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APPLICATIONS OF THE INTEGRATION OF ESSENTIALLY BOUNDED FUNCTIONS AND CLASSIFICATION OF ASYMMETRIC SPACES

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Resumen

Este trabajo corresponde a una Tesis Doctoral para la obtención del grado de Doctor en Ciencias de la Ingeniería, mención Modelación Matemática. El presente trabajo está inscrito en la vasta área del Análisis Variacional y Funcional, y exhibe una síntesis de los resultados disponibles en los artículos publicados "Linear Structure of Functions with Maximal Clarke Subdifferential" [30] y "Index of symmetry and topological classification of asymmetric normed spaces" [11], junto al preprint ArXiv "On an identification of the Lipschitz-free spaces over subsets of $\mathbb{R}^n$" [35].

El presente documento está dividido en dos partes. La Parte I trata principalmente con funciones Lipschitz y se centra en el estudio de propiedades estructurales de los espacios de funciones Lipschitz definidas sobre un subconjunto no vacío, abierto y convexo de un espacio de dimensión finita. En el Capítulo 1 se muestra que el espacio vectorial de las funciones Lipschitz que se anulan en un punto predeterminado del dominio anterior, dotado de la norma Lipschitz, es isométrico a un subespacio específico de funciones esencialmente acotadas con valores en el dual del dominio de las funciones Lipschitz. La propiedad que define dicho subespacio recuerda a la condición de Poincaré clásica, asegurando la integrabilidad de un campo vectorial. El Capítulo 2 trata con el concepto de espacio Lipschitz-libre y la isometría mencionada anteriormente es usada para mostrar que el espacio Lipschitz-libre en el mismo contexto es isométrico a un cuociente específico de las funciones integrables con valores en el dominio de las funciones Lipschitz. El subespacio cerrado que define este cuociente está formado por aquellas funciones integrables cuya divergencia no suave es igual a cero, mostrando una conexión de estos dos capítulos con una especie de Cálculo Multivariado no suave. En el Capítulo 3 se continúa tratando con funciones Lipschitz, pero en este caso el estudio se centra en las propiedades del subdiferencial de Clarke y los conceptos de lineabilidad y espaciabilidad. Más específicamente, se muestra que el conjunto de funciones Lipschitz (definidas sobre el mismo tipo de dominio que antes) cuyo subdiferencial de Clarke es maximal en todo punto (en el sentido que es tan grande como es posible) de hecho contiene una copia de ℓ^∞, mostrando que este conjunto es "algebráicamente grande".

Motivado por el desarrollo reciente de estructuras asimétricas y la existencia de isometrías canónicas de espacio quasimétricos en espacios normados asimétricos, en la Parte II se analiza el concepto de espacios normados asimétricos y se entregan nociones análogas para su contraparte métrica, los espacios quasimétricos. En el Capítulo 4, se busca una forma de clasificar espacios normados asimétricos en términos del grado de asimetría de sus normas. Para ello se introduce la noción de *índice de simetría*, la cual resume en un número entre cero y uno que tan simétrica es la norma. En términos de dicho índice se muestra que cada vez que éste es positivo, la norma es suficientemente simétrica, es decir, la topología del espacio puede ser obtenida por una norma clásica. Esto muestra que los casos de mayor importancia son aquellos donde el índice de simetría es igual a cero. Así, existen dos tipos de espacios, donde la principal diferencia entre ellos es el grado de separación de sus topologías. Esto a su vez cambia completamente la estructura de los espacios duales. Esta clasificación es particularmente interesante en espacios de dimensión infinita, donde varios espacios definidos de manera natural tienen índice de simetría igual a cero, pero sus topologías pueden o no ser Hausdorff, en contraste con el caso de dimensión finita, donde un índice de simetría igual a cero sólo es posible cuando la topología no es Hausdorff (de hecho, ni siquiera T_1).

Abstract

This work corresponds to a Doctorate Thesis dissertation for obtaining the PhD Degree in Engineering Science (Mention: Mathematical Modelling). The dissertation is inscribed in the broad area of Variational and Functional Analysis and presents a synthesis of results that are available in the published articles "Linear Structure of Functions with Maximal Clarke Subdifferential" [30] and "Index of symmetry and topological classification of asymmetric normed spaces" [11], together with the ArXiv preprint "On an identification of the Lipschitz-free spaces over subsets of $\mathbb{R}^n$" [35].

The document is divided in two parts. Part I deals mainly with Lipschitz functions and focuses on the study of structural properties for the spaces of Lipschitz functions defined over a nonempty open convex subset of a finite-dimensional space. In Chapter 1 we show that the vector space of Lipschitz functions that vanish at a prescribed point of the above domain, endowed with the Lipschitz norm, is actually isometric to a specific subspace of essentially bounded functions with values in the dual of the domain of the Lipschitz functions. The property that determines this subspace is reminiscent of the classical Poincaré condition ensuring the integrability of a vector field. Chapter 2 deals with the concept of Lipschitz-free space and the aforementioned isometry is used to show that the Lipschitz-free space in the same framework is isometric to a specific quotient of integrable functions with values in the domain of the Lipschitz functions. The closed subspace that appears in this quotient is formed by the integrable functions whose nonsmooth divergence is equal to zero, showing a connection of these two chapters with a kind of Nonsmooth Multivariate Calculus. In Chapter 3 we continue dealing with Lipschitz functions, but in this case we focus on the properties of Clarke subdifferential and the concepts of lineability and spaceability. More specifically, we show that the set of Lipschitz functions (defined in the same type of domain as before) whose Clarke subdifferential is maximal at every point (in the sense that it is as big as it can be) in fact contains a copy of ℓ^∞, showing that this set is "algebraically big".

Motivated by recent developments on asymmetric structures and the existence of a canonical isometric embedding of a quasimetric space to an asymmetric normed space (its free quasimetric space), in Part II we analyse the concept of asymmetric normed spaces and give some related definitions for their metric counterpart, quasimetric spaces. In Chapter 4, we look for a way of classifying asymmetric normed spaces in terms of the degree of asymmetry of their norms. In order to do that, we introduce the notion of *index of symmetry* for an asymmetric normed space, which resumes in a number between zero and one how symmetric the norm is. In terms of the aforementioned index, we show that whenever this index is greater than zero, the norm is sufficiently symmetric, meaning that the associated topology for the space can be obtained by some classical norm. This shows that the cases of utmost importance are those where the index of symmetry is equal to zero. Therefore, there are two types of spaces, where the main difference between them is the separation properties for their topologies. This, in turn, changes completely the structure of their dual spaces. This classification is particularly interesting in infinite dimensions, where many naturally defined spaces turn out to have index of symmetry equal to zero, but their topologies might or might not Hausdorff, in contrast to the case of finite dimensions, where an index of symmetry equal to zero is only possible when the topology is non-Hausdorff (actually, not even T_1).

A mi familia y amigos, parte fundamental de este proceso

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Contents

Introduction **1**
Framework and basic tools 1
Lipschitz-free spaces: Definitions and previous results 4
Definitions 5
State-of-the-Art 5
Lineability and spaceability: An alternative way to measure the extent of a property 8
Asymmetric normed spaces: State-of-the-Art 9
Definitions 10
Previous results 12

I Integration of essentially bounded functions **15**

1 Identification of the space of Lipschitz functions in finite-dimensional spaces **16**
1.1 One-dimensional integration over $\mathcal{U}$ 18
1.2 The space of Lipschitz-compatible functions 20
1.3 Identification of the space of Lipschitz functions 22

2 Identification of the Lipschitz-free space of finite-dimensional spaces **25**
2.1 Identification of the Lipschitz-free space 26

3 Clarke-saturated Lipschitz functions on finite-dimensional spaces **31**
3.1 Context and previous results 33
3.2 Main result 34
3.2.1 The one-dimensional case 34
3.2.2 The higher dimensional case 38
3.2.3 The space of Clarke-saturated functions 41

II Classification of asymmetric normed spaces **45**

4 Index of symmetry for asymmetric normed spaces **46**
4.1 Definitions and notation 50
4.2 Index of asymmetric normed space 51
4.3 The main results 53

4.3.1 The first main result and consequences. 54
4.3.2 The second main result 59
4.3.3 Classification and examples 61

Conclusions **64**

Bibliography **68**

Introduction

This work corresponds to the PhD Thesis elaborated between 2017 - 2020 at the Department of Mathematical Engineering of the University of Chile for the obtention of the Degree of Doctor in Mathematical Modeling.

The document is divided in two independent parts. The first part deals with the space of Lipschitz functions, viewed as a nonlinear dual of a given Euclidean space (or subset of it) as well as of a first-order determination of them via derivatives or subdifferentials. The second part deals with the study of asymmetrical normed spaces, more precisely with a classification for the aforementioned spaces in terms of their asymmetry. This is done using a specific coefficient which leads to a variety of properties of the spaces and their dual spaces depending on its value.

The original results presented in this document are contained in three ArXiv preprints that gave place to two published articles: "Linear Structure of Functions with Maximal Clarke Subdifferential" in *SIAM Journal of Optimization* [30] and "Index of symmetry and topological classification of asymmetric normed spaces" in *Rocky Mountains Journal of Mathematics* [11]. The third ArXiv preprint, which is not published, is titled "On an identification of the Lipschitz-free spaces over subsets of $\mathbb{R}^n$" [35].

In this introduction, the guidelines for each part as well as the framework and necessary tools to follow the ideas are stated.

Framework and basic tools

In this section we give the main definitions and results from Measure Theory and Functional Analysis, which are prerequisites for several parts of the present work. Most of the notation used throughout this document is standard, but it will be precised if necessary to avoid confusion. As usual, $(\mathbb{R}^d, \|\cdot\|)$ stands for the d-dimensional space endowed with the norm $\|\cdot\|$. While topologically the choice of this norm is not important, in terms of isometries it is. In this sense, when $\mathbb{R}^d$ is endowed with some of the classical p-norms for $p \in [0, \infty]$ we will denote the resulting Banach space simply as ℓ_p^d. Since a great part of this work is related to the study of metric properties, the corresponding norm will always be specified.

Recall that a nonempty set M endowed with a function $d : M \times M \to R_{\geq 0}$ which satisfies the following properties for every $x, y, z \in M$

i) $d(x,y) = 0$ if and only if $x = y$,

ii) $d(x,y) = d(y,x)$, and

iii) $d(x,y) \leq d(x,z) + d(z,y)$

is called a metric space, in which case we call d a distance over M. It is important to notice that, in general, there is no algebraic structure on the set M. Because of that, and in constrast to what happens in the study of normed vector spaces, the natural morphisms between metric spaces are no longer linear operators, since linearity has no sense in a general metric space. Instead, the functions that arise as natural morphisms between metric spaces are Lipschitz functions. For a metric space (M,d), we say that a function $f : M \to \mathbb{R}$ is Lipschitz if

$$(\exists L > 0)(\forall x, y \in M) \quad f(x) - f(y) \leq L\, d(x,y),$$

and by its Lipschitz constant we mean the greatest lower bound among all constants L which satisfy the aforementioned property for f, or equivalently

$$\mathrm{Lip}(f) := \sup_{\substack{x,y \in M \\ x \neq y}} \frac{f(x) - f(y)}{d(x,y)}.$$

We denote by $\mathrm{Lip}(M)$ the linear space of all real-valued Lipschitz functions over M. Most of the present work is closely related to Lipschitz functions, or more precisely, to an specific class of Lipschitz functions. A metric space (M,d) is called pointed if it has a distinguished point, which in general we will call 0_M (or simply 0 if there is no ambiguity).

If M is a pointed metric space, we denote by $\mathrm{Lip}_0(M)$, the linear space of all real-valued Lipschitz functions that vanish at 0. In topological terms, both spaces $\mathrm{Lip}(M)$ and $\mathrm{Lip}_0(M)$ can be endowed with the function $\|f\|_L := \mathrm{Lip}(f)$, which turns out to be a seminorm over $\mathrm{Lip}(M)$ and a norm over $\mathrm{Lip}_0(M)$. In this latter case, $(\mathrm{Lip}_0(M), \|\cdot\|_L)$ is a Banach space.

In case that the metric space is (isometrically identified to) a subset of $\mathbb{R}^d$, an essencial property of Lipschitz functions is given in the following theorem, which will be paramount for many proofs in the sequel.

Theorem 1 (Rademacher) *Let $\mathcal{U} \subseteq \mathbb{R}^d$ be an nonempty open set and $f : \mathcal{U} \to \mathbb{R}$ a Lipschitz function. Then, the set $\mathcal{D}_f \subseteq \mathcal{U}$ where f is differentiable has full measure (with respect to the Lebesgue measure over $\mathcal{U}$).*

Remark It is a well known fact that Hadamard and Fréchet differentiability are equivalent in finite dimensional spaces. Moreover, for Lipschitz functions over any Banach space Gâteaux and Hadamard differentiability are also equivalent. This being said, the above mentioned types of differentiability are the same for Lipschitz functions defined over finite-dimensional spaces. We refer to [24] for this and other properties of Lipschitz functions.

For a normed linear space X, its dual space is defined as follows

$$X^* = \{\varphi : X \to \mathbb{R} \,|\, \varphi \text{ is linear and continuous}\}.$$

It is easily seen that X^* is a vector space, with becomes a Banach space when endowd with the norm

$$\|\varphi\|_* := \sup_{\|x\|\leq 1} \varphi(x), \quad \forall \varphi \in X^*,$$

which is well defined thanks to the continuity of φ. Moreover, in virtue of the linearity of $\varphi \in X^*$, it is easy to verify that

$$\|\varphi\|_* := \sup_{\substack{x,y\in X \\ x\neq y}} \frac{\varphi(x) - \varphi(y)}{\|x - y\|}, \quad \forall \varphi \in X^*,$$

from which we see that this norm simply computes the Lipschitz constant for the linear functional φ. In other words, when dealing with normed linear spaces, we define its dual using their natural morphisms with real values, that is, linear functionals, which reflect the linear structure inside the space. On the other hand, it is natural to consider a dual space for metric spaces using its natural morphisms, that is, real valued Lipschitz functions, which reflect the metric structure of the space. No matter the case we deal with, the norm over those natural real valued morphisms is defined in the same way.

In this sense, considering the pointed metric space X with $0 \in X$ its distinguished point, the space $\mathrm{Lip}_0(X)$ is often called the Lipschitz-dual of X (in opposition to the dual space, which in this context we may call linear-dual).

We are also going to use notions and results from measure theory. To this end, let (Ω, Σ, μ) be a measure space and X a Banach space. A function $f : \Omega \to X$ is called μ-measurable (or simply measurable when there is no confusion) if it is almost everywhere the pointwise limit of a sequence of simple functions, that is functions of the form

$$\omega \mapsto \sum_{k=1}^{n} x_k \mathbb{1}_{A_k}(\omega),$$

where $A_k \subset \Omega$, for $i = 1, \ldots, n$ and $n \in \mathbb{N}$. A μ-measurable function f is said to be Bochner-integrable if the function $\omega \mapsto \|f(\omega)\|$ is integrable as a real-valued measurable function. For relevant definitions and results on vector-valued measurable functions, we refer to [33]. The following spaces which arise from measurable functions will be useful in terms of finding isometries in Part I of the present work.

Definition 1 Let $p \in [1, \infty)$ and a measure space (Ω, Σ, μ). The Lebesgue-Bochner space $L^p(\Omega, \Sigma, \mu; X)$ is the space given by the (equivalence classes of) μ-Bochner-integrable functions $f : \Omega \to X$ such that

$$\|f\|_p := \left(\int_\Omega \|f\|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

We also define the Lebesgue-Bochner space $L^\infty(\Omega, \Sigma, \mu; X)$ as the space given by the (equivalence classes of) μ-measurable essentially bounded functions, that is such that

$$\|f\|_\infty := \operatorname*{esssup}_{\Omega} \|f\| < \infty.$$

These spaces endowed with the corresponding p-norm become Banach spaces.

Remark Whenever there is no confusion with the σ-algebra and the measure in this definition, we will simply write $L^p(\Omega; X)$. In the case that Ω is a Lebesgue-measurable subset of $\mathbb{R}^d$ and the considered measure is the Lebesgue measure, we simply denote these spaces as $L^p(\Omega)$.

Several properties can be found in the literature for these spaces, in particular for the case that $X = \mathbb{R}$. We give now some of these essential properties in full generality.

Definition 2 (RNP space) We say that a Banach space X has the Radon-Nikodym property (RNP) if for every σ-finite measure space (Ω, Σ, μ) we have that for every X-valued absolutely continuous measure $\nu : \Sigma \to X$ of bounded variation, there exists a function $f \in L^1(\Omega, \Sigma, \mu; X)$ such that

$$\nu(A) = \int_A f d\mu \, , \forall A \in \Sigma.$$

The Radon-Nikodym property has several equivalences related to martingale convergence, geometry of Banach spaces, etc., which we will not state here. An important result in this sense is the Dunford-Pettis theorem, which is inscribed in our framework, that is, finite-dimensional spaces.

Theorem 2 (Dunford-Pettis) *Let X be a separable dual space. Then, X has the RNP. In particular, every reflexive Banach space has the RNP.*

As a final important result, we state the following theorem which generalizes a well known duality result.

Theorem 3 *Let (Ω, Σ, μ) be a measure space and X be a Banach space such that its dual X^* has the Radon-Nikodym property. Then*

$$L^p(\Omega, \Sigma, \mu; X)^* \equiv L^q(\Omega, \Sigma, \mu; X^*),$$

where $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Since every finite-dimensional Banach space is reflexive (under any norm), the above result applies in our framework and will be used in the sequel to establish isometries.

Lipschitz-free spaces: Definitions and previous results

Lipschitz-free spaces have been extensively studied in recent literature, but their structure is far from being completely understood. Nevertheless, there are specific cases where some results are available. Here we will make a selection of those results and the required definitions. These definitions will be given in full generality, but in the subsequent chapters we will focus on the case where the underlying metric space is an open convex subset of a finite-dimensional Banach space. Most of the preliminary results can be found in [42]. Throughout this document, we shall also refer to other relevant works for results concerning the Lipschitz-free space of particular metric spaces.

Definitions

The idea behind Lipschitz-free spaces is to define a Banach space related to a pointed metric space M such that its dual space coincides with the space $\mathrm{Lip}_0(M)$ of real-valued Lipschitz functions which vanish at a distinguished point $x_0 \in M$, which is often called base point and denoted by 0 ($\mathrm{Lip}_0(M)$). In order to so this, we focus on an specific subset of $\mathrm{Lip}_0(M)$ as follows. We define the evaluation function $\delta_M : M \to \mathrm{Lip}_0(M)^*$ as the function such that for every $x \in M$

$$\langle \delta_M(x), f \rangle = f(x).$$

Again, when there is no confusion on the subjacent metric space, we denote this function simply as δ. The Lipschitz-free space over M, denoted by $\mathcal{F}(M)$, is defined as the subspace of $\mathrm{Lip}_0(M)^*$ given by

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) \,:\, x \in M \setminus \{0\}\}.$$

It is easy to see that the set $\{\delta(x) \,:\, x \in M \setminus \{0\}\}$ is linearly independent. Also, it can be shown that this space verifies that $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$, that is, there exists a linear isometry between these spaces.

The definition of the Lipschitz-free spaces can be understood as follows. Using $\mathrm{Lip}_0(M)^*$ as a host space, we assign to every $x \in M \setminus \{0\}$ a different direction in this space, while $0 \in M$ is mapped to the origin. Then, $\mathcal{F}(M)$ is the smallest Banach space (up to isometry) which contains an isometric copy of M, where the associated isometry is the evaluation function δ.

State-of-the-Art

To get a better understanding of the definition of a Lipschitz-free space, we now present a result concerning the mentioned embedding on Lipschitz-free spaces

Lemma 1 Let M, N be two metric spaces, each one with a base point (0_M and 0_N, respectively) and $F : M \to N$ a Lipschitz function such that $F(0_M) = 0_N$. Then, there exists a unique linear operator $\hat{F} : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $Lip(F) = \|\hat{F}\|$ and $\delta_N \circ F = \hat{F} \circ \delta_M$, that is, the following diagram conmutes

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\delta_M"] & N \arrow[d, "\delta_N"] \\ \mathcal{F}(M) \arrow[r, "\hat{F}"] & \mathcal{F}(N) \end{tikzcd}$$

Lemma 1 shows that the whole Lipschitz structure present in a given metric space is fully absorbed by its Lipschitz-free space, since it assigns a different direction to every point of the metric space. It is worth noticing that if M is a metric subspace of N, Lemma 1 also says that $\mathcal{F}(M)$ can be seen as a subspace of $\mathcal{F}(N)$. Considering these facts, this gives a way of linearizing problems, in the sense that they can be seen as linear problems in the respective Lipschitz-free spaces. The main problem is that the structure of Lipschitz-free space is not yet fully understood. As an example, it is known that $\mathcal{F}(\mathbb{R})$ is isometric to $L^1(\mathbb{R})$ (later on, we will revisit this proof), but A. Naor and G. Schechtman proved on [56] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L^1. Also, in the separable case it is still open the relation between Lipschitz-equivalence and isomorphisms, that is, whether two Banach spaces X, Y are Lipschitz-equivalent (there exists a Lipschitz homeomorfism $F : X \to Y$) implies they are isomorphic.

In the case where the pointed metric space for which we are interested to determine its Lipschitz-free space is contained in a finite-dimensional normed space there are some results concerning the metric structure of the associated metric space. In this sense, it is known that both $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^d)$ admit monotone finite-dimensional Schauder decompositions (Lancien-Pernecká, [52]). This last result has further been improved, showing that those spaces in fact have a Schauder basis (Hájek-Pernecká, [45]). In the case that $\mathbb{R}^d$ is equipped with an arbitrary norm, we also know that if $M \subset R^d$ is convex and compact, then the space $\mathcal{F}(M)$ has the metric approximation property (Pernecká-Smith [59]).

To finish this section we review the proof that states a simple identification for the Lipschitz-free space in the case that the subjacent metric space is $\mathbb{R}$, which can be seen for example on [65]. In order to do this, suppose that $\mathcal{U}$ is a non-empty open interval of the real line and fix $x_0 \in \mathcal{U}$. We shall show that $\mathrm{Lip}_0(\mathcal{U})$ and $L^\infty(\mathcal{U})$ are isometric.

Let $T : L^\infty(\mathcal{U}) \to \mathrm{Lip}_0(\mathcal{U})$ be the linear operator defined by

$$Tg(x) = \int_{x_0}^{x} g(t)dt, \quad \text{for all } x \in \mathcal{U}.$$

The function is well-defined, being the integral of an essentially bounded function over a bounded set. Moreover, it is Lipschitz since for every $x, y \in \mathcal{U}$

$$|Tg(x) - Tg(y)| \leq \int_{y}^{x} |g(t)|dt \leq \|g\|_\infty |x - y|$$

and it vanishes as x_0. We deduce from here that T is well defined and continuous, with $\|Tg\| \leq \|g\|_\infty$. We show that this operator defines a bijective isometry between the spaces.

- T is injective: Suppose that $Tg = 0$. In particular, we have that for every interval $a, b \in \mathcal{U}$

$$\int_{a}^{b} g(t)dt = 0,$$

  which implies that $g(t) = 0$ almost everywhere over $\mathcal{U}$, or equivalently $g = 0$. With this, T is injective.

- T is surjective: It suffices to notice that if $f \in \mathrm{Lip}_0(\mathcal{U})$, then f' is well-defined almost everywhere over $\mathcal{U}$ (thanks to Theorem 1) and that $\|f'(x)\|_\infty \leq \|f\|_L$ for every differentiability point of f, yielding $f' \in L^\infty(\mathcal{U})$. Therefore

$$Tf'(x) = \int_{x_0}^{x} f'(t)dt = f(x) - f(x_0) = f(x).$$

from which we deduce that T is surjective.

From these last observations, we have that T is bijective. Moreover, its inverse is given by the operator $D : \mathrm{Lip}_0(\mathcal{U}) \to L^\infty(\mathcal{U})$ given by $Df = f'$. From the inequality used in the proof of the surjectivity, we deduce easily that D is continuous, with $\|Df\|_\infty \leq \|f\|_L$. It follows directly from the estimates of the continuity of both operators T and D that T is an isometry.

Remark In fact, since $\mathcal{U} \subset \mathbb{R}$. the use of Theorem 1 can be avoided by noticing that every Lipschitz functions is absolutely continuous which implies that its derivative is defined a.e. on $\mathcal{U}$.

To finish the required identification, we recall the following theorem.

