition 4.2.3 For $\mathrm{d} \geq 2, k \geq 1$ and $n > k$, let $f_n(z) = \prod_{j=1}^{\mathrm{d}} ((z+n-1)\mathrm{d} + j)$. Let also $B_k = \prod_{n \geq k+1} \left(1 - \frac{\mathrm{d}!}{f_n(0)}\right)$, and

$$\varphi(z) = \frac{\mathrm{d}! B_k}{f_{k+1}(z)} \prod_{n \geq k+1} \frac{f_n(z)}{f_n(z) - \mathrm{d}!}, \qquad z \in D, \tag{4.2.8}$$

where $D = \mathbb{C} \setminus \mathcal{P}_{\Delta^{\mathrm{d}}}$. Then φ is the analytic continuation of the sequence $b_j^{(k)}$ in (4.2.6) to the domain D. Furthermore, the singularities of φ are isolated poles, elements of the countable set $\mathcal{P}_{\Delta^{\mathrm{d}}}$.

Proof. First we show that φ is well defined and is analytic in D. Note that $f_n(0) = \frac{(\mathrm{d}n)!}{(\mathrm{d}(n-1))!}$, then $B_k < \infty$ if and only if the series $\sum_{n \geq k} \frac{1}{(\mathrm{d}n+1)\cdots(\mathrm{d}n+\mathrm{d})} < \infty$, which holds because $\mathrm{d} \geq 2$.

We now check that φ is analytic on D. To that end it we observe that

$$\prod_{n \geq k+1} \frac{f_n(z)}{f_n(z) - \mathrm{d}!} = \lim_{m \to \infty} \prod_{n=k+1}^{m} \frac{f_n(z)}{f_n(z) - \mathrm{d}!} = \lim_{m \to \infty} \prod_{n=k+1}^{m} \left(1 + \frac{\mathrm{d}!}{f_n(z) - \mathrm{d}!}\right),$$

where the limit on the rhs of the expression above exists and is analytic if the series $\sum_{n \geq k+1} \frac{1}{|f_n(z) - \mathrm{d}!|}$ converges uniformly on compact sets contained in D. So, let V be a disk of radius R contained in D. Choose n_0 such that $n_0 > 1 + R + (\mathrm{d}-1)!$, then, for $n \geq n_0$, we have $|f_n(z) - \mathrm{d}!| > (\mathrm{d}(n-1) + 1 - \mathrm{d}R)^{\mathrm{d}} - 1$ and we conclude that φ is analytic on D.

Furthermore, from formula (4.2.8) we can see that the zeros of $f_n(z) - \mathrm{d}!$ are the potential singularities of φ, for $n \geq k+1$, but not all of them are poles. First, we observe that $f_n(-n+1) = \mathrm{d}!$, so $\{-k, -k-1, -k-2\ldots\}$ is a set of potential poles. However, noting that $f_{n-1}(-n+1) = \prod_{\mathrm{i}=1}^{\mathrm{d}} (\mathrm{i} - \mathrm{d}) = 0$ we have that only $-k$ and $-k-1$ are actual poles, the rest are not, because of cancellation.

We illustrate this with $-k-2$, noting that in the expression of φ we have

$$f_{k+2}(z) = (z+k+2)\mathrm{d} \prod_{j=1}^{\mathrm{d}-1} ((z+k+1)\mathrm{d} + j) = 0$$

in the numerator and

$$f_{k+3}(z) - \mathrm{d}! = (z + k + 2) \prod_{\lambda \neq -k-2} (z - \lambda) = 0$$

in the denominator. So, we apparently have a ratio of type $0/0$ if $z = -k - 2$ but in fact, it is well defined because of the cancellation of $((z + k + 1)\mathrm{d} + j)$. The same idea applies to $-k - 3, -k - 4$, etc.

Additionally, observe that $f_n(-\mathrm{d}^{-1} - n) = (-1)^{\mathrm{d}}\mathrm{d}!$ and so, if d is even, we see that this is another potential pole, but the same cancellation noted above also takes place in this case. Moreover, all these candidates cancel out. To see this, observe that $f_{n+1}(-\frac{1}{\mathrm{d}} - n) = \prod_{\mathrm{i}=1}^{\mathrm{d}}(\mathrm{i} - 1) = 0$, for $n \geq k + 1$. Finally if λ is a not real root of $p(z)$, then $\lambda/\mathrm{d} + 1 - n$ is a complex zero of $f_n(z) - \mathrm{d}!$. There are $\mathrm{d} - 1$ or $\mathrm{d} - 2$ such zeros, depending on the parity of d and there is no cancellation in this case. So $\mathcal{P}_{\Delta^{\mathrm{d}}}$ is the set of poles of φ, as stated, and φ is analytic in $\mathbb{C} \setminus \mathcal{P}_{\Delta^{\mathrm{d}}}$.

To conclude, we verify that φ interpolates the sequence $b_j^{(k)}$. Indeed, for $j \geq 0$, we have

$$\begin{aligned}\varphi(j) =& \frac{\mathrm{d}! B_k}{\prod_{\mathrm{i}=1}^{\mathrm{d}}(\mathrm{d}(j + k) + \mathrm{i})} \prod_{n \geq k+1} \frac{f_n(z)}{f_n(z) - \mathrm{d}!} \\ =& \frac{\mathrm{d}!}{\prod_{\mathrm{i}=1}^{\mathrm{d}}(\mathrm{d}(j + k) + \mathrm{i})} \prod_{n=k+1}^{k+j} \left(1 - \mathrm{d}! \prod_{\mathrm{i}=1}^{\mathrm{d}}(\mathrm{d}(n - 1) + \mathrm{i})^{-1}\right) = b_j^{(k)}.\end{aligned}$$

□

We continue our brief study of the roots of polynomial $p(z)$, of Definition 4.2.1. This analysis will provide elements to check the conditions of Theorem 3.2.1. The analytic theory of zeros and critical points of polynomials is a well known and extensively developed topic. For additional information, the interested reader can consult the references [9, 46, 48]. To avoid trivialities, we assume in the rest of this chapter that $\mathrm{d} \geq 2$. Before stating the next lemma, we need a general property due to Grace-Heawood, that establishes a connection between the roots of a polynomial and their critical points. We quote the version of the result that appears in [6]; see also [59].

Theorem 4.2.1 (Grace-Heawood) Let $z_1 \neq z_2$ be zeros of a polynomial of degree $\mathrm{d} \geq 2$. Then the polynomial has a critical point in the region

$$\left\{z \in \mathbb{C} : \left|z - \tfrac{z_1+z_2}{2}\right| \leq \tfrac{1}{2}|z_1 - z_2| \cot\left(\tfrac{\pi}{\mathrm{d}}\right)\right\}.$$

Proof. See theorem 4.3.1 in page 126 of [48]. □

Now we provide some facts about the roots of p of Definition 4.2.1. and a localization property related to r_1 of Lemma 4.2.3.

Lemma 4.2.4 With the notation of Definition 4.2.1 and Lemma 4.2.3, the following properties hold.

(i) $|\lambda| \leq \mathrm{d}+1$, $\Re(\lambda) \leq 0$ and $|\Im(\lambda)| \leq (\mathrm{d}!)^{1/\mathrm{d}}$, for $\lambda \in \mathcal{R}(p)$,
(ii) $|\Re(\lambda)| \geq \alpha_{\mathrm{d}}(r_1)$, for $\lambda \in \mathcal{R}(p), \lambda \neq 0$,

where

$$\alpha_{\mathrm{d}}(r_1) = 2|r_1| + \mathrm{d}\cot^2\left(\tfrac{\pi}{\mathrm{d}}\right) - \sqrt{\left(2|r_1| + \mathrm{d}\cot^2\left(\tfrac{\pi}{\mathrm{d}}\right)\right)^2 - 4r_1^2}. \tag{4.2.9}$$

Proof. We reason by contradiction. In (i) we show that if any of the statements does not hold, for some $\lambda_0 \in \mathcal{R}(p)$, then $|q(\lambda_0)| > \mathrm{d}!$. Indeed, if $|\lambda_0| > \mathrm{d}+1$ then

$$|q(\lambda_0)| = \prod_{j=1}^{\mathrm{d}} |\lambda_0 + j| \geq \prod_{j=1}^{\mathrm{d}} (|\lambda_0| - j) > \prod_{j=1}^{\mathrm{d}} (\mathrm{d}+1-j) = \mathrm{d}!.$$

If $\Re(\lambda_0) > 0$ then $|q(\lambda_0)| = \prod_{j=1}^{\mathrm{d}} |\Re(\lambda_0) + j| > \prod_{j=1}^{\mathrm{d}} j = \mathrm{d}!$. Finally, if $|\Im(\lambda_0)| > (\mathrm{d}!)^{1/\mathrm{d}}$, then $|q(\lambda_0)| = \prod_{j=1}^{\mathrm{d}} |\Im(\lambda_0)| > \prod_{j=1}^{\mathrm{d}} (\mathrm{d}!)^{1/\mathrm{d}} = \mathrm{d}!$.

For (ii) we assume first $\mathrm{d} > 2$. Suppose that $|\Re(\lambda_0)| < \alpha_{\mathrm{d}}(r_1)$, for some $\lambda_0 \in \mathcal{R}(p), \lambda_0 \neq 0$. Then $|\Re(\lambda_0)| < 2|r_1|$. From Lemma 4.2.3 and Theorem 4.2.1, with $z_1 = 0$ and $z_2 = \lambda_0$, there exists a critical point of p (hence, of q), say $r \in \mathbb{R}$ (possibly $r \neq r_1$), such that

$$\left| r - \frac{\lambda_0}{2} \right| \leq \frac{1}{2}|\lambda_0| \cot\left(\frac{\pi}{\mathrm{d}}\right). \tag{4.2.10}$$

A geometrical analysis, illustrated in Figure 4.1, yields the inequalities

$$0 < |r_1| - \frac{|\Re(\lambda_0)|}{2} \leq \left| r_1 - \frac{\lambda_0}{2} \right| \leq \left| r - \frac{\lambda_0}{2} \right| \leq \frac{1}{2}|\lambda_0| \cot\left(\frac{\pi}{\mathrm{d}}\right). \tag{4.2.11}$$

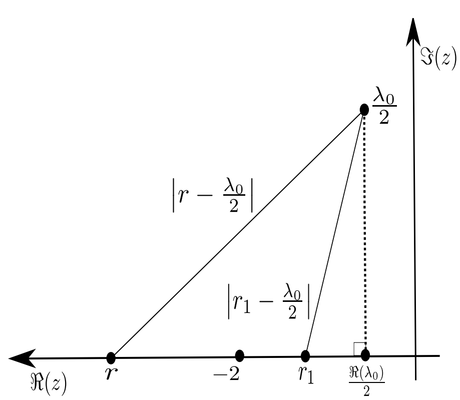


Figure 4.1: Geometric visualization of inequalities in (4.2.11).

Moreover, noting that

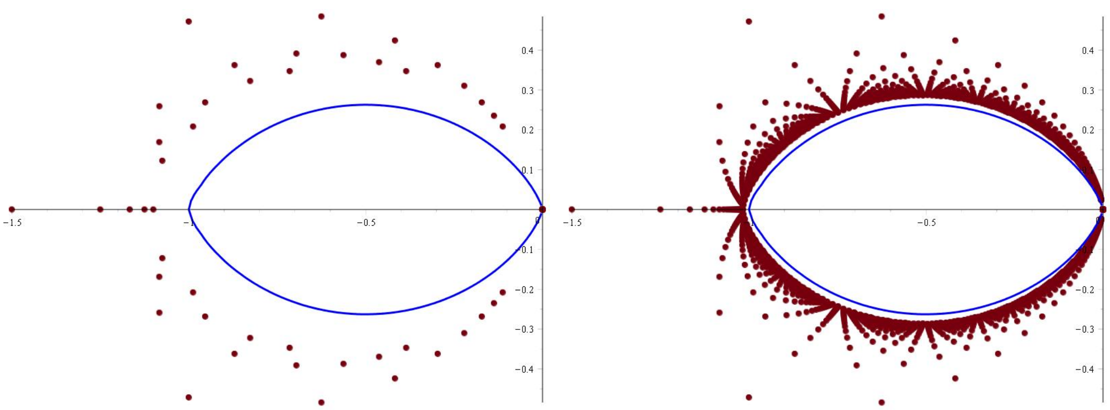
$$|q(\lambda_0)| = \prod_{k=1}^{\mathrm{d}} (k^2 + 2k\Re(\lambda_0) + |\lambda_0|^2)^{1/2} \geq \prod_{k=1}^{\mathrm{d}} (k^2 - 2\mathrm{d}|\Re(\lambda_0)| + |\lambda_0|^2)^{1/2},$$

from (4.2.11) we have, since $\cot^2(\pi/\mathrm{d}) > 0$ for $\mathrm{d} > 2$,

$$\begin{aligned} -2\mathrm{d}|\Re(\lambda_0)| + |\lambda_0|^2 &\geq -2\mathrm{d}|\Re(\lambda_0)| + \frac{(2|r_1| - |\Re(\lambda_0)|)^2}{\cot^2(\pi/\mathrm{d})} \\ &= \frac{4|r_1|^2 - |\Re(\lambda_0)|(4|r_1| + 2\mathrm{d}\cot^2(\pi/\mathrm{d})) + |\Re(\lambda_0)|^2}{\cot^2(\pi/\mathrm{d})} > 0. \end{aligned} \tag{4.2.12}$$

so $|q(\lambda_0)| > \mathrm{d}!$ and we are done. For the positivity in (4.2.12) we can argue by considering the numerator in the second line, as the quadratic polynomial $R(x) := x^2 - bx + c$, with $x = |\Re(\lambda_0)|, b = 4|r_1| + 2\mathrm{d}\cot^2(\pi/\mathrm{d})$ and $c = 4|r_1|^2$. Also, R has real roots x_1, x_2, with $x_1 = (b - \sqrt{b^2 - 4c})/2 = \alpha_{\mathrm{d}}(r_1) < x_2$, since clearly, $b^2 > 4c$. Moreover, $R(x) > 0$, for $x < x_1$, that is, for $|\Re(\lambda_0)| < \alpha_{\mathrm{d}}(r_1)$, which is our assumption. If $\mathrm{d} = 2$, we have $\mathcal{R}(p) = \{-3, 0\}$, $\cot(\pi/\mathrm{d}) = 0$, $\lambda = -3$, $r_1 = -3/2$ and $\alpha_{\mathrm{d}}(r_1) = 2|r_1| = 3$. So, as stated, $|\Re(\lambda)| = \alpha_{\mathrm{d}}(r_1)$. □

Remark 4.2.1 The interest of Lemmas 4.2.3 and 4.2.4 is to allow a precise estimation of ε_0 in Proposition 4.2.4 below. As pointed out in [41], the polynomial $p(z)$ appears in a variety of problems in combinatorial analysis. For example, in the study of random increasing k trees or packing problems, see additional references in [41]. Furthermore, a graphical analysis of the distribution of zeros of $p(z)$ is also given in [41]. This study suggests, as d increases, the roots approach the implicit curve $|z^{-z}(1+z)^{1+z}| = 1$. This can be interesting in order to complement and to expand our study of zeros and critical points, shown in Lemmas 4.2.3 and 4.2.4, if d is large. We illustrate the situation in Figure 4.2 below



(a) Roots of p, $\mathrm{d} = 2, \ldots, 10$. (b) Zeros for $\mathrm{d} = 2, \ldots, 60$.

