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THE ROLE OF PRIMORDIAL SYMMETRIES IN THE DETERMINATION OF THE NON-GAUSSIAN PROPERTIES OF OUR UNIVERSE

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS, MENCIÓN FÍSICA

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In this thesis, we show how future cosmological observations can provide us access to unveil the mechanism responsible for the origin of the structure in our universe. The primordial perturbations sourced by quantum vacuum fluctuations in the early universe, are stretched to cosmological scales during inflation, thus generating the seeds for the structure formation. Observations of the temperature anisotropies in the cosmic microwave background have revealed that the primordial spectrum of perturbations is adiabatic and almost scale-invariant. These non-trivial features are remarkably in agreement with the predictions of cosmic inflation. Furthermore, observations are consistent with a highly Gaussian distribution of primordial perturbations. On the other hand, a generic prediction of cosmic inflation is the production of primordial gravitational waves that can later be detected as polarization in the cosmic microwave background. Nevertheless, deviations from Gaussianity —known as non-Gaussianities— nor primordial gravitational waves have been detected yet. Both observables carry exquisite information about the mechanism that nature chose to produce the structures that we see today in the universe.

Throughout this work, we face problems related to the aforementioned observables. We study the production of primordial local non-Gaussianity in canonical single-field models of inflation, and we delve into the implications of detecting sizable primordial gravitational waves on the attempts of incorporating cosmic inflation in a quantum theory of gravitation. In particular, by using a new class of symmetries, we derive a novel soft-theorem for the 3-point function that generalizes the well-known Maldacena's consistency relation, being valid for attractor and non-attractor models of inflation. This relation allows one to derive the well-known violation of the consistency relation found in ultra slow-roll, where the curvature perturbations grow on super-horizon scales. Then, we study the observability of primordial local non-Gaussianity, where we show that, independently of whether inflation is attractor or non-attractor, the size of the observable primordial local non-Gaussianity vanishes. Also, we show how to overcome the well-known swampland distance conjecture by taking advantage that it only applies to geodesic distances. We build a multi-field model of inflation, characterized for having a non-geodesic inflationary trajectory, that allows us to establish a relation between geodesic and non-geodesic field displacements. Together with the Lyth bound, such relation allows us to explain a signal of primordial gravitational waves, with large values of the tensor-to-scalar ratio r, without invoking super-Planckian field displacements. We also show how to constrain the parameter space of multi-field models by extending the distance conjecture to the theory of perturbations and combining it with non-Gaussianity bounds.

Our results allow claiming that a detection of local primordial non-Gaussianity would rule out single-field canonical models of inflation and that detection of sizable primordial gravitational waves would not necessarily imply super-Planckian inflaton field displacements, thus allowing inflation to be incorporated into a theory of quantum gravity.

EL ROL DE LAS SIMETRÍAS PRIMORDIALES EN LA DETERMINACIÓN DE LAS PROPIEDADES NO-GAUSSIANAS DE NUESTRO UNIVERSO

En esta tesis mostramos como futuras observaciones cosmológicas pueden darnos acceso a develar el mecanismo responsable del origen de la estructura en nuestro universo. Las perturbaciones primordiales creadas por fluctuaciones cuánticas del vacío en el universo temprano, son estiradas a escalas cosmológicas durante inflación, generando así las semillas para la formación de estructuras. Las observaciones de las anisotropías de temperatura en el fondo de radiación cósmico, nos han revelado que el espectro de perturbaciones primordiales es adiabático y casi invariante de escala. Estas características no triviales están notablemente en acuerdo con las predicciones de inflación. Más aún, las observaciones son consistentes con una distribución de perturbaciones primordiales altamente Gaussiana. Por otro lado, una predicción genérica de la inflación cósmica es la producción de ondas gravitacionales primordiales que posteriormente pueden ser detectadas como una polarización en el fondo de radiación cósmico. Sin embargo, todavía no se han detectado desviaciones a la Gaussianidad (no-Gaussianidades) ni ondas gravitacionales primordiales. Ambos observables, contienen información exquisita del mecanismo que la naturaleza escogió para producir las estructuras que hoy día vemos en el universo.

A lo largo de este trabajo estudiamos la producción de no-Gaussianidad primordial local en modelos canónicos de inflación con un campo, y ahondamos en las implicancias que tiene una posible detección de ondas gravitacionales primordiales de gran amplitud, en los intentos de incorporar la inflación cósmica en una teoría cuántica de la gravitación. En particular, usando una nueva clase de simetrías, derivamos un novedoso teorema suave para la función de tres puntos, el cuál generaliza la bien conocida relación de consistencia de Maldacena, siendo válido para modelos atractores y no atractores de inflación. Esta relación permite derivar la conocida violación de la relación de consistencia encontrada en *ultra slow-roll*, en donde las perturbaciones de curvatura crecen en escalas super horizonte. Luego, estudiamos la observabilidad de las no-Gaussianidades, en donde mostramos que, independiente si inflación es atractora o no, la magnitud de la no-Gaussianidad primordial local desaparece. Adicionalmente, mostramos como evadir la bien conocida conjetura de la distancia del *swampland* al aprovechar que ésta solo aplica a distancias geodésicas. Construimos un modelo de inflación con múltiples campos, caracterizado por tener una trayectoria inflacionaria no geodésica la cuál permite establecer una relación entre desplazamientos geodésicos y no geodésicos. Tal relación en conjunto con la cota de Lyth, permite explicar una señal de ondas gravitacionales primordiales de gran amplitud, sin invocar desplazamientos super Planckianos del campo. También mostramos como restringir el espacio de parámetros de modelos con múltiples campos al extender la conjetura de la distancia a la teoría de perturbaciones, y combinándola con cotas en las no-Gaussianidades.

Nuestros resultados permiten afirmar que una detección de no-Gaussianidad primordial local descartaría a los modelos canónicos de inflación con un campo, y que una detección de ondas gravitaciones primordiales de gran amplitud, no necesariamente implica desplazamientos super Planckianos del campo, permitiendo así que inflación pueda ser incorporada en una teoría de gravedad cuántica.

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Contents

Introduction

The claim that we can describe the universe rationally is undoubtedly ambitious. Having a cosmos rather than chaos is an idea that had its origins in ancient greek natural philosophy. Back then, symmetry considerations such as beauty, harmony, and unity have implicitly guided the attempts to develop theories of nature. For instance, Platonic solids were used to describe the four natural elements given the proportions they contain and the beauty of their forms. Moreover, in those days, the shape of the universe was thought to be that of a dodecahedron. Even after that, scientists like Kepler used these figures as essential ingredients in the physical description of planetary architecture.

The remarkable progress that occurred in cosmology in the last decades owes as much to the observations as to the theory that has been developed. Since the first detection of anisotropies in the cosmic microwave background (CMB) by the COBE [1] satellite and its later refinements by the WMAP [2] and PLANCK [3] satellites, cosmology has evolved to become a high precision science with the capacity to explain plenty of observations. Although technological advances have allowed us to refine observations, bringing us access to exquisite details, in the theoretical developments, we continue employing the same concepts of symmetry used by the ancients, which have also given us a deeper understanding of nature.

Thanks to theoretical and observational efforts, it has been possible to establish the ΛCDM standard model for cosmology. Through this model, we have found that our universe originated approximately 13.8 billion years ago, that it is homogeneous, isotropic, and extremely flat on large scales. Its composition consists of 5% baryonic matter, and 25% cold dark matter (CDM), the nature of which is still a mystery. Furthermore, recently our universe began to experience an accelerated expansion phase, produced by a type of dark energy, parameterized by a cosmological constant Λ, which occupies the remaining 70% of the total energy budget. The existence of structure in the universe necessarily reveals to us that, at a certain scale, the universe ceases to be homogeneous and isotropic. Galaxies are grouped into galaxy clusters, and these, in turn, are organized into galaxy superclusters, which are parts of a network of filaments known as the large-scale structure (LSS) of the universe. The ΛCDM model requires that the large-scale structure of the universe be the result of small inhomogeneities that had their origin in the early universe. These inhomogeneities evolved as the universe expanded to become, through gravitational collapse, stars, galaxies, and then galaxy clusters.