Theorem 4 *Let X, Y be two Banach spaces. Let $T : Y^* \to X^*$ be a linear bounded operator. Suppose that T is w^*-w^* continuous. Then, there exists a linear bounded operator $S : X \to Y$ such that $S^* = T$. Moreover, if T is a bijective isometry, so is S.*

It is known (thanks to Grothendieck's Theorem, see [43]) that $L^1(\mathcal{U})$ is the unique predual of $L^\infty(\mathcal{U})$ (up to isometry). We show that the operator T defined above is actually w^*-w^* continuous, considering $L^\infty(\mathcal{U})$ and $\mathrm{Lip}_0(\mathcal{U})$ as the dual spaces of $L^1(\mathcal{U})$ and $\mathcal{F}(\mathcal{U})$, respectively. Let $(g_\lambda)_{\lambda\in\Lambda}$ be a w^*-convergent net on $L^\infty(\mathcal{U})$ and let $g \in L^\infty(\mathcal{U})$ be its limit. Then, for each $x \in \mathcal{U}$ we have that

$$\langle Tg_\lambda, \delta(x)\rangle = \int_{x_0}^{x} g_\lambda(t)dt = \langle g_\lambda, \mathbb{1}_{[x_0,x]}\rangle.$$

Since $\mathbb{1}_{[x_0,x]} \in L^1(\mathcal{U})$ and $(g_\lambda)_{\lambda\in\Lambda}$ is w^*-convergent, we deduce that

$$\langle Tg_\lambda, \delta(x)\rangle \to \langle g, \mathbb{1}_{[x_0,x]}\rangle = \int_{0}^{x} g(t)dt = \langle Tg, \delta(x)\rangle$$

Then, passing through linear combinations and limits, we conclude that $Tg_\lambda \overset{*}{\rightharpoonup} Tg$ whenever $g_\lambda \overset{*}{\rightharpoonup} g$. Then, thanks to Theorem 4 there exists an linear operator $S : \mathcal{F}(\mathcal{U}) \to L^1(\mathcal{U})$ identifying both spaces and such that its adjoint operator is T. Moreover, since T is a bijective isometry, so is S, which implies that $\mathcal{F}(\mathcal{U}) \equiv L^1(\mathcal{U})$.

In Chapter 1 we generalize the ideas of the proof of the above result to the case where $\mathcal{U}$ is a nonempty open convex subset of $\mathbb{R}^d$ endowed with an arbitrary norm. This relies mainly on identifying, among the essentially bounded functions those which are gradients of some Lipschitz function. In Chapter 2 we search for a predual of the aforementioned subspace of functions in order to use it to identify the associated Lipschitz-free space. In Chapter 3 we use the ideas developed on Chapter 1 to study the set of functions whose Clarke-subdifferential is big in the sense that it coincides with the closed dual ball of radius $\|f\|_L$ at every point (the exact definition will be given in the sequel).

Lineability and spaceability: An alternative way to measure the extent of a property

In this section, we recall the notions of lineability and spaceability, concepts that have been extensively studied in several areas and relies on the concept of finding linear structures inside classes of functions satisfying certain properties. Examples of this line of research can be found in [8], [9], [15], [16], [10] and [44].

Definition 3 Suppose that $\mathcal{S}$ is a set of functions $f : X \to Y$ which satisfy a certain property $\mathcal{P}$, where X is a set and Y is a normed space. We say that $\mathcal{S}$ is

- Lineable if $\mathcal{S} \cup \{0\}$ contains a linear subspace of Y^X.
- κ-lineable, for a cardinal κ if $\mathcal{S} \cup \{0\}$ contains a linear subspace of Y^X with $\dim(\kappa)$.
- Spaceable if $\mathcal{S} \cup \{0\}$ is contained in a normed space and contains a copy of an infinite-dimensional Banach space.

In this context, several properties have been studied in the search for linearity. As examples, we have that the set of differentiable functions on $\mathbb{R}$ that are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$ ([10], Theorem 2.4) and that the set of continuous everywhere surjective functions on $\mathbb{R}$ is $2^{\mathfrak{c}}$-lineable ([10], Theorem 4.3)

In this sense, both results say that in a particular sense the sets are big. More precisely, they are algebraically big. To use as a point of comparison, we recall that we can say that a set is topologically big if it contains a dense G_δ set. Notice that for lineability there is no topology involved, which allows us to study the algebraic size of sets in a more general setting. Spaceability deals exactly with the cases where we can do more, that is, finding a linear space inside the set which is also (isometric to) a Banach space.

It is important to state that the search of linearity on sets is not an easy task, since there are cases where a rich linear structure is not found inside a set, as is shown in [44], where it is proven that the set of continuous functions on $[0, 1]$ which attain their maxima at exactly one point is a dense G_δ in $\mathcal{C}([0, 1])$ but it does not contain any linear subspace with dimension greater that 1. As we can see, even when the definition of the set relies on a simple property and is topologically big, it has a very poor linear structure.

The start point for Chapter 3 relies on the following result, which shows that our set of interest is topollogically big. This result is given in a more general setting, but we state it in the particular case which will serve as motivation: Let X be a Banach space and $C \subset X$. For a real valued Lipschitz function f defined over C, the Clarke directional derivative is defined as

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y + tv) - f(y)}{t}, \quad \text{for every } x \in C, v \in X.$$

It is easily verifiable that this directional derivative is well defined, thanks to f being Lipschitz. Moreover, for every $x \in C$, $v \mapsto f^\circ(x; v)$ is finite, positively homogeneous, subadditive on X and satisfies $|f^\circ(x;v)| \leq \|f\|_L\|v\|$. In terms of this generalized derivative, the Clarke subdifferential is defined (for f as before) as

$$\partial^\circ f(x) = \{\varphi \in X^* \mid \langle \varphi, v\rangle \leq f^\circ(x;v) \quad \forall v \in X\}, \quad \text{for every } x \in C,$$

which is a nonempty, convex, weak*-compact subset of X^*, which is contained in the closed ball centered at the origin of X^*, with radius equal to $\|f\|_L$, the Lipschitz constant of f.

For fixed $K > 0$, the set of Lipschitz functions such that $\partial^\circ f$ equals K times the dual closed unit ball over C is generic in the set of Lipschitz functions with constant at most K endowed with the metric of uniform convergence over bounded sets.

We see that the above result reveals that the set of Lipschitz functions whose Clarke subdifferential is everywhere maximal (that is, is the biggest it can be at any point of the domain) contains a G_δ dense set, meaning that it is topologically big under the right topology over the space of Lipschitz functions with a fixed maximum Lipschitz constant.

In Chapter 3 we use the tools developed on Chapter 1 in order to obtain a sequence of linearly independent Lipschitz functions which have the property of maximality of their Clarke subdifferentials, as mentioned before, which in turn are suited to build inside the space of all Lipschitz functions a Banach space containing only functions which satisfy the same property, obtaining a spaceability result in line with the result mentioned from [19]. For this, it will be necessary to change both the host set and the metric. The change of the host set is obvious from the fact that we are looking for a linear space, while in the set of Lipschitz functions with constant at most $K > 0$ the biggest linear space we can find is that defined by the constant functions. Moreover, we change also the domain of definition for the functions in order to obtain the result using Chapter 1, which deals with Lipschitz functions defined over convex subsets of $\mathbb{R}^d$. In terms of the metric used in [19], which was that of uniform convergence over bounded subsets, will be also changed, since it is not well adapted to the space of all Lipschitz functions (there is no completeness given the lack of bounds for the Lipschitz constants). It is worth mentioning that a first attempt was done by directly using the norm given by the Lipschitz constants in order to replicate the functions exposed in [19], which failed since the metric of uniform convergence over bounded sets allows to move the slopes of the functions as needed not losing the convergence, which is impossible with the aforementioned norm, since it completely controls those slopes. However, our result will be based in a explicit construction while the result from [19] relies on Baire category theorem, which shows in particular that the approach taken is completely different.

Asymmetric normed spaces: State-of-the-Art

The last chapter of this thesis deals with a topic of different nature, which has to do with asymmetric structures. We give here the main definitions and some examples of asymmetric normed spaces. We refer to [25], where the developement of Functional Analysis on these spaces is detailed.

The idea behind asymmetric normed spaces is to emancipate from the classical symmetry assumption of classical norms while preserving the rest of its structure. This leads to topologies whose properties will can naturally be expected to depend on the level of asymmetry of the norm.

Definitions

Asymmetric normed spaces (as can be deduced by the name) are linear spaces endowed with a non-negative function which satisfies all usual properties of a norm, except for absolute homogeneity and identifying 0, properties that are replaced by positive homogeneity and requiring both x and $-x$ to evaluate to 0 in order to conclude that $x = 0$. Such a functional is called asymmetric norm, whose main property is that x and $-x$ do not necessarily have the same norm, hence the term asymmetric. More precisely

Definition 4 Let X be a real linear space. We say that a functional $\|\cdot| : X \to [0, \infty)$ is an asymmetric norm over X if

i) $\|x| = \|-x| = 0 \implies x = 0$.

ii) $\|\lambda x| = \lambda \|x|$ for every $x \in X$ and $\lambda \geq 0$.

iii) $\|x + y| \leq \|x| + \|y|$ for every $x, y \in X$.

When endowed with an asymmetric norm, we say that X is an asymmetric normed space.

We can easily see that every norm satisfies trivially the above definition, therefore it is also an asymmetric norm and the difference between an asymmetric norm and a norm is that the equality $\|-x| = \|x|$ is not necessarily true for every $x \in X$. The topology of an asymmetric normed space is defined in the same way as for normed spaces, that is, a subset $\mathcal{U}$ of X is open if for every $x \in \mathcal{U}$ there exist a ball centered at x and radius r which is completely contained in $\mathcal{U}$. Then, we first need to define the notion of a ball as follows:

Definition 5 For an asymmetric normed space, we define the open and closed balls centered at $x \in X$ of radius $r > 0$ as

$$B(x, r) = \{y \in X \,:\, \|y - x| < r\} \quad \text{and} \quad \overline{B}(x, r) = \{y \in X \,:\, \|y - x| \leq r\},$$

respectively.

Remark Given the asymmetry of the norm and contrary to the normed case, the order $\|y - x|$ (instead of $\|x - y|$) of the difference in the definition is important. Changing this order will in general give different sets. In the literature can also be found this definition and the analogous with the difference in the other sense as forward and backward balls.

To understand how the lack of symmetry affects the topology of these spaces we shall begin by studying the most elementary asymmetric normed space, which in turn will be useful to state the definition of duality over these spaces. Over the real line, it is not hard to see that the function $\|t|_0 = \max\{0, t\}$ defines an asymmetric norm. Moreover, the open unit ball is given by $B(0,1) = (-\infty, 1)$, which raises a topology which is not Hausdorff. Nevertheless there are cases where the asymmetric norm is not a norm, but the induced topology is still Hausdorff. The following example shows this.

Example Let $\alpha > 0$. On $\mathbb{R}$ consider the asymmetric norm given by $\|t|_\alpha = \max\{-\alpha t, t\}$. We can easily see that $B(0,1) = (-1/\alpha, 1)$, which implies that the induced topology is the same as the usual topology over $\mathbb{R}$. As a consequence, every topological result for $\mathbb{R}$ endowed with the absolute value remains true for this asymmetric norm. Moreover, it is not hard to notice that both spaces are actually linearly isomorphic, which is not the case for the asymmetric norm defined before this example.

In the spirit of this example, we focus on studying the degree of asymmetry of an asymmetric norm, which will give raise to an index and a classification of these spaces in terms of that index. In order to do this, an important structure will be the one that we call the symmetrization of an asymmetric normed space.

Definition 6 Let X be an asymmetric normed space. By its symmetrization we mean the space X endowed with the norm $\|x\| := \max\{\|x|, \|-x|\}$. We denote by X_s the symmetrization of X.

Another concept related to asymmetry is that of quasi-metric spaces, which can be understood as asymmetric metric spaces. We give the the definition of these spaces for completeness.

Definition 7 (Quasi-metric space) A quasi-metric space is a pair (M, d), where M is a nonempty set and $d : M \times M \to \mathbb{R}_{\geq 0}$ is a function satisfying

i) $d(x,x) = 0$ for every $x \in M$,

ii) $d(x,y) = d(y,x) = 0$ implies $x = y$, for every $x, y \in M$, and

iii) $d(x,y) \leq d(x,z) + d(z,y)$, for every $x, y, z \in M$.

In general, a quasi-metric space needs not to be T_2 (neither T_1), which is also the case for asymmetric normed spaces. Nevertheless, several examples of quasi-metric spaces which are T_2 can be found in the literature, such as Finsler manifolds, for which we refer to [31]. In the same way that we can define the symmetrization of an asymmetric normed space, we can also define the symmetrization for a quasi-metric space, which is always a metric space (hence T_2).

The study of continuous linear functionals is of paramount importance in Functional Analysis. In this sense, the space of countinuous linear functionals has been studied in the context of asymmetric normed spaces.

Definition 8 Let X be an asymmetric normed space. We say that a linear functional $\varphi : X \to \mathbb{R}$ is bounded if there exists a constant $C > 0$ such that

$$\varphi(x) \leq C\|x| \quad \forall x \in X.$$

It is known that for normed spaces boundedness and continuity of linear functionals are equivalent, which is not directly true in the case for asymmetric normed spaces, as we will see in the following section. To finish this section, we define the asymmetric dual of an asymmetric normed space.

Definition 9 Let X be an asymmetric normed space. We define its asymmetric dual as the set of all bounded linear functionals, that is

$$X^\flat := \{\varphi : X \to \mathbb{R} \,:\, \varphi \text{ is linear and bounded}\}.$$

Consider also the function $\| \cdot |_\flat : X_s^* \to \mathbb{R}_{\geq 0}$ given by

$$\|\varphi|_\flat := \sup_{\|x| \leq 1} \varphi(x)$$

It follows directly from the definition that in general $X^\flat$ is a cone contained in X_s^*. Moreover, the function defined above is actually an extended asymmetric norm over X_s^*, that is, an asymmetric norm which takes infinity as a value. It is clear that φ belongs to the asymmetric dual of X if and only if φ belongs to X_s^* and $\|\varphi|_\flat$ is finite. From this we trivially see that if X is normed, then $X^\flat$ coincides with X^*. This fact is something that we will revisit during Chapter 4.

Previous results

In this section we state some properties of asymmetric normed spaces. More precisely, we announce some results that will be useful during the development of Chapter 4. At the end of the previous section we gave the definition for bounded linear functionals and the asymmetric dual for asymmetric normed spaces. The first consequence of this is that continuity of linear functional needs to be treated carefully. We give details of this on the following proposition.

Proposition 1 Let X be an asymmetric normed space and $\varphi : X \to \mathbb{R}$ a linear functional. The following are equivalent.

i) φ is continuous from X to $(\mathbb{R}, \| \cdot |_\mathbb{R})$.

ii) φ is continuous at 0.

iii) φ is bounded.

iv) φ is upper semicontinuous from X to $(\mathbb{R}, | \cdot |)$.

Considering this proposition we can see that if $\mathbb{R}$ is endowed with $\|\cdot|_{\mathbb{R}}$, its asymmetric dual is given by the interval $[0,\infty)$. From this, the main difference with the classical dual is evident, since in this case the asymmetric dual is not a linear space. The general result in this sense is given in the following proposition.

Proposition 2 Let X be an asymmetric normed space. Then, $X^\flat$ is a convex cone contained in the dual of X_s.

Another important fact is that $X^\flat$ is not necessarily strictly contained in the dual of X_s. A simple example of that is when X is actually normed. A more elaborated example is in the case of $\mathbb{R}$ endowed with an asymmetric norm of the form $\|t|_\alpha := \max\{-\alpha t, t\}$, with $\alpha > 1$.

In the case of quasi-metric spaces we can also define the analogous of the Lipschitz-dual for metric spaces, which is given by the Semi-Lipschitz functions, which are defined as follows

Definition 10 (Semi-Lipschitz function) Let (M,d) be a quasi-metric space. A function $f : M \to \mathbb{R}$ is called semi-Lipschitz if there exists a constant $L \geq 0$ such that

$$f(y) - f(x) \leq L\, d(y,x), \quad \text{for all } x, y \in M.$$

The infimum of the above constants $L \geq 0$ is called the semi-Lipschitz constant of f, that is,

$$\|f|_L := \sup_{d(y,x)>0} \frac{f(y) - f(x)}{d(y,x)}.$$

We denote by $\mathrm{SLip}(M)$ the set of semi-Lipschitz functions on (M,d).

The analogy in the definitions of $X^\flat$ and $\mathrm{SLip}(M)$ is evident, and a result in the line of Proposition 2 is easy to prove and is given for completeness.

Proposition 3 Let M be a quasi-metric space. Then, $\mathrm{SLip}(M)$ is a convex cone contained in $\mathrm{Lip}(M_s)$, the space of real valued Lipschitz functions defined over the symmetrization of M, M_s.

The importance of this concept in our work relies in the fact that an analogous theory around free spaces exists for quasi-metric spaces: Every quasi-metric space can be injected in a asymmetric normed space, the so-called semi-Lipschitz free space. We now state the related definitions and properties, both for completeness and to see the analogy with the definitions given by G. Godefroy and N. J. Kalton [42].

Let (M,d) be a pointed quasi-metric space. For $x \in M$ we consider the corresponding evaluation mapping

$$\delta_x : \mathrm{SLip}_0 \to \mathbb{R} \text{ defined by } \delta_x(f) = f(x), \forall f \in \mathrm{SLip}_0(M).$$

An easily verifiable fact is that δ_x is a linear mapping over the cone $\mathrm{SLip}_0(M)$. A first result on δ_x is that it belongs to the linearity part of the dual cone $(\mathrm{SLip}_0(M))^*$, that is, to a vector space contained in that dual cone, which we state in the following proposition

Proposition 4 For each $x \in M$, both the evaluation functional δ_x and its opposite $-\delta_x$ belong to the dual cone $(\mathrm{SLip}_0(M))^*$.

To make clear the analogy with the classic framework of Lipschitz-free space, notice that in the construction for those spaces the same evaluation functional is used, but the fact that both δ_x and δ_x is trivially assured, since in that case δ_x belongs to the dual of a normed space. We proceed to state a final proposition before giving the definition of the semi-Lipschitz free space of M.

Proposition 5 The mapping

$$\delta : M \to \mathrm{SLip}_0(M)^*,$$

defined by $\delta(x) = \delta_x$ is an isometry onto its image.

We now take the asymmetric normed space $(\mathrm{span}(\delta(M)), \| \cdot |_*)$ (which is contained in the normed cone $(\mathrm{SLip}_0, \| \cdot |_*)$), and we define the semi-Lipschitz free space to be the bicompletion of $(\mathrm{span}(\delta(M)), \| \cdot |_*)$.

Definition 11 (The semi-Lipschitz free space) Let (M, d) be a quasi-metric space with a base point x_0. The semi-Lipschitz free space over (M, d), denoted by $\mathcal{F}_a(M)$, is the (unique) bicompletion of the asymmetric normed space $(\mathrm{span}(\delta(M)), \| \cdot |_*)$, where $\| \cdot |_*$ is the restriction of the norm of $\mathrm{SLip}_0(M)^*$.

It should be now clear the analogy between this definition and that of Lipschitz-free space in the symmetric case. For more detailed information on this topic, we refer to [32].

In Chapter 4 we study the problem of determining whether $X^\flat$ is a linear space, classifying in this way the asymmetric normed spaces in terms of an index which depends only on the asymmetric norm given.

Part I

Integration of essentially bounded functions

Chapter 1

Identification of the space of Lipschitz functions in finite-dimensional spaces

In this part, we work around the concept of Lipschitz-free spaces. More precisely, we study the properties and structure of Lipschitz functions over finite-dimensional spaces and then using those properties to identify the Lipschitz-free spaces over finite-dimensional spaces. In the beginning, the study of those Lipschitz functions was done in order to find the aforementioned identification. However, the same results developed there bootstrap on the study of deeper properties in the structure of the space of Lipschitz functions.

The study of Lipschitz-free spaces goes back to [65], where more precisely Arens-Eells spaces are analysed. These spaces are defined in a constructive way starting from what is called the Lipschitz dual of a Banach space, that is, the space of all Lipschitz functions that vanishes at the origin. It is worth noticing that it is not necessary for the definition of these spaces that the host space (that is, the domain of the Lipschitz functions) is a Banach space. It suffices for it to be a pointed metric space, that is, a metric space with a fixed distinguishable point, which takes the role of the origin, that is, we ask our functions to vanish at that point. In any case, the set of Lipschitz functions which vanishes at the origin is in fact a Banach space when endowed with the norm given by the Lipschitz constant.

From here, the construction of the Lipschitz-free space is straightforward: It is defined as the closed linear span of the evaluation functionals $\delta(x) : \mathrm{Lip}_0(X) \to \mathbb{R}$ (that is, $\langle \delta(x), f\rangle = f(x)$). It can be easily proven that the space defined in this way is actually a predual of $\mathrm{Lip}_0(X)$, and that it contains an isometric copy of the host space, where the isometry is actually the operator $\delta : X \to \mathcal{F}(X)$. Notice that, even in the case where X is a Banach space, δ is not linear.

A general problem involving these ideas arises on the study of Banach spaces. More precisely, the following question is considered for two Banach spaces X and Y: Is it true that X and Y are linearly isomorphic whenever they are Lipschitz isomorphic (i.e. there exists a bijective bi-Lipschitz map $F : X \to Y$)? It is already known that in full generality this is false, but the case where the spaces are separable still remains as an open problem. For this and related topics, we refer to [14].

A detailed study of these concepts was done by J. Kalton (e.g. [49], [42]). These works are the current starting point to any research in this topic.

In this chapter, we take the ideas on the identification of the space of Lipschitz functions over the real line exposed on the introduction and generalize them in order to obtain an identification for the case where those functions are defined over non-empty open convex subsets of finite-dimensional spaces. As was already mentioned in the introduction of the present work, the structure of Lipschitz-free spaces in a general setting is not yet fully understood. In order to find some properties of an specific Lipschitz-free space $\mathcal{F}(M)$ we can begin by the study of its dual space, namely the space of Lipschitz functions which vanish at a fixed point, $\mathrm{Lip}_0(M)$. Having this idea in mind, during this chapter we search for a way of generalizing the proof for the case where the asociated metric space is simply a non-empty open real interval.

In this context, we fix a non-empty open convex set $\mathcal{U} \subset \mathbb{R}^d$ and without loss of generality we assume that $0 \in \mathcal{U}$. It is not difficult to see that this is not a restrictive condition, since the space of Lipschitz functions over $\mathcal{U}$ which vanish at a fixed point $x_0 \in \mathcal{U}$ and the space of Lipschitz functions over $\mathcal{U} - x_0$ which vanish at 0 are linearly isometric. In fact, the operator $f \mapsto f(x_0 + \cdot)$ defines the aforementioned isometry.

Making use of Rademacher theorem, we first observe that the gradient of any Lipschitz function defined over $\mathcal{U}$ is well defined almost everywhere. Moreover, over the set of its differentiability points it is also bounded. This allows us to study the problem from the same starting point as in the case of dimension 1. In this sense, the first problem we encounter is that (contrary to the case one-dimensional case) not every vector-valued essentially bounded function is almost everywhere equal to the gradient of a Lipschitz function, as is shown in the following example.

Example Suppose that $\mathcal{U} = (-1, 1)^2$ and consider the function $g : \mathcal{U} \to \mathbb{R}^2$ given by $g(x_1, x_2) = (0, x_1)$. It is clear that this function belongs to $L^\infty(\mathcal{U}; \mathbb{R}^2)$ and also that it is not the gradient of any Lipschitz function (in fact, it is not the gradient of any function).

Considering the previous example it becomes necessary to find a stronger condition to assure that a vector-valued essentially bounded functions is in fact almost everywhere equal to the gradient of a Lipschitz function. The rest of this chapter deals with this problem and is divided in three main sections. First we study the structure of $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$ in terms of the properties of sets derived from their members and the use of them on integrability. Then we go deeper on the structure of Lipschitz functions over $\mathcal{U}$ in order to obtain a necessary and sufficient condition for the elements of $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$ to be the gradient of a Lipschitz function. Finally, using the preceding parts we obtain the desired isometry to identify the space of Lipschitz functions over $\mathcal{U}$ which vanish at 0 with a particular subset of $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$. The results that we present in this chapter can also be found in the author's Master Thesis ([36], in Spanish).

1.1 One-dimensional integration over $\mathcal{U}$

Following the ideas of the one-dimensional case, we would like to define Lipschitz functions from essentially bounded functions via integration over curves. The first complication of this process is that piecewise smooth curves have null Lebesgue measure, which may lead to ill-defined functions. In order to overcome this difficulty we proceed to study some properties of full measure sets and we apply them to obtain an operator on the class of essentially bounded functions, which will be the starting point for the final result of this chapter. With this in mind, let us start with the following proposition.

Proposition 1.1 Suppose that $A \subseteq \mathcal{U}$ has full measure in $\mathcal{U}$. Then, for every $x \in \mathcal{U}$

$$A_x := \{y \in \mathcal{U} \,:\, x + t(y-x) \in A \text{ a.e. on } [0,1]\}$$

has full measure in $\mathcal{U}$.