Figure 4.2: Behavior of roots (red dots) of p as d increases. The limiting implicit curve $|z^{-z}(1+z)^{1+z}| = 1$ is in blue.

Proposition 4.2.4 With the notation of Definition 4.2.1, Proposition 4.2.3 and Lemmas 4.2.3, 4.2.4, let

$$\gamma_m = \{z \in \mathbb{C} : |z| = m + \varepsilon\},$$

for $m \geq 1$, $\varepsilon > 0$. Then there exist $m_0 \geq k+1, \varepsilon_0 > 0$ such that $\gamma_m \cap \mathcal{P}_{\Delta^{\mathrm{d}}} = \emptyset$, for all $m \geq m_0, \varepsilon \leq \varepsilon_0$.

Proof. We show that, for sufficiently large m and sufficiently small ε, it holds that $f_n(z) \neq \mathrm{d}!$, for all $z \in \gamma_m$, $n \geq 2$.

Setting $z = (m+\varepsilon)\mathrm{e}^{\mathrm{i}\theta}$, with $\theta \in [0, 2\pi]$ and $a = (n-1)\mathrm{d} + j$, we have

$$
\begin{aligned}
|f_n(z)| &= \prod_{j=1}^{\mathrm{d}} (|z|^2\mathrm{d}^2 + 2a\mathrm{d}\Re(z) + a^2)^{\frac{1}{2}} \\
&= \prod_{j=1}^{\mathrm{d}} ((m+\varepsilon)^2\mathrm{d}^2 + 2a\mathrm{d}(m+\varepsilon)\cos\theta + a^2)^{\frac{1}{2}} \\
&\geq \prod_{j=1}^{\mathrm{d}} |(m+\varepsilon)\mathrm{d} - a| \\
&= \prod_{j=1}^{\mathrm{d}} |(m-n+1+\varepsilon)\mathrm{d} - j|.
\end{aligned}
\tag{4.2.13}
$$

Depending on the relation between n and m, we separate the analysis into two cases. If $|m-n+1+\varepsilon| \geq 2$, from (4.2.13) we have

$$|f_n(z)| \geq (|m-n+1+\varepsilon|\mathrm{d} - \mathrm{d})^{\mathrm{d}} \geq \mathrm{d}^{\mathrm{d}} > \mathrm{d}!.$$

On the other hand, if $\varepsilon < 1$, the condition $|m-n+1+\varepsilon| < 2$ is equivalent to $m-n \in \{-3,-2,-1,0\}$. If $m-n=-3$, then (4.2.13) yields

$$|f_n(z)| \geq \prod_{j=1}^{\mathrm{d}} |(\varepsilon-2)\mathrm{d} - j| = \prod_{j=1}^{\mathrm{d}} |(2-\varepsilon)\mathrm{d} + j| > \mathrm{d}!.$$

If $m-n=-2$ the analysis is similar. In the two remaining cases ($n = m+1$ and $n = m$), inequality (4.2.13) is not useful so we prove directly that $|f_n(z) - \mathrm{d}!| = |p((z+n-1)\mathrm{d})| \neq 0$. That is, we have to show that $(z+n-1)\mathrm{d} = ((m+\varepsilon)\mathrm{e}^{\mathrm{i}\theta}+n-1)\mathrm{d}$ is not a root of p. Recalling that $\lambda_j, j = 1, \ldots, \mathrm{d}$ are the roots of p and letting $n = m+1$ and $u = (z+m)\mathrm{d}$, we have

$$|f_n(z) - \mathrm{d}!|^2 = |p(u)|^2 = \prod_{j=1}^{\mathrm{d}} |u - \lambda_j|^2.$$

Let $Q_j(\theta) = |u-\lambda_j|^2$, for $j = 1, \ldots, \mathrm{d}$. Then, as a function of θ, $Q_j(\theta)$ is continuous and so reaches its minimum in $[0, 2\pi]$. For the boundary points we have the value

$$
\begin{aligned}
Q_j(0) = Q_j(2\pi) &= |\lambda_j - (2m+\varepsilon)\mathrm{d}|^2 \\
&= |\lambda_j|^2 - 2(2m+\varepsilon)\mathrm{d}\Re(\lambda_j) + (2m+\varepsilon)^2\mathrm{d}^2 \\
&\geq (2m+\varepsilon)^2\mathrm{d}^2,
\end{aligned}
$$

since, by 1 of Lemma 4.2.4, $\Re(\lambda_j) \leq 0$.

If the minimum of Q_j is reached at $\theta^* \in (0, 2\pi)$, then θ^* is a critical point of Q_j and we compute $Q_j'(\theta) = \frac{\partial}{\mathrm{d}\theta}|u-\lambda_j|^2$. Noting that $\Re(u) = ((m+\varepsilon)\cos\theta + m)\mathrm{d}$ and $\Im(u) = \mathrm{d}(m+\varepsilon)\sin\theta$, and letting $\alpha := \Im(\lambda_j)$ and $\beta := m\mathrm{d} - \Re(\lambda_j)$, we have

$$Q_j(\theta) = \mathrm{d}^2(m+\varepsilon)^2 + 2\mathrm{d}(m+\varepsilon)(\beta\cos\theta - \alpha\sin\theta) + \beta^2 + \alpha^2$$

and, consequently,

$$Q'_j(\theta) = -2\mathrm{d}(m+\varepsilon)\alpha\cos\theta - 2\mathrm{d}(m+\varepsilon)\beta\sin\theta.$$

Hence, $Q'_j(\theta^*) = 0$ is equivalent to $\beta\sin\theta^* = -\alpha\cos\theta^*$ (note that $\beta > 0$ because $\Re(\lambda_j) \leq 0$).

Now, replacing θ by θ^* in $Q_j(\theta)$, we get

$$\begin{aligned} Q_j(\theta^*) &= \mathrm{d}^2(m+\varepsilon)^2 + 2\mathrm{d}(m+\varepsilon)\Big(\beta + \frac{\alpha^2}{\beta}\Big)\cos\theta^* + \beta^2 + \alpha^2 \\ &= \mathrm{d}^2(m+\varepsilon)^2 \pm 2\mathrm{d}(m+\varepsilon)\sqrt{\alpha^2+\beta^2} + \beta^2 + \alpha^2 \\ &= \Big(\mathrm{d}(m+\varepsilon) \pm \sqrt{\alpha^2+\beta^2}\Big)^2. \end{aligned}$$

In the expression above we clearly must pick the minus sign (that is, $\cos\theta^* \geq 0$), which yields $Q_j(\theta^*) < Q_j(0)$. Hence, for all $\theta \in [0, 2\pi]$,

$$Q_j(\theta) \geq Q_j(\theta^*) = \Big(\mathrm{d}(m+\varepsilon) - \sqrt{\alpha^2+\beta^2}\Big)^2$$

and there remains to check that $Q_j(\theta^*) > 0$ for all $j = 1, \ldots, \mathrm{d}$, in order to reach the conclusion $|f_n(z) - \mathrm{d}!| > 0$. To that end, we observe that

$$\lim_{m\to\infty} \Big(\mathrm{d}(m+\varepsilon) - \sqrt{\alpha^2+\beta^2}\Big)^2 = (\Re(\lambda_j) + \mathrm{d}\varepsilon)^2. \tag{4.2.14}$$

The result above indicates that for m large enough and ε sufficiently small, we have $Q_j(\theta^*) > 0$, for $j = 1, \ldots, \mathrm{d}$. Hence we have the conclusion in the case $n = m+1$. Finally, the remaining case $n = m$ is dealt with in a similar fashion and details are thus omitted. □

Proposition 4.2.5 There exists a positive constant c such that $|\varphi(z)| < c$, for all $z \in \gamma_m$, for all $m \geq m_0$, where φ is defined in (4.2.8).

Proof. Let $z = (m+\varepsilon)\mathrm{e}^{\mathrm{i}\theta} \in \gamma_m$ then, from inequality (4.2.13), with $n = k+1$, and taking $m \geq k+1$, we have

$$\begin{aligned} \big|f_{k+1}\big((m+\varepsilon)\mathrm{e}^{\mathrm{i}\theta}\big)\big| &= \prod_{j=1}^{\mathrm{d}} \Big|\mathrm{d}((m+\varepsilon)\mathrm{e}^{\mathrm{i}\theta} + k) + j\Big| \\ &\geq \prod_{j=1}^{\mathrm{d}} \Big|\mathrm{d}(m-k+\varepsilon) - j\Big| = \prod_{j=1}^{\mathrm{d}} \Big|\mathrm{d}(m-k) - j + \mathrm{d}\varepsilon\Big| \geq (\mathrm{d}\varepsilon)^{\mathrm{d}}. \end{aligned} \tag{4.2.15}$$

Moreover, observe that

$$\begin{aligned} \left|\prod_{n\geq k+1} \frac{f_n(z)}{f_n(z) - \mathrm{d}!}\right| &= \prod_{n\geq k+1} \left|1 + \frac{\mathrm{d}!}{f_n(z) - \mathrm{d}!}\right| \leq \prod_{n\geq k+1} \left(1 + \frac{\mathrm{d}!}{|f_n(z) - \mathrm{d}!|}\right) \\ &= \exp\left(\sum_{n\geq k+1} \log\left(1 + \frac{\mathrm{d}!}{|f_n(z) - \mathrm{d}!|}\right)\right) \\ &\leq \exp\left(\mathrm{d}! \sum_{n\geq k+1} \frac{1}{|f_n(z) - \mathrm{d}!|}\right). \end{aligned}$$

So, if we bound $\sum_{n\geq k+1} \frac{1}{|f_n(z)-\mathrm{d}!|}$ uniformly on m, the result follows. We decompose the sum for $n = k+1, \ldots, m-1$; then for $n = m, n = m+1, n \geq m+2$ and bound all the terms.

Note from (4.2.13) that, if $n \leq m-1$, we have

$$|f_n(z)| \geq \prod_{j=1}^{\mathrm{d}} ((m-n+1+\varepsilon)\mathrm{d} - j) \geq (m-n+\varepsilon)^{\mathrm{d}}\mathrm{d}^{\mathrm{d}}$$

and so, $|f_n(z) - \mathrm{d}!| \geq (m-n+\varepsilon)^{\mathrm{d}}\mathrm{d}^{\mathrm{d}} - \mathrm{d}!$. Then, provided that $m_0 \geq k+2$,

$$\sum_{n=k+1}^{m-1} \frac{1}{|f_n(z) - \mathrm{d}!|} \leq \sum_{j=1}^{\infty} \frac{1}{(j+\varepsilon)^{\mathrm{d}}\mathrm{d}^{\mathrm{d}} - \mathrm{d}!} < \infty. \tag{4.2.16}$$

Now if $n \geq m+2$ then, from (4.2.13),

$$\begin{aligned} |f_n(z)| &\geq \prod_{j=1}^{\mathrm{d}} |(m-n+1+\varepsilon)\mathrm{d} - j| \\ &= \prod_{j=1}^{\mathrm{d}} ((n-m-1-\varepsilon)\mathrm{d} + j) \\ &\geq (n-m-1-\varepsilon)^{\mathrm{d}}\mathrm{d}^{\mathrm{d}} + \mathrm{d}!. \end{aligned}$$

Hence, $|f_n(z) - \mathrm{d}!| \geq \mathrm{d}^{\mathrm{d}}(n-m-1-\varepsilon)^{\mathrm{d}}$ and

$$\sum_{n\geq m+2} \frac{1}{|f_n(z) - \mathrm{d}!|} \leq \frac{1}{\mathrm{d}^{\mathrm{d}}} \sum_{j=2}^{\infty} \frac{1}{(j-1-\varepsilon)^{\mathrm{d}}} < \infty. \tag{4.2.17}$$

The cases $n = m+1$ and $n = m$ follow from the corresponding analysis in the proof of Proposition 4.2.4. For example, if $n = m+1$, m large and ε small enough, from (4.2.14) we have

$$|f_{m+1}(z) - \mathrm{d}!| \geq \frac{1}{2} \prod_{j=1}^{\mathrm{d}} |\Re(\lambda_j) + \mathrm{d}\varepsilon| > 0, \tag{4.2.18}$$

so $|f_{m+1}(z) - \mathrm{d}!|$ is bounded by a positive constant independent of m. The case $n = m$ is analogous.

