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CONTINUOUS EIGENVALUES FOR REPETITIVE MEYER SYSTEMS

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA.

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CONTINUOUS EIGENVALUES FOR REPETITIVE MEYER SYSTEMS.

In this Ph.D. thesis, we make some contributions to the study of continuous eigenvalues for dynamical systems associated with some particular repetitive discrete sets: Meyer sets and inter-model sets. Specifically, the main theorem (Theorem A) in Chapter 3 gives a dynamical characterization of inter-model sets with Euclidean internal space. The characterization is similar to previous results for general inter-model sets obtained independently by Baake, Lenz and Moody, and Aujogue, but with an additional condition written in terms of the address map introduced by Lagarias. Using our characterization and the characterization given by Baake, Lenz, and Moody in [BLM07, Theorem 5], we obtain as a corollary a characterization for regular model sets with Euclidean internal space (see Theorem B). Also as a corollary of our characterization, we give another characterization for regular inter-model sets with Euclidean internal space in terms of the minimal window (Theorem C).

For finitely generated Delone set in $\mathbb{R}^{\mathrm{d}}$, we can associate a coordinate map called the address map [L99]. We use it to construct an equicontinuous morphism for the dynamical hull system of a repetitive Meyer set. This construction is given in the proof of Proposition A. From [KS14, Theorem 1.3] we know that the dynamical hull system associated with a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ has d continuous eigenvalues. Our construction of an equicontinuous morphism for this dynamical system gives us a method to find at least d continuous eigenvalues for the hull system of a repetitive Meyer set.

In Chapter 4 we study linearly repetitive Meyer sets. In [So99] B. Solomyak proved that for a Delone set constructed from a self-affine substitution with some additional properties, its hull system has only continuous eigenvalues. For linearly repetitive Meyer systems we find a sufficient condition for the dynamical hull system has only continuous eigenvalues. This condition is about the algebraic structure of the return vectors, and some of their combinatorial properties. This result is Theorem D, and it is the main theorem of §4.

In a similar way to Lagarias in [L99], we use almost linear sequences to construct Meyer sets in the real line. We used Proposition A, to verify that there are Meyer sets in the real line with two continuous eigenvalues rationally independent. Also, we show some examples to check that some hypotheses in Theorem D are necessary.

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VALORES PROPIOS CONTINUOUS PARA EL SISTEMA DINÁMICO ASOCIADO A UN CONJUNTO DE MEYER REPETITIVO.

Esta tesis doctoral realiza una contribución al estudio de valores propios para sistemas dinámicos asociados a algunos conjuntos discretos y repetitivos en $\mathbb{R}^d$: Delone, Meyer, inter-model y model. Específicamente, el teorema principal en el Capítulo 3 (Teorema A), da una caracterización de los conjuntos inter-model con espacio interno Euclidiano. Esta caracterización es similar a los resultados previos obtenidos independientemente por Baake, Lenz y Moody, y Aujogue, pero con una condición adicional escrita en términos de la address map introducida por Lagarias. Usando nuestra caracterización y la caracterización dada por Baake, Lenz y Moody en [BLM07, Theorem 5], obtenemos como un corolario una caracterización para conjuntos model regulares con espacio interno Euclidiano (ver Teorema B). También como un corolario de nuestra caracterización, es posible dar una para conjuntos inter-model regulares con espacio interno Euclidiano en términos de una ventana minimal (Teorema C).

Para un conjunto de Delone finitamente generado de $\mathbb{R}^d$, podemos asociar una función coordenada llamada address map [L99]. Usaremos esta para construir un morfismo equicontinuo del sistema dinámico asociado a un conjunto de Meyer repetitivo. Esta construcción aparece en la demostración de la Proposición A. Por [KS14, Theorem 1.3], sabemos que el sistema dinámico asociado a un conjunto de Meyer repetitivo en $\mathbb{R}^d$ tiene d valores propios continuos. Nuestra construcción de un morfismo equicontinuo para este sistema dinámico, nos da un método para encontrar al menos d valores propios continuos para el sistema dinámico asociado a un conjunto de Meyer repetitivo.

El Capítulo 4 se refiere a sistemas dinámicos para un conjunto de Meyer linealmente repetitivo. En [So99], B. Solomyak probó que para un conjunto de Delone construido a partir de una substitución auto-afine con algunas propiedades adicionales, su sistema dinámico tiene solo valores propios continuos. Para conjuntos de Meyer linealmente repetitivos el principal resultado del Capítulo 4, Teorema D, da una condición suficiente para que su sistema dinámico tenga solo valores propios continuos. Esta condición es sobre la estructura algebraica de los vectores de retorno y algunas de sus propiedades combinatorias.

Similarmente a Lagarias en [L99], usamos secuencias casi-lineales para construir conjuntos de Meyer en la recta real. Mediante la Proposición A, mostramos un conjunto de Meyer en la recta real con dos valores propios continuos racionalmente independientes. También veremos ejemplos para chequear que algunas hipótesis en el Teorema D son necesarias.

Contents

Glossary of Notation

Sets

$\emptyset$	empty set
$\mathbb{N}$, $\mathbb{Z}$	the sets of natural and integers numbers
$\mathbb{Z}^{\mathrm{d}}$	cartesian product of d copies of integers numbers
$\mathbb{R}^{\mathrm{d}}$	the d-dimensional Euclidean space
$\mathbb{C}$, $\mathbb{S}^1$	the set of complex numbers, and its unit circle
$\mathbb{T}^{\mathrm{d}}$	the d-dimensional torus
$B(x,r)$	open Euclidean ball centered at x with radius r
$L^2(X;H)$	H-valued Lebesgue square-integrable functions over X
$\overline{S}$, $\mathrm{cl}(S)$	the topological closure of S
$S' \subseteq S$	S' is a subset of S
S^{c}	the complement set of S
$\mathrm{int}(S)$, S°	the topological interior of S
$S - S$	the set of differences of S
$S \sqcup S'$	the disjoint union of the sets S and S'

Operations and Symbols

$:=$ and $=:$	equal by definition
$\sim$	equivalence relation
$[x]_{\sim}$	the class of x in the equivalence relation $\sim$
$X/\sim$	the quotient space of X with the equivalence relation $\sim$
$X \times Y$	the cartesian product of the spaces X and Y
$\\|\cdot\\|_{\mathrm{d}}$, $\langle\cdot,\cdot\rangle$	norm and inner product in the Euclidean space $\mathbb{R}^{\mathrm{d}}$
$\|\cdot\|$	norm in the complex space $\mathbb{C}$
$\\|\cdot\\|_{op}$	operator norm for bounded linear operators in $L^2(X;H)$
$\vert\vert\vert\cdot\vert\vert\vert$	distance to the nearest integer
$x \to \bar{x}$	x converges to $\bar{x}$

Functions and Mappings

$\mathrm{e}xp(\cdot), \mathrm{e}^{(\cdot)}$	exponential function in $\mathbb{C}$
$\sup_{x\in I} f(x)$	supremum of $f(x)$ on I
$f : X \to Y$	map from X to Y
$\mathcal{G}(f)$	the graph of f
$f \circ g$	the composition of functions f and g
$\mathrm{d}(\cdot), r(\cdot)$	domain and range maps for groupoids
$A_{\mathrm{i},\cdot}$	ith row of the matrix A
$A_{\cdot,j}$	jth column of the matrix A

Chapter 1

Introduction

1.1 General Introduction.

In this thesis, we study the set of eigenvalues of some mathematical models for quasicrystals. We start by modeling the atomic positions of a d-dimensional crystalline solid via a discrete subset Λ of $\mathbb{R}^{\mathrm{d}}$. We can associate a dynamical system $(X, \mathbb{R}^{\mathrm{d}})$ that allows us to know Λ and its translations via the topology on X. We are interested in the study of spectral properties for the translation action of $\mathbb{R}^{\mathrm{d}}$ on $L^2(X, \mathbb{C})$. We focus on the pure point part of the spectrum, the eigenvalues for the associated dynamical system. Specifically, on the set of continuous eigenvalues, when it is possible to choose the associated eigenfunction being continuous. The set of continuous eigenvalues takes relevance, from the dynamical point of view, because every continuous eigenvalue defines a topological factor from the associated dynamical system into a rotation in a torus. This allows us to understand the dynamics of X a little better. In the context of diffraction theory, the spectrum of the dynamical system is related to a measure known as mathematical diffraction. This measure is the mathematical version of X-ray diffraction in crystallography. To have a better understanding of the context, we will start with a few words in crystallography: some of its history and its mathematical point of view. After, we give the principal definitions to state the results of this thesis.

Figure 1.1.1: A picture of the sunstone from Getty images, and a picture of the cave in Naica-Mexico from www.infobae.com

1.1.1 Crystallography.

When the matter is in solid-state the atoms can be arranged from a regular geometric lattice to the worse case; in a completely amorphous way (see Figure 1.1.2). Many people have worked on this subject to understand how the atoms are arranged to form solids (or other elements). One of them is the ancient Greek philosopher Plato. He proposed, in his dialogue Timaeus, that the elements in nature were made of some convex, regular polyhedron. These types of solids are known as Platonic solids and many people worked on this topic, for example, Aristotle and Euclid. Later, the German astronomer Johannes Kepler used the Platonic solids to give a model of the Solar System. But the initial idea of Plato has a flaw. If we assume that the atoms in a solid are arranged to fill a portion of the space, we need that the geometrical shapes have no overlaps and gaps. But the cube is the unique Plato solid that ensure this.

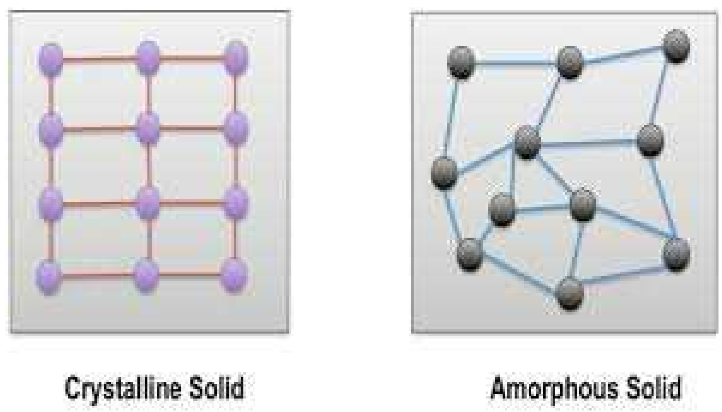


Figure 1.1.2: Example of atoms arranged in 2-D (image from www.askiitians.com).

The pioneer in geology Nicolas Steno was the first to work on the symmetries of the "crystals" by the year 1669. In 1801 the French priest and mineralogist René Just Haüy, who is considered the founder of crystallography, publishes his "Traité de minéralogie", where he wrote

"*The observation I have just noted is that which has served to develop my ideas on the structure of crystals. It presented itself in the case of a crystal that the citizen Defrance was kind enough to give me just after it had broken off from a group this enlightened amateur was showing me, and which formed part of his mineralogical collection. The prism had a single fracture along one of the edges of the base, by which it had been attached to the rest of the group. Instead of placing it in the collection I was then forming, I tried to divide it in other directions, and I succeeded, after several attempts, in extracting its rhomboid nucleus.*"

For this reason, he proposed that a solid material is a *crystal* if it is composed of regular polygon-shaped blocks in a regular way to construct the crystal figure. Later, the solids material whose atoms are arranged to form a lattice that extends in all directions was called *crystalline* solids. The word crystal comes from the Greek word *krustallos* and means *ice* or *rock crystal*.

The most common example of a crystal is Sodium chloride (as a mineral: Halite) or

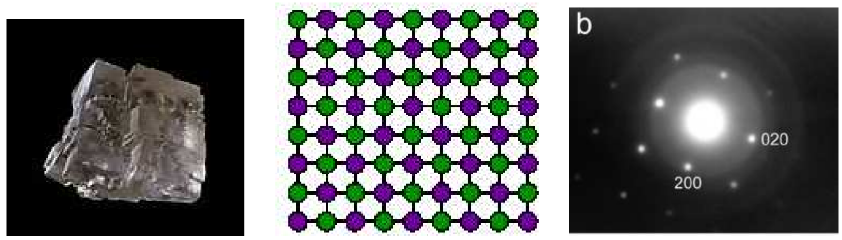


Figure 1.1.3: A picture of Halite in solid-state from crystal-information.com, an image of its atoms arrange from www.123rf.com and an image of its diffraction pattern published by Ian Freestone.

commonly known as salt. In Figure 1.1.3 we can observe the crystalline solid Halite, a scheme of its atoms arrange, and its diffraction pattern. But we can find crystalline solids everywhere, for example, some types of carbon, diamond, quartz, and the Iceland spar*(a variety of calcite). The Iceland spar has the property of being birefractive. The Vikings knew the Iceland spar as *Sunstone* by this property. They used it to locate the sun in the sky when it is cloudy, and thus orient themselves in the open sea during their travels around the world. Another example of natural crystals can be found in Mexico, where there is a cave with large crystals of selenite, see Figure 1.1.1. But there are also examples of crystals that come from space. In 1971 in Haverö - Sweden, a meteorite crashed the earth and when the researchers studied a piece of this they found two new forms of carbon. One of them is similar to the diamond, and it was predicted to exist years ago but had never before seen in nature. The hardness of the diamond is due to the carbon atoms inside it are arranged in a tetrahedron-shaped lattice but the crystalline carbon found in Haverö has its atoms arranged in a rhombohedral lattice (see Figure 1.1.4).

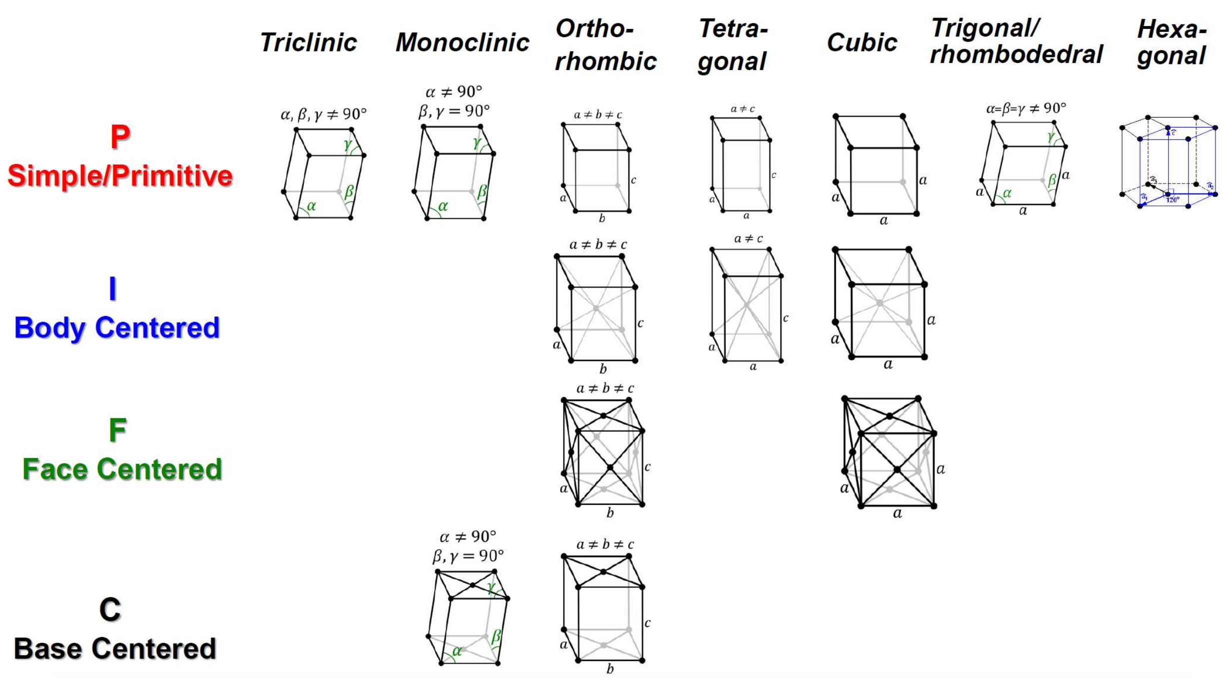


Figure 1.1.4: Bravais lattices in 3D (Image from https://www.quora.com).

*http://www.bbc.com/earth/story/20150623-ten-crystals-with-magic-powers

Consider regular polygon-shaped blocks, matching vertex to vertex, to construct the crystal figure. If we used only one type of polygon, is clear that triangles, squares, or hexagons can fill the whole Euclidean plane with no gaps and overlaps. But using different regular polygons we have eight ways more to do this. A natural question about the crystal figure is can this have symmetries? In the Euclidean plane, some geometric facts imply that crystalline solids can possess only two, three, four, and six-fold rotational symmetries, and thus only finitely many ways to arranged the atoms are possible. This result is known as the crystallographic restriction theorem.

In 1850, Auguste Bravais wrote his "Mémoire sur les systèmes formés par les points distribués régulièrement sur un plan ou dans l'espace". Where he studied lattices regularly distributed on the Euclidean plane and the Euclidean space. Geometrical facts imply there exist 7 different lattices that bond atoms in 2D. By symmetries, we obtain 14 possible lattices, called Bravais lattices. In Figure 1.1.4 we see a representation of the Bravais lattices in the Euclidean space. In 1891, Yevgrof Fedorov proved that for the 2-dimensional Euclidean space there are only 17 symmetry groups. Although many years ago, ancient civilizations knew about these symmetry groups. An example is the Alhambra Palace, where the Nasrid architects and craftsmen used these groups to decorate de walls (see Figure 1.1.5). After the study of the symmetries groups in the Euclidean plane, Fedorov together with Arthur Schönflies worked to obtain a classification of the symmetry group in the Euclidean space. They used the Bravais lattices to conclude that the Euclidean space has 230 symmetry groups.

Figure 1.1.5: Some walls on Alhambra's palace painted using symmetry groups.

Specifically, the science of crystallography studies the mean position of the atoms in crystalline solids[†]. For certain crystalline solids, we can deduce from the mean atomic positions, some chemical and physical properties of the solid. For example, chemical bonds or their crystallographic disorder. To study the mean position of the atoms in crystalline solids, some crystallographic methods consist of analyzing the *Diffraction patterns*. These patterns are produced by a beam (commonly X-rays) of a crystalline solid sample. These methods are used since 1895 when Wilhem Rontgen discovered X-rays. After studying and improving this method, many contributions were obtained in different fields of science. For example, in 1952, the chemist Rosalind Franklin[‡] used the diffraction patterns analysis method to determine the molecular conformations of DNA as a double helix.

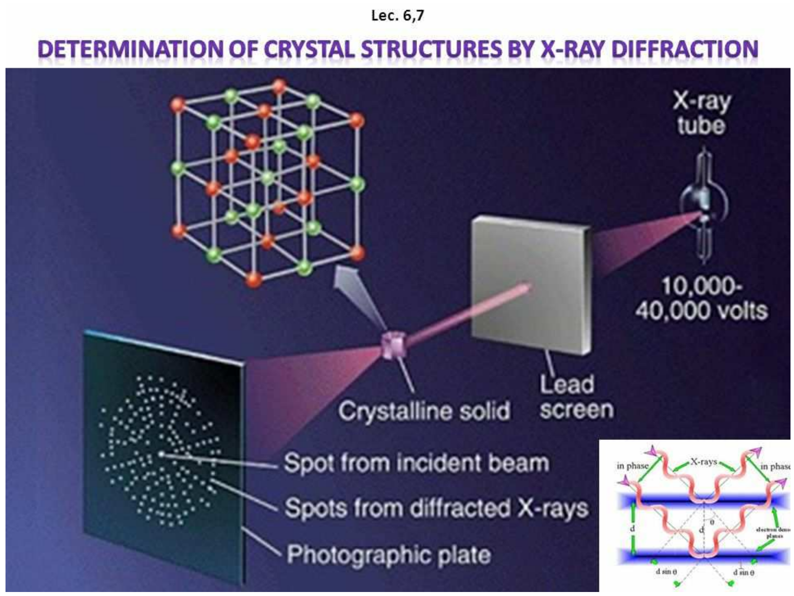


Figure 1.1.6: Diffraction experiments.

In 1984 Dan Shechtman et al. [SBGC84] observe that certain alloys of aluminum and manganese, rapidly cooled, produced an unusual diffraction pattern. Unusual because this had ten-fold rotational symmetries (or five-fold rotational symmetries) and this is not possible to crystals, see Figure 1.1.7. After this, a lot of effort was put into understanding the physical and geometrical properties of solids without the usual symmetries of crystals. In 1992, for the impact of Shechtman's research, the International Union of Crystallography (IUCr) modified the definition of cristal. They redefined *crystal* as solid having a discrete points diffraction pattern with the possibility of it is ordered is periodically or not. Thus, when the solid has not periodic diffraction pattern was called *quasiperiodic crystal* or *quasicrystal*. The diffraction spectrum of a quasicrystal shows bright spots that are not compatible with the rotational symmetries of crystals, and from this one deduces that the atomic positions are ordered but not in a periodic way. By its discovery, Shechtman received the Nobel prize in 2011.

There are two types of quasicrystals known. The quasicrystals with some axis of symmetry (8, 10, or 12-fold symmetry) and another axis quasiperiodic, and the quasicrystals that are aperiodic in all directions. Until the year 2004, only about 100 solids are known to form

[†]https://www.britannica.com/science/crystallography
[‡]http://www.dnaftb.org/19/bio-3.html

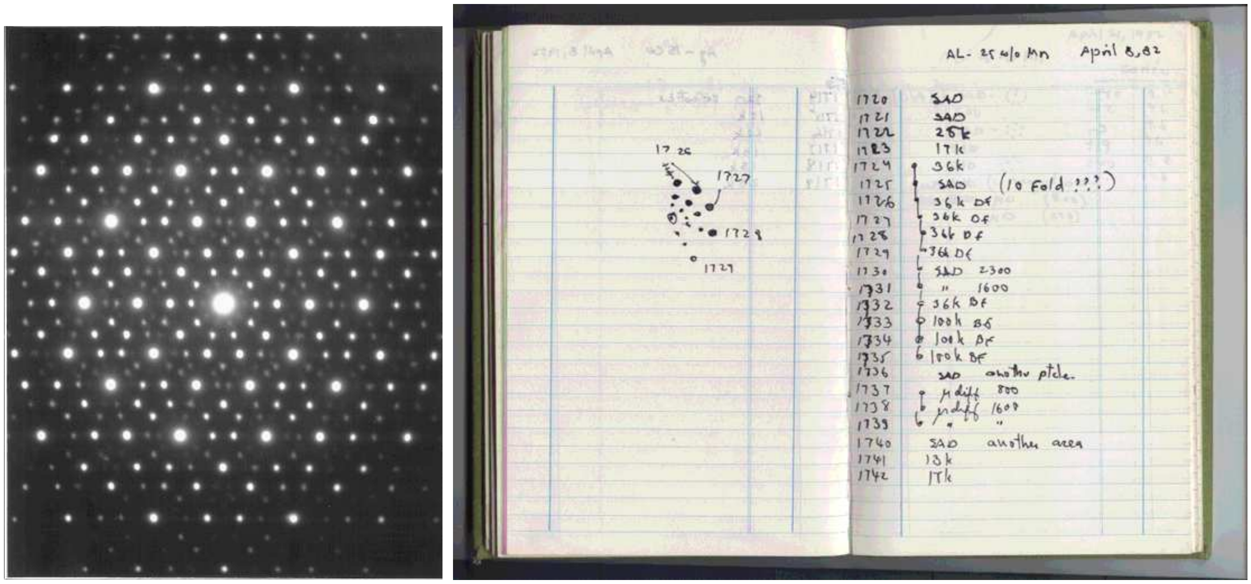


Figure 1.1.7: Pattern diffraction wasfound by Dan Shechtman and a picture of his notebook.

quasicrystals, and about 400,000 form periodic crystals. In 2009 we had the first evidence of the existence of a possible natural quasicrystal, the *Icosahedrite*. This mineral (quasicrystal) was found in the Khatyrka River in eastern Russia, but one year later some analysis indicates it may be meteoritic in origin. The diffraction pattern of the icosahedrite is similar to the first picture in Figure 1.1.7.

1.1.2 Mathematical point of view.

In the previous subsection, we comment that the atomic positions of a crystal in $\mathbb{R}^{\mathrm{d}}$ can be modeled by a discrete set Λ in $\mathbb{R}^{\mathrm{d}}$. But when we think about modeling the atomic position for solids in $\mathbb{R}^{\mathrm{d}}$, we should have two physic considerations. First, each atom occupies a place in space, and second, the atoms extend in all directions. For this reason, a model Λ in $\mathbb{R}^{\mathrm{d}}$ to the atomic position of a solid must be:

Uniformly discrete: there exists $r > 0$ such that for all $x, y \in \Lambda$ with $x \neq y$ we have $\|x - y\|_{\mathrm{d}} > r$.

Relatively dense: there exists a radio $R > 0$ such that every open ball in $\mathbb{R}^{\mathrm{d}}$ of radio R contains at least one point of Λ.

The sets in $\mathbb{R}^{\mathrm{d}}$ that are uniformly discrete and relatively dense are called *Delone* sets. These types of sets can be used to model the atomic positions in a quasicrystals, amorphous solids, and crystals too (since each lattice in $\mathbb{R}^{\mathrm{d}}$ is a Delone set). These sets were called Delone set in honor of the Russian mathematician Boris Delone (or Delauney).

In mathematics, a crystalline solid is a Delone set Λ in $\mathbb{R}^{\mathrm{d}}$ such that its set of periods $\{t \in \mathbb{R}^{\mathrm{d}} \mid \Lambda - t = \Lambda\}$ is a co-compact discrete subgroup of $\mathbb{R}^{\mathrm{d}}$, i.e. Λ is a *lattice* in $\mathbb{R}^{\mathrm{d}}$. In this case, we say that Λ is a *periodic* set. When the set of periods is just the null vector, i.e.

$\{t \in \mathbb{R}^{\mathrm{d}} \mid \Lambda = \Lambda - t\} = \{0\}$, the set Λ is called *aperiodic*. In other cases, we say that Λ is *not periodic*.

To study the structure of a cristal Λ we can study the lattice properties. But to study the chemical and physical properties of the solid we need a mathematical notion of diffraction. The mathematical diffraction of a solid modeled via a Delone set Λ in $\mathbb{R}^{\mathrm{d}}$ can be described following [H95, BLM07]. Let $\{A_n\}_{n\in\mathbb{N}}$ be a *van Hove* sequence, this means that for all n in $\mathbb{N}$ we have that $A_n \subseteq \mathbb{R}^{\mathrm{d}}$ verifies for every compact set K in $\mathbb{R}^{\mathrm{d}}$ that

$$\lim_{n\to\infty} \frac{\mathrm{Leb}\left(((A_n + K) \setminus \mathrm{int}(A_n)) \cup ((\overline{A_n^c} - K) \cap A_n)\right)}{\mathrm{Leb}(A_n)} \quad = \quad 0 \, .$$

For every van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathrm{d}}$, we consider the limit (in the weak*-topology)

$$\gamma = \lim_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sum_{x,y\in\Lambda\cap A_n} \delta_{x-y} \, , \tag{1.1.1}$$

where δ_{x-y} denotes the Dirac measure supported in the set $\{x - y\}$. When this limit exists, γ is called the *autocorrelation measure* of Λ associated with the van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$. Although there could be more than one accumulation point, thus each accumulation point in the weak*-topology, of the expression in (1.1.1) is called autocorrelation measure of Λ.

A Delone set Λ in $\mathbb{R}^{\mathrm{d}}$ has *uniform patch frequencies* if for each finite subset P of Λ and for all $t_0 \in \mathbb{R}^{\mathrm{d}}$, we have that the expression

$$\frac{\mathrm{card}\{t \in \mathbb{R}^{\mathrm{d}} \mid t + P \subseteq \Lambda \cap (t_0 + A_n)\}}{\mathrm{Leb}(A_n)},$$

converges uniformly in t_0 for every van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$. If Λ has uniform patch frequencies the autocorrelation measure exists and does not depend on the choice of the van Hove sequence $\{A_n\}_{n\in\mathbb{N}}$. If the autocorrelation measure exists, its Fourier transform $\widehat{\gamma}$ is a measure called mathematical *diffraction*. By Lebesgue's decomposition theorem, this measure can be decomposed into three parts: pure point, singular continuous, and absolutely continuous.

$$\widehat{\gamma} = \widehat{\gamma}_{pp} + \widehat{\gamma}_{sc} + \widehat{\gamma}_{ac} \, .$$

Following the definition of crystal given by the IUCr in 1992, a good mathematical model to quasicrystal should have mathematical diffraction being pure point, i.e. $\widehat{\gamma} = \widehat{\gamma}_{pp}$.

If Λ is a lattice, using the Poisson summation formula, we have

$$\widehat{\gamma} = \widehat{\gamma}_{pp} = I \cdot \sum_{x\in\Lambda} \delta_x \, ,$$

where I is the diffraction intensity (the number of lattice points per unit volume) see Sect.4 in [H95].

The mathematics of aperiodic Delone set is before quasicrystals. In the 1930s, the Russian school proposed Delone set as the fundamental object to study in crystallography [DAP34]. Since for each tiling in $\mathbb{R}^{\mathrm{d}}$, we can construct a Delone set in $\mathbb{R}^{\mathrm{d}}$ and using Voronoi cell (see

[BBG06, Section 2.2]) it is possible to associate for each Delone set a tiling. The mathematical development of the Delone set theory is related to tiling theory. In 1976 Roger Penrose discover (using tilings) a two-dimensional aperiodic Delone set with five-fold symmetry. One year later, Alan Mackay [Ma82] showed that the mathematical diffraction of this set is as in the Figure 1.1.8.

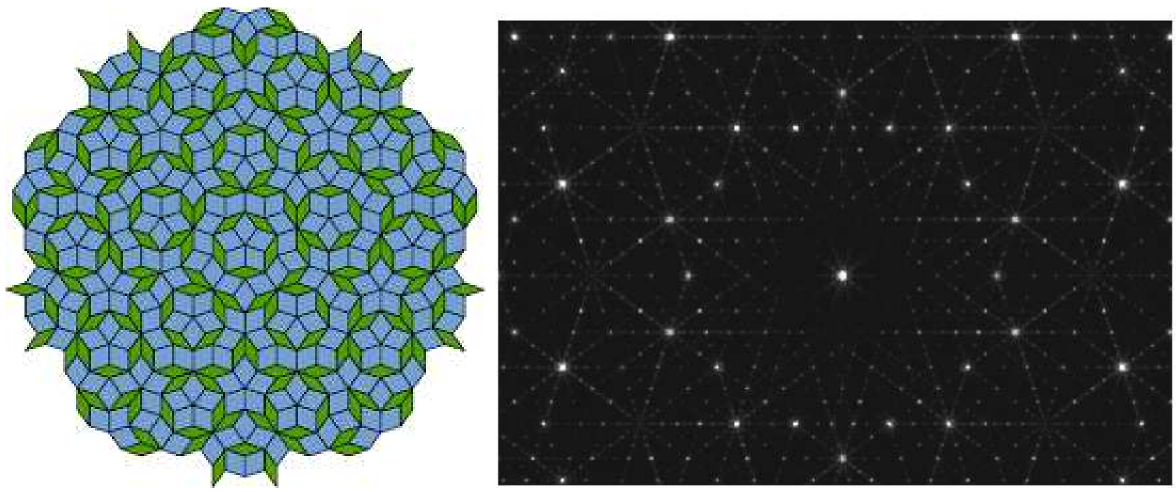


Figure 1.1.8: Penrose tiling and its mathematical diffraction pattern.

One way to study the structure of a Delone set Λ, is done through the study of the topology of its *hull* Ω_Λ. It is the collection of all Delone sets whose patterns are up to translation the same as Λ. See §2.2.1 for more details. The hull is a metric space, where two sets are closed if they agree on a large closed ball up to small translation. On Ω_Λ the group $\mathbb{R}^{\mathrm{d}}$ acts continuously by translation and it defines a dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ called the *hull system*, see §2.2.1 for more details and references. If we suppose that we have a homogeneous sample of the solid and without impurities, we can consider two additional hypotheses for Λ. First, we assume that for every radio $r > 0$ exists only a finite number (up to translation) of patterns of diameter at most r in Λ. This property is called *finite local complexity*, or FLC for short. The metric space Ω_Λ is compact if and only if Λ has the FLC property. Second, we consider that every pattern that appears in Λ it appears with some regularity in Λ. In other words, we suppose Λ is *repetitive*. It means, for every compact K in $\mathbb{R}^{\mathrm{d}}$ the set $\{t \in \mathbb{R}^{\mathrm{d}} \mid (\Lambda - t) \cap K = \Lambda \cap K\}$ is relatively dense in $\mathbb{R}^{\mathrm{d}}$. When Λ has FLC, repetitivity of Λ is equivalent to the minimality of the dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

What is the relation between the mathematical diffraction of a solid with atomic position given by a Delone set Λ and its hull system? The answer is related to the concept of the eigenvalue for the dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ that we define below. Let μ be an $\mathbb{R}^{\mathrm{d}}$-invariant, ergodic measure on Ω_Λ. An *eigenvalue* for the measurable dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}}, \mu)$ is a vector λ in $\mathbb{R}^{\mathrm{d}}$ such that there is a function f in $L^2(\Omega_\Lambda, \mu, \mathbb{C})$ such that for every t in $\mathbb{R}^{\mathrm{d}}$ and μ-almost every $x \in \Omega_\Lambda$ we have

$$f(x - t) = \mathrm{e}^{\mathrm{i}2\pi\langle \lambda, t\rangle} f(x).$$

The function f associated with the eigenvalue λ is called *eigenfunction*. We recall that the dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}}, \mu)$ has a *pure point dynamical spectrum* if the set of eigenfunctions is dense in $L^2(\Omega_\Lambda, \mu, \mathbb{C})$. The following relation between the dynamical spectrum of

$(\Omega_\Lambda, \mathbb{R}^d, \mu)$ and the mathematical diffraction was proved in 2002.