Proof. Thanks to the invariance under translations of the Lebesgue measure, we can assume that $x = 0$. It suffices to prove that for every $R > 0$, the equality $\lambda(B_R \cap A_0) = \lambda(B_R)$ holds, where $B_R = B_2(0,R) \cap \mathcal{U}$ and $B_2(0,R)$ is the ball of $\mathbb{R}^d$ endowed with the Euclidean norm. Since A has full measure, we know that $\lambda(B_R \cap A) = \lambda(B_R)$. Then, using spherical coordinates (where dv denotes the surface measure over $\mathbb{S}^{d-1}$) we have that

$$\int_{\mathbb{S}^{d-1}} \int_0^{R(v)} r^{d-1} dr dv = \int_{\mathbb{S}^{d-1}} \int_0^{R(v)} \mathbb{1}_A(rv) r^{d-1} dr dv,$$

where $R(v) = \sup\{r \in [0,R] \,:\, rv \in \mathcal{U}\}$. This yields that necessarily for almost every $v \in \mathbb{S}^{d-1}$

$$\int_0^{R(v)} \mathbb{1}_A(rv) r^{d-1} dr = \int_0^{R(v)} r^{d-1} dr.$$

Let $\Sigma \subseteq \mathbb{S}^{d-1}$ be the set of directions v where the last equality is true. Then for every $v \in \Sigma$, $rv \in A$ for almost every $r \in [0, R(v)]$, or equivalently $R(v)v \in A_0$. We easily see that A_0 is star-shaped, so actually we have that $rv \in A_0$ for every $r \in [0, R(v)]$, whenever $v \in \Sigma$. Considering this, we deduce that

$$\lambda(B_R \cap A_0) = \int_{\mathbb{S}^{d-1}} \int_0^{R(v)} \mathbb{1}_{A_0}(rv) r^{d-1} dr dv$$

$$= \int_\Sigma \int_0^{R(v)} r^{d-1} dr dv = \int_{\mathbb{S}^{d-1}} \int_0^{R(v)} r^{d-1} dr dv = \lambda(B_R),$$

for every $R > 0$. We deduce that A_0 has full measure. $\square$

In virtue of Proposition 1.1, we see that for every $x \in \mathcal{U}$ the set

$$A_x = \{y \in \mathcal{U} \,:\, \|g(x + t(y-x))\|_* \leq \|g\|_\infty \text{ a.e. on } [0,1]\}$$

have full measure, which is obtained directly from the fact that $\|g(z)\|_* \leq \|g\|_\infty$ almost everywhere over $\mathcal{U}$. This remark allows us to integrate essentially bounded functions over almost every straight line with endpoints $x, y \in \mathcal{U}$. This fact will become more precise in the next definition.

Definition 1.1 For $x \in \mathcal{U}$, we define the operator

$$T_x : L^\infty(\mathcal{U}; (\mathbb{R}^d)^*) \to \mathbb{R}^{\mathcal{U}} / \sim,$$

where $\mathbb{R}^U / \sim$ stands for the quotient of $\mathbb{R}^{\mathcal{U}}$ with respect to equality almost everywhere, as the equivalence class of the function defined as

$$f(y) = \begin{cases} \int_0^1 \langle g(x + t(y - x)), y - x \rangle dt & , \quad \text{if the integral is well defined} \\ 0 & , \quad \text{otherwise} \end{cases} .$$

In other words, the operator T_x takes an essentially bounded function g over $\mathcal{U}$, and gives a real-valued function which is defined almost everywhere as the integral of g over the straight line going from x to the point in which we are evaluating. We can see the resemblance with the one-dimensional case, where an essentially bounded function over an interval is used to obtain a real-valued function integrating from 0 to the evaluation point. In the latter case, it is clear that the resulting function is well-defined and Lipschitz. For higher dimensions this is not straightforward anymore. Our task is to show that this operator can be used in a similar way as in the one-dimensional case, which means that we need to find its relation with Lipschitz functions. It is worth noticing that the integral given in the definition is well defined for almost every $y \in \mathcal{U}$ thanks to the remark after Proposition 1.1. This fact will be paramount for assuring that the operator is well defined. To see how this operator acts, consider the following example.

Example Suppose that $\mathcal{U} = (-1, 1)^2$ and consider the function $g : \mathcal{U} \to \mathbb{R}^2$ given by $g(x_1, x_2) = (0, x_1)$. It is clear that this function belongs to $L^\infty(\mathcal{U}; \mathbb{R}^2)$. Notice that for every $x \in \mathcal{U}$ the integral in Definition 1.1 is well defined for every $y \in \mathcal{U}$. With this in mind, we have that T_0 is given by the equivalence class of the function over $\mathcal{U}$ given by

$$f(y_1, y_2) = \int_0^1 t y_1 y_2 dt = \frac{y_1 y_2}{2}.$$

Notice that the function obtained from this is actually Lipschitz over $\mathcal{U}$ and it vanishes at 0. If change the domain of definition of g to all of $\mathbb{R}^2$, the function f will be the same, and hence it will not be Lipschitz.

The last remark of the previous example tells us that it is necessary to set a condition over essentially bounded functions in order to obtain Lipschitz functions making use of the operator given in Definition 1.1. Before going deeper on that, we need to assure that this operator is actually well defined, that is, it does not depend on the chosen representative of the equivalence class in the domain.

Proposition 1.2 For every $x \in \mathcal{U}$, T_x is well-defined and is linear.

Proof. Let $g, h \in L^\infty(\mathcal{U}; (\mathbb{R}^n)^*)$ be such that $g = h$ almost everywhere. We want to prove that $T_x g = T_x h$. Thanks to Proposition 1.1 we know that the following sets have full measure

- $\{y \in \mathcal{U} \,:\, \|g(x+t(y-x))\|_* \leq \|g\|_\infty$ a.e. on $[0,1]\}$.
- $\{y \in \mathcal{U} \,:\, \|h(x+t(y-x))\|_* \leq \|h\|_\infty$ a.e. on $[0,1]\}$.
- $\{y \in \mathcal{U} \,:\, g(x+t(y-x)) = h(x+t(y-x))$ a.e. on $[0,1]\}$.

Then, for every $y \in \mathcal{U}$ belonging to the intersection of these three sets (which again has full measure) we have that

$$\int_0^1 \langle g(x+t(y-x)), y-x\rangle dt = \int_0^1 \langle h(x+t(y-x)), y-x\rangle dt,$$

where both integrals are well defined. We deduce that $T_x g = T_x h$. The linearity of T_x is trivial. $\square$

Once we have established that the operator T_x is well defined, we shall use it in a similar manner as in the one-dimensional case. In the next section, we study thoroughly a specific property of Lipschitz functions which will allow us to restrict the domain of these operators in order to obtain always Lipschitz functions as the image.

1.2 The space of Lipschitz-compatible functions

Recall that every $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$ (that is, φ is a smooth compactly supported function over $\mathcal{U}$) is Lipschitz. We begin by looking closely at this class of functions in order to obtain a general property of Lipschitz functions which will allow us to restrict the domain of the operators T_x defined in the previous section. Let $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$ and consider its gradient $\nabla\varphi$, which defines a bounded function over $\mathcal{U}$. Suppose that Γ is a piecewise smooth and regular curve in $\mathcal{U}$ of finite length and take $\gamma : [a,b] \to \mathcal{U}$ any parametrization of Γ. Then

$$\oint_\Gamma \nabla\varphi d\vec{r} = \int_a^b \langle \nabla\varphi(\gamma(t)), \gamma'(t)\rangle dt = \int_a^b (\varphi \circ \gamma)'(t) dt = \varphi(y) - \varphi(x),$$

where $x = \gamma(a)$ and $y = \gamma(b)$. In the same way, we can now go further. Suppose now that f is a Lipschitz function over $\mathcal{U}$, take Γ and γ as before. We see that $f \circ \gamma$ is Lipschitz, hence its derivative is well defined almost everywhere on (a,b). Then if g is any essentially bounded function such that $g(x) = \nabla f(x)$ for every $x \in \mathcal{D}_f$ and $\langle g(\gamma(t)), \gamma'(t)\rangle = f'(\gamma(t); \gamma'(t))$ almost everywhere on (a,b), we deduce that

$$\oint g d\vec{r} = \int_a^b \langle g(\gamma(t)), \gamma'(t)\rangle dt = \int_a^b f'(\gamma(t), \gamma'(t)) dt = \int_a^b (f \circ \gamma)'(t) dt = f(y) - f(x),$$

where again the points $x, y \in \mathcal{U}$ are the endpoints of Γ. We see that given $x, y \in \mathcal{U}$, the definition of g depends a priori on the chosen curve going from x to y and its parametrization. Nevertheless we see that the function g obtained as described is always equal almost everywhere to ∇f. This observation imposes a restriction over the elements of $L^\infty(U; (\mathbb{R}^d)^*)$ in order to be the gradient of a Lipschitz function, which is detailed in the following definition.

Definition 1.2 We say that an essentially bounded function g over $\mathcal{U}$ is Lipschitz-compatible if for almost every $x, y \in \mathcal{U}$ (with respect to the product measure over $\mathcal{U} \times \mathcal{U}$)

$$(T_0 g)(y) - (T_0 g)(x) = (T_x g)(y),$$

where $T_0 g$ and $T_x g$ are seen as any representative of the equivalence class.

Remark It is not difficult to see that there are non-trivial functions that are Lipschitz-compatible. More precisely, in the beginning of this section we saw that for every $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$, $\nabla\varphi$ is Lipschitz-compatible. The property of Lipschitz-compatibility can be understood as the fact that the functions defined by T_x and T_0 are the same (up to a constant) for almost every $x \in \mathcal{U}$. Moreover, it is easy to verify that this is actually a property of the class, since T_x is linear. Another way to understand this property is noticing that this is the same as saying that for almost every triangle (hence, every polygonal closed curve) the integral of g over it is equal to zero, that is, Lipschitz-compatible functions behave like conservative fields.

We would like to determine the image of Lipschitz-compatible functions via the operator T_0. To study the mentioned image, we need to introduce the next definition

Definition 1.3 The space of essentially Lipschitz functions is defined as

$$\mathcal{L}ip(\mathcal{U}) := \{f \in \mathbb{R}^{\mathcal{U}} \,:\, \mathcal{L}(f) < +\infty\},$$

where the essential Lipschitz constant is defined as

$$\mathcal{L}(f) = \operatorname*{esssup}_{\substack{x,y\in\mathcal{U}\\ x\neq y}} \frac{f(y) - f(x)}{\|y - x\|}.$$

Here the essential supremum is taken with respect to the product measure over $\mathcal{U}\times\mathcal{U}$. We will also say that $f \in \mathcal{L}ip_0(\mathcal{U})$ if $f \in \mathcal{L}ip(\mathcal{U})$ and there exists $K \geq \mathcal{L}(f)$ such that $|f(x)| \leq K\|x\|$ almost everywhere over $\mathcal{U}$.

The first question that arises is which is the link between this concept of essentially Lipschitz functions and the space of Lipschitz functions. In this sense, the next lemma gives us the answer and also shows the way to follow in order to prove the main result of this chapter.

Lemma 1.1 Let $f \in \mathcal{L}ip_0(\mathcal{U})$. We have that

i) If $h = f$ almost everywhere, then $\mathcal{L}(h) = \mathcal{L}(f)$.

ii) There exists a unique $h \in \mathrm{Lip}_0(\mathcal{U})$ such that $h = f$ almost everywhere.

In particular, $\mathrm{Lip}_0(\mathcal{U})$ is linearly isometric to the quotient of $\mathcal{L}ip_0(\mathcal{U})$ with respect to equality almost everywhere.

Proof. First we notice that $\mathcal{L}(\cdot)$ defines a seminorm on $\mathcal{L}ip_0(\mathcal{U})$. This is analogous to the fact that

$$\operatorname*{esssup}_{\omega\in\Omega} |g(\omega)|$$

defines a seminorm on $\mathcal{L}^\infty(\Omega)$. This directly proves the first part, since $\mathcal{L}$ is zero only on almost everywhere null functions. For the second part, we have that there exists a set $F \subseteq \mathcal{U}$ of full measure such that

$$\frac{|f(x) - f(y)|}{\|x - y\|} \leq \mathcal{L}(f)$$

for every $x, y \in F$. Since F is dense in $\mathcal{U}$, let us define h as the unique Lipschitz extension of $f|_F$. It is clear that $h(0) = 0$, because $f \in \mathcal{L}ip_0(\mathcal{U})$.

For the last part, we notice from the proof of the second part that we can also deduce that $\mathcal{L}(\cdot)$ defines a norm on the mentioned quotient and the operator that maps every $f \in \mathcal{L}ip_0(\mathcal{U})$ to its unique Lipschitz representative is linear, since it is defined by density. We deduce that the mentioned operator is in fact a linear isometry thanks to the first part of the lemma. □

In the last section of this chapter we use the tools developed above to show that the space of Lipschitz-compatible functions of $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$ contains nothing but the gradients of Lipschitz functions, thus giving us the desired identification of the space $\mathrm{Lip}_0(\mathcal{U})$.

1.3 Identification of the space of Lipschitz functions

In the previous section we studied some structural properties of Lipschitz functions over $\mathcal{U}$, which lead us to the definition of Lipschitz-compatible functions. Moreover, we gave an intermediate definition, namely the space of essentially Lipschitz functions, which will help us to state the connection between Lipschitz-compatible and Lipschitz functions. Considering that this will be key in the sequel, denote by Z the subspace of $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$ defined by the Lipschitz-compatible elements. We give now the proof of the main result of the present chapter.

Theorem 1.1 *The spaces Z and $\mathrm{Lip}_0(\mathcal{U})$ are linearly isometric.*

Proof. In virtue of Proposition 1.1 and the definition of Lipschitz-compatibility, we have that for almost every $x, y \in \mathcal{U}$

$$|(T_0 g)(y) - (T_0 g)(x)| = |(T_x g)(y)| \leq \|g\|_\infty \|y - x\|.$$

Again thanks to Proposition 1.1, we also have that for almost every $x \in \mathcal{U}$

$$|(T_0 g)(x)| \leq \|g\|_\infty \|x\|.$$

From this, we deduce that $T_0 g \in \mathcal{L}ip_0(\mathcal{U})$. Accordingly to Lemma 1.1, we define the linear operator $T : Z \to \mathrm{Lip}_0(\mathcal{U})$ as follows: Tg is the only continuous representative of $T_0 g$. From this and the first inequality it also follows that $\|T\| \leq 1$, which implies that T is continuous.

Consider now the operator $D : \mathrm{Lip}_0(U) \to Z$ given by $Rf = \nabla f$. Using Theorem 1 together with Proposition 1.1, we see that this operator is well-defined (that is, ∇f is Lipschitz-compatible) and it is clearly linear with $\|R\| \leq 1$, since $\|\nabla f(x)\|_* \leq \|f\|_L$ for every $x \in \mathcal{D}_f$. Moreover, we have that $TD = \mathrm{Id}_{\mathrm{Lip}_0(\mathcal{U})}$, since for every $f \in \mathrm{Lip}_0(\mathcal{U})$ we have that

$$TDf = T\nabla f = T_0(\nabla f) = T_0 g = f,$$

where the function $g : \mathcal{U} \to \mathbb{R}^d$ is defined as follows:

$$g(x) := \begin{cases} \nabla f(x) & , \ x \in \mathcal{D}_f \\ f'(x;x) \cdot \frac{u_x}{\|x\|} & , \ x \notin \mathcal{D}_f \wedge x \neq 0 \wedge f'(x;x) \text{ exists} \\ 0 & , \ \text{otherwise} \end{cases}$$

where $u_x \in \mathbb{R}^d$ is such that $\|u_x\|_* = 1$ and $\langle u_x, x\rangle = \|x\|$. We see that $g = \nabla f$ almost everywhere and for any $x \in \mathcal{U} \setminus \{0\}$, $\langle g(tx), x\rangle = f'(tx;x)$ almost everywhere on $[0,1]$, and then $T_0 g = f$.

In particular, we have that T is surjective. Suppose now that there exists $x \in \mathcal{U}$ such that $Tg(x) > 0$. Since $f := Tg \in \mathrm{Lip}_0(\mathcal{U})$, there exists $\delta > 0$ such that for almost every $y \in B(x,\delta)$

$$0 < f(y) = \int_0^1 \langle g(ty), y\rangle dt,$$

that is, for almost every $y \in B(x,\delta)$ there exists a non-null subset of $[0,1]$ such that $g(ty) \neq 0$ on that subset. This implies that there exists a non-null subset of $\mathcal{U}$ such that $g \neq 0$ on that subset. Then, T is injective. We deduce that T is bijective with $T^{-1} = D$. With this T is a linear isometry between Z and $\mathrm{Lip}_0(\mathcal{U})$. □

This last proposition goes in line with the procedure for the one-dimensional case. It is worth to clarify at this point the main difference that appears on higher dimensions. If we take all that we have done to this point during this chapter in the case $d = 1$ we can see that every function of $L^\infty(\mathcal{U})$ is Lipschitz compatible, which is a trivial consequence of the definition. Hence in that case, since every Lipschitz function is absolutely continuous, the link between Lipschitz functions derivatives and essentially bounded functions is evident. In higher dimensions, since we have other directions to go, this link is broken and it becomes necessary to study more in detail the structure of Lipschitz functions. We now prove the main result of this section, which states an identification for the space $\mathrm{Lip}_0(\mathcal{U})$.

We finish this chapter by mentioning that the last characterization was also proven in [28] with a very similar statement.

Proposition 1.3 (M. Cúth, O. Kalenda, P. Kaplický) For any $f \in \mathrm{Lip}_0(\mathcal{U})$ set $Df = \nabla f$. Then, the following hold

i) D is a linear isometry of $\mathrm{Lip}_0(\mathcal{U})$ into $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$.

ii) The range of D is

$$Z(\mathcal{U}) = \{g \in L^\infty(\mathcal{U}; (\mathbb{R}^d)^*) \ : \ \partial_i g_j = \partial_j g_i \text{ for } i,j = 1,\dots,d\},$$

where the derivatives are considered in the sense of distributions on $\mathcal{U}$.

iii) The inverse operator $D^{-1} : Z(\mathcal{U}) \to \mathrm{Lip}_0(\mathcal{U})$ is defined by

$$D^{-1}g(x) = \lim_{k\to\infty} \int_0^1 \langle (g * u_k)(tx), x\rangle dt,$$

where $(u_k)_{k\in\mathbb{N}}$ is a smooth mollifier.

The main difference between these results is the way of describing the space of Lipschitz-compatible elements of $L^\infty(\mathcal{U};(\mathbb{R}^d)^*)$. We have described it in terms of integrals over closed polygonal curves, asking for this value to be 0, while in the aforementioned result it is described in terms of derivatives in the sense of distributions. It is not hard to notice that we can see a direct relation between these results: We can consider $L^\infty(\mathcal{U};(\mathbb{R}^d)^*)$ as vector fields. Then, in one hand we are asking these fields to be conservative over $\mathcal{U}$ while in the other case we are asking them to have null curl, which recalls a classical result for smooth vector fields on $\mathbb{R}^d$.

Chapter 2

Identification of the Lipschitz-free space of finite-dimensional spaces

We have already seen in the previous chapter that even in the case where the domain of the Lipschitz functions is some appropiate subset of a finite-dimensional space, the structure of those functions is not simple, and they are connected to the space of essentially bounded functions in a way that is not as direct as it is in the one-dimensional case. Before going further on our discussion, a few known results for Lipschitz-free spaces are presented as a starting point in order to understand the objective behind finding a way of identifying the Lipschitz-free spaces for finite-dimensional spaces. For more details on these results, we refer to [27], [42], [49], [45], [52] and [59].

Suppose that X, Y are Banach space and that $L : X \to Y$ is a Lipschitz map such that $L(0) = 0$. A basic lemma tells us that in this case, there exists a unique linear map $\hat{L} : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\hat{L}\delta_X = \delta_Y L$ and $\|\hat{L}\| = \|L\|_{\mathrm{Lip}}$. In other words, we can think of Lipschitz-free spaces as those spaces that not only contain a copy of the host space, but also absorb all the Lipschitz structure, "linearizing" Lipschitz functions. A straightforward consequence of the previous lemma is that whenever X is a subspace of Y, with $\iota : X \to Y$ the canonical embedding, then $\hat{\iota} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is an isometric embedding.

In this sense, the question of how to distinguish the spaces $\mathcal{F}(\mathbb{R}^n)$ for $n \geq 1$ arises. More precisely, a first natural question is whether or not the spaces $\mathcal{F}(\mathbb{R}^{n+1})$ and $\mathcal{F}(\mathbb{R}^n)$ are isometric. A partial answer to this was given on [56], where it is proven that $\mathcal{F}(\mathbb{R})$ is strictly embedded in $\mathcal{F}(\mathbb{R}^2)$. Nevertheless, the general case remains an open problem, which leads to the study of these spaces in a general setting.

But even considering that the aforementioned connection between those spaces remains an open question, there are numerous results concerning their internal structure. More precisely, in [52], G. Lancien and E. Pernecká proved that $\mathcal{F}(\mathbb{R}^d)$ admits monotone finite-dimensional Schauder decompositions when $\mathbb{R}^d$ is endowed with the norm $\|\cdot\|_1$. Moreover, from a result of P. Hájek and E. Pernecká [45] we know that the space $\mathcal{F}(X)$ has a Schauder basis whenever X is the product of countably many closed intervals in $\mathbb{R}$, with endpoints in $\mathcal{Z} \cup \{-\infty, \infty\}$, considered as a metric subspace of ℓ_1 equipped with the inherited metric, which in particular shows that $\mathcal{F}(\mathbb{R}^d)$ has a Schauder basis when $\mathbb{R}^d$ is endowed with the norm $\|\cdot\|_1$. We readily see that in both cases the choice of the norm is important, which is a point that we will consider in the following.

In a more general setting, E. Pernecká and R. J. Smith proved in [59] that independently of the choice of the norm, the space $\mathcal{F}(M)$ has the metric approximation property whenever $M \subset \mathbb{R}^d$ satisfies certain structural conditions. In particular, this is true when M is a compact and convex subset of $\mathbb{R}^d$. In a sense, this result allows us to state the question of the structure of the Lipschitz-free space of a subset U of $\mathbb{R}^d$ at least keeping the convexity for U, expecting to develop a technique that works no matter the subjacent metric structure.

In order to do this we use the main result of Chapter 1 (Theorem 1.1) to identify the Lipschitz-free space for any non-empty open convex subset $\mathcal{U}$ of $\mathbb{R}^d$, which without loss of generality contains 0. As we already saw during the motivation for the definition of Lipschitz-compatibility, the fact that every smooth function compactly supported over $\mathcal{U}$ is Lipschitz gives a great insight to the structure of the space of Lipschitz functions. In this sense we will continue with the study of those functions in order to obtain an isometry for $\mathcal{F}(\mathcal{U})$ with an appropiate space related to Z, the space of Lipschitz-compatible functions.

Recall that in the one-dimensional case once we obtain the isometry between $\mathrm{Lip}_0(\mathcal{U})$ and $L^\infty(\mathcal{U})$, the identification for $\mathcal{F}(\mathcal{U})$ follows easily, since it is known that $L^\infty(\mathcal{U})$ has a unique predual, up to isometry (thanks to Grothendieck's Theorem, see [43]). To do this, the use of Theorem 4 was useful to obtain the desired isometry. It is worth noticing that the use of that theorem can easily be avoided in the one-dimensional case, while for our purpose in higher dimensions will be quite useful.

2.1 Identification of the Lipschitz-free space

In analogy to the one-dimensional case we would like to use the operator T from the previous chapter to obtain a linear isometry between $\mathcal{F}(\mathcal{U})$ and a predual space for Z (the space of Lipschitz-compatible elements of $L^\infty(\mathcal{U};(\mathbb{R}^d)^*)$). For the one-dimensional case we used the known fact that $L^1(\mathcal{U})$ is the unique (up to isometry) predual of $L^\infty(\mathcal{U})$ for any open interval $\mathcal{U} \subseteq \mathbb{R}$. In our case we only know that X is a dual space. To find a predual space for $\mathcal{U}$, we first recall a classical result on Banach spaces theory

Proposition 2.1 Let X be a Banach space and Y a closed subspace of X. Then the dual space $(X/Y)^*$ is linearly isometric to the annihilator of Y

$$Y^\perp = \{x^* \in X^* \,:\, \langle x^*, y\rangle = 0, \forall y \in Y\}.$$

Considering this, we want to compute the space $Y := Z^\perp \cap L^1(\mathcal{U};\mathbb{R}^d)$. In order to do this, we state a relation between $\mathcal{C}_0^\infty(\mathcal{U})$ and Z. To this end, we need the next lemma

Lemma 2.1 Let $g \in Z$ be any function and $(\varphi_k)_{k\in\mathbb{N}}$ a sequence on $\mathcal{C}_0^\infty(\mathcal{U})$ with $\varphi_k(0) = 0$ for every $k \in \mathbb{N}$. Then, the following are equivalent *i)* $(\exists f \in \mathrm{Lip}_0(\mathcal{U}), K > 0)\, \varphi_k \xrightarrow{p.w.} f$, $\|\varphi_k\|_L \leq K$ and $g = \nabla f$.

ii) $\nabla \varphi_k \overset{*}{\rightharpoonup} g$ on $L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$.