Nowadays, we are privileged to be living in an extraordinary epoch on the understanding of our universe without any doubt. During the last five years, two major breakthroughs in physics had happened; the first direct detection of gravitational waves by the LIGO interferometer [4], and the EHT collaboration released the very first image of a black hole [5]. The theoretical ground of both phenomena relies on the notion of the geometrical description of spacetime through a metric tensor, whose dynamics is well described by Einstein's equations. Black holes are typically *background* solutions of these equations, while gravitational waves are *perturbations* in the curvature of spacetime that propagates on itself. These milestones, aside from opening new ways of observing the universe, have given us the possibility to test our current notions about the nature of spacetime and gravity in extreme conditions. However, we still know nothing about the quantum behavior of gravity. Besides all the theoretical issues that arise when building a quantum theory of gravitation, we also have to consider that there are not many phenomenological scenarios where this behavior may be tested.

Fortunately, early universe cosmology give us a chance to test at least perturbative effects of quantum gravity. In this context, the paradigm[1]of cosmic inflation [7–12], provide us sensitivity to genuine quantum gravitational effects that may have been imprinted on various cosmological observables. On the one hand, a general prediction of inflation is the generation of primordial gravitational waves sourced by quantum vacuum fluctuations. These waves imprint a polarization in the CMB known as B-modes. Therefore, its detection would confirm the treatment of gravity as a quantum phenomenon. On the other hand, if the statistical profile of temperature fluctuations in the CMB it deviates from an exact Gaussian distribution, therefore it is possible that such deviation —called non-Gaussianity— was sourced by additional degrees of freedom that typically arise in theories of quantum gravity. Nevertheless, what both effects mentioned above have in common, is that they are sensitive to ultra-violet (UV) physics and arise as a consequence of the evolution of different kinds of perturbations.

Perturbations are at the core of modern cosmology, since all the inhomogeneities that we observe in the universe come from a single primordial perturbation that evolved through the different stages and scales of the universe's history. Nowadays, based on strong theoretical and phenomenological evidence, this primordial perturbation is believed to be a quantum fluctuation present in the early universe, which was stretched by inflation from microphysical to cosmological scales. On large scales, these vacuum fluctuations become classical and induce energy density fluctuations, which therefore generate the temperature anisotropies that we observe in the cosmic microwave background and the matter fluctuations that we observe in the large-scale structure of the universe.

In order to characterize and connect perturbations with observations we make use of their correlation functions. Most of cosmological observations are made by doing correlations of different quantities. Quite safely, we can say that the main goal of modern cosmology is to develop a consistent history of the universe that explains these correlations. An important and powerful tool that we have for this purpose are symmetries. It is widely known that the use of symmetries in physics has been tremendously fruitful, it is enough to mention the role of gauge symmetries in the construction of the standard model of particle physics to make this manifest. In cosmology, symmetries has been used to develop several theories. For instance, effective field theory approaches to inflation [13], dark energy [14] and large scale structure [15], to mention a few, have been built by demanding the breaking of some symmetry

[1]Here we are referring to inflation as a paradigm in the sense introduced by Kuhn in [6] rather than a single theory.

in the underlying theory. More to the point, the use of symmetries as a tool to compute or constrain correlation functions in inflation has been thoroughly studied, see [16–21] for an incomplete list.

Despite all our modern understanding and new observational results, of course, there are many open questions and problems in cosmology waiting to be addressed. For instance, we still know almost nothing about the nature of dark matter and dark energy, apparently, we do not know well the expansion rate of the universe, we do not know the statistics of primordial fluctuations and we still do not have a complete theory of its generation. In this thesis, we will address issues related to the last two. In particular we will study the generation and phenomenological implications of primordial (local) non-Gaussianities arising from the simplest —and most favored by data— models of inflation, namely, canonical single-field models. Also, we will study the consequences of having a possible measurement of primordial gravitational waves in the near future, on the construction of a theory of inflation consistent with the requirements demanded by a theory of quantum gravity. In order to achieve this, we will use symmetries in a twofold way. The generation of non-Gaussianities will be done by exploiting (explicitly) a new class of symmetries to compute correlation functions. The mechanism we will use for inflation to make sense in a quantum theory of gravity, relies in a theory with high degree of symmetry (implicitly).

This thesis is organized as follows: In Chapter 1 we review the theory of cosmological perturbations. We begin by defining the backgrounds on which perturbations evolve around, studying how the matter content of the universe affects its dynamics. Then, we present one of the causality problems that give rise to the idea of cosmic inflation, and we introduce the dynamics of the homogeneous scalar field(s). Subsequently, the treatment of relativistic perturbations in an expanding universe is presented, we show the issues related with the gauge-invariance and we introduce a few gauge-invariant perturbations. Finally, we move to the perturbations generated during inflation, where we show how to quantize them and compute statistical quantities such as the power spectrum and the bispectrum. Additionally, we sketch the computation of the tree-point correlation function in single-field inflation, and we end by introducing the primordial local non-Gaussianity.

In Chapter 2 we introduce the very first consistency relation that gives account of the bispectrum in ultra slow-roll models of inflation and generalizes the well-known Maldacena's non-Gaussian consistency relation, as shown in [22]. We begin by introducing a new symmetry of inflationary backgrounds that absorbs a super-horizon growing long-mode in spacetime coordinates, allowing us to compute correlation functions in a modified background. Then, we show how to derive the consistency relation that recovers the bispectrum for ultra slow-roll models. Finally, we discuss the reliability of our results.

In Chapter 3, based on [23], we study the production of observable primordial local non-Gaussianity in two opposite regimes of canonical single-field inflation: attractor and non attractor. We use the so-called Conformal Fermi Coordinates in order to compute genuinely gauge-invariant correlation functions measured by inertial observers in a perturbed FLRW spacetime. The main claim of this chapter is that, independently of whether inflation is attractor or non-attractor, the size of the *observable* primordial local non-Gaussianity vanishes. In appendix A we include details of some computations made in this chapter.

In Chapter 4, we present a novel soft-theorem for the three point correlation function of single-field inflation based on [24]. We found a new class of symmetry of a perturbed FLRW spacetime constituted by a spacetime diffeomorphism and field redefinitions. Then, we use the fact that a super-horizon long-wavelength mode is adiabatic during inflation, no matter the background is attractor or not, to absorb it in a transformation of coordinates. Therefore we compute correlation functions in this new coordinates, which subsequently give rises to a robust generalized consistency relation valid at all orders in slow-roll parameters, no matter the size of them. In addition, we show that it is always possible to write the perturbed metric in conformal Fermi coordinates, independently of whether the inflationary background is attractor or non-attractor, allowing us to compute the physical squeezed limit of the bispectrum as observed by local inertial observers. We show that slow-roll (attractor) inflation predicts vanishing local non-Gaussianity to all orders in slow-roll parameters (a result previously understood up to first order in slow-roll). In addition, we find that in the absence of sharp transitions (from attractor and non-attractor regimes) observable local non-Gaussianity is generically suppressed. Our results show that large local non-Gaussianity is not a generic consequence of non-attractor backgrounds. In appendix B we make explicit the computation of the temporal part of the diffeomorphism.

In Chapter 5, we study how low-energy bounds set by quantum gravity considerations, affect the implications of measuring a sizable signal of primordial gravitational waves from inflation as shown in [25]. First, we review how the upper bound —also known as the distance conjecture— emerges by considering spatial compactifications in string theory and its relation with the Lyth bound. Since the distance conjecture applies to *geodesic* distances and the Lyth bound to the inflationary trajectory, we engineer a mechanism within multi-field inflation that allows to establish a relation between geodesic and non-geodesic distances. Considering that the inflationary trajectory in multi-field inflation typically follows a non-geodesic path in the field space, we show that it is possible to evade the distance conjecture. Also, we show how to constrain the parameter space of multi-field inflation by extending the distance conjecture to the theory of perturbations and considering bounds coming from non-Gaussianities. In appendices C and D we detail aspects of the coordinates and the geometry of the field space used in the derivation of our results.

Finally we outline the general conclusions of this work and we point out future research directions.

Conventions

Along this thesis, we will be working in a system of units where the speed of light in vacuum is set to $c = 1$, as well as the reduced Planck constant $\hbar = 1$. G stands for Newton's universal gravitational constant.

The Fourier transform and its inverse are defined as

$$f(x) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} f_k(k),$$

$$f_k(k) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} f(x).$$

The signature of the metric tensor will be mostly positive, $(-, +, +, +)$.