Theorem 1.1.1. *[LMS02, Theorem 3.2] Let Λ be a Delone set with FLC and uniform patch frequencies. Then $\widehat{\gamma}$ is a pure point measure if and only if the hull system has pure point dynamical spectrum.*

This is one of the motivations to study the set of eigenvalues and eigenfunctions of $(\Omega_\Lambda, \mathbb{R}^d, \mu)$. But Delone sets are a class so wild that we cannot ensure that autocorrelation (and hence the diffraction) exists. As the diffraction is a limit of Dirac measures supported in $\Lambda - \Lambda$, we could assume certain regularity in $\Lambda - \Lambda$. A Delone set Λ in $\mathbb{R}^d$ is called a *Meyer* set if $\Lambda - \Lambda$ is a Delone set too. In 1972, Yves Meyer defined these sets based on harmonic analysis and proposed these as models for the atomic positions in a quasicrystals. There are different ways to characterizes Meyer set, see for example [Me72, M97, L99]. For example, in [KS14] the authors characterize the hull system of a Meyer set as the hull system of a Delone set with some dynamical properties.

Thinking about modeling quasicrystals, Meyer sets have their complications. There exist examples of Meyer sets where the autocorrelation does not exist [L99]. There are also examples such that the autocorrelation exists but this is not pure point diffractive [FS07]. Thus, not all Meyer sets are good models for quasicrystals. Because of this, many authors study *model* sets [Me72, H95, L99, S00, FHK02, LM97, A16b]. These sets are a sub-class of Meyer sets that in particular are always pure point diffractive. This was proved by Hof [H95] for repetitive regular inter-model sets with Euclidean internal space. These result, makes the inter-model set a good candidate to model quasicrystals. Inter-model sets in $\mathbb{R}^d$ are a subset of the projection of part of a lattice $\mathcal{L}$ in the *embedding space* $\mathbb{R}^d \times H$, injectively on the *physical space* $\mathbb{R}^d$. Where H is a locally compact, σ-compact Abelian group called *internal space* and the projection of the lattice in H is dense. Meyer and model sets are related by the following fact proved by Yves Meyer in 1972 [Me72]. A discrete set Λ in $\mathbb{R}^d$ is a Meyer set if and only if exists a model set M such that $\Lambda \subseteq M$. In 1981 deBruijn proved that the set of vertices of the Penrose tiling form an inter-model set and thus is a Meyer set [dB81].

1.2 Our contributions.

In the 70's Meyer introduced some discrete sets in $\mathbb{R}^d$ in connection with his work in harmonic analysis, and he observed that each one of these sets, now called Meyer sets can be embedded into another type of discrete set called model set. This last collection is a sub-class of Meyer sets and these sets are defined by a simple geometric construction: they are the projection on the first coordinate of some part of a lattice in $\mathbb{R}^d \times H$ where H, called the internal space, is a locally compact Abelian group.

After the discovery of quasicrystals by D. Shechtman et al. [SBGC84], model sets with Euclidean internal space was proposed as geometric models for the atomic positions in a quasicrystal. Euclidean model sets and their associated dynamical systems played an important role in the mathematical diffraction theory of quasicrystals. Hof in [H95] proved that every repetitive regular inter-model set (see definition in §2) has pure point diffraction and then, Schlottmann in [S00] generalize this result to repetitive regular inter-model sets with arbitrary locally compact Abelian group as internal space.

In a previous work [S98], Schlottmann gave a necessary and sufficient condition on a Delone set for being a general non-singular model set (see definition in §2) in terms of the recurrence structure of the separated net and he asked for a characterization of non-singular model sets with well-behaved internal space for example $\mathbb{R}^n$. We recall that every non-singular model set is a repetitive inter-model set.

Dynamical characterization of repetitive regular inter-model sets was given by Baake, Moody and Lenz in [BLM07] and then, Aujogue [A16] extended this characterization to arbitrary repetitive inter-model sets not necessarily regular. Both results apply to general repetitive inter-model sets but left open the question of characterizing repetitive inter-model sets with Euclidean internal space.

In this thesis, we answer this question by adding an algebraic and a dynamical property to the previous characterizations in [BLM07] and [A16]. The first condition is given in terms of the rank of the Abelian group generated by the set of differences of the Delone set and the second condition is written in terms of a flow on a torus constructed from the address map introduced by Lagarias in [L99], we call this flow the address system.

We recall that every inter-model set is a Meyer set, and all the previous characterizations of inter-model sets are written in the form of what we need to add to a Meyer set to have an inter-model set. Our result state that all the information needed for being an inter-model set with Euclidean internal space is encoded in the rank of the group of differences and the dynamical relation between the dynamical system associated with the Meyer set and the address system.

The results in this thesis are named with capital letters A, B, C, etc. In order to give a more detailed statement of our results we recall some definitions, see §2 for details.

A discrete subset Λ of $\mathbb{R}^{\mathrm{d}}$ is a *Delone set* if it is uniformly discrete and relatively dense. It is *finitely generated* if the Abelian group generated by $\Lambda - \Lambda$ is finitely generated, and it is *repetitive* if every pattern in Λ appears with bounded gaps.

Given a Delone set Λ its *hull* Ω_Λ is defined as the collection of all Delone sets whose local patterns agree with those of Λ up to translation. If Λ has *finite local complexity*, then the hull can be endowed with a topology that is metrizable and compact. The subset of the hull of all Delone sets containing 0 is called the *canonical transversal* of Ω_Λ and we denote it by Ξ_Λ.

The group $\mathbb{R}^{\mathrm{d}}$ acts on the hull continuously by translation, giving a (topological) dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Some combinatorial properties of the Delone set to translate into dynamical properties. For example, repetitivity of Λ is equivalent to minimality of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

It is well known in dynamical systems theory that there is a dynamical system with an equicontinuous action of $\mathbb{R}^{\mathrm{d}}$ that is a factor (semi-conjugacy) of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ and it is maximal to these properties. This dynamical system is unique up to topological conjugacy and we call it the *maximal equicontinuous factor* of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

It is known that repetitivity implies finite local complexity, see for instance [BG13], and that finite local complexity implies finitely generated, see [L99]. A Delone set Λ in $\mathbb{R}^{\mathrm{d}}$ is a *Meyer* set if the set of difference $\Lambda - \Lambda$ is a Delone set.

Let Λ be a finitely generated Delone set in $\mathbb{R}^{\mathrm{d}}$. The *rank* of Λ is the rank of the Abelian group generated by Λ as a subset of $\mathbb{R}^{\mathrm{d}}$. We denote this group by $\langle\Lambda\rangle$, and by s its rank. Let $\mathcal{B}$ be a basis of $\langle\Lambda\rangle$. The *address map* for Λ associated with $\mathcal{B}$, is the coordinate map from $\langle\Lambda\rangle$ to $\mathbb{Z}^s$ with respect to the basis $\mathcal{B}$. Notice that since $\langle\Lambda\rangle$ is an Abelian group and $\langle\Lambda - \Lambda\rangle \subseteq \langle\Lambda\rangle$, we have that: $\langle\Lambda\rangle$ is finitely generated if and only if $\langle\Lambda - \Lambda\rangle$ is finitely generated. Also, observe that if Λ is a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ then for every Λ_0 in Ξ_Λ we have that

$$\langle\Lambda_0\rangle = \langle\Lambda_0 - \Lambda_0\rangle = \langle\Lambda - \Lambda\rangle.$$

Given a basis $\mathcal{B}$ of $\langle\Lambda - \Lambda\rangle$, let $\varphi : \langle\Lambda - \Lambda\rangle \to \mathbb{Z}^s$ be the coordinate map with respect to the basis $\mathcal{B}$. We have that for every Λ_0 in Ξ_Λ the address map of Λ_0 is equal to φ.

Lagarias proved in [L99] that if Λ is a Meyer set then, there is a linear map from $\mathbb{R}^{\mathrm{d}}$ to $\mathbb{R}^s$ whose distance to the address map of Λ is uniformly bounded on the points of Λ. Indeed, this property characterizes Meyer sets. Our first result gives the existence of one linear map that approximates the address map of all Delone sets in Ξ_Λ, and it also gives a linear flow on a torus that we use to characterize inter-model sets with Euclidean internal space.

Put $\|x\|_s$ for the Euclidean norm of x in $\mathbb{R}^s$, and $[x]_{\mathbb{Z}^s}$ to denote the equivalent class of $x \in \mathbb{R}^s$ in $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$.

Proposition A (Address system)**.** *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let s be the rank of $\langle\Lambda - \Lambda\rangle$. Let $\mathcal{B}$ be a basis of $\langle\Lambda - \Lambda\rangle$ and let $\varphi : \langle\Lambda - \Lambda\rangle \to \mathbb{Z}^s$ be the coordinate map with respect to the basis $\mathcal{B}$. There are an injective linear map $\ell : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ and a constant $C > 0$ such that for every Λ_0 in Ξ_Λ and every $t \in \Lambda_0$ we have*

$$\|\varphi(t) - \ell(t)\|_s \leq C.$$

Moreover, there is a linear flow $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ defined by

$$(w, t) \in \mathbb{T}^s \times \mathbb{R}^{\mathrm{d}} \longmapsto w + [\ell(t)]_{\mathbb{Z}^s},$$

and there is a homomorphism $\pi_{Ad} : \Omega_\Lambda \to \mathbb{T}^s$ such that for every Λ' in Ω_Λ and every t in $\mathbb{R}^{\mathrm{d}}$ we have $\pi_{Ad}(\Lambda' - t) = \pi_{Ad}(\Lambda') + [\ell(t)]_{\mathbb{Z}^s}$.

Notice that the dynamical system $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ and the homomorphism π_{Ad} in Proposition A depend on the basis $\mathcal{B}$ chosen, however, if we change the basis then the new system is topologically conjugate to the previous one. We call any of these dynamical systems, the *address system* of Λ, and to the map π_{Ad} the *address homomorphism* of Λ, which are well defined up to topological conjugacy. Observe that each coordinate of π_{Ad} in Proposition A gives a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ onto the circle $\mathbb{T}$, however, the address system is not necessarily a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. The minimality of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ implies that the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ if and only if the address system of

Λ is minimal. Finally, notice that if we denote by A the representative matrix of ℓ in the canonical basis and by A^T the transpose then we have that the address system is minimal if and only if $\mathrm{Ker}(A^T) \cap \mathbb{Z}^s = \{0\}$, which gives a simple way to check the minimality of the address system.

The next theorem is the main result of §3, it characterizes inter-model sets with Euclidean internal space.

Theorem A. *A repetitive Meyer set Λ in $\mathbb{R}^{\mathrm{d}}$ is an inter-model set with Euclidean internal space if and only if rank$(\langle \Lambda - \Lambda \rangle) > \mathrm{d}$ and the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ such that there is one point with a unique preimage under the factor map.*

After some comments from J. Kellendonk and using a result of Paul [P73, Proposition 1.1], we can observe that the address system of a Euclidean inter-model set Λ, is the maximal equicontinuous factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

From Theorem A and [BLM07, Theorem 5], we obtain the following characterization for regular inter-model sets with Euclidean internal space. Observe that if the address system of Λ is minimal it is also uniquely ergodic.

Theorem B. *A repetitive Meyer set Λ in $\mathbb{R}^{\mathrm{d}}$ is a regular inter-model set with Euclidean internal space if and only if rank$(\langle \Lambda - \Lambda \rangle) > \mathrm{d}$ and the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ such that the set of points in the address system with unique preimages under the factor map has full measure for the unique ergodic measure.*

For the proof of Theorem A, given a Meyer set we construct a cut and project scheme with a Euclidean internal space and a window, which we call the "Lagarias CPS" and the "minimal window", respectively. What we actually prove in Theorem A is that if Λ satisfies the necessary condition then it is an inter-model set generated by the Lagarias CPS and the minimal window. Using again [BLM07, Theorem 5] we can give a more explicit version of Theorem B.

Theorem C. *A repetitive Meyer set Λ in $\mathbb{R}^{\mathrm{d}}$ is a regular inter-model set with Euclidean internal space if and only if rank$(\langle \Lambda - \Lambda \rangle) > \mathrm{d}$, the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ such that there is one point with unique preimage under the factor map and the boundary of minimal window of Λ has measure zero.*

In order to put in context our results we mention an application to the theory of unimodular Pisot substitution tilings. In this setting, one can prove that the address system corresponds to the canonical torus and it is a topological factor of the hull of the substitution tiling. We can also prove that the minimal window is the image of the Rauzy fractal by a linear isomorphism. Using Theorems A and C and the fact that the Rauzy fractal has zero measure boundary (see for instance [BST10]) one can give another proof of the following known characterization of pure point unimodular Pisot substitution tilings as regular model sets with Euclidean internal space (see Theorem 7.3, Corollary 9.4, and Remark 18.6 in [BK06]).

Theorem 1.2.1. *Let Ω_Λ be the hull of a unimodular Pisot substitution tiling Λ in $\mathbb{R}$. The* *following are equivalent:*

(i) Ω_Λ *has pure point dynamical spectrum.*

(ii) Ω_Λ *is the hull of a regular model set with Euclidean internal space.*

(iii) There is a point in the address system of Λ *with a unique preimage under the factor map.*

In Chapter 4 we restrict our attention to linearly repetitive Meyer systems. This is a subclass of Meyer system where there exists a constant $L > 1$, such that each ball of radio $L \times \rho$ contains each ρ-patch of Λ. We give a more detailed definition in §4.1. In general, for Meyer systems, there are examples without continuous eigenvalues up to the trivial one. Even in the case of linearly repetitive Meyer system. For example, by associating equal lengths to the scrambled Fibonacci sequence, we can obtain a Meyer system that is pure point diffractive with one continuous eigenvalue (see [KS14]). Also, it is possible to construct another example of a linearly repetitive Meyer system assigning equal lengths in a symbolic sequence constructed in [BDM05]. This system has no continuous eigenvalues up to the trivial one. In general is very difficult to have only continuous eigenvalues for a large class of Delone sets. In [So99], the author proved that each Delone system from a self-affine substitution with some additional properties has only continuous eigenvalues. The main result in §4 is Theorem D. It gives a sufficient condition for a linearly repetitive Meyer system has only continuous eigenvalues. We give some definitions after we state the result in detail.

Let Λ be a repetitive Delone set in $\mathbb{R}^{\mathrm{d}}$. Denote by $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ and Ξ_Λ its hull system and transversal space, respectively. A *clopen* set in Ξ_Λ is a closed and open subset of Ξ_Λ. For every Λ' in Ω_Λ and a clopen set C in Ξ_Λ, we define the set of *return vectors* of Λ' to C as

$$\mathcal{R}_C(\Lambda') := \{t \in \mathbb{R}^{\mathrm{d}} \mid \Lambda' - t \in C\}.$$

This set is a repetitive Delone set in $\mathbb{R}^{\mathrm{d}}$ [C11], and if Λ' is in Ξ_Λ then $\mathcal{R}_C(\Lambda') \subseteq \Lambda'$. In [C09] it is proven the following facts.

Theorem 1.2.2. *Let* Λ *be a linearly repetitive aperiodic Delone set in* $\mathbb{R}^{\mathrm{d}}$ *containing* 0 *and let* μ *be the unique invariant measure for* $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$*. For every* Λ_0 *in* Ξ_Λ*, there is a decreasing sequence of clopen sets* $(C_n)_{n\in\mathbb{N}}$ *in* Ξ_Λ *containing* Λ_0 *and a sequence of finite sets* $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^{\mathrm{d}}$ *verifying that for every* $n \in \mathbb{N}$ *the set*

$$\vec{\mathcal{F}}_n \subseteq \mathcal{R}_{C_n}(\Lambda_0) - \mathcal{R}_{C_n}(\Lambda_0) \text{ and } \mathcal{R}_{C_n}(\Lambda_0) \subseteq \langle \vec{\mathcal{F}}_n \rangle,$$

such that the following property holds: If α *in* $\mathbb{R}^{\mathrm{d}}$ *is an eigenvalue for* $(\Omega_\Lambda, \mu, \mathbb{R}^{\mathrm{d}})$ *then the series*

$$\sum_{n=1}^{\infty} \max_{v\in\vec{\mathcal{F}}_n} \| \langle \alpha, v \rangle \|^2$$

converges. Moreover, one can choose the sequences $(C_n)_{n\in\mathbb{N}}$ *and* $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ *satisfying the following property: There is* M *in* $\mathbb{N}$ *such that for every* n *in* $\mathbb{N}$ *we have that every vector* v *in* $\vec{\mathcal{F}}_{n+1}$ *can be written as an integer linear combination with less than* M *vectors in* $\vec{\mathcal{F}}_n$*.*

We used this result to proved the main theorem in §4.1. This theorem is stated below.

Theorem D. *Let Λ be a linearly repetitive aperiodic Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let μ be the unique invariant measure for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Let Λ_0 be in Ξ_Λ and let $(C_n)_{n\in\mathbb{N}}$ and $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ be sequences associated to Λ_0 as in the statement of Theorem 1.2.2. If for every n in $\mathbb{N}$ there is a base of the group $\langle \mathcal{R}_{C_n}(\Lambda_0)\rangle$ included in $\vec{\mathcal{F}}_n$, then every eigenvalue of $(\Omega_\Lambda, \mu, \mathbb{R}^{\mathrm{d}})$ is continuous.*

Finally, we proved two corollaries from Theorem A. One of them characterizes the maximal equicontinuous factor of the hull system Ω_{MS} associated with a repetitive inter-model set with Euclidean internal space. See §2.3 and §2.2.1 for definitions about model set and dynamical systems.

Proposition B. *Let Ω_{MS} be the hull of the repetitive inter-model sets generated by a Euclidean cut and project scheme $(\mathbb{R}^n, \Gamma, s_{\mathbb{R}^n})$ over $\mathbb{R}^{\mathrm{d}}$ and a window W. Then, for every Λ in Ω_{MS} we have that the group $\langle \Lambda - \Lambda\rangle$ is equal to Γ and its rank is $\mathrm{d} + n$. Moreover, the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugate to the address system of Λ.*

The last corollary is about the continuous eigenvalues for a repetitive Euclidean inter-model set in $\mathbb{R}^{\mathrm{d}}$ with rank $s > \mathrm{d}$. Denote by Ω_{MS} its associated hull. For $\widetilde{\Lambda}$ in the transversal space of Ω_{MS} we denote by ℓ the linear map defined in Proposition A. Its matrix representation is denoted by A. We use a result of minimality for torus flows (see Lemma J), to obtain that the rows of A are rationally independent. Thus, we get the following consequence.

Corollary A. *Let Ω_{MS} be the hull of the repetitive inter-model sets generated by a Euclidean CPS $(\mathbb{R}^n, \Gamma, s_{\mathbb{R}^n})$ over $\mathbb{R}^{\mathrm{d}}$. Consider $\widetilde{\Lambda}$ in the transversal space of Ω_{MS} and denote A the matrix representation of ℓ. Then the rows of A are $n + \mathrm{d}$ rationally independent continuous eigenvalues for $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$.*

The fact that each rank s, repetitive, Euclidean inter-model set has s continuous eigenvalues is well known. For example, it can be obtained from Theorem 2.3.1. This corollary adds an explicit description of these s continuous eigenvalues as the rows of the matrix A.

1.3 Outline of the text.

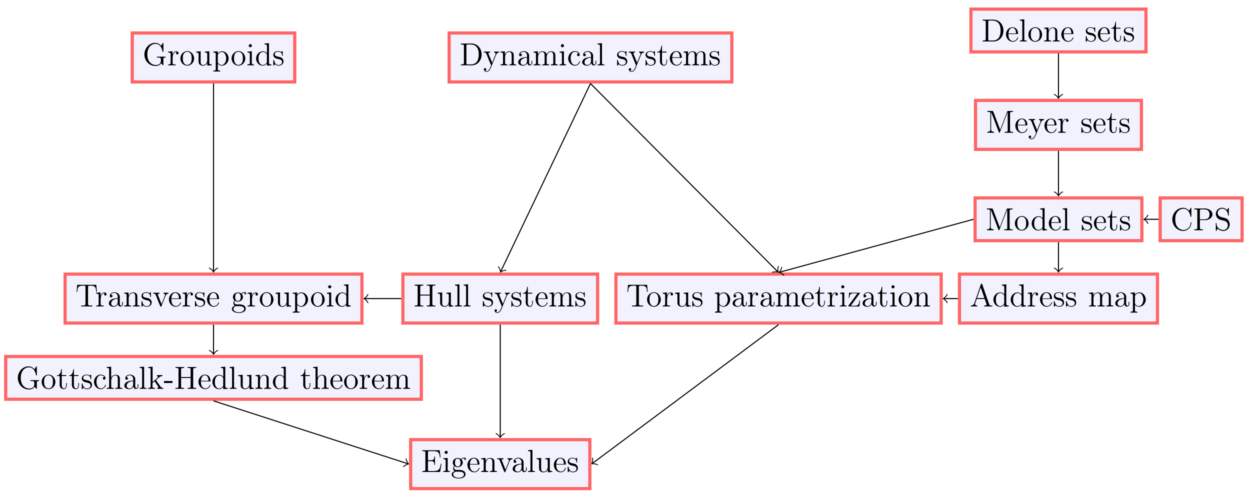
The text is organized as follows: In Chapter 2 we give some basic facts about Delone sets, and we define some objects that we use in the text. In §2.1.1 we define the address map associated with a finitely generated Delone set. After in §2.2.1, we recall the definition of a dynamical system and how we can obtain a dynamical system from a Delone set. Particularly, we define the dynamical hull system associated with a Delone set. In §2.3.1 we define the torus parametrization for some dynamical systems.

In Chapter 3 the main results are stated: Theorem A and Proposition A. The proof of the Proposition A is in §3.1. For its proof, we define the transverse groupoid in §2.2.1. We used Proposition D, proved in §3.1.1, and the address map to define a cocycle in the groupoid. By Lemma C, proved in §2.2.1, we have the minimality of the transversal groupoid. Then we apply a version of the Gottschalk-Hedlund theorem for groupoids to obtain eigenvalues and eigenfunctions for the transverse groupoid. Using Lemma A we extend the eigenfunctions to whole the hull. The proof of this lemma is in §2.2.2. The proof of the Theorem A is in §3.2. The necessary condition uses a characterization of the maximal equicontinuous factor for the hull of a Euclidean model set. This characterization is stated in Proposition B and its proof is in §3.2. The proof of the sufficient condition is in §3.2.2 and it uses two construction: the Lagarias CPS and the Main Technical Lemma. In §3.2.2, we explain the construction of the Lagarias CPS. It associates for each Meyer set a model set that contains it. The Main Technical lemma 3.2.2 is proved at the end of this chapter in §3.3.

In Chapter 4 the main result is Theorem D. The chapter begins with an introduction and comments about Delone sets and topological pure point spectrum (pure point spectrum with only continuous eigenfunctions). After that, we recall some definitions of linearly repetitive Delone sets, and we mention a result that we use in the proof of Theorem D. This result was proved by Daniel Coronel in his Ph.D. thesis [C09]. Finally, in §4.2 is the proof of the main theorem of this chapter. To prove this we define returns vector for a decreasing sequence of clopen in the associated hull system. Using the return vectors we define topological factors of the hull system. To conclude the proof we assume α an eigenvalue of the hull system. We used the recurrence structure and Proposition A to conclude that α is a continuous eigenvalue for some of the topological factors, and then α is a continuous eigenvalue for the hull system.

We show some examples in Chapter 5. These illustrate the technique in Proposition A to find continuous eigenvalues. In §5.1, we find continuous eigenvalues for a Meyer set in the real line. They are obtained from associating rationally independent lengths to an almost linear sequence in m symbols. After in §5.2, we associated lengths for an almost linear sequence to obtain two examples of Meyer sets in $\mathbb{R}$. One of them with one continuous eigenvalue, and another with two rationally independent continuous eigenvalues. In §5.3, we consider a Meyer set from associating lengths for a fixed point of a primitive substitution in m symbols. Finally, we work two examples to show that hypotheses in Theorem D are necessary §5.4.

Some ideas for future work are stated in Chapter 6. Finally, in the appendix, we define a $\mathbb{R}^{\mathrm{d}}$-flow in the s-torus §7. We characterize when this flow is minimal in Lemma J. In Corollary A, we used this to prove that every repetitive, Euclidean inter-model set in $\mathbb{R}^{\mathrm{d}}$ with rank $s > \mathrm{d}$ has s continuous eigenvalues.


Groupoids
Dynamical systems
Delone sets
Meyer sets
Model sets
CPS
Transverse groupoid
Hull systems
Torus parametrization
Address map
Gottschalk-Hedlund theorem
Eigenvalues

Chapter 2

Notations and main concepts

Let $\mathbb{R}^{\mathrm{d}}$ be the Euclidean d-space endowed with its Euclidean norm that we denote by $\|\cdot\|_{\mathrm{d}}$.

2.1 Delone sets.

A subset Λ of $\mathbb{R}^{\mathrm{d}}$ is called a Delone set if it is *uniformly discrete*, meaning that there is $r > 0$ such that every closed ball of radius r intersects Λ in at most one point; and *relatively dense*, which means that there is $R > 0$ such that every closed ball of radius R intersects Λ in at least one point.

Let Λ be a Delone set in $\mathbb{R}^{\mathrm{d}}$. For every $t \in \mathbb{R}^{\mathrm{d}}$, we denote by $\Lambda - t$ the Delone set $\{x - t \mid x \in \Lambda\}$.

For every $\rho > 0$ and every t in $\mathbb{R}^{\mathrm{d}}$ denote by $B(t, \rho)$ the *open ball* in $\mathbb{R}^{\mathrm{d}}$ of radius ρ and center t. A *ρ-patch* of Λ centered at $t \in \mathbb{R}^{\mathrm{d}}$ is the set $\Lambda \cap \overline{B(t, \rho)}$. We consider two notions of long-range order for Delone sets: The first one states that a Delone set Λ has *finite local complexity* if for every $\rho > 0$ it has a finite number of ρ-patches up to translation; and the second says that Λ is *repetitive* if for each $\rho > 0$ there is a number $M > 0$ such that each closed ball of radius M contains the center of a translated copy of every possible ρ-patch of Λ. Observe that every repetitive Delone set has finite local complexity (see [BG13, Proposition 5.6]).

A Delone set Λ is *finitely generated* if the Abelian group generated by $\Lambda - \Lambda$ is finitely generated. We denote by $\langle \Lambda - \Lambda \rangle$ this group. Observe that if $\langle \Lambda - \Lambda \rangle$ has finite rank the group $\langle \Lambda \rangle$ also has finite rank. For every finitely generated Delone set, we define its *rank* as the rank of the group $\langle \Lambda \rangle$. We recall the following proposition proved in [L99].

Proposition 2.1.1. *Let Λ be a Delone set.*

1. *Λ has finite local complexity if and only if $\Lambda - \Lambda$ is a discrete and closed subset of $\mathbb{R}^{\mathrm{d}}$.*
2. *If Λ has finite local complexity if and only if it is finitely generated.*

2.1.1 Meyer sets and address map.

Let Λ be a Delone set in $\mathbb{R}^{\mathrm{d}}$. We recall that Λ is a *Meyer set* if $\Lambda - \Lambda$ is a Delone set. Following [L96], it definition is equivalent to the fact that, there is a finite set F in $\mathbb{R}^{\mathrm{d}}$ such that

$$\Lambda - \Lambda \subseteq \Lambda + F. \tag{2.1.1}$$

Let Λ be a Meyer set in $\mathbb{R}^{\mathrm{d}}$ with rank s, and let $\mathcal{B} \coloneqq \{v_1, \ldots, v_s\}$ be a basis of $\langle\Lambda\rangle$. We recall that the address map for Λ (associated to the basis $\mathcal{B}$) is the map $\varphi : \langle\Lambda\rangle \to \mathbb{Z}^s$ such that to every x in $\langle\Lambda\rangle$ assigns its coordinates in the basis $\mathcal{B}$. The following characterization of Meyer set is used in the proofs of Proposition A and Main Theorem.

Theorem 2.1.1. *[L99, Theorem 3.1] A Delone set Λ in $\mathbb{R}^{\mathrm{d}}$ is a Meyer set if and only if it finitely generated and every address map*

$$\varphi : \langle\Lambda\rangle \to \mathbb{Z}^s,$$

is almost linear, that is, there are a unique linear map $\ell : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ and a constant $C > 0$ such that for every x in Λ we have

$$\|\varphi(x) - \ell(x)\|_s \leq C. \tag{2.1.2}$$

Remark 2.1.2. In the proof of [L99, Theorem 3.1] it was proved that ℓ is some kind of "ideal address map" in the sense that if $\{v_1, \ldots, v_s\}$ is the basis of $\langle\Lambda\rangle$ that we used to define the address map of Λ then for every t in $\mathbb{R}^{\mathrm{d}}$ we have

$$\sum_{\mathrm{i}=1}^{s} \ell_{\mathrm{i}}(t) v_{\mathrm{i}} = t, \tag{2.1.3}$$

where $\ell_{\mathrm{i}}(t)$ denotes the i-coordinate of $\ell(t) \in \mathbb{R}^s$.

2.2 Dynamical systems.

This section is about dynamical system theory. We define a dynamical system and some basic concepts. We explain briefly, how to get a dynamical system from a Delone set. After we give some comments on continuous eigenvalues for dynamical systems.

We consider a *dynamical system* (or *topological dynamical system*) as a couple (X, G) where X is a compact metric space and G is an Abelian group acting on X continuously by homeomorphisms. For each g in G, we denote by $g^* : X \to X$ the homeomorphism that it defines. This homeomorphism is defined by $g^*(x) \coloneqq x - g$, where $x - g$ denotes the action of g in x. This notation is due to the fact that, in this text, we mainly use translation actions. The dynamical system theory studies the set of orbits for this action, i.e. for every $x \in X$ the set $\{x - g \mid g \in G\}$.

A dynamical system (Y, G) is called a *topological factor* of (X, G) if there is an onto and continuous map $\pi : X \to Y$, such that for all $g \in G$ and $x \in X$ we have $\pi(x - g) = \pi(x) - g$.

Let d be a metric on X. The dynamical system (X, G) is called *equicontinuous* if the family $G^* := \{g^* \mid g \in G\}$ is equicontinuous. This means that for every positive real number ϵ there exists $\delta > 0$ such that $\mathrm{d}(x, x') < \delta$ implies $\mathrm{d}(x - g, x' - g) < \epsilon$ for all $g \in G$.

We recall that every topological dynamical system admits a *maximal equicontinuous factor*. That is, a topological factor with an equicontinuous action such that any other equicontinuous factor is a topological factor of it (see for instance [K, BKS12, BK13]). For a topological dynamical system (X, G) we denote by (X_{me}, G) its maximal equicontinuous factor. Given two dynamical systems (X, G) and (Y, G), and a factor map $\pi : (X, G) \to (Y, G)$ we say that π is an *almost automorphic extension*, or that (X, G) is an *almost automorphic extension of* (Y, G), if there is a point in Y with a unique preimage under π.