Proof. Suppose that $\|\varphi_k\|_L \leq K$ and that φ_k converges pointwise to f, with f such that $\nabla f = g$. It suffices to show that for every open bounded d-dimensional cube H

$$\int_H \frac{\partial \varphi_k}{\partial x_1} d\lambda \longrightarrow \int_H \frac{\partial f}{\partial x_1} d\lambda.$$

Let $I = (a, b)$ be an interval and H' an open bounded $(d-1)$-dimensional cube such that $H = I \times H'$. Then

$$\int_H \frac{\partial \varphi_k}{\partial x_1} d\lambda^{(n)} = \int_{H'} \int_a^b \frac{\partial \varphi_k}{\partial x_1}(t, y) dt dy = \int_{H'} \left(\varphi_k(b, y) - \varphi_k(a, y)\right) dy.$$

We see that for every $y \in H'$, $\varphi_k(b, y) - \varphi_k(a, y) \longrightarrow f(b, y) - f(a, y)$. Moreover, we have that

$$|\varphi_k(b, y) - \varphi_k(a, y)| \leq \|\varphi_k\|_L (b - a)\|e_1\| \leq K(b - a)\|e_1\|.$$

Since H' has finite measure, in virtue of the Dominated Convergence Theorem we have that

$$\int_H \frac{\partial \varphi_k}{\partial x_1} d\lambda \longrightarrow \int_{H'} f(b, y) - f(a, y) dy,$$

but we also have that

$$\int_{H'} f(b, y) - f(a, y) dy = \int_{H'} \int_a^b \frac{\partial f}{\partial x_1}(t, y) dt dy = \int_H \frac{\partial f}{\partial x_1} d\lambda.$$

Repeating the same procedure over every coordinate, the direct implication is proven.

For the converse, suppose that $\nabla \varphi_k \overset{*}{\rightharpoonup} g$ on $L^\infty(U; (\mathbb{R}^d)^*)$. We want to prove that φ_k converges pointwise to some Lipschitz function. Let $x \in U \setminus \{0\}$ be any point and consider $V = \{x\}^\perp$. For $\varepsilon > 0$, we denote by $B_V(\varepsilon)$ the restriction to V of the ball of radius ε centered at 0. Let λ' be the $(d-1)$-dimensional Lebesgue measure over V. Then, in virtue of Lebesgue Differentiation Theorem, we have that

$$\begin{aligned}
\varphi_k(x) &= \lim_{\varepsilon \to 0} \frac{1}{\lambda'(B_V(\varepsilon))} \int_{B_V(\varepsilon)} \varphi_k(x + y) - \varphi_k(y) dy \\
&= \lim_{\varepsilon \to 0} \frac{1}{\lambda'(B_V(\varepsilon))} \int_{B_V(\varepsilon)} \int_0^1 \langle \nabla \varphi_k(y + tx), x \rangle dt dy \\
&= \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \left\langle \nabla \varphi_k, \frac{x}{\|x\|_2 \lambda'(B_V(\varepsilon))} \right\rangle d\lambda = \lim_{\varepsilon \to 0} \langle \nabla \varphi_k, f_\varepsilon \rangle,
\end{aligned}$$

where

$$H_\varepsilon = \{z \in U \,:\, (\exists y \in B_V(\varepsilon), t \in [0, 1])\, z = y + tx\}$$

and

$$f_\varepsilon = \frac{x}{\|x\|_2 \lambda'(B_V(\varepsilon))} \mathbb{1}_{H_\varepsilon}.$$

We easily deduce that $f_\varepsilon \in L^1(U;\mathbb{R}^d)$. In fact, it suffices to see that given the definition of H_ε, $\lambda(H_\varepsilon) = \|x\|_2 \lambda'(B_V(\varepsilon))$. Now, for $k, j \in \mathbb{N}$ we have that

$$|\varphi_k(x) - \varphi_j(x)| = \lim_{\varepsilon\to 0} |\langle \nabla\varphi_k - \nabla\varphi_j, f_\varepsilon\rangle|.$$

Let $\eta > 0$. Then, there exists $\varepsilon > 0$ be such that

$$\lim_{\varepsilon\to 0} |\langle \nabla\varphi_k - \nabla\varphi_j, f_\varepsilon\rangle| \leq |\langle \nabla\varphi_k - \nabla\varphi_j, f_\varepsilon\rangle| + \eta.$$

But the sequence $(\langle \nabla\varphi_k, f_\varepsilon\rangle)_{i\in\mathbb{N}}$ is Cauchy, hence there exists $N \in \mathbb{N}$ such that for every $j, k \geq N$

$$|\varphi_k(x) - \varphi_j(x)| \leq |\langle \nabla\varphi_k - \nabla\varphi_j, f_\varepsilon\rangle| + \eta \leq 2\eta.$$

We deduce that for every $x \in U$, $(\varphi_k(x))_{k\in\mathbb{N}}$ is a Cauchy (hence convergent) sequence.

In the following, we denote by f the pointwise limit of this sequence. It is clear that $f(0) = 0$. Since $\nabla\varphi_k$ is w^*-convergent, $\|\nabla\varphi_k\|_\infty = \|\varphi_k\|_L$ are bounded, say by $K > 0$ and then

$$|f(x) - f(y)| = \lim_{k\to\infty} |\varphi_k(x) - \varphi_k(y)| \leq \limsup_{k\to\infty} \|\varphi_k\|_L \|x - y\| \leq K\|x - y\|,$$

which leads to $f \in \mathrm{Lip}_0(U)$. We now see that $\nabla\varphi_k \overset{*}{\rightharpoonup} \nabla f$ because of the direct implication and we conclude that $g = \nabla f$. □

Once stated this close link between w^* convergence on $L^\infty(U;(\mathbb{R}^d)^*)$ and pointwise convergence, we must highlight two facts. On one hand, the last lemma reveals the presence of $\mathcal{F}(U)$ in the structure of Z, since pointwise convergence and w^*-convergence on $\mathrm{Lip}_0(U)$ are closely related. On the other hand, it says that it suffices to focus on a class of functions with good properties to study the space of Lipschitz-compatible functions. A more detailed approach of this last remark is given in the following corollary.

Corollary 2.1 *The subspace $\tilde{Z} := \{\nabla\varphi \,:\, \varphi \in \mathcal{C}_0^\infty(U)\}$ of $L^\infty(U;(\mathbb{R}^d)^*)$ is w^*-dense on Z.*

Using this consequence and the class of compactly supported smooth functions we can now get a first description of the desired predual of Z.

Proof. For $g \in Z$ let $f = Tg$. For a mollifier $(u_k)_{k\in\mathbb{N}} \in \mathcal{C}_0^\infty(U)$, define $\varphi_k = u_k * f - u_k * f(0)$. Then, apply Lemma 2.1. On the other hand, if $\nabla\varphi_k \overset{*}{\rightharpoonup} g$ on $L^\infty(U;(\mathbb{R}^d)^*)$ we again apply Lemma 2.1, since we can always assume that $\varphi_k(0) = 0$. □

Proposition 2.2 Let $h \in L^1(U;\mathbb{R}^d)$ be any function. Then $h \in Z^\perp$ if and only if $\nabla \cdot h = 0$ in the sense of distributions.

Proof. For every $\varphi \in \mathcal{C}_0^\infty(U)$ we have that

$$\langle \nabla\varphi, h\rangle = \int_U \langle \nabla\varphi, h\rangle d\lambda = \sum_{k=1}^d \int_U \frac{\partial\varphi}{\partial x_k} h_k d\lambda = -\int_U \varphi(\nabla \cdot h) d\lambda.$$

As the derivatives of $\mathcal{C}_0^\infty(U)$ functions are w^*-dense on Z, we conclude the desired equivalence. $\square$

Consider now the subspace of $L^1(U;\mathbb{R}^d)$ given by

$$Y = \{f \in L^1(U;\mathbb{R}^d) \,:\, \nabla \cdot f = 0 \text{ in the sense of distributions}\} = Z^\perp \cap L^1(U;\mathbb{R}^d).$$

Using Proposition 2.1, now we know that $(L^1(U;\mathbb{R}^n)/Y)^*$ is linearly isometric to

$$Y^\perp = (Z^\perp \cap L^1(U;\mathbb{R}^d))^\perp = \overline{Z}^{w^*} = Z.$$

In other words, $L^1(U;\mathbb{R}^d)/Y$ is a predual for Z. To conclude, we show that the operator T defined in Chapter 1 is continuous when we equip $\mathrm{Lip}_0(U)$ and Z with their w^*-topologies, looking at these spaces as the dual spaces of $\mathcal{F}(U)$ and $L^1(U;\mathbb{R}^d)/Y$, respectively.

Proposition 2.3 The linear isometry $T : X \to \mathrm{Lip}_0(U)$ is w^*-w^* continuous.

Proof. This is equivalent to prove that $D = T^{-1}$ is w^*-w^* continuous. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathrm{Lip}_0(U)$ such that $f_\lambda \overset{*}{\rightharpoonup} f$. We want to prove that $\nabla f_\lambda \overset{*}{\rightharpoonup} \nabla f$, considering Z as the dual of $L^1(U;\mathbb{R}^d)/Y$. That is, for every $[h] \in L^1(U;\mathbb{R}^d)/Y$

$$\langle \nabla f_\lambda, [h]\rangle \to \langle \nabla f, [h]\rangle.$$

Noticing that this is equivalent to $\langle \nabla f_\lambda, h\rangle \to \langle \nabla f, h\rangle$ for every $h \in L^1(U;\mathbb{R}^d)$, we can use Lemma 2.1. Then, it suffices to prove pointwise convergence and boundedness of f_k. These two are trivial from the fact that $f_\lambda \overset{*}{\rightharpoonup} f$, since $\mathrm{Lip}_0(U) \equiv \mathcal{F}(U)^*$. $\square$

Using the last proposition together with Proposition 4 we conclude that

$$L^1(U;\mathbb{R}^d)/Y \equiv \mathcal{F}(U).$$

We summarize the main result of this chapter in the following theorem.

Theorem 2.1 *Let U be a nonempty open convex subset of $\mathbb{R}^d$. Then, $\mathcal{F}(U)$ is linearly isometric to $L^1(U;\mathbb{R}^d)/Y$, where Y is the subspace of $L^1(U;\mathbb{R}^d)$ given by the functions with null divergence in the sense of distributions. Moreover, if S is the preadjoint of T and $\Psi : U \to \mathcal{C}_0^\infty(U)^*$ is such that*

$$\langle \Psi(x), \varphi\rangle = \varphi(x)\,,\ \forall x \in U, \varphi \in \mathcal{C}_0^\infty(U),$$

then $S\delta(x) = [f]$ if and only if $\nabla \cdot f = \Psi(0) - \Psi(x)$.

Proof. The first part is direct from Proposition 2.3. For the final part, let $f \in L^1(U;\mathbb{R}^d)$ and $x \in U$. Then

$$I\delta(x) = [f] \Leftrightarrow (\forall \varphi \in \mathcal{C}_0^\infty(U))\, \langle \nabla\varphi, I\delta(x)\rangle = \langle \nabla\varphi, f\rangle.$$

Then, let $\varphi \in \mathcal{C}_0^\infty(U)$ be any function. We see that

$$\langle \nabla\varphi, I\delta(x)\rangle = \langle TT^{-1}(\varphi - \varphi(0)), \delta(x)\rangle = \varphi(x) - \varphi(0) = \langle \Psi(x) - \Psi(0), \varphi\rangle.$$

On the other hand

$$\langle \nabla\varphi, f\rangle = \sum_{k=1}^{d} \int_U \frac{\partial\varphi}{\partial x_k} f_k d\lambda = -\int_U \varphi \sum_{k=1}^{d} \frac{\partial f_k}{\partial x_k} d\lambda = -\langle \nabla \cdot f, \varphi\rangle.$$

Then, we have that

$$I\delta(x) = [f] \Leftrightarrow \nabla \cdot f = \Psi(0) - \Psi(x).$$

□

Just like in the end of Chapter 1, it is important to mention that also in [28], M. Cúth, O. Kalenda and P. Kaplický obtained the same identification for the Lipschitz-free space of non-empty open convex subsets of a finite-dimensional space, starting from their description of what we have called Lipschitz-compatible functions during the present work.

The results of this Chapter give us a way to deal with the Lipschitz-free spaces of metric spaces isometric to nonempty open convex subsets of a finite dimensional space. It is important to consider that these results are completely independent of the chosen norm in $\mathbb{R}^d$, but this does not mean that the Lipschitz-free space of a nonempty open convex set $\mathcal{U} \subset \mathbb{R}^d$ is the same for every norm. Actually, the dependence of this norm is present in the L^∞ and L^1 spaces which host the identifications for both the spaces $\mathrm{Lip}_0(\mathcal{U})$ and $\mathcal{F}(\mathcal{U})$.

Recall that a question involving the Lipschitz-free spaces of finite dimensional spaces is that if it is true or not that $\mathcal{F}(\mathbb{R}^d)$ is isometric to $\mathcal{F}(\mathbb{R}^k)$ whenever $d \neq k$. This has already been answered negatively in the particular case where $d = 1$ and $k = 2$, but it still remains an open question in the general case. This identifications and the study of their structure may serve as a way to further study and answer this question.

Several of the arguments given during the proofs relied strongly in the fact that we were working in a finite-dimensional space, notably Rademacher's Theorem. It is not clear how to adapt these ideas to a more general framework, for which a different approach is needed.

Chapter 3

Clarke-saturated Lipschitz functions on finite-dimensional spaces

Lineability and spaceability are concepts that can be summarized as the search of linear structure or even normed spaces inside a certain set defined by some property. Our goal is to make use of the tools developed during Chapter 1 together with differentiability properties for Lipschitz functions in order to find a linear structure inside the set of Lipschitz functions whose Clarke subdifferential is as big as it can be. For this, we first need to clarify the meaning of this last statement.

The concepts of differentiability and subdifferentials are in the core of convex analysis and convex optimization. In this sense, recall that for a convex function $f : \mathcal{U} \subset X \to \mathbb{R}$ the directional derivatives, given by

$$f'(x; d) := \lim_{t \to 0^+} \frac{f(x+td) - f(x)}{t},$$

are well defined for every $x \in \mathcal{U}$ and $d \in X$, where U is an open subset of a Banach space X. In the same way, the convex subdifferential of f at $x_0 \in \mathcal{U}$, defined as

$$\partial f(x_0) := \{x^* \in X^* : \langle x^*, d\rangle \leq f'(x_0; d) \, \forall d \in X\}$$

becomes a nonempty convex-w^* closed subset of X^*. As mentioned before, a straightforward consequence of these definitions is that the function $f'(x_0; \cdot)$ is sublinear for every $x_0 \in \mathcal{U}$ and that $z \in \mathcal{U}$ defines a minimum of f if and only if $0 \in \partial f(z)$, which is nothing but a generalization of the well known optimality condition for differentiable convex functions over finite dimensional spaces.

For the case of locally Lipschitz functions, a generalization of these definitions is made having in mind the bound given by the Lipschitz constant. More precisely, the Clarke derivative and subdifferential are defined as follows

$$f^\circ(x_0; d) := \limsup_{y \to x, t \to 0^+} \frac{f(y+td) - f(y)}{t},$$

$$\partial^\circ(x_0) := \{x^* \in X^* : \langle x^*, d\rangle \leq f^\circ(x_0; d) \, \forall d \in X\},$$

where $x_0 \in \mathcal{U}$ and $d \in X$. As was the case with the directional derivative of a convex function, the function $f^\circ(x_0;\cdot)$ is sublinear for every $x_0 \in \mathcal{U}$ and at every $x_0 \in \mathcal{U}$ the Clarke subdifferential is a nonempty convex w^*-compact subset of X^*. While these definitions share these basic properties with the convex case, it is well known that there exist non-constant Lipschitz functions even over $\mathbb{R}$ for which 0 belongs to the Clarke subdifferential at every point. To see the real implications of the aforementioned anomaly, we must first understand why these definitions are actually a generalization from the convex case. Recall that if a function f is convex, then if it is bounded above in a neighbourhood of a point of its domain, then it is locally Lipschitz over all its domain. In this case, we have that both definitions of directional derivatives (hence, of subdifferentials) coincide.

The idea of finding a linear structure as mentioned at the beginning of this chapter comes arises in the following context. Let X be a separable Banach space and $\mathcal{U}$ a nonempty open subset of X. We denote by $\overline{B}_*$ the closed unit ball of the dual space X^* and by $||f||_{\text{Lip}}$ the Lipschitz constant of a Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ below). We also denote by $\text{Lip}^{[k]}(\mathcal{U})$ the set of Lipschitz functions f defined on $\mathcal{U}$ of Lipschitz constant $||f||_{\text{Lip}} \leq k$. This space, when endowed with the metric of uniform convergence over bounded subsets of $\mathcal{U}$, is complete.

In the above setting J. Borwein and X. Wang have shown in [19]–[20], that the set of Lipschitz functions with maximal Clarke subdifferential (that is, $\partial^\circ f(x) \equiv ||f||_{\text{Lip}}\, \overline{B}_*$ for all $x \in \mathcal{U}$) is generic in $\text{Lip}^{[k]}(\mathcal{U})$. The result has been obtained via a standard application of Baire's category theorem. However, this result highly depends on the chosen metric, the reason being that wild functions with oscillating derivatives can be obtained as uniform limits of well-behaved ones (piecewise linear or quadratic). An explicit construction of such a wild function with maximal Clarke subdifferential is given in [18].

Therefore, in some generic sense, most Lipschitz functions are *Clarke-saturated* (see forthcoming Definition 3.1), but this genericity is strongly related to the chosen topology. To illustrate further this fact, let us fix a nonempty compact subset K of $\mathcal{U}$ and let us consider $\text{Lip}^{[k]}(K)$ as a closed subset of the Banach space $(\mathcal{C}(K), ||\cdot||_\infty)$ (a uniform limit of Lipschitz continuous functions of Lipschitz constant bounded by k is Lipschitz). Then $||\cdot||_\infty$-limits of piecewise polynomial functions in $\text{Lip}^{[k]}(K)$ may give rise to Lipschitz functions with maximal Clarke subdifferentials. A completely different behaviour appears if one uses instead, the Lipschitz norm to describe convergence: in this case $||\cdot||_{\text{Lip}}$-limits of (piecewise) polynomials are (piecewise) $\mathcal{C}^1$-functions (therefore $\partial^\circ f(x) \equiv \{df(x)\}$, for all $x \in K$). The reason is that for smooth functions the Lipschitz norm $||\cdot||_{\text{Lip}}$ coincides with the norm of uniform convergence of the derivatives and under this norm $\mathcal{C}^1(K)$ is a Banach subspace of $\text{Lip}(K)$.

If $X = \mathbb{R}^d$, then important subclasses of Lipschitz functions, such as semialgebraic (more generally, o-minimal) Lipschitz functions or finite selections of $\mathcal{C}^d$-smooth functions have small Clarke subdifferentials: indeed, the aforementioned classes satisfy a Morse-Sard theorem for their generalized critical values, see [17, Corollary 5(ii)] and [13, Theorem 5] respectively, while every point (and consequently every value) of a Clarke-saturated Lipschitz function is critical.

In this chapter we complement the results [18], [19], [20] by establishing a topology-independent result (Theorem 3.1(i)), namely, that the set of Clarke-saturated Lipschitz functions contains an infinite dimensional linear space of uncountable dimension; in particular it is *lineable*, according to the terminology of [44], and consequently algebraically large. Moreover, surprisingly, $(\mathrm{Lip}(K), ||\cdot||_{\mathrm{Lip}})$ contains a closed non-separable subspace of Clarke-saturated functions, hence this set is also *spaceable*. We refer to [9] for related terminology and an exposition on the state of the art of this trend, nowadays known as lineability and spaceability. We also refer to [8], [55], and the expository paper [16], for recent results. In some sense, our results have been anticipated in [16, page 114].

3.1 Context and previous results

Throughout the rest of this chapter, the functions used will be defined over a nonempty open convex subset $\mathcal{U}$ of $\mathbb{R}^d$. In this case, we have that the Clarke subdifferential is given by

$$\partial^\circ f(x) := \overline{\mathrm{co}}\{\lim_{k\to\infty} \nabla f(x_k) \,:\, (x_k)_{k\in\mathbb{N}} \subset \mathcal{D}_f \setminus N \text{ and } x_k \to x\}, \tag{3.1}$$

where $\mathcal{D}_f$ stands for the set of differentiability points of f and $N \subset \mathcal{U}$ is any null set. In other words, $\partial^\circ f(x)$ is the closed convex hull of all the accumulation points of the gradient of f around x. It can be proven that the choice of N does not change the result and hence, thanks to Rademacher theorem [1], this set is well defined at every $x \in \mathcal{U}$ and coincides with the general definition. As mentioned earlier, the Clarke subdifferential is always a nonempty convex w^*-compact subset of X^* whenever f is locally Lipschitz. We state this more precisely in the following proposition.

Proposition 3.1 For every $f \in \mathrm{Lip}(\mathcal{U})$ and $x \in \mathcal{U}$, $\partial^\circ f(x)$ is a non-empty convex compact subset of $\mathbb{R}^d$. Moreover, $\partial^\circ f(x)$ is contained in the closed dual ball centered at 0 of radius $\|f\|_L$.

Recall the result from J. Borwein and X. Wang [19, Theorem 1], which states the genericity of Lipschitz functions that attain the inclusion with equality in the context of the metric of uniform convergence. We deal now with the question of whether a similar result can be obtained in the space of all Lipschitz functions which does not depend on this specific metric, but is based, instead, on the concepts of lineability and spaceability. More precisely, we ask ourselves if the set of functions with big subdifferentials contains a linear structure and how rich is that structure.

We begin by stating the definition of Clarke-saturated (Lipschitz continuous) functions which will be the starting point for this investigation.

Definition 3.1 (Clarke-saturated function) Let $\mathcal{U} \subset \mathbb{R}^d$. We say that $f \in \mathrm{Lip}(\mathcal{U})$ has a maximal Clarke subdifferential at $x \in \mathcal{U}$ whenever $\partial^\circ f(x) = \|f\|_L \overline{B}_*$, that is, the Clarke subdifferential equals to the closed ball of $(\mathbb{R}^d)^*$ centered at 0 and with radius $\|f\|_L$. If this is true for every $x \in \mathcal{U}$, we say that f is Clarke-saturated.

The first example of a Clarke-saturated Lipschitz function in one-dimension has been given (up to obvious modifications) by G. Lebourg in [53, Proposition 1.9]. The function was given by an explicit formula based on a *splitting* subset A of $\mathbb{R}$ with respect to the family of nontrivial intervals of $\mathbb{R}$, that is, a measurable subset A satisfying

$$0 < \lambda(A \cap I) < \lambda(I), \qquad \text{for every (nontrivial) interval } I \subset \mathbb{R}, \tag{3.2}$$

where λ denotes the Lebesgue measure. An explicit construction of such a splitting set can be found in [50] in a general setting (atomless measure space). In the next section we shall enhance this construction to the particular case of a real line and come up with a countable family of disjoint spitting sets. This family will be paramount for the proof of our main result.

Finally, recall Theorem 1.1 and Proposition 1.3 which state an isometry for the space $\mathrm{Lip}_0(\mathcal{U})$ in terms of the gradient of the functions. More precisely, the operator

$$D : \mathrm{Lip}_0(\mathcal{U}) \to L^\infty(\mathcal{U}; (\mathbb{R}^d)^*)$$

given by $Df = \nabla f$ is a linear isometry whose image is given by

$$Z = \{g \in L^\infty(\mathcal{U}; (\mathbb{R}^d)^*) \,:\, \partial_i g_j = \partial_j g_i \text{ for every } i, j \in \{1, \dots, d\}\},$$

where $\partial_i g_j$ stands for the partial derivative with respect to x_i of the j-th component of g in the sense of distributions. That is, the required equality goes as follows

$$\int_U g_j \frac{\partial \varphi}{\partial x_i} d\lambda = \int_{\mathcal{U}} g_i \frac{\partial \varphi}{\partial x_j} d\lambda, \quad \text{for every } \varphi \in \mathcal{C}_0^\infty(\mathcal{U}).$$

For better understanding and for justifying the constructions, the following section deals with the cases $d = 1$ and $d > 1$ separately. Nevertheless, both constructions rely on the same principle, with a slight modification for the case $d > 1$.