Summarizing, the conclusion follows from the bounds in (4.2.15), (4.2.16), (4.2.17) and (4.2.18). □

Theorem 4.2.2 For sufficiently large n,

$$\lambda_{n+1}^{(k)} = -(-1)^n \sum_{s\in\mathcal{P}_{\Delta^{\mathrm{d}}}} \operatorname*{Res}_{z=s} \left[\frac{n!\varphi(z)}{z(z-1)\ldots(z-n)} \right], \tag{4.2.19}$$

where $\operatorname*{Res}_{z=s} \psi(z)$ is the residue of $\psi(z)$ at $z = s$, and φ, $\mathcal{P}_{\Delta^{\mathrm{d}}}$ are defined in Proposition 4.2.3.

Proof. The result follows from Theorem 3.2.1, whose hypotheses are verified in Propositions 4.2.4 and 4.2.5. □

Computation of residues

We proceed to evaluate the residues in the right-hand side of (4.2.19).

Proposition 4.2.6 Let $\Psi(z) = \frac{n!\varphi(z)}{z(z-1)\dots(z-n)}$. Then, for the real poles the residues are

$$\begin{aligned} \operatorname*{Res}_{z=-k} \Psi(z) &= \frac{(-1)^{n+1}}{k\mathrm{d}H_{\mathrm{d}}} \binom{n+k}{k}^{-1} \prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!}, \\ \operatorname*{Res}_{z=-k-1} \Psi(z) &= \frac{(-1)^{n}}{(n+k+1)\mathrm{d}H_{\mathrm{d}}} \binom{n+k}{k}^{-1} \prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!}, \end{aligned} \tag{4.2.20}$$

where $H_{\mathrm{d}} = \sum_{\mathrm{i}=1}^{\mathrm{d}} \frac{1}{\mathrm{i}}$, and $f_j(0) = \prod_{\mathrm{i}=1}^{\mathrm{d}}(\mathrm{d}(j-1)+\mathrm{i})$. On the other hand, in the case of complex poles, we have, for $\lambda \in \mathcal{R}_{\Im}$, $m \geq k+1$,

$$\begin{aligned} \operatorname*{Res}_{z=1-m+\frac{\lambda}{\mathrm{d}}} \Psi(z) = &\frac{n!(-1)^{n-1}}{(1-m+\frac{\lambda}{\mathrm{d}})_{n+1}} \frac{B_k}{\mathrm{d}!\mathrm{d}\sum_{j=1}^{\mathrm{d}} \frac{1}{\lambda+j}} \frac{1}{f_{k+1}(1-m+\frac{\lambda}{\mathrm{d}})} \\ &\times \prod_{\substack{j\geq k+1 \\ j\neq m}} \frac{f_j(1-m+\frac{\lambda}{\mathrm{d}})}{f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!}, \end{aligned} \tag{4.2.21}$$

where $(s)_{n+1} = s(s-1)\cdots(s-n)$ denotes the falling factorial.

Proof. Usual computation of residues, as in the proof in Proposition 3.2.5. □

The next step is to obtain the asymptotic behavior of residues in Proposition 4.2.6.

Lemma 4.2.5 As $n \to \infty$,

$$\begin{aligned} \operatorname*{Res}_{z=-k} \Psi(z) &\sim (-1)^{n+1} \frac{(k-1)!}{\mathrm{d}H_{\mathrm{d}}} \left(\prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!} \right) n^{-k} \\ \operatorname*{Res}_{z=-k-1} \Psi(z) &\sim (-1)^{n} \frac{k!}{\mathrm{d}H_{\mathrm{d}}} \left(\prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!} \right) n^{-(k+1)} \end{aligned} \tag{4.2.22}$$

Proof. The results follow from using Stirling's approximation in equations (4.2.20). □

We turn our attention to the complex poles of which, as we know, there are countable many. So we study the asymptotic order (as $n \to \infty$) of the series

$$\sum_{\lambda\in\mathcal{R}_{\Im}} \sum_{m\geq k+1} \operatorname*{Res}_{z=1-m+\frac{\lambda}{\mathrm{d}}} \Psi(z). \tag{4.2.23}$$

where $\mathcal{R}_{\Im}$ is the set of roots of $p(z) = \prod_{\mathrm{i}=1}^{\mathrm{d}}(z+\mathrm{i}) - \mathrm{d}!$ with $\Im(\lambda_l) \neq 0$, See Definition 4.2.1. For technical reasons, we deal separately with the first term, namely $m = k+1$.

Lemma 4.2.6 As $n \to \infty$,

$$\sum_{\lambda \in \mathcal{R}_{\Im}} \left| \operatorname*{Res}_{z=-k+\frac{\lambda}{\mathrm{d}}} \Psi(z) \right| \leq C n^{-(k+\eta(\mathrm{d}))}, \tag{4.2.24}$$

where C is a positive constant and $\eta(\mathrm{d}) = \min\{|\Re(\lambda)| : \lambda \in \mathcal{R}_{\Im}\} > 0$.

Proof. We consider the formula (4.2.21), with $m = k+1$ and bound its modulus. First, note that, by Stirling's approximation, as $n \to \infty$,

$$\frac{n!(-1)^{n-1}}{(\frac{\lambda}{\mathrm{d}} - k)_{n+1}} \sim \Gamma\Big(k - \frac{\lambda}{\mathrm{d}}\Big) n^{\frac{\lambda}{\mathrm{d}}-k},$$

where $(s)_{n+1} = s(s-1)\cdots(s-n)$ is the falling factorial, for any $s \in \mathbb{C}$. Furthermore, we have that $f_{k+1}\Big(\frac{\lambda}{\mathrm{d}} - k\Big) = \prod_{\mathrm{i}=1}^{\mathrm{d}} (\lambda + \mathrm{i}) \neq 0$ and noting that $f_j(\frac{\lambda}{\mathrm{d}} - k) \neq \mathrm{d}!$, for $j \geq k+2$, then

$$\operatorname*{Res}_{z=\frac{\lambda}{\mathrm{d}}-k} \Psi(z) \sim \Gamma\Big(k - \frac{\lambda}{\mathrm{d}}\Big) n^{\frac{\lambda}{\mathrm{d}}-k} \frac{\mathrm{d}! B_k \prod_{\mathrm{i}=1}^{\mathrm{d}} (\lambda + \mathrm{i})^{-1}}{\mathrm{d}\big(\psi(\lambda_{\mathrm{d}} + \mathrm{d} + 1) - \psi(\lambda_{\mathrm{d}})\big)} \prod_{j \geq k+2} \frac{f_j(\frac{\lambda}{\mathrm{d}} - k)}{f_j(\frac{\lambda}{\mathrm{d}} - k) - \mathrm{d}!}. \tag{4.2.25}$$

Now observe that $|n^{\frac{\lambda}{\mathrm{d}}-k}| = n^{-k}|n^{\frac{\lambda}{\mathrm{d}}}| = n^{-k} n^{\Re(\frac{\lambda}{\mathrm{d}})}$. Hence, by taking modulus, collecting constants and summing over λ, the result follows. □

Now we estimate the contribution of poles with $m \geq k+2$ in (4.2.23).

Lemma 4.2.7 For $m \geq k+2$ and $\lambda \in \mathcal{R}_{\Im}$,

$$\left| \operatorname*{Res}_{z=1-m+\frac{\lambda}{\mathrm{d}}} \Psi(z) \right| \leq C \binom{n+m-1}{m-1}^{-1}, \tag{4.2.26}$$

for sufficiently large n, where C is a positive constant not depending on n nor m.

Proof. We proceed to bound the terms in (4.2.21). First note that, since $\Re(\frac{\lambda}{\mathrm{d}}) \leq 0$ (see Lemma 4.2.4) it holds that $|j - \frac{\lambda}{\mathrm{d}}|^2 = j^2 - 2j\Re(\frac{\lambda}{\mathrm{d}}) + |\frac{\lambda}{\mathrm{d}}|^2 > j^2$, and hence,

$$\left| \frac{n!(-1)^{n-1}}{(1 - m + \frac{\lambda}{\mathrm{d}})_{n+1}} \right| = n! \prod_{j=m-1}^{n+m-1} \frac{1}{|j - \frac{\lambda}{\mathrm{d}}|} \leq n! \prod_{j=m}^{n+m} \frac{1}{j-1} \leq \binom{n+m-1}{m-1}^{-1}. \tag{4.2.27}$$

Also, it is easy to see that, as $m \to \infty$,

$$\left| f_{k+1}(1 - m + \tfrac{\lambda}{\mathrm{d}}) \right|^{-1} = \prod_{j=1}^{\mathrm{d}} \Big(\big(\mathrm{d}(1 - m + k) + j + \Re(\lambda)\big)^2 + \Im(\lambda)^2 \Big)^{-1/2} \to 0. \tag{4.2.28}$$

On the other hand, for the infinite product in (4.2.25), we have

$$F(m) := \left| \prod_{\substack{j\geq k+1\\ j\neq m}} \frac{f_j(1-m+\frac{\lambda}{\mathrm{d}})}{f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!} \right| \leq \prod_{\substack{j\geq k+1\\ j\neq m}} \left(1+\frac{\mathrm{d}!}{\left|f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!\right|}\right) \leq \exp\left(\mathrm{d}! \sum_{\substack{j\geq k+1\\ j\neq m}} \frac{1}{\left|f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!\right|}\right). \tag{4.2.29}$$

So, we need to bound the last term of (4.2.29). To that end, first note that any root z_q of $q(z) = \prod_{\mathrm{i}=1}^{\mathrm{d}}(z+\mathrm{i})$ belongs to $\{-\mathrm{d},\dots,-2,-1\}$, so $0 < |z_q| \leq \mathrm{d}$. Then, every root z_p of $p(z) = q(z) - \mathrm{d}!$ satisfies $|z_p| \leq \mathrm{d} + (\mathrm{d}!)^{1/\mathrm{d}} < 2\mathrm{d}$ (see theorem 1.1 in [6] or [59]). This allows one to write $\lambda = \rho e^{\mathrm{i}\theta}$ with $0 < \rho \leq \mathrm{d} + (\mathrm{d}!)^{1/\mathrm{d}}$ and $\theta \in [0, 2\pi]$, so

$$\left|f_j\big(1-m+\tfrac{\lambda}{\mathrm{d}}\big)\right| = \prod_{l=1}^{\mathrm{d}}((\mathrm{d}(j-m)+l)^2 + 2\rho\cos(\theta)(\mathrm{d}(j-m)+l)+\rho^2)^{1/2}. \tag{4.2.30}$$

Now we split the analysis into cases:

1. If $j \geq m+2$ and noting that $\rho < 2\mathrm{d}$, we have

$$\left|f_j\big(1-m+\tfrac{\lambda}{\mathrm{d}}\big)\right| \geq \prod_{l=1}^{\mathrm{d}} |\mathrm{d}(j-m)-\rho+l| \geq (\mathrm{d}(j-m)-\rho)^{\mathrm{d}} + \mathrm{d}!.$$

   Therefore

$$\begin{aligned}\sum_{j\geq m+2} \frac{1}{\left|f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!\right|} &\leq \frac{1}{\mathrm{d}^{\mathrm{d}}} \sum_{j\geq m+2} \frac{1}{(j-m-\rho/\mathrm{d})^{\mathrm{d}}}\\ &= \frac{1}{\mathrm{d}^{\mathrm{d}}}\left(\frac{1}{(2-\rho/\mathrm{d})^{\mathrm{d}}} + \sum_{n\geq 1}\frac{1}{(n+2-\rho/\mathrm{d})^{\mathrm{d}}}\right)\\ &\leq \frac{1}{\mathrm{d}^{\mathrm{d}}}\left(\frac{1}{(2-\rho/\mathrm{d})^{\mathrm{d}}} + \zeta(\mathrm{d})\right),\end{aligned} \tag{4.2.31}$$

   where $\zeta(\mathrm{d}) = \sum_{n\geq 1}\frac{1}{n^{\mathrm{d}}}$.
2. If $j \leq m-2$ and since for any $\lambda \in \mathcal{R}_{\mathfrak{I}}$ it holds $\Re(\lambda) \leq 0$, we have

$$\begin{aligned}\left|f_j\big(1-m+\tfrac{\lambda}{\mathrm{d}}\big)\right| &= \prod_{l=1}^{\mathrm{d}}((\mathrm{d}(j-m)+l)^2+\rho^2+2\rho\cos(\theta)(\mathrm{d}(j-m)+l))^{1/2}\\ &\geq \prod_{l=1}^{\mathrm{d}}\left|\mathrm{d}(j-m)+l\right| = \prod_{l=1}^{\mathrm{d}}\big(\mathrm{d}(m-j)-l\big)\\ &= \prod_{l=1}^{\mathrm{d}}\big(\mathrm{d}(m-j-1)+\mathrm{d}-l\big)\\ &\geq (\mathrm{d}^{\mathrm{d}}-\mathrm{d}!)(m-j-1)^{\mathrm{d}}+\mathrm{d}!.\end{aligned}$$

Hence, provided that $m \geq k+3$,

$$\begin{aligned}
\sum_{j=k+1}^{m-2} \frac{1}{\left|f_j(1-m+\frac{\lambda}{\mathrm{d}})-\mathrm{d}!\right|} &\leq \frac{1}{\mathrm{d}^{\mathrm{d}}-\mathrm{d}!} \sum_{j=k+1}^{m-2} \frac{1}{(m-j-1)^{\mathrm{d}}} \\
&= \frac{1}{\mathrm{d}^{\mathrm{d}}-\mathrm{d}!} \sum_{n=1}^{m-k-2} \frac{1}{n^{\mathrm{d}}} \\
&\leq \frac{1}{\mathrm{d}^{\mathrm{d}}-\mathrm{d}!}\zeta(\mathrm{d}).
\end{aligned} \tag{4.2.32}$$

3. Finally, for the cases $j = m-1$, $j = m+1$, and observing that $\lambda \pm \mathrm{d}$ is not an integer, we have

$$\begin{aligned}
f_{m-1}\big(1-m+\tfrac{\lambda}{\mathrm{d}}\big) &= \prod_{l=1}^{\mathrm{d}}(-\mathrm{d}+\lambda+l) = q(\lambda-\mathrm{d}) \neq \mathrm{d}!, \\
f_{m+1}\big(1-m+\tfrac{\lambda}{\mathrm{d}}\big) &= \prod_{l=1}^{\mathrm{d}}(\mathrm{d}+\lambda+l) = q(\lambda+\mathrm{d}) \neq \mathrm{d}!.
\end{aligned} \tag{4.2.33}$$

So, from (4.2.31), (4.2.32) and (4.2.33), we conclude that $F(m)$, defined in (4.2.29), is bounded as

$$F(m) \leq \exp\left(\mathrm{d}!\left[\tfrac{1}{\mathrm{d}^{\mathrm{d}}}\left(\tfrac{1}{(2-\rho/\mathrm{d})^{\mathrm{d}}}+\zeta(\mathrm{d})\right)+\tfrac{1}{\mathrm{d}^{\mathrm{d}}-\mathrm{d}!}\zeta(\mathrm{d})+\tfrac{1}{|q(\lambda-\mathrm{d})-\mathrm{d}!|}+\tfrac{1}{|q(\lambda+\mathrm{d})-\mathrm{d}!|}\right]\right) \tag{4.2.34}$$

From (4.2.27), (4.2.28) and (4.2.34), the result follows. □

Corollary 4.2.1 As $n \to \infty$,

$$T_n := \left|\sum_{\lambda\in\mathcal{R}_{\mathfrak{I}}}\sum_{m\geq k+2} \operatorname*{Res}_{z=1-m+\frac{\lambda}{\mathrm{d}}} \Psi(z)\right| \leq C n^{-(k+1)}, \tag{4.2.35}$$

where C is a positive constant.