2.2.1 Hull systems and transverse groupoid.

Let $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ be a Delone set with finite local complexity. The *hull* of Λ is the collection of all Delone sets in $\mathbb{R}^{\mathrm{d}}$ whose ρ-patches, for every $\rho > 0$, are also ρ-patches of Λ up to translation. We denote this set by Ω_Λ. There is a natural metrizable topology on Ω_Λ. Roughly speaking, two Delone sets are close in this topology if they agree on a large ball around the origin up to a small translation. In particular, for every Λ' in Ω_Λ a basis of open neighborhoods for Λ' is given by the following sets: First, for every $R > 0$ put

$$T(\Lambda', R) := \{\widetilde{\Lambda} \in \Omega_\Lambda \mid \widetilde{\Lambda} \cap \overline{B(0, R)} = \Lambda' \cap \overline{B(0, R)}\},$$

and for every $0 < \varepsilon < R/2$ we define the open neighborhood $N(\Lambda', \varepsilon, R)$ of Λ' by

$$N(\Lambda', \varepsilon, R) := \{\Lambda'' \in \Omega_\Lambda \mid \exists \widetilde{\Lambda} \in T(\Lambda', R), \exists t \in B(0, \varepsilon), \Lambda'' = \widetilde{\Lambda} - t\},$$

for more details see for example [S00, FHK02, LM97, KL13]. If Λ has finite local complexity then its hull Ω_Λ is compact. Observe that the action by translation of $\mathbb{R}^{\mathrm{d}}$ on Ω_Λ is continuous. Thus, we obtain a topological dynamical system denote by $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. The *orbit* of x in Ω_Λ is the set $\{x - t \mid t \in \mathbb{R}^{\mathrm{d}}\}$, and a subset A of Ω_Λ is called *invariant* if it is invariant by the action of $\mathbb{R}^{\mathrm{d}}$. The dynamical systems $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ is minimal if and only if the only closed invariant sets are the empty set and the whole space. It is well known that minimality is equivalent to the fact every point has a dense orbit, and in the context of Delone sets repetitivity is equivalent to minimality [LP03, Theorem 3.2].

The *transversal* of the hull is the closed subset

$$\Xi_\Lambda := \{x \in \Omega_\Lambda \mid 0 \in x\} \subseteq \Omega_\Lambda.$$

In general, the restriction of the action of $\mathbb{R}^{\mathrm{d}}$ to Ξ_Λ is not defined. For this reason, to study the dynamical properties of the transversal we introduce the *transverse groupoid*,

$$\mathfrak{G}_\Lambda = \{(x, t) \in \Xi_\Lambda \times \mathbb{R}^{\mathrm{d}} \mid x - t \in \Xi_\Lambda\} \subseteq \Xi_\Lambda \times \mathbb{R}^{\mathrm{d}}.$$

This set, endowed with the induced topology from the product space $\Xi_\Lambda \times \mathbb{R}^{\mathrm{d}}$, has the structure of a topological groupoid (see [R80] for the abstract definition of topological groupoids).

Two elements (x,t) and (z,s) in $\mathfrak{G}_\Lambda$ are composable if and only if $x-t=z$, and the composition of (x,t) and (z,s) is defined by

$$(x,t)\cdot(z,s)=(x,t+s).$$

The inverse map $\cdot^{-1}:\mathfrak{G}_\Lambda\to\mathfrak{G}_\Lambda$ is defined by $(x,t)^{-1}=(x-t,-t)$ and the *domain* $\mathrm{d}:\mathfrak{G}_\Lambda\to\Xi_\Lambda$ and *range* $r:\mathfrak{G}_\Lambda\to\Xi_\Lambda$ maps are defined by

$$\mathrm{d}(x,t)=x \quad \text{and} \quad r(x,t)=x-t.$$

Notice that $\mathrm{d}(\mathfrak{G}_\Lambda)=r(\mathfrak{G}_\Lambda)=\Xi_\Lambda$. In this context, the set Ξ_Λ is called the *unit space* of $\mathfrak{G}_\Lambda$.

We say that a subset E of the unit space is *invariant* by the groupoid $\mathfrak{G}$ if $E=r(\mathrm{d}^{-1}(E))$. We recall the following definition from [R80].

Definition 2.2.1. A groupoid is *minimal* if the only open invariant subsets of its unit space are the empty set and the unit space itself.

Some dynamical properties of the hull system can be set in terms of the transverse groupoid. The following result states the minimality of $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$ from the minimality of the transverse groupoid.

Proposition C. *The topological groupoid $\mathfrak{G}_\Lambda$ is minimal if and only if the dynamical system $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$ is minimal.*

Proof. First, observe that for every subset E of Ξ_Λ we have

$$r\left(\mathrm{d}^{-1}(E)\right)=\{x-t\in\Xi_\Lambda \mid x\in E, t\in x\}. \tag{2.2.1}$$

Assume that the dynamical system $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$ is minimal. Suppose, by contradiction, that $E\subseteq\Xi_\Lambda$ invariant by the groupoid $\mathfrak{G}_\Lambda$. Define

$$\widehat{E}:=\{x-t\in\Omega_\Lambda \mid x\in E, t\in\mathbb{R}\}.$$

We have that $\widehat{E}$ is open in Ω_Λ and by (2.2.1) it is invariant for the $\mathbb{R}^{\mathrm{d}}$-action on Ω_Λ. Then, the complement of $\widehat{E}$ is an invariant non-empty closed set strictly contained in Ω_Λ which contradicts the minimality of $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$.

Reciprocally, suppose that $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$ is not minimal. Let $C\subseteq\Omega_\Lambda$ be an invariant non-empty closed set strictly contained in Ω_Λ. Put $E=C^c\cap\Xi_\Lambda$. By (2.2.1) we have

$$E\subseteq r\left(\mathrm{d}^{-1}(E)\right).$$

Since C is invariant $\mathbb{R}^{\mathrm{d}}$-action, we get $r\left(\mathrm{d}^{-1}(E)\right)=E$. So, E is a non empty open set strictly contained in Ξ_Λ invariant by the groupoid and thus $\mathfrak{G}_\Lambda$ is not minimal. $\square$

2.2.2 Continuous eigenvalues.

We use continuous eigenvalues to define the factor map induced by the address map. In this subsection we recall the definition of continuous eigenvalue for $(\Omega_\Lambda,\mathbb{R}^{\mathrm{d}})$ and for $\mathfrak{G}_\Lambda$, and we study the relationship between them.

Denote by S^1 the circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and by $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^{\mathrm{d}}$. We say that α in $\mathbb{R}^{\mathrm{d}}$ is a *continuous eigenvalue* for the dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ if there is a continuous function $f : \Omega_\Lambda \to S^1$ such that for all $t \in \mathbb{R}^{\mathrm{d}}$ and $x \in \Omega_\Lambda$,

$$f(x - t) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t \rangle} f(x). \tag{2.2.2}$$

The function f is called the *continuous eigenfunction* for the eigenvalue α.

For the transverse groupoid $\mathfrak{G}_\Lambda$, we say that $\alpha \in \mathbb{R}^{\mathrm{d}}$ is a *continuous eigenvalue* for $\mathfrak{G}_\Lambda$, if there is a continuous function $f : \Xi_\Lambda \to S^1$ such that for every $(x, t) \in \mathfrak{G}_\Lambda$,

$$f \circ r(x, t) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t \rangle} f \circ \mathrm{d}(x, t).$$

In this case, we say that f is the *continuous eigenfunction* for α. Observe that each continuous eigenvalue for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ is also a continuous eigenvalue for $\mathfrak{G}_\Lambda$, where the associated eigenfunction is the same function restricted to the transversal. Indeed, we will prove in the following lemma that the set of continuous eigenvalues of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ and $\mathfrak{G}_\Lambda$ are the same.

Some spectral properties can be extended from the transversal space to whole the hull using the structure of transverse groupoid.

Lemma A. *Let $f : \Xi_\Lambda \to S^1$ be a continuous eigenfunction on $\mathfrak{G}_\Lambda$ for the eigenvalue $\alpha \in \mathbb{R}^{\mathrm{d}}$, then f extends to a continuous eigenfunction $\widehat{f} : \Omega_\Lambda \to S^1$ for α on the dynamical system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.*

Proof. Let $\alpha \in \mathbb{R}^{\mathrm{d}}$ be a continuous eigenvalue for $\mathfrak{G}_\Lambda$ with continuous eigenfunction $f : \Xi_\Lambda \to S^1$. We extend this function to a function defined on all Ω_Λ satisfying (2.2.2).

For all $y \in \Omega_\Lambda$, there are $x \in \Xi_\Lambda$ and $t \in \mathbb{R}^{\mathrm{d}}$ such that $y = x - t$. Define $\widehat{f} : \Omega_\Lambda \to S^1$, by

$$\widehat{f}(y) := \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t \rangle} f(x).$$

We prove that $\widehat{f}$ is well defined. Take $y \in \Omega_\Lambda$ and suppose that there are $x_1, x_2 \in \Xi_\Lambda$ and vectors $t_1, t_2 \in \mathbb{R}^{\mathrm{d}}$ such that $y = x_1 - t_1 = x_2 - t_2$. So, $\widehat{f}(y) = \widehat{f}(x_1 - t_1) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t_1 \rangle} f(x_1)$ and since $x_2 - (t_2 - t_1) = x_1 \in \Xi_\Lambda$, we have

$$\widehat{f}(y) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t_1 \rangle} f(x_2 - (t_2 - t_1)) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t_1 \rangle} \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t_2 - t_1 \rangle} f(x_2) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, t_2 \rangle} f(x_2).$$

Therefore, $\widehat{f}$ is well defined. To prove the continuity of $\widehat{f}$, consider a sequence $(y_n)_{n \in \mathbb{N}} \subseteq \Omega_\Lambda$ converging to $y \in \Omega_\Lambda$. By definition of the topology on Ω_Λ, there is a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to 0, and there is a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{\mathrm{d}}$ with $t_n \in B(0, \epsilon_n)$ such that

$$(y_n - t_n) \cap \overline{B(0, \epsilon_n^{-1})} = y \cap \overline{B(0, \epsilon_n^{-1})}.$$

In particular, there exist $n_0 \in \mathbb{N}$ and $t \in \mathbb{R}^{\mathrm{d}}$ such that for all integer $n > n_0$ we have $y_n - (t_n + t), y - t \in \Xi_\Lambda$ and

$$(y_n - (t_n + t)) \cap \overline{B(0, \epsilon_n^{-1} - \|t\|_{\mathrm{d}})} = (y - t) \cap \overline{B(0, \epsilon_n^{-1} - \|t\|_{\mathrm{d}})}.$$

If we denote $x_n = y_n - (t_n + t)$ and $x = y - t$, the previous argument implies that x_n converge to x in Ξ_Λ and as f is continuous in Ξ_Λ we have

$$\lim_{n\to+\infty} f(x_n) = f(x).$$

Since $x_n \in \Xi_\Lambda$ we get $\widehat{f}(x_n) = f(x_n)$ and

$$\lim_{n\to+\infty} \widehat{f}(y_n) = \lim_{n\to+\infty} \mathrm{e}^{2\pi\mathrm{i}\langle\alpha,-(t_n+t)\rangle}\widehat{f}(x_n) = \mathrm{e}^{2\pi\mathrm{i}\langle\alpha,-t\rangle} f(x) = \widehat{f}(x+t) = \widehat{f}(y).$$

This concludes the continuity of $\widehat{f}$. Finally, for all $y \in \Omega_\Lambda$ and $t' \in \mathbb{R}^{\mathrm{d}}$ if we write $y = x - t$ for some x in Ξ_Λ and t in $\mathbb{R}^{\mathrm{d}}$ then we have

$$\widehat{f}(y - t') = \widehat{f}(x - (t + t')) = \mathrm{e}^{2\pi\mathrm{i}\langle\alpha,t'\rangle}\mathrm{e}^{2\pi\mathrm{i}\langle\alpha,t\rangle} f(x) = \mathrm{e}^{2\pi\mathrm{i}\langle\alpha,t'\rangle}\widehat{f}(y).$$

Therefore, $\widehat{f}$ is a continuous eigenfunction for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ associated with the eigenvalue α. $\square$

2.3 Cut and project scheme and inter-model sets.

We recall that a model set is the projection on the first coordinate of some part of a lattice in $\mathbb{R}^{\mathrm{d}} \times H$ where H, called the internal space, is a locally compact Abelian group. Each model set is a Meyer set [Me72], an example of a Meyer set that is not a model set can be found in [K13]. In Theorem A we characterize repetitive Meyer set that are Euclidean model sets, via dynamical properties.

A *cut and project scheme* (CPS) over $\mathbb{R}^{\mathrm{d}}$ is the data (H, L) of a locally compact σ-compact Abelian group H, a discrete set $L \subseteq \mathbb{R}^{\mathrm{d}} \times H$ with compact quotient $(\mathbb{R}^{\mathrm{d}} \times H)/L$ whose first coordinate projection on $\mathbb{R}^{\mathrm{d}}$ is one-to-one and whose second coordinate projection on H is dense. A compact subset W of H that is the closure of its interior is called a *window* for the CPS. In the CPS the space $\mathbb{R}^{\mathrm{d}}$ is called *the physical space*, the locally compact Abelian group H is called *the internal space* and the set L *the lattice*. Following [A16b], we have that a CPS can also be described as a triple (H, Γ, s_H) where H is a locally compact σ-compact Abelian group, Γ a countable subgroup of $\mathbb{R}^{\mathrm{d}}$ and $s_H : \Gamma \to H$ a group morphism with range $s_H(\Gamma)$ dense in H such that the graph

$$\mathcal{G}(s_H) \coloneqq \{(\gamma, s_H(\gamma)) \in \mathbb{R}^{\mathrm{d}} \times H \mid \gamma \in \Gamma\}$$

is a *lattice*, that is, a discrete and co-compact set. Recall that L is a co-compact set if $(\mathbb{R}^{\mathrm{d}} \times H)/L$ is a compact set for the quotient topology. When H is a Euclidean space $\mathbb{R}^n$, for some positive integer n, we say that (H, Γ, s_H) is a *Euclidean CPS.*

Let (H, Γ, s_H) be a CPS with window W. For every w in H, the projection on $\mathbb{R}^{\mathrm{d}}$ of the set $\mathcal{G}(s_H) \cap (\mathbb{R}^{\mathrm{d}} \times (w + W))$ is called a *model set.* More generally, for every window W in H and every w in H denote by $\curlywedge(w + W)$ the model set

$$\curlywedge(w + W) \coloneqq \{t \in \Gamma \mid s_H(t) \in w + W\}.$$

In general, for every subset V of H, we put $\curlywedge(V) = \{t \in \Gamma \mid s_H(t) \in V\}$. Let (H, Γ, s_H) be a CPS and consider $V \subseteq H$ a relatively compact set. By definition, there exists a compact set C in H containing V. Recall that $\mathcal{G}(s_H)$ is a lattice. For each closed ball of radius $r > 0$, B_r, the rectangle $B_r \times C$ contains finitely many points of $\mathcal{G}(s_H)$. Because $\mathcal{G}(s_H)$ has FLC, there are only a finite number of different patches (up to translations) in a rectangle of the type $B_r \times C$. Hence for each $r > 0$, there is a finite number of different r-patches in $\curlywedge(V)$ (up to translations). Thus, for each relatively compact set V the set $\curlywedge(V)$ has finite local complexity (FLC). In particular, $\curlywedge(V)$ is uniformly discrete.

Denote by p_1 and p_2 the projections in the first and second coordinate for the CPS. Let V be an open set in H. By density of $p_2(\mathcal{G}(s_H))$ in H, there is a compact set $K \subseteq \mathbb{R}^{\mathrm{d}}$ such that $\mathbb{R}^{\mathrm{d}} \times H = \mathcal{G}(s_H) + (K \times V)$. Hence $\mathbb{R}^{\mathrm{d}} = \curlywedge(V) + K$, and we have that $\curlywedge(V)$ is relatively dense. We will assume that V is relatively compact set, with non-empty interior. Both conditions ensure that $\curlywedge(V)$ is uniformly discrete and relatively dense (a Delone set).

Remark 2.3.1. For each Delone set $\Lambda \subset \curlywedge(V)$ we have that $\Lambda - \Lambda \subseteq \curlywedge(V) - \curlywedge(V) \subseteq \curlywedge(V - V)$. Since V is relatively compact, $V - V$ is relatively compact and hence $\Lambda - \Lambda$ is uniformly discrete. In a similar way, any set of the form $\Lambda \pm \cdots \pm \Lambda$ (with finitely many terms) is uniformly discrete.

Observe that for any Delone set $\Lambda \subseteq \curlywedge(V)$, there exists a compact set $K \subseteq \mathbb{R}^{\mathrm{d}}$ such that

$$\Lambda + K = \mathbb{R}^{\mathrm{d}}. \tag{2.3.1}$$

Using Remark 2.3.1, the set $F := (\Lambda - \Lambda - \Lambda) \cap K$ is finite. By (2.3.1), for all x, y in Λ there exist $z \in \Lambda$ and $k \in K$ such that $x - y = z + k$. Thus $k = x - y - z$, and so k is in $\Lambda - \Lambda - \Lambda$. We conclude that for each pair of elements x, y in Λ there exist $z \in \Lambda$ and $k \in F$ such that $x - y = z + k$. Thus $\Lambda - \Lambda \subseteq \Lambda + F$, and therefore Λ is a Meyer set.

If V is a closed set, $\overline{s_H(\curlywedge(V))} \subseteq \overline{V} = V$. When V is the closure of its interior, we have that $\overline{s_H(\curlywedge(V))} = V$. In fact, let V be a set that is the closure of its interior. As a consequence V° is non-empty. By definition of CPS we have $s_H(\curlywedge(V)) = p_2(\mathcal{G}(s_H)) \cap V$. This gives

$$p_2(\mathcal{G}(s_H)) \cap V^\circ = s_H(\curlywedge(V)) \cap V^\circ \subseteq s_H(\curlywedge(V)).$$

Since V is the closure of its interior, we get $V = \overline{p_2(\mathcal{G}(s_H)) \cap V^\circ} \subseteq \overline{s_H(\curlywedge(V))}$. In what follows we assume that V is compact and it is the closure of its interior. In particular $\curlywedge(V)$ is a Meyer set such that $\overline{s_H(\curlywedge(V))} = V$.

Example 2.3.2. Let L be the lattice in $\mathbb{R}^2$ generated by vectors $\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$. Note that the first coordinate projection is one-to-one and the second coordinate projection is dense. Denote $\Gamma = \{n\sqrt{2} + m \in \mathbb{R} \mid n, m \in \mathbb{Z}\}$ and $W = [-1, 1] \subseteq \mathbb{R}$. Consider the group morphism $s_{\mathbb{R}} : \Gamma \to \mathbb{R}$ defined by $s_{\mathbb{R}}(n\sqrt{2} + m) = n - m\sqrt{2}$. An example of model set is

$$\begin{aligned} \curlywedge(W) &:= \{t \in \Gamma \mid s_H(t) \in W\} \\ &= \left\{ n\sqrt{2} + m \in \mathbb{R} \mid -1 \leq n - m\sqrt{2} \leq 1 \right\}. \end{aligned}$$

If $V = [1, \infty)$ the set $\curlywedge([0, \infty))$ is non uniformly discrete and dense. We observe that $\curlywedge(V)$ is not a Delone set.

Definition 2.3.3. Let (H, Γ, s_H) be a CPS over $\mathbb{R}^{\mathrm{d}}$ with window W. A Delone set $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ is called *inter-model* set if exist $t \in \mathbb{R}^{\mathrm{d}}$ and $w \in H$ such that

$$\curlywedge(w + \operatorname{int}(W)) - t \subseteq \Lambda \subseteq \curlywedge(w + W) - t.$$

We say that an inter-model set Λ is *non-singular* or *generic* if there is (t, w) in $\mathbb{R}^{\mathrm{d}} \times H$ such that

$$\curlywedge(w + \operatorname{int}(W)) - t = \Lambda = \curlywedge(w + W) - t.$$

Observe that this is equivalent to the fact that the boundary of $w + W$ does not intersect the projection of $\mathcal{G}(s_H)$ in H. Additionally, if the boundary of $w + W$ has zero Haar measure we say the inter-model set is *regular*.

Remark 2.3.4. Notice that ∂W has an empty interior, and thus for each γ^* in $S_H(\Gamma)$ the complement of $\gamma^* - \partial W$ is an open and dense set. Since $s_H(\Gamma)$ is countable, Baire Theorem implies that for all t in $\mathbb{R}^{\mathrm{d}}$ the set

$$NS := H \setminus \bigcup_{\gamma^* \in s_H(\Gamma)} \gamma^* - \partial W = \bigcap_{\gamma^* \in s_H(\Gamma)} (\gamma^* - \partial W)^c$$

is a dense G_δ-set in H. Moreover, for every w in H, the boundary of $w + W$ does not intersect the projection of $\mathcal{G}(s_H)$ in H if and only if $w \in NS$. In particular, for every (t, w) in $\mathbb{R}^{\mathrm{d}} \times H$, the set $\curlywedge(w + W) - t$ is a non-singular inter-model set if and only if $w \in NS$.

The following two results are folklore.

Proposition 2.3.5. *Let $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ be a CPS over $\mathbb{R}^{\mathrm{d}}$ with window W. The class of generic model sets generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and window W gives a unique hull, denoted by Ω_{MS}. Every inter-model set generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and the window W is repetitive if and only if it belongs to Ω_{MS}. In particular, for every repetitive inter-model set Λ generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and the window W we have that $\Omega_\Lambda = \Omega_{MS}$, and the dynamical system $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is minimal.*

Let (H, Γ, s_H) be a CPS in $\mathbb{R}^{\mathrm{d}}$ and consider the set $\mathbb{T}_{\mathcal{G}} := (\mathbb{R}^{\mathrm{d}} \times H)/\mathcal{G}(s_H)$ with an action of $\mathbb{R}^{\mathrm{d}}$ given by translation on the first coordinate. More precisely, for every $u \in \mathbb{R}^{\mathrm{d}}$ and every $[(t, w)] \in \mathbb{T}_{\mathcal{G}}$ the action of u on $[(t, w)]$ is

$$[(t, w)] \cdot u := [(t, w)] + [(u, 0)].$$

We say that W' is *irredundant* if the equation $W' + w = W'$ holds only for $w = 0$ in H.

Theorem 2.3.1. *Let $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ be a CPS over $\mathbb{R}^{\mathrm{d}}$, let W be an irredundant window, and let Ω_{MS} be the hull of the repetitive inter-model sets generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and W. Then, every point in Ω_{MS} is an inter-model set, and there exists a factor map $\pi : \Omega_{MS} \to \mathbb{T}_{\mathcal{G}}$ such that for every Λ' in Ω_{MS} there is (t, w) in $\mathbb{R}^{\mathrm{d}} \times H$ such that $\pi(\Lambda') = [(t, w)]$ if and only if*

$$\curlywedge(w + int(W)) - t \subseteq \Lambda' \subseteq \curlywedge(w + W) - t. \tag{2.3.2}$$

Moreover, the map π is injective precisely on the subset of non-singular inter-model sets in Ω_{MS} and the dynamical system $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^{\mathrm{d}})$ is the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$.

The proof of the Theorem 2.3.1 is mainly in [S00]. The proof that $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^{\mathrm{d}})$ is the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ follows from the fact that $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^{\mathrm{d}})$ is an equicontinuous factor and from the existence of points where π is injective (see for instance [ABKL15, Lemma 3.11]).

2.3.1 Torus parametrization.

The notion of torus parametrization was introduced in [BHP97]. Here, we recall its definition and some properties we will use later. Let X be a compact space and let $(X, \mathbb{R}^{\mathrm{d}})$ be a topological dynamical system under the action of $\mathbb{R}^{\mathrm{d}}$ by the homeomorphisms $\{\rho_t\}_{t\in\mathbb{R}^{\mathrm{d}}}$. Consider a compact Abelian group $\mathbb{K}$ with a minimal action of $\mathbb{R}^{\mathrm{d}}$ by homeomorphisms $\{\kappa_t\}_{t\in\mathbb{R}^{\mathrm{d}}}$. A *torus parametrization* is a continuous map $\pi : X \to \mathbb{K}$ such that for all $t \in \mathbb{R}^{\mathrm{d}}$ and $x \in X$ we have

$$\kappa_t \circ \pi(x) = \pi \circ \rho_t(x).$$

For more details see [BLM07, S00]. We recall the following lemma.

Lemma 2.3.6. *[BLM07, Lemma 1] If $\pi : X \to \mathbb{K}$ is a torus parametrization then π is onto.*

Let $\pi : X \to \mathbb{K}$ be a torus parametrization. A *section* of π is a map $s : \mathbb{K} \to X$ such that $\pi \circ s$ is the identity on $\mathbb{K}$. A point $x \in X$ is called *singular* if the fiber $\pi^{-1}(\pi(x))$ contains more than one element. Otherwise, $x \in X$ is *called non-singular*. The set of non-singular points of X for π is denoted $R_\pi(X)$. The following proposition was proved in [BLM07].

Proposition 2.3.7. *[BLM07, Proposition 3] Let $\pi : X \to \mathbb{K}$ be a torus parametrization and let s be a section of π. Then s is continuous at all points of $\pi(R_\pi(X))$.*

Chapter 3

Euclidean Model sets and continuous eigenvalues for Meyer sets

In this chapter, we prove Proposition A and Theorem A. The proof of the main result (Theorem A) is in §3.2 and it uses two results: the Address system (Proposition A) and the Main Technical Lemma in §3.3. In §3.1 we use the address map to define a continuous cocycle in the transverse groupoid. Using Gottschalk-Hedlund's Theorem for groupoids [R12], we proved Proposition A. In §3.2.2 we describe the Lagarias CPS. In the last section §3.3, we use the Lagarias CPS and some ideas from [A16b] to prove a general version of the Main Technical Lemma.

3.1 The Address system.

In this section, we prove the following result.

Proposition A. *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let s be the rank of $\langle \Lambda - \Lambda \rangle$. Let $\mathcal{B}$ be a basis of $\langle \Lambda - \Lambda \rangle$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^s$ be the coordinate map for the basis $\mathcal{B}$. There are an injective linear map $\ell : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ and a constant $C > 0$ such that for every Λ_0 in Ξ_Λ and every $t \in \Lambda_0$ we have*

$$\|\varphi(t) - \ell(t)\|_s \leq C.$$

Moreover, there is a linear flow $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ defined by

$$(w, t) \in \mathbb{T}^s \times \mathbb{R}^{\mathrm{d}} \longmapsto w + [\ell(t)]_{\mathbb{Z}^s},$$

and there is a homomorphism $\pi_{Ad} : \Omega_\Lambda \to \mathbb{T}^s$ such that for every Λ' in Ω_Λ and every t in $\mathbb{R}^{\mathrm{d}}$ we have $\pi_{Ad}(\Lambda' - t) = \pi_{Ad}(\Lambda') + [\ell(t)]_{\mathbb{Z}^s}$.

Given a repetitive Meyer set Λ in $\mathbb{R}^{\mathrm{d}}$, we start defining a continuous and bounded cocycle in the transverse groupoid of Λ. We use a version of Gottschalk-Hedlund's theorem for groupoids to show that this cocycle is a coboundary. This gives d continuous eigenvalues of the transverse groupoid. We use these continuous eigenvalues and their associated eigenfunctions to construct an equicontinuous dynamical system and homomorphism from $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ into this equicontinuous system.

3.1.1 Defining a cocycle on the groupoid.

Let $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ be a repetitive Meyer set. If $\mathcal{B} = \{v_1, \ldots, v_s\} \subseteq \mathbb{R}^{\mathrm{d}}$ be a basis for $\langle \Lambda - \Lambda \rangle \subseteq \langle \Lambda \rangle$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^{\mathrm{d}}$ be the coordinate map for the basis $\mathcal{B}$. Recall that by the repetitivity of Λ for every $x \in \Xi_\Lambda$ we have that $\langle x \rangle = \langle x - x \rangle = \langle \Lambda - \Lambda \rangle$ and thus, the address map of x associated to $\mathcal{B}$ is equal to φ. Note that for all t and t' in $\langle \Lambda - \Lambda \rangle$ we have

$$\varphi(t + t') = \varphi(t) + \varphi(t'). \tag{3.1.1}$$

From Theorem 2.1.1, for every $x \in \Xi_\Lambda$ there is a unique linear map $\ell_x : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ approaching the images of φ in the point of x, i.e. such that

$$\xi_x \coloneqq \sup_{t \in x} \|\varphi(t) - \ell_x(t)\|_s < +\infty. \tag{3.1.2}$$

We define the maps $\Phi : \mathfrak{G}_\Lambda \to \mathbb{Z}^s$ and $L : \mathfrak{G}_\Lambda \to \mathbb{R}^s$ as follows: for every $(x, t) \in \mathfrak{G}_\Lambda$,

$$\Phi(x, t) := \varphi(t) \text{ and } L(x, t) \coloneqq \ell_x(t).$$

This subsection aims to show that $L - \Phi$ define a continuous cocycle on $\mathfrak{G}_\Lambda$. For this, we first prove that L does not depend on the first coordinate. The proof of the continuity is at the end of the subsection.

Proposition D. *There is a linear map $\ell : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ such that for all $(x, t) \in \mathfrak{G}_\Lambda$ we have $L(x, t) = \ell(t)$.*

The proof of this proposition is given at the end of this subsection after some lemmas.

Lemma B. *Let Λ' be a relatively dense set in $\mathbb{R}^{\mathrm{d}}$. The set $\left\{ \frac{t}{\|t\|_{\mathrm{d}}} \mid t \in \Lambda' \right\}$ is dense in the boundary of the Euclidean unitary ball centered on the origin. In particular, for all linear map $T : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ we have that*

$$\|T\|_{op} = \sup_{t \in \Lambda'} \left\| T\left(\frac{t}{\|t\|_{\mathrm{d}}} \right) \right\|_s,$$

where $\|\cdot\|_{op}$ is the operator norm.

Proof. Put $D \coloneqq \left\{ \frac{t}{\|t\|_{\mathrm{d}}} \mid t \in \Lambda' \right\}$. By contradiction suppose the set D is not dense in the boundary of $B(0, 1)$. So, there exists an open set in the relative topology which contains no elements of D. If we project this open set towards infinity, it generates a cone that contains Euclidean balls of size arbitrarily large and where there are no points of Λ'. This contradicts the fact that Λ' is relatively dense. □

Lemma C. *For all $(x,t) \in \mathfrak{G}_\Lambda$, we have $\ell_x = \ell_{x-t}$.*

Proof. Fix (x,t) in $\mathfrak{G}_\Lambda$. Let $u \in \mathbb{R}^{\mathrm{d}}$ be such that $u \in x - t$. In particular, $t + u \in x$. By (3.1.2) we have

$$\|\varphi(u) - \ell_{x-t}(u)\|_s \leq \xi_{x-t} \quad \text{and} \quad \|\varphi(u) - \ell_x(t+u)\|_s \leq \xi_x.$$

Using these inequalities and (3.1.1), we get

$$\begin{aligned}\|\ell_x(t+u) - \ell_{x-t}(u)\|_s &\leq \|\varphi(t+u) - \ell_x(t+u)\|_s + \|\varphi(t+u) - \ell_{x-t}(u)\|_s \\ &\leq \xi_x + \|\varphi(t) + \varphi(u) - \ell_{x-t}(u)\|_s \\ &\leq \xi_x + \|\varphi(t)\|_s + \xi_{x-t}.\end{aligned}$$

Dividing by $\|u\|_{\mathrm{d}}$ into both sides of this last inequality, we obtain

$$\left\|\ell_x\left(\frac{t}{\|u\|_{\mathrm{d}}}\right) + \ell_x\left(\frac{u}{\|u\|_{\mathrm{d}}}\right) - \ell_{x-t}\left(\frac{u}{\|u\|_{\mathrm{d}}}\right)\right\|_s \leq \frac{\xi_x + \|\varphi(t)\|_s + \xi_{x-t}}{\|u\|_{\mathrm{d}}}.$$

Taking the limit when $\|u\|_{\mathrm{d}} \to +\infty$ we have

$$\lim_{\substack{\|u\|_{\mathrm{d}} \to +\infty \\ u \in x-t}} \left\|(\ell_x - \ell_{x-t})\left(\frac{u}{\|u\|_{\mathrm{d}}}\right)\right\|_s = 0.$$

This together with Lemma B implies that $\|\ell_x - \ell_{x-t}\|_{op} = 0$, and thus, concludes the proof of the lemma. $\square$

Proof of Proposition D. Fix y in Ξ_Λ. We prove that for every x in Ξ_Λ we have $\ell_x = \ell_y$. By (3.1.1), (3.1.2) and Lemma C, for t' in y we have

$$\begin{aligned}\xi_{y-t'} &= \sup_{t \in y-t'} \|\varphi(t) - \ell_{y-t'}(t)\|_s \\ &= \sup_{t \in y-t'} \|\varphi(t) - \ell_y(t)\|_s \\ &= \sup_{t+t' \in y} \|\varphi(t+t') - \varphi(t') - \ell_y(t)\|_s \\ &= \sup_{t+t' \in y} \|\varphi(t+t') - \varphi(t') - \ell_y(t+t'-t')\|_s \\ &= \sup_{t+t' \in y} \|\varphi(t+t') - \varphi(t') - \ell_y(t+t') + \ell_y(t')\|_s \\ &\leq 2\,\xi_y.\end{aligned} \tag{3.1.3}$$

Fix x in Ξ_Λ. By minimality, there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathrm{d}}$ such that $y - t_n$ converges to x in Ξ_Λ. Fix $t \in x$ and consider $\epsilon > 0$ such that $\|t\| \leq \frac{1}{\epsilon}$. There is $N \in \mathbb{N}$ such that for all $n > N$ we have

$$(y - t_n) \cap \overline{B\left(0, \frac{1}{\epsilon}\right)} = x \cap \overline{B\left(0, \frac{1}{\epsilon}\right)}.$$

In particular, for all $n > N$ we get $t \in y - t_n$. Then, using Lemma C and (3.1.3), for every t in x we have

$$\|\varphi(t) - \ell_y(t)\|_s = \|\varphi(t) - \ell_{y-t_n}(t)\|_s \leq 2\,\xi_y.$$

By uniqueness of the map ℓ_x, we conclude the proof of the proposition.