Chapter 1

Cosmological Perturbation Theory

Nowadays, it is widely known that the inhomogeneities —such as galaxy clusters— that form the large-scale structure of the universe have grown from small initial perturbations via gravitational instabilities. A natural way to provide a generation mechanism for these perturbations, is to consider that they were created on microscopic scales as quantum vacuum fluctuations. The evolution of these perturbations is such that their wavelengths become much larger than the horizon (Hubble radius) for an extended period of cosmic evolution via an accelerated expansion of the background geometry. Eventually, they re-enter the horizon and evolve through different epochs and scales of the history of the universe. For an accurate description of the generation and evolution of these fluctuations, quantum mechanics and gravity have to be taken into account. This is because, we are interested in computing the correlation functions of quantum fields on a dynamical spacetime. The suitable framework that allows us to develop such a purpose is the theory of cosmological perturbations.

In this context, correlation functions are of crucial importance, since they allow us to do phenomenology. Observational data have shown us that the primordial spectrum of inhomogeneities is adiabatic and almost scale-invariant. These non-trivial features are remarkably in agreement with the predictions of cosmic inflation. On the one hand, by construction, the dynamical degree of freedom of inflation, *i.e.*, the primordial curvature perturbation, is adiabatic. On the other hand, the introduction of this quantum field forces the geometry of the expanding universe to be a quasi de Sitter one. And therefore, by exploiting the dilation isometry of this spacetime, one immediately constrains the shape of the two-point correlation function —and the power spectrum— to be nearly scale-invariant. These facts are simply impressive, since they are not the main reason why inflation was actually introduced, they just come off the construction.

In the following, we will give an overview of the theory of cosmological perturbations, paying special attention to cosmic inflation. The main goal of this chapter is to introduce the important quantities and fix the notation that will be used in the following chapters.

1.1 Background Dynamics

To properly understand the behavior of the perturbations, first of all, it is essential to introduce the backgrounds on which they evolve around. In order to do so, it is useful to establish and make use of the cosmological principle [26]. This principle, basically tell us that the properties of the universe are the same for all the observers. Which means that, at large scales, the distribution of matter in the universe is homogeneous and isotropic, bringing us a high amount of symmetry to describe it. These statements, which in principle may sound whimsical, are strongly supported by observations [27, 28]. Moreover, observations tell us that the universe is in expansion[1] and its geometry is flat [31].

The spacetime which encodes all the aforementioned properties, is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric[2]

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{1.1}$$

where $d\mathbf{x}^2 \equiv \delta_{ij}dx^i dx^j$ and the function $a(t)$ is the scale factor which inform us about the expansion history of the universe. The dynamics of the spacetime is governed by Einstein field equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor describing the geometry of the universe, $R_{\mu\nu}$ is the Ricci tensor and $R = R^{\sigma}{}_{\sigma}$ is the Ricci scalar. The Ricci tensor is defined as $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$, where $R^{\sigma}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho}$ is the Riemann curvature tensor, which is built from the Christoffel connection $\Gamma^{\lambda}_{\mu\nu}$, defined in terms of the metric tensor $g_{\mu\nu}$ as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{1.3}$$

In the Right hand side of (1.2), $T_{\mu\nu}$ is the energy-momentum tensor which give us account about the matter (or energetic) content of the universe. To be consistent with the symmetries of the spacetime, the energy-momentum tensor has to be diagonal and its spatial components must be equal, a simple choice for this requirements is the perfect fluid, characterized by a time-dependent density $\rho(t)$ and pressure $P(t)$, *i.e.*, $T^{\mu}_{\nu} = \mathrm{diag}(\rho, -P, -P, -P)$. The form of equations (1.2) for the perfect fluid with the ansatz (1.1) are known as the Friedmann equations, their 00 and ij components respectively are

$$H^2 = \frac{8\pi G}{3}\rho, \quad \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P). \tag{1.4}$$

Here $H(t) \equiv \dot{a}/a$ is the Hubble expansion rate of the universe, which unlike the scale factor is a measurable quantity. In general, one will be interested in finding how the spacetime —by means of $a(t)$ or $H(t)$— reacts to some specific energetic content. Another useful equation, for this purpose, can be obtained combining both Friedmann equations or equivalently using the Bianchi identities for the conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, yielding

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{1.5}$$

[1] Currently there is an ongoing debate about the expansion rate of the universe H_0, dubbed as the Hubble tension, see for instance [29, 30]

[2] In this context we are using metric tensor, $g_{\mu\nu}$ and line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ interchangeably.

This is the so called continuity equation, which can be easily integrated after specifying an equation of state for the perfect fluid, for instance with the parametrization $P = w\rho$, with $w =$ constant one obtains

$$\rho \sim a^{-3(1+w)}, \tag{1.6}$$

and after inserting this result in the Friedmann equations, we find the following behavior for the scale factor

$$a(t) \propto \begin{cases} t^{\frac{2}{3}(1+\omega)} & \text{if } w \neq -1, \\ e^{Ht} & \text{if } w = -1. \end{cases} \tag{1.7}$$

The cases $w = 0$ and $w = 1/3$ recover the well known results for a matter and radiation dominated universe respectively. Here, when $t = 0$ the scale factor takes the value $a(0) = 0$, which is known as the Big Bang singularity or simply the Big Bang. The case $w = -1$ corresponds to a constant energy density where the universe experiences an exponential expansion and the geometry of the spacetime is called de Sitter space. This is useful to describe a dark energy dominated era as well as the period of cosmic inflation.

For a long time, it was believed that the Big Bang singularity marks the beginning of the universe; due to the incompatibilities with observations and the problems that this entails, now we know this cannot be the case. It is worth mentioning one of those problems to see the issues explicitly and also to present a few useful concepts. Let us begin by introducing the conformal time τ defined as

$$\tau = \int \frac{dt}{a(t)}, \tag{1.8}$$

which allow us to rewrite the FLRW metric as one conformal to Minkowski spacetime

$$ds^2 = a^2(\tau)(-d\tau^2 + d\mathbf{x}^2). \tag{1.9}$$

Writing the metric in this way is quite convenient to analyze the causal structure of the expanding universe, since like in flat space, null geodesics can be represented by straight lines at 45° in the $\tau - \mathbf{x}$ plane, and therefore the maximum distance that light can travel in an interval of time, is simply given by the particle horizon,

$$\chi_{ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)}. \tag{1.10}$$

Another important quantity, useful to study the causal structure of FLRW is the comoving Hubble radius[3] $(aH)^{-1}$, which for a matter or radiation dominated universe is always an increasing function. Then, from the Big Bang onwards, the horizon increases as the universe expands.

Despite (1.9) shares the same causal structure of Minkowski spacetime, there exist an important difference between them. In Minkowski spacetime, every past light cone of an event always will intersect another one at some finite time; therefore, any two separated regions in space always have been in causal contact at some time in the past. On the other hand, for a universe dominated by radiation or matter, the FLRW metric has a finite lifetime due to the Big-Bang singularity. This imposes a time where past light cones of events cannot

[3]We will refer to this quantity simply as the horizon.

be extended beyond this point, which means that in a radiation or matter-dominated universe there are spatial regions causally disconnected.

In particular, the earliest physical event in the history of our universe that we can observe, is the epoch of recombination, described by the process $p + e^- \to H + \gamma$, which occurred about 380 000 years after the Big Bang. We are able to detect the remnants of this epoch as the photons of the CMB radiation, which exhibits an almost perfect black body spectrum corresponding to an isotropic and homogeneous temperature of $T = 2.725$K with an accuracy of one part in ten thousand. If we denote the time where recombination occurs as t_{rec}, we directly realize that the CMB defines a spacelike slide in the $\tau - \mathbf{x}$ plane in which many points are causally disconnected between the Big Bang at $t = 0$ and t_{rec}. In particular, the intersection of our current past light cone with the CMB spacelike surface, corresponds to two opposite points in the observed CMB. These points, share the same temperature up to deviations of order 10^{-5}, but the Big Bang singularity does not allow those points to have been in causal contact. Moreover, it is possible to show [32] that there are $\mathcal{O}(10^4)$ regions in the sky that share the same issue. Then, the natural question is: if there wasn't enough time between the Big Bang at $t = 0$ and the CMB formation at t_{rec} for these regions to communicate, why do they look so similar with such high precision? The above, is the so called *horizon problem*[4]. Facing this situation, we have two options: if one tries to solve it while maintaining the Big Bang singularity, then we need to fine-tune the initial conditions for these 10^4 regions to be the same on super-causal scales at the 10^{-5} level, which sounds quite unnatural. Or, we can consider a period of expansion in the early universe in order to avoid the singularity and naturally allow causal contact for these regions. This last option is what cosmic inflation actually does.