Proof. Directly from Lemma 4.2.7 we obtain

$$\begin{aligned}
T_n &\leq \sum_{\lambda\in\mathcal{R}_{\mathfrak{I}}} C(\lambda) \sum_{m\geq k+2} \binom{m+n-1}{m-1}^{-1} \\
&= \frac{k+1}{n-1}\binom{n+k}{k}^{-1} \sum_{\lambda\in\mathcal{R}_{\mathfrak{I}}} C(\lambda) \leq C n^{-(k+1)},
\end{aligned}$$

as $n \to \infty$. □

Corollary 4.2.2 For $k \geq 1$,

$$n^k \lambda_n^{(k)} = \frac{(k-1)!}{\mathrm{d}H_{\mathrm{d}}} \prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!} + O(n^{-\eta(\mathrm{d})}), \tag{4.2.36}$$

as $n \to \infty$.

Proof. The result follows from the formula in Theorem 4.2.2 and collecting the asymptotic results of Lemmas 4.2.5, 4.2.6 and Corollary 4.2.1 □

4.2.2 Weak convergence of record heights

Theorem 4.2.3 $n\mathcal{H}_n \xrightarrow{\mathcal{D}} \mathcal{H}$, where $\mathcal{H}$ is a random variable with distribution characterized by its moments

$$\mathbb{E}(\mathcal{H}^k) = \frac{(k-1)!}{\mathrm{d}H_\mathrm{d}} \prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!}, k \geq 1.$$

In the case $\mathrm{d} = 2$, $\mathcal{H} \overset{\mathcal{D}}{=} U^2 V$, where U is uniformly distributed in $[0,1]$, V is exponential with mean 1 and U, V are independent.

Proof. Let $\nu_k^{(\mathrm{d})} = \frac{(k-1)!}{\mathrm{d}H_\mathrm{d}} \prod_{j=2}^{k} \frac{f_j(0)}{f_j(0)-\mathrm{d}!}$. Then, from Corollary 4.2.2 and noting that $-\eta(\mathrm{d}) < 0$, $\mathbb{E}(n\mathcal{H}_n)^k = n^k \lambda_n^{(k)} \to \nu_k^{(\mathrm{d})}$. We have convergence of moments and so, the result follows if we show that the sequence $(\nu_k^{(\mathrm{d})})$ determines a distribution. To that end we check Carleman's condition, namely that $\sum_{k\geq 1} (\nu_{2k}^{(\mathrm{d})})^{-\frac{1}{2k}} = \infty$. Note that $\nu_k^{(\mathrm{d})}$ is decreasing as a function of d, so

$$\nu_k^{(\mathrm{d})} \leq \nu_k^{(2)} = \frac{(k-1)!}{2H_2} \frac{k}{2k+1} = \frac{k!}{2k+1}.$$

Hence $(\nu_k^{(\mathrm{d})})^{-\frac{1}{k}} \geq (\nu_k^{(2)})^{-\frac{1}{k}} \sim \mathrm{e}k^{-1}$, from the Stirling's approximation of $k!$, and the series diverges. For $\mathrm{d} = 2$, if U is uniform in $[0,1]$ random and V is exponential, with parameter 1, then $\mathbb{E}((U^2)^k) = \frac{1}{2k+1}$, $\mathbb{E}(V^k) = k!$, so $\nu_k^{(2)} = \mathbb{E}((U^2V)^k)$. □

4.3 Study of $\mathcal{N}_n$

Recall that $\mathcal{N}_n$ is the number of chain records within the first n observations. The asymptotic analysis of this object will be similar to the analysis in the case of $[0,1]^2$, in Chapter 3. The recursion of moments in (4.2.1) and the relation between $\mathcal{N}_n$ and the sum of $\mathcal{H}_n$ are our main tools in this section.

4.3.1 Preliminaries and previous results

Our goal is to prove that $\frac{3}{\log n} \sum_{\mathrm{i}=1}^{n} \mathcal{H}_\mathrm{i} \xrightarrow{L_2} 1$. Under suitable modifications, we proceed as in the case of the model $\mathbf{U}([0,1]^2)$.

Lemma 4.3.1 The following asymptotic results hold

$$\begin{aligned}\sum_{\mathrm{i}=1}^{n}\mathbb{E}(\mathcal{H}_{\mathrm{i}}) &\sim \frac{1}{3}\log n,\\ \mathbb{E}(\mathcal{N}_n) &\sim \frac{1}{3}\log n,\\ \sum_{\mathrm{i}=1}^{n}\mathrm{Var}(\mathcal{H}_{\mathrm{i}}) &= O(1),\\ \sum_{1\le \mathrm{i}<j\le n}\mathbb{E}(\mathcal{H}_{\mathrm{i}})\mathbb{E}(\mathcal{H}_j) &= \frac{1}{18}H_n^2+O(\log n),\end{aligned} \tag{4.3.1}$$

where H_n is the nth harmonic number.

Proof. The first expression in (4.3.1) follows from $\mathbb{E}(\mathcal{H}_{\mathrm{i}})=\lambda_{\mathrm{i}}^{(1)}=\frac{1}{3(\mathrm{i}+1)}$, due to (4.2.5). The second follows from $\mathbb{E}(I_{\mathrm{i}}\,|\,\mathcal{F}_{\mathrm{i}-1})=\mathcal{H}_{\mathrm{i}-1}$, which implies $\mathbb{E}(I_{\mathrm{i}})=\mathbb{E}(\mathcal{H}_{\mathrm{i}-1})$, so $\mathbb{E}(\mathcal{N}_n)=\frac{1}{3}H_n\sim\frac{1}{3}\log n$. Moreover, since

$$\mathrm{Var}(\mathcal{H}_{\mathrm{i}})=\lambda_{\mathrm{i}}^{(2)}-\big(\lambda_{\mathrm{i}}^{(1)}\big)^2=\frac{2}{5}\frac{1}{(\mathrm{i}+2)(\mathrm{i}+1)}-\frac{1}{9}\frac{1}{(\mathrm{i}+1)^2},$$

the third assertion is proved. Finally the last one follows by arguing as in the proof in Lemma 3.4.3. □

Observe that the recurrences associated to the moments in $[0,1]^2$ and Δ^2 are structurally the same, so we can repeat the procedure of Proposition 3.4.2 to obtain

Proposition 4.3.1 For $n,k\ge 1$, we have

$$\mathbb{E}(\mathcal{H}_n\mathcal{H}_{n+k})=\frac{1}{3}\sum_{j=0}^{k-1}\binom{n-1}{j}(-1)^j\left[\frac{2j+3}{(j+1)(2j+5)}\binom{n+j+2}{j+2}^{-1}-\frac{2j+5}{(j+2)(2j+7)}\binom{n+j+3}{j+3}^{-1}\right] \tag{4.3.2}$$

Proof. See proof of Proposition 3.4.2. □

Lemma 4.3.2 Let

$$q_1(\mathrm{i},k)=\frac{4}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{2j+5}\binom{\mathrm{i}+j+2}{j+2}^{-1},\quad q_2(\mathrm{i},k)=\frac{1}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j+1}\binom{\mathrm{i}+j+2}{j+2}^{-1}, \tag{4.3.3}$$

and

$$q_3(\mathrm{i},k)=\frac{4}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{2j+7}\binom{\mathrm{i}+j+3}{j+3}^{-1},\quad q_4(\mathrm{i},k)=\frac{1}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j+2}\binom{\mathrm{i}+j+3}{j+3}^{-1}. \tag{4.3.4}$$

Then

$$q_3(\mathrm{i},k)=q_1(\mathrm{i},k)-q_1(\mathrm{i},k+1),\quad q_4(\mathrm{i},k)=q_2(\mathrm{i},k)-q_2(\mathrm{i},k+1) \tag{4.3.5}$$

and $q_2(\mathrm{i},k)=\frac{4}{9}p_1(\mathrm{i},k)$ and $q_4(\mathrm{i},k)=\frac{4}{9}p_3(\mathrm{i},k)$, where p_1,p_3 as in (3.4.5) and (3.4.6). Moreover

$$\mathbb{E}(\mathcal{H}_{\mathrm{i}}\mathcal{H}_{\mathrm{i}+k}) = [q_1 - q_3 + q_2 - q_4](\mathrm{i}, k). \tag{4.3.6}$$

PROOF. See proof of Lemma 3.4.5. □

Lemma 4.3.3 The sums $q_1(\mathrm{i}, k)$ and $q_3(\mathrm{i}, k)$ are positive for every $\mathrm{i}, k \geq 1$.

PROOF. Reasoning as in Lemma 3.4.6, we have $q_1(\mathrm{i}, k) = \frac{4}{9}\mathbb{E}\big(\mathcal{H}_{\mathrm{i}}^2(1-\mathcal{H}_{\mathrm{i}})^{k-1}\big)$ and $q_3(\mathrm{i}, k) = \frac{4}{9}\mathbb{E}\big(\mathcal{H}_{\mathrm{i}}^3(1-\mathcal{H}_{\mathrm{i}})^{k-1}\big)$, and the conclusion follows. □

Lemma 4.3.4 Let q_2 be as in (4.3.3), then

$$\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-1} q_2(\mathrm{i}, k+1) = \frac{1}{18}H_n^2 + O(1).$$

PROOF. We proceed as in Lemma 3.4.7. Using the relation $q_2(\mathrm{i}, k+1) = \frac{4}{9}p_1(\mathrm{i}, k+1)$, we have

$$\begin{aligned}
\frac{4}{9}\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} p_1(\mathrm{i}, k+1) &= \frac{4}{9}\sum_{l=2}^{n}\sum_{k=1}^{l-1} p_1(l-k, k+1) = \frac{1}{9}\sum_{l=2}^{n}\sum_{k=1}^{l-1}\frac{2l-k+2}{(l+1)(l+2)(l-k+1)}\\
&= \frac{1}{9}\sum_{l=2}^{n}\sum_{k=1}^{l-1}\left[\frac{1}{(l+2)(l-k+1)} + \frac{1}{(l+1)(l+2)}\right]\\
&= \frac{1}{9}\sum_{l=2}^{n}\left[\frac{H_l}{(l+2)} - \frac{2}{(l+1)(l+2)}\right]\\
&= \frac{1}{9}\sum_{l=1}^{n}\frac{H_l}{l+2} - \frac{1}{9} + \frac{2}{9(n+2)}\\
&= \frac{1}{18}\left[H_{n+1}^2 - H_{n+1}^{(2)}\right] + \frac{1}{9(n+2)} - \frac{2}{9} + \frac{2}{9(n+2)} = \frac{1}{18}H_n^2 + O(1).
\end{aligned}$$

In the expression above we used a formula from [13], page 2223, and the identity $H_{n+1} = H_n + \frac{1}{n+1}$. □

Lemma 4.3.5 Let q_1 be as in (4.3.3), then $\sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} q_1(\mathrm{i}, k+1) \leq \frac{4}{27}(H_n - 1)$.