$\square$

Let H be an Abelian group. A *cocycle* on the topological groupoid $\mathfrak{G}_\Lambda$ with values in H is a map $c : \mathfrak{G}_\Lambda \to H$ such that for all composable pairs (x, t) and (z, s) in $\mathfrak{G}_\Lambda$ one has

$$c((x,t) \cdot (z,s)) = c((x,t)) + c((z,s)).$$

Lemma D. *The map $L - \Phi$ is a continuous cocycle on $\mathfrak{G}_\Lambda$.*

Proof. By (3.1.1) and Proposition D we have that $L - \Phi$ is a cocycle. Now we prove the continuity of $L - \Phi$. Consider a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ in $\mathfrak{G}_\Lambda$ that converges to (x, t) in $\mathfrak{G}_\Lambda$. By definition of convergence in the groupoid, we have that $\{x_n\}_{n\in\mathbb{N}} \subseteq \Xi_\Lambda$ converges to $x \in \Xi_\Lambda$, and $\{t_n\}_{n\in\mathbb{N}}$ converges to t in $\mathbb{R}^{\mathrm{d}}$. Let ϵ be a positive real number less than the uniformly discrete radius of Λ such that $\|t\|_{\mathrm{d}} < \frac{1}{2\epsilon}$. There is a positive integer N such that for all $n \geq N$ we have

$$x_n \cap \overline{B\left(0, \frac{1}{\epsilon}\right)} = x \cap \overline{B\left(0, \frac{1}{\epsilon}\right)}, \|t_n - t\|_{\mathrm{d}} < \epsilon \text{ and } \|t_n\|_{\mathrm{d}} < \frac{1}{\epsilon}. \tag{3.1.4}$$

By definition of the groupoid $\mathfrak{G}_\Lambda$, for all n in $\mathbb{N}$ we have that $t_n \in x_n$, and also $t \in x$. By (3.1.4), for every $n \geq N$ we get $t_n = t$. Then, for every $n \geq N$ we have

$$L(t_n) = \ell(t) \quad \text{and} \quad \Phi(x_n, t_n) = \varphi(t_n) = \varphi(t) = \Phi(x, t),$$

which implies the continuity of $L - \Phi$.

$\square$

3.1.2 Proof of Proposition A.

We use the following version of Gottschalk-Hedlund's Theorem, due to Jean Renault, to find continuous eigenvalues of $\mathfrak{G}_\Lambda$. This version is adapted to our context from [R80, Theorem 1.4.10] and it appears in [R12].

Theorem 3.1.1. *Let G be a minimal topological groupoid with compact unit space X. For a continuous cocycle $c : G \to \mathbb{R}^{\mathrm{d}}$ the following properties are equivalent:*

1. *There exists a continuous function $g : X \to \mathbb{R}^{\mathrm{d}}$ such that*

$$c = g \circ r - g \circ \mathrm{d}.$$

2. *There exists $x \in X$ such that $c(\mathrm{d}^{-1}(x))$ is relatively compact.*
3. *$c(G)$ is relatively compact.*

Proof of Proposition A. Let $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ be a repetitive Meyer set. Let $\mathcal{B} = \{v_1, \dots, v_s\} \subseteq \mathbb{R}^{\mathrm{d}}$ be a basis for $\langle \Lambda - \Lambda \rangle$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^{\mathrm{d}}$ be the coordinate map for the basis $\mathcal{B}$. Let L and Φ be as in §3.1.1. We check that $\mathfrak{G}_\Lambda$ and the cocycle $L - \Phi : \mathfrak{G}_\Lambda \to \mathbb{R}^s$ verify the hypotheses of Theorem 3.1.1). By Proposition C the groupoid is minimal. By Lemma D, the map $L - \Phi$ is a continuous cocycle. Let ℓ be the linear map given by Proposition D. By (3.1.2), for every $x \in \Xi_\Lambda$ the set

$$(L - \Phi)(\mathrm{d}^{-1}(x)) = \{\ell(t) - \varphi(t) \mid t \in x\}$$

is bounded. By Theorem 3.1.1, there is a continuous map $F : \Xi_\Lambda \to \mathbb{R}^s$ such that for every (x, t) in $\mathfrak{G}_\Lambda$ we have

$$\ell(t) - \varphi(t) = L(x,t) - \Phi(x,t) = F \circ r(x,t) - F \circ \mathrm{d}(x,t) = F(x-t) - F(x). \tag{3.1.5}$$

Since F is continuous and the space Ξ_Λ is compact there is a constant $C > 0$ such that the inequality in the first part of Proposition A holds.

Now we check that ℓ is injective. By contradiction suppose that the kernel of ℓ has dimension greater than one. Hence, there is an infinite subset of Λ such that the address map is bounded on this infinite set, which gives a contradiction.

Finally, we construct the address system. Denote by $\mathbb{T}^s$ the torus $\mathbb{R}^s/\mathbb{Z}^s$. Since ℓ is linear the following map defines an equicontinuous action of $\mathbb{R}^{\mathrm{d}}$ on $\mathbb{T}^s$:

$$(w, t) \in \mathbb{T}^s \times \mathbb{R}^{\mathrm{d}} \longmapsto w + [\ell(t)]_{\mathbb{Z}^s}.$$

Now we define $\pi_{\mathrm{Ad}} : \Omega_\Lambda \to \mathbb{T}^s$ as follows: For every $y \in \Omega_\Lambda$ there exist $x \in \Xi_\Lambda$ and $t \in \mathbb{R}^{\mathrm{d}}$ such that $y = x - t$, put

$$\pi_{\mathrm{Ad}}(y) \coloneqq [F(x)]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s}.$$

We verify that π_{Ad} is well defined. Indeed, suppose that for $y \in \Omega_\Lambda$ there are $x_1, x_2 \in \Xi_\Lambda$ and $t_1, t_2 \in \mathbb{R}^{\mathrm{d}}$ such that $y = x_1 - t_1 = x_2 - t_2$. Thus, $x_1 = x_2 - (t_2 - t_1)$, and by (3.1.5) we have that

$$F(x_1) = F(x_2) + \ell(t_2 - t_1) - \varphi(t_2 - t_1),$$

which is equivalent to

$$F(x_1) + \ell(t_1) = F(x_2) + \ell(t_2) - \varphi(t_2 - t_1).$$

Together with the fact that $\varphi(t_2 - t_1) \in \mathbb{Z}^s$, this implies that π_{Ad} is well defined. Now we prove the continuity of π_{Ad}. Fix $y \in \Omega_\Lambda$ and suppose that $y = x - t$ for some $x \in \Xi_\Lambda$ and $t \in \mathbb{R}^{\mathrm{d}}$. For every y' close to y there is x' in Ξ_Λ close to x and there is t' close to t such that $y' = x' - t'$. By the continuity of F and ℓ, the map $\widetilde{\pi}_{\mathrm{Ad}}$ defined in a sufficiently small neighborhood of y by $\widetilde{\pi}_{\mathrm{Ad}}(y') = F(x') + \ell(t')$ is continuous. By the continuity of the canonical projection of $\mathbb{R}^s$ onto $\mathbb{T}^s$ we conclude that π_{Ad} is continuous at y. It remains to check that for every y in Ω_Λ and every t in $\mathbb{R}^{\mathrm{d}}$ we have $\pi_{\mathrm{Ad}}(y - t) = \pi_{\mathrm{Ad}}(y) + [\ell(t)]_{\mathbb{Z}^s}$. Fix y in Ω_Λ and fix t in $\mathbb{R}^{\mathrm{d}}$. There are x_1 and x_2 in Ξ_Λ and t_1 and t_2 in $\mathbb{R}^{\mathrm{d}}$ such that $y = x_1 - t_1$ and $y - t = x_2 - t_2$. Then, $x_2 = x_1 - (t_1 - t_2 + t)$. Using this, (3.1.5) and the fact that $\varphi(t_1 - t_2 + t) \in \mathbb{Z}^s$ we get that

$$\begin{aligned}
\pi_{\mathrm{Ad}}(y - t) &= [F(x_2)]_{\mathbb{Z}^s} + [\ell(t_2)]_{\mathbb{Z}^s} \\
&= [F(x_1) + \ell(t_1 - t_2 + t) - \varphi(t_1 - t_2 + t)]_{\mathbb{Z}^s} + [\ell(t_2)]_{\mathbb{Z}^s} \\
&= [F(x_1) + \ell(t_1)]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s} = \pi_{\mathrm{Ad}}(y) + [\ell(t)]_{\mathbb{Z}^s},
\end{aligned}$$

which concludes the proof of the proposition. □

Remark 3.1.1. Since ℓ is injective, we have that image of ℓ has dimension d. Thus there are at least d rows linearly independent of A, the representative matrix of ℓ. Let $(\Omega_\Lambda.\mathbb{R}^{\mathrm{d}})$ be the hull system of a repetitive Meyer set $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ with rank $s \geq \mathrm{d}$. Each row of A is a continuous eigenvalue of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. The number of continuous eigenvalues obtained from the address system is the number of rationally independent rows of matrix A. In particular, since the image of A has dimension d, we have that $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ has at least d linearly independent continuous eigenvalues.

3.2 Proof of Theorem A.

In this section, we prove the following theorem.

Theorem A. *A repetitive Meyer set Λ in $\mathbb{R}^{\mathrm{d}}$ is an inter-model set with Euclidean internal space if and only if rank$(\langle \Lambda - \Lambda \rangle) > \mathrm{d}$ and the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ such that there is one point with a unique preimage under the factor map.*

After some comments from J. Kellendonk and using a result of Paul [P73, Proposition 1.1], we can observe that the address system of a Euclidean inter-model set Λ, is the maximal equicontinuous factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

First, we prove a characterization of the maximal equicontinuous factor for a Euclidean CPS and then, we prove the necessary condition. After this, we use the address map to construct a Euclidean CPS and that we use in the proof of the sufficient condition. Finally, we prove the sufficient condition assuming the Main Technical Lemma. This lemma is stated in §3.2.2, and it is proved in §3.3.

We assume in this section that $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ is a repetitive Meyer set and denote by $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ its associated dynamical system.

3.2.1 Necessary condition.

Let Λ be an inter-model set for a Euclidean CPS over $\mathbb{R}^{\mathrm{d}}$ with internal space $\mathbb{R}^n$, lattice L, and window W. Denote by Ω_{MS} the hull of the generic set generated by these data. Repetitivity of Λ and Proposition 2.3.5 imply that $\Omega_{MS} = \Omega_\Lambda$. By [A16, Theorem 8.1], the associated dynamical system $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is almost automorphic (see also [S00, FHK02]). The remaining part of the proof of the necessary condition follows directly from the following proposition.

Proposition B. *Let Ω_{MS} be the hull of the repetitive inter-model sets generated by a Euclidean cut and project scheme $(\mathbb{R}^n, \Gamma, s_{\mathbb{R}^n})$ over $\mathbb{R}^{\mathrm{d}}$ and a window W. Then, for every Λ in Ω_{MS} we have that the group $\langle \Lambda - \Lambda \rangle$ is equal to Γ and its rank is $\mathrm{d} + n$. Moreover, the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugate to the address system of Λ.*

Proof of Proposition B

Denote by p_1 and by p_2 the orthogonal projections from $\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ onto $\mathbb{R}^{\mathrm{d}}$ and $\mathbb{R}^n$, respectively, and put $L := \mathcal{G}(s_{\mathbb{R}^n})$. Fix Λ in Ω_{MS}. By [M97, Proposition 2.6 (ii)] for every w in $\mathbb{R}^n$ we have that

$$\langle \curlywedge(w + W)\rangle = \Gamma.$$

In particular, $\langle \curlywedge(w + W) - \curlywedge(w + W)\rangle = \Gamma$. By Proposition 2.3.5 there is w in NS such that $\curlywedge(w + W)$ is in Ω_{MS} and thus, by repetitivity

$$\langle \Lambda - \Lambda\rangle = \langle \curlywedge(w + W) - \curlywedge(w + W)\rangle = \Gamma.$$

Now we prove that the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugate to the address system of Λ. Fix a basis $\mathcal{B} = \{\widetilde{v}_1, \dots, \widetilde{v}_s\}$ of L. Let ℓ be the linear map given by Proposition A applied to Λ with the basis $p_1(\mathcal{B})$ for Γ and let $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ be the address system. Denote by $\psi : \mathbb{R}^s \to \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ the linear isomorphism sending the canonical basis of $\mathbb{R}^s$ onto $\{\widetilde{v}_1, \dots, \widetilde{v}_s\}$, i.e.

$$\psi(u_1, \dots, u_s) = u_1\widetilde{v}_1 + \cdots + u_s\widetilde{v}_s.$$

By (2.1.3) for every $t \in \mathbb{R}^{\mathrm{d}}$ we have

$$p_1(\psi(\ell(t))) = t. \tag{3.2.1}$$

Define the map $\Psi : \mathbb{T}^s \to \mathbb{T}_{\mathcal{G}}$ by $\Psi([w]_{\mathbb{Z}^s}) = [\psi(w)]_L$. Note that Ψ is an homeomorphism. By (3.2.1), for all $t \in \mathbb{R}^{\mathrm{d}}$ and $[w] \in \mathbb{T}^s$, we have

$$\begin{aligned}\Psi([w]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s}) &= \Psi([w + \ell(t)]_{\mathbb{Z}^s})\\ &= [\psi(w + \ell(t))]_L = [\psi(w)]_L + [\psi(\ell(t))]_L\\ &= [\psi(w)]_L + [(p_1(\psi(\ell(t))), p_2(\psi(\ell(t))))]_L\\ &= [\psi(w)]_L + [(t, p_2(\psi(\ell(t))))]_L\,.\end{aligned}$$

For proving that Ψ conjugates the address system with the maximal equicontinuous factor $(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n/L, \mathbb{R}^{\mathrm{d}})$, we need to show that for every $t \in \mathbb{R}^{\mathrm{d}}$,

$$p_2(\psi(\ell(t))) = 0.$$

By Remark 2.3.4 and the fact that the window W has a non-empty interior, there is w in NS such that $0 \in w + W$ and the set $\curlywedge(w + W)$ is non-singular. Put $\Lambda_0 := \curlywedge(w + W)$. We have that Λ_0 is in Ξ_Λ. Let φ be the address map for Λ_0 associated to the basis $p_1(\mathcal{B})$. By Proposition A there is a constant $\widehat{C} > 0$ such that for every $t \in \Lambda_0$ we have

$$\|p_2(\psi(\varphi(t))) - p_2(\psi(\ell(t)))\|_{\mathrm{d}} \le \widehat{C}.$$

Together with the fact that $p_2(\psi(\varphi(\Lambda_0))) = p_2(s_{\mathbb{R}^n}(\Lambda_0)) \subseteq w + W$ this implies that the map $p_2 \circ \psi \circ \ell$ is uniformly bounded on Λ_0. Using that Λ_0 is relatively dense in $\mathbb{R}^{\mathrm{d}}$ and that $p_2 \circ \psi \circ \ell$ is linear, we get that $p_2(\psi(\ell(\mathbb{R}^{\mathrm{d}})))$ is bounded, which implies that $p_2(\psi(\ell(\mathbb{R}^{\mathrm{d}}))) = 0$. We conclude that $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ and $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^{\mathrm{d}})$ are topologically conjugated finishing the proof of the proposition.

3.2.2 Sufficient condition.

The Lagarias cut and project scheme

Let Λ be a Meyer set in $\mathbb{R}^{\mathrm{d}}$ and suppose that $\langle \Lambda - \Lambda \rangle$ have rank $s > \mathrm{d}$. Let $\mathcal{B}$ be a basis of $\langle \Lambda - \Lambda \rangle$ formed by vectors $\{v_1, \ldots, v_s\} \subseteq \mathbb{R}^{\mathrm{d}}$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^s$ be the coordinate map for the basis $\mathcal{B}$. Fix Λ_0 in Ξ_Λ. Remember that since $0 \in \Lambda_0$, we have $\langle \Lambda - \Lambda \rangle = \langle \Lambda_0 - \Lambda_0 \rangle = \langle \Lambda_0 \rangle$ and that φ is also the address map for Λ_0. Let $\ell : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^s$ be the linear map given by Proposition A. Define $\phi : \mathbb{R}^s \to \mathbb{R}^{\mathrm{d}}$ by $\phi(u_1, \ldots, u_s) = u_1 v_1 + \cdots + u_s v_s$. By (2.1.3) for every $t \in \mathbb{R}^{\mathrm{d}}$ we have

$$\phi \circ \ell(t) = t. \tag{3.2.2}$$

In particular,

$$\mathrm{Ker}(\ell) = \{0\} \qquad \text{and} \qquad \mathrm{Im}(\phi) = \mathbb{R}^{\mathrm{d}}. \tag{3.2.3}$$

Put $n \coloneqq s - \mathrm{d}$ and note that the dimension of $\mathrm{Ker}(\phi)$ is n. Let $\mathcal{B}' \coloneqq \{k_1, \ldots, k_n\}$ be an orthonormal basis for $\mathrm{Ker}(\phi)$. Notice that for every $1 \le j \le s$ we have that the vector $w_j \coloneqq \ell(v_j) - \mathrm{e}_j$ belongs to $\mathrm{Ker}(\phi)$, where e_j is the jth-canonical coordinate vector. For every $j \in \{1, \ldots, s\}$ denote by $(\alpha_{j,1}, \ldots \alpha_{j,n})$ the coordinates of w_j in the basis $\mathcal{B}'$, and define for every $j \in \{1, \ldots, s\}$ the vectors

$$v_j^\star \coloneqq (\alpha_{j,1}, \ldots \alpha_{j,n})^t \text{ and } \widetilde{v}_j \coloneqq (v_j, v_j^\star).$$

In the proof of [L99, Theorem 3.1], Lagarias proved that the set $\widetilde{\mathcal{B}} \coloneqq \{\widetilde{v}_1 \ldots, \widetilde{v}_s\}$ is $\mathbb{Z}$-linearly independent in $\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ and it generates a full rank lattice. Denote by $\widetilde{L}$ the lattice generated by $\widetilde{\mathcal{B}}$. Denote by p_1 and p_2 the orthogonal projections of $\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ onto $\mathbb{R}^{\mathrm{d}}$ and $\mathbb{R}^n$, respectively. By construction, p_1 is injective on $\widetilde{L}$ and its image is $\langle \Lambda - \Lambda \rangle$. Denote by $\psi : \mathbb{R}^s \to \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ the linear isomorphism sending the canonical basis of $\mathbb{R}^s$ onto $\{\widetilde{v}_1, \ldots, \widetilde{v}_s\}$, i.e.

$$\psi(u_1, \ldots, u_s) = u_1 \widetilde{v}_1 + \cdots + u_s \widetilde{v}_s.$$

In the proof of [L99, Theorem 3.1], it was proved that for every t in $\langle \Lambda - \Lambda \rangle$ we have

$$\| p_2(\psi(\varphi(t))) \|_n = \| \varphi(t) - \ell(t) \|_s. \tag{3.2.4}$$

Lemma E. *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$. If the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$, then $p_2(\widetilde{L})$ is dense in $\mathbb{R}^n$.*

Proof. The proof is by contradiction. Assume that $p_2(\widetilde{L})$ is not dense. Then, there is a non-empty closed ball $V \subseteq \mathbb{R}^n$ such that $p_2(\widetilde{L}) \cap V = \{\emptyset\}$. In particular,

$$\widetilde{L} \cap (\mathbb{R}^{\mathrm{d}} \times V) = \{\emptyset\}. \tag{3.2.5}$$

By Proposition A and (3.2.4) there is a constant $\widehat{C} > 0$ such that for every $t \in \Lambda_0$ we have

$$\max\{\| p_2(\psi(\varphi(t))) \|_n, \| p_2(\psi(\varphi(t))) - p_2(\psi(\ell(t))) \|_n\} \le \widehat{C}.$$

Therefore the linear map $p_2 \circ \psi \circ \ell$ is uniformly bounded on Λ_0, which is relatively dense. Then, for all $t \in \mathbb{R}^{\mathrm{d}}$ we have

$$p_2 \circ \psi \circ \ell(t) = 0. \tag{3.2.6}$$

Consider the dynamical system defined on the space $(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$ with the following $\mathbb{R}^{\mathrm{d}}$-action: for every $t \in \mathbb{R}^{\mathrm{d}}$ and every $w \in (\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$,

$$w \cdot t := w + [(t, 0)]_{\widetilde{L}}.$$

Define the map $\Psi : \mathbb{T}^s \to (\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$ by $\Psi([w]_{\mathbb{Z}^s}) = [\psi(w)]_{\widetilde{L}}$ for every $[w]_{\mathbb{Z}^s}$ in $(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$. By (3.2.6) the map Ψ is a topological conjugacy between the address system of Λ and the dynamical system just defined $((\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^{\mathrm{d}})$. Let π_{Ad} be the address homomorphism defined in Proposition A. Since we are assuming that π_{Ad} is a factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$, we have that the map $\Psi \circ \pi_{\mathrm{Ad}}$ is also a factor from $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ to $((\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^{\mathrm{d}})$. By the repetitivity of Λ we have that $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ is minimal and then, the factor $((\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^{\mathrm{d}})$ is also minimal. But the set $[\mathbb{R}^{\mathrm{d}} \times V]_{\widetilde{L}}$ is closed and $\mathbb{R}^{\mathrm{d}}$-invariant, and by (3.2.5), it is strictly contained in $(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$, which is a contradiction to the minimality of $((\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^{\mathrm{d}})$. □

Put $s_{\mathbb{R}^n} := p_2 \circ \psi \circ \varphi$ on $\langle \Lambda - \Lambda \rangle$. By Lemma E if the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ the triple $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ is a CPS and we call it the *Lagarias CPS for* Λ.

Recall that a window is irredundant if its redundancies group is trivial (see §2.3). By definition, every compact set in $\mathbb{R}^n$ is irredundant. Using Theorem 2.1.1 and (3.2.4), the set $W = \overline{s_{\mathbb{R}^n}(\Lambda_0)} \subseteq \mathbb{R}^n$ is a compact and irredundant set. In particular, W is a relatively compact set.

Assume that the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. For each $\Lambda_0 \in \Xi_\Lambda$, the compact and irredundant set $W = \overline{s_{\mathbb{R}^n}(\Lambda_0)} \subseteq \mathbb{R}^n$, and $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ form a CPS in the sense of [S00]. From this, we can observe that W is the closure of its interior. By [S00, Lemma 4.1], for each Λ' in Ω_Λ such that $\Lambda' \subseteq \langle \Lambda - \Lambda \rangle$, the set

$$\bigcap \{ s_{\mathbb{R}^n}(t) - W \mid t \in \Lambda' \}$$

contains exactly one element $c_{\Lambda'} \in H$ and $\overline{s_{\mathbb{R}^n}(\Lambda')} = c_{\Lambda'} + W$. For each Λ' in Ω_Λ there is s in $\mathbb{R}^n$ such that $\Lambda' + s \subseteq \langle \Lambda - \Lambda \rangle$. We use this to define a map $\beta : \Omega_\Lambda \to (\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\widetilde{L}$ by $\beta(\Lambda') = [(s, c_{\Lambda'+s})]_{\widetilde{L}}$, where s is such that $\Lambda' + s \subseteq \langle \Lambda - \Lambda \rangle$. From [S00, Proposition 4.3], β is well-defined, continuous, and onto. Note that $\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n$ is σ-compact and $p_2(\widetilde{L})$ is a countable set in $\mathbb{R}^n$, by Remark 2.3.4, there exists γ in $\mathbb{R}^n$ such that $(\gamma + \partial W) \cap p_2(\widetilde{L}) = \emptyset$. Because β is onto, there is Λ' in Ω_Λ such that $c_{\Lambda'} = \gamma$. Hence $\overline{s_{\mathbb{R}^n}(\Lambda')} = \gamma + W$. By construction, $s_{\mathbb{R}^n}(\Lambda')$ is contained in the interior of $\gamma + W$. This implies that $W = \overline{W^\circ}$, i.e. W equals the closure of its interior.

Thus we obtain the following result.

Lemma F. *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by $\Lambda - \Lambda$. Put $n = rank(\langle \Lambda - \Lambda \rangle) - \mathrm{d}$ and assume that $n > 0$. Also, assume that the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ be the Lagarias CPS for Λ. For every Λ_0 in Ξ_Λ the set $\overline{s_{\mathbb{R}^n}(\Lambda_0)}$ is an irredundant window.*

From the proof of Lemma E and by Lemma F we obtain the following lemma.

Lemma G. *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by $\Lambda - \Lambda$. Put $n = rank(\langle \Lambda - \Lambda \rangle) - \mathrm{d}$ and assume that $n > 0$. Also assume that the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ be the Lagarias CPS for Λ. For every Λ_0 in Ξ_Λ, let Ω_{MS} be the hull of the generic inter-model sets generated by $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ and the window $\overline{s_{\mathbb{R}^n}(\Lambda_0)}$. Then the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugated to the address system of the Lagarias CPS for Λ.*

Proof of sufficient condition

The main technical step in the proof of the sufficient condition is the following lemma that we state below. Its proof will be given in §3.3. We recall that $\Omega_{MS,\mathrm{me}}$ denotes the maximal equicontinuous factor of the dynamical system $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$, see 2.2.

Main Technical Lemma. *Let $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ be a repetitive Meyer set and let Γ be the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and suppose that $W' = \overline{s_{H'}(\Lambda)}$ is a window. Let Ω_{MS} be the hull of the generic model sets generated by $(H', \Gamma, s_{H'})$ and W'. Then, there is a factor map*

$$\widetilde{\pi} : \Omega_\Lambda \to \Omega_{MS,me},$$

such that if $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ is an almost automorphic extension of $(\Omega_{MS,me}, \mathbb{R}^{\mathrm{d}})$ for $\widetilde{\pi}$, then there are Λ_0 in Ω_Λ and a non-singular inter-model set Λ_1 in Ω_{MS} such that $\Lambda_0 = \Lambda_1$.

Proof of sufficient condition in Theorem A. Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by $\Lambda - \Lambda$. Assume that $\mathrm{rank}(\langle \Lambda - \Lambda \rangle) = s > \mathrm{d}$, that the address system is a topological factor $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ and that $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ is almost automorphic extension of the address system.

Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ be the Lagarias CPS for Λ where $n = s - \mathrm{d}$. Fix Λ_* in Ξ_Λ and recall that by the repetitivity of Λ we have that $\Omega_\Lambda = \Omega_{\Lambda_*}$. By Lemma F the set $W' = \overline{s_{\mathbb{R}^n}(\Lambda_*)}$ is an irredundant window. Denote by Ω_{MS} the hull of generic inter-model sets generated by $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$ and W'. By Lemma G the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugated to the address system of Λ which agrees with the address system of Λ_* by Proposition A. By hypothesis, the dynamical system $(\Omega_{\Lambda_*}, \mathbb{R}^{\mathrm{d}})$ is an almost automorphic extension of the address systems of Λ_* and then, it is also an almost automorphic extension of $(\Omega_{MS,me}, \mathbb{R}^{\mathrm{d}})$. By the Main Technical Lemma applied to Λ_* and $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_{\mathbb{R}^n})$, there are $\Lambda_0 \in \Omega_{\Lambda_*}$ and $\Lambda_1 \in \Omega_{MS}$ such that $\Lambda_0 = \Lambda_1$. By the minimality of $(\Omega_{\Lambda_*}, \mathbb{R}^{\mathrm{d}})$ we have that Ω_{Λ_*} is equal to the hull of Λ_0 which is equal to the hull of the generic model sets generated by a Euclidean CPS. Since W' is irredundant and $\Omega_\Lambda = \Omega_{\Lambda_*}$ by Theorem 2.3.1 we conclude that Λ is an inter-model set generated by a CPS with Euclidean internal space, finishing the proof of the sufficient condition.

□

3.3 Proof of Main Technical Lemma.

In this section, we prove the Main Technical Lemma used in the proof of Theorem A. Indeed, we prove a more detailed version of the Main Technical Lemma for future references.

Main Technical Lemma'. *Let $\Lambda \subseteq \mathbb{R}^{\rm d}$ be a repetitive Meyer set and let Γ the subgroup of $\mathbb{R}^{\rm d}$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and suppose that $W' = \overline{s_{H'}(\Lambda)}$ is a compact, irredundant window in H'.*

Let Ω_{MS} be the hull of the generic inter-model sets for the CPS $(H', \Gamma, s_{H'})$ and window W'. Let π_0 be the maximal equicontinuous factor map from Ω_{MS} to $\Omega_{MS,me}$, and denote by $R_{\pi_0}(\Omega_{MS})$ the set of non-singular points in Ω_{MS} for π_0. Then, there is a factor map

$$\widetilde{\pi} : \Omega_\Lambda \to \Omega_{MS,me}.$$

Put $\Omega^0_\Lambda := \widetilde{\pi}^{-1}(\pi_0(R_{\pi_0}(\Omega_{MS})))$. There is a continuous map

$$\pi_1 : \Omega^0_\Lambda \to R_{\pi_0}(\Omega_{MS})$$

such that for every $\Lambda_0 \in \Omega^0_\Lambda$ we have

$$\pi_1(\Lambda_0 - t) = \pi_1(\Lambda_0) - t \quad \text{and} \quad \widetilde{\pi}(\Lambda_0) = \pi_0 \circ \pi_1(\Lambda_0).$$

Moreover, for every Λ_1 in $R_{\pi_0}(\Omega_{MS})$ we have

$$\Lambda_1 = \bigcup_{\Lambda' \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda'. \tag{3.3.1}$$

Besides, if $\widetilde{\pi} : \Omega_\Lambda \to \Omega_{MS,me}$ is an almost automorphic extension then

$$\pi_0(R_{\pi_0}(\Omega_{MS})) \cap \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$$

is a residual set in $\Omega_{MS,me}$, and for every Λ_1 in $R_{\pi_0}(\Omega_{MS})$ such that $\pi_0(\Lambda_1) \in \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$ we have that Λ_1 is in Ω^0_Λ.

The proof of the lemma will be given in §3.3.2 after recalling the definition of optimal CPS of a Meyer set introduced in [A16].

3.3.1 The optimal CPS and the optimal window

Let Λ be a repetitive Meyer set in $\mathbb{R}^{\rm d}$ and let Γ be the subgroup of $\mathbb{R}^{\rm d}$ generated by Λ. Define Ξ^Γ as the collection of all $\Lambda' \in \Omega_\Lambda$ having support into Γ,

$$\Xi^\Gamma := \{\Lambda' \in \Omega_\Lambda \mid \Lambda' \subseteq \Gamma\}.$$

Observe that $\Xi_\Lambda \subseteq \Xi^{\Gamma'}$. We consider the *combinatorial topology* on Ω_Λ, which is obtained from the distance

$$\mathrm{dist}(\Lambda', \Lambda'') = \left\{ \frac{1}{R+1} \mid \Lambda' \cap \overline{B(0,R)} = \Lambda'' \cap \overline{B(0,R)} \right\}.$$

The combinatorial topology is always strictly finer than the usual topology on Ω_Λ and on the transversal Ξ_Λ both topologies coincide. We endow $\Xi^{\Gamma'}$ with the combinatorial topology. We say that Λ' and Λ'' in Ω_Λ are *strongly regionally proximal*, denoted $\Lambda' \sim_{\rm srp} \Lambda''$, if for each $R > 0$ there are $\Lambda_1, \Lambda_2 \in \Omega_\Lambda$ and $t \in \mathbb{R}^{\rm d}$ such that

$$\begin{aligned}\Lambda' \cap \overline{B(0,R)} &= \Lambda_1 \cap \overline{B(0,R)} \\ \Lambda'' \cap \overline{B(0,R)} &= \Lambda_2 \cap \overline{B(0,R)} \\ (\Lambda_1 - t) \cap \overline{B(0,R)} &= (\Lambda_2 - t) \cap \overline{B(0,R)}.\end{aligned}$$

Since Λ is a repetitive Meyer set we have that the strongly regionally proximal relation is a closed $\mathbb{R}^{\rm d}$-invariant relation on Ω_Λ. The quotient $\Omega_\Lambda / \sim_{\rm srp}$ gives the maximal equicontinuous factor [BK13].

In Proposition 3.3.1 below, we recall some results in [A16] which allow us to introduce the optimal CPS and optimal window for a Meyer set. More precisely, part (1) is deduced by [A16, Proposition 4.4 and Lemma 4.5], part (2) comes from [A16, Proposition 6.1 and Definition 6.2] and finally, part (3) is in [A16, Theorem 7.1].