In order to avoid the Big-Bang singularity and allow causal contact for all the CMB regions, we can consider a phase in the early universe in which the comoving Hubble radius decreases as the universe expands, *i.e.,*

$$\frac{d}{dt}(aH)^{-1} < 0. \tag{1.11}$$

In this scenario, the integral (1.10) is dominated by the lower limit, and the Big Bang singularity now is pushed to infinite negative conformal time, $\tau_i \to -\infty$. Therefore, now we have sufficient enough conformal time to allow the separated regions of the CMB to communicate. By considering this period, now the conformal time $\tau = 0$ is not anymore the initial singularity, instead it becomes a transition point known as reheating.

The expression (1.11) can be re arranged as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} < 1, \tag{1.12}$$

which implies a slowly-varying Hubble expansion rate. Also, we have introduced the first slow roll parameter ϵ to quantify this. Another useful way to express the condition (1.11) is the following

$$\ddot{a} > 0. \tag{1.13}$$

[4]There are another problems in cosmology related with the existence of the Big Bang singularity: the flatness and the monopole problems. Both are solved with the same solution proposed for the horizon problem.

Here, we realize that this period where the comoving Hubble radius decreases is equivalent to a period of accelerated expansion in the early universe, dubbed as cosmic inflation. As soon as inflation ends at $\tau = 0$, the reheating occurs and then the radiation dominated era begins.

The duration of inflation has to be at least sufficient to solve the horizon problem. To achieve this, it is necessary to keep ϵ small for a sufficient number of Hubble times. This requirement is satisfied by mantaining a second slow roll parameter small, namely

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H}, \quad |\eta| < 1. \tag{1.14}$$

Basically, η tell us how quickly ϵ evolves. In this case, the fractional change of ϵ per Hubble time is small and inflation persists.

To close this subsection, let us note that it is possible to insert (1.12) on the Friedmann (1.4) and the continuity (1.5) equations, and conclude that inflation requires a negative pressure $w < -\frac{1}{3}$, and a nearly constant energy density $\left|\frac{d\ln\rho}{d\ln a}\right| = 2\epsilon < 1$ for it to happen. Conventional matter sources cannot produce both conditions, therefore we need to introduce uncommon sources of matter so that inflation can occur.

1.2 Inflationary Backgrounds

Since we have concluded that a period of cosmic inflation in the early universe is capable of solving the horizon problem, now we should concern about incorporating the kind of matter that can satisfy the conditions for this to occur. The simplest candidate to realize inflation, is a cosmological constant Λ with energy momentum tensor $T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$ which describes a fluid with parameter of state $w = -1$ and constant energy density ρ. Nevertheless, if this is the case, inflation never ends and we cannot recover the universe in which we live in. Therefore we need to introduce a kind of matter which can mimic the cosmological constant and at the same time provide a mechanism to end inflation. In the following we will see that this is achieved by introducing a scalar field ϕ [33, 34].

1.2.1 Single-field inflation

For simplicity, let us begin with a single scalar field ϕ with canonical kinetic term and a potential V, minimally coupled to gravity on a FLRW background described by the action,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.15}$$

where we have introduced the Planck mass as $M_{\rm Pl}^2 = \frac{1}{8\pi G}$. In this context, the scalar field ϕ is dubbed as the inflaton, and by consistency with the symmetries of FLRW spacetime it is required that the background value of the inflaton only depends on time, $\phi = \phi(t)$. The corresponding energy momentum tensor of the inflaton is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\sigma\rho}\partial_\sigma\phi\partial_\rho\phi + V(\phi)\right]. \tag{1.16}$$

Their 00 and ij components can be matched to energy density and pressure respectively as

$$\rho = \frac{1}{2}\dot{\phi}^2 + V, \quad P = \frac{1}{2}\dot{\phi}^2 - V, \tag{1.17}$$

where it is direct to see that the parameter of state is

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{1.18}$$

Therefore, if the inflaton potential is dominant compared to its kinetic energy, the effective parameter of state corresponds to $w \approx -1$, then, the universe expands quasi-exponentially. In a while, we will see that this assumption follows naturally from the definition of inflation.

The equations of motion derived from the action (1.15), are the Friedmann equation and the Klein-Gordon equation for a curved spacetime, namely

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right], \tag{1.19}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{1.20}$$

Additionally, the continuity equation (1.5) in this setup is

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{\rm Pl}^2}, \tag{1.21}$$

which can be rewritten in terms of the scale factor with the help of equation (1.19) as

$$\frac{\ddot{a}}{a} = -\frac{(\dot{\phi}^2 - V)}{3M_{\rm Pl}^2}. \tag{1.22}$$

Recalling the condition for inflation (1.13), we then can conclude that, in order to have a period of inflation sourced by a theory described by the action (1.15) it is necessary that this satisfy

$$\dot{\phi}^2 < V. \tag{1.23}$$

As we anticipate, inflation requires that the potential energy of the scalar field dominate over its kinetic energy, this situation is called slow-roll inflation. By taking the time derivative of (1.23), we can simplify equation (1.20) as

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{1.24}$$

which has attractor solutions in the dynamical phase space $\{\phi, \dot{\phi}\}$. Additionally, (1.24) allows us to replace the dynamics of ϕ in terms of the derivative of the potential. Here, it is useful to introduce the potential slow roll parameters

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2, \quad \eta_V \equiv \frac{M_{\rm Pl}^2}{V}\frac{d^2V}{d\phi^2}, \tag{1.25}$$

which after considering (1.23), are related to (1.12) and (1.14) as

$$\epsilon = \epsilon_V, \quad \eta = 4\epsilon_V - 2\eta_V. \tag{1.26}$$

If $\epsilon_V, \eta_V \ll 1$, then the potential energy dominates, therefore we can say that the field is slowly rolling down the potential. This kind of model is the simplest realization of inflation, and until now is in good agreement with observational data [35].

There are situations where it is possible to consider an intermediate or previous stage to slow roll inflation in which the potential is exactly flat. During this period, known as ultra-slow roll inflation, the equation of motion for the inflaton (1.20) is

$$\ddot{\phi} + 3H\dot{\phi} = 0, \tag{1.27}$$

which has non-attractor solutions. Since $\dot{\phi} \propto a^{-3}$, the slow roll parameters in this model behaves as

$$\epsilon \propto a^{-6}, \quad \eta \simeq -6. \tag{1.28}$$

In this non-attractor phase of inflation, the second slow roll parameter is characterized for having a large value. This kind of behavior only has physical sense, if there is a transition between the non-attractor and the attractor phases of inflation. Some, supergravity inspired models of inflation can provide such scenario [36].

To conclude this subsection, let us briefly mention about the amount of inflation. During inflation, the universe has expanded a number N of e-folds defined by

$$N = \int_t^{t_{end}} dtH. \tag{1.29}$$

In single field slow roll inflation, the number of e-folds can be written in terms of the potential solely as

$$N \approx \frac{1}{M_{\text{Pl}}^2} \int_{\phi_i}^{\phi_{end}} \frac{V}{V'} d\phi. \tag{1.30}$$

In most theories of inflation it is necessary about $N \sim 40 - 60$ e-folds of expansion in order to solve causality problems and explain the homogeneity of the CMB.

Finally, inflation ends when the potential steepens and the inflaton acquires kinetic energy, then, this energy is transferred to the particles of the Standard Model through the process of reheating in which the inflaton decays. The particles produced by the decay of the inflaton will create a soup of particles which eventually will reach the thermal equilibrium at the reheating temperature.

1.2.2 Multi-field inflation

So far, we have seen that the inclusion of a scalar field can satisfy the conditions to provide a successful inflation period. Nevertheless, we have restricted our analysis considering only a single field driving inflation. Several reasons suggest that inflation is likely to be described in the context of theories beyond the standard model of particle physics, and that has to have UV-completion in a theory of quantum gravity. A common feature of those theories, like, for instance, string theory, is that they typically predict the existence of several degrees of freedom[5], providing the possibility of having multiple scalar fields. Furthermore, the existence of other degrees of freedom gives us an exquisite phenomenology to exploit [37].

In the following, we will see that several scalar fields can also produce a successful cosmic inflation period.