PROOF. Recalling that $q_1(\mathrm{i}, k) = \frac{4}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{2j+5}\binom{\mathrm{i}+j+2}{j+2}^{-1}$, we define an auxiliary term in order to derive a relationship between them. Let $q_0(\mathrm{i}, k) = \frac{4}{9}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{2j+3}\binom{\mathrm{i}+j+1}{j+1}^{-1}$ and note that $q_0(\mathrm{i}, k) = \frac{4}{9}\mathbb{E}\big(\mathcal{H}_{\mathrm{i}}(1-\mathcal{H}_{\mathrm{i}})^{k-1}\big) > 0$. Also, it is easy to see that $(\frac{9}{4}q_0(\mathrm{i}, k))_k$ is the Euler transform of the sequence $(a_j)_j$, with $a_j = \frac{1}{2j+3}\binom{\mathrm{i}+j+1}{j+1}^{-1}$ and $(\frac{9}{4}q_1(\mathrm{i}, k))_k$ is the Euler transform of the sequence $(a_{j+1})_j$. So, by Lemma 3.4.4, we have $q_1(\mathrm{i}, k) = q_0(\mathrm{i}, k) - q_0(\mathrm{i}, k+1)$. Then, taking into account that $q_0(\mathrm{i},k)$ is positive and is clearly decreasing in k, we get

$$\sum_{k=1}^{n-\mathrm{i}} q_1(\mathrm{i},k+1) = q_0(\mathrm{i},2) - q_0(\mathrm{i},n-\mathrm{i}+2) \le q_0(\mathrm{i},2) \le q_0(\mathrm{i},1) = \frac{4}{27(\mathrm{i}+1)},$$

and the conclusion follows. □

Lemma 4.3.6

$$\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i},\mathcal{H}_j) = o(\log^2 n). \tag{4.3.7}$$

Proof. We have

$$\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i},\mathcal{H}_j) = \sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i}\mathcal{H}_j) - \sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i})\mathbb{E}(\mathcal{H}_j).$$

From (4.3.1), $\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_\mathrm{i})\mathbb{E}(\mathcal{H}_j) = \frac{1}{18}H_n^2 + O(\log n)$. Moreover, from Lemmas 4.3.2, 4.3.4 and 4.3.5 , we get

$$\begin{aligned}\sum_{1\le \mathrm{i}<j\le n} \mathbb{E}(\mathcal{H}_{\mathrm{i},}\mathcal{H}_j) &= \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} (q_1-q_3)(\mathrm{i},k) + \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} (q_2-q_4)(\mathrm{i},k)\\ &= \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} q_1(\mathrm{i},k+1) + \sum_{\mathrm{i}=1}^{n-1}\sum_{k=1}^{n-\mathrm{i}} q_2(\mathrm{i},k+1)\\ &= \frac{1}{18}H_n^2 + O(\log n).\end{aligned}$$

Hence, $\sum_{1\le \mathrm{i}<j\le n} \mathrm{Cov}(\mathcal{H}_\mathrm{i},\mathcal{H}_j)$ is at most $O(\log n)$ and the conclusion follows. □

4.3.2 Law of large numbers for $\mathcal{N}_n$

As consequence of results in previous sections we have

Theorem 4.3.1 The record heights $\mathcal{H}_\mathrm{i}$, for the uniform model on Δ^2 satisfy the following law of large numbers,

$$\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n} \mathcal{H}_\mathrm{i} \xrightarrow{L_2} \frac{1}{3}. \tag{4.3.8}$$

Furthermore

$$\frac{\mathcal{N}_n}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{3}. \tag{4.3.9}$$

Proof. Note that

$$\mathbb{E}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i} - \frac{1}{3}\right)^2 = \mathrm{Var}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_\mathrm{i}\right) + \left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathbb{E}(\mathcal{H}_\mathrm{i}) - \frac{1}{3}\right)^2.$$

The second term in the rhs of the decomposition above goes to 0 because of (4.3.1). Furthermore,

$$\mathrm{Var}\left(\frac{1}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}}\right)=\frac{1}{\log^2 n}\sum_{\mathrm{i}=1}^{n}\mathrm{Var}(\mathcal{H}_{\mathrm{i}})+\frac{2}{\log^2 n}\sum_{1\le \mathrm{i}<j\le n}\mathrm{Cov}(\mathcal{H}_{\mathrm{i}},\mathcal{H}_j)\to 0,$$

because of (4.3.1) and (4.3.7), hence (4.3.8) holds. For (4.3.9), we have from (2.3.3) that

$$\frac{\mathcal{N}_n}{\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}}}\to 1,$$

a.s. Then, (4.3.9) follows at once from (4.3.8). □

4.3.3 Asymptotic normality for $\mathcal{N}_n$

As in Chapter 3, we verify the conditions of Theorem 3.4.2.

Theorem 4.3.2 The following convergence to the normal distribution holds.

$$\frac{\mathcal{N}_n-\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}}}{\sqrt{\frac{1}{3}\log n}}\xrightarrow{\mathcal{D}} N(0,1). \tag{4.3.10}$$

Proof. Consider the sequence (ξ_{i}) defined by $\xi_{\mathrm{i}}=I_{\mathrm{i}}-\mathcal{H}_{\mathrm{i}-1}$, for $\mathrm{i}\ge 1$, with $\mathcal{H}_0:=1$. Note that $\mathbb{E}(I_{\mathrm{i}}\,|\,\mathcal{F}_{\mathrm{i}-1})=\mathcal{H}_{\mathrm{i}-1}$, so $\mathbb{E}(\xi_{\mathrm{i}}\,|\,\mathcal{F}_{\mathrm{i}-1})=0$ and that $\mathbb{E}(\xi_{\mathrm{i}}^2\,|\,\mathcal{F}_{\mathrm{i}-1})=\mathcal{H}_{\mathrm{i}-1}-\mathcal{H}_{\mathrm{i}-1}^2$.

We conclude by checking the conditions (1) and (2) in Theorem 3.4.2, with $b_n=\sqrt{\frac{1}{3}\log n}$. For (1) we have

$$\frac{1}{b_n^2}\sum_{\mathrm{i}=1}^{n}\mathbb{E}(\xi_{\mathrm{i}}^2\,|\,\mathcal{F}_{\mathrm{i}-1})=\frac{3}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}-1}-\frac{3}{\log n}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}-1}^2\xrightarrow{\mathbb{P}} 1,$$

by (4.3.8) and because, for any $\varepsilon>0$ we get

$$\mathbb{P}\Big(\frac{1}{b_n^2}\sum_{\mathrm{i}=1}^{n}\mathcal{H}_{\mathrm{i}-1}^2>\varepsilon\Big)\le\frac{3}{\varepsilon\log n}\sum_{\mathrm{i}=1}^{n}\frac{2}{5\mathrm{i}(\mathrm{i}+1)}\to 0. \tag{4.3.11}$$

Condition (2) is direct because the ξ_{i} are bounded. Hence, (4.3.10) follows.

□

Chapter 5

Work in progress

In this chapter we consider some preliminary ideas, partial results and conjectures about the convergence in distribution of chain-maxima $\mathcal{M}_n$, for iid observations, uniformly distributed on $[0,1]^2$.

5.1 Convergence of $\mathcal{M}_n$

From Proposition 2.4.5 we know that $\mathcal{M}_n$ converges a.s. to $(1,1)$ and we would like to know how fast this convergence takes place. In this perspective we conjecture the existence of a real and deterministic sequence $A_n \uparrow \infty$ such that

$$A_n(1 - \mathcal{M}_n^{(1)}, 1 - \mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} (X, Y), \tag{5.1.1}$$

where (X, Y) is a nondegenerate random vector, concentrated on $\mathbb{R}_+^2$.

We use the fact that $n\mathcal{H}_n = n(1 - \mathcal{M}_n^{(1)})(1 - \mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} Z$, where Z is distributed as UV, with U uniform in $[0,1]$ and V is exponential with parameter 1, mutually independent (Theorem 3.2.3, with d = 2). The first easy conclusion of (5.1.1), due to symmetry, is that $X \stackrel{\mathcal{D}}{=} Y$. Another simple conclusion, based on the continuous mapping theorem (see e.g [11]), is that $A_n^2(1 - \mathcal{M}_n^{(1)})(1 - \mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} XY$ that we compare with Z.

Proposition 5.1.1 (i) If (5.1.1) holds then the distribution of (X, Y) cannot concentrate on any (nondegenerate) hyperbola. In other words, $\mathbb{P}(XY = c) < 1$, for any constant $c > 0$.

(ii) If $\mathbb{P}(XY = 0) = 1$, then $A_n = o(\sqrt{n})$.

Proof. (i) Suppose $XY = c > 0$ a.s., then $A_n^2(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \to c$ in probability. Hence, by Slutsky's theorem and knowing that Z is nondegenerate, we have the contradiction

$$\frac{n}{A_n^2} = \frac{n(1 - \mathcal{M}_n^{(1)})(1 - \mathcal{M}_n^{(2)})}{A_n^2(1 - \mathcal{M}_n^{(1)})(1 - \mathcal{M}_n^{(2)})} \xrightarrow{\mathcal{D}} \frac{Z}{c}.$$

(ii)

$$\frac{A_n^2}{n}[n(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)})]^2 = A_n^2(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)})n(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \to 0,$$

so $A_n/n^2 \to 0$ because $[n(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)})]^2 \xrightarrow{\mathcal{D}} Z^2$. □

From the above results we can say that convergence in distribution to a probability concentrated on the axes is still possible but in such case, the normalizing sequence A_n is $o(\sqrt{n})$. So

$$\sqrt{n}(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \not\xrightarrow{\mathcal{D}} (X,Y),$$

if $\mathbb{P}(XY=0)=1$. If all the mass is drifting to the axes, then we have no proper convergence in distribution on the axes and possibly, the mass is escaping to $+\infty$.

We ask if we can have convergence in distribution to non degenerate random vectors in both cases, that is, with $A_n = o(\sqrt{n})$, assuming $XY=0$, and with $\sqrt{n}$. This cannot happen, as seen below.

The knowledge about the asymptotic behavior of moments is informative about the possible normalizing sequence A_n. In Chapter 3, we give a general treatment for mixed moments. From Corollary 3.3.2 we have

$$\mu_n^{k,l} = \mathbb{E}\left[(1-\mathcal{M}_n^{(1)})^k(1-\mathcal{M}_n^{(2)})^l\right] \sim Cn^{z_1^+} = Cn^{-\frac{k+l}{2}+r_{kl}-1},$$

where $r_{kl} = \frac{1}{2}\sqrt{(k-l)^2+4}$ and C is a constant. We show that sequences $A_n = n^\alpha$, with $\alpha < 1/2$ are not admissible for convergence in distribution to a nondegenerate random vector.

Observe that $\mathbb{E}(n^\alpha(1-\mathcal{M}_n^{(1)}))^k \sim Cn^{-k(\frac{1}{2}-\alpha)+\sqrt{k^2+4}/2-1}$. We consider the existence of k such that the exponent of n is negative. Let $\alpha = 1/2-\varepsilon$, with $0<\varepsilon<1/2$, and let $s_\varepsilon(k) = \sqrt{k^2+4}/2 - k\varepsilon - 1$.

Proposition 5.1.2 (i) $\sqrt{n}(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)})$ and $A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)})$, with $A_n = o(\sqrt{n})$, cannot converge simultaneously, in distribution, to nondegenerate random vectors.

(ii) For any $\varepsilon \in (0,1/2)$ there exists $k_\varepsilon > 0$ such that $s_\varepsilon(k_\varepsilon) < 0$.

Proof. (i) If $\sqrt{n}(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} (X,Y)$ and $A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} (X',Y')$ then, for some α,β, we have $\alpha\sqrt{n}(1-\mathcal{M}_n^{(1)})+\beta\sqrt{n}(1-\mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} \alpha X+\beta Y$ non degenerate and $\alpha A_n(1-\mathcal{M}_n^{(1)})+\beta A_n(1-\mathcal{M}_n^{(2)}) \to \alpha X'+\beta Y'$ nondegenerate. Then, by the convergence of types theorem, we have $A_n/\sqrt{n} \to A > 0$, which is a contradiction.

(ii) Note that $s_\varepsilon(k)$ is continuous and that $s_\varepsilon(0)=0$. Also, its derivative $s'_\varepsilon(k) = \frac{k}{2\sqrt{k^2+4}} - \varepsilon$ yields $s'_\varepsilon(0) < 0$ and so, $s_\varepsilon(k)$ is negative on some interval $(0,k_\varepsilon^*)$. In fact, k_ε^* is solution of $s_\varepsilon(k)=0$, given by $k_\varepsilon^* = \frac{8\varepsilon}{1-4\varepsilon^2} > 0$. We can choose $k_\varepsilon \in (0,k_\varepsilon^*)$ arbitrarily. □

Corollary 5.1.1 Let $\varepsilon \in (0,1/2)$, $\alpha = 1/2-\varepsilon$ and $k_\varepsilon \in (0,k_\varepsilon^*)$. Then there exists a constant C such that

(i) $\mathbb{E}(n^\alpha(1-\mathcal{M}_n^{(1)}))^{k_\varepsilon} \sim Cn^{s_\varepsilon(k_\varepsilon)} \to 0$.

(ii) $n^\alpha(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \to (0,0)$ in probability.

Proof. The first assertion follows from Proposition 5.1.2 (ii); the second, from Markov's inequality. □

Observe that, for $\varepsilon = 0$ (that is $n^\alpha = n^{1/2}$), we have $s_0(k) > 0$ for all $k > 0$, so this implies $\mathbb{E}(n^{k/2}(1-\mathcal{M}_n^{(1)})^k) \to \infty$. We now consider the density.

Proposition 5.1.3 Let $Z_n = n^\alpha(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)})$ and $\mathbf{h}_n^\alpha(x,y)$ its density function. Then $\mathbf{h}_n^\alpha(x,y) \to 0$, for all $(x,y) \in \mathbb{R}_+^2$, $0 < \alpha < 1/2$.

Proof. Let $x, y, h \in \mathbb{R}_+$. If $\mathbf{h}_n^\alpha(x,y) \not\to 0$ then, for some $\delta > 0$, $\mathbf{h}_{n'}^\alpha(x,y) > \delta$ along a subsequence (n'). More over, since $\mathbf{h}_n^\alpha$ is decreasing in both coordinates, we have $\mathbf{h}_{n'}^\alpha(s,t) > \delta$, for all $s \le x, t \le y$. This implies that $Z_{n'} \not\to (0,0)$ in probability, which contradicts (ii) of Corollary 5.1.1. □

If A_n grows faster than $\sqrt{n}$ we see, not surprisingly, mass escaping to infinity in some sense. Let then $A_n \uparrow \infty$ such that $n/A_n^2 \to 0$. We have $A_n^2(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \to \infty$ in probability, i.e. $\mathbb{P}(A_n^2(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \le c) \to 0$ for all $c > 0$. So, for any finite rectangle $R \subset \mathbb{R}_+^2$, $\mathbb{P}(A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \in R) \to 0$, that is, the joint df is converging to 0.

We come back to an arbitrarily slow sequence $A_n = o(\sqrt{n})$. In this case we necessarily have $A_n^2(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \to 0$ in probability because $n(1-\mathcal{M}_n^{(1)})(1-\mathcal{M}_n^{(2)}) \xrightarrow{\mathcal{D}} Z$. This means that all the mass is going to the axes, but we do not know if $A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)})$ is converging or not in distribution; it may well happen that part of the mass goes to ∞ along the axes. It is clear though that, for any finite rectangle R with edges away from both axes, $\mathbb{P}(A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)}) \in R) \to 0$, which is the same conclusion obtained above, for sequences growing faster than $\sqrt{n}$. We can argue as in the proof of Proposition 5.1.3 to prove that, for general sequences (not just powers of n) A_n growing faster or slower than $\sqrt{n}$, the density of $A_n(1-\mathcal{M}_n^{(1)}, 1-\mathcal{M}_n^{(2)})$ goes to 0 (excluding the axes in the slow case).