3.2 Main result

In this section we establish our main result which consists in exhibiting a linear space of uncountable dimension of Clarke-saturated Lipschitz functions, whenever $\mathcal{U} \subseteq \ell_d^1$ is a nonempty open convex set. More precisely, endowing $\mathrm{Lip}_0(\mathcal{U})$ with the Lipschitz norm $\|\cdot\|_L$ we obtain a closed subspace of Clarke-saturated elements, which in turn yields the result thanks to Baire theorem (which ensures that every infinite dimensional Banach space has uncountable dimension). Our technique is as follows: we will first prove the result for the one-dimensional case and then we extend the construction for the d-dimensional case. In both cases, we first obtain countably many linearly independent Clarke-saturated functions in $\mathrm{Lip}_0(\mathcal{U})$ and in the final subsection we use these functions to obtain the final result.

3.2.1 The one-dimensional case

We begin by studying the simplest case, given by $d = 1$. To this end, we show how to construct Clarke-saturated functions using a known technique and with some modifications, how to obtain a countable family of such functions which are also linearly independent. As mentioned earlier, the construction for the aforementioned family of functions relies on some basic results concerning Lebesgue measure. Let us start with a typical example of a subset of $[0, 1]$ which is closed, nowhere dense and has positive measure.

Definition 3.2 (Smith-Volterra-Cantor set) Consider the subsets $F_n \subset [0,1]$ defined as follows:

- $F_0 = [0,1]$
- F_n is obtained by removing the middle open interval of length $\frac{1}{2\cdot 4^n}$ from each of the 2^n closed intervals whose union is F_n.

Let $F = \bigcap_{n\geq 0} F_n$. Then F is closed and contains no intervals. Moreover, F is Lebesgue measurable with measure $1/2$.

From now on we shall use the term fat Cantor set for any Cantor-type set (that is, a set built in this way) with positive measure. It is clear that this procedure can be carried out over any (open or closed) interval, thanks to the homogeneity and invariance of the Lebesgue measure. Related to this concept, we have the following.

Definition 3.3 (everywhere positive-measured set) A subset A of $\mathbb{R}$ is called everywhere positive-measured, if it intersects any nontrivial interval in a set of positive measure.

Notice that a set A has the splitting property (3.2) for the family of intervals of $\mathbb{R}$ if both A and $\mathbb{R}\setminus A$ are everywhere positive-measured. The following lemma asserts the existence of a countable partition of $\mathbb{R}$ into splitting sets.

Lemma 3.1 (countable splitting partition) There exists a countable partition $\{A_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$ each of which splits the family of intervals.

Proof. Let us first notice that it suffices to obtain a partition of $[0,1)$ with the above property, since we can translate those sets over every interval of the form $[m, m+1)$, $m \in \mathbb{Z}$. To this end, let $\{I_n\}_{n\in\mathbb{N}}$ be an enumeration of the subintervals of $(0,1)$ with rational end points, say $I_n = (a_n, b_n)$. We split I_1 into two open contiguous intervals, that is, we take $c \in (a_1, b_1)$ and consider the intervals (a_1, c) and (c, b_1). Then let $T_1^{(1)}$ and $B^{(1)}$ be two fat Cantor sets over (a_1, c) and (c, b_1) respectively. Since $T_1^{(1)} \cup B^{(1)}$ is nowhere dense, there exists $(a_2', b_2') \subseteq I_2$ such that

$$(a_2', b_2') \bigcap \left(T_1^{(1)} \cup B^{(1)}\right) = \emptyset.$$

We now proceed inductively as follows: Given $T_k^{(i)}$, $B^{(i)}$ for $1 \leq k \leq i \leq n-1$, since their union is a nowhere dense closed subset of $(0,1)$, there exists a subinterval (a_n', b_n') of I_n which is disjoint from this union. We now split the interval (a_n', b_n') into $n+1$ contiguous open intervals and define $T_k^{(n)}, B^{(n)}$ (where $k \in \{1, \ldots, n\}$) to be fat Cantor sets over each one of these intervals. In this way we obtain inductively disjoint fat Cantor subsets $T_k^{(n)}$, $B^{(n)}$ of $(0,1)$ where $1 \leq k \leq n$, and $n \in \mathbb{N}$. We then define

$$A_k = \bigcup_{n\geq k} T_k^{(n)} \quad ; \quad A_0 = [0,1) \setminus \left(\bigcup_{k\geq 1} T_k \right) \quad ; \quad B = \bigcup_{n\geq 1} B^{(n)}.$$

We claim that the family $\{A_k\}_{k\geq 0}$ is the partition of $[0,1)$ we are looking for. Indeed, the sets $\{A_k\}_{k\geq 1}$ are mutually disjoint: Let $1 \leq k < k'$ and assume towards a contradiction that $x \in A_k \cap A_{k'}$. Then, there exists $n \geq k$ and $n' \geq k'$ such that $x \in T_k^{(n)}$ and $x \in T_{k'}^{(n')}$, which is impossible by construction. Notice further that $B \subseteq A_0$ (the argument is the same as before) and that $A_0 \subseteq [0,1) \setminus A_k$, for every $k \geq 1$. Now, let $[a,b] \subseteq [0,1)$ be any interval. For $k \geq 1$, let $n \geq k$ such that $I_n \subseteq [a,b]$. It follows that

$$\lambda(A_k \cap [a,b]) \geq \lambda(A_k \cap I_n) \geq \lambda(T_k^{(n)} \cap I_n) = \lambda(T_k^{(n)}) > 0.$$

On the other hand

$$\begin{aligned} \lambda(A_0 \cap [a,b]) &\geq \lambda(B \cap [a,b]) \geq \lambda(B \cap I_n) \\ &\geq \lambda(B^{(n)} \cap I_n) = \lambda(B^{(n)}) > 0, \end{aligned}$$

yielding the result.

□

Let now $\mathcal{U} \subseteq \mathbb{R}$ be a nontrivial open interval and suppose without loss of generality that $0 \in \mathcal{U}$. Define the family of functions given by

$$g_k(x) = \mathbb{1}_{A_{2k+1}}(x) - \mathbb{1}_{A_{2k}}(x), \quad x \in \mathcal{U} \tag{3.3}$$

and set

$$f_k(x) = \int_0^x g_k(t)dt. \tag{3.4}$$

We list below some properties of the family $\mathcal{F} = \{f_k : k \in \mathbb{N}\}$ of functions defined by (3.4). In what follows, we denote by c_{00} the space of finitely supported sequences, that is, $\mu = (\mu_n)_{n\in\mathbb{N}}$ if and only if $\mathrm{supp}(\mu) := \{n \in \mathbb{N} : \mu_n \neq 0\}$ is finite.

i) $\mathcal{F} \subset \mathrm{Lip}_0(\mathcal{U})$. In particular, for every $k \in \mathbb{N}$, f_k is Lipschitz, with $\|f_k\|_L = 1$.

This is straightforward from the fact that the functions $g_k = f_k'$ belong to $L^\infty(\mathcal{U})$, with $\|g\|_\infty = 1$.

ii) The family $\mathcal{F}$ is linearly independent.

Let $\mu \in c_{00}$. Then

$$\sum_{k\in\mathbb{N}} \mu_k f_k = 0 \iff \int_0^x \left(\sum_{k\in\mathbb{N}} \mu_k g_k(t) \right) dt = 0, \quad \forall x \in \mathcal{U}.$$

In virtue of Rademacher theorem and Lebesgue differentiation theorem, the above equality yields that

$$\sum_{k\in\mathbb{N}} \mu_k g_k(x) = 0, \quad \text{almost everywhere on } \mathcal{U}.$$

Since $\{A_k\}_{k\in\mathbb{N}}$ are disjoint, everywhere positive-measured sets, we can for every $k \in \mathbb{N}$ choose $x_k \in A_{2k+1} \cap \mathcal{U}$, . Then $x_k \notin A_{2k}$, and in view of (3.3) we have $g_k(x_k) = 1$ and $g_k(x_{k'}) = 0$ for $k \neq k'$. From this, we deduce that for every $k \in \mathbb{N}$, $\mu_k = 0$. Therefore $\{f_k\}_{k\in\mathbb{N}}$ is a linearly independent family.

iii) The functions f_k are Clarke-saturated, for every $k \in \mathbb{N}$.

Since $f_k' = g_k$ almost everywhere on $\mathcal{U}$, it follows that f_k' takes each one of the values $\{-1, 0, 1\}$ over an everywhere positive-measured (and a fortiori in a dense) subset of $\mathcal{U}$. It follows by (3.1) that $\partial f_k^\circ(x) = [-1, 1] = \overline{B}_*(0, 1)$ for every $x \in \mathcal{U}$.

Considering this observations, we would like to use these functions somehow as a basis for the desired linear space. For this is necessary that the Clarke-saturation of these functions is preserved under linear combinations. Before giving details of this, is necessary to recall a property of the Clarke-subdifferential, which will show that this required property is far from being trivial.

Proposition 3.2 Suppose that $f, g : \mathcal{U} \subset \mathbb{R}^d \to \mathbb{R}$ are Lipschitz. Then,

$$\partial^\circ(f + g)(x) \subset \partial^\circ f(x) + \partial^\circ g(x) \quad , \quad \forall x \in \mathcal{U}.$$

From this last property, it is clear that Clarke-saturation is not trivially preserved. Indeed, if the Lipschitz constant of $f + g$ is equal to the sum of both Lipschitz constants, since the right will be simply the ball centered at 0 with radius equal to the sum of those constants whenever f, g are Clarke-saturated, then the inclusion is the same obtained by the properties of the Clarke subdifferential, but equality is not assured. Nevertheless, only sums of functions can bring this problem, as shown in the following proposition.

Proposition 3.3 Suppose that $f : \mathcal{U} \subset \mathbb{R}^d \to \mathbb{R}$ is Lipschitz an let $\lambda \in \mathbb{R}$. Then,

$$\partial^\circ(\lambda f)(x) = \lambda \partial^\circ f(x) \quad , \quad \forall x \in \mathcal{U}.$$

In particular, if f is Clarke-saturated, so is λf.

We proceed now to show that the property of Clarke-saturation is indeed inherited under linear combinations of the family $\mathcal{F}$.

Proposition 3.4 (Lineability in the one-dimensional case) Every linear combination of the functions $\{f_k\}_{k\in\mathbb{N}}$ has maximal Clarke subdifferential.

Proof. Let $\mu \in c_{00}$ and set $f = \sum_{k\in\mathbb{N}} \mu_k f_k$ (that is, a finite linear combination). Then it holds almost everywhere on $\mathcal{U}$

$$f'(x) = \sum_{k\in\mathbb{N}} \mu_k\, f_k'(x) = \sum_{k\in\mathbf{supp}(\mu)} \mu_k\, g_k(x).$$

Notice that for a given $x \in \mathcal{U}$ there exists at most one $k \in \mathbb{N}$ such that $g_k(x) \neq 0$ (namely, $g_k(x) = 1$ or -1). Therefore f' can only take the values $\{\pm\mu_k\}_{k\in\mathbb{N}}$ and 0. Using the same argument as before, we deduce that each of these values is taken on a dense subset of $\mathcal{U}$. Therefore

$$\partial^\circ \left(\sum_{k\in\mathbb{N}} \mu_k f_k \right)(x) = \|\mu\|_\infty[-1,1] = \overline{B}_*(0, \|\mu\|_\infty), \quad \text{for every } x \in \mathcal{U}.$$

Moreover,

$$\|f\|_L = \left\| \sum_{k\in\mathbb{N}} \mu_k f_k \right\|_L = \left\| \sum_{k\in\mathbb{N}} \mu_k g_k \right\|_\infty = \|\mu\|_\infty.$$

We conclude that this linear combination has maximal Clarke subdifferential everywhere, that is, it is Clarke-saturated. $\square$

The above results provide an efficient way to find a linear structure inside the set of Clarke-saturated functions. Moreover, in the previous proof we have also dealt with a metric structure inside this set, which will be useful in the proof of spaceability. Indeed, we have implicitly shown that for the constructed family of functions, the Lipschitz norm of any linear combination coincides with the supremum norm of the coeficients defining that linear combination. Taking this into account, we can construct an explicit isometry between this class of Clarke-saturated functions and the space c_{00}. We give details of this after generalizing this method for higher dimensions.

3.2.2 The higher dimensional case

We now proceed to study the general case of higher dimension. As was mentioned before, a special care is required in order to make an analog construction. The details and consequences for this construction will be developed below. For technical reasons which will be clear in the construction, we work over the space ℓ_1^d, that is, the d-dimensional space $\mathbb{R}^d$ equipped with the $\|\cdot\|_1$-norm. Then, the dual space is ℓ_∞^d, that is, the d-dimensional space $\mathbb{R}^d$ equipped with the $\|\cdot\|_\infty$-norm. This allows us to extend the aforementioned construction easily in order to obtain Clarke-saturation. We do not know whether or not this result remains true under a different choice of the norm.

Let $\mathcal{U} \subseteq \ell_1^d$ be a non-empty open convex set and let D stand for the isometry in Theorem 1.1. For $k \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathcal{U}$ we define the function $G^k : \mathcal{U} \to \ell_\infty^d$ as

$$G^k(x) := (g_k(x_1), \ldots, g_k(x_d)) = (\mathbb{1}_{A_{2k+1}}(x_1) - \mathbb{1}_{A_{2k}}(x_1), \ldots, \mathbb{1}_{A_{2k+1}}(x_d) - \mathbb{1}_{A_{2k}}(x_d)). \quad (3.5)$$

In other words,

$$\langle G^k(x), e_i \rangle = g_k(\langle x, e_i \rangle),$$

where the functions g_k are given by (3.3) and $\{e_i\}_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$.

Let us first show that the functions $\{G^k\}_{k\in\mathbb{N}}$ are "derivatives" of functions of $\mathrm{Lip}_0(\mathcal{U})$ in the appropiate sense. This part relies on Proposition 1.3.

Proposition 3.5 (G^k are derivatives) For every $k \in \mathbb{N}$, $G^k \in D(\mathrm{Lip}_0(\mathcal{U}))$.

PROOF. Let $i, j \in \{1, \ldots, d\}$ with $i \neq j$ and $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$. Then

$$\int_{\mathcal{U}} \partial_j G_i^k \varphi d\lambda = -\int_{\mathcal{U}} G_i^k \frac{\partial \varphi}{\partial x_j} d\lambda = -\int_{\mathcal{U}} g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) dx.$$

Since $\varphi \in \mathcal{C}_0^\infty(U)$, thanks to Fubini theorem we can first integrate over the variable x_j and deduce that the above integral is equal to 0. Therefore, $\partial_i G_j^k = 0$ whenever $i \neq j$. In particular, $\partial_i G_j^k = \partial_j G_i^k$ in the sense of distributions, and according to Theorem 1.1 we deduce that $G^k \in D(\mathrm{Lip}_0(\mathcal{U}))$. □

In view of the above proposition, we can define the family

$$\mathcal{F} = \{f_k\}_{k \geq 0} \subset \mathrm{Lip}_0(\mathcal{U})$$

as the inverse images of the family $\{G^k\}_{k \geq 0}$, that is,

$$f_k := D^{-1}(G^k), \quad \text{for every } k \in \mathbb{N}. \tag{3.6}$$

Remark From this last definition, is not difficult to see that

$$f_k(x) = \sum_{i=1}^{d} h_k(x_i),$$

where the functions h_k are the ones used in the one-dimensional case, that is, the one that verifies $h_k' = g_k$ over the appropiate domain.

We now verify the same properties as in the previous section for the above functions.

i) $\mathcal{F} \subset \mathrm{Lip}_0(\mathcal{U})$. In particular, $\|f_k\|_L = 1$.

Notice that the values of G^k are vectors $v \in \mathbb{R}^d$ whose components take the values $\{-1, 0, 1\}$, each of them over everywhere positive-measured sets. Therefore $\|G^k\|_\infty = 1$ and the result follows from the fact that D is an isometry.

ii) The family $\mathcal{F}$ is linearly independent.

It suffices to prove that the family $\{G^k\}_{k \geq 0}$ is linearly independent, since D is an isometry. Let $\mu \in c_{00}$ (compactly supported sequence) and assume

$$\sum_{k \in \mathbb{N}} \mu_k G^k = 0, \quad \text{that is,} \quad \sum_{k \in \mathbb{N}} \mu_k G^k = 0 \text{ a.e. on } \mathcal{U}.$$

For every $k \geq 0$ let $x^k \in (A_k \times \ldots \times A_k) \cap \mathcal{U}$. Given the definition of the functions G^k, we have that for $i \in \{1, \ldots, d\}$

$$\left(\sum_{k\in\mathbb{N}} \mu_k G^k(x^k)\right)_i = \begin{cases} \mu_{2n+1}, & \text{if } k = 2n+1 \\ -\mu_{2n}, & \text{if } k = 2n. \end{cases}$$

Since $(A_k \times \ldots \times A_k) \cap \mathcal{U}$ has positive measure everywhere, we conclude that $\mu = 0$. Therefore $\{G^k\}_{k\geq 0}$ is linearly independent and the assertion follows.

iii) The functions f_k are Clarke-saturated.

Notice that every extreme point of the unit ball of ℓ_∞^d is attained as a value of G^k on a subset of $\mathcal{U}$ which has positive measure everywhere. Since $Df_k = G^k$, G^k is equal to ∇f almost everywhere on $\mathcal{U}$. We conclude that $\partial^\circ f_k(x) = \overline{B}_*$, for all $x \in \mathcal{U}$.

After stating these facts, it becomes clear why the host space for the domain has been chosen to be ℓ_1^d, since its dual is ℓ_∞^d and it was crucial to obtain the Clarke-saturation via the use of the extreme points of the dual ball. Similarly to the one-dimensional case we now establish that Clarke-saturation is preserved under linear combinations of elements of $\mathcal{F}$.

Proposition 3.6 (Lineability) Every linear combination of the functions $(f_k)_{k\in\mathbb{N}}$ has maximal Clarke subdifferential.

Proof. Let $\mu \in c_{00}$. Then we have

$$\nabla\left(\sum_{k\in\mathbb{N}} \mu_k f_k\right)(x) = \sum_{k\in\mathbb{N}} \mu_k G^k(x), \quad \text{for a.e. } x \in \mathcal{U}.$$

The values of this last function are exclusively vectors $v \in \mathbb{R}^d$ with components in the set $\{\pm\mu_k \,:\, k \geq 0\}$. Moreover, each component takes each one of the values $\{\pm\mu_k\}_{k\in\mathbb{N}}$ over subsets of $\mathcal{U}$ which have everywhere positive measure. It follows readily from (3.1) that for every $x \in \mathcal{U}$

$$\partial^\circ\left(\sum_{k\in\mathbb{N}} \mu_k f_k\right)(x) = \|\mu\|_\infty \overline{B}_*.$$

In addition, using the isometry D we deduce that

$$\|f\|_L = \left\|\sum_{k\in\mathbb{N}} \mu_k f_k\right\|_L = \left\|\sum_{k\in\mathbb{N}} \mu_k G^k\right\|_\infty = \|\mu\|_\infty,$$

which completes the proof. □

An important consequence which is implicit in the proof of the aforementioned facts is that, similarly to the one-dimensional case, there is a inherent metric asociated. More precisely, the Lipschitz norm of any linear combination of the family $\mathcal{F}$ coincides with the supremum norm of the coeficients defining that linear combination, just as before. Having in mind that this property is present in all previous cases, we are in condition to state the main result of this chapter.

3.2.3 The space of Clarke-saturated functions

In the previous sections we constructed for every $d \geq 1$ a countable family of linearly independent Clarke-saturated functions f_k belonging to $\mathrm{Lip}_{x_0}(\mathcal{U})$, where $\mathcal{U} \subseteq \ell_1^d$ is a nonempty open convex set and $x_0 \in \mathcal{U}$. We shall now describe in terms of the isometry D (Theorem 1.1) the closure of the space generated by these functions. In what follows we denote by $\ell^\infty(\mathbb{N})$ the (nonseparable) Banach space of bounded sequences.

Proposition 3.7 Let $T : \ell^\infty(\mathbb{N}) \to L^\infty(\mathcal{U}; \ell_\infty^d)$ given by

$$T\mu = \sum_{k \geq 0} \mu_k G^k, \quad \text{for all } \mu = (\mu_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}).$$

Then T is well defined and establishes a linear isometric injection of $\ell^\infty(\mathbb{N})$ into $L^\infty(\mathcal{U}; \ell_\infty^d)$.

Proof. Let $\{A_k\}_{k\in\mathbb{N}}$ be the countable partition of $\mathbb{R}$ given by Lemma 3.1. Let $x \in \mathcal{U}$. Since each A_k is everywhere positive measured, there exists $j_1, \ldots, j_d \geq 0$ such that $x_i \in A_{j_i}$, for $i \in \{1, \ldots, d\}$. This implies that the sum

$$\sum_{k\geq 0} \mu_k G^k(x)$$

is finite for every $x \in \mathcal{U}$, with norm less than or equal to $\|\mu\|_\infty$. Therefore $T\mu \in L^\infty(\mathcal{U}; \ell_\infty^d)$, with $\|T\mu\|_\infty \leq \|\mu\|_\infty$. Moreover, if $x \in (A_{2n+1} \times \ldots \times A_{2n+1})$ and $x' \in (A_{2n} \ldots \times A_{2n})$ then

$$T\mu(x) = -T\mu(x') = (\mu_k, \ldots, \mu_k),$$

which leads to $\|T\mu\|_\infty = \|\mu\|_\infty$. Since T is obviously linear, it follows that T is a linear isometry between $\ell^\infty(\mathbb{N})$ and $T(\ell^\infty(\mathbb{N}))$. $\square$

Before proceeding with the next results that will lead to the final theorem, notice that the previous proposition implies in particular the results from the previous sections. Indeed, it suffices to consider the restriction of T to the subspace of $\ell^\infty(\mathbb{N})$ given by the finitely supported sequences. Considering that, the previous proposition can be understood broadly as taking infinite linear combinations in an appropiate way. Now, we state the relation between $T(\ell^\infty(\mathbb{N}))$ and $D(\mathrm{Lip}_{x_0}(\mathcal{U}))$. This relation is obtained in a similar way as in the case of linear combinations studied in the previous sections.

Proposition 3.8 $T(\ell^\infty(\mathbb{N})) \subseteq D(\mathrm{Lip}_{x_0}(\mathcal{U}))$.

Proof. Let $\mu \in \ell^\infty(\mathbb{N})$. We need to prove that $T\mu$ is the gradient of some Lipschitz function. Let $i, j \in \{1, \ldots, d\}$ with $i \neq j$. Then

$$(T\mu)_i(x) = \sum_{k\geq 0} \mu_k g_k(x_i).$$

If $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$, we have that

$$\langle \partial_j (T\mu)_i, \varphi \rangle = -\int_{\mathcal{U}} \left(\sum_{k\geq 0} \mu_k g_k(x_i) \right) \frac{\partial \varphi}{\partial x_j}(x) dx = -\int_{\mathcal{U}} \sum_{k\geq 0} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right) dx.$$

We define for $n \geq 0$

$$\psi_n(x) = \sum_{k=0}^{n} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right) \quad \text{and} \quad \psi(x) = \sum_{k\geq 0} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right).$$

Notice that for $x = (x_1, \ldots, x_d) \in \mathcal{U}$ and $i \in \{1, \ldots, d\}$ we have

$$g_k(x_i) \neq 0 \iff x_i \in A_{2k+1} \cup A_{2k}$$

and in this case $g_{k'}(x_i) = 0$, for all $k' \neq k$. Therefore, there exists some $N \geq 0$ large enough such that

$$\psi_n(x) = \sum_{k=0}^{n} \mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) = \begin{cases} 0, & n < N \\ \mu_N \; g_N(x_i) \; \frac{\partial \varphi}{\partial x_j}(x), & n \geq N \end{cases}$$

yielding

$$\psi_n \to \psi \;\; \text{(pointwise)} \quad \text{and} \quad |\psi_n| \leq ||\mu||_\infty \left| \frac{\partial \varphi}{\partial x_j} \right| \in L^1(\mathcal{U}).$$

In virtue of the Lebesgue dominated convergence theorem, we have that

$$\langle \partial_j (T\mu)_i, \varphi \rangle = -\sum_{k\geq 0} \left(\int_{\mathcal{U}} \mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) dx \right).$$

But thanks to Fubini theorem, we can integrate first with respect to the x_j variable and since φ has compact support, we conclude that all the integrals are equal to 0. Then $\partial_j (T\mu)_i = 0$ whenever $i \neq j$, which leads to $T\mu \in D(\mathrm{Lip}_{x_0}(\mathcal{U}))$. The proof is complete. □

As the last proposition shows, even passing through infinite linear combinations of the functions G^k we obtain gradients of Lipschitz functions in the sense studied in Chapter 1, that is, via the isometry D. Just as in the case of finite linear combinations studied in the previous sections, it only suffices now to prove that the Lipschitz functions obtained as the inverse images through D of these gradients still preserve the property of Clarke saturation. This is stated in the following proposition.