Let us consider a set of scalar fields minimally coupled to Einstein's gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R - \frac{1}{2}g^{\mu\nu}\gamma_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi)\right]. \tag{1.31}$$

Here ϕ^a, with $a = 1, \ldots, N$, are scalar fields spanning a $N-$dimensional scalar manifold $\mathcal{M}_\phi$ equipped with a scalar sigma model metric $\gamma_{ab}(\phi)$. The Friedmann and the continuity equations emerging from (1.31) are the same as those of the single-field case, namely

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0^a)\right], \tag{1.32}$$

$$\dot{H} = -\frac{\dot{\phi}_0^2}{2M_{\rm Pl}^2}, \tag{1.33}$$

where $\dot{\phi}_0^2 \equiv \gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b$ with $\phi_0^a = \phi_0^a(t)$ being the background value of ϕ^a. The Klein-Gordon equation for the scalar fields can be found by varying the action respect to ϕ^a, yielding

$$D_t\dot{\phi}_0^a + 3H\dot{\phi}^a + \gamma^{ab}V_b = 0, \tag{1.34}$$

where $V_b \equiv \partial V/\partial\phi_0^b$ and $D_tX^a \equiv \dot{X}^a + \Gamma^a_{bc}X^b\dot{\phi}_0^a$, is a time covariant derivative in the field-space with the usual Christoffel symbols Γ^a_{bc} built from the field space metric γ_{ab}. It is essential to mention that the fields ϕ^a plays the role of coordinates on $\mathcal{M}_\phi$; therefore, we are free to choose or transform to a convenient basis to continue working. A convenient choice is the local orthogonal frame,

$$e_I^a e_J^b\gamma_{ab} = \delta_{IJ}, \quad e_I^a e_J^b\delta^{IJ} = \gamma^{ab}, \tag{1.35}$$

where the veilbein $e_I^a(t)$ maps from an arbitrary basis denoted by the index a to an orthogonal one denoted by the subscript I. A useful orthogonal basis is the kinematic basis, which is set along and perpendicular to the trajectory of ϕ^a in the field-space. It is possible to define an unitary tangent vector

$$T^a \equiv \frac{\dot{\phi}_0^a}{\dot{\phi}_0}, \tag{1.36}$$

which points along the direction of the trajectory of ϕ^a. Additionally, an orthonormal vector to the trajectory, N^a, is introduced such that $\gamma_{ab}T^aN^b = 0$. Naturally, N^a is proportional to the time covariant derivative of T^a, more concretely

$$D_tT^a = -\Omega N^a, \tag{1.37}$$

[5]For instance, in string theory, after compactifying spatial dimensions one end up with a moduli described by a multi-scalar field action, we will return to this point in detail in Chapter 5.

where Ω is an angular velocity parametrizing the rate of bending of the trajectory. Now, we can project equation (1.34) along T^a and N^a, yielding respectively,

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V_\phi = 0, \tag{1.38}$$

$$\Omega = \frac{V_N}{\dot{\phi}_0}, \tag{1.39}$$

where $V_\phi \equiv T^a V_a$ and $V_N \equiv N^a V_a$ are the projections of the potential derivative onto the tangential and perpendicular direction to the trajectory respectively. Notice that the background equation (1.38) is precisely the same as that in single field inflation (1.20), this point will be important when we discuss perturbations. The second equation (1.39) give us the value of Ω in terms of the slope of the potential along the normal direction N^a.

The slow roll conditions and the relation between the Hubble[6] and potential slow roll parameters is a little subtle in multi-field setups. In single field inflation one encounters that $\epsilon = \epsilon_V$ as indicated in (1.26), but this equivalence no longer holds in multi-field, because the inflationary trajectory does not necessarily align with the gradient flow of the potential. To appreciate this, first notice that the definition of the potential first slow roll parameter in multi-field is

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\frac{V^a V_a}{V^2}, \tag{1.40}$$

and the Hubble first slow roll parameter after using (1.33) becomes

$$\epsilon = \frac{\dot{\phi}_0^2}{2M_{\rm Pl}^2 H^2}. \tag{1.41}$$

By using the background equations of motion, and the previous definitions, it is possible to show [38, 39] that the relation between ϵ and ϵ_V in multi-field setups is given by

$$\epsilon_V = \epsilon\left(1 + \frac{\Omega^2}{9H^2}\right). \tag{1.42}$$

Recall that in order to have inflation one needs $\epsilon \ll 1$, but in multi-field scenarios if the gradient flow of the potential is large and the trajectory is orthogonal to it, then $\Omega^2/H^2 \gg 1$. That leads to situations where $\epsilon_V \gtrsim \mathcal{O}(1)$ and inflation keeps occurring.

In the case of the second slow roll parameter, one can define the following quantity

$$\eta^a \equiv -\frac{1}{H\dot{\phi}_0}\frac{D\dot{\phi}_0^a}{dt}, \tag{1.43}$$

which similarly to V, can also be projected along T^a and N^a as

$$\eta^a = \eta_\parallel T^a + \eta_\perp N^a, \tag{1.44}$$

$$\eta_\parallel \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0}, \quad \eta_\perp \equiv \frac{V_N}{H\dot{\phi}_0}. \tag{1.45}$$

[6]We are referring as Hubble slow roll parameters as those defined in terms of H as it is shown in (1.12) and (1.14).

Here, it is possible to show that in order to keep ϵ small for a sufficient amount of time it is equivalent to demand that $|\eta_{\parallel}| \ll 1$. A potential second slow roll parameter is defined as

$$\eta_V \equiv \text{min eigenvalue}(\nabla_a \nabla_b V)/V, \tag{1.46}$$

and its relation with the the parameter η is non-trivial and model dependent. However, we can say that similarly to (1.42), that there can be situations in which due to large values of Ω/H, one can have $|\eta| \ll 1$ while $|\eta_V| \gg 1$ without affecting the duration of inflation.

1.3 Relativistic Perturbations

Since we have already defined the background quantities that compose the homogeneous universe, now we are in place to continue with the perturbations that make it looks inhomogeneous at small scales. To do so, it is necessary to allow space and time dependence on the fields, making Einstein's equations more complicated. One of the major problems we will face, is the gauge redundancy of general relativity; the diffeomorphism invariance of the theory, *i.e.,* the freedom to choose the coordinates, does not allow a clear distinction between gauge artifacts and genuine perturbations. Therefore, in order to have gauge-invariant physical quantities that we later can connect to observables, it is necessary to define suitable combinations of perturbations and/or fix the gauge before doing computations [40].

The starting point is to perturb Einstein's equations as

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{1.47}$$

Let us first work with the left-hand side of the equation. The main idea is to consider small metric perturbations $\delta g_{\mu\nu}$ around the FLRW metric $\bar{g}_{\mu\nu}$ defined in (1.1). A general way to write the metric fluctuations in a FLRW spacetime is the following

$$ds^2 = -(1+2A)dt^2 + 2a(t)B_i dx^i dt + a^2(t)(\delta_{ij} + h_{ij}), \tag{1.48}$$

where $\delta g_{00} = -2A$ is a scalar perturbation, $\delta g_{0i} = a(t)B_i$ and $\delta g_{ij} = a^2(t)h_{ij}$ are vector and tensor perturbations that can be decomposed[7] as

$$B_i = \partial_i B + F_i, \tag{1.49}$$

$$h_{ij} = 2C\delta_{ij} + D_{ij}E + 2\partial_{(i}G_{j)} + \gamma_{ij}, \tag{1.50}$$

with $D_{ij} \equiv (\partial_i\partial_j - \frac{1}{3}\nabla^2)$ and $\partial_{(i}G_{j)} \equiv \frac{1}{2}(\partial_i G_j + \partial_j G_i)$. In order to ensure that the previous decomposition is irreducible, the vector and tensor perturbations are imposed to satisfy the following conditions

$$\partial^i F_i = 0, \tag{1.51}$$

$$\partial^i G_i = 0, \tag{1.52}$$

$$\partial^i \gamma_{ij} = 0, \tag{1.53}$$

$$\gamma^i{}_i = 0, \tag{1.54}$$

[7] This is known as the scalar-vector-tensor (SVT) decomposition. In representation theory this corresponds to decomposing the perturbations under the group of spatial rotations.

where the indices are lowered and raised with δ_{ij}. Therefore, we end up with four scalar perturbations A, B, C and E, two transverse vector perturbations F_i and G_i and one transverse-traceless tensor perturbation γ_{ij}. Notice that, as in any gauge theory, the number of components in the metric is larger than the genuine physical degrees of freedom. The advantage of using the SVT decomposition is the fact that the Einstein's equations for scalar, vectors and tensors do not mix at linear order and can therefore be treated separately.