Proposition 3.3.1. *Let Λ be a repetitive Meyer set in $\mathbb{R}^{\rm d}$ and let Γ the subgroup of $\mathbb{R}^{\rm d}$ generated by Λ.*

1. *If $\Lambda' \in \Xi^\Gamma$ then its equivalence class $[\Lambda']_{srp}$ is contained into Ξ^Γ.*
2. *The set $H := \Xi^\Gamma / \sim_{srp}$ with the quotient topology admits a structure of locally compact abelian group such that $[\Lambda]_{srp}$ is the neutral element, the map $s_H : \Gamma \to H$ defined by $s_H(\gamma) = [\Lambda - \gamma]_{srp}$ is a group morphism and $\overline{s_H(\Gamma)} = H$.*

We remark that Aujogue defined s_H in [A16] with a sign plus instead of a minus as we did. So, some results that we use from [A16] and [A16b] look slightly different since we need to do a correction in the sign. From Proposition 3.3.1, the triple (H, Γ, s_H) is a CPS. Moreover, by [A16, Theorem 6.3], the set $[\Xi_\Lambda]_{\rm srp}$ is a window for (H, Γ, s_H). The CPS (H, Γ, s_H) and the window $[\Xi_\Lambda]_{\rm srp}$ are called the *optimal CPS* and the *optimal window* for Λ, respectively. Indeed, in [A16b], the author proved that the model set that it defines,

$$\underline{\Lambda} := \{\gamma \in \mathbb{R}^{\rm d} \mid s_H(\gamma) \in [\Xi_\Lambda]_{\rm srp}\},$$

satisfies that for every model set M that includes Λ we have $\Lambda \subseteq \underline{\Lambda} \subseteq M$.

Finally, we recall some results in [A16b] that we use in the proof of the Main Technical Lemma'. The first result allows us to prove that a compact and irredundant set is a window.

Proposition 3.3.2. *[A16b, Proposition 3.3] Let Λ be a repetitive Meyer set in $\mathbb{R}^{\rm d}$ and let Γ the subgroup of $\mathbb{R}^{\rm d}$ generated by Λ. Let (H, Γ, s_H) and W be optimal CPS and window for Λ, respectively. Suppose that $(H', \Gamma, s_{H'})$ is a CPS such that the closure W' of the set $s_{H'}(\Lambda)$ is compact and irredundant in H'. Then, there is a continuous, open, and onto morphism*

$$\theta : H \to H'$$

such that $s_{H'} = \theta \circ s_H$ on Γ. Moreover, the set W' is a window in H' and $W' = \theta([\Xi_\Lambda]_{srp})$.

In the following result, we recall the definition of a map that we use to construct the maps π_1 and $\widetilde{\pi}$ in the statement of the Main Technical Lemma'.

Lemma 3.3.3. *[A16b, Lemma 3.4, 3.5, 3.6] Let Λ be a repetitive Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let Γ the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by Λ. Suppose that $(H', \Gamma, s_{H'})$ is a CPS such that the closure W' of the set $s_{H'}(\Lambda)$ is compact and irredundant in H'. We have that each Λ' in Ξ^{Γ} defines a unique element $w_{\Lambda'}$ through*

$$\{w_{\Lambda'}\} = \bigcap_{\gamma \in \Lambda'} s_{H'}(\gamma) - W'.$$

Define the map

$$\begin{aligned} \omega : \ & \Xi^{\Gamma} \to H' \\ & \Lambda' \mapsto w_{\Lambda'}. \end{aligned}$$

We have that ω is uniformly continuous for the combinatorial topology, and for all $\Lambda' \in \Xi^{\Gamma}$ and $\gamma \in \Gamma$ we have

1. *$\omega(\Lambda' - \gamma) = \omega(\Lambda') - s_{H'}(\gamma)$.*
2. *$\omega(\Lambda') = -\theta([\Lambda']_{srp})$, where θ is the morphism in Proposition 3.3.2.*

3.3.2 Proof of Main Technical Lemma'

Let $\Lambda \subseteq \mathbb{R}^{\mathrm{d}}$ be a repetitive Meyer set and let Γ be the subgroup of $\mathbb{R}^{\mathrm{d}}$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and assume that $W' = \overline{s_{H'}(\Lambda)}$ is a compact and irredundant window in H'. Let Ω_{MS} be the hull of inter-model sets generated by $(H', \Gamma, s_{H'})$ and W'. Recall that the maximal equicontinuous factor $\Omega_{MS,\mathrm{me}}$ can be obtained by the quotient $(\mathbb{R}^{\mathrm{d}} \times H')/\mathcal{G}(s_{H'})$ and denote by π_0 be the maximal equicontinuous factor map from Ω_{MS} to $\Omega_{MS,\mathrm{me}}$.

Construction of $\widetilde{\pi}$

Now we construct the map $\widetilde{\pi} : \Omega_{\Lambda} \to \Omega_{MS,\mathrm{me}}$. For every (t, w) in $\mathbb{R}^{\mathrm{d}} \times H'$ we denote by $[(t, w)]$ its equivalent class in $\Omega_{MS,\mathrm{me}}$. For every $\widetilde{\Lambda}$ in Ω_{Λ} there is $t \in \mathbb{R}^{\mathrm{d}}$ such that $\widetilde{\Lambda} - t$ is in Ξ^{Γ}, define $\widetilde{\pi}(\widetilde{\Lambda})$ by

$$\widetilde{\pi}(\widetilde{\Lambda}) := [(-t, \omega(\widetilde{\Lambda} - t))] \in \Omega_{MS,\mathrm{me}}.$$

We verify that $\widetilde{\pi}$ is well defined. Assume that there is s in $\mathbb{R}^{\mathrm{d}}$ such that $\widetilde{\Lambda} - s$ is in Ξ^{Γ}. Observe that $t - s$ is in Γ. By part (1) in Lemma 3.3.3, we have that

$$\begin{aligned} (-t, \omega(\widetilde{\Lambda} - t)) &= (-t + s - s, \omega(\widetilde{\Lambda} - (t + s - s))) \\ &= (-s - (t - s), \omega(\widetilde{\Lambda} - s) - s_{H'}(t - s)) \\ &= (-s, \omega(\widetilde{\Lambda} - s)) - (t - s, s_{H'}(t - s)). \end{aligned}$$

Since $(t - s, s_{H'}(t - s))$ belongs to $\mathcal{G}(s_{H'})$, we have that

$$[(-t, \omega(\widetilde{\Lambda} - t))] = [(-s, \omega(\widetilde{\Lambda} - s))],$$

and hence $\widetilde{\pi}$ is well defined.

Now we check that $\widetilde{\pi}$ commutes with the $\mathbb{R}^{\mathrm{d}}$ action on Ω_Λ and on $\Omega_{MS,\mathrm{me}}$. Let $\widetilde{\Lambda}$ be in Ω_Λ and t be in $\mathbb{R}$. There are s and s' in $\mathbb{R}^{\mathrm{d}}$ such that $\widetilde{\Lambda}-s$ and $(\widetilde{\Lambda}-t)-s'=\widetilde{\Lambda}-(t+s')$ are in Ξ^Γ. Notice that $t+s'-s$ belongs to Γ. Again, by part (1) in Lemma 3.3.3, we have

$$\begin{aligned}(-s',\omega((\widetilde{\Lambda}-t)-s')) &= (-s',\omega((\widetilde{\Lambda}-s)-(t+s'-s)))\\ &= (-s',\omega(\widetilde{\Lambda}-s)-s_{H'}(t+s'-s))\\ &= (-s'+(t+s'-s),\omega(\widetilde{\Lambda}-s))-(t+s'-s,s_{H'}(t+s'-s))\\ &= (t-s,\omega(\widetilde{\Lambda}-s))-(t+s'-s,s_{H'}(t+s'-s)).\end{aligned}$$

Since $(t+s'-s,s_{H'}(t+s'-s))$ is in $\mathcal{G}(s_{H'})$ we have

$$\widetilde{\pi}(\widetilde{\Lambda}-t)=[(-s',\omega((\widetilde{\Lambda}-t)-s'))]=[(-s,\omega(\widetilde{\Lambda}-s))]+[(t,0)]=\widetilde{\pi}(\widetilde{\Lambda})+[(t,0)].$$

Now we prove that $\widetilde{\pi}$ is continuous. Let Λ' be Ω_{MS} and let U be a neighborhood of 0 in $\Omega_{MS,\mathrm{me}}$. We can assume that $U=[B(0,r_0)\times U_{H'}]$ where $r_0>0$ and $U_{H'}$ is a neighborhood of 0 in H'. There exists $t'\in\Lambda'$ such that $\Lambda'-t'\in\Xi_\Lambda\subseteq\Xi^\Gamma$. For $r>0$, denote

$$C_r=\bigcap_{\gamma\in(\Lambda'-t')\cap\overline{B(0,r)}} s_{H'}(\gamma)-W',$$

and observe that for $r>r'$ we have $C_r\subseteq C_{r'}$. By Lemma 3.3.3,

$$\bigcap_{r>0} C_r=\{\omega(\Lambda'-t')\}. \tag{3.3.2}$$

Now we prove that there is $r'>0$ such that for every $r\geq r'$

$$C_r\subseteq\omega(\Lambda'-t')+U_{H'}. \tag{3.3.3}$$

By contradiction suppose that there is an increasing sequence $(r_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ of positive real numbers converging to infinity as i goes to infinity, such that $(C_{r_{\mathrm{i}}}-\omega(\Lambda'-t'))\cap U^c_{H'}\neq\emptyset$. Then, for every $\mathrm{i}\in\mathbb{N}$ there is

$$x_{\mathrm{i}}\in(C_{r_{\mathrm{i}}}-\omega(\Lambda'-t'))\cap U^c_{H'}.$$

Since for every i,j in $\mathbb{N}$ with $j\geq\mathrm{i}$ we have $C_{r_j}\subseteq C_{r_{\mathrm{i}}}$. By compactness of C_{r_1} there is an accumulation point $\tilde{x}$ of $(x_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ in $U^c_{H'}$ and thus, $\tilde{x}\neq 0$. But $\tilde{x}$ also belongs to $\bigcap_{r>0}C_r-\omega(\Lambda'-t')$ which is $\{0\}$ by (3.3.2), giving the desired contradiction.

Put $R\coloneqq\|t'\|_{\mathrm{d}}+r'+r_0$ and consider set

$$T\coloneqq\{\widetilde{\Lambda}\in\Omega_\Lambda\mid\Lambda'\cap\overline{B(0,R)}=\widetilde{\Lambda}\cap\overline{B(0,R)}\}.$$

For every $\varepsilon>0$ sufficiently small the set

$$V_\varepsilon\coloneqq\{\Lambda''\in\Omega_\Lambda\mid\exists\widetilde{\Lambda}\in T,\exists t\in B(0,\varepsilon),\Lambda''=\widetilde{\Lambda}-t\}$$

is an open neighborhood of Λ'. Fix $\varepsilon<r_0$. By the definition of R, for every Λ'' in V_ε there are t in $B(0,\varepsilon)$ and $\widetilde{\Lambda}$ in T such that

$$(\Lambda''-(t'-t))\cap\overline{B(0,r')}=(\widetilde{\Lambda}-t')\cap\overline{B(0,r')}=(\Lambda'-t')\cap\overline{B(0,r')}.$$

Put $t'' := t' - t$, we have $\|t' - t''\|_{\mathrm{d}} < r_0$ and since $\Lambda' - t'$ is in Ξ_Λ we also have that $\Lambda'' - t''$ is in $\Xi_\Lambda \subseteq \Xi^\Gamma$. Then,

$$\bigcap_{\gamma \in (\Lambda'' - t'') \cap \overline{B(0,r')}} s_{H'}(\gamma) - W' = \bigcap_{\gamma \in (\Lambda' - t') \cap \overline{B(0,r')}} s_{H'}(\gamma) - W'.$$

Together with (3.3.3), this implies $\omega(\Lambda'' - t'') \in \omega(\Lambda' - t') + U_{H'}$. Therefore, $\widetilde{\pi}(\Lambda'') = [-t'', \omega(\Lambda'' - t'')]$ is included in

$$\begin{aligned}[-t' + (t' - t''), \omega(\Lambda' - t') + U_{H'}] &\subseteq [-t' + B(0,\delta), \omega(\Lambda' - t') + U_{H'}] \\ &= [-t', \omega(\Lambda' - t')] + [B(0,\delta), U_{H'}],\end{aligned}$$

showing the continuity of $\widetilde{\pi}$ at Λ' in Ω_{MS}.

Finally, since the $\mathbb{R}^{\mathrm{d}}$-action on $\Omega_{MS,\mathrm{me}}$ is minimal we have that $\widetilde{\pi}$ is surjective, which concludes the proof that $\widetilde{\pi}$ is a factor map.

Definition of π_1

Recall that $R(\Omega_{MS})$ denotes the set of non-singular points of Ω_{MS} for π_0 as defined in §2.3.1. By definition, all sections of π_0 agree on $\pi_0(R(\Omega_{MS}))$. Let $\widetilde{s} : \Omega_{MS,\mathrm{me}} \to \Omega_{MS}$ be a section of π_0. Put $\Omega^0_\Lambda := \widetilde{\pi}^{-1}(\pi_0(R(\Omega_{MS}))$, and define the surjective map $\pi_1 : \Omega^0_\Lambda \to R(\Omega_{MS})$ by $\pi_1 := \widetilde{s} \circ \widetilde{\pi}$.

By the continuity of $\widetilde{\pi}$ and Proposition 2.3.7, the map π_1 is also continuous. Since $\widetilde{s}$ is a section of π_0, for every Λ' in Ω^0_Λ we have

$$\widetilde{\pi}(\Lambda') = \pi_0 \circ \pi_1(\Lambda'). \tag{3.3.4}$$

Since $\widetilde{s}$ commutes with the action of $\mathbb{R}^{\mathrm{d}}$ on the set $\pi_0(R(\Omega_{MS}))$ we get that for every Λ' in Ω^0_Λ and t in $\mathbb{R}^{\mathrm{d}}$,

$$\pi_1(\Lambda' - t) = \pi_1(\Lambda') - t.$$

Proof of (3.3.1)

Fix Λ_1 in $R_{\pi_0}(\Omega_{MS})$. We prove that (3.3.1) holds. First, we assume that Λ_1 is in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$. By Theorem 2.3.1 if $\pi_0(\Lambda_1) = [(t, w)]$ then

$$\curlywedge(w + \mathrm{int}(W')) = \Lambda_1 + t = \curlywedge(w + W'). \tag{3.3.5}$$

Observe that by definition of $\widetilde{\pi}$ for every Λ' in $\Omega^0_\Lambda \cap \Xi^\Gamma$ we have $\widetilde{\pi}(\Lambda') = [(0, \omega(\Lambda')]$. In addition, if Λ' satisfies that $\pi_1(\Lambda') = \Lambda_1$ then using (3.3.4), we get

$$\pi_0(\Lambda_1) = \pi_0 \circ \pi_1(\Lambda') = \widetilde{\pi}(\Lambda') = [(0, \omega(\Lambda'))].$$

Together with (3.3.5) implies that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$, we have

$$\Lambda_1 = \{\gamma \in \Gamma \mid s_{H'}(\gamma) \in \omega(\Lambda') + W'\}. \tag{3.3.6}$$

By Proposition 3.3.2 and part (2) of Lemma 3.3.3, we have

$$-\omega(\Xi_\Lambda) = \theta([\Xi_\Lambda]_{\mathrm{srp}}) = W'. \tag{3.3.7}$$

Since $\Lambda' \in \Xi^\Gamma$, and for every $\gamma \in \Lambda'$ we have $\Lambda' - \gamma \in \Xi_\Lambda$, using part (1) of Lemma 3.3.3, we get $\omega(\Lambda' - \gamma) = \omega(\Lambda') - s_{H'}(\gamma)$. Together with (3.3.6) and (3.3.7) this implies that for every γ in Λ' we have

$$\begin{aligned}\omega(\Lambda' - \gamma) \in \omega(\Xi_\Lambda) \Longleftrightarrow s_{H'}(\gamma) \in \omega(\Lambda') - \omega(\Xi_\Lambda)&\\ \Longleftrightarrow s_{H'}(\gamma) \in \omega(\Lambda') + W' \Longleftrightarrow \gamma \in \Lambda_1.&\end{aligned}$$

Therefore, for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$ we have

$$\Lambda' \subseteq \Lambda_1. \tag{3.3.8}$$

On the other hand, fix γ in Λ_1. By (3.3.6) for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$ we have

$$s_{H'}(\gamma) \in \omega(\Lambda') + W' \Leftrightarrow \omega(\Lambda') \in \omega(\Xi_\Lambda + \gamma).$$

Thus, there is Λ'' in $\Xi_\Lambda + \gamma \subseteq \Xi^\Gamma$ such that $\omega(\Lambda'') = \omega(\Lambda')$. Then $\Lambda'' - \gamma$ is in Ξ_Λ, and hence γ is in Λ''. Therefore,

$$\Lambda_1 \subseteq \bigcup_{\Lambda'' \in \Omega^0_\Lambda \cap \Xi^\Gamma \text{ s.t. } \omega(\Lambda'') = \omega(\Lambda')} \Lambda''. \tag{3.3.9}$$

Observe that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ and every $\Lambda'' \in \Xi^\Gamma$ such that $\omega(\Lambda') = \omega(\Lambda'')$ we have that $\widetilde{\pi}(\Lambda') = \widetilde{\pi}(\Lambda'')$ and thus, $\Lambda'' \in \Omega^0_\Lambda \cap \Xi^\Gamma$. In particular, $\pi_1(\Lambda') = \pi_1(\Lambda'')$, which together with (3.3.9) implies

$$\Lambda_1 \subseteq \bigcup_{\pi_1(\Lambda'') = \Lambda_1} \Lambda''. \tag{3.3.10}$$

Now we prove that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$, and every $\Lambda'' \in \Omega^0_\Lambda$ such that $\pi_1(\Lambda') = \pi_1(\Lambda'')$, we have that

$$\Lambda'' \in \Xi^\Gamma. \tag{3.3.11}$$

First, observe that the definition of π_1 for all Λ' and Λ'' in Ω^0_Λ we have that $\pi_1(\Lambda') = \pi_1(\Lambda'') \Leftrightarrow \widetilde{\pi}(\Lambda'') = \widetilde{\pi}(\Lambda')$. Now, let $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ and $\Lambda'' \in \Omega^0_\Lambda$ be such that $\widetilde{\pi}(\Lambda'') = \widetilde{\pi}(\Lambda')$. By definition of $\widetilde{\pi}$ this holds if and only if there exists t in $\mathbb{R}^{\mathrm{d}}$ such that $\Lambda'' - t \in \Xi^\Gamma$ and $[(-t, \omega(\Lambda'' - t))] = [(0, \omega(\Lambda'))]$, which is equivalent to the existence of γ in Γ such that

$$(-t, \omega(\Lambda'' - t)) - (0, \omega(\Lambda')) = (\gamma, s_{H'}(\gamma)).$$

Then $-t = \gamma \in \Gamma$, and we get $\Lambda'' \subseteq \Gamma - \gamma = \Gamma$, which proves (3.3.11). By (3.3.8), (3.3.10) and (3.3.11) we conclude that

$$\Lambda_1 = \bigcup_{\pi_1(\Lambda'') = \Lambda_1} \Lambda'',$$

which is equivalent to

$$\Lambda_1 = \bigcup_{\Lambda' \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda'. \tag{3.3.12}$$

If Λ_1 is not in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$, then there is t in $\mathbb{R}^{\mathrm{d}}$ such that $\Lambda_1 - t$ is in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$. By (3.3.12) we have that

$$\Lambda_1 - t = \bigcup_{\widetilde{\Lambda} \in\, \widetilde{\pi}^{-1}(\pi_0(\Lambda_1 - t))} \widetilde{\Lambda}.$$

Since $\widetilde{\pi}(\widetilde{\Lambda}) = \pi_0(\Lambda_1 - t)$ if and only if $\widetilde{\pi}(\widetilde{\Lambda} - (-t)) = \pi_0(\Lambda_1)$, we conclude that

$$\Lambda_1 = \bigcup_{\widetilde{\Lambda} \in\, \widetilde{\pi}^{-1}(\pi_0(\Lambda_1 - t))} \widetilde{\Lambda} - (-t) = \bigcup_{\Lambda' \in\, \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda', \tag{3.3.13}$$

which finishes the proof of (3.3.1).

$(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ almost automorphic extension of $(\Omega_{MS,\mathbf{me}}, \mathbb{R}^{\mathrm{d}})$

Finally, if $\widetilde{\pi}$ is an almost automorphic extension of $(\Omega_{MS,\mathrm{me}}, \mathbb{R}^{\mathrm{d}})$. By [V70, Lemma 4.1], we have that the set

$$\widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda)) = \{x \in \Omega_{MS,\mathrm{me}} \mid \widetilde{\pi}^{-1}(x) \text{ is a singleton}\}$$

is a residual set in $\Omega_{MS,\mathrm{me}}$ and by [A16], the set $\pi_0(R_{\pi_0}(\Omega_{MS}))$ is also a residual set in $\Omega_{MS,\mathrm{me}}$. Then,

$$\pi_0(R_{\pi_0}(\Omega_{MS})) \cap \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$$

is also a residual set in $\Omega_{MS,\mathrm{me}}$. By (3.3.13) for every Λ_1 in $R_{\pi_0}(\Omega_{MS})$ such that $\pi_0(\Lambda_1) \in \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$ we have that Λ_1 is in Ω^0_Λ, which concludes the proof of the Main Technical Lemma'.

Chapter 4

Eigenvalues for linearly repetitive Meyer systems

4.1 Introduction.

In this chapter, we give a condition on linearly repetitive aperiodic Meyer sets in $\mathbb{R}^{\mathrm{d}}$ that ensure that every eigenfunction can be chosen continuous. The condition is given in terms of first return vectors to a decreasing sequence of clopen sets in the canonical transversal. The condition is inspired in the work with towers systems and Bratteli diagrams [CDHM, BDM05, AC11]. We also consider an additional condition relating first return vectors and the base of the group generated by the first return vectors. The main result in this chapter is Theorem D. For general repetitive Meyer sets in $\mathbb{R}^{\mathrm{d}}$ it is not true that every eigenfunction can be choosen continuous. In [KS14] a repetitive Meyer set in $\mathbb{Z}$ is constructed such that, it has a pure point dynamical spectrum (or diffraction measure) but with some non-continuous eigenfunctions. Also is constructed a Meyer set in $\mathbb{R}^{\mathrm{d}}$ where each eigenfunction is non-continuous, except those associated with the trivial eigenvalue.

We start recalling a result that we use in the proof of Theorem D. Let Λ be a repetitive Delone set in $\mathbb{R}^{\mathrm{d}}$ and let C be a closed and open set (clopen set) in Ξ_Λ. For every Λ' in Ω_Λ we define the *set of return vectors to* C as

$$\mathcal{R}_C(\Lambda') := \{t \in \mathbb{R}^{\mathrm{d}} \mid \Lambda' - t \in C\}. \tag{4.1.1}$$

It is known that $\mathcal{R}_C(\Lambda')$ is a repetitive Delone set in $\mathbb{R}^{\mathrm{d}}$ (see [C11]), and if Λ' is in Ξ_Λ then $\mathcal{R}_C(\Lambda') \subseteq \Lambda'$.

Repetitivity of Λ implies that for each $\rho > 0$ there exists $M > 0$ such that each closed ball of radius M contains the center of an occurrence of every ρ-patch of Λ. We denote by $M_\Lambda(\rho)$ the smallest of such radius $M > 0$ for each ρ.

Definition 4.1.1. A repetitive Delone set Λ is called *linearly repetitive* if there exists $L > 0$ such that for each $\rho > 0$ we have $M_\Lambda(\rho) \leq L\,\rho$.

Given v in $\mathbb{R}$, we denote by $\|v\|$ the distance of v to the nearest integer. For a vector $\vec{v} = (v_1, \dots, v_{\rm d}) \in \mathbb{R}^{\rm d}$ we write

$$\|\vec{v}\| = \max_{1 \leq j \leq {\rm d}} \|v_j\|.$$

In [C09] it is proven the following facts.

Theorem 1.2.2. *Let Λ be a linearly repetitive aperiodic Delone set in $\mathbb{R}^{\rm d}$ containing 0 and let μ be the unique invariant measure for $(\Omega_\Lambda, \mathbb{R}^{\rm d})$. For every Λ_0 in Ξ_Λ, there is a decreasing sequence of clopen sets $(C_n)_{n\in\mathbb{N}}$ in Ξ_Λ containing Λ_0 and a sequence of finite sets $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\rm d}$ verifying that for every $n \in \mathbb{N}$ the set*

$$\vec{\mathcal{F}}_n \subseteq \mathcal{R}_{C_n}(\Lambda_0) - \mathcal{R}_{C_n}(\Lambda_0) \text{ and } \mathcal{R}_{C_n}(\Lambda_0) \subseteq \langle \vec{\mathcal{F}}_n \rangle,$$

such that the following property holds: If α in $\mathbb{R}^{\rm d}$ is an eigenvalue for $(\Omega_\Lambda, \mu, \mathbb{R}^{\rm d})$ then the series

$$\sum_{n=1}^{\infty} \max_{v\in\vec{\mathcal{F}}_n} \|\langle \alpha, v\rangle\|^2$$

converges. Moreover, one can choose the sequences $(C_n)_{n\in\mathbb{N}}$ and $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ satisfying the following property: There is M in $\mathbb{N}$ such that for every n in $\mathbb{N}$ we have that every vector v in $\vec{\mathcal{F}}_{n+1}$ can be written as an integer linear combination with less than M vectors in $\vec{\mathcal{F}}_n$.

Using the decreasing sequence of clopen in Theorem 1.2.2, we can define a sequence of topological factors of $(\Omega_\Lambda, \mathbb{R}^{\rm d})$. Each one of them defines a finitely generated Abelian group. Controlling the generators of these groups we proved that a measurable eigenvalue of $(\Omega_\Lambda, \mathbb{R}^{\rm d})$ is a continuous eigenvalue for some of these factors. Thus we give a proof of Theorem D. It gives a condition on Meyer sets which ensure that every continuous eigenvalue is continuous.

Theorem D. *Let Λ be a linearly repetitive aperiodic Meyer set in $\mathbb{R}^{\rm d}$ and let μ be the unique invariant measure for $(\Omega_\Lambda, \mathbb{R}^{\rm d})$. Let Λ_0 be in Ξ_Λ and let $(C_n)_{n\in\mathbb{N}}$ and $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ be sequences associated to Λ_0 as in the statement of Theorem 1.2.2. If for every n in $\mathbb{N}$ there is a base of the group $\langle \mathcal{R}_{C_n}(\Lambda_0)\rangle$ included in $\vec{\mathcal{F}}_n$, then every eigenvalue of $(\Omega_\Lambda, \mathbb{R}^{\rm d})$ is continuous.*

We remark that Theorem D applies to every linearly repetitive aperiodic Delone set in $\mathbb{Z}^{\rm d}$ since in this case, the Meyer condition is automatically satisfied.

Now we give a more detailed version of Theorem 1.2.2 and Theorem D in dimension 1, that are more suitable for applications. Let $\mathcal{A} = \{1, \dots, q\}$ be a finite alphabet and consider $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology. Let X be a subset of $\mathcal{A}^{\mathbb{Z}}$, closed an invariant for the shift action. We denote by $(X, \mathbb{Z})$ the dynamical system given by the shift action σ on X, and we assume that it is minimal. We say that $(X, \mathbb{Z})$ has a Kakutani-Rokhlin partition $\mathcal{P}$ (KR-partition for short), if it has a partition that can be described for some positive integer t by

$$\mathcal{P} = \{\sigma^{-j} B({\rm i}) \mid {\rm i} \in \{1, \dots, t\}, 0 \leq j < h({\rm i})\},$$

where $B(1), \dots, B(t)$ are clopen subsets of X and $h({\rm i})$ is a positive integer. The *base* of $\mathcal{P}$ is the set $B = \cup_{{\rm i}=1}^{t} B({\rm i})$. For every ${\rm i} \in \{1, \dots, t\}$ the set $\mathcal{T}({\rm i}) = \cup_{j=0}^{h({\rm i})-1} \sigma^{-j} B({\rm i})$ is called the i*th tower* of *base* $B({\rm i})$. From [HPS92], the dynamical system $(X, \mathbb{Z})$ has a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of KR-partitions

$$\mathcal{P}_n = \{\sigma^{-j} B_n({\rm i}) \mid {\rm i} \in \{1, \dots, t_n\}, 0 \leq j < h_n({\rm i})\},$$

satisfying the following conditions,

[KR0] For $n = 0$ we have $t_0 = q$, and for every $1 \leq \mathrm{i} \leq q$ we have $h_n(\mathrm{i}) = 1$. The base $B_n(\mathrm{i}) = [\mathrm{i}]_0$ is the cilinder defined by the i symbol on the 0 coordinate.

[KR1] For each non-negative integer n we have $B_{n+1} \subseteq B_n$,

[KR2] For all C in $\mathcal{P}_{n+1}$ there exists C' in $\mathcal{P}_n$ such that $C \subseteq C'$.

[KR3] $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton.

[KR4] The sequence of partitions $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X.

[KR5] For all $n \geq 1, 1 \leq k \leq t_{n-1}$ and $1 \leq l \leq t_n$, there exists $0 \leq j < h_n(l)$ such that $\sigma^{-j} B_n(l) \subseteq B_{n-1}(k)$.

[KR6] For each $n \geq 1$, we have $B_n \subseteq B_{n-1}(1)$.

In particular we associate to $(\mathcal{P}_n)_{n\in\mathbb{N}}$ a sequence of incidence matrices $(M(n))_{n\geq 1}$, where $M(n)$ is the $t_n \times t_{n-1}$ matrix of positive integers given by

$$m_{l,k}(n) = \#\{0 \leq j < h_n(l) \mid \sigma^{-j} B_n(l) \subseteq B_{n-1}(k)\}.$$

For $0 \leq m < n$, we define

$$P(n, m) = M(n)M(n-1)\cdots M(m+1) \quad \text{and} \quad P(n) = P(n, 0).$$

There is a notion of linearly recurrent for sequences similar to Definition 4.1.1. We say that x in $\mathcal{A}^{\mathbb{Z}}$ is linearly recurrent if there is $L > 0$ such that for every positive integer k, each word of lenght k in x appears in a window of lenght Lk in x. By minimality, if there is a linearly recurrent sequence in X then every sequence in X is linearly recurrent. In this case, we say that $(X, \mathbb{Z})$ is linearly recurrent. It is know that when the dynamical system $(X, \mathbb{Z})$ is linearly recurrent, there is a sequence of KR-partitions verifying $[KR1] - [KR6]$ and also the following property

[LR] there exists $L > 0$ such that for all $n \geq 1, 1 \leq l \leq t_n$ and $1 \leq k \leq t_{n-1}$ we have

$$h_n(l) \leq L\, h_{n-1}(k). \tag{4.1.2}$$

For each symbol a in $\mathcal{A}$ we asign a lenght $l_a > 0$ such that for different symbols a, a' we have $l_a \neq l_{a'}$. Denote by $\mathcal{L}_0$ the set of lenghts $\{l_1, \ldots, l_q\}$. For all x in X we put the lenghts from $\mathcal{L}_0$ into the symbols in $\mathcal{A}$, to obtain a finitely generated Delone set $\Lambda(x)$ in the real line. Fix $x_0 \in X$ and denote by $\Omega_{\Lambda(x_0)}$ its associated hull space defined in §2.2.1. We recall that $(\Omega_{\Lambda(x_0)}, \mathbb{R})$ is a dynamical system, and $\Xi_{\Lambda(x_0)}$ is the transversal space. If $(X, \mathbb{Z})$ is linearly recurrent, then $\Lambda(x_0)$ is linearly repetitive and thus $(\Omega_{\Lambda(x_0)}, \mathbb{R})$ has a unique, invariant probability measure μ.

Since the lenghts are all different, the process of assigning lenghts is reversible for Delone set in $\Xi_{\Lambda(x_0)}$. This means that, for all Delone set $\widetilde{\Lambda}$ in $\Xi_{\Lambda(x_0)}$ there exists a unique sequence $x_{\widetilde{\Lambda}}$ in X such that $\Lambda(x_{\widetilde{\Lambda}}) = \widetilde{\Lambda}$. Indeed, the topological spaces X and $\Xi_{\Lambda(x_0)}$ are homeomorphic.

For all $n \geq 1$ and $1 \leq \mathrm{i} \leq t_n$, we define $C_n(\mathrm{i}) = \{\widetilde{\Lambda} \in \Xi_{\Lambda(x_0)} \mid x_{\widetilde{\Lambda}} \in B_n(\mathrm{i})\}$, and denote

$$C_n = \bigcup_{\mathrm{i}=1}^{t_n} C_n(\mathrm{i}). \tag{4.1.3}$$

By $[KR1]$, for each n we have $C_n \subseteq C_{n-1}$. In particular, $\{C_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of clopen sets in $\Xi_{\Lambda(x_0)}$. Following (4.1.1), for every n there is a repetitive Delone set of finite type called the set of return vectors $\mathcal{R}_{C_n}(\Lambda(x_0)) \coloneqq \{t \in \mathbb{R} \mid \Lambda(x_0) - t \in C_n\}$.