Proposition 3.9 Let $f \in \mathrm{Lip}_{x_0}(\mathcal{U})$ be such that $Df = T\mu$. Then f is Clarke-saturated.

Proof. It suffices to notice that

$$\|f\|_L = \|Df\|_\infty = \|T\mu\|_\infty = \|\mu\|_\infty$$

and that for every extreme point v of the dual ball $\overline{B}_*$ and $k \geq 0$ there exists an everywhere positive-measured set $A \subset \mathcal{U}$ such that

$$Df(x) = T\mu(x) = \mu_k v \quad \text{for every } x \in A.$$

Since f is differentiable almost everywhere, we conclude that

$$\partial^\circ f(x) = \|\mu\|_\infty \overline{B}_* = \overline{B}_*(0, \|f\|_L),$$

which finishes the proof. □

Using all the previous tools, we are now ready to state the main result of the present chapter. However, it is important to summarize everything to this point. Recall that until now we are supposing that $\mathcal{U} \subset \ell^d_\infty$. This is important in the construction of the functions, since in order to obtain the operator T in a constructive way, it was important the use of the extreme points of the unit ball of ℓ^d_∞. With this in mind, we state now the main result of the chapter, as well as a corollary involving a related result in the case that the host space for the domain of the functions is endowed with a different norm.

Theorem 3.1 (Spaceability of Clarke-saturated functions) *Let $d \geq 1$ and $\mathcal{U} \subseteq \ell^1_d$ be a nonempty open convex set. Then,*

i) (lineability) *The space* $\mathrm{Lip}(\mathcal{U})$ *of Lipschitz functions contains a linear subspace of Clarke-saturated functions of uncountable dimension.*

ii) (spaceability) *For any $x_0 \in \mathcal{U}$, the Banach space* $\mathrm{Lip}_{x_0}(\mathcal{U}), \|\cdot\|_L)$ *contains a (proper) linear subspace of Clarke-saturated functions isometric to $\ell^\infty(\mathbb{N})$.*

In particular, if $\mathcal{F} = \{f_k : k \in \mathbb{N}\}$ is the family defined in (3.6), then $\mathrm{span}\{f_k\}$ *is isometric to c_{00}, while* $\overline{\mathrm{span}}\{f_k\}$ *is isometric to $c_0(\mathbb{N})$ (the Banach space of null sequences).*

Proof. Thanks to Propositions 3.7 and 3.8, we deduce that $\ell^\infty(\mathbb{N})$ is isometric to the subspace

$$Z = D^{-1}(T(\ell^\infty(\mathbb{N})))$$

of $\mathrm{Lip}_{x_0}(\mathcal{U})$. This subspace is proper, since any strictly differentiable not null function h belonging to $\mathrm{Lip}_{x_0}(\mathcal{U})$ does not belong to Z. This proves $ii)$, and yields directly that Clarke-saturated functions contain a linear subspace of uncountable dimension. Therefore $i)$ holds, since $\mathrm{Lip}_{x_0}(\mathcal{U})$ is a linear subspace of $\mathrm{Lip}(\mathcal{U})$. Finally, an easy computation shows that if $\mu \in c_{00}$, then $D^{-1}T\mu \in \mathrm{span}\{f_k\}$, whence c_{00} is isometric to $\mathrm{span}\{f_k\}$. It follows readily by the continuity of the operators that $c_0(\mathbb{N})$ is isometric to $\overline{\mathrm{span}}\{f_k\}$. □

Before finishing this chapter, we give the following straightforward consequences of Theorem 3.1.

Corollary 3.1 *Let $p \in \mathbb{R}^d$ and $r > 0$. Then, there exists $f \in \mathrm{Lip}(\mathcal{U})$ such that $\partial^\circ f(x) = \overline{B}_*(p, r)$ for every $x \in \mathcal{U}$.*

Proof. Let $\mu \in \ell^\infty(\mathbb{N})$ be such that $\|\mu\|_\infty = r$. Set $h_1 = D^{-1}T\mu$ and $h_2 = \langle p, \cdot\rangle$. Then $\partial^\circ h_1(x) = \overline{B}_*(0, r)$ and $\partial^\circ h_2(x) = \{p\}$ for every $x \in \mathcal{U}$, where we used that h_2 is strictly differentiable. Again thanks to that, if $f = h_1 + h_2$ then for every $x \in \mathcal{U}$

$$\partial^\circ f(x) = \partial^\circ(h_1 + h_2)(x) = \partial^\circ h_1(x) + \partial^\circ h_2(x) = \overline{B}_*(p, r).$$

The proof is complete. $\square$

Corollary 3.2 *Suppose that $\mathcal{U} \subset \mathbb{R}^d$ endowed with any norm. Then the following property is lineable in* $\mathrm{Lip}(\mathcal{U})$ *and spaceable in* $(\mathrm{Lip}_{x_0}(\mathcal{U}), \|\cdot\|_L)$*:*

$$(\exists \delta > 0)\,(\forall x \in \mathcal{U}) \quad \overline{B}_*(0,\delta) \subset \partial^\circ f(x)$$

Proof. Recall that every pair of norms over $\mathbb{R}^d$ are equivalent and that the gradients of functions are independent of the chosen norm. Hence, δ arises from the equivalence for the dual norm of the chosen norm over $\mathbb{R}^d$ with the norm of ℓ_∞^d. $\square$

As indicated at the beginning of the present chapter, it is now clear that the pathological situation described in Definition 3.1 is actually present over an algebraically large subset of Liscphitz continuous functions. In this sense, it is important to have in mind that not only there is a subspace of uncountable dimension contained in the set of Clarke-saturated functions, but in addition, this subspace is also non-separable, which reveals even more clearly that in this case, the subdifferential does not give any information on the function, unless we have additional assumptions on the function. As mentioned at the beginning, it is straightforward to see that adding convexity to the function, the presence of the zero subgradient in the subdifferential, that is, $0 \in \partial f(x)$ for all $x \in \mathcal{U}$, leads directly to the global minimum of the function. Having said all the above, it should be noted that in general, for concrete subclasses of interest, the Clarke subdifferential is not necessarily pathological as it has been called throughout this chapter, but it gives some insight on how rich the space of Lipschitz functions can be, even when defined over $\mathbb{R}$.

It is important to have in mind that throughout this chapter, in order to establish our results it was necessary to work with the 1-norm in the domain. This does not necessarily mean that an analogous statement would be false in full generality, but rather reveals the limitations of the current technique. Therefore, it seems that another approach should be taken, or at least modify our construction in an appropiate way.

A first idea is to deal with the case of polyhedral norms over $\mathbb{R}^d$. Recall that a norm on $\mathbb{R}^d$ is called polyhedral if the closed unit ball for this norm is a polytope. It is not difficult to see that in this case the dual norm is also polyhedral. We think of these norms as a natural starting point to further develop our results since our constructions strongly relied in the extreme points of the unit ball of $\mathbb{R}^d$ endowed with the 1-norm, which is clearly polyhedral.

A question on another direction is that of the case where the domain is not of finite dimension. More precisely, if there are a set of properties for a Banach space X in order to obtain lineability for the set of Lipschitz functions with maximal Clarke subdifferential. It is worth noticing that in order to analyse this case it is necessary to take a completely different approach, since we no longer have the characterization for the Clarke subdifferential which we use in the finite-dimensional case, which is also the case for the results of Chapter 1.

Part II

Classification of asymmetric normed spaces

Chapter 4

Index of symmetry for asymmetric normed spaces

In this last chapter we focus on the study of asymmetric normed spaces. Most of the definitions given in this chapter can be found in [25].

It is difficult to track down the first moment when the concept of asymmetry has been introduced, but in 1968 on a paper by R. J. Duffin and L. A. Karlovitz [34] it was already proposed. More recently, especially in papers coming from the Polytechnic University of Valencia and others Spanish universities, there has been a systematic study of this concept. The importance of this subject relies not only on its intrinsic interest, but also on its applications on Computer Science [63] and the Markov moment problem [51].

The concept of asymmetric normed spaces arises naturally when considering non-reversible Finsler manifolds ([23, 31, 58]) and meet applications in Physics ([47]) as well as in Game Theory ([2, 41]). As mentioned before, these spaces have been studied by several authors, emphasizing on its main similarities as well as differences with the classical framework of normed spaces. To clearify the state-of-the-art, we refer to [25], where the classic results on Functional Analysis are studied, giving its counterparts in the asymmetric frame.

It is worth noticing that the addition of asymmetry is not exclusive for the study over linear spaces. An associated concept is that of quasi-metric space, which in a few words gets rid of the symmetry assuption for the distance. We refer to the recent works [31] and [32] where quasi-metric spaces are used together with come to be the natural morphisms between those spaces, called semi-Lipschitz functions. Notice that we follow a similar approach in our work: Studying linear operators in the framework of asymmetric normed spaces. Finally, semi-Lipschitz functions are used for the definition of semi-Lipschitz free spaces, which are the analog for Lipschitz-free spaces when asymmetry is taken into account.

Our goal is to focus on finding a way to classify these spaces, since it is clear in the literature that there are at least two types of asymmetric normed spaces: Those that behave just as a normed space and the limit case, where the asymmetry is big enough to change its behaviour.

More details of this are given in the main results of this chapter.

An asymmetric normed space is a real vector space X equipped with a positive, subadditive and positively homogeneous function $\|\cdot|_X$ satisfying

$$\|x|_X = \|-x|_X = 0 \Longleftrightarrow x = 0.$$

A more general concept related to asymmetric normed spaces are quasi-metric spaces. We say that a set M endowed with a mapping $\rho : M \times M \to [0, \infty)$ is a quasi-metric space if $\rho(x, x) = 0$ for every $x \in M$ and ρ satisfies the triangle inequality, that is

$$\rho(x, y) \leq \rho(x, z) + \rho(z, x) \quad , \quad \forall x, y, z \in M.$$

In this case, we say that ρ is a quasi-metric. In brief, we require from ρ all properties of a usual metric except from the symmetry. It follows readily from the above definitions that every asymmetric normed space is trivially asociated to a quasi-metric space, where $\rho(x, y) = \|x - y|_X$, for every $x, y \in X$.

The main difference between a classical norm and an asymmetric norm, is that the equality $\|-x|_X = \|x|_X$ does not always hold. In the literature (see [4, 5] and [61]) asymmetric norms are also called a quasi-norms. Every asymmetric normed space $(X, \|\cdot|_X)$ can be associated to a normed space $X_s := (X, \|\cdot\|_s)$ with the norm defined by

$$\|x\|_s := \max\{\|x|_X, \|-x|_X\}.$$

It is not hard to see that any real vector space can be endowed with an asymmetric norm. Such a functional can be obtained as the Minkowski gauge functional of an absorbing convex subset of the space, dropping the usual symmetry condition over this subset, which in turn implies the lack of symmetry for the functional.

During this chapter we will focus on the degree of symmetry of these spaces, defining the so-called index of symmetry. As we will see, this index does not simply quantifies that property, but also allows to state a classification of these spaces and, in particular, the properties for the dual space that arise directly from that value.

In order to do this, we need to state in this framework the concept of dual space, for which we refer to [25]. The dual $X^\flat$ of an asymmetric normed space X is formed by all linear continuous functionals from $(X, \|\cdot|_X)$ to $(\mathbb{R}, \|\cdot|_\mathbb{R})$, or equivalently, by all linear upper semicontinuous functionals from $(X, \|\cdot|_X)$ to $(\mathbb{R}, |\cdot|)$. In contrast to the usual case, the dual $X^\flat$ and the set of continuous linear operators $L_c(X, Y)$ are not necessarily linear spaces, but merely convex cones contained respectively in X^* and $L(X_s, Y_s)$. But as far as we know, there is no characterization in the literature for the asymmetric normed spaces X for which $L_c(X, Y)$ is also an asymmetric normed space.

It is clear that it is simply the lack of symmetry of the unit ball which gives this bizarre behaviour of the dual space. With this in mind, for an asymmetric normed space $(X, \|\cdot|)$, the index of symmetry of X is defined as

$$c(X) := \inf_{\|x|=1} \|-x| \in [0, 1],$$

which quantifies the amount of deviation of the unit ball from a perfect ball, meaning a ball obtained from a normed space.

The aim of this chapter is to prove that the case $c(X) = 0$ is exactly the situation where the convex cone $L_c(X, Y)$ (in particular the dual $X^\flat$) has no linear structure, for every asymmetric normed space Y. As a consequence, from a topological point of view, the case where $c(X) = 0$ turns out to be the only interesting case in the theory of asymmetric normed spaces. Indeed, we prove in Corollary 4.3 that the following are equivalent

i) $X^\flat$ is a vector space

ii) $(X, \|\cdot|_X)$ is isomorphic to its associated normed space

iii) $L_c(X, Y)$ is a vector space isomorphic to $L(X_s, Y_s)$, for every asymmetric normed space Y

iv) $c(X) > 0$

These equivalences indicate that the case where $c(X) > 0$ refers to the classical framework of normed spaces. The most challenging implication is $i) \Longrightarrow ii)$, which uses the Baire category theorem. These statements are consequences of the first main result, Theorem 4.1. The second main result (Theorem 4.2) shows that an asymmetric normed space X is a T_1 space (that is, every singleton is closed) if and only if its dual $X^\flat$ is w^*-dense in (X^*, w^*).

Considering the preceding statements, it is clear that this study naturally leads to a topological classification for asymmetric normed spaces. More precisely, recall that a topological space X is said to be T_1 if every pair $x, y \in X$ can be separated by an open set, that is, for every $x, y \in X$ there exists an open set containing x but not containing y. With this in mind, the aforementioned classification goes as follows.

Definition 4.1 Let X be an asymmetric normed space. We say that

i) X is of type I if $c(X) > 0$ (necessarily a T_1 space).

ii) X is of type II if $c(X) = 0$ and X is a T_1 space.

iii) X is of type III if X is not a T_1 space (necessarily $c(X) = 0$).

As it will be clear in the discusion, from a topological point of view asymmetric normed spaces of type I present no new interest compared to the classical theory of normed spaces, since these spaces are isomorphic to their associated normed spaces and the same holds for their duals (see Corollary 4.3). Moreover, the class of finite dimensional T_1 asymmetric normed spaces is contained in the class of type I (see Theorem 4.3). Spaces of type II and III are the interesting cases, since they differ from the framework of classical normed spaces. Spaces of type III include finite dimensional spaces (consider for example the space $(\mathbb{R}, \|\cdot|_{\mathbb{R}})$ defined in the Introduction of the present work), while spaces of type II only include infinite dimensional spaces (see Proposition 4.4).

In order to fix the ideas behind these definitions, we show some examples of these spaces.

Example *Finite dimensional space of type III:* Let $X = \mathbb{R}$ and $\|t|_{\mathbb{R}} := \max\{0, t\}$ for all $t \in \mathbb{R}$. Then, $(\mathbb{R}, \|\cdot|_{\mathbb{R}})$ is an asymmetric normed space for which $c(X) = 0$ and $(X^\flat, \|\cdot|_\flat)$ is not a vector space.

Example *Infinite dimensional space of type III:* Let $X = C_0[-1, 1]$ the space of all continuous functions from $[-1, 1]$ to $\mathbb{R}$ such that $f(0) = 0$. We define on X the following asymmetric norm

$$\|f| := \sup_{x \in [-1,1]} f(x) \leq \|f\|_\infty = \max\{\|f|, \|-f|\}.$$

It is easy to see that $c(X) = 0$. Let us denote by $\delta_x : X \to \mathbb{R}$ the evaluation map associated to $x \in [-1, 1]$ defined by $\delta_x(f) = f(x)$ for all $f \in X$. Clearly, $\delta_x \in X^\flat$ and $\|\delta_x|_\flat = 1$ for all $x \in [-1, 1]$. However, $-\delta_x \notin X^\flat$, for all $x \in [-1, 1] \setminus \{0\}$. It follows that $X^\flat$ is not a vector space.

Example *Space of type II:* Let $X = l^\infty(\mathbb{N}^*)$ equipped with the asymmetric norm $\|\cdot|_\infty$ defined by

$$\|x|_\infty = \sup_{n \in \mathbb{N}^*} \|x_n|_{\frac{1}{n}} \leq \|x\|_\infty,$$

where for each $t \in \mathbb{R}$ and each $n \in \mathbb{N}^*$, $\|t|_{\frac{1}{n}} = t$ if $t \geq 0$ and $\|t|_{\frac{1}{n}} = -\frac{t}{n}$ if $t \leq 0$.

Then, clearly $\hat{S}_X = S_X$ since $\|x|_\infty = 0 \iff \|-x|_\infty = 0 \iff x = 0$. Thus, $(X, \|x|_\infty)$ is a T_1 asymmetric normed space. On the other hand, for each $n \in \mathbb{N}^*$, we have $\|e_n|_\infty = 1$ and $\|-e_n|_\infty = \frac{1}{n}$, where (e_n) is the canonical basis of $c_0(\mathbb{N}^*)$. It follows that $c(l^\infty(\mathbb{N}^*)) = 0$ and $(l^\infty(\mathbb{N}^*))^\flat$ is not a vector space (see Theorem 4.1).

Example *Space of type I:* Let $(X, \|\cdot|)$ be an asymmetric normed space. Define a new asymmetric norm on X as follows: $\|x|_1 = \|x| + \|x\|_s$, where $\|x\|_s = \max\{\|x|, \|-x|\}$, for all $x \in X$. Then, the index of symmetry $c(X, \|\cdot|_1)$ of X for the asymmetric norm $\|\cdot|_1$, satisfies $0 < c(X, \|\cdot|_1) < 1$. First, we see that $c(X, \|\cdot|_1) < 1$ since $\|\cdot|_1$ is not a norm. Suppose that $c(X, \|\cdot|_1) = 0$, there exists $(x_n) \subset X$ such that $\|x_n| + \|x_n\|_s = 1$ for all $n \in \mathbb{N}$ and $\|-x_n| + \|-x_n\|_s \to 0$. This implies that $\|x_n\|_s = \|-x_n\|_s \to 0$. Since $\|x_n| \leq \|x_n\|_s$, it follows that $\|x_n| + \|x_n\|_s \to 0$, which is a contradiction. Recall that, in every asymmetric normed space, the condition $c(X) > 0$ implies that $\hat{S}_X = S_X$ (see Proposition 4.2). The dual of $(X, \|\cdot|_1)$ is a vector space by Corollary 4.3.

In brief, the interest of asymmetric normed space theory (from a topological point of view) concerns only the following cases:

i) Infinite dimensional spaces which are T_1 with $c(X) = 0$ (spaces of type II).

ii) Finite and infinite dimensional spaces X which are not T_1 (spaces of type III, where necessarily $c(X) = 0$).

Types II and III (corresponding to the case of $c(X) = 0$) are exactly the situations where the dual $X^\flat$ is not a vector space. Moreover, an asymmetric normed space X of type I will be always isomorphic to its associated normed space and $L_c(X,Y) \simeq L(X_s, Y_s)$ for every asymmetric normed space Y. Examples illustrating the three types of spaces will be given at the end of Section 4.3.3.

The rest of the present chapter is organized as follows. In Section 4.1 we recall definitions and notation from the literature. In Section 4.2, we give some basic properties of the index of symmetry. In Section 4.3, we state and prove our main results (Theorem 4.1, Corollary 4.3 and Theorem 4.2) and some consequences. Finally, in Section 4.3.3, we give the proofs of Theorem 4.3 and Proposition 4.4 which justify (making use of the already defined index of symmetry) the classification given above.

4.1 Definitions and notation

In this section, we recall known properties of asymmetric normed spaces that are going to be used in the sequel.

Definition 4.2 Let X be a real linear space. We say that $\|\cdot| : X \to \mathbb{R}^+$ is an asymmetric norm on X if the following properties hold.

i) For every $\lambda \geq 0$ and every $x \in X$, $\|\lambda x| = \lambda \|x|$.

ii) For every $x, y \in X$, $\|x + y| \leq \|x| + \|y|$.

iii) For every $x \in X$, if $\|x| = \|-x| = 0$ then $x = 0$.

Let $(X, \|\cdot|_X)$ and $(Y, \|\cdot|_Y)$ be two asymmetric normed spaces. A linear operator between these spaces $T : (X, \|\cdot|_X) \to (Y, \|\cdot|_Y)$ is said to be bounded if there exists $C \geq 0$ such that

$$\|T(x)|_Y \leq C\|x|_X, \;\; \forall x \in X.$$

In this case, we denote $\|T|_{L_c} := \sup_{\|x|_X \leq 1} \|T(x)|_Y$. It is known (see [25, Proposition 3.1]) that a linear operator T is bounded if and only if it is continuous, which in turn is equivalent to being continuous at 0. Also, we know from [25, Proposition 3.6] that the constant $\|T|_{L_c}$ can be calculated also by the formula

$$\|T|_{L_c} = \sup_{\|x|_X = 1} \|T(x)|_Y.$$

It is worth noticing that continuity in this framework must be treated carefully, since the asymmetry of the norm (hence, the asymmetry of the balls) can make some operators continuous, even if they are not continuous on a classical normed sense.

We can see that $L_c(X,Y)$ is a convex cone included in $L(X_s, Y_s)$ (see [25, Proposition 3.3]) but is not a vector space in general. Note that for each $T \in L_c(X,Y)$ we have that

$$\|T\|_{L_s} \leq \|T|_{L_c},$$

where $\|\cdot\|_{L_s}$, denotes the usual norm of $L(X_s, Y_s)$, where X_s and Y_s stand for the symmetrization of the spaces X and Y.. The function $\|\cdot|_{L_c}$ defines an asymmetric norm on $L_c(X,Y)\cap(-L_c(X,Y))$. Recall that in the case where $Y = (\mathbb{R}, \|\cdot|_{\mathbb{R}})$, where $\|t|_{\mathbb{R}} = \max\{0,t\}$ for all $t \in \mathbb{R}$, we denote $X^\flat := L_c(X, \mathbb{R})$, called the dual of the asymmetric normed space X. The topological dual of the associated normed space $X_s := (X, \|\cdot\|_s)$ of X is denoted X^* and is equipped with the usual dual norm denoted $\|p\|_* = \sup_{\|x\|_s\leq 1}\langle p, x\rangle$, for all $p \in X^*$. From [25, Theorem 2.2.2] (which is a straightforward consequence of Hahn-Banach theorem) we have that the convex cone $X^\flat$ is not trivial, that is, $X^\flat \neq \{0\}$ whenever $X \neq \{0\}$. Moreover, it follows directly from the definition that

$$X^\flat \subset X^* \text{ and } \|p\|_* \leq \|p|_\flat, \text{ for all } p \in X^\flat.$$

We say that $(X, \|\cdot|_X)$ and $(Y, \|\cdot|_Y)$ are isomorphic, denoted by $(X, \|\cdot|_X) \simeq (Y, \|\cdot|_Y)$, if there exists a bijective linear operator $T : (X, \|\cdot|_X) \to (Y, \|\cdot|_Y)$ such that T and T^{-1} are bounded.

4.2 Index of asymmetric normed space

In this section, we begin the study of the already defined index of symmetry for asymmetric normed spaces. Recall that for an asymmetric normed space X, its index of symmetry is defined as

$$c(X) = \inf_{\|x|=1} \|-x|.$$

Consider

$$S_X := \{x \in X : \|x| = 1\} \quad \text{and} \quad \hat{S}_X := \{x \in S_X : \|-x| \neq 0\}.$$

It is not difficult to see that

$$\hat{S}_X = S_X \text{ if and only if } \forall x \in X, \|x| = 0 \Longleftrightarrow x = 0.$$

It is known that a topological space X is T_1 if and only if for every $x \in X$, the singleton $\{x\}$ is closed. We begin by stating a few propositions relying on the topological structure of X, which will allow us to relate those properties with the value of $c(X)$.

Proposition 4.1 Let $(X, \|\cdot|_X)$ be an asymmetric normed space. Then, X is a T_1 space if and only if $\hat{S}_X = S_X$.

Proof. Suppose that X is not T_1. Then, there exists $a \in X$ such that $X \setminus \{a\}$ is not open. Thus, there exists $b \in X \setminus \{a\}$ such that for every $\varepsilon > 0$, $a \in B_{\|\cdot|_X}(b, \varepsilon)$. In other words, we have that $\|a - b|_X = 0$. Thus, we have that $a - b \neq 0$ and $\|a - b|_X = 0$. It follows that $\|b - a|_X \neq 0$. Let us set $e := \frac{b-a}{\|b-a|_X}$. Then, we have that $e \in S_X$ and $\|-e|_X = 0$. Hence, $\hat{S}_X \neq S_X$. Conversely, suppose that $\hat{S}_X \neq S_X$ and let $e \in S_X$ be such that $\|-e|_X = 0$. This implies that the singleton $\{0\}$ is not closed in X. Hence, X is not T_1. $\square$

Proposition 4.1 makes evident that the asymmetry of the norm readily changes the separation properties of the spaces. More precisely, it is known that every normed linear space is T_2, while in our case, T_1 is only assured when the unit ball is bounded for the associated normed space X_s. In this sense, we focus first on the case where $c(X) > 0$, where we have that X will be a T_1 space.