5.2 Solving the recurrence for $\mathbf{f}_n$

Here we provide an analysis of the recurrence satisfied by $\mathbf{f}_n$, the density of $\mathcal{M}_n$. From (2.2.8) we have $\mathbf{f}_1(x,y) = 1_{[0,1]^2}(x,y)$ and, for $n \ge 1$,

$$\mathbf{f}_{n+1}(x,y) = (1-(1-x)(1-y))\mathbf{f}_n(x,y) + \mathbf{F}_n(x,y)1_{[0,1]^2}(x,y),\ n \ge 1, \tag{5.2.1}$$

Recurrence (5.2.1) can be solved for small values of n. For example, using the software wxMaxima we obtain

$$\begin{aligned}
\mathbf{f}_2(x,y) &= x+y,\\
\mathbf{f}_3(x,y) &= \frac{1}{2}(x+y)(2(x+y)-xy),\\
\mathbf{f}_4(x,y) &= (x+y)^3 + \frac{xy}{12}\left[5xy(x+y) - 14(x+y)^2 - 2xy\right].
\end{aligned}$$

We list below some elementary properties of $\mathbf{f}_n$.

Proposition 5.2.1 The solution $\mathbf{f}_n$ of recurrence (5.2.1) satisfies

1. $\mathbf{f}_n$ is non-negative and integrates 1 in $[0,1]^2$.
2. $\mathbf{f}_n$ is increasing in both x and y.
3. $\mathbf{f}_n(0,0)=0, \mathbf{f}_n(1,1)=n$.
4. $\mathbf{f}_n$ is a polynomial function only of xy and $x+y$, with no constant coefficient.
5. The (sum) degree of $\mathbf{f}_n$ is $2n-3$ and the lowest degree of a monomial is $n-1$.
6. Let $\mathbf{f}_n$ be written as
$$\mathbf{f}_n(x,y) = \sum_{\mathrm{i},j\geq 0} a^n_{\mathrm{i}j} x^{\mathrm{i}} y^j.$$
Then the coefficients $a^n_{\mathrm{i}j}, \mathrm{i}, j \geq 0, n \geq 1$, are symmetrical, in the sense that $a^n_{j\mathrm{i}} = a^n_{\mathrm{i}j}$, and satisfy the recurrence
$$a^n_{\mathrm{i}j} = a^{n-1}_{\mathrm{i}-1,j} + a^{n-1}_{\mathrm{i},j-1} - a^{n-1}_{\mathrm{i}-1,j-1} K_{\mathrm{i}j}, \tag{5.2.2}$$
for $n\geq 2$, where $K_{\mathrm{i}j} = 1 - \frac{1}{\mathrm{i}j}$, if $\mathrm{i}, j > 0$, and $K_{\mathrm{i}j}=0$ otherwise. Further, $a^1_{00}=1$ and $a^1_{\mathrm{i}j}=0$, if $\mathrm{i}\neq 0$ or $j\neq 0$.
7. For $n\geq 2$, $a^n_{\mathrm{i}j}=0$ if $(\mathrm{i},j)\notin\{(\mathrm{i},j)\in\mathbb{N}^2 : n-1\leq \mathrm{i}+j\leq 2n-3\}$.
8. Some particular cases of $a^n_{\mathrm{i}j}$:
$$a^n_{n-1-\mathrm{i},\mathrm{i}} = \binom{n-1}{\mathrm{i}},$$
for $\mathrm{i}=0,\ldots,n-1$.
$$a^n_{n-1,n-2} = (-1)^n K_{12}K_{23}\cdots K_{n-2,n-1} = (-1)^n \frac{1}{2}\frac{5}{6}\frac{11}{12}\cdots\left(1-\frac{1}{(n-1)(n-2)}\right).$$

5.3 Delannoy numbers

We observe that a similarity exists between the $a^n_{\mathrm{i}j}$ coefficients, which satisfy recurrence (5.2.2), and *Delannoy numbers* $\mathrm{d}(m,k)$, which satisfy

$$\mathrm{d}(m,k) = \mathrm{d}(m-1,k) + \mathrm{d}(m,k-1) + \mathrm{d}(m-1,k-1). \tag{5.3.1}$$

The value $\mathrm{d}(m,k)$ can be seen as the number of increasing paths from $(0,0)$ to (m,k) in $\mathbb{Z}^2$. The movements along a path can be horizontal, vertical or diagonal. See Figure 5.1. The boundary conditions are $\mathrm{d}(0,0)=\mathrm{d}(m,0)=\mathrm{d}(0,k)=1$.

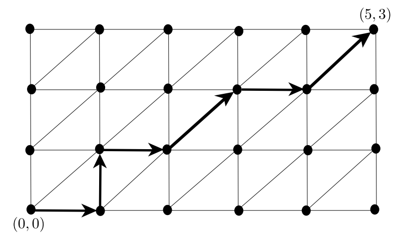


Figure 5.1: A Delannoy path from $(0,0)$ to $(5,3)$.

These numbers, first studied by Henri Delannoy [15], arise in many combinatorial problems, in particular the *central Delannoy numbers* ($m = k$) have been intensively studied; see [8, 55, 56] for further information.

Observe that (5.2.2) suggests a generalization of the Delannoy numbers when one considers attaching weights to the segments, which are added to define the weight of each path. See, for example, [23, 45] and [19]. However, among the above cited works, none considers paths with weights depending on the current state, which is our situation.

We define a directed graph G with vertices $(\mathrm{i}, j), \mathrm{i}, j = 0, 1, \ldots$ and edges $(\mathrm{i}, j) \to (\mathrm{i}+1, j)$, called E (for east), $(\mathrm{i}, j) \to (\mathrm{i}, j+1)$, called N (for north) and $(\mathrm{i}, j) \to (\mathrm{i}+1, j+1)$, called NE (for north-east); see Figure 5.2.

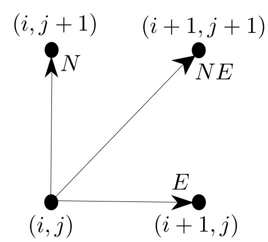


Figure 5.2: Possible movements in the graph.

The edges of graph G have weights as follows: all N and E edges have weight 1 and the NE edge $(\mathrm{i}-1, j-1) \to (\mathrm{i}, j)$ has weight $-K_{\mathrm{i}j}, \mathrm{i}, j > 0$. Let us consider two vertexes with positive integer coordinates, (i, j) and (I, J) where $\mathrm{i} \leq I, j \leq J$. A path from (i, j) to (I, J) is defined as a collection of edges $(\mathrm{i}_k, j_k) \to (\mathrm{i}_{k+1}, j_{k+1})$, for $k = 1, \ldots, n-1$, such that $(\mathrm{i}_1, j_1) = (\mathrm{i}, j), (\mathrm{i}_n, j_n) = (I, J)$. The length of a path is defined as the number of its edges; a path with length l is called l-path; the weight of a path is defined as the product of the weights of all its edges.

Proposition 5.3.1 The value of $a^n_{\mathrm{i}j}$ is equal to the sum of weights of all Delannoy paths of length $n-1$, from $(0,0)$ to (i, j).

5.4 Stick- and board-breaking models

We consider a continuous-time model, based on ideas from Brennan and Durrett [12]. These authors develop a stick-breaking model and pay attention (among other things) to the evolution of a tagged interval, using tools from Renewal Theory. We propose a two-dimensional extension of Brennan and Durrett's model that we call the board-breaking model.

5.4.1 Stick-breaking

Consider a stick of length L which, after an exponential time of rate L^α, for $\alpha > 0$, breaks into two new sticks, without loss of mass. The resulting sticks have lengths that can be represented by the random variables LU and $L(1-U)$, where U is a $[0,1]$-valued random variable with df F. The same procedure is applied to the new segments, using independent copies of U, and so on.

Among other processes studied in [12], we are interested in the length at time t, denoted L_t, of the *left-most interval*. We choose this particular interval because its evolution is surprisingly similar to that of records in $[0,1]$.

We collect below some properties of the process (L_t), as presented in [12], with some changes of notation. We assume that $L_0 = L = 1$, for simplicity.

Note first that (L_t), for $t \geq 0$, is a Markov jump process, with state space $[0,1]$. At every jump time, L_t gets multiplied by an independent copy of U, so that the successive values of L_t are $1, U_1, U_1U_2, \ldots$ and the holding times are exponential random variables with parameters $1, U_1^\alpha, (U_1U_2)^\alpha, \ldots$, where $U_1, U_2, \ldots$ are independent copies of U.

The range of $\{-\log L_t : t \geq 0\}$ is given by $0 = S_0 < S_1 < \cdots$, where $S_n = -\sum_{\mathrm{i}=1}^n \log U_\mathrm{i}$, for $n \geq 1$, are the arrival epochs of renewal process, with interarrival times distributed as $-\log U$. More precisely,

$$\mathbb{P}(S_{n+1} - S_n \leq x) = \mathbb{P}(-\log U \leq x) = \hat{F}(x),$$

where $\hat{F}(x) = 1 - F(\mathrm{e}^{-x}-)$. Conditionally on $S_1, S_2, \ldots$, the holding times of $-\log L_t$ at each S_n are independent, exponential random variables, with rates $\mathrm{e}^{-\alpha S_n}$.

Let $M_y = \sup\{n \geq 0 : S_n \leq -\log y\}$, for $y \in (0,1)$ and let $(\xi'_n)_{n\geq 0}$ be an independent sequence of iid exponential random variables, with unit mean. Then

$$\mathbb{P}(L_t < y) = \mathbb{P}\Big(\sum_{n=0}^{M_y} \exp(\alpha S_n)\xi'_n \leq t\Big).$$

Note that

$$\sum_{\mathrm{i}=0}^{M_y} \exp(\alpha S_\mathrm{i})\xi'_\mathrm{i} = \sum_{\mathrm{i}=0}^{M_y} \exp(\alpha S_{M_y-\mathrm{i}})\xi'_{M_y-\mathrm{i}}. \tag{5.4.1}$$

Then, letting $T_n(y) = -\log y - S_{M_y - n}$, we obtain

$$\sum_{i=0}^{M_y} \exp(-\alpha T_i(y) - \alpha \log y)\xi'_{M_y - i} = y^{-\alpha} \sum_{i=0}^{M_y} \exp(-\alpha T_i(y))\xi'_{M_y - i}. \tag{5.4.2}$$

Now we replace $(\xi'_{M_y-i})_i$ by new iid sequence $(\xi_i)_i$ of exponential random variables, with unit mean, to obtain

$$\mathbb{P}\Big(\sum_{i=0}^{M_y} \exp(\alpha S_i)\xi'_i \le t\Big) = \mathbb{P}\Big(\sum_{i=0}^{M_y} \exp(-\alpha T_i(y))\xi_i \le y^{\alpha} t\Big). \tag{5.4.3}$$

From Renewal Theory (see [50]), we know that, if $\hat{F}$ has finite mean μ,

$$\{T_n(y) \,:\, n \ge 0\} \xrightarrow{\mathcal{D}} \{T_n \,:\, n \ge 0\}, \quad \text{as} \quad y \to 0^+, \tag{5.4.4}$$

where $\{T_n \,:\, n \ge 0\}$ is the equilibrium renewal process generated by $\hat{F}$. That is, T_0 has distribution $\frac{1}{\mu}\int_0^x (1 - \hat{F}(t))\mathrm{d}t$ and the remaining random variables have distribution $\hat{F}$, where $\mu := \int_0^\infty (1 - \hat{F}(s))\mathrm{d}s < \infty$. With the change of variable $y = xt^{-1/\alpha}$ in (5.4.3), we get

$$\lim_{t\to\infty} \mathbb{P}\Big(t^{1/\alpha} L_t < x\Big) = \mathbb{P}\Big(\sum_{i\ge 0} \exp(-\alpha T_i)\xi_i \le x^{\alpha}\Big).$$

Finally, letting $Y_\alpha = \sum_{i\ge 0} \exp(-\alpha T_i)\xi_i$, we conclude that, as $t \to \infty$,

$$t^{1/\alpha} L_t \xrightarrow{\mathcal{D}} (Y_\alpha)^{1/\alpha}.$$

5.4.2 Board-breaking

We present here a two-dimensional analog of the process described in the previous section. The process starts with an initial (unit) square board which breaks into four rectangles, without loss of material, defined by the edges of the square and two orthogonal lines passing through a point, randomly chosen within the square. Each of the four resulting rectangles is then submitted to the same breaking process so that, after each iteration, there are four times more fragments, which become obviously smaller and smaller. As in the one-dimensional model, the breaks occur after exponentially distributed times. We are interested in the evolution of a tagged rectangle, namely in how fast its dimensions (side length, area) converge to 0.

Let (U_n) and (V_n) be iid sequences of $[0, 1]$-valued random variables, independent of each other. Let F denote the df of the U's and G that of the V's. We consider the bivariate process $\{(L_t^F, L_t^G) : t \ge 0\}$, where L_t^F, L_t^G represent respectively, the lengths of the horizontal and the vertical sides of a tagged rectangle, with $L_0^F = L_0^G = 0$. We choose the tagged rectangle as the one located at the left-lower corner of $[0, 1]^2$. In this manner, the tagged rectangle in the board-splitting process corresponds, by analogy, to the left-most interval in the stick-breaking model.

After the first break we have four rectangles, with sides given by $(U_1, V_1), (1 - U_1, V_1)$, $(1 - U_1, 1 - V_1)$ and $(U_1, 1 - V_1)$. The dimensions of the tagged rectangle are U_1, V_1. See Figure 5.3.