For all $n \in \mathbb{N}, \mathrm{i} \in \{1, \ldots, t_n\}$ and for all x and y in $B_n(\mathrm{i})$, we have

$$x_{[-(h_n(\mathrm{i})-1),0]} = y_{[-(h_n(\mathrm{i})-1),0]}. \tag{4.1.4}$$

For all integer $n \geq 1$ and i in $\{1, \ldots, t_n\}$, choose $x \in B_n(\mathrm{i})$ and define

$$L_n(\mathrm{i}) = \sum_{j=-(h_n(\mathrm{i})-1)}^{0} l_{x_j}.$$

By (4.1.4), we have that $L_n(\mathrm{i})$ does not depend on $x \in B_n(\mathrm{i})$. For all $n \geq 1$, we define the vectors of lenghts by

$$\vec{L}(n) = \begin{bmatrix} L_n(\mathrm{i}) \\ \vdots \\ L_n(t_n) \end{bmatrix} \quad \text{and} \quad \vec{L}(0) = \begin{bmatrix} l_1 \\ \vdots \\ l_q \end{bmatrix}.$$

Observe that for all $n \geq 1$ we have $\vec{L}(n) = M(n)\vec{L}(n-1)$, and thus $\vec{L}(n) = P(n)\vec{L}(0)$. Denote by $\mathcal{L}_n$ the set of lenghts $\{L_n(1), \cdots, L_n(t_n)\}$. Notice that

$$\mathcal{L}_n \subseteq \mathcal{R}_{C_n}(\Lambda(x_0)) - \mathcal{R}_{C_n}(\Lambda(x_0)) \quad \text{and} \quad \mathcal{R}_{C_n}(\Lambda(x_0)) \subseteq \langle \mathcal{L}_n \rangle.$$

Using the same strategy of the proof in [CDHM, Theorem 10], one can give a proof of the following result.

Proposition 4.1.2. *Let x_0 be a linearly repetitive, aperiodic sequence in $\mathcal{A}^{\mathbb{Z}}$, and denote by $\Omega_{\Lambda(x_0)}$ the hull associated to the Delone set $\Lambda(x_0)$. Let μ be the unique invariant measure for $(\Omega_{\Lambda(x_0)}, \mathbb{R})$. Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of KR-partitions verifying $[KR0] - [KR6]$ and $[LR]$. Let α be a real number. If α is an eigenvalue of $(\Omega_{\Lambda(x_0)}, \mathbb{R}, \mu)$, then $\sum_{n\geq 2} |||\alpha P(n)\vec{L}(0)|||^2 < \infty$.*

Using Proposition 4.1.2 we get the following theorem analogous to Theorem D.

Theorem D'. *Let x_0 be a linearly repetitive, aperiodic sequence in $\mathcal{A}^{\mathbb{Z}}$. Denote by $X = \overline{\{\sigma^n(x_0) \mid n \in \mathbb{Z}\}}$, and by $\Omega_{\Lambda(x_0)}$ the hull associated to the Delone set $\Lambda(x_0)$. Let μ be the unique invariant measure for $(\Omega_{\Lambda(x_0)}, \mathbb{R})$. Let $(C_n)_{n\in\mathbb{N}}$ the sequence of clopen associated to a sequence of KR-partitions for $(X, \mathbb{Z})$ satisfying the properties $[KR0] - [KR6]$ and $[LR]$. If $\Lambda(x_0)$ is a Meyer set and for every n in $\mathbb{N}$ there is a base of the group $\langle \mathcal{R}_{C_n}(\Lambda(x_0)) \rangle$ included in $\mathcal{L}_n$, then every eigenvalue of $(\Omega_{\Lambda(x_0)}, \mathbb{R}, \mu)$ is continuous.*

The proof of this result is the same of Theorem D.

4.2 Proof of Theorem D.

The idea of the proof is simple. We used a decreasing sequence of clopen in the transversal to obtain topological factors of the hull system. Finally, we use the second part of Theorem 1.2.2

and Proposition A to get that each eigenvalue of the hull system is a continuous eigenvalue for some of these factors.

Let Λ be a linearly repetitive aperiodic Meyer set in $\mathbb{R}^{\mathrm{d}}$ and let μ be the unique invariant measure for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Fix α in $\mathbb{R}^{\mathrm{d}}$ being an eigenvalue for $(\Omega_\Lambda, \mu, \mathbb{R}^{\mathrm{d}})$. We want to prove that α is a continuous eigenvalue. We recall that for each Λ' in Ω_Λ and every clopen set C in Ξ_Λ, the set of return vectors to C is defined as

$$\mathcal{R}_C(\Lambda') \coloneqq \{t \in \mathbb{R}^{\mathrm{d}} \mid \Lambda' - t \in C\}.$$

We used Theorem 1.2.2 for Λ_0 in Ξ_Λ. Hence there is $(C_n)_{n\in\mathbb{N}}$ a decreasing sequence of clopen sets in Ξ_Λ containing Λ_0, and a sequence of finite sets $(\vec{\mathcal{F}}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathrm{d}}$. These sets verify that for every $n \in \mathbb{N}$ we have

$$\vec{\mathcal{F}}_n \subseteq \mathcal{R}_{C_n}(\Lambda_0) - \mathcal{R}_{C_n}(\Lambda_0) \quad \text{and} \quad \mathcal{R}_{C_n}(\Lambda_0) \subseteq \langle \vec{\mathcal{F}}_n \rangle. \tag{4.2.1}$$

By [AC11, Lemma 2.2] for each clopen set C in Ξ_Λ and $\widetilde{\Lambda} \in \Omega_\Lambda$, the set $\mathcal{R}_C(\widetilde{\Lambda})$ is a repetitive Delone set. Actually, $\mathcal{R}_C(\widetilde{\Lambda})$ is a Meyer set because Λ is a Meyer set. Thus, for every n the set $\mathcal{R}_{C_n}(\Lambda_0)$ is a repetitive Meyer set contained in Λ_0. By hypothesis, we assume that $\vec{\mathcal{F}}_n$ contains a basis of the Abelian generated by $\mathcal{R}_{C_n}(\Lambda_0)$.

Repetitivity of Λ implies that the hull systems associated for Λ and Λ_0 are conjugated. The following result provides a factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$, from the hull system associated with $\mathcal{R}_{C_n}(\Lambda_0)$.

Lemma H. *Let C be a clopen in Ξ_Λ and fix $\widetilde{\Lambda}$ in Ω_Λ. The dynamical system $(\Omega_{\mathcal{R}_C(\widetilde{\Lambda})}, \mathbb{R}^{\mathrm{d}})$ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. In particular, if α in $\mathbb{R}^{\mathrm{d}}$ is a continuous eigenvalue for $(\Omega_{\mathcal{R}_C(\widetilde{\Lambda})}, \mathbb{R}^{\mathrm{d}})$ then α is a continuous eigenvalue for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.*

Proof. By repetitivity of Λ the system $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$ is topologically conjugated to $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Thus is sufficient to prove that $(\Omega_{\mathcal{R}_C(\widetilde{\Lambda})}, \mathbb{R}^{\mathrm{d}})$ is a topological factor of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$. Define the map $\Pi : \Omega_{\widetilde{\Lambda}} \to \Omega_{\mathcal{R}_C(\widetilde{\Lambda})}$ by

$$\Pi(x) = \{t \in \mathbb{R}^{\mathrm{d}} \mid x - t \in C\}.$$

By definition we have $\Pi(\widetilde{\Lambda}) = \mathcal{R}_C(\widetilde{\Lambda})$. Observe that for each $R > 0$ big enough there is a positive number $r < R$, such that for all $x, y \in \Omega_{\widetilde{\Lambda}}$ with $x \cap B(0,R) = y \cap B(0,R)$ we have

$$\Pi(x) \cap B(0,r) = \Pi(y) \cap B(0,r).$$

Moreover, r goes to infinity when R goes to infinity. Thus, Π is well-defined and clearly continuous. For all $x \in \Omega_{\widetilde{\Lambda}}$ and t in $\mathbb{R}^{\mathrm{d}}$ we have

$$\begin{aligned}
\Pi(x-t) &= \{w \in \mathbb{R}^{\mathrm{d}} \mid (x-t) - w \in C\} \\
&= \{w \in \mathbb{R}^{\mathrm{d}} \mid x - (t+w) \in C\} \\
&= \{u \in \mathbb{R}^{\mathrm{d}} \mid x - u \in C\} - t \\
&= \Pi(x) - t.
\end{aligned}$$

In particular, since $\overline{\{\Pi(\widetilde{\Lambda}) - t \mid t \in \mathbb{R}\}} = \Omega_{\mathcal{R}_C(\widetilde{\Lambda})}$, this and the continuity of Π implies that Π is surjective. We conclude that Π define a topological factor between $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$ and $(\Omega_{\mathcal{R}_C(\widetilde{\Lambda})}, \mathbb{R}^{\mathrm{d}})$.

Let $\alpha \in \mathbb{R}^{\mathrm{d}}$ be a continuous eigenvalue for $(\Omega_{\mathcal{R}_C(\widetilde{\Lambda})}, \mathbb{R}^{\mathrm{d}})$. By definition, there exists a continuous function $f : \Omega_{\mathcal{R}_C(\widetilde{\Lambda})} \to \mathbb{C}$ such that for all $x \in \Omega_{\mathcal{R}_C(\widetilde{\Lambda})}$ and $t \in \mathbb{R}^{\mathrm{d}}$, we have

$$f(x-t) = \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot t} f(x).$$

Using the factor map Π, define the continuous function $\hat{f} : \Omega_{\widetilde{\Lambda}} \to \mathbb{C}$ by $\hat{f}(w) := f(\Pi(w))$. Clearly, for all $w \in \Omega_{\widetilde{\Lambda}}$ and $t \in \mathbb{R}^{\mathrm{d}}$ we have

$$\hat{f}(w-t) = f(\Pi(w-t)) = f(\Pi(w)-t) = \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot t} f(\Pi(w)) = \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot t}\hat{f}(w).$$

Concluding that α is a continuous eigenvalue of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$. $\square$

For each n, denote by s_n the rank of $\mathcal{R}_{C_n}(\Lambda_0)$. Using the hypothesis, there is a subset of $\vec{\mathcal{F}}_n$ with s_n rationally independent vectors $\mathcal{V}_n = \{v_{n,1}, \ldots, v_{n,s_n}\}$ such that $\langle \mathcal{R}_{C_n}(\Lambda_0)\rangle = \mathbb{Z}[\mathcal{V}_n]$. Observe that $\vec{0} \in \mathbb{R}^{\mathrm{d}}$ belongs to $\mathcal{R}_{C_n}(\Lambda_0)$, hence $\langle \mathcal{R}_{C_n}(\Lambda_0) - \mathcal{R}_{C_n}(\Lambda_0)\rangle = \langle \mathcal{R}_{C_n}(\Lambda_0)\rangle$, and by (4.2.1), we have

$$\langle \mathcal{R}_{C_n}(\Lambda_0)\rangle = \langle \vec{\mathcal{F}}_n\rangle.$$

Note that, for every positive integer n we have that $\mathcal{R}_{C_{n+1}}(\Lambda_0) \subseteq \mathcal{R}_{C_n}(\Lambda_0)$. This implies that there is a finite collection $\{z_{\mathrm{i},j}^{(n)} \in \mathbb{Z} \mid 1 \le \mathrm{i} \le s_n,\ 1 \le j \le s_{n+1}\}$, such that for every index $1 \le j \le s_{n+1}$ we have

$$v_{n+1,j} = z_{1,j}^{(n)}\ v_{n,1} + \cdots + z_{s_n,j}^{(n)}\ v_{n,s_n}. \tag{4.2.2}$$

We write $V(n) = [v_{n,1} \cdots v_{n,s_n}]$ the matriz of size $\mathrm{d} \times s_n$ whose columns are the vectors $\{v_{n,1}, \ldots, v_{n,s_n}\}$, and by $Z(n)$ the matrix of size $s_n \times s_{n+1}$ with integer coefficients form by the collection $(z_{\mathrm{i},j}^{(n)})_{\substack{1\le \mathrm{i}\le s_n\\ 1\le j\le s_{n+1}}}$. Then the following relation is satisfies

$$V(n+1) = V(n)\cdot Z(n), \quad \text{in particular} \quad V(n) = V(1)\cdot Z(1)\cdots Z(n-1). \tag{4.2.3}$$

Remark 4.2.1. From the hypothesis of Theorem D, for each n the basis $\mathcal{V}_n$ is contained in $\vec{\mathcal{F}}_n$. The last part of Theorem 1.2.2, implies that each vector in $\mathcal{V}_{n+1}$ is an integer linear combination with less than M vectors in $\vec{\mathcal{F}}_n$. From this and (4.2.1), each vector in $\mathcal{V}_{n+1}$ is an integer linear combination with less than $2M$ vectors in $\mathcal{V}_n$. Hence the sequence $\{\|Z(n)\|_{op}\}_{n\in\mathbb{N}}$ is bounded.

For any pair of positive integers $1 \le m < n$, we denote by $Q(m,n)$ the matrix with integer coefficients of size $s_m \times s_n$ given by

$$Q(m,n) = Z(m)\cdot Z(m+1)\cdots Z(n-1), \text{ and } Q(n) = Q(1,n).$$

Observe that by definition and (4.2.3), we have

$$Q(n)\cdot Z(n) = Q(n+1) \quad \text{and} \quad V(n) = V(1)\cdot Q(n).$$

Lemma 4.2.2. *Let Λ be a linearly repetitive Delone set. Let u be a real vector such that $\|u^T \cdot Q(n)\|$ converges to zero as $n \to \infty$. Then, there are $m \in \mathbb{N}$, an integer vector w, and a real vector v such that $u^T \cdot Q(m) = w^T + v^T$. The vector v^T verifies that $||v^T \cdot Q(m,n)||$ converges to 0 when n goes to infinity.*

Proof. By hypothesis, for all $n \in \mathbb{N}$ exist $w_n \in \mathbb{Z}^{s_n}$ and $\epsilon_n \in \mathbb{R}^{s_n}$ such that

$$u^T \cdot Q(n) = w_n^T + \epsilon_n^T \quad \text{and} \quad \|\epsilon_n\|_{s_n} \xrightarrow{n\to\infty} 0.$$

Note that $\|\epsilon_n^T \cdot Z(n) - \epsilon_{n+1}^T\|_{s_{n+1}} \leq \|\epsilon_n\|_{s_n} \cdot \|Z(n)\|_{op} + \|\epsilon_{n+1}\|_{s_{n+1}}$. By 4.2.1, the set $\{\|Z(n)\|_{op}\}_{n\in\mathbb{N}}$ is bounded. This implies that

$$\|\epsilon_n^T \cdot Z(n) - \epsilon_{n+1}^T\|_{s_{n+1}} \xrightarrow{n\to\infty} 0.$$

Thus $\epsilon_{n+1}^T + w_{n+1}^T = u^T \cdot Q(n+1) = u^T \cdot Q(n) \cdot Z(n) = \epsilon_n^T \cdot Z(n) + w_n^T \cdot Z(n)$, and we have

$$\epsilon_n^T \cdot Z(n) - \epsilon_{n+1}^T = w_{n+1}^T - w_n^T \cdot Z(n) \in \mathbb{Z}^{s_{n+1}}.$$

Hence, the sequence $\{\epsilon_n^T \cdot Z(n) - \epsilon_{n+1}^T\}_{n\in\mathbb{N}}$ is eventually zero. Concluding that there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we get $\epsilon_m^T \cdot Z(m) = \epsilon_{m+1}^T$. This implies that for all $n > m \geq m_0$ we have

$$\epsilon_{n+1}^T = \epsilon_m^T \cdot Q(m,n) \quad \text{and} \quad \|\epsilon_m^T \cdot Q(m,n)\|_{s_{n+1}} = \|\epsilon_{n+1}^T\|_{s_{n+1}} \xrightarrow{n\to\infty} 0.$$

Finally, we conclude the lemma for $w = w_{m_0}$ and $v = \epsilon_{m_0}$. ☐

By definition, for every n in $\mathbb{N}$, the set $\mathcal{V}_{n+1}$ form a basis for $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$ as Abelian group. Thus, there is a unique way to write elements of $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$ using vectors in $\mathcal{V}_{n+1}$. Hence, for every t in the group $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$ there exists a unique collection $\{k_{j,n+1}\}_{1\leq j\leq s_{n+1}}$ of integer numbers such that

$$t = \sum_{j=1}^{s_{n+1}} k_{j,n+1}\, v_{n+1,j}.$$

The address map $\varphi_{n+1} : \langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle \to \mathbb{Z}^{s_{n+1}}$ of $\mathcal{R}_{C_{n+1}}(\Lambda_0)$, introduced in §2.1.1, is defined by

$$t = \sum_{j=1}^{s_{n+1}} k_{j,n+1}\, v_{n+1,j} \longmapsto \varphi_{n+1}(t) = \begin{bmatrix} k_{1,n+1} \\ \vdots \\ k_{s_{n+1},n+1} \end{bmatrix}.$$

Observe that every t in the group $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$ verifies that $t = V(n+1)\cdot\varphi_{n+1}(t)$. For t in $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$, by (4.2.2) we have

$$t = \sum_{j=1}^{s_{n+1}} k_{j,n+1}\, v_{n+1,j} = \sum_{j=1}^{s_{n+1}} k_{j,n+1} \left(\sum_{\mathrm{i}=1}^{s_n} z_{\mathrm{i},j}^{(n)}\, v_{n,\mathrm{i}} \right) = \sum_{\mathrm{i}=1}^{s_n} \left(\sum_{j=1}^{s_{n+1}} k_{j,n+1}\, z_{\mathrm{i},j}^{(n)} \right) v_{n,\mathrm{i}},$$

which implies that $k_{\mathrm{i},n} = \sum_{j=1}^{s_{n+1}} k_{j,n+1}\, z_{\mathrm{i},j}^{(n)}$. Hence, for every n in $\mathbb{N}$ and each t in the Abelian group $\langle \mathcal{R}_{C_{n+1}}(\Lambda_0)\rangle$ we have

$$\varphi_n(t) = Z(n)\cdot\varphi_{n+1}(t). \tag{4.2.4}$$

Since for all n the set $\mathcal{R}_{C_n}(\Lambda_0)$ is a repetitive Meyer set, there is a linear map $\ell_n : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{s_n}$ that approximates the address map φ_n (see §2.1.1). By Theorem 2.1.1 this linear map is unique, and we define

$$\xi_n \coloneqq \sup_{t\in\mathcal{R}_{C_n}(\Lambda_0)} \|\varphi_n(t) - \ell_n(t)\|_{s_n} < +\infty.$$

Denote by $A(n)$ the real matrix of size $s_n \times \mathrm{d}$, that represent the linear map ℓ_n in the canonical bases. Recall that for each $t \in \mathcal{R}_{C_{n+1}}(\Lambda_0) \subset \mathcal{R}_{C_n}(\Lambda_0)$. By (4.2.4) we have

$$\begin{aligned} \|A(n)\cdot t - Z(n)\cdot A(n+1)\cdot t\|_{s_n} &\leq \|A(n)\cdot t - \varphi_n(t)\|_{s_n} + \|\varphi_n(t) - Z(n)\cdot A(n+1)\cdot t\|_{s_n} \\ &\leq \xi_n + \|Z(n)\|_{op}\cdot \xi_{n+1} \end{aligned}$$

Thus, dividing by the norm of t and using Lemma B, we have

$$\|A(n) - Z(n)\cdot A(n+1)\|_{op} \leq \sup_{t\in\mathcal{R}_{C_{n+1}}(\Lambda_0)} \frac{\xi_n + \|Z(n)\|_{op}\cdot\xi_{n+1}}{\|t\|_{\mathrm{d}}}.$$

From Remark 4.2.1 the sequence of real positive numbers $\{\|Z(n)\|_{op}\}_{n\in\mathbb{N}}$ is bounded. Hence we have that

$$\|A(n) - Z(n)\cdot A(n+1)\|_{op} = 0, \quad \text{and} \quad A(n) = Z(n)\cdot A(n+1).$$

Observe that for every n, we have $V(n)\cdot A(n) = V(n)\cdot Z(n)\cdot A(n+1) = V(n+1)\cdot A(n+1)$. For all $x \in \langle \mathcal{R}_{C_1}(\Lambda_0)\rangle$ we have that

$$\|x - V(1)\cdot A(1)\cdot x\|_{\mathrm{d}} = \|V(1)\,\varphi_1(x) - V(1)\cdot A(1)\cdot x\|_{\mathrm{d}} \leq \|V(1)\|_{op}\cdot\xi_1.$$

Dividing by $\|x\|_{\mathrm{d}}$, from Lemma B, we get that the sequence of matrices $\{V(n)\cdot A(n)\}$ is constant and equal to the identity matrix $\mathrm{Id} = V(n)\cdot A(n)$ of size $\mathrm{d}\times\mathrm{d}$. For all $1 \leq m < n$, the following relation is satisfied.

$$A(m) = Q(m,n)\cdot A(n). \tag{4.2.5}$$

When Λ is linearly repetitive, we can prove that the family of matrices $A(n)$ is bounded in the operator norm. This is stated in the following result.

Lemma I. *If Λ is a linearly repetitive Meyer set, then the family matrices $\{A(n)\}_{n\in\mathbb{N}}$ defined before has bounded coefficients.*

Proof. We descompose $Z(n) = Z(n)^+ - Z(n)^-$ and $A(n) = A(n)^+ - A(n)^-$, where the matrices $Z(n)^+, Z(n)^-, A(n)^+, A(n)^-$ has non-negative coefficients. Consider the matrices $\tilde{Z}(n) \in M_{2s_n\times 2s_{n+1}}(\mathbb{Z}_0^+)$ and $\tilde{A}(n) \in M_{2s_n\times\mathrm{d}}(\mathbb{R}_0^+)$, defined by

$$\tilde{Z}(n) = \begin{bmatrix} Z(n)^+ & Z(n)^- \\ Z(n)^- & Z(n)^+ \end{bmatrix} \quad \text{and} \quad \tilde{A}(n) = \begin{bmatrix} A(n)^+ \\ A(n)^- \end{bmatrix}.$$

Since $A(n) = Z(n)\cdot A(n+1) = (Z(n)^+ - Z(n)^-)\cdot(A(n+1)^+ - A(n+1)^-)$ we have that

$$A(n)^+ = Z(n)^+\cdot A(n+1)^+ + Z(n)^-\cdot A(n+1)^-,$$

and $A(n)^- = Z(n)^+\cdot A(n+1)^- + Z(n)^-\cdot A(n+1)^+$. Hence,

$$\begin{aligned} \tilde{Z}(n)\cdot\tilde{A}(n+1) &= \begin{bmatrix} Z(n)^+ & Z(n)^- \\ Z(n)^- & Z(n)^+ \end{bmatrix}\cdot\begin{bmatrix} A(n+1)^+ \\ A(n+1)^- \end{bmatrix} \\ &= \begin{bmatrix} Z(n)^+\cdot A(n+1)^+ + Z(n)^-\cdot A(n+1)^- \\ Z(n)^+\cdot A(n+1)^- + Z(n)^-\cdot A(n+1)^+ \end{bmatrix} \\ &= \tilde{A}(n). \end{aligned}$$

Thus for all $n \in \mathbb{N}$, the coefficients of $\tilde{A}(n)$ are a positive integer linear combination of the coefficients of $\tilde{A}(n+1)$. We conclude that the absolute value of the highest coeficient of $\tilde{A}(n)$ form a non-increasing sequence of positive real numbers. We conclude the lemma since the highest coefficient of $\tilde{A}(n)$ is equal to the highest absolute value of the coefficients of $A(n)$. □

Finally, we prove Theorem D. We assume that α is an eigenvalue, possibly non-continuous eigenvalue, for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. Using Theorem 1.2.2 and the structure of the Meyer sets $\mathcal{R}_{C_n}(\Lambda_0)$, we will conclude that it must be a continuous eigenvalue for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$.

Proof of Theorem D. Let α be an eigenvalue for $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$. We assume that for every integer $n \geq 1$ there is a basis $\mathcal{V}_n$ of the group $\langle \mathcal{R}_{C_n}(\Lambda_0) \rangle$ included in $\vec{\mathcal{F}}_n$.

By definition of the matrices $V(n)$, and since $\mathcal{V}_n \subseteq \vec{\mathcal{F}}_n$ we have

$$\|\alpha^T \cdot V(1) \cdot Q(n)\| = \|\alpha^T \cdot V(n)\| = \max_{v \in \mathcal{V}_n} \|\alpha^T \cdot v\| \leq \max_{v \in \vec{\mathcal{F}}_n} \|\alpha^T \cdot v\|.$$

Theorem 1.2.2 applied to the linearly repetitive Meyer set Λ, implies that

$$\lim_{n \to \infty} \max_{v \in \vec{\mathcal{F}}_n} \|\alpha^T \cdot v\| = 0.$$

Using Lemma 4.2.2 for $u^T = \alpha^T \cdot V(1)$, we have there is a positive integer m, an integer vector w, and a real vector ϵ such that

$$\alpha^T \cdot V(1) \cdot Q(m) = w^T + \epsilon^T, \tag{4.2.6}$$

with $\lim_{n \to \infty} \|\epsilon^T \cdot Q(m,n)\| = 0$. Recall that the sequence of matrices $\{V(n) \cdot A(n)\}$ is constant and equal to the identity matrix. Multiplying (4.2.6) by the matrix $A(m)$, and using (4.2.5), we have

$$\alpha^T = \alpha^T \cdot V(1) \cdot A(1) = \alpha^T \cdot V(1) \cdot Q(m) \cdot A(m) = w^t \cdot A(m) + \epsilon^t \cdot A(m). \tag{4.2.7}$$

Moreover, for each $n > m+1$ we have

$$\|\epsilon^t \cdot A(m)\|_{\mathrm{d}} = \|\epsilon^t \cdot Q(m,n) \cdot A(n)\|_{\mathrm{d}} \leq \|\epsilon^t \cdot Q(m,n)\|_{s_n} \cdot \|A(n)\|_{op}.$$

Hence, by Lemma 4.2.2 and Lemma I, taking limit when n goes to $+\infty$ we conclude that $\epsilon^t \cdot A(m) = 0$. Thus, by (4.2.7) we have

$$\alpha^t = w^t \cdot A(m).$$

This implies that α is an integer linear combination of the rows of $A(m)$. By Remark 3.1.1, we have that α is a continuous eigenvalue of $(\Omega_{\mathcal{R}_{C_m}(\Lambda_0)}, \mathbb{R}^{\mathrm{d}})$. Using Lemma H, we conclude the proof. □

Chapter 5

Examples

In this chapter, we show some examples where we can compute the continuous eigenvalues using the method of the address system. In the first example §5.1, we consider a sequence in $m+1$ symbols. We tile the real line with this sequence, by assigning lengths to each symbol. We put a condition in the sequence to obtain a Meyer set, and we find some continuous eigenvalues of its associated hull system. In §5.2, we used the previous example to construct Meyer sets in the real line with different sets of eigenvalues. Specifically, in §5.2.1 we construct a Meyer set whose associated hull system has two rationally independent continuous eigenvalues. While in §5.2.2, we obtain a Meyer set with a hull system where the continuous eigenvalue group has rank one. In §5.3 we used the fixed point of a primitive, Pisot substitution to obtain a Delone set. Using a result of Adamczewski [A04] and Corollary B, we describe some continuous eigenvalues for the associated dynamical system. In §5.4 we give two examples to show that some hypotheses are necessary for Theorem D.

5.1 Almost linear sequences.

For this example, we construct Delone sets in $\mathbb{R}$ using sequence in $m+1$ symbols and associating lengths to each symbol. For some sequences, we obtain a Meyer set with rank $m+1$. This construction of Meyer sets from a sequence with two symbols can be found in [L99]. Then we apply Proposition A, to construct $m+1$ rationally independent continuous eigenvalues of the associated hull system.

Let $s = (s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}}$ be a sequence in $m+1$ symbols that we denoted by $\{0, \ldots, m\}$. For every $\mathrm{i} \in \mathbb{Z}$ and each symbol a in $\{0, \ldots, m\}$, we define

$$sign(\mathrm{i}) := \begin{cases} 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n \leq -1 \end{cases} \quad \text{and} \quad S_{\mathrm{i}}(a) := \begin{cases} \sum_{k=0}^{\mathrm{i}-1} \mathbb{1}_{\{a\}}(s_k) & \text{if } \mathrm{i} \geq 1, \\ 0 & \text{if } \mathrm{i} = 0, \\ \sum_{k=\mathrm{i}}^{-1} \mathbb{1}_{\{a\}}(s_k) & \text{if } \mathrm{i} \leq -1. \end{cases}$$

Definition 5.1.1. A sequence $s = (s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \{0, \ldots, m\}^{\mathbb{Z}}$ is *almost linear* if there exists a finite collection of real positive numbers $\{\gamma_a\}_{a=0}^{m}$ and some constant $C > 0$ such that for all $\mathrm{i} \in \mathbb{Z}$ we have $\max_{0\leq a\leq m} |S_{\mathrm{i}}(a) - \mathrm{i}\; sign(\mathrm{i})\; \gamma_a| \leq C$.

Let $\{\alpha_0, \ldots, \alpha_m\}$ be a finite collection of real positive numbers. We define the Delone set $D_{\alpha_0,\ldots,\alpha_m}(s)$, as the collection of real numbers $(t_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}}$ defined by

$$t_{\mathrm{i}} = sign(\mathrm{i}) \sum_{a=0}^{m} S_{\mathrm{i}}(a)\ \alpha_a.$$

In [L99], Lagarias proved (for $m = 1$) the following theorem that characterizes when $D_{\alpha_0,\alpha_1}(s)$ is a Meyer set.

Theorem 5.1.1. *[L99, Theorem 5.1] Let $s = (s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ be a symbol sequence. If s is almost linear, then $D_{\alpha_0,\alpha_1}(s)$ is a Meyer set for all pair of real positive numbers α_0 and α_1. If s is not almost linear, then $D_{\alpha_0,\alpha_1}(s)$ is a Delone set of finite type and satisfies:*

1. *If α_0 and α_1 are rationally dependent then $Rank(D_{\alpha_0,\alpha_1}(s)) = 1$, and $D_{\alpha_0,\alpha_1}(s)$ is a Meyer set.*
2. *If α_0 and α_1 are rationally independent then $Rank(D_{\alpha_0,\alpha_1}(s)) = 2$, and $D_{\alpha_0,\alpha_1}(s)$ is not a Meyer set.*

We extend the first part of this result for $m+1$ symbols and using Proposition A we give an explicit description of some rationally independent continuous eigenvalues for the hull system $(\Omega_{D_{\alpha_0,\ldots,\alpha_m}(s)}, \mathbb{R})$.