Before moving to the issue of gauge invariance, let us work the right-hand side of (1.47). Firs, notice that the background energy momentum tensor, $\bar{T}_{\mu\nu}$, of the perfect fluid can be written as

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{P})\bar{u}_\mu \bar{u}_\nu + \bar{P}\bar{g}_{\mu\nu}, \tag{1.55}$$

with $\bar{u}_\mu = (-1, 0, 0, 0)$ being the timelike 4-velocity vector in the rest frame of the fluid, such that $\bar{u}_\mu \bar{u}^\mu = -1$. Now, if we introduce the perturbation $\delta T_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu}$, we need to consider the full energy-momentum tensor

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{1.56}$$

with

$$u_\mu = \bar{u}_\mu + \delta u_\mu, \tag{1.57}$$

$$\rho = \bar{\rho} + \delta\rho, \tag{1.58}$$

$$P = \bar{P} + \delta P, \tag{1.59}$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{1.60}$$

where the perturbed quantities with the δ are functions of the space and time. In order to get an explicit expression for the components of $\delta T_{\mu\nu}$,

$$\delta T_{\mu\nu} = (\delta\rho + \delta P)\bar{u}_\mu \bar{u}_\nu + (\bar{\rho} + \bar{P})(\delta u_\mu \bar{u}_\nu + \bar{u}_\mu \delta u_\nu) + \bar{P}\delta g_{\mu\nu} + \delta P \bar{g}_{\mu\nu}, \tag{1.61}$$

first we note that since $u_\mu u^\mu = -1$ then $\delta u_0 = \frac{1}{2}\delta g_{00}$. Therefore, it follows that

$$\delta u_0 = \delta u_0, \tag{1.62}$$

$$\delta u^i = a^{-2}(\delta u_i - \delta g_{i0}). \tag{1.63}$$

The previous relations allow us to write the components of $\delta T_{\mu\nu}$ as

$$\delta T_{00} = \delta\rho - \bar{\rho}\delta g_{00}, \tag{1.64}$$

$$\delta T_{0i} = \bar{P}\delta g_{0i} - (\bar{\rho} + \bar{P})\delta u_i, \tag{1.65}$$

$$\delta T_{ij} = \bar{P}\delta g_{ij} + a^2 \delta P \delta_{ij}. \tag{1.66}$$

Additionally, δu_i also can be decomposed into its scalar and vector parts as

$$\delta u_i = \partial_i \delta u + \delta u_i^V, \quad \partial^i \delta u_i^V = 0. \tag{1.67}$$

For completeness, let us mention that in order to match the degrees of freedom of δg_{ij} with those of δT_{ij}, it is possible to add an anisotropic stress tensor Π_{ij} to (1.66) which can be SVT decomposed as

$$\Pi_{ij} = D_{ij}\Pi + \partial_{(i}\Pi_{j)} + \hat{\Pi}_{ij}, \tag{1.68}$$

where

$$\partial^i \Pi_i = 0, \quad \partial^i \hat{\Pi}_{ij} = 0, \quad \delta^{ij}\hat{\Pi}_{ij} = 0. \tag{1.69}$$

However, the anisotropic stress will always be negligible in this thesis.

1.3.1 Gauge Transformations

Having decomposed the perturbations into their scalar, vector, and tensorial parts have the advantage that, at linear order, Einstein's equations do not mix these components between them. Nevertheless, the metric perturbations $\delta g_{\mu\nu}$ are not uniquely defined. They depend on the choice of coordinates or gauge. Different choices of coordinates can change the values of perturbations; moreover, in the worst scenario, a change of coordinates can induce fictitious perturbations known as gauge artifacts. In order to perturbations describe genuine physical observables, it is necessary to consider all possible perturbations in both the matter and the metric. Then, changing gauge allow us to exchange metric perturbations for matter perturbations and vice versa, keeping untouched the physics described by Einstein's equations.

Let us begin by considering an infinitesimal coordinate transformation of the form

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(t, \mathbf{x}), \tag{1.70}$$

where the spatial components of ξ_i can be decomposed as

$$\xi_i = \partial_i \xi + \xi_i^V, \quad \partial^i \xi_i^V = 0. \tag{1.71}$$

Under the previous transformation, the metric perturbations $\delta g_{\mu\nu}$ transform as

$$\delta g_{\mu\nu} \to \delta g'_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}, \tag{1.72}$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ, such that $\mathcal{L}_\xi \bar{g}_{\mu\nu} \equiv 2\nabla_{(\mu}\xi_{\nu)}$. The previous expression in components read as,

$$\delta g'_{00} = \delta g_{00} - 2\dot{\xi}_0, \tag{1.73}$$
$$\delta g'_{0i} = \delta g_{0i} - \dot{\xi}_i - \partial_i \xi_0 + 2H\xi_i, \tag{1.74}$$
$$\delta g'_{ij} = \delta g_{ij} - \partial_{(i}\xi_{j)} + 2a^2 H \xi_0 \delta_{ij}. \tag{1.75}$$

The matter perturbations $\delta T_{\mu\nu}$, under (1.70) transform as,

$$\delta T_{\mu\nu} \to \delta T'_{\mu\nu} = \delta T_{\mu\nu} - \xi_\sigma \nabla^\sigma \bar{T}_{\mu\nu} - \bar{T}_\mu{}^\sigma \nabla_\nu \xi_\sigma - \bar{T}_\nu{}^\sigma \nabla_\mu \xi_\sigma, \tag{1.76}$$

which in components is

$$\delta T'_{00} = \delta T_{00} + \dot{\bar{\rho}}\xi_0 + 2\bar{\rho}\dot{\xi}_0, \tag{1.77}$$
$$\delta T'_{0i} = \delta T_{0i} - \bar{P}(\dot{\xi}_i - 2H\xi_i) + \bar{\rho}\partial_i \xi_0, \tag{1.78}$$
$$\delta T'_{ij} = \delta T_{ij} + a^2 \xi_0 (\dot{\bar{P}} + 2H\bar{P})\delta_{ij} - \bar{P}\delta_{ij}. \tag{1.79}$$

Now we are in position to write how each component of the metric and the energy-momentum tensor transforms. Considering the SVT decomposition, the metric components transform as

$$A' = A + \dot{\xi}_0, \tag{1.80}$$

$$B' = B - \frac{1}{a}\left(\dot{\xi} + \xi_0 - 2H\xi\right), \tag{1.81}$$

$$C' = C + H\xi_0 - \frac{1}{3a^2}\nabla^2\xi, \tag{1.82}$$

$$E' = E - \frac{2}{a^2}\xi, \tag{1.83}$$

$$F_i' = F_i - \frac{1}{a}\left(\dot{\xi}_i^V - 2H\xi_i^V\right), \tag{1.84}$$

$$G_i' = G_i - \frac{1}{a^2}\xi_i^V, \tag{1.85}$$

$$\gamma_{ij}' = \gamma_{ij}. \tag{1.86}$$

And the components of the energy-momentum tensor as

$$\delta\rho' = \delta\rho + \dot{\bar{\rho}}\xi_0, \tag{1.87}$$

$$\delta P' = \delta P + \dot{\bar{P}}\xi_0, \tag{1.88}$$

$$\delta u' = \delta u - \xi_0, \tag{1.89}$$

$$\delta u_i'^V = \delta u_i^V, \tag{1.90}$$

$$\Pi_{ij}' = \Pi_{ij}. \tag{1.91}$$

It is possible to note that the quantities γ_{ij}, δu_i^V and Π_{ij} are gauge-invariant, not so the remaining ones. Therefore, we are interested in building gauge invariant combinations of the perturbations. For instance, it is possible to show [41] that the combination

$$\Phi = A + \frac{d}{dt}\left(aB - \frac{a^2}{2}\dot{E}\right), \tag{1.92}$$

$$\Psi = -C + \frac{1}{6}\nabla^2 E - aHB + \frac{a^2}{2}H\dot{E}, \tag{1.93}$$

$$\hat{\Phi}_i = F_i - a\dot{G}_i, \tag{1.94}$$

is gauge invariant. Here Φ, Ψ and $\hat{\Phi}_i$ are known as Bardeen variables, and they can be considered as genuine spacetime perturbations, since they cannot be removed by a gauge transformation. Similarly, one can define gauge-invariant matter perturbations of the form

$$\delta\rho_{\mathrm{gi}} \equiv \delta\rho + \dot{\bar{\rho}}\,\Xi, \tag{1.95}$$

$$\delta P_{\mathrm{gi}} \equiv \delta P + \dot{\bar{P}}\,\Xi, \tag{1.96}$$

$$\delta g_{\mathrm{gi}} \equiv \delta u + \Xi, \tag{1.97}$$

with $\Xi \equiv aB - \frac{a^2}{2}\dot{E}$. Additionally, sometimes it is possible to combine quantities from the metric and matter perturbations to build gauge-invariant quantities. In this context, there are two important gauge-invariant variables. One of them, is the curvature perturbation on uniform-density hypersurfaces, defined as

$$\zeta \equiv C - \frac{1}{6}\nabla^2 E - H\frac{\delta\rho}{\dot{\bar{\rho}}}. \tag{1.98}$$