Proposition 4.2 Let $(X, \| \cdot |)$ be an asymmetric normed space. Suppose that $c(X) > 0$. Then, X is a T_1 space (equivalently, $\hat{S}_X = S_X$). Moreover, we have

$$\frac{1}{c(X)} = \sup_{x \in S_X} \| -x| = \sup_{\|x|=1, \|p|_\flat=1; \langle -p, x\rangle > 0} \langle -p, x\rangle.$$

Therefore, we have that

$$(\sup_{x \in S_X} \| -x|)(\inf_{x \in S_X} \| -x|) = 1,$$

and so $c(X) \in [0, 1]$.

Proof. Let $x \in X$ such that $\|x| \neq 0$, then $\frac{x}{\|x|} \in S_X$ and $\|\frac{-x}{\|x|}| \geq c(X) > 0$. It follows that $\| -x| \neq 0$. Equivalently, $\| -x| = 0 \Longrightarrow \|x| = 0$ and so $x = 0$. It follows that $\hat{S}_X = S_X$ and so X is a T_1 space. On the other hand,

$$x \in S_X \Longleftrightarrow (\exists z \in S_X)\, x = -\frac{z}{\| -z|}.$$

Indeed, it sufices to take $z = -\frac{x}{\|-x|}$. From this, we have that

$$\sup_{x \in S_X} \| -x| = \sup_{z \in S_X} \frac{1}{\| -z|} = \frac{1}{\inf_{z \in S_X} \| -z|} = \frac{1}{c(X)}.$$

Now, by the Hahn-Banach theorem (see [25, Corollary 2.2.4]), we have

$$\sup_{\|x|=1} \| -x| = \sup_{\|x|=1} \sup_{\|p|_\flat=1} \langle p, -x\rangle = \sup_{\|x|=1, \|p|_\flat=1; \langle -p, x\rangle > 0} \langle -p, x\rangle.$$

□

The importance of this last result relies on the fact that it allows us to obtain bounds for the asymmetric norms in terms of the norm of X_s, exposing readily part of our main results, more presicely, the isomorphism between an asymmetric space and its symmetric version in the case where $c(X) > 0$.

Proposition 4.3 Let $(X, \| \cdot |_X)$ and $(Y, \| \cdot |_Y)$ be asymmetric normed spaces. Suppose that $c(X) > 0$. Then, we have the following formulas:

$$c(X)\|x| \leq \| -x| \leq \frac{1}{c(X)}\|x|, \ \forall x \in X, \tag{4.1}$$

$$c(X)\|x\|_s \leq \|x| \leq \|x\|_s, \ \forall x \in X, \tag{4.2}$$

$$c(X)\|T|_{L_c} \leq \| - T|_{L_c} \leq \frac{1}{c(X)}\|T|_{L_c}, \;\; \forall T \in L_c(X,Y), \tag{4.3}$$

and,

$$\|T\|_{L_s} \leq \|T|_{L_c} \leq \frac{1}{c(X)}\|T\|_{L_s}, \;\; \forall T \in L_c(X,Y). \tag{4.4}$$

As a consequence, we have that $(X, \|\cdot|_X) \simeq (X, \|\cdot\|_s)$, $(L_c(X,Y), \|\cdot|_{L_c})$ is an asymmetric normed space, with $(L_c(X,Y), \|\cdot|_{L_c}) \simeq (L(X_s,Y_s), \|\cdot\|_{L_s})$. Moreover, $c(L_c(X,Y)) \geq c(X)$.

Proof. Formula (4.1) follows easily from Proposition 4.2, the rest of the assertions are simple consequences of this formula and the definitions for the norms and asymmetric norms involved. □

Considering all the already proven propositions, we see that in the case where $c(X) > 0$, the spaces $(X, \|\cdot|_X)$ and $(X, \|\cdot\|_s)$ (and also $(L_c(X,Y), \|\cdot|_{L_c})$ and $(L(X_s,Y_s), \|\cdot\|_{L_s})$) are topologically the same. This last statement leads us to focus our study mainly on the case where $c(X) = 0$, since on the contrary, even if the unit balls are not symmetric and maybe too different from a ball obtained by a classical norm, these distortions are not enough to change the topology of the space, hence, we can simply study the properties of the space directly seeing its associated normed space, for which we can use every tool from the theory of normed spaces to obtain topological properties.

Remark We proved in Proposition 4.2 that if $c(X) > 0$, then $\hat{S}_X = S_X$. The converse of this fact is not true in general, see for instance Example 4.3.

4.3 The main results

This section is devoted to the main results Theorem 4.1, Theorem 4.2 and Corollary 4.3 and their consequences. In order to do this, we need first to state some basic definitions related to the topology of asymmetric normed spaces. Let $(X, \|\cdot|_X)$ and $(Y, \|\cdot|_Y)$ be asymmetric normed spaces. We call the open and closed unit balls of L_c, respectively, the sets

$$B_{L_c}(0,1) := \{T \in L_c(X,Y) : \|T|_{L_c} < 1\}$$

and

$$\overline{B}_{L_c}(0,1) := \{T \in L_c(X,Y) : \|T|_{L_c} \leq 1\}.$$

An asymmetric normed space $(Y, \|\cdot|_Y)$ is called biBanach , if its associated normed space $(Y, \|\cdot\|_s)$ is a Banach space. In this case, $(L(X_s,Y_s), \|\cdot\|_{L_s})$ becomes a Banach space. We need the following lemma.

Lemma 4.1 Let $(X, \|\cdot|_X)$ be an asymmetric normed space and $(Y, \|\cdot|_Y)$ be an asymmetric normed biBanach space. Then, $\overline{B}_{L_c}(0,1)$ is a closed subset of $(L(X_s,Y_s), \|\cdot\|_{L_s})$ (which is a Banach space) and the open unit ball $B_{L_c}(0,1)$ is dense in $\overline{B}_{L_c}(0,1)$ for the norm $\|\cdot\|_{L_s}$.

Proof. Let (T_n) be a sequence in $\overline{B}_{L_c}(0,1)$ that converges to $T \in L(X_s, Y_s)$ for the norm $\|\cdot\|_{L_s}$ and let us prove that $T \in \overline{B}_{L_c}(0,1)$. Indeed, for all $n \in \mathbb{N}$ and all $x \in X$, we have

$$\begin{aligned}
\|T(x)|_Y &\leq \|(T-T_n)(x)|_Y + \|T_n(x)|_Y \\
&\leq \|T-T_n\|_{L_s}\|x\|_s + \|T_n|_{L_c}\|x|_X \\
&\leq \|T-T_n\|_{L_s}\|x\|_s + \|x|_X.
\end{aligned}$$

Sending n to $+\infty$, we get that $\|T(x)|_Y \leq \|x|_X$, for all $x \in X$. It follows that $T \in \overline{B}_{L_c}(0,1)$ which implies that $\overline{B}_{L_c}(0,1)$ is a closed subset of $(L(X_s, Y_s), \|\cdot\|_{L_s})$. To see that $B_{L_c}(0,1)$ is dense in $\overline{B}_{L_c}(0,1)$ for the norm $\|\cdot\|_{L_s}$, let $T \in \overline{B}_{L_c}(0,1)$ and consider the sequence $T_n = (1-\frac{1}{n})T$ so that $\|T_n|_{L_c} \leq 1-\frac{1}{n} < 1$. Then, $T_n \in B_{L_c}(0,1)$ for all $n \in \mathbb{N}$ and $\|T-T_n\|_{L_s} = \frac{1}{n}\|T\|_{L_s} \to 0$. □

Remark Since $\overline{B}_{L_c}(0,1)$ is a closed subset of $(L(X_s, Y_s), \|\cdot\|_{L_s})$, then $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$ is a complete metric space and being so, the Baire category theorem applies. However, we do not know if the whole space $(L_c(X,Y), \|\cdot\|_{L_s})$ is a Baire space. In general it is not closed in $(L(X_s, Y_s), \|\cdot\|_{L_s})$ (see Corollary 4.6 in the case where $(Y, \|\cdot|_Y) = (\mathbb{R}, \|\cdot|_{\mathbb{R}})$).

4.3.1 The first main result and consequences.

Our first main result is the following theorem, which gives a necessary and sufficient condition so that $L_c(X,Y)$ is not a vector space. This case corresponds to $c(X) = 0$, since it was already proven that whenever $c(X) > 0$, $L_c(X,Y)$ and $L(X_s, Y_s)$ are isomorphic. This is the main reason why the case $c(X) = 0$ is worth studying, since it is in this case where the linear structure of $L_c(X,Y)$ may be lost, which in turn makes the theory of asymmetric normed spaces interesting, since it shows a behaviour where the structure of this set is not as expected in the framework of classical norms.

Theorem 4.1 *Let $(X, \|\cdot|_X)$ be asymmetric normed space. Then, the following assertions are equivalent.*

i) $c(X) = 0$.

ii) For every biBanach asymmetric normed space $(Y, \|\cdot|_Y)$ for which there exists $y \in Y$ such that $\|y|_Y = 1$ and $\|-y|_Y = 0$ (that is, Y is not a T_1 space) and every $H \in L_c(X,Y)$, the set

$$\mathcal{G}(H) := \{T \in \overline{B}_{L_c}(0,1) : -(H+T) \notin L_c(X,Y)\},$$

is a G_δ dense subset of $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$. In particular, $L_c(X,Y)$ is not a vector space whenever $(Y, \|\cdot|_Y)$ is not T_1.

iii) There exists a biBanach asymmetric normed space $(Y, \|\cdot|_Y)$ such that the convex cone $L_c(X,Y)$ is not a vector space.

Proof. $(i) \Longrightarrow (ii)$ For each $k \in \mathbb{N}$, let us set

$$O_k := \{T \in \overline{B}_{L_c}(0,1) \,|\, (\exists x_k \in X)\, \|-(H+T)(x_k)|_Y > k\|x_k|_X\}.$$

Clearly, we have that $\cap_{k\in\mathbb{N}} O_k = \mathcal{G}(H)$. By the Baire theorem, $\mathcal{G}(H)$ will be a G_δ dense subset of $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$ whenever, for each $k \in \mathbb{N}$, the set O_k is open and dense in the complete metric space $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$ (see Lemma 4.1).

Let us prove that O_k is open in $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$, for each $k \in \mathbb{N}$. Let $T \in O_k$ and $0 < \varepsilon < \frac{\|-(H+T)(x_k)|_Y - k\|x_k|_X}{\|x_k\|_s}$. Let $S \in \overline{B}_{L_c}(0,1)$ such that $\|S-T\|_{L_s} < \varepsilon$. We have that

$$\begin{array}{rcl} \|-(H+S)(x_k)|_Y & \geq & \|-(H+T)(x_k)|_Y - \|(S-T)(x_k)|_Y \\ & \geq & \|-(H+T)(x_k)|_Y - \|(S-T)(x_k)\|_s \\ & > & \|-(H+T)(x_k)|_Y - \|S-T\|_{L_s}\|x_k\|_s \\ & > & \|-(H+T)(x_k)|_Y - \varepsilon\|x_k\|_s \\ & > & k\|x_k|_X. \end{array}$$

Thus, $S \in O_k$ for every $S \in \overline{B}_{L_c}(0,1)$ such that $\|S-T\|_{L_s} < \varepsilon$. Hence, O_k is open in $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$.

Now, let us prove that O_k is dense in $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$, for each $k \in \mathbb{N}$. Since the set $(B_{L_c}(0,1), \|\cdot\|_{L_s})$ is dense in $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$ (by Lemma 4.1), it suffices to prove that O_k is dense in $(B_{L_c}(0,1), \|\cdot\|_{L_s})$. Let $T \in B_{L_c}(0,1)$ and $0 < \varepsilon < 1 - \|T|_{L_c}$. Since $c(X) = 0$, there exists a sequence $(a_n) \subset X$ such that $\|a_n|_X = 1$ for all $n \in \mathbb{N}$ and $\|-a_n|_X \to 0$. Let us set $I := \{n \in \mathbb{N} : \|-a_n|_X = 0\}$. We have two cases:

Case 1. $I = \emptyset$. In this case, for all $n \in \mathbb{N}$, let $z_n := \frac{-a_n}{\|-a_n|_X}$. We see that $\|z_n|_X = 1$ and $-z_n = \frac{a_n}{\|-a_n|_X}$. Using the Hahn-Banach theorem [25, Theorem 2.2.2], for each $n \in \mathbb{N}$, there exists $p_n \in X^\flat$ such that $\|p_n|_\flat = 1$ and $\langle p_n, -z_n\rangle = \|-z_n|_X > 0$. Now, let $e \in Y$ such that $\|e|_Y = 1$ and consider the operator $T + \varepsilon p_n e : x \mapsto T(x) + \varepsilon\langle p_n, x\rangle e$. We have that, $\|T + \varepsilon p_n e|_{L_c} \leq \|T|_{L_c} + \varepsilon\|p_n e|_{L_c} = \|T|_{L_c} + \varepsilon\|p_n|_\flat = \|T|_{L_c} + \varepsilon < 1$, which leads us to $T + \varepsilon p_n e \in B_{L_c}(0,1) \subset \overline{B}_{L_c}(0,1)$. On the other hand,

$$\begin{array}{rcl} \|-(H+T+\varepsilon p_n e)(z_n)|_Y & = & \|(H+T+\varepsilon p_n e)(-z_n)|_Y \\ & \geq & \|\varepsilon p_n(-z_n)e|_Y - \|(H+T)(z_n)|_Y \\ & = & \varepsilon\|-z_n|_X - \|(H+T)(z_n)|_Y \\ & = & \dfrac{\varepsilon}{\|-a_n|_X} - \|(H+T)(z_n)|_Y \\ & \geq & \dfrac{\varepsilon}{\|-a_n|_X} - \|H+T|_{L_c}. \end{array}$$

Since $\|-a_n|_X \to 0$, when $n \to +\infty$, there exists a subsequence (a_{n_k}) such that for each $k \in \mathbb{N}$ $\frac{\varepsilon}{\|-a_{n_k}|_X} - \|H+T|_{L_c} > k$. Hence, for each $k \in \mathbb{N}$, we have that $\|z_{n_k}|_X = 1$ and

$$\|-(H+T+\varepsilon p_{n_k} e)(z_{n_k})|_Y \quad > \quad k = k\|z_{n_k}|_X. \tag{4.5}$$

From formula (4.5), we have that $T + \varepsilon p_{n_k} \in O_k$. Since,

$$\|(T+\varepsilon p_{n_k} e) - T\|_{L_s} = \|\varepsilon p_{n_k} e\|_{L_s} \leq \varepsilon\|p_{n_k}|_\flat = \varepsilon,$$

it follows that O_k is dense in the space $(\overline{B}_{L_c}(0,1), \|\cdot\|_{L_s})$.

Case 2. $I \neq \emptyset$. In this case, there exists $n_0 \in I$ such that $\|a_{n_0}|_X = 1$ and $\| - a_{n_0}|_X = 0$. Using the Hahn-Banach theorem [25, Theorem 2.2.2], let $p \in X^\flat \setminus \{0\}$ such that $\|p|_\flat = 1$ and $\langle p, a_{n_0}\rangle = \|a_{n_0}|_X = 1$. Thus, we have that

$$\begin{aligned}
\| - (H + T + \varepsilon pe)(-a_{n_0})|_Y &= \|(-H - T - \varepsilon pe)(-a_{n_0})|_Y \\
&\geq \|\varepsilon\langle -p, -a_{n_0}\rangle e|_Y - \| - (H + T)(a_{n_0})|_Y \\
&= \varepsilon\|\langle p, a_{n_0}\rangle e|_Y - \|(H + T)(-a_{n_0})|_Y \\
&\geq \varepsilon - \|H + T|_{L_c}\| - a_{n_0}|_X \\
&= \varepsilon \\
&> 0 = k\| - a_{n_0}|_X.
\end{aligned}$$

On the other hand, we have that $\|T+\varepsilon pe|_{L_c} \leq \|T|_{L_c}+\varepsilon\|pe|_{L_c} = \|T|_{L_c}+\varepsilon\|p|_\flat = \|T|_{L_c}+\varepsilon < 1$, so that $T + \varepsilon pe \in B_{L_c}(0, 1) \subset \overline{B}_{L_c}(0, 1)$. Thus, $T + \varepsilon pe \in O_k$ and

$$\|(T + \varepsilon pe) - T\|_{L_s} = \|\varepsilon pe\|_{L_s} \leq \varepsilon.$$

Hence, O_k is dense in $(\overline{B}_{L_c}(0, 1), \| \cdot \|_{L_s})$.

Hence, in both cases, we have that $\cap_{k\in\mathbb{N}} O_k = \mathcal{G}$ is a G_δ dense subset of $(\overline{B}_{L_c}(0, 1), \| \cdot \|_{L_s})$.

$(ii) \implies (iii)$ is trivial.

$(iii) \implies (i)$ Suppose that there exists a biBanach asymmetric normed space $(Y, \| \cdot |_Y)$ such that the convex cone $L_c(X, Y)$ is not a vector space. Then, there exists $T \in L_c(X, Y) \setminus \{0\}$, such that $-T \notin L_c(X, Y)$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$\| - T(x_n)|_Y > n\|x_n|_X.$$

It follows that, for all $n \in \mathbb{N}$

$$\|T|_{L_c}\| - x_n|_X \geq \|T(-x_n)|_Y = \| - T(x_n)|_Y > n\|x_n|_X.$$

Let us set $z_n = \frac{-x_n}{\|-x_n|_X}$ (since $\| - x_n|_X \neq 0$) for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N} \setminus \{0\}$, we have $\|z_n|_X = 1$, $\| - z_n| = \frac{\|x_n|}{\|-x_n|} < \frac{\|T|_{L_c}}{n} \to 0$. Hence, $c(X) = 0$. □

Remark It does not seem obvious to establish the validity of the implication $(i) \implies (iii)$ directly without the use Baire theorem or, in other words, without passing through the implication $(i) \implies (ii)$.

We readily see that whenever $c(X) = 0$, the set $X^\flat$ does not have linear structure. Moreover, the set of linear functionals $\varphi \in X^\flat$ for which $-\varphi \notin X^\flat$ is quite big, since the set of those functionals with norm less than or equal to 1 is G_δ dense on in the closed unit ball of X^*_s. In other words, we see that passing from the case where the index of symmetry is positive to the case where it is equal to 0 makes a huge difference, going from an empty to a G_δ dense set.

A straightforward consequence of Theorem 4.1 is that, whenever Y is not T_1, $L_c(X, Y)$ is not a vector space if and only if $c(X) = 0$. Since Y is not T_1, in particular we have that

$c(Y) = 0$. The converse is not true in general: There exists spaces which are T_1 and have index of symmetry equal to 0, which necessarily are infinite-dimensional spaces. Then, for the finite-dimensional case, we have the following lemma.

Lemma 4.2 Let $(Y, \|\cdot|_Y)$ be an finite-dimensional asymmetric normed space. Then, $c(Y) = 0$ if and only if Y is not a T_1 space

Proof. The "if" part is clear, since Y is not a T_1 space if and only if there exists $y \in Y$ such that $\|y|_Y = 1$ and $\|-y|_Y = 0$. For the other implication, suppose that $c(Y) = 0$. Then, there exists a sequence $(y_n)_n \subset Y$ such that $\|y_n|_Y = 1$ for every $n \in \mathbb{N}$ and $\|y_n|_Y \to 0$. Without loss of generality, we can assume that $\|y_n|_Y < 1$ for every $n \in \mathbb{N}$, which implies that $\|y_n\|_s = 1$, for every $n \in \mathbb{N}$. Since $(Y, \|\cdots\|_s)$ is a finite-dimensional normed space, there exists a subsequence $(y_{n_k})_k$ converging to some $y \in Y$ for the norm $\|\cdot\|_s$. We see that $\|y|_Y = 0$. Indeed,

$$\|-y|_Y \leq \|y_{n_k} - y|_Y + \|-y_{n_k}|_Y \leq \|y_{n_k} - y\|_s + \|-y_{n_k}|_Y \to 0.$$

Thus, $\|-y|_Y = 0$ and $\|y|_Y = \|y\|_s = 1$. Hence, Y is not a T_1 space. □

In virtue of the previous lemma, a complete characterization for $L_c(X,Y)$ being a vector space can be stated, in the case where Y is finite-dimensional.

Corollary 4.1 *Let $(X, \|\cdot|_X)$ and $(Y, \|\cdot|_Y)$ be asymmetric normed spaces and suppose that Y is finite-dimensional. Then, $Lc(X,Y)$ is not a vector space if and only if $c(X) = c(Y) = 0$. The "only if" part remains true even if Y is infinite-dimensional.*

Proof. To see the "only if" part, we follow the proof of part $iii) \implies i)$ of Theorem 4.1. Indeed, suppose that $L_c(X,Y)$ is not a vector space. Then, there exists $T \in L_c(X,Y) \setminus \{0\}$, such that $-T \notin L_c(X,Y)$. Thus, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\|-T(x_n)|_Y > n\|x_n|_X$. It follows that, for all $n \in \mathbb{N}$

$$n\|x_n|_X < \|-T(x_n)|_Y = \|T(-x_n)|_Y \leq \|T|_{Lc}\|-x_n|_X.$$

Let us set $z_n = -x_n/\|-x_n|_X$ (since $\|-x_n|_X \neq 0$) for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N} \setminus \{0\}$, we have $\|z_n|_X = 1$, $\|-z_n|_X < \|T|_{L_c}/n \to 0$. Hence, $c(X) = 0$. It remains to show that $c(Y) = 0$. Indeed, since $T \in L_c(X,Y) \setminus \{0\}$, then $\|T(x_n)|_Y \leq \|T|_{L_c}\|x_n|_X$. Thus, using the above inequality, we get

$$\|T(x_n)|_Y < (\|T|_{L_c}/n)\|-T(x_n)|_Y.$$

This implies in particular that $\|-T(x_n)|_Y \neq 0$ for all $n \in \mathbb{N} \setminus \{0\}$. Let $y_n = (-T(x_n))/\|-T(x_n|_Y \in Y$. Then we have that $\|y_n|_Y = 1$ for all $n \in \mathbb{N} \setminus \{0\}$ and $\|-y_n|_Y < \|T|_{L_c}/n \to 0$. Hence $c(Y) = 0$.

The "if" part follows from Theorem 4.1 provided that the condition $c(Y) = 0$ implies the existence of $y \in Y$ such that $\|y|_Y = 1$ and $\|-y|_Y = 0$. This is true since Y is finite-dimensional (necessarily a biBanach space), by Lemma 4.2. □

Using these results, we can also now state a density result for the asymmetric norm $\| \cdot |_{L_c}$.

Corollary 4.2 *Let $(X, \| \cdot |_X)$ be an asymmetric normed space with $c(X) = 0$ and $(Y, \| \cdot |_Y)$ be a biBanach asymmetric normed space for which there exists $y \in Y$ such that $\|y|_Y = 1$ and $\| -y|_Y = 0$. Then, the set of elements $H \in L_c(X,Y)$ such that $-H \notin L_c(X,Y)$ is dense in $L_c(X,Y)$ for the asymmetric norm $\| \cdot |_{L_c}$.*

Proof. Noticing that part *ii*) of Theorem 4.1 is valid for any radius for the balls, we have that for every $\varepsilon > 0$ and $H \in L_c(X,Y)$, there exists $T \in \overline{B}_{L_c}(0,\varepsilon)$ such that $-(H+T) \notin L_c(X,Y)$ and $H+T \in L_c(X,Y)$, which finishes the proof. $\square$

Making use of Theorem 4.1 and Proposition 4.3, we state in the following corollary a complete characterization for the convex cone $L_c(X,Y)$ to be a vector space. The non trivial part of the following corollary is the implication $v) \Longrightarrow i)$, which is a consequence of Theorem 4.1. Notice also that, thanks to Proposition 4.3, we do not need to assume that Y is biBanach in the following corollary, since the condition of biBanach used in Theorem 4.1 is implicitly verified by the space $(\mathbb{R}, \| \cdot |_{\mathbb{R}})$ in part v).

Corollary 4.3 *Let $(X, \| \cdot |_X)$ be an asymmetric normed space. Then, the following assertions are equivalent.*

i) $c(X) > 0$.

ii) *$(X, \| \cdot |_X)$ is isomorphic to its associated normed space.*

iii) *For every asymmetric normed space $(Y, \| \cdot |_Y)$, $L_c(X,Y)$ is an asymmetric normed space isomorphic to the space $L(X_s, Y_s)$.*

iv) *$(X^\flat, \| \cdot |_\flat)$ is an asymmetric normed space isomorphic to the Banach space $(X^*, \| \cdot \|_*)$.*

v) *$X^\flat$ is a vector space.*

The following result shows that if an asymmetric normed space X is a dual of some asymmetric normed space, then necessarily it is isomorphic to its associated normed space, in other words, necessarily $c(X) > 0$. In simple words, we prove that duality for asymmetric normed spaces preserves the index of symmetry of the spaces.