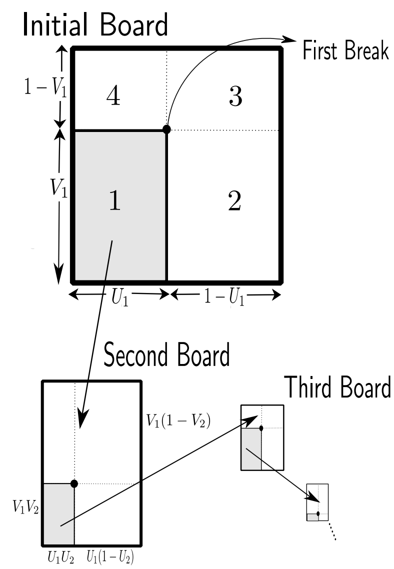


Figure 5.3: Evolution of a tagged rectangle in board breaking process.

For the breaking times we assume that the rectangle with dimensions L_t^F, L_t^G breaks after an exponential time of parameter $g(L_t^F, L_t^F)$, where $g(x, y)$ is a positive valued function such as, for example, $g(x, y) = (xy)^\alpha$ or $g(x, y) = x^\alpha y^\beta$, with α, β, suitable parameters.

Observe that the sides of the tagged rectangle evolve, respectively as the sequences $1, U_1, U_1U_2, \cdots$ and $1, V_1, V_1V_2, \cdots$, which correspond to independent one-dimensional stick breaking processes, as described in the previous section. So, the board breaking process seems to have no theoretical interest, because its evolution is that of two independent stick breaking processes. We will see below that the situation is not that simple when we take into account the continuous-time evolution of (L_t^F, L_t^G).

The continuous-time evolution of (L_t^F, L_t^G) is Markovian. The holding times are exponential with rates that depend on L_t^F and L_t^G. Hence, in general, the marginal processes (L_t^F), (L_t^G) are not independent stick-breaking processes, and not even Markovian.

As in the stick-breaking process, the range of $\{-\log L_t^F : t \geq 0\}$ is given by $0 = S_0^F < S_1^F < \cdots$, where $S_n^F = -\sum_{\mathrm{i}=1}^n \log U_\mathrm{i}$, for $n \geq 1$, are the arrival epochs of renewal process, with interarrival times distributed as $-\log U$. More precisely,

$$\mathbb{P}(S_{n+1}^F - S_n^F \leq x) = \mathbb{P}(-\log U \leq x) = \hat{F}(x),$$

where $\hat{F}(x) = 1 - F(\mathrm{e}^{-x}-)$. Analogously, the range of $\{-\log L_t^G : t \geq 0\}$ is given by $0 = S_0^G < S_1^G < \cdots$, where $S_n^G = -\sum_{\mathrm{i}=1}^n \log V_\mathrm{i}$, for $n \geq 1$, are the arrival epochs of renewal process, with interarrival times distributed as $-\log V$ and

$$\mathbb{P}(S_{n+1}^G - S_n^G \leq x) = \mathbb{P}(-\log V \leq x) = \hat{G}(x),$$

where $\hat{G}(x) = 1 - G(\mathrm{e}^{-x}-)$. Conditionally on $S_1^F, S_2^F, \dots$ and $S_1^G, S_2^G, \dots$, the holding times of $(-\log L_t^F, -\log L_t^G)$ at each (S_n^F, S_n^G) are independent, exponential random variables, with rates $g(\mathrm{e}^{-S_n^F}, \mathrm{e}^{-S_n^G})$. For simplicity, we use $g(x,y) = (xy)^\alpha$ hereon, with $\alpha > 0$, so the exponential holding times have parameters $\mathrm{e}^{-\alpha(S_n^F + S_n^G)}$. We introduce the variables $S_n^H = -\sum_{\mathrm{i}=1}^n (\log U_\mathrm{i} + \log V_\mathrm{i})$ and we have

$$\mathbb{P}(S_{n+1}^H - S_n^H \leq x) = \mathbb{P}(-\log U - \log V \leq x) = \hat{H}(x),$$

where $\hat{H} = \hat{F} \star \hat{G}$ ($\star$ denotes convolution). Now, the rates of the exponential holding times can be written as $\mathrm{e}^{-\alpha S_n^H}$.

Let also $M_y^F = \sup\{n \geq 0 : S_n^F \leq -\log y\}$ and $M_y^G = \sup\{n \geq 0 : S_n^G \leq -\log y\}$, for $y \in]0,1[$. In addition, we consider an iid sequence $(\xi'_\mathrm{i})_\mathrm{i}$ of exponential random variables, with unit mean. So, for $t > 0$ and $y_1, y_2 \in]0,1[$, we have

$$\mathbb{P}(L_t^F < y_1,\, L_t^G < y_2) = \mathbb{P}\left(\sum_{\mathrm{i}=0}^{M_{y_1}^F} \exp(\alpha S_\mathrm{i}^H)\xi'_\mathrm{i} \leq t, \sum_{\mathrm{i}=0}^{M_{y_2}^G} \exp(\alpha S_\mathrm{i}^H)\xi'_\mathrm{i} \leq t \right), \tag{5.4.5}$$

noting that $(\xi'_\mathrm{i})_\mathrm{i}$ is the same for both coordinates. Then the probability above is equal to

$$\mathbb{P}\left(\sum_{\mathrm{i}=0}^{M_{y_1}^F} \exp(\alpha S_{M_{y_1}^F - \mathrm{i}}^H)\xi'_{M_{y_1}^F - \mathrm{i}} \leq t, \sum_{\mathrm{i}=0}^{M_{y_2}^G} \exp(\alpha S_{M_{y_2}^G - \mathrm{i}}^H)\xi'_{M_{y_2}^G - \mathrm{i}} \leq t \right).$$

Now, in the sums above we make the changes of variable $T_\mathrm{i}^{HF}(y) := -\log y - S_{M_y^F - \mathrm{i}}^H$ and $T_\mathrm{i}^{HG}(y) := -\log y - S_{M_y^G - \mathrm{i}}^H$, to obtain

$$\mathbb{P}(L_t^F < y_1, L_t^G < y_2) = \mathbb{P}\Big(\sum_{\mathrm{i}=0}^{M_{y_1}^F} \exp(-\alpha T_\mathrm{i}^{HF}(y_1))\xi_\mathrm{i} \leq t y_1^\alpha, \sum_{\mathrm{i}=0}^{M_{y_2}^G} \exp(-\alpha T_\mathrm{i}^{HG}(y_2))\xi_\mathrm{i} \leq t y_2^\alpha \Big). \tag{5.4.6}$$

Now, as in the one-dimensional model, if (from Renewal Theory?)

$$\{T_n^{HF}(y) : n \geq 0\} \xrightarrow{\mathcal{D}} \{T_n^{HF} : n \geq 0\}, \quad \{T_n^{HG}(y) : n \geq 0\} \xrightarrow{\mathcal{D}} \{T_n^{HG} : n \geq 0\}, \quad \text{as} \quad y \to 0^+, \tag{5.4.7}$$

where $\{T_n^{HF} : n \geq 0\}, \{T_n^{HG} : n \geq 0\}$ are some kind of "equilibrium renewal processes", and setting $y_1 = x_1 t^{-1/\alpha}$ and $y_2 = x_2 t^{-1/\alpha}$ in (5.4.6), we obtain

$$\lim_{t\to\infty} \mathbb{P}(t^{\frac{1}{\alpha}} L_t^F < x_1,\, t^{\frac{1}{\alpha}} L_t^G < x_2) = \mathbb{P}\Big(\sum_{\mathrm{i}\geq 0} \exp(-\alpha T_\mathrm{i}^{HF})\xi_\mathrm{i} \leq x_1^\alpha, \sum_{\mathrm{i}\geq 0} \exp(-\alpha T_\mathrm{i}^{HG})\xi_\mathrm{i} \leq x_2^\alpha \Big).$$

Finally, letting $Y_\alpha^{HF} = \sum_{\mathrm{i}\geq 0} \exp(-\alpha T_\mathrm{i}^{HF})\xi_\mathrm{i}, Y_\alpha^{HG} = \sum_{\mathrm{i}\geq 0} \exp(-\alpha T_\mathrm{i}^{HG})\xi_\mathrm{i}$, we conclude that, as $t \to \infty$,

$$t^{1/\alpha}(L_t^F, L_t^G) \xrightarrow{\mathcal{D}} ((Y_\alpha^{HF})^{1/\alpha}, (Y_\alpha^{HG})^{1/\alpha}).$$

The key of the reasoning above is precisely the conjectured convergence of $T_n^{HF}(y), T_n^{HG}(y)$, as $y \to 0$. This is an open problem.

Final comments

In this dissertation we study chain-maxima and related processes, from a sample of iid random vectors. Chain-maxima $\mathcal{M}_n$ are a new multidimensional type of maxima, whose definition is based on the usual component-wise partial order in $\mathbb{R}^{\mathrm{d}}$. This notion preserves the recursive structure that one-dimensional maxima have and, consequently, a Markovian structure and similar properties naturally arise.

Chain-records $(\mathcal{R}_n)$ appear as a random subsequence of $(\mathcal{M}_n)$ and also as the points where the sequence $(\mathcal{M}_n)$ jumps. Our research about these objects is centered into two main aspects, namely distributional and asymptotic results. First, we develop a general theory where we present properties of chain-maxima. Then we apply the general results to particular cases, namely the hypercube $[0,1]^{\mathrm{d}}$ and the d-simplex Δ^{d}. Finally, we present work in progress and open questions. Some may serve as source for future work.

In Chapter 2 we derive, under reasonable assumptions, distributional results about $(\mathcal{M}_n)$ and $(\mathcal{R}_n)$ in a general context. The recursive relation (2.1.1) yields the Markovian structure of these objects. Other related Markov chains emerge as well. That is the case of chain-record-times and chain-record-values (see e.g. Corollary 2.2.1 and Proposition 2.2.3). Interestingly, many of these facts are extensions of one-dimensional results found, for example, in [53]. With respect to asymptotic results, we can mention a martingale related to inter record-times, which leads to a useful connection between the number of chain-records and the sum of record heights $\mathcal{H}_n$, defined as the conditional probability that an observation $\mathbf{X}$ becomes a record $(\mathbf{X} \succ \mathcal{M}_n)$. In this direction, there is an important logarithmic growth result that connects the sequences (Δ_n) and $\overline{\mathbf{F}}(\mathcal{R}_n)$. In this case, we also have a generalization of lemma 2 in [52], but in contrast to this result, we don't need a condition on the speed of $\overline{\mathbf{F}}(\mathcal{R}_n) \to 0$. Furthermore, from this treatment, additional a.s. results about waiting times of chain-records are obtained. They generalize results in [40].

It is quite natural to see chain-records values as a point process in $\mathbb{R}^{\mathrm{d}}$. So, we briefly establish and discuss some notions, such as the multivariate counting process and the relation with the corresponding marginal counting processes. We believe this can be an interesting line of future research.

Under the additional assumption of continuity and independence of components of observations, we get the natural Markovian structure of the marginals of $\mathcal{R}_n$ and their transitions probabilities, concluding that each one behaves as usual record values and that the multivariate point process of chain-records is distributed as the minimum of d independent non-homogeneous Poisson processes. Another interesting feature is the distribution-free property of the counting process of chain-records $\mathcal{N}_n$, chain-record times $\mathcal{T}_k$ and inter record-times.

Finally, Chapter 2 ends with a short discussion about generalizing chain-records by changing the dominance relation $\preceq$. For example, we consider the partial order induced by a cone $\mathcal{K}$ in $\mathbb{R}^{\mathrm{d}}$. Another extension is related to strict chain-records, based on a strict dominance relation (2.5.2). We found that this definition allows a simplified analysis in the case of observations with discrete independent components.

Chapter 3 is devoted to the study of chain-maxima and chain-records from the model $\mathbf{U}([0,1]^{\mathrm{d}})$. The distribution of chain-record values is obtained from the general result for independent components. Also, an explicit form of the marginal density is obtained. A useful recursive iterative scheme as a form to generate chain records values in $[0,1]^{\mathrm{d}}$ is provided. It is shown that $\mathcal{R}_n \to (1,\ldots,1) \in [0,1]^{\mathrm{d}}$ a.s., according to Proposition 2.4.5.

Record heights $\mathcal{H}_n$ are random variables related to the behavior of $\mathcal{M}_n$. We develop a full study of these variables, for observations from the $\mathbf{U}([0,1]^{\mathrm{d}})$ model. First, a recurrence for the moments of $\mathcal{H}_n$ is derived and solved explicitly in dimension $\mathrm{d} = 2$. In general, for any dimension d, the solution is an alternating sum of a given sequence (b_n). To study the asymptotic behavior of this kind of sum, we apply a technique due to Flajolet and Sedgewick, presented in [22]. The key idea is to study a sum of residues of an analytic continuation φ of (b_n), over its poles, where only the real ones are significant in the asymptotic analysis. In this part we got involved in long computations in order to construct φ and to check its growth and boundedness. This leads to establishing a weak convergence of $\mathcal{H}_n$, suitably normalized. We mention that this result was obtained by Gnedin [25] as well, using completely different tools. We carry out a similar analysis to establish the asymptotic behavior of mixed moments and thus provide more information about $\mathcal{M}_n$.

For the counting process of chain-records we obtain a law of large numbers in quadratic mean, for $\mathrm{d} = 2$. We take advantage of the a.s. equivalence between $\mathcal{N}_n$ and the sum of record height proved in Lemma 2.3.2. This connection is useful because we have an expression about the moments of $\mathcal{H}_n$, hence the study of the asymptotic variance and covariances can be done. In this study, we came across with sums involving harmonic numbers that required careful treatment. Properties about alternating sums turned out to be very useful, as well.