The other one, is the comoving curvature perturbation defined as

$$\mathcal{R} \equiv C - \frac{1}{6}\nabla^2 E - H\delta u. \tag{1.99}$$

The last one, is particularly useful to study inflationary perturbations. From Einstein's equations, these two quantities are related by

$$\zeta = \mathcal{R} + \mathcal{O}\left(k^2/(aH)^2\right), \tag{1.100}$$

with k being the momentum in Fourier space.

To conclude the discussion of gauge transformations, let us mention that instead of using gauge-invariant quantities to remove gauge artifacts, it is also possible to choose a particular gauge and keep track of all the perturbations. A particularly important gauge choice for the study of inflation, is the *comoving-gauge*, in which ξ^μ is defined such that

$$\delta u' = 0, \quad E' = 0, \quad G_i' = 0. \tag{1.101}$$

And the non-vanishing perturbations are

$$C = \mathcal{R}, \quad A = \delta N, \quad B = \frac{1}{a}\left(\chi - \frac{1}{H}\mathcal{R}\right). \tag{1.102}$$

The meaning of δN and χ will be clarified in the context of inflationary perturbations.

1.3.2 Adiabatic and Isocurvature Perturbations

The comoving Hubble radius, allows us to separate perturbations that live inside and outside of it, in terms of their wave-number k. The ones that are inside, are known as sub-horizon and are defined by $k \gg aH$. The ones that are outside, are known as super-horizon, defined by $k \ll aH$. And finally, we can define the horizon-crossing as $k = aH$.

In general, one can consider a universe with several constituents and different energy densities ρ_i, therefore its perturbations are written as

$$\delta\rho_i(t, \mathbf{x}) = \rho_i(t, \mathbf{x}) - \bar{\rho}_i(t). \tag{1.103}$$

If we consider the limit of super-horizon scales, the perturbations are such that $\rho_i(t, \mathbf{x})$ satisfies the same background equations of motion respected by $\bar{\rho}_i(t)$. Namely, it is possible to write $\rho_i(t, \mathbf{x})$ as if it were a background solution of the form

$$\rho_i(t, \mathbf{x}) = \bar{\rho}_i(t + \delta t(t, \mathbf{x})) \tag{1.104}$$

where δt is a time-shift that tell us that at different Hubble patches the background is slightly different but is still solution of the background equations of motion. Comparing (1.103) with (1.104) one notes that

$$\frac{\delta\rho_i(t, \mathbf{x})}{\dot{\bar{\rho}}_i} = \delta t(t, \mathbf{x}). \tag{1.105}$$

It is interesting to note that the time shift δt is independent of the constituent, therefore it is possible to write

$$\frac{\delta\rho_i}{\dot{\bar{\rho}}_i} = \frac{\delta\rho_j}{\dot{\bar{\rho}}_j}, \quad i \neq j. \tag{1.106}$$

Perturbations that satisfy the previous relation are called *adiabatic*. In general, given a generic scalar quantity $\mathcal{X}$, its perturbations can be described by a unique perturbation in expansion with respect to the background as

$$H\delta t = H\frac{\delta\mathcal{X}}{\dot{\mathcal{X}}}. \tag{1.107}$$

Conversely, if two scalar quantities $\mathcal{X}$ and $\mathcal{Y}$ are such that

$$\frac{\delta\mathcal{X}}{\dot{\mathcal{X}}} \neq \frac{\delta\mathcal{Y}}{\dot{\mathcal{Y}}}, \tag{1.108}$$

they are called *isocurvature* perturbations. For a set of fluids the isocurvature perturbation are defined as

$$S_{ij} = 3H\left(\frac{\delta\rho_i}{\dot{\rho}_i} + \frac{\delta\rho_j}{\dot{\rho}_j},\right). \tag{1.109}$$

Finally, let us mention that with the definition of adiabaticity, it is possible to show that the comoving (1.99) and the constant-density (1.98) curvature perturbations are adiabatic.

1.4 Inflationary Perturbations

Up to now, when discussing inflation, we just have considered a scalar field ϕ and its classical evolution to provide a period of accelerated expansion in the early universe. However, inflation not only solves the otherwise finely tuned initial conditions like the horizon problem, but also provides the seed of subsequent structure formation in the universe.

As we mentioned at the beginning of this chapter, the inhomogeneous distribution of galaxies at small scales, as well as the anisotropies in the temperature fluctuations of the CMB had their origin in a primordial perturbation. Our best guess is that, subsequently, such a perturbation had a quantum-mechanical origin as a vacuum fluctuation during a primordial epoch in spacetime history.

In the following, we will see that considering quantum fluctuations $\delta\phi(t, \mathbf{x})$ around the homogeneous scalar field $\phi(t)$, which drives inflation, can provide the initial conditions for our universe. More concretely, we have to consider the entire scalar field written as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}), \tag{1.110}$$

where $\phi_0(t)$ is its background value, governed by the classical equations of Section 1.2. Then, as we have seen, since gravity talks to any component of the universe, the perturbation $\delta\phi(t, \mathbf{x})$ is intimately related to the metric fluctuations. In particular, with the curvature perturbations $\mathcal{R}$ and ζ. Therefore, a prediction of inflation is that all of the structure we see in the universe is a result of quantum-mechanical fluctuations $\delta\phi(t, \mathbf{x})$ mediated and amplified by gravity.

To begin the discussion of perturbations during inflation, it is useful to adopt the approach outlined in the seminal work of Maldacena [42]. First, let us write the perturbed line element using the ADM formalism as

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{1.111}$$

where N and N^i are the so called lapse and shift functions respectively, that also act as Lagrange multipliers. It is possible to write the action[8] of single-field inflation (1.15) for $\phi(t, \mathbf{x})$ using the previous metric as,

$$S = \frac{1}{2}\int d^4x\sqrt{h}\Big(N\,{}^{(3)}R - \frac{1}{N}(\bar{E}^2 - \bar{E}_{ij}\bar{E}^{ij}) - 2NV(\phi) + \frac{1}{N}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\Big), \tag{1.112}$$

where $\bar{E} = \bar{E}^i{}_i$, $\bar{E}_{ij} = NK_{ij}$ and K_{ij} is the extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right], \tag{1.113}$$

and ${}^{(3)}R$ is the 3-dimensional Ricci scalar defined on the hypersurfaces with induced metric h_{ij}, which is related with the 4-dimensional Ricci scalar as

$$R = {}^{(3)}R - K^2 + K^{ij}K_{ij}, \tag{1.114}$$

up to a boundary term. Now, we can vary the action (1.112) respect to N and N^i giving respectively,

$$ {}^{(3)}R + \frac{1}{N^2}(\bar{E}^2 - \bar{E}^{ij}\bar{E}^{ij}) - 2V(\phi) - \frac{1}{N^2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi = 0, \tag{1.115}$$

$$\nabla_i[N^{-1}(\bar{E}\delta^i_j - \bar{E}^i{}_j)] - 2(\dot{\phi} - N^j\partial_j\phi)\partial_i\phi = 0. \tag{1.116}$$