Proposition E. *If $s = (s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \{0,\ldots,m\}^{\mathbb{Z}}$ is an almost linear sequence, then for every finite collection of rationally independent positive numbers $\alpha_0, \ldots, \alpha_m$, the set $D_{\alpha_0,\ldots,\alpha_m}(s)$ is a Meyer set with rank $m+1$.*

Proof. By hypothesis, the numbers $\alpha_0, \ldots, \alpha_m$ are rationally independent. Thus the rank of $D_{\alpha_0,\ldots,\alpha_m}(s)$ is equal to $m+1$, and the Delone set $D_{\alpha_0,\ldots,\alpha_m}(s)$ is finitely generated. The address map $\varphi : \langle D_{\alpha_0,\ldots,\alpha_m}(s)\rangle \to \mathbb{Z}^{m+1}$ obtained from the basis $\{\alpha_0, \ldots, \alpha_m\}$ is defined by

$$\varphi(t_{\mathrm{i}}) = sign(\mathrm{i}) \begin{bmatrix} S_{\mathrm{i}}(0) \\ \vdots \\ S_{\mathrm{i}}(m) \end{bmatrix}.$$

By Theorem 2.1.1, $D_{\alpha_0,\ldots,\alpha_m}(s)$ is a Meyer set if and only if $D_{\alpha_0,\ldots,\alpha_m}(s)$ is finitely generated and the address map is almost linear on $D_{\alpha_0,\ldots,\alpha_m}(s)$. Hence, we consider a linear map $L : \mathbb{R} \to \mathbb{R}^{m+1}$ given by

$$L(t) = \begin{bmatrix} \beta_1\ t \\ \vdots \\ \beta_{m+1}\ t \end{bmatrix}.$$

For every k in $\{1, \ldots, m+1\}$, we define $\beta_k = \frac{\gamma_{k-1}}{\sum_{a=0}^{m} \gamma_a\ \alpha_a}$. We proved that there is a constant $\eta > 0$ such that for every integer number i we have

$$\|\varphi(t_{\mathrm{i}}) - L(t_{\mathrm{i}})\|_{m+1} \leq \eta. \tag{5.1.1}$$

For $k \in \{1, \dots, m+1\}$, note that the k-coordinate of $\varphi(t_{\rm i}) - L(t_{\rm i})$ is given by the expression

$$(\varphi(t_{\rm i}) - L(t_{\rm i}))_k = sign({\rm i})\ S_{\rm i}(k-1) - \beta_k\ sign({\rm i}) \sum_{a=0}^{m} S_{\rm i}(a)\ \alpha_a.$$

Hence, for every k in $\{1, \dots, m+1\}$ we have

$$\begin{aligned}
|(\varphi(t_{\rm i}) - L(t_{\rm i}))_k| &= \left| S_{\rm i}(k-1) - \frac{\gamma_{k-1}}{\sum_{b=0}^{m} \gamma_b \alpha_b} \sum_{a=0}^{m} S_{\rm i}(a) \alpha_a \right| \\
&= \frac{1}{\sum_{b=0}^{m} \gamma_b \alpha_b} \left| S_{\rm i}(k-1) \sum_{b=0}^{m} \gamma_b \alpha_b - \gamma_{k-1} \sum_{a=0}^{m} S_{\rm i}(a) \alpha_a \right| \\
&\leq \frac{1}{\sum_{b=0}^{m} \gamma_b \alpha_b} \sum_{a=0}^{m} |S_{\rm i}(k-1) \gamma_a \alpha_a - \gamma_{k-1} S_{\rm i}(a) \alpha_a| \qquad (5.1.2)
\end{aligned}$$

Observe that

$$S_{\rm i}(k-1)\gamma_a\alpha_a - \gamma_{k-1}S_{\rm i}(a)\alpha_a = \gamma_a\alpha_a\{S_{\rm i}(k-1) - {\rm i}\ sign({\rm i})\gamma_{k-1}\} - \gamma_{k-1}\alpha_a\{S_{\rm i}(a) - {\rm i}\ sign({\rm i})\gamma_a\}.$$

Thus, by definition of an almost linear sequence, we have

$$|S_{\rm i}(k-1)\gamma_a\alpha_a - \gamma_{k-1}S_{\rm i}(a)\alpha_a| \leq \{\gamma_a + \gamma_{k-1}\}\alpha_a C.$$

Denote $\Theta \coloneqq \max\{\alpha_a \mid 0 \leq a \leq m\}$ and $\Upsilon \coloneqq \max\{\beta_k \mid 1 \leq k \leq m+1\}$. By (5.1.2) we conclude that

$$|(\varphi(t_{\rm i}) - L(t_{\rm i}))_k| \leq \frac{1}{\sum_{b=0}^{m} \gamma_b \alpha_b}\ \sum_{a=0}^{m} \{\gamma_a + \gamma_{k-1}\}\Theta C \leq 2(m+1)\Upsilon\Theta C.$$

Thus we have that (5.1.1) is satisfied for $\eta = 2(m+1)\Upsilon\Theta C$. It implies that the address map of $D_{\alpha_0,\dots,\alpha_m}(s)$ is almost linear. By Theorem 2.1.1, we conclude that $D_{\alpha_0,\dots,\alpha_m}(s)$ is a Meyer set. □

Remark 5.1.2. In the previous proposition if $\alpha_0, \dots, \alpha_m$ are not rationally independent, one can prove that the set $D_{\alpha_0,\dots,\alpha_m}(s)$ is also a Meyer set but with rank strictly smaller than $m+1$. In fact, suppose that $\alpha_0, \dots, \alpha_m$ are not rationally independent. In particular, there are only $k < m+1$ rationally independent values. We denote by $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_k$ these k values. Thus there exists a matrix M with integer coefficients, k rows, and $m+1$ columns such that changes coordinates in $\alpha_0, \dots, \alpha_m$ to coordinates in $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_k$. The address map $\widetilde{\varphi} : \langle D_{\alpha_0,\dots,\alpha_m}(s)\rangle \to \mathbb{Z}^k$ obtained from the basis $\{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_k\}$, and associated for the finitely generated Delone set $D_{\alpha_0,\dots,\alpha_m}(s)$, is given by

$$\widetilde{\varphi}(t_{\rm i}) = M \cdot sign({\rm i}) \begin{bmatrix} S_{\rm i}(0) \\ \vdots \\ S_{\rm i}(m) \end{bmatrix}.$$

Hence we used the same argument in Proposition E, for the linear map $\widetilde{L} : \mathbb{R}^{\rm d} \to \mathbb{R}^k$ defined by $\widetilde{L}(t) = M \cdot L(t)$. It allows us to conclude that

$$\sup_{{\rm i}\in\mathbb{Z}} \|\widetilde{\varphi}(t_{\rm i}) - \widetilde{L}(t_{\rm i})\|_{m+1} \leq \|M\|_{op} \cdot \sup_{{\rm i}\in\mathbb{Z}} \|\varphi(t_{\rm i}) - L(t_{\rm i})\|_{m+1} \leq \|M\|_{op} \cdot 2(m+1)\Upsilon\Theta C < \infty.$$

This implies that if $s = (s_{\rm i})_{{\rm i}\in\mathbb{Z}} \in \{0, \dots, m\}^{\mathbb{Z}}$ is an almost linear sequence, then for every finite collection of positive numbers $\alpha_0, \dots, \alpha_m$, the set $D_{\alpha_0,\dots,\alpha_m}(s)$ is a Meyer set.

Observe that for sequences, repetitivity is not a consequence of almost linearity. An example of this is the sequence $(s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ defined by

$$s_{\mathrm{i}} := \begin{cases} 1 & \text{if} \quad \mathrm{i}\cdot sign(\mathrm{i}) \geq 2 \text{ and i is even}, \\ 0 & \text{if not.} \end{cases}$$

This sequence is represented by

$$\ldots 010101010.0010101010\ldots$$

Clearly, this sequence is almost linear with $\gamma_0 = \gamma_1 = \frac{1}{2}$. Because the patch 000 appears once, this sequence is not repetitive (and not linear repetitive). In general, for sequences, linearly repetitive does not imply almost linearity. Although, from the Lagarias-Pleasants Theorem [LP03], we have the following close relation.

Theorem 5.1.2. *[AC11, Theorem 1.2] Let s a linearly repetitive sequence in m symbols. There exist $\delta > 0$ and a collection of real positive numbers $\{\gamma_1, \ldots, \gamma_m\}$ such that for each symbol a we have*

$$\left| \frac{S_{\mathrm{i}}(a)}{\mathrm{i}\ sign(\mathrm{i})} - \gamma_a \right| = O((\mathrm{i}\ sign(\mathrm{i}))^{-\delta}).$$

Assuming repetitivity of the sequence, we can use Proposition A to observe that for all $1 \leq j \leq m+1$ the real numbers

$$\beta_j = \frac{\gamma_{j-1}}{\sum_{a=0}^{m} \gamma_a\ \alpha_a},$$

are continuous eigenvalue of the hull system $(\Omega_{D_{\alpha_0,\ldots,\alpha_m}(s)}, \mathbb{R})$. A consequence of this is the following result.

Corollary B. *Let $s = (s_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \{0,\ldots,m\}^{\mathbb{Z}}$ be a repetitive and almost linear symbol sequence. Use this sequence to tile the real line associating rationally independent lengths $\alpha_0,\ldots,\alpha_m$ to each symbol $0,\ldots,m$, respectively. Then we have that*

1. *The linear map approximating the address map of $D_{\alpha_0,\ldots,\alpha_m}(s)$ is given by*

$$\ell(t) = \begin{bmatrix} \beta_1\ t \\ \vdots \\ \beta_{m+1}\ t \end{bmatrix},$$

   where for each j in $\{1,\ldots,m+1\}$ we have $\beta_j = \frac{\gamma_{j-1}}{\sum_{a=0}^{m}\gamma_a\ \alpha_a}$.

2. *The set of continuous eigenvalues of $(\Omega_{D_{\alpha_0,\ldots,\alpha_m}(s)}, \mathbb{R})$ contains the Abelian group*

$$\mathbb{Z}[\beta_1,\ldots,\beta_{m+1}].$$

5.2 Two examples of Meyer sets in the real line.

In this section, we show two examples of Meyer set in $\mathbb{R}$. We used a repetitive, and almost linear sequence in $\{0,1\}^{\mathbb{Z}}$ to tile the real line, by assigning lengths to each symbol. By

Theorem 5.1.1, the vertices of these tilings form Meyer sets. When the associated lengths are rationally independent, these Meyer sets have rank equal 2.

Let x be a repetitive and almost linear sequence in two symbols such that γ_1 is an irrational number, and $\gamma_0 = 1 - \gamma_1$. We assigned length 1 to the symbol 0, and length α for the symbol 1 to construct a Delone set $D_{1,\alpha}(x)$. We used Theorem 5.1.1 and Proposition A to compute some eigenvalues of $(\Omega_{D_{1,\alpha}(x)}, \mathbb{R})$.

For each real positive number α, Theorem 5.1.1 implies that $D_{1,\alpha}(x)$ is a Meyer set. If 1 and α are rationally independent, $D_{1,\alpha}(x)$ has rank equal 2 and it is finitely generated. Using Corollary B, we have that the linear map approximating the address map of $D_{1,\alpha}(x)$ is given by

$$\ell(t) = \begin{bmatrix} \beta_1 \, t \\ \beta_2 \, t \end{bmatrix},$$

where $\beta_1 = \frac{\gamma_0}{\gamma_0 + \alpha\gamma_1}$ and $\beta_2 = \frac{\gamma_1}{\gamma_0 + \alpha\gamma_1}$. Thus Proposition A implies that the set of continuous eigenvalues for $(\Omega_{D_{1,\alpha}(x)}, \mathbb{R})$ contains the set

$$\{n \cdot \beta_1 + m \cdot \beta_2 \mid n, m \in \mathbb{Z}\}.$$

Now, we choose different values for γ_1 (see Definition 5.1.1), to obtain a Meyer set with one continuous eigenvalue, and another with two rationally independent continuous eigenvalues.

5.2.1 A Meyer set in the real line with two continuous eigenvalues.

In this example, we construct a Meyer set such that its hull dynamical system has two rationally independent continuous eigenvalues. Suppose that γ_1 is a positive irrational number. We want to show that β_1 and β_2 are rationally independent.

By contradiction, suppose that β_1 and β_2 are not rationally independent. This means that there exist integer numbers A and B that are not null such that

$$A \cdot \frac{\gamma_0}{\gamma_0 + \alpha\gamma_1} + B \cdot \frac{\gamma_1}{\gamma_0 + \alpha\gamma_1} = 0. \tag{5.2.1}$$

Observe that if A and B are equal, then they must be equal to zero. If $A \neq B$, then (5.2.1) implies that

$$A\gamma_0 + B\gamma_1 = 0.$$

Since $\gamma_0 = 1 - \gamma_1$, we have $\gamma_1 = \frac{A}{A-B}$. This is a contradiction because γ_1 is an irrational number. We conclude that the set of continuous eigenvalues for $(\Omega_{D_{1,\alpha}(x)}, \mathbb{R})$ contains the group $\{n \cdot \beta_1 + m \cdot \beta_2 \mid n, m \in \mathbb{Z}\}$, and it has rank equal 2.

5.2.2 A Meyer set in the real line with one continuous eigenvalue.

In this example, we follow the same strategy in the previous example to construct a Meyer set. But now, we suppose that γ_1 is a positive rational number. We show that in this case, β_1 and β_2 are not rationally independent.

Suppose that $\gamma_1 = \frac{p}{q}$ with p and q being coprime. Observe that β_1 and β_2 in §5.2 are not rationally independent. Because $\gamma_0 = 1 - \gamma_1$, and

$$(-p) \cdot \frac{1-\gamma_1}{\gamma_0 + \alpha\gamma_1} + (q-p) \cdot \frac{\gamma_1}{\gamma_0 + \alpha\gamma_1} = 0.$$

Thus, using $A = -p$ and $B = q - p$, we observe that β_1 and β_2 are rationally dependent. By Proposition A, the set of continuous eigenvalues for $(\Omega_{D_{1,\alpha}(x)}, \mathbb{R})$ contains the group of rank 1 given by

$$\{m \cdot \beta_2 \mid m \in \mathbb{Z}\}.$$

5.3 Meyer sets from primitive, Pisot substitutions.

Consider $u = (u_\mathrm{i})_{\mathrm{i}\in\mathbb{Z}}$ be a fixed point of a primitive substitution σ defined over the alphabet $\mathcal{A} = \{a_0, \ldots, a_m\}$, and denote by θ the highest eigenvalues of this substitution. Define

$$P := \max\{|\theta_2| \mid \theta_2 \text{ is an eigenvalue for the substitution with } \theta_2 \neq \theta\}.$$

A substitution is called *Pisot* if we have $P < 1 < |\theta|$. In this context, Adamczewski proved the following result [A04].

Theorem 5.3.1. *Let $u = (u_\mathrm{i})_{\mathrm{i}\in\mathbb{Z}}$ be a fixed point of a primitive substitution σ defined over he alphabet $\mathcal{A}$ and θ the highest eigenvalue of σ. Consider μ the natural probability measure associated with u and denotes $\mathbb{1}_a$ the characteristic function of the set $\{a\}$. If the substitution is Pisot, then exists $C > 0$ such that*

$$\Delta_N(u) := \max_{a\in\mathcal{A}} \left| \sum_{k=0}^{N-1} \mathbb{1}_a(u_k) - N \cdot \mu(\{a\}) \right| < C.$$

If we tile the real line, using the method in the first example for the fixed point u of the substitution, and associating lengths

$$a_0 \to 1 = \alpha_0, \quad a_1 \to \alpha_1, \quad \ldots, \quad a_n \to \alpha_n$$

we obtain a finitely generated Delone set denoted by $\widehat{\Lambda}(u) := D_{\alpha_0,\ldots,\alpha_m}(s)$.

Assuming the hypothesis in Theorem 5.3.1, we have that u is an almost linear sequence in $\mathcal{A}^{\mathbb{Z}}$. Thus, by Proposition E, the Delone set $\widehat{\Lambda}(u)$ is actually Meyer. Hence, Corollary B implies the following result.

Theorem E. *Let $u = (u_\mathrm{i})_{\mathrm{i}\in\mathbb{Z}}$ be a fixed point of a primitive substitution σ defined over the alphabet $\mathcal{A}$ and denote by θ the highest eigenvalues of this substitution. If the substitution is Pisot, and the lengths $\{\alpha_\mathrm{i}\}_{0\le\mathrm{i}\le m}$ are rationally independent. Then set of continuous eigenvalues for $(\Omega_{\widehat{\Lambda}(u)}, \mathbb{R})$ contains the set of real numbers*

$$\left\{ \frac{\mu(\{k-1\})}{\sum_{a=0}^{m} \mu(\{a\})\, \alpha_a} \mid 1 \le k \le m+1 \right\}.$$

5.4 Meyer sets from a sequence of substitutions.

This example is inspired by [BDM05, Section 6], where the authors exhibit a minimal Cantor system with all their eigenvalues not continuous except the trivial one. We obtain Delone sets in the real line from this system. In §5.4.1, we study the case of a Delone set with rank 2 with tile lengths rationally independent. We proved that the unique continuous eigenvalue for this dynamical system is 0. From [KS14, Theorem 1.1], we have that this Delone set is not Meyer. This implies that the hypothesis of a Meyer set is necessary for Theorem D. In §5.4.2, we have a Meyer set with rank 1 and tile lengths rationally dependent. We prove that the continuous eigenvalues are in $\mathbb{Z}$ and this system has non-continuous eigenvalues. This implies that the hypothesis that relates the group of return vectors and the set of first return vectors is necessary in Theorem D.

Consider the matrices $A = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Observe they commute, and have a common basis of eigenvectors $\left\{ \begin{pmatrix} \phi \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \phi \end{pmatrix} \right\}$. If we denote the eigenvalues of A by $\{\alpha_A, \beta_A\}$ and the eigenvalues of B by $\{\alpha_B, \beta_B\}$, we have the following relations

$$0 < \beta_B < 1 < \beta_A < \alpha_B < \alpha_A.$$

For each of these matrices we can associate substitutions $\sigma_A, \sigma_B : \{1,2\} \to \{1,2\}^\star$ defined by

$$\sigma_A : \begin{cases} \sigma_A(1) = 2211111 \\ \sigma_A(2) = 22211 \end{cases} \quad \text{and} \quad \sigma_B : \begin{cases} \sigma_B(1) = 211 \\ \sigma_B(2) = 21 \end{cases}.$$

In [BDM05], the authors used these substitutions to construct a sequence $\xi \in \{1,2\}^{\mathbb{Z}}$. We recall that construction. Define the sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ inductively by $v_1 = 1$, and

$$v_{n+1} = \begin{cases} \beta_A \cdot v_n \text{ ; if } n \cdot v_n \leq 1, \\ \beta_B \cdot v_n \text{ ; if } n \cdot v_n > 1. \end{cases}$$

This sequence verifies that for each positive integer n, we have $\beta_B \leq n\, v_n \leq 2\beta_A$. Hence,

$$\sum_{n\in\mathbb{N}} v_n = \infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} v_n^2 < \infty. \tag{5.4.1}$$

Now we used this sequence to define a new sequence of substitutions $(\sigma_n)_{n\in\mathbb{N}} : \{1,2\} \to \{1,2\}^\star$ by $\sigma_1 = \mathrm{Id}$ and $\sigma_{n+1} = \sigma_n \circ \sigma_{M(n)}$, where

$$M(n+1) = \begin{cases} A & \text{; if } \quad n \cdot v_n \leq 1, \\ B & \text{; if } \quad n \cdot v_n > 1. \end{cases}$$

Observe that the words $\sigma_n(1)$ and $\sigma_n(2)$ satisfies that

$$\sigma_{n+1}(1) = \sigma_n \circ \sigma_{M(n)}(1) = \ldots \sigma_n(1),$$

$$\sigma_{n+1}(2) = \sigma_n \circ \sigma_{M(n)}(2) = \sigma_n(2) \ldots.$$

The lengths of these words converges to infinity, and there exists $\xi \in \{1,2\}^{\mathbb{Z}}$ such that

$$\lim_{n\to\infty} {}^{\infty}1\sigma_n(1).\sigma_n(2)2^{\infty} = \xi.$$

For the dynamical system $(X_\xi, \mathbb{Z})$ there exists a sequence of KR-partitions such that for each level we have only two towers. The bases of these towers are describe for each integer $n \geq 1$ by

$$B_{n+1}(1) \coloneqq [\sigma_n(1)\sigma_{n+1}(1).\sigma_n(2)] \quad \text{and} \quad B_{n+1}(2) \coloneqq [\sigma_n(1)(\sigma_{n+1}(2).\sigma_n(2)].$$

The sequence of matrices $(M(n+1))_{n\in\mathbb{N}}$ agree with the sequence of incidence matrices that defines the sequence of KR-partition.

Remark 5.4.1. Briefly, we will discuss informally an idea of the proof that the linearly recurrent sequence ξ, is linearly repetitive. By the linearly recurrent property, there exists a sequence of Kakutani-Rokhlin partitions verifying some topological and combinatorial properties. We denote by $h_k(\mathrm{i})$ the height of the ith tower of kth level of the partition, and L the linear recurrent constant. By definition of towers system, for each integer $n > 0$, there is a positive integer number k such that each tower of kth level we have $n \leq h_k(\mathrm{i})$, and there is some tower of $(k-1)$th level with $h_{k-1}(\mathrm{i}_0) < n$. Let k_0 the smallest of these k. Thus, for each word w in ξ of length n, we have two possibilities. If a translated copy of w is in some tower of k_0th level, then it appears in a ball of radius $\max_{\mathrm{i}\in I} h_{k_0+1}(\mathrm{i})$. If each translated copy of w is in the concatenation of towers of k_0th level, then w is in some tower of (k_0+1)th level. This implies that w appears in a ball of radius $\max_{\mathrm{i}\in I} h_{k_0+2}(\mathrm{i})$. Recall that $M_\xi(n)$ is the smallest radius $M > 0$ such that each closed ball of radius M contains the center of an occurrence of every n-patch of ξ. Hence, using $[LR]$ property that appears in (4.1.2), we have

$$M_\xi(n) \leq \max_{\mathrm{i}\in I} h_{k_0+2}(\mathrm{i}) \leq L^3 h_{k_0-1}(\mathrm{i}_0) \leq L^3 \cdot n.$$

We conclude that the sequence ξ is linearly repetitive.

Put lengths l_1 and l_2 in the configuration ξ to obtain a linearly repetitive Delone set which we will denote by $\Lambda_{l_1,l_2}(\xi)$.

5.4.1 Rationally independent lengths.

In this example we associate rationally independent lengths $l_1 = 1$ and $l_2 = \phi = \frac{1+\sqrt{5}}{2}$ for symbols in ξ, to obtain a Delone set $\Lambda_{l_1,l_2}(\xi)$. We recall that the Delone set $\Lambda_{l_1,l_2}(\xi)$ is linearly repetitive and the dynamical system $(\Omega_{\Lambda_{l_1,l_2}(\xi)}, \mathbb{R})$ is minimal. Since the determinants of the matrices A and B are non zero, the associated lengths at each level n of the return vectors are rationally independent. Thus at each level, the generators of the Abelian group $\langle \mathcal{R}_{C_n}(\Lambda_{l_1,l_2}(\xi))\rangle$ generated by return vectors are contained in the finite set $\mathcal{L}_n$ of lengths of level n. We will prove that the dynamical system $(\Omega_{\Lambda_{l_1,l_2}(\xi)}, \mathbb{R})$ has only a continuous eigenvalue 0, and the set of eigenvalues is given by

$$E = \left\{ \alpha \in \mathbb{R} \mid \alpha = \left(\frac{1}{2}, \frac{\phi-1}{2}\right) \cdot A^{-l} \cdot w,\ l \geq 0,\ w \in \mathbb{Z}^2 \right\}.$$

By [KS14, Theorem 1.1] $\Lambda_{l_1,l_2}(\xi)$ is not Meyer. This will show that the hypothesis $\Lambda_{l_1,l_2}(\xi)$ is a Meyer set in Theorem D is necessary

For $n > m \geq 0$ we denote $P(n,m) = M(n)M(n-1)\cdots M(m+1)$ and $P(n) = P(n,0)$. The authors in [BDM05] proved the following lemma that we have written in our context.

Lemma 5.4.2. *Take $v \in \mathbb{R}^2$ such that $\lim_{n\to\infty}\|P(n)v\| = 0$. Then v is orthogonal to the vector $\mu(0) = (\mu_1, \mu_2)$, where $\mu_1 + \mu_2\phi = 1$.*

Proof. Let $v \in \mathbb{R}^2$ such that $\lim_{n\to\infty}\|P(n)v\| = 0$. Denote by $\mu(n) := (\mu(B_n(\mathrm{i})))_{1\leq \mathrm{i}\leq t_n}$ the vector of measures of the bases at level n, and observe that $\mu(0) = P^t(n)\mu(n)$. For $n \geq 1$ we have $|\langle \mu(0), v\rangle| = |\langle P^t(n)\mu(n), v\rangle| = |\langle \mu(n), P(n)v\rangle| \leq \|P(n)v\|$. $\square$

Consider the vector of initial lengths $\vec{L}(0) = \begin{pmatrix} 1 \\ \phi \end{pmatrix}$, and denote $\vec{L}(m) = P(m)\vec{L}(0)$. Using the previous lemma we proved the following.

Proposition 5.4.3. *Take $\alpha \in \mathbb{R}$. In the previous context are equivalents:*

1. $\lim_{n\to\infty} \|\!|P(n)(\alpha\vec{L}(0))\|\!| = 0$.
2. $\alpha \in E$.

Moreover, if $\alpha \in E$ there exists a constant $c > 0$ sucht that for every positive integer n we have $\|\!|P(n)(\alpha\vec{L}(0))\|\!| = c\|P(n)v\|$.

Proof. Suppose that $\lim_{n\to\infty} \|\!|P(n)(\alpha\vec{L}(0))\|\!| = 0$. Using [CDHM, Lemma 12], there exist $m \in \mathbb{N}$, $w \in \mathbb{Z}^2$ and $v' \in \mathbb{R}^2$ sucht that $P(m)(\alpha\vec{L}(0)) = w + v'$ and $\lim_{n\to\infty} \|P(n,m)v'\| = 0$. In particular when n goes to infinity, we have that $\|P(n,m)v'\| = \|P(n)P(m)^{-1}v'\|$ goes to 0. By Lemma 5.4.2, $P(m)^{-1}v'$ is orthogonal to μ. On the other hand, the vector

$$v = -(\phi-2)\begin{pmatrix} -1 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi-2 \\ \phi-1 \end{pmatrix}$$

verifies that $P(n)v = v_n \cdot v$. Since

$$\sum_{n\in\mathbb{N}} \|P(n)v\|^2 = \sum_{n\in\mathbb{N}} \|v_n v\|^2 = \|v\| \sum_{n\in\mathbb{N}} |v_n|^2 < \infty,$$

we have $\lim_{n\to\infty} \|P(n)v\| = 0$. By Lemma 5.4.2, v is orthogonal to μ. Thus, there exists a constant $k \in \mathbb{R}$ such that $P(m)^{-1}v' = kv$.

We write $P(m)(\alpha\vec{L}(0)) = P(m)kv + w$. If $k = 0$, then $\alpha\vec{L}(m) = P(m)(\alpha\vec{L}(0)) = w \in \mathbb{Z}^2$. Since $\det(A)$ and $\det(B)$ are not zero, for all $m \in \mathbb{N}$ the coordinates of $\vec{L}(m)$ are rationally independent. Hence, we have $\alpha = 0$. If $k \neq 0$, we write $P(m)(\alpha\vec{L}(0)) = P(m)[kv + P(m)^{-1}w]$. Thus we get the equation

$$\alpha\vec{L}(0) = kv + P(m)^{-1}w. \tag{5.4.2}$$

Replacing $\vec{L}(0) = \begin{pmatrix} 1 \\ \phi \end{pmatrix}$, $v = \begin{pmatrix} \phi-2 \\ \phi-1 \end{pmatrix}$ and $P(m)^{-1}w = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2$, we obtain

$$k = \frac{1}{2}(\phi(a-b) + a).$$

Using this value in (5.4.2), we can write

$$\alpha\vec{L}(0) := \begin{pmatrix} \alpha \\ \alpha\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\phi-1}{2} \\ \frac{\phi}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

The determinants of the matrices A and B are 11 and 1 respectively. In particular, B^{-1} has integers coefficients. If we write $P(m) = A^{l_m} B^{q_m}$, for some positive integers l_m and q_m. We conclude that for some $w' \in \mathbb{Z}^2$ we have

$$\alpha = \left(\frac{1}{2}, \frac{\phi-1}{2}\right) \cdot A^{-l_m} B^{-q_m} w = \left(\frac{1}{2}, \frac{\phi-1}{2}\right) \cdot A^{-l_m} w'.$$

Hence, α is in E.

Reciprocally, suppose that $\alpha \in E$. Thus, there exist $l \geq 0$ and $w \in \mathbb{Z}^2$ such that

$$\alpha\vec{L}(0) = \begin{pmatrix} \alpha \\ \alpha\phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\phi-1}{2} \\ \frac{\phi}{2} & \frac{1}{2} \end{pmatrix} A^{-l} w.$$

Let $u = \phi \begin{pmatrix} \phi \\ 1 \end{pmatrix}$ be an eigenvector of A and B and recall that $v = \begin{pmatrix} \phi-2 \\ \phi-1 \end{pmatrix}$ verifies that $P(n)v = v_n \cdot v$. Note that the real number $k = \frac{1}{2}\langle u, A^{-l}w\rangle$ verifies

$$\frac{1}{2}\begin{pmatrix} 1 & \phi-1 \\ \phi & 1 \end{pmatrix} A^{-l} w = kv + A^{-l}w.$$

Hence, $P(n)(\alpha\vec{L}(0)) = P(n)\left\{kv + A^{-l}w\right\} = kP(n)v + P(n)A^{-l}w$. Since $P(n)A^{-l}w \in \mathbb{Z}^2$ for n large enough, we conclude

$$\lim_{n\to\infty} \|\!| P(n)(\alpha\vec{L}(0)) |\!\| = \lim_{n\to\infty} \|kP(n)v\| = k \lim_{n\to\infty} \|P(n)v\| = 0.$$

$\square$

Finally, let α be an eigenvalue of $(\Omega_{\Lambda_{l_1,l_2}(\xi)}, \mathbb{R})$. By Theorem 1.2.2, we have

$$\sum_{n\geq 2} \|\!| P(n)(\alpha\vec{L}(0)) |\!\|^2 < \infty.$$

In particular $\lim_{n\to\infty} \|\!| P(n)(\alpha\vec{L}(0)) |\!\| = 0$, and by Proposition 5.4.3, we have $\alpha \in E$.

Again by Proposition 5.4.3, if $\alpha \in E$ then $\lim_{n\to\infty} \|\!| P(n)(\alpha\vec{L}(0)) |\!\| = 0$. Thus, by (5.4.1) we have

$$\sum_{n\geq 2} \|\!| P(n)(\alpha\vec{L}(0)) |\!\|^2 = \sum_{n\geq 2} k\|P(n)v\|^2 = k\sum_{n\geq 2} \|v_n v\|^2 < \infty.$$

But by (5.4.1),

$$\sum_{n\geq 2} \|\!| P(n)(\alpha\vec{L}(0)) |\!\| = \sum_{n\geq 2} k\|P(n)v\| = k\sum_{n\geq 2} \|v_n v\| = \infty.$$

We conclude that $\Lambda_{l_1,l_2}(\xi)$ is a linearly repetitive Delone set, such that a base of the group $\langle \mathcal{R}_{C_n}(\Lambda_{l_1,l_2}(\xi))\rangle$ is contained in $\mathcal{L}_n$, and all eigenvalues are not continuous except for the trivial one. This proves that the hypothesis that the set $\Lambda_{l_1,l_2}(\xi)$ is a Meyer set, is necessary for Theorem D.

5.4.2 Rationally dependent lengths.

Consider the same context as the previous example with $\vec{L}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This means that the associated initial lengths are integer numbers. Thus the Delone set $\Lambda_{l_1,l_2}(\xi) \subseteq \mathbb{Z}$ is actually a Meyer set. The Abelian group generated by return vectors at each level is isomorphic to $\mathbb{Z}$. And for big enough levels, the set of returns vector not contains the generator of $\mathbb{Z}$.

In this case, clearly, all $\alpha \in \mathbb{Z}$ is a continuous eigenvalue. Moreover, in the same way, that in the previous example, we have that the set of eigenvalues is given by

$$\widetilde{E} = \mathbb{Z} \cup \left\{ \alpha \in \mathbb{R};\ \alpha = \left(\frac{2\phi - 1}{5}, \frac{3 - \phi}{5} \right) \cdot A^{-l} w,\ l \geq 0,\ w \in \mathbb{Z}^2 \right\}.$$

Where the only continuous eigenvalues are $\alpha \in \mathbb{Z}$. Thus the hypothesis that the generators of the Abelian groups at each level are contained in the set of returns vector, is necessary for Theorem D.

Chapter 6

Future work

In this chapter, we discuss some ideas on the topic of eigenvalues for dynamical systems. Principally, about Bratteli diagram systems §6.1 and a groupoid associated with some dynamical system §6.2. We give definitions, relations with the work in this thesis, and what we want to study soon about these contexts.

6.1 Eigenvalues for Bratteli diagrams from hull systems.

In this section we refer to Bratteli diagrams. These diagrams was introduced in [Br72] to classify some operators algebras. We define a Bratteli driagram, and we comment a necessary and sufficient condition to be a measurable eigenvalue in Bratelli-Vershik system. This condition appears in [DFM19]. After this, we give an idea about how we can obtain these diagrams from a hull space. Another construction of Bratteli diagrams from dynamical systems can be found in [BJS10]. Our goal is to obtain a similar condition for Bratteli diagrams from a hull space, and thus obtain a necessary a sufficient condition to be a measurable eigenvalue of a hull system. A necessary a sufficient condition to be a continuous eigenvalue for some hull system from tilings can be found in [FS14].

A *Bratteli diagram* is a graph $B = (V, E)$ with set of vertices V and edges E that are partitioned into disjoint, and finite subsets

$$V = \cup_{n\geq 0} V_n \quad \text{and} \quad E = \cup_{n\geq 0} E_n.$$

The set V_0 is a single point called the *root* of the diagram. There exist a range map $r : E \to V$ and a source map $s : E \to V$ such that, for every integer positive number n we have

$$r(E_n) = V_n \quad \text{and} \quad s(E_n) = V_{n-1}.$$

We assume that for all $v \in V$ and $v' \in V \setminus V_0$, we have that $s^{-1}(v) \neq \emptyset$ and $r^{-1}(v') \neq \emptyset$. We say that the Bratteli diagram is of *finite rank* if there exists a positive integer number d such that for each n we have $\#V_n \leq \mathrm{d}$. A Bratteli diagram $B' = (V', E')$ is a *telescoping* or *contraction* of $B = (V, E)$, if there exists a strictly increasing sequence of integers $(n_k)_{k\geq 0}$ such that $V'_k = V_{n_k}$ and $E'_k = E_{n_k}$.