Corollary 4.4 *Let $(X, \| \cdot |)$ be an asymmetric normed space and suppose that $c(X) = 0$. Then, X can not be the dual of an asymmetric normed space. The converse is false in general (ex. the Banach space $X = (c_0(\mathbb{N}), \| \cdot \|_\infty)$, is not a dual space but $c(X) = 1$).*

Proof. Suppose that there exists an asymmetric normed space Y which is the predual of X, that is $(Y^\flat, \| \cdot |_\flat) = (X, \| \cdot |)$. We prove that $c(Y) = 0$. Indeed, suppose by contradiction that $c(Y) > 0$, then by formula (4.3) of Proposition 4.3 (applied with the spaces $(Y, \| \cdot |_Y)$ and

$(\mathbb{R}, \| \cdot |_{\mathbb{R}})$), we have that

$$c(Y)\|p\|_\flat \leq \| -p|_\flat \leq \frac{1}{c(Y)}\|p\|_\flat, \ \ \forall p \in Y^\flat = X.$$

This implies that $c(X) \geq c(Y) > 0$, which contradict the fact that $c(X) = 0$. Hence, $c(Y) = 0$. Now, using Theorem 4.1, we get that $Y^\flat$ is not a vector space which contradict the fact that $X = Y^\flat$ is a vector space. Finally, X cannot be the dual of an asymmetric normed space. $\square$

To finish this section, and as a consequence of the previous results, we state the relation between the sets $L_c(X, Y)$ and $-L_c(X, Y)$, which finally relies only on the value of the index of symmetry of X.

Corollary 4.5 *Let X be an asymmetric normed space. Then, either*

$$L_c(X, Y) \cap (-L_c(X, Y)) = L_c(X, Y) = L(X_s, Y_s)$$

or $L_c(X, Y) \cap (-L_c(X, Y))$ is of first Baire category in $(L(X_s, Y_s), \| \cdot \|_{L_s})$.

Proof. If $c(X) > 0$, then by Corollary 4.3, we have that $L_c(X, Y)$ is an asymmetric normed space isomorphic to $L(X_s, Y_s)$, thus we have that

$$L_c(X, Y) \cap (-L_c(X, Y)) = L_c(X, Y) = L(X_s, Y_s).$$

Otherwise, we have that $c(X) = 0$. In this case, to see that $L_c(X, Y) \cap (-L_c(X, Y))$ is of first Baire category in the space $(L(X_s, Y_s), \| \cdot \|_{L_s})$, it suffices to observe, using Theorem 4.1 with $H = 0$, that we have

$$L_c(X, Y) \cap (-L_c(X, Y)) = \cup\{n(\overline{B}_{L_c}(0, 1) \setminus \mathcal{G}(H)) : n \in \mathbb{N}\},$$

so that, it is of first Baire category in $(L(X_s, Y_s), \| \cdot \|_{L_s})$, being the countable union of first Baire category sets. $\square$

4.3.2 The second main result

From the previous section, we obtained the conclusion that $X^\flat$ has a linear structure if and only if $c(X) > 0$, in which case $X^\flat$ and X_s^* are isomorphic. From this naturally arises the question of the relation between those sets in the case where $c(X) = 0$, that is, asking if there is some sort of topological relation between them, even considering the lack of linear structure in $X^\flat$. In this sense, we focus on the density of the dual $X^\flat$ in X_s^*. By $\overline{X^\flat}^{w^*}$, we denote the weak-star closure of $X^\flat$ in (X_s^*, w^*).

Theorem 4.2 *Let $(X, \| \cdot |)$ be an asymmetric normed space. Then, X is a T_1 space if and only if $\overline{X^\flat}^{w^*} = X_s^*$.*

Proof. Assume that X is a T_1 space. Suppose by contradiction that $\overline{X^\flat}^{w^*} \neq X^*_s$ and fix $p \in X^*_s \setminus \overline{X^\flat}^{w^*}$. By the classical Hahn-Banach theorem in the Hausdorff locally convex vector space (X^*_s, w^*), there exists $x_0 \in X \setminus \{0\}$ and $\alpha \in \mathbb{R}$, such that

$$\langle p, x_0 \rangle > \alpha \geq \langle q, x_0 \rangle, \text{ for all } q \in \overline{X^\flat}^{w^*}. \tag{4.6}$$

Since X is T_1 space and $x_0 \neq 0$, we have that $\|x_0| > 0$. Using [25, Theorem 2.2.2], there exists $q_0 \in X^\flat$ such that $\|q_0|_\flat = 1$ and $\langle q_0, x_0 \rangle = \|x_0|$. Since $X^\flat \subset \overline{X^\flat}^{w^*}$ is a convex cone, we obtain using (4.6) that for all $n \in \mathbb{N}$,

$$\langle p, x_0 \rangle > \alpha \geq \langle nq_0, x_0 \rangle = n\|x_0|.$$

This implies that $\|x_0| = 0$ which is impossible. Hence, $\overline{X^\flat}^{w^*} = X^*_s$. Conversely, suppose that $\overline{X^\flat}^{w^*} = X^*$. We need to show that $\|x| > 0$ whenever $x \neq 0$. Indeed, let $x \neq 0$. By the Hahn-Banach theorem (in X^*_s), there exists $p \in X^*$ such that $\|p\|_* = 1$ and $\langle p, x \rangle = \|x\|_s > 0$. On the other hand, $p \in \overline{X^\flat}^{w^*} = X^*_s$, thus, for every $\varepsilon > 0$, there exists $q_\varepsilon \in X^\flat$ such that

$$\langle q_\varepsilon, x \rangle + \varepsilon \geq \langle p, x \rangle = \|x\|_s.$$

Suppose by contradiction that $\|x| = 0$. It follows that for every $\varepsilon > 0$, $\langle q_\varepsilon, x \rangle \leq \|q_\varepsilon|_\flat \|x| = 0$. So using the above formula, we get that $\|x\|_s \leq \varepsilon$ for every $\varepsilon > 0$ which implies that $x = 0$ and gives a contradiction. Hence, $\|x| > 0$ for every $x \neq 0$, which implies that X is a T_1 space. $\square$

Remark Following the same arguments as in the last proof, we have that in general

$$\overline{\mathrm{span}(X^\flat)}^{w^*} = X^*,$$

even if X is not a T_1 space.

We know from [40, Theorem 4.] (see also [25, Proposition 2.4.2.]) that $\overline{B}_\flat(0,1)$ is always weak-star closed in (X^*_s, w^*) (in fact, weak-star compact). On the other hand, $\overline{B}_\flat(0,1)$ is always norm closed in $(X^*_s, \|\cdot\|_*)$ (see Lemma 4.1). These results are not always true for the whole space $X^\flat$ when $c(X) = 0$. We have the following characterization.

Corollary 4.6 *Let $(X, \|\cdot|)$ be a T_1 asymmetric normed space. Then, $X^\flat$ is weak-star closed in (X^*_s, w^*) if and only if, $c(X) > 0$ if and only if X is isomorphic to its associated normed space.*

Proof. If $c(X) = 0$, by Theorem 4.1 we know that $X^\flat \neq -X^\flat$. It follows that $X^\flat \neq X^*_s$ and so by Theorem 4.2, $X^\flat \neq \overline{X^\flat}^{w^*} = X^*_s$. Equivalently, $X^\flat = \overline{X^\flat}^{w^*}$, implies that $c(X) > 0$. Conversely, $c(X) > 0$ is equivalent to the fact that $X^\flat$ is isomorphic to X^*_s by Corollary 4.3 and so it is in particular is weak-star closed in (X^*_s, w^*). $\square$

Remark If we assume that X_s^* is a reflexive space, then we can replace the w^*-closure by the $\|\cdot\|_*$-closure. This follows from the well-known Mazur's theorem on the coincidence of weak and norm topologies on convex sets (see [21]), so we have that

$$\overline{X^\flat}^{w^*} = \overline{X^\flat}^{w} = \overline{X^\flat}^{\|\cdot\|_*},$$

since weak-star and weak topologies coincide in reflexive spaces.

4.3.3 Classification and examples

There are several topological studies of asymmetric normed spaces, see for instance [3], [25] and [48]. Our study leads to the classification given in Definition 4.1 and the already mentioned consequences. Recall that the two possible situations which go beyond the classical framework of normed spaces are:

i) Infinite dimensional spaces which are T_1 with $c(X) = 0$ (spaces of type II).

ii) Finite and infinite dimensional spaces X which are not T_1 (spaces of type III, necessarily $c(X) = 0$).

These affirmations are consequences of Corollary 4.3, Proposition 4.4 and Theorem 4.3.

Let $(X, \|\cdot|)$ be an asymmetric normed linear space endowed with the topology $\tau_{\|\cdot|}$ induced by the quasi-metric defined by

$$d_{\|\cdot|}(x, y) := \|y - x|, \forall x, y \in X.$$

The closed unit ball $\overline{B}_{\|\cdot|_X}(0, 1)$ is the set $\{y \in X : \|y| \leq 1\}$. A set $K \subset X$ is said to be compact if it is compact considered as a subspace of X with the induced topology, that is, $(K, \|\cdot|)$ is compact with respect to the topology $\tau_{\|\cdot|_X}$. A set K of X is compact if every sequence in K has a convergent subsequence whose limit is in K.

The following proposition shows that a finite dimensional asymmetric normed space can not be of type II.

Proposition 4.4 Let $(X, \|\cdot|_X)$ be an asymmetric normed space. Suppose that X is of type II. Then, the closed unit balls $\overline{B}_{\|\cdot|_X}(0, 1)$ and $\overline{B}_{\|\cdot\|_s}(0, 1)$ of X and its associated normed space respectively, are not compact. In consequence X is infinite dimensional.

Proof. From the definition of spaces of type II, there exists a sequence $(x_n) \subset X$ such that $\|x_n|_X = 1$ for all $n \in \mathbb{N}$ and $0 < \|-x_n|_X \to 0$. We can assume without loss of generality that $0 < \| - x_n|_X < 1$ so that $\|x_n\|_s = \|x_n|_X = 1$ for all $n \in \mathbb{N}$. Suppose by contradiction that $B_{\|\cdot|_X}(0, 1)$ is compact. Let (x_{n_k}) be a subsequence converging for $\|\cdot|_X$ to some $a \in X$.Then,

$$\| - a|_X \leq \|x_{n_k} - a|_X + \| - x_{n_k}|_X \to 0,$$

which implies that $\| - a|_X = 0$. Since X is a T_1 space, then $a = 0$. This contradict the fact that $\|x_{n_k} - a|_X = \|x_{n_k}|_X = \|x_{n_k}\|_s = 1$ for each $k \in \mathbb{N}$. Hence, the sequence (x_n) has no convergent subsequence neither for $\|\cdot|_X$ nor for $\|\cdot\|_s$ (since $\|\cdot|_X \leq \|\cdot\|_s$). Thus, the closed unit balls $B_{\|\cdot|_X}(0,1)$ and $B_{\|\cdot\|_s}(0,1)$ are not compact. In particular X is of infinite dimension. $\square$

Following the same idea, the following theorem shows that a T_1 space of finite dimension is necessarily isomorphic to its associated normed space, or equivalently, it is of type I.

Theorem 4.3 *Let $(X, \|\cdot|_X)$ be an asymmetric normed space of finite dimension. Then, X is T_1 if and only if X is of type I, if and only if X is isomorphic to its associated normed space.*

Proof. Suppose that X is T_1. Then, X is not of type III. Since spaces of type II are infinite dimensional by Proposition 4.4, it follows that X, is of type I. Hence, equivalently, by Corollary 4.3, X is isomorphic to its associated normed space. The converse is trivial. $\square$

For finishing this chapter, we show that using the tools developed we recover the result of García Raffi in [38, Theorem 13.]. This is done in the following corollary.

Corollary 4.7 *The closed unit ball of a T_1 asymmetric normed space X is compact, if and only if it is finite dimensional.*

Proof. Suppose that X is finite dimensional. Since X is T_1, then by Theorem 4.3, X is isomorphic to its associated normed space. Thus, the closed unit ball of $(X, \|\cdot|_X)$ is compact. Conversely, suppose that the closed unit ball of $(X, \|\cdot|_X)$ is compact. Then, by Proposition 4.4, X is not of type II. Since, X is T_1, then X is of type I and so $(X, \|\cdot|_X)$ is isomorphic to $(X, \|\cdot\|_s)$ (by Corollary 4.3), which is finite dimentional by Riesz's theorem, since its closed unit ball is compact. $\square$

It becomes evident that it is not only the presence or absence of asymmetry what is worth considering at the moment of working over asymmetric normed spaces, but also the degree of this asymmetry. A concrete way of doing this is using the index of symmetry defined during this chapter, which relies only in the structure of the asymmetric norm of the space.

It is also important to take into account that the structure of the associated spaces, say $X^\flat$ and $L(X,Y)$ (where Y is an asymmetric normed space), depends only in the structure of X, which is not obvious from the beginning. It is exactly through a deep study of the defined index that this property becomes evident.

Recall that asymmetric normed spaces have its metric counterpart, the so-called quasi-metric spaces. Since the definition for the index of symmetry of an asymmetric normed space relies only in the asymmetric norm of the space, a similar approach can be taken over quasi-metric spaces in order to obtain results in the line of those present in this chapter. More precisely, the idea is to describe the structure of the set of real valued Lipschitz functions defined over the quasi-metric space, which is also called the Lipschitz-dual. We can go even further and study the next natural space associated to it, that is the semi-Lipschitz free space, which is the analog for Lipschitz-free space when asymmetry is taken into account, and describe their structures in terms of the degree of asymmetry of the base quasi-metric space, quantified by the index of symmetry adapted to that end. An expected result in this line is that for a given quasi-metric space, the index of symmetry of its free quasi-metric space (which is an asymmetric normed space) coincides with the index of symmetry of the quasi-metric space (now in the sense of the generalized definition for quasi-metric spaces), showing that this index not only shows some properties of the space and its Lipschitz-dual, but is also preserved during the construction of the asymmetric predual of the Lipschitz-dual.

In this same framework, the study of the behaviour of the index of symmetry of a quasi-metric space may be done in the following context. In [32], an specific asymmetrization of a metric space is done in the following way. Let (M,d) be a metric space and $\mathcal{F}(M)$ its free-Lipschitz space. Define over $\mathcal{F}(M)$ the asymmetric norm given by

$$\|\mu| := \sup_{\|f\|_L=1, f\geq 0} \langle f, \mu\rangle, \quad \text{for every } \mu \in \mathcal{F}(M).$$

The asymmetry of this norm is easily proven. Recall that M is isometrically embedded in $\mathcal{F}(M)$, so we can now use the asymmetric norm defined over this linear space to define an asymmetric distance over M, in a way such that the operator $\delta : M \to \mathcal{F}(M)$ continues to be an isometry, now between the quasi-metric space M and the asymmetric normed space $\mathcal{F}(M)$. Given this, a natural question arises: Which is the relation between the index of symmetry of both spaces and the structure of the cone $\{f \in \mathrm{Lip}_0(M) : f \geq 0\}$? Given a number $c \in [0,1]$, is it possible to find a cone in the linear space $\mathrm{Lip}_0(M)$ such that the resulting quasi-metric space has index of symmetry equal to c? These and others questions that might arise are still not answered, and will be revisited in future works.

Conclusions

If we were to resume the present work in one phrase and propose an alternative title, the following sentence would definitely fit to both purposes: "Structural properties of linear spaces related to some classes of Lipschitz functions".

During the development of Chapters 1 and 2, we focused on the internal structure of Lipschitz functions defined over a nonempty open convex subset of $\mathbb{R}^d$ taking as an starting point the behaviour of the same functions in the case $d = 1$. The first important point was to isolate the difficulties added when passing from one to multiple dimensions, which lead us to a more detailed study of essentially bounded vector valued functions in terms of how we could treat them in order to view them as gradients of Lipschitz functions, in analogy to the one dimensional case.

Our approach was somehow close to Multivariate Calculus, but from a nonsmooth viewpoint. More precisely, we considered regular Lipschitz functions as potentials. Therefore, their gradients must have null curl, which corresponds to a (nonsmooth) Poicaré condition. It is precisely this which lead us to study the integrals of essentially bounded functions over some specific closed paths. As simple as this idea is, it was not free of difficulties, the main one being that we where using a one dimensional integral for functions which are essentially bounded for the d-dimensional Lebesgue measure, that is, we integrated over null sets. It was precisely this technique that naturally gave rise to the presence of what we called essentially Lipschitz functions, since we needed to consider the behaviour not only of the chosen class representative on the space of essentially bounded functions, but of the whole class.

Considering our approach, we can resume the main result of Chapter 1 by saying that we can identify the space of Lipschitz functions over a nonempty open convex subset of $\mathbb{R}^d$ with the space of conservative essentially bounded vector fields, or equivalently, with the space of essentially bounded vector fields with null curl. This idea is further developed throughout Chapter 2, where we turn our attention to compactly supported smooth functions over the same set as before, which are clearly Lipschitz functions. This allows us on one hand to work easily with their gradients, but also to take a closer look to the predual of the subspace of essentially bounded functions obtained in the previous chapter.

Here, an approach more centered in Functional Analysis has been more natural, mostly since we needed to directly study duality properties in order to finally identify the Lipschitz-free space. But the advantage of working with compactly supported functions relies not only in the possibility of using the gradients, but also in the known density of these functions in the spaces we where using. But even if we left aside the Multivariate Calculus approach, we also obtained a description for the space identifying the Lipschitz-free space, more precisely, it is related to integrable vector fields with null divergence. It is worth noticing that in both description of the spaces in terms of known concepts of Multivariate Calculus, all the derivatives are not taken in the classical sense, but in the sense of distributions, which shows again the importance of working with compactly supported functions for obtaining the desired results.

In Chapter 3 we left aside Lipschitz-free spaces to focus on another property of Lipschitz functions. More precisely, we centered our attention in the results on genericity for Lipschitz functions from J. Borwein and X. Wang [19]. The concepts appearing during this Chapter were not even considered during the first two chapters, in particular, Clarke's subdifferential was not present, even when is one of the main concepts associated to Lipschitz functions defined over normed spaces. It is worth noticing that in the beginning, the use of Clarke's directional derivative for the development of the results of Chapters 1 and 2 was considered, but ultimately this had to be revised in the light of possible pathological behaviour, as has already revealed in the PhD Thesis of X. Wang [64]. Nevertheless, Clarke's subdifferential appeared once again thanks to the aforementioned result on genericity for Lipschitz functions, which in a few words states that in the set of Lipschitz functions with Lipschitz constant at most $K > 0$ endowed with the metric of uniform convergence over bounded subsets, Lipschitz functions with maximal Clarke subdifferential are generic.

Our first goal was to check if a similar proof could be carried out now in the vector space of all Lipschitz functions endowed with the metric of uniform convergence over bounded sets, which was rapidly discarded, since such metric space is not complete and that was ultimately necessary for the use of Baire's Theorem in the proof. It seemed much more natural in order to regain completeness to work with the Lipschitz norm, recalling in particular, that this is the choice of norm for spaces of Lipschitz functions, and directly relates to the theory of Lipschitz-free. However, althought completeness is ensured, the use of this norm leads to another problem: Contrary to the metric of uniform convergence over bounded subsets, the Lipschitz norm measures and controls the slopes of the functions. Having freedom over those slopes is essential for the proof of J. Borwein and X. Wang. At this point, is was clear that an extension of the result for the whole space was not a simple task and maybe a new approach was necessary.

This is where the notion of lineability comes into play and undertakes a central role. We change the concept of being "topologically big" associated to genericity to that of being "algebraically big", meaning that we should look for a Banach subspace of Lipschitz functions with maximal subdifferential instead of a G_δ dense set of those functions. From here on our work relies on the results from Chapter 1, taking a constructive approach of that subspace instead of the Baire's Theorem approach, even if completeness is still present. But it is necessary to clarify that even if the results of Chapter 1 are valid no matter the norm considered in $\mathbb{R}^d$, the construction obtained in Chapter 3 works only in the case where we use the 1-norm in $\mathbb{R}^d$, which is used in the proof for constructing suitable functions whose Clarke's subdifferentials are balls in $\mathbb{R}^d$ endowed with the infinity norm. A natural question that arises is if is it possible to make a similar construction for any norm in order to obtain the same result. The answer to this is not obvious and after this PhD thesis, it still remains open.

In Chapter 4, which is does not directly rely on the previous part of this work, we dealt with the concept of asymmetry. More precisely, we left aside Lipschitz functions to focus our efforts in linear spaces endowed with asymmetric norms and mainly in the consequences of this on the structure of the natural spaces associated to them, that is the dual space and the space of linear operators with values in another asymmetric normed space. An important part here was to comprehend profoundly the tools already developed around this concept of asymmetry, which is not reserved only for linear spaces, but also exists in their metric counterpart: Quasi-metric spaces. Given the particularities found for the set of linear functionals defined over an asymmetric normed spaces, such as that they have no linear structure in general, our goal was to determine how and when the aforementioned structure could be found. In other words, our goal was to determine which degree of asymmetry was allowed for a space for the set of linear functionals to have linear structure. This lead us to the main definition of Chapter 4, which is that of index of symmetry for asymmetric normed spaces.

In terms of its definition, the index of symmetry of an asymmetric normed space is simply a number between 0 and 1 that quantifies the asymmetry of the space, or in other words, how far away is the unit ball of being symmetrical. Starting from this it became evident that the case where that index is equal to zero was precisely the case of interest: Whenever that index is greater than 0 it is easily proven that the resulting topology for the space is the same obtained for some norm. Moreover, this index is equal to 1 only for normed spaces. Before noticing these consequences, the name "index of symmetry" was still not adopted, but this name arose naturally when this was clear: The closer to 1 this index is, the less difference there is between the unit ball of the space and the unit ball of a certain normed space.

Once it becomes evident that the case of interest were those spaces whose index of symmetry is equal to 0 (which we could understand as "completely asymmetrical spaces"), we focus our attention in analyzing some examples corresponding to natural asymmetrical versions of $\mathbb{R}^d$, ℓ^p, and L^p. Using these constructions to comprehend the behaviour of linear functionals over them, an important observation was that in finite dimension there necessarily exists a point different from the origin whose asymmetric norm is equal to zero. Let us point out, that this is far from being the case of infinite dimensions. In other words, it is possible for an asymmetric normed space to have index of symmetry equal to zero while keeping the property of positivity of the norm for any point except for the origin.

Therefore, it becomes clear that as long as the study of the index of symmetry is concerned, there were two possibilities, which are resumed on the topology being T_1 or not. This distinction was made since when analizing the aforementioned examples of asymmetric normed spaces, the property of having a dual with linear structure and having a T_1 topology arose. This naturally lead us to the mains results of the chapter, which give a characterization for the linear structure of dual spaces and spaces of linear operators in terms of the index of symmetry.

Even if this last chapter seems to be completely disconected from the rest of the work, when we observe that similar results could be obtained in the framework of quasi-metric spaces and the sets of semi-Lipschitz functions defined over them (real valued and with values in another quasi-metric space) we see that there is an ultimate connection between both parts.

The choice of dealing with the case of asymmetric normed spaces instead of quasi-metric spaces was a choice of convenience: It was simpler to get a better understanding of the effect of asymmetry when the linear structure is also present. Using this as an starting point, we can develop these ideas in a more general framework.

To finish this work, we detail some ideas for future work starting from the concepts reviewed above. One of the main reasons that inspired the development of Chapters 1 and 2 was the question of whether or not the Lipschitz-free spaces of $\mathbb{R}^d$ are isometric in general. The identifications made for those spaces could serve as a way to answer that question in full generality, but it does not seem as an easy task, since those description of the spaces are made using quotiens of L^1 spaces, each of them of functions with values in spaces of different dimensions. A more detailed study of these spaces is needed in order to, in the best case, use them to that end. As for Chapter 3, as it was already mentioned, the results are still incomplete. It would be ideal to have the same results, but valid for any norm on $\mathbb{R}^d$. As of the moment of writing this documents, the various attempts made in order to do this have failed, but the fact that the results of Chapter 1 are valid no matter the norm shows no reason to believe that the choice of the norm in Chapter 3 cannot be changed. Finally, Chapter 4 is the one that gives more possibilities for future work. In this case we can go from studying even in more detail the case where the index of symmetry is equal to zero to obtain further properties for the structure of its dual spaces, changes of the value of this index when passing through isomorphisms, finding a relation between this index and the Banach-Mazur distance, etc., to center our efforts in the adaptation of the results already obtained to quasi-metric spaces, and its applications in the analysis of the spaces of Lipschitz spaces over then and the so-called semi-Lipschitz free spaces.

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