In Chapter 4 we focus on the model $\mathbf{U}(\Delta^{\mathrm{d}})$. We start with a distributional study chain-records values $(\mathcal{R}_n)$, which are shown to be a Markov chain, with explicit transition probabilities. From this fact we derive a recurrence for the corresponding density function, that can be solved for small values of n and in dimension $\mathrm{d} = 2$. We emphasize that observations from this model do not have independent components. As in Chapter 3, we construct a recursive scheme that allows us to analyze the asymptotic behavior of $\mathcal{R}_n$. In this framework $\mathcal{R}_n$ satisfies a stochastic equation whose solution in distributional sense we investigate. As preliminary result, we show that $\mathcal{R}_n$ converges a.s. to a random point $\mathcal{R}_\infty \in \{\mathbf{x} \in \mathbb{R}_+^{\mathrm{d}} : \sum_{\mathrm{i}=1}^{\mathrm{d}} x^{(\mathrm{i})} = 1\}$. A perpetuity representation is also used to investigate further properties of $\mathcal{R}_\infty$. Results from [37] and [58] allow us to find that the solution is a non-degenerate random vector, whose distribution is of pure type (singular continuous or absolutely continuous). Furthermore, if $\mathrm{d} = 2$, the problem is reduced to a one-dimensional stochastic equation, whose solution is the density function of the stationary law of a Markov chain (after a suitable transformation). The solution happens to be a well know probability density.

Record heights $\mathcal{H}_n$ are analyzed using the same techniques of the uniform model on $[0,1]^{\mathrm{d}}$. A study of their moments leads to a recurrence whose solution has the form of an alternating sum (or Euler's transform) of a certain sequence b_j. Again, the asymptotic analysis of this expression is done by applying the methods of [22]. Nevertheless, this case is much more complicated than the uniform $\mathbf{U}[0,1]^{\mathrm{d}}$. The roots of a polynomial related to the singularities of φ (the meromorphic extension of (b_j) cannot be given in explicit form, in general dimension d. This fact motivates us to develop a deeper study of these zeros. In this sense, we apply analytical tools about localization of zeros and critical points of polynomials. We derive properties that allow to gain knowledge about the distribution of the poles and also to get a controlled growth of φ, over a suitable set. These conditions are critical to apply Flajolet and Segewick's theorem. As a consequence we show weak convergence of record-heights $\mathcal{H}_n$, in any dimension d. The weak limit in $\mathrm{d}=2$ is characterized as product of two independent random variables, with well-known distributions.

Chapter 4 ends with the asymptotic analysis of $(\mathcal{N}_n)$, the counting process of chain-records in Δ^{d}. We obtain an L_2 convergence of this object, by means of non-trivial computations of variances and covariances of $\mathcal{H}_n$. The a.s. connection between the number of chain-records and the sum of record heights is the main tool involved here.

In Chapter 5 we present some unfinished research topics. Some of them, are work in progress and others lead to open questions that may serve as source for future work. The problem of weak convergence of $\mathcal{M}_n$ is tackled from different points of view. Some of them reveal interesting features of chain-maxima $\mathcal{M}_n$. First we concentrate on finding the sequence (A_n) such that $A_n((1,1)-\mathcal{M}_n)$ converges in distribution to a non-degenerated limit law. We derive some results, using the symmetry of components and the fact that $\mathcal{H}_n$ is normalized by n. We investigate the recurrence for the density function $\mathbf{f}_n$ of $\mathcal{M}_n$. From this we find that $\mathbf{f}_n$ is a two variable polynomial whose coefficients $a_{\mathrm{i},j}$ can be interpreted as a kind of Delannoy numbers. Second, we propose a continuous-time stochastic model, motivated by results from [12]. The board-breaking model arises as an analog in continuous-time of the evolution of chain-record in dimension $\mathrm{d}=2$. We are interested in the evolution of the lengths of the sides of a tagged rectangular fragment. We have an argument, based on tools from renewal theory, which mimics the corresponding argument for the stick-breaking process. Unfortunately, it is not clear whether convergence in distribution holds or not.

Ideas for future work

This dissertation leaves a number of unanswered questions and unsolved problems, which may well become a natural source of material for future work in this topic and other related ones, that may arise.

1. We plan to extend the result of Chapter 4, about the law of $\mathcal{R}_\infty$, as the invariant distribution of a Markov chain, to dimension $\mathrm{d}>2$.
2. It could be interesting to explore the chain records in other geometries such as, for example, the unit circle $\{(x,y)\in\mathbb{R}^2_+ : x^2+y^2\leq 1\}$.
3. The problem of normalizing the random vector $(1,\ldots,1)-\mathcal{M}_n$ in $[0,1]^{\mathrm{d}}$ is open. In-

deed, we conjecture that the only candidate, due to symmetry and our experience in dimension d = 2, is $n^{1/\mathrm{d}}$. The escape-of-mass phenomenon might also be present here.

4. Another interesting problem is to expand the brief study of Chapter 2, about the multivariate point process generated by the chain record values. A first attempt could be to obtain something similar to the decomposition of a one-dimensional Shorrock process.
5. We plan to explore more chain-maxima and related processes generated by discrete observations. It would be interesting to check if martingales are useful and compare with results from one-dimensional, integer-valued random variables.
6. Another potentially interesting idea (briefly explored in Chapter 2) is to investigate about chain-maxima and chain-records when the classical order is changed, for example, by a cone-generated order.
7. It also looks promising to explore extensions of chain-records in the direction of near-records. For example, we could introduce and study notions such as δ-chain records or similar, see [31, 33].

Bibliography

[1] M. Ahsanullah, *Record statistics*, Nova Science Publishers, 1995.

[2] N.I. Akhiezer, *The classical moment problem: And some related questions in analysis*, University mathematical monographs, Oliver & Boyd, 1965.

[3] H. Alzer, D. Karayannakis, and H. M. Srivastava, *Series representations for some mathematical constants*, J. Math. Anal. Appl. **320** (2006), no. 1, 145–162. MR 2230462

[4] B.C Arnold, N. Balakrishnan, and H.N Nagaraja, *Records*, vol. 768, John Wiley & Sons, 2011.

[5] Zhi-Dong Bai, Hsien-Kuei Hwang, and Wen-Qi Liang, *Normal approximation of the number of records in geometrically distributed random variables*, Random Structures and Algorithms **13** (1998), 319–334.

[6] R Bakić, *Generalization of the Grace-Heawood theorem*, Publ. Inst. Math. (Beograd) (N.S.) **93(107)** (2013), 65–67. MR 3089077

[7] N. Balakrishnan, A. Stepanov, and V.B. Nevzorov, *North-east bivariate records*, to appear in Metrika (2019), 1–15.

[8] C. Banderier and S. Schwer, *Why Delannoy numbers?*, J. Statist. Plann. Inference **135** (2005), no. 1, 40–54. MR 2202337

[9] E. J. Barbeau, *Polynomials*, Problem Books in Mathematics, Springer-Verlag, New York, 1995, Corrected reprint of the 1989 original. MR 1409810

[10] I. Bayramoglu, *On the records of multivariate random sequences*, Metrika **79** (2016), no. 6, 725–747. MR 3518584

[11] P. Billingsley, *Probability and measure*, third ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1995, A Wiley-Interscience Publication. MR 1324786

[12] M.D Brennan and R. Durrett, *Splitting intervals ii: Limit laws for lengths*, Probability theory and related fields **75** (1987), no. 1, 109–127.

[13] J. Choi and H. M. Srivastava, *Some summation formulas involving harmonic numbers*

and generalized harmonic numbers, Math. Comput. Modelling **54** (2011), no. 9-10, 2220–2234. MR 2834625

[14] L. de Haan and A. Ferreira, *Extreme value theory*, Springer Series in Operations Research and Financial Engineering, Springer, New York, 2006, An introduction. MR 2234156

[15] H. Delannoy, *Employ d'échiquier pour la résolution de certains problèmes de probabilités*, Assoc. Franc. Bordeaux (1895), no. 24, 70–90.

[16] J. Deuschel and O. Zeitouni, *Limiting curves for i.i.d. records*, Ann. Probab. **23** (1995), no. 2, 852–878. MR 1334175

[17] P. Diaconis and D. Freedman, *Iterated random functions*, SIAM review **41** (1999), no. 1, 45–76.

[18] C. Dombry and M. Zott, *Multivariate records and hitting scenarios*, Extremes **21** (2018), no. 2, 343–361. MR 3800302

[19] M. Dziemiańczuk, *Generalizing Delannoy numbers via counting weighted lattice paths*, Integers **13** (2013), Paper No. A54, 33. MR 3106100

[20] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling extremal events: for insurance and finance*, vol. 33, Springer Science & Business Media, 2013.

[21] M. Falk, A. Khorrami Chokami, and S. A. Padoan, *On multivariate records from random vectors with independent components*, J. Appl. Probab. **55** (2018), no. 1, 43–53. MR 3780381

[22] P. Flajolet and R. Sedgewick, *Mellin transforms and asymptotics: finite differences and Rice's integrals*, Theoret. Comput. Sci. **144** (1995), no. 1-2, 101–124, Special volume on mathematical analysis of algorithms. MR 1337755

[23] R. D. Fray and D. P. Roselle, *Weighted lattice paths*, Pacific J. Math. **37** (1971), 85–96. MR 0302463

[24] A. Gnedin, *Records from a multivariate normal sample*, Statist. Probab. Lett. **39** (1998), no. 1, 11–15. MR 1649331

[25] A. Gnedin, *The chain records*, Electron. J. Probab. **12** (2007).

[26] C.M. Goldie and R. Grübel, *Perpetuities with thin tails*, Adv. in Appl. Probab. **28** (1996), no. 2, 463–480. MR 1387886

[27] C.M. Goldie and S. Resnick, *Records in a partially ordered set*, Ann. Probab. **17** (1989), no. 2, 678–699. MR 985384

[28] ———, *Many multivariate records*, Stochastic Process. Appl. **59** (1995), no. 2, 185–216. MR 1357651

[29] R. Gouet, F.J. López, and M. San Miguel, *A martingale approach to strong convergence*

of the number of records, Adv. in Appl. Probab. **33** (2001), no. 4, 864–873. MR 1875783

[30] R. Gouet, F.J. López, and G. Sanz, *Central limit theorems for the number of records in discrete models*, Adv. in Appl. Probab. **37** (2005), no. 3, 781–800. MR 2156560

[31] ———, *Asymptotic normality for the counting process of weak records and δ-records in discrete models*, Bernoulli **13** (2007), no. 3, 754–781. MR 2348749

[32] ———, *Asymptotic normality for the number of records from general distributions*, Adv. in Appl. Probab. **43** (2011), no. 2, 422–436. MR 2848383

[33] R Gouet, FJ López, and G Sanz, *On geometric records: rate of appearance and magnitude*, Journal of Statistical Mechanics: Theory and Experiment **2012** (2012), no. 01, P01005.

[34] R. Gouet, F.J. López, and G. Sanz, *Records from stationary observations subject to a random trend*, Adv. in Appl. Probab. **47** (2015), no. 4, 1175–1189. MR 3433301

[35] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science. MR 1397498

[36] U. Grenander, *A limit theorem for sums of minima of stochastic variables*, Ann. Math. Statist. **36** (1965), 1041–1042. MR 0176514

[37] A. K. Grincevicius, *The continuity of the distribution of a certain sum of dependent variables that is connected with independent walks on lines*, Teor. Verojatnost. i Primenen. **19** (1974), 163–168. MR 0345178

[38] P. Hall and C. Heyde, *Martingale limit theory and its application*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980, Probability and Mathematical Statistics. MR 624435

[39] P. Hitczenko and J. Wesolowski, *Renorming divergent perpetuities*, Bernoulli **17** (2011), no. 3, 880–894.

[40] P. Holmes and W. Strawderman, *A note on the waiting times between record observations*, J. Appl. Probability **6** (1969), 711–714. MR 0258151

[41] Hsien-Kuei Hwang and Tsung-Hsi Tsai, *Multivariate records based on dominance*, Electron. J. Probab. **15** (2010), no. 60, 1863–1892. MR 2738341

[42] A.F. Karr, *Point processes and their statistical inference*, second ed., Probability: Pure and Applied, vol. 7, Marcel Dekker, Inc., New York, 1991. MR 1113698

[43] J. Neveu, *Discrete-parameter martingales*, revised ed., North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975, Translated from the French by T. P. Speed, North-Holland Mathematical Library, Vol. 10. MR 0402915

[44] V.B. Nevzorov, *Records*, Theory of Probability & Its Applications **32** (1988), no. 2, 201–228.

[45] R. Noble, *Asymptotics of the weighted Delannoy numbers*, Int. J. Number Theory **8** (2012), no. 1, 175–188. MR 2887889

[46] V. Prasolov, *Polynomials*, Algorithms and Computation in Mathematics, vol. 11, Springer-Verlag, Berlin, 2010, Translated from the 2001 Russian second edition by Dimitry Leites, Paperback edition [of MR2082772]. MR 2683151

[47] S. Purkait and B. Sury, *Some vanishing sums involving binomial coefficients in the denominator*, Albanian J. Math. **2** (2008), no. 1, 27–32. MR 2383491

[48] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press, Oxford University Press, Oxford, 2002. MR 1954841

[49] S. Resnick, *Extreme values, regular variation and point processes*, Springer, 2013.

[50] S.M. Ross, *Stochastic processes*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1996. MR 1373653

[51] J. San Martín, *Teoría de la medida*, primera ed., Textos Universitarios, Editorial Univesitaria, Santiago, Chile, 2018.

[52] R. Shorrock, *A limit theorem for inter-record times*, J. Appl. Probability **9** (1972), 219–223. MR 0295421

[53] ———, *On record values and record times*, J. Appl. Probability **9** (1972), no. 02, 316–326. MR 0350869

[54] M. Spivey, *Combinatorial sums and finite differences*, Discrete Math. **307** (2007), no. 24, 3130–3146. MR 2370116

[55] R. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112

[56] R.A. Sulanke, *Objects counted by the central Delannoy numbers*, J. Integer Seq. **6** (2003), no. 1, Article 03.1.5, 19. MR 1971435

[57] W. Vervaat, *Limit theorems for records from discrete distributions*, Stochastic Processes Appl. **1** (1973), 317–334. MR 0362457

[58] ———, *On a stochastic difference equation and a representation of non–negative infinitely divisible random variables*, Advances in Applied Probability **11** (1979), no. 4, 750–783.

[59] J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer.

Math. Soc. **24** (1922), no. 3, 163–180. MR 1501220

[60] D. Williams, *Probability with martingales*, Cambridge University Press, 1997.