An *infinite path* in a Bratteli diagram $B = (V, E)$ is a sequence of edges $(\mathrm{e}_n)_{n\leq 0}$ such that for every positive integer number n we have $\mathrm{e}_n \in E_n$ and $r(\mathrm{e}_n) = s(\mathrm{e}_{n+1})$. We assume that for every $v \in V$ there are at least two distinct infinite paths through v. We denote by X_B the set of infinite paths in B, starting at the root of B. There is a natural topology on X_B generated by the cylinder sets

$$U_{\mathrm{e}_1,\ldots,\mathrm{e}_k} = \{x \in X_B \mid x_n = \mathrm{e}_n \text{ for every } 1 \leq n \leq k\},$$

where $(\mathrm{e}_1, \ldots, \mathrm{e}_k)$ is a finite path in B. In [BJS10, Proposition 3.4], the authors comment that X_B doted with this topology is a Cantor set. If we put an order $\prec$ in the edges, and thus an order in the finite paths, there is a natural map $T : X_B \to X_B$ called *Vershik* map. Denote by (X_B, T) this dynamical system that we called *Bratteli-Vershik* system. Using the Vershik map, we can construct for each $n \geq 0$ a Kakutani-Rohklin partition of X_B that we denote by

$$\mathcal{P}_n = \{T^{-j}B_n(v) \mid \; v \in V_n, \quad 0 \leq j < h_n(v)\}.$$

Where $B_n(v)$ is a clopen set in X_B, and $h_n(v)$ is the length of the tower v of $\mathcal{P}_n$. Associated to these partitions we have, for each $n \geq 1$, a sequence of matrices M_n and maps τ_n and ρ_n called the *tower map* and the *first entrance vector map to the base*, respectively.

For $0 \leq m < n$, denote by $P_{m,n} = M_m \cdot M_{m+1} \cdots M_n$ and by $E_{m,n}$ the set of finite paths in X_B from a vertex u in V_m to a vertex v in V_n. For each $x \in X_B$ and integers $0 \leq m < n$, we denote the *suffix vector* at coordinate $u \in V_m$ by

$$s_{m,n}(x, u) = \#\{\mathrm{e} \in E_{m,n} \mid (x_{m+1}, \ldots, x_n) \prec \mathrm{e}, \; s(\mathrm{e}) = u\}.$$

For each $0 \leq m < n, \; u \in V_m$ and $v \in V_n$ we define

$$S_{m,n}(u, v) = \{s_{m,n}(x) \mid \tau_m(x) = u \quad \text{and} \quad \tau_n(x) = v\}.$$

If B has finite rank, by [BKMS13, Theorem 3.3], there is a contraction of B and $\delta > 0$ such that

1. for any ergodic, invariant measure μ on X_B there exists $I_\mu \subseteq \{1, \ldots, \mathrm{d}\}$ verifying:
   - for every $v \in I_\mu$ and $n \geq 1$ we have $\mu(\tau_n = v) \geq \delta$, and
   - for each $v \notin I_\mu$ we have $\lim_{n\to\infty} \mu(\tau_n = v) = 0$.
2. If μ and ν are different ergodic, invariant measures, then $I_\mu \cap I_\nu \neq \emptyset$.

A Bratteli diagram with this property is called *clean*. In this context, there exists a necessary and sufficient condition to be a measurable eigenvalue for (X_B, T).

Theorem 6.1.1. *[DFM19, Theorem 11] Let (X_B, T) be a clean Bratteli-Vershik system of finite rank. Let μ be an ergodic invariant measure in (X_B, T). For some Bratteli-Vershik systems called* propers *we have the following. The value $\lambda = \mathrm{e}^{2\pi \mathrm{i}\alpha}$ is an eigenvalue of (X_B, T) for μ if and only if one of the following conditions holds:*

1. *for all $v \in I_\mu$,*

$$\sum_{u \in I_\mu} \frac{h_m(u)}{h_n(v)} \left| \sum_{s \in S_{m,n}(u,v)} \lambda^{\langle s, h_m \rangle} \right| \xrightarrow{m \to \infty} 1,$$

uniformly for $n > m$.

2. *For all $u \in \{1, \ldots, \mathrm{d}\}$ and $v \in I_\mu$,*

$$\frac{h_m(u)}{h_n(v)} \left[P_{m,n}(u,v) - \left| \sum_{s \in S_{m,n}(u,v)} \lambda^{\langle s, h_m \rangle} \right| \right] \xrightarrow{m \to \infty} 1,$$

uniformly for $n > m$.

We knew that the transversal in the hull system has a structure of Bratteli diagram. Hence, we want to prove a similar condition to be a measurable eigenvalue but in the context of hull systems.

Let Λ be a Delone set in $\mathbb{R}^{\mathrm{d}}$ and denote by $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ its associated hull system. We construct a Bratteli diagram from this hull system in the following way. Denote by Ξ_Λ the transversal of Ω_Λ.

Theorem 6.1.2. *[AC11, Theorem 3.6] Let Λ' be a repetitive aperiodic Delone set in $\mathbb{R}^{\mathrm{d}}$. The hull system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ admit a Kakutani-Rohlin towers system.*

From this result, there exists a towers system in Ξ_Λ that we denote

$$(\mathcal{P}_n = \{C_{n,\mathrm{i}} + v \mid \mathrm{i} \in \{1, \ldots, t_n\}, v \in R_{n,\mathrm{i}}\})_{n \geq 0}.$$

Recall that for every index $\mathrm{i} \in \{1, \ldots, t_n\}$ and each index j in $\{1, \ldots, t_{n-1}\}$ we denote by

$$R_{n,\mathrm{i},j} = \{v \in R_{n,\mathrm{i}} \mid C_{n,\mathrm{i}} - v \subseteq C_{n-1,j}\}.$$

For every $n \geq 1$, we define the finite sets

$$V_n = \{1, \ldots, t_n\} \quad \text{and} \quad E_n = \cup_{\mathrm{i} \in \{1,\ldots,t_n\}} \cup_{j \in \{1,\ldots,t_{n-1}\}} R_{n,\mathrm{i},j}.$$

Denote by $B_\Lambda = (V, E)$ the associated Bratteli diagram to the Delone set Λ. Observe that, by definition of towers system, each infinite path $x = (x_n)_{n \geq 0}$ in X_{B_Λ} defines a unique Delone set D_x in Ξ_Λ by

$$D_x = \cap_{n \geq 1} C_{n, r(x_n)} + x_n.$$

It is possible to prove that $\psi : X_{B_\Lambda} \to \Xi_\Lambda$ defined by $\phi(x) = D_x$ is a homeomorphism. This implies that the associated Bratteli diagram is a representation of the transversal space Ξ_Λ. But in this context (at least for $\mathrm{d} > 1$) we don't have a Vershik map. We have only an equivalent relation. We say that $x = (x_n)_{n \geq 0}$ and $y = (y_n)_{n \geq 0}$ in X_B are *tail* equivalent if there exists a positive integer number n_0 such that for all $n \geq n_0$ we have $x_n = y_n$. This defines an equivalence relation in X_B called *tail* or *cofinal* equivalence relation. Since the edges of B_Λ is a subset of $\mathbb{R}^{\mathrm{d}}$, we represent the elements in E by vectors in $\mathbb{R}^{\mathrm{d}}$. We use this structure to represent in the space X_{B_Λ}, the groupoid action of $\mathbb{R}^{\mathrm{d}}$ in Ξ_Λ. The tail equivalent class of each x in X_{B_Λ} is equal to the orbit of the groupoid action of $\mathbb{R}^{\mathrm{d}}$ in X_{B_Λ}.

We want to use the Bratteli diagram X_{B_Λ} with this groupoid action of $\mathbb{R}^{\mathrm{d}}$ to prove a necessary and sufficient condition to be a measurable eigenvalue of $\mathfrak{G}_\Lambda$ (and by Lemma A, a measurable eigenvalue for the hull system $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$). To use Theorem 6.1.1 we need to prove or assume conditions such that X_{B_Λ} is a proper, clean, and of finite rank Bratteli diagram.

6.2 Eigenvalues for Groupoids.

This section intended to motivate our investigation of eigenvalues for groupoids. The concept of groupoid was introduced implicity by H. Brandt in the decade of the 20s via semigroups [B27]. This concept generalizes the notion of a group in many ways. The groupoid theory provides us a general frame to understand the action of a subset of a group in some space. This context is weaker than the context of dynamical system, because we don't have necessarily a group action. That is the case of the restricted action of $\mathbb{R}^{\mathrm{d}}$ in the transversal of a hull system. For this reason, we think that it is the natural context to work in the transversal of a hull system. We give definitions about groupoids and we explain what we want to study. See [R80, Mac66] for more details about the groupoid theory.

A *groupoid* is a set G endowed with

1. a product map $(x, y) \to x \cdot y \in G$ from a subset G^2 of $G \times G$ called the set of *composable pairs*,
2. and an inverse map $x \to x^{-1}$ from G to G such that the following relations are satisfied,
   (a) $(x^{-1})^{-1} = x$,
   (b) for all $(x, y), (y, z)$ in G^2 we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
   (c) if (x^{-1}, x) and (x, y) are in G^2, then $x^{-1} \cdot (x \cdot y) = y$,
   (d) if (x, x^{-1}) and (y, x) are in G^2, then $(y \cdot x) \cdot x^{-1} = y$.

Associated with the groupoid G we have two maps defined from G in itself. For each x in the groupoid the *domain* map $\mathrm{d}(x) = x \cdot x^{-1}$, and the *range* map $r(x) = x^{-1} \cdot x$. The image of these maps $G^0 = \mathrm{d}(G) = r(G)$ is a subset of G called the *unit space*. Observe that every group is a groupoid with $G^2 = G \times G$. Each equivalence relation in a set X has a structure of groupoid. Some works used groupoid to study properties of Bratteli diagrams and dynamical systems (see [R12, BJS10, FHK02]).

In the context of hull dynamical systems, let Λ be a Delone set in $\mathbb{R}^{\mathrm{d}}$. Denote by $(\Omega_\Lambda, \mathbb{R}^{\mathrm{d}})$ its hull system, and Ξ_Λ its transversal space. If we restrict the action of $\mathbb{R}^{\mathrm{d}}$ on Ω_Λ at the transversal space, we obtain a structure of groupoid. This groupoid was defined in §2.2.1 and its denoted by

$$\mathfrak{G}_\Xi = \{(x, t) \in \Xi_\Lambda \times \mathbb{R}^{\mathrm{d}} : x - t \in \Xi_\Lambda\} \subset \Xi_\Lambda \times \mathbb{R}^{\mathrm{d}}.$$

It is endowed with the induced topology from $\Xi_\Lambda \times \mathbb{R}^{\mathrm{d}}$. In this context, the first coordinate of each (x, t) in the groupoid denotes an element x in the transversal space. The second coordinate, t denotes the homeomorphism

$$\eta_t : A \to \Xi_\Lambda \quad \text{defined by} \quad \eta_t(y) = y - t,$$

and where A is a clopen set in Ξ_Λ containing x. Thus, the range map in (x,t) in $\mathfrak{G}_\Xi$ is the image of x by the homeomorphism η_t, and the domain map in (x,t) is the preimage of $\eta_t(x)$ by the homeomorphism η_t. This means that $r : \mathfrak{G}_\Xi \to \mathfrak{G}_\Xi$ and $\mathrm{d} : \mathfrak{G}_\Xi \to \mathfrak{G}_\Xi$ are defined by

$$r(x,t) = (x-t, 0) \quad \text{and} \quad \mathrm{d}(x,t) = (x,0).$$

We can identify the transversal space with the unit space $G^0 = r(\mathfrak{G}_\Xi) = \mathrm{d}(\mathfrak{G}_\Xi)$ via

$$G^0 = \{(x,t) \in \mathfrak{G}_\Xi \mid t = 0\} = \Xi_\Lambda \times \{0\} \subseteq \mathfrak{G}_\Xi.$$

We inspire in this way to write the action in the transversal space, to study some types of groupoids. Let X be a compact metric space. Suppose that $(G, \star)$ is a group acting by homeomorphism $\{\eta_t\}_{t\in G}$ on X. Let C be a closed subset of X. To study the dynamics that the action of G defines in C we have some problems. Principally, the action of some $t \in G$ at $x \in C$ may not be in C. Or maybe, the orbit of x by the elements of G never come back to C. For this reason, for each t in G, we need to consider an open subset O_t of C where the action by t is well defined. It is related to the partial action of a group in a set. These types of actions were used in [GGS17, Ex94] to study dynamical properties and are related to C^*-algebras associated with dynamical systems [Ex98]. A *partial* action of G in C is a pair $\theta = (\{O_t\}_{t\in G}, \{\eta_t\}_{t\in G})$, where for each t in G, we have that O_t is an open set in C and $\eta_t : O_{t^{-1}} \to O_t$ is a homeomorphism such that:

1. for e the identity in G, the set $O_\mathrm{e} = C$ and η_e is the identity map on C,
2. $\eta_t(O_{t^{-1}} \cap O_s) = O_t \cap O_{t\star s}$ for all t, s in G.
3. $\eta_t(\eta_s(x)) = \eta_{t\star s}(x)$ for all x in $O_{s^{-1}} \cap O_{s^{-1}\star t^{-1}}$ and t, s in G.

Consider that $(G, \star)$ defines a partial action in C. This defines a groupoid by

$$\mathfrak{G}_C = \{(x,t) \in C \times G : \eta_t(x) \in C\} \subset X \times G.$$

In effect, a pair of elements (x,t) and (y,s) are composable if and only if $\eta_t(x) = y$. The composition is defined by

$$(x,t) \cdot (y,s) = (x, t \star s).$$

The inverse map $(\cdot)^{-1} : \mathfrak{G}_C \to \mathfrak{G}_C$ is defined by $(x,t)^{-1} = (\eta_t(x), t^{-1})$. If we identify the topological space C with the pairs $(x,0)$ in $\mathfrak{G}_C$ we have that the range and domain maps are defined by

$$r(x,t) = \eta_t(x) \quad \text{and} \quad \mathrm{d}(x,t) = x.$$

Observe that for a dynamical system (X,G), the restricted action of G in a compact subset $C \subseteq X$ defines a partial action and then a groupoid in the sense above. We want to study what information we can obtain from the dynamical system from this groupoid. In particular, we will study the relationship between the eigenvalues of (X,G) and the eigenvalues of the groupoid $\mathfrak{G}_C$ for some particular set C where the dynamic is concentrated. After, we want to use this groupoid to associate groups and/or algebras to the dynamical system. These groups and algebras give us some information that we want to study, about topology or measurable structure of the dynamical systems.

6.3 Dynamical systems and C*-algebras.

In this section, we explain how we can obtain a groupoid $\mathfrak{G}$ from a dynamical system (X, G). After, we define C^*-algebras and specifically some C^*-algebras associated with groupoids. Finally, we establish some relationships between dynamical systems and C^*-algebras that we want to study with more details in the future.

Let (X, G) be a dynamical system and denote by e the identity element in G. Then the set

$$\mathfrak{G} = \{(x, h^{-1}g, x') \mid x, x' \in X, g.x = h.x'\},$$

is a topological groupoid where the unit space $\mathfrak{G}^0$ may be identified with X via the map $x \mapsto (x, \mathrm{e}, x)$. The range and source map $r, s : \mathfrak{G} \to X$ are defined respectively by

$$r(x, g, x') = x \quad \text{and} \quad s(x, g, x') = x'.$$

The composition of (x, g, x') and (y, h, y') is possible when $x' = y$, and it is given by the formula $(x, g, x') \cdot (y, h, y') = (x, hg, y')$. The inverse map is $(x, g, x')^{-1} = (x', g^{-1}, x)$, and for each $x \in \mathfrak{G}^0$ we denote

$$\mathfrak{G}^x = r^{-1}(x), \quad \mathfrak{G}_x = s^{-1}(x) \quad \text{and} \quad \mathfrak{G}_x^x = s^{-1}(x) \cap r^{-1}(x).$$

Now we define a C^*-algebra, and we construct a C^*-algebra for the groupoid $\mathfrak{G}$. After, we define a state of a C^*-algebra. These algebras were introduced by I. E. Segal in 1947 to describe some subalgebras of $B(H)$, the space of bounded operators on a Hilbert space H. A C^*-algebra A is a Banach algebra over the complex field, with an involution map $x \mapsto x^*$ that verifies for all $x, y \in A$ and $\lambda \in \mathbb{C}$

- $(x^*)^* = x$,
- $(x + y)^* = x^* + y^*$,
- $(xy)^* = y^*x^*$,
- $(\lambda x)^* = \overline{\lambda}x^*$,
- $\|xx^*\| = \|x\|^2$.

The Gelfand-Naimark-Segal theorem characterizes each C^*-algebra as a closed in norm, self-adjoint, algebra of bounded operators on a Hilbert space. Where self-adjoint means a subalgebra of bounded operators that is invariant by the adjoint map $x \mapsto x^*$.

Let $\mathfrak{G}$ be a topological groupoid and consider $C_c(\mathfrak{G})$ the space of continuous and compactly supported function on $\mathfrak{G}$. An involution for f in $C_c(\mathfrak{G})$ is given for each g in the groupoid $\mathfrak{G}$ by $f^*(g) = \overline{f(g^{-1})}$. The product is defined by

$$(f_1 \cdot f_2)(g) = \sum_{h \in \mathfrak{G}^{r(g)}} f_1(h) f_2(h^{-1}g).$$

The *full groupoid* C^*-algebra $C^*(\mathfrak{G})$ is the completion of $C_c(\mathfrak{G})$ for the norm

$$\|f\| = \sup_{\pi} \|\pi(f)\|,$$

where the supremum is taken over all $*$-representations π of $C_c(\mathfrak{G})$ on a Hilbert space.

Let A be a C^*-algebra, a linear and bounded functional $\omega : A \to \mathbb{C}$ is called *state* if it verifies that for each a in A we have $\omega(aa^*) \geq 0$, and $\|\omega\| = 1$. Denote by $Aut(A)$ the set of automorphisms of A. Let $\alpha : \mathbb{R} \to Aut(A)$ be a strongly continuous action, i.e. for each t in $\mathbb{R}$ the map $\alpha_t : A \to A$ is an automorphism, and for every a in A the map $t \mapsto \alpha_t(a)$ is continuous. An element a in A is called *analytic* (or entire) for α, if the function $t \mapsto \alpha_t(a)$ has an extension to an analytic function $\xi \mapsto \alpha_\xi(a)$ with ξ in $\mathbb{C}$.

Definition 6.3.1. Let A be a C^*-algebra and consider $\alpha : \mathbb{R} \to Aut(A)$ be a strongly continuous action. For a real number $\beta \neq 0$, a state $\omega : A \to \mathbb{C}$ satisfies the *KMS-condition* for α at inverse temperature β if $\omega(b\alpha_{i\beta}(a)) = \omega(ab)$ for all a, b in A with a analytic for α.

When the C^*-algebra comes from a dynamical system, this condition gives us some dynamical information. Let $(X, \mathbb{Z})$ be a dynamical system given by a local homeomorphism $T : X \to X$. Denote by $\mathfrak{G}$ the associated groupoid. Every ϕ in $C(X, \mathbb{R})$ defines a continuous one-cocycle by the formula

$$c_\phi(x, m-n, y) = \sum_{i=0}^{m-1} \phi(T^i(x)) - \sum_{j=0}^{n-1} \phi(T^j(y)). \tag{6.3.1}$$

Each continuous one-cocycle c defines a strongly continuous action $\alpha^c : \mathbb{R} \to Aut(C^*(\mathfrak{G}))$ defined for $t \in \mathbb{R}$ and f in $C^*(\mathfrak{G})$ by

$$(\alpha_t^c(f))(x) = \exp(itc(x))f(x) \quad \text{for } x \in \mathfrak{G}.$$

A continuous surjection $T : X \to X$ is called *positively expansive* if there is an $\epsilon > 0$ such that if $x \neq y$, then the distance between $T^n(x)$ and $T^n(y)$ is greater than ϵ for some integer n. In this context, A. Kumjian and J. Renault proved the following result that relates KMS states of the full groupoid C^*-algebra of $(X, \mathbb{Z})$ and the topological pressure of $(X, \mathbb{Z})$.

Theorem 6.3.1. *[KR06] Let $T : X \to X$ be a local homeomorphism that is positively expansive and exact*[*]*. Consider ϕ in $C(X, \mathbb{R})$ and let α be the strongly continuous action associated with the cocycle c_ϕ, see 6.3.1. Then there is a KMS state for α at inverse temperature β in $\mathbb{R}$ if and only if $P(T, -\beta\phi) = 0$. Where $P(T, \cdot)$ denotes the topological pressure*

In the context of quadratic maps, Klaus Thomsen related the KMS states with conformal measures [T12]. For hyperbolic diffeomorphisms on a compact metric space, David Ruelle related the KMS states with Gibbs states [R88]. Clearly, there are more relations between C^*-algebras and dynamical systems. We want to study these relations to understand in a better way how we can use these tools in operator algebras to solve problems in dynamical systems or vice versa.

[*] T is called exact if for every non-empty open set $U \subseteq X$ there is a positive integer number n such that $T^n(U) = X$

Chapter 7

Appendix

In this chapter, we explain previous results in dynamical systems theory, that we use to understand some consequences of the results in this thesis.

Minimality of flows on the torus.

We recall that a finite family of vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^s$ is rationally independent, if for any nonzero integer vector $(k_1, \ldots, k_n)$ we have $\sum_{\mathrm{i}=1}^{n} k_{\mathrm{i}}\ v_{\mathrm{i}} \neq \vec{0}$. Recall that $\mathbb{T}^s$ denotes the s-dimensional torus

$$\mathbb{R}^s/\mathbb{Z}^s = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{s \text{ times}}.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_s)^t$ be a vector in $\mathbb{R}^s$.The translational flow $\{T_\gamma^t\}_{t\in\mathbb{R}}$ in the s-torus is defined for $t \in \mathbb{R}$ and $w \in \mathbb{T}^s$ by

$$T_\gamma^t(w) \coloneqq w + [\gamma \cdot t]_{\mathbb{Z}^s}.$$

The next result characterizes minimality of the translational $\mathbb{R}$-flow T_γ^t in the s-torus.

Proposition 7.0.1. *[KH, Proposition 1.5.1] The flow $\{T_\gamma^t\}_{t\in\mathbb{R}}$ in $\mathbb{T}^s$ is minimal if and only if for any nonzero integer vector $(k_1, \ldots, k_s)$ we have $\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\ \gamma_{\mathrm{i}} \neq 0$.*

For a repetitive, Euclidean inter-model set in $\mathbb{R}$ with rank s. The previous result and Proposition A implies that its hull dynamical system has s continuous eigenvalues rationally independent in $\mathbb{R}$. Now, we give an idea of how to prove this. After, for $1 \leq \mathrm{d} \leq s$, we state a similar result for minimal $\mathbb{R}^{\mathrm{d}}$-flows on the s-torus and we give the proof.

Let Λ be a repetitive, Euclidean inter-model set in $\mathbb{R}$ with rank $s > 1$. By Proposition A, there is an address system $(\mathbb{T}^s, \mathbb{R})$. The action is defined for $t \in \mathbb{R}$ and w in $\mathbb{T}^s$ by

$$w \bullet_\ell t \coloneqq w + [A \cdot t]_{\mathbb{Z}^s},$$

where A is the $s \times 1$ matrix that represents the linear map ℓ in the canonical bases. On the proof of Proposition A, we saw that the rows of A are continuous eigenvalues for $(\Omega_\Lambda, \mathbb{R})$. Proposition B implies that $(\mathbb{T}^s, \mathbb{R})$ is actually a minimal factor of $(\Omega_\Lambda, \mathbb{R})$. Using Proposition 7.0.1 we have that the rows of A are rationally independent. This implies that the rows of A give us s continuous eigenvalues rationally independent of $(\Omega_\Lambda, \mathbb{R})$.

Now, for integer numbers $1 \leq \mathrm{d} \leq s$ we define an $\mathbb{R}^{\mathrm{d}}$-flow on the torus $\mathbb{T}^s$. Fix integer numbers $1 \leq \mathrm{d} \leq s$ and consider a matrix A with real coefficients of size $s \times \mathrm{d}$. For every t in $\mathbb{R}^{\mathrm{d}}$ we can define a translation $T_{A\cdot t}$ on the s-torus. It is defined, for every $w \in \mathbb{T}^s$ by

$$T_{A\cdot t}(w) := w + [A \cdot t]_{\mathbb{Z}^s}.$$

If the kernel of A is bigger than $\{\vec{0}\}$, this translation has many fixed points. We are interested in minimal translation flows on $\mathbb{T}^s$. Thus, we assume that the kernel of A is $\{\vec{0}\}$. For every $1 \leq \mathrm{i} \leq s$, denote by $\vec{\alpha}_{\mathrm{i}}$ the i-row of A.

Lemma J. *The flow $\{T_{A\cdot t}\}_{t\in\mathbb{R}^{\mathrm{d}}}$ on $\mathbb{T}^s$ is minimal if and only if the rows of A are rationally independent, i.e. for any nonzero integer vector $(k_1, \ldots, k_s)$ we have $\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \vec{\alpha}_{\mathrm{i}} \neq \vec{0} \in \mathbb{R}^{\mathrm{d}}$.*

Recall that the flow defined by $T_{A\cdot t}$ is minimal if for some $t_0 \in \mathbb{R}^{\mathrm{d}}$ the map $T_{A\cdot t_0}$ is minimal in $\mathbb{T}^s$. By [KH, Proposition 1.4.1] this is equivalent to that, for every nonzero integer vector $(k_1, \ldots, k_s)$ we have

$$\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \langle \vec{\alpha}_{\mathrm{i}}, t_0 \rangle \notin \mathbb{Z}.$$

Observe that if for some t in $\mathbb{R}^{\mathrm{d}}$ there are integer numbers $k_1, \ldots, k_s, k$ such that

$$\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \langle \vec{\alpha}_{\mathrm{i}}, t \rangle = k.$$

Then t belongs to the hyperplane $H_{k_1,\ldots,k_s,k}$ defined by the equation

$$\left\langle \sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \vec{\alpha}_{\mathrm{i}}\ ,\ t \right\rangle = k.$$

Note that $\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \vec{\alpha}_{\mathrm{i}} \neq \vec{0}$ implies that the hyperplanes $H_{k_1,\ldots,k_s,k}$ are not empty. Otherwise, if $\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \vec{\alpha}_{\mathrm{i}} = \vec{0}$ then the hyperplanes $H_{k_1,\ldots,k_s,k}$ are empty except for $k = 0$ where we have

$$H_{k_1,\ldots,k_s,k} = \mathbb{R}^{\mathrm{d}}.$$

Proof of Lemma J.. For sufficient condition, suppose that the rows of A are rationally independent and $T_{A\cdot t}$ is not minimal. By rationally independent condition of the rows of A, the hyperplanes $H_{k_1,\ldots,k_s,k}$ are not empty. Since $T_{A\cdot t}$ is not minimal, for every $t \in \mathbb{R}^{\mathrm{d}}$ there are integer numbers $k_1, \ldots, k_s, k$ (depending on t) such that

$$\sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\, \langle \vec{\alpha}_{\mathrm{i}}, t \rangle = k.$$

This implies that every t in $\mathbb{R}^{\mathrm{d}}$ belongs to some hyperplane $H_{k_1,\ldots,k_s,k} \subseteq \mathbb{R}^{\mathrm{d}}$. Then,

$$\mathbb{R}^{\mathrm{d}} = \bigcup_{k_1,\ldots,k_s,k\in\mathbb{Z}} H_{k_1,\ldots,k_s,k}.$$

For each finite collection of integer numbers $k_1,\ldots,k_s,k$, the complement of $H_{k_1,\ldots,k_s,k}$ is an open and dense set in $\mathbb{R}^{\mathrm{d}}$. Then, Baire's theorem implies that the set

$$\bigcap_{k_1,\ldots,k_s,k\in\mathbb{Z}} H^c_{k_1,\ldots,k_s,k},$$

is an open and dense set in $\mathbb{R}^{\mathrm{d}}$. This implies a contradiction, because

$$\bigcap_{k_1,\ldots,k_s,k\in\mathbb{Z}} H^c_{k_1,\ldots,k_s,k} = \left[\bigcup_{k_1,\ldots,k_s,k\in\mathbb{Z}} H_{k_1,\ldots,k_s,k}\right]^c = [\mathbb{R}^{\mathrm{d}}]^c = \emptyset.$$

For the necessary condition, suppose that there is a nonzero integer vector $\vec{K} = (\hat{k}_1,\ldots,\hat{k}_s)$ such that $\vec{K}\cdot A = \sum_{\mathrm{i}=1}^{s} \hat{k}_{\mathrm{i}}\, \vec{\alpha}_{\mathrm{i}} = \vec{0} \in \mathbb{R}^{\mathrm{d}}$. Define $\phi : \mathbb{T}^s \to \mathbb{R}$, for $w \in \mathbb{T}^s$ by

$$\phi(w) = \sin\left(2\pi \sum_{\mathrm{i}=1}^{s} \hat{k}_{\mathrm{i}}\, w_{\mathrm{i}}\right) = \sin(2\pi\langle \vec{K}\ ,\ w\rangle).$$

Note that ϕ is a continuous and nonconstant map. For every $t \in \mathbb{R}^{\mathrm{d}}$ and each $w \in \mathbb{T}^s$ we have

$$\phi(T_{A\cdot t}(w)) = \phi(w + [A\cdot t]_{\mathbb{Z}^s}) = \sin(2\pi\langle \vec{K}\ ,\ w + [A\cdot t]_{\mathbb{Z}^s}\rangle).$$

But $\langle \vec{K}\ ,\ [A\cdot t]_{\mathbb{Z}^s}\rangle = \vec{K}^t \cdot A \cdot t = \langle \sum_{\mathrm{i}=1}^{s} k_{\mathrm{i}}\ \vec{\alpha}_{\mathrm{i}}\ ,\ t\rangle = 0$. Hence, ϕ is invariant by the flow defined by $T_{A\cdot t}$, which contradicts the minimality of the flow. $\square$

As a consequence of this, we can prove Corollary A.

Proof of Corollary A.

Let Ω_{MS} be the hull of the repetitive inter-model sets generated by a Euclidean CPS $(\mathbb{R}^n, \Gamma, s_{\mathbb{R}^n})$ over $\mathbb{R}^{\mathrm{d}}$. Fix $\widetilde{\Lambda}$ in Ω_{MS} with rank $n + \mathrm{d}$.

By Proposition B the address system $(\mathbb{T}^s, \mathbb{R}^{\mathrm{d}})$ is topologically conjugated to the maximal equicontinuous factor of $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$. From Theorem 2.3.1, this maximal equicontinuous factor is given by the dynamical system $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^{\mathrm{d}})$. Where the set $\mathbb{T}_{\mathcal{G}}$ is equal to $(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^n)/\mathcal{G}(s_{\mathbb{R}^n})$, and the action of $\mathbb{R}^{\mathrm{d}}$ is defined for all $u \in \mathbb{R}^{\mathrm{d}}$ and $[(t,w)] \in \mathbb{T}_{\mathcal{G}}$ by

$$[(t,w)]\cdot u := [(t,w)] + [(u,0)].$$

Let $\widetilde{\Lambda}$ in the transversal space of Ω_{MS}. Since $\widetilde{\Lambda}$ is a Meyer set with rank s the address map is almost linear, and there is a matrix A with s rows and d columns that represent the linear approximation ℓ of the address map. By repetitivity and Proposition 2.3.5 we have that $\Omega_{\widetilde{\Lambda}} = \Omega_{MS}$. Thus the address system is topologically conjugated to the maximal equicontinuous factor of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$. Recall that the address system of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$ is given by an action of $\mathbb{R}^{\mathrm{d}}$ on the torus $\mathbb{T}^s$. This action is defined by

$$(w, t) \in \mathbb{T}^s \times \mathbb{R}^{\mathrm{d}} \longmapsto w + [\ell(t)]_{\mathbb{Z}^s}.$$

Observe that for the matrix A, the flow T_A that it defines is topologically conjugated to the maximal equicontinuous factor of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$. In particular, the flow defined by T_A is minimal. Using Lemma J, we have that the rows of A are rationally independent. By Remark 3.1.1 we have s rationally independent continuous eigenvalues of $(\Omega_{\widetilde{\Lambda}}, \mathbb{R}^{\mathrm{d}})$. Thus $(\Omega_{MS}, \mathbb{R}^{\mathrm{d}})$ has s rationally independent continuous eigenvalues.

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