Before continuing, let us do a counting of degrees of freedom. As we have seen in Section 1.3, there are four scalar modes of the metric A, B, C and E together with the scalar field fluctuation $\delta\phi$ are in total five scalar degrees of freedom. There are two vector modes G_i and F_i which can be removed by gauge invariance and by the Einstein's equation constraints, therefore, there are not propagating vector degrees of freedom in (1.112). Additionally, gauge invariance also removes two scalar degrees of freedom and Einstein's equations removes another two, which leave us with one scalar degree of freedom. Since γ_{ij} is gauge invariant, we cannot remove it, therefore we end up with its two degrees of freedom described by its polarizations. Finally, we expect to (1.112) describes three propagating physical degrees of freedom, one from the scalar field and two from gravity. Hence, it is convenient to choose a suitable gauge to describe the dynamics of these three degrees of freedom. A useful gauge choice, is the so-called comoving gauge, defined by

$$\delta\phi = 0, \quad h_{ij} = a^2(t)e^{\zeta}(t, \mathbf{x})(\delta_{ij} + \gamma_{ij}), \quad \partial^i\gamma_{ij} = 0, \quad \gamma^i{}_i = 0. \tag{1.117}$$

where ζ is the scalar gauge-invariant comoving curvature perturbation[9] and γ is a traceless-transverse tensorial perturbation, which can be identified as the graviton. Both are first order quantities and they are the physical degrees of freedom. Note that, Einstein's equations allowed us to move the matter perturbation $\delta\phi$ to a metric perturbation ζ, which are related by $\zeta = -\frac{H}{\dot{\phi}_0}\delta\phi$.

We are interested in finding an action for ζ and γ_{ij}. That can be achieved by solving perturbatively the constraint equations (1.115) and (1.116) in the comoving gauge. For the moment, we will first find the quadratic action. If we write

$$N = 1 + \delta N, \quad N_i = \partial_i\psi + N_i^T, \quad \partial^i N_i^T = 0, \tag{1.118}$$

[8] For this section we have fixed $M_{\rm Pl}^2 = 1$.

[9] For historical reasons, here we are using ζ instead of $\mathcal{R}$ as the comoving curvature perturbation.

it is possible to show that their solution is given by

$$\delta N = \frac{\dot{\zeta}}{H}, \quad \psi = -\frac{\zeta}{H} + \chi, \quad \partial^2\chi = a^2\frac{\dot{\phi}_0^2}{2H^2}\dot{\zeta}, \quad N_T^i = 0. \tag{1.119}$$

Now, replacing back the solutions of the constraints on (1.112) we can find the quadratic action. First, we note that

$$\sqrt{h} = a^3 e^{3\zeta}, \tag{1.120}$$

$$\bar{E}_{ij} = \partial_i\partial_j\psi - (H + \dot{\zeta})\delta_{ij}, \tag{1.121}$$

$$^{(3)}R = -\frac{2e^{-2\zeta}}{a^2}\left(2\partial^2\zeta + \partial_i\zeta\partial^i\zeta\right). \tag{1.122}$$

After replacing those quantities and performing several integration by parts, it is possible to arrive to the quadratic action for ζ

$$S_\zeta^{(2)} = \int dt d^3\mathbf{x}\frac{\dot{\phi}_0}{2H^2}a^3\left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial\zeta)^2\right], \tag{1.123}$$

note that using the background equations of motion, we can write $\epsilon = \dot{\phi}_0^2/2H^2$, then, the second order action for ζ is proportional to ϵ. In a purely de Sitter universe, *i.e.,* $H =$ constant, and the action vanishes. This is because in de Sitter space the scalar mode is simply a gauge mode that can be removed by the background constraints and therefore no primordial fluctuations are generated. Similarly, it is possible to find the quadratic action for the tensor fluctuations, and it is given by

$$S_\gamma^{(2)} = \int dt d^3\mathbf{x}\frac{a^3}{8}\left[\dot{\gamma}_{ij}\dot{\gamma}^{ij} - \frac{1}{a^2}(\partial_k\gamma_{ij})^2\right]. \tag{1.124}$$

Up to now, ζ and γ has been treated classically, but we will see that they can be quantized and predict statistical quantities than can be latter measured in cosmological surveys.

Let us begin with (1.123) introducing the Mukhanov variable $v \equiv z\zeta$ with $z^2 \equiv a^2\frac{\dot{\phi}_0^2}{H^2} = 2a^2\epsilon$, and changing to conformal time $d\tau = dt/a$, the action becomes

$$S_v^{(2)} = \int d\tau d^3\mathbf{x}\left[v'^2 + (\partial_i v)^2 + \frac{z''}{z^2}v^2\right], \tag{1.125}$$

where the primes $(\ )'$ indicate derivatives respect to conformal time. After varying the action and moving to Fourier space the equation of motion is

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \tag{1.126}$$

which is known as the Mukhanov-Sasaki equation. In general, the previous equation is not straightforward to solve due to the time dependence on z. Nevertheless, we can solve it in different physical regimes of inflation as well as making use of the de Sitter limit and the slow roll approximation.

First, let us consider the de Sitter limit, where the function

$$\frac{z''}{z} = (aH)^2 \left[2 - \epsilon + \frac{3}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2 + \frac{1}{2}\eta\tilde{\xi}\right], \tag{1.127}$$

with $\tilde{\xi} = \frac{\dot{\eta}}{\eta H}$ being the third slow roll parameter, becomes

$$\frac{z''}{z} = \frac{2}{\tau^2}. \tag{1.128}$$

This is because, in de Sitter space $aH = -1/\tau$ and we can neglect the slow roll parameters in (1.127) and capture its time evolution through the definition of v. In this case, the solution of (1.126) is given by the mode function,

$$v_k = C_1 \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau} + C_2 \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) e^{ik\tau}. \tag{1.129}$$

The integration constants C_1 and C_2 are fixed after the quantization of the theory and by choosing an appropriate vacuum state.

1.4.1 Quantization

In order to perform a canonical quantization on v, it is necessary to define its canonical conjugated momentum, $\pi = \partial\mathcal{L}/\partial v' = v'$, with $\mathcal{L}$ defined through (1.125) as $S_v^{(2)} = \int d\tau\mathcal{L}$. Then, we promote them as quantum operators $\hat{v}$ and $\hat{\pi}$ satisfying the equal-time commutation relations

$$[\hat{v}(\tau, \mathbf{x}), \hat{v}(\tau, \mathbf{y})] = 0, \tag{1.130}$$
$$[\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = 0, \tag{1.131}$$
$$[\hat{v}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{1.132}$$

In Fourier space, the fluctuations are expanded in terms of bosonic creation and annihilation operators as

$$\hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\mathbf{k}}, \tag{1.133}$$

with $v_k(\tau)$ being its classical solution given by (1.129) and $v^*_k(\tau)$ its complex conjugate. The operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ satisfy the bosonic algebra

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0, \tag{1.134}$$
$$\left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = 0, \tag{1.135}$$
$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{1.136}$$

Quantum states in the Hilbert space are built by defining the vacuum state such that,

$$\hat{a}_{\mathbf{k}}|0\rangle = 0, \tag{1.137}$$

and the excited states are generated by the repeated application of $\hat{a}^\dagger_{\mathbf{k}}$ in the vacuum,

$$|m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle = \frac{1}{\sqrt{m!n!\ldots}}(\hat{a}^\dagger_{\mathbf{k}_1})^m(\hat{a}^\dagger_{\mathbf{k}_2})^n \ldots |0\rangle. \tag{1.138}$$

Additionally, the commutators (1.132) and (1.136) imply a normalization on the mode functions of the form

$$v_k^* v_k' - v_k'^* v_k = -i. \tag{1.139}$$

In order to the vacuum state will be well defined it is necessary to fix the mode function. For that, we can note that at sufficiently early times, *i.e.,* $\tau \to -\infty$, all the modes were deep inside the horizon $k/aH \sim |k\tau| \gg 1$, and the Mukhanov-Sasaki equation can be reduced to

$$v_k'' + k^2 v_k \approx 0, \tag{1.140}$$

which is nothing but the equation of motion for a free field in Minkowski space. The positive frequency mode solution of this equation, $v_k \propto e^{-ik\tau}$ defines the vacuum state to be the ground state of its correspondent Hamiltonian. We will employ the same choice to define the vacuum of inflation, which means that the solution of the Mukhanov-Sasaki equation should satisfy

$$\lim_{\tau \to -\infty} v_k(\tau) = \frac{C_1}{\sqrt{2k}} e^{-ik\tau}. \tag{1.141}$$

The previous initial condition defines a particular set of mode functions and a unique vacuum state known as the Bunch-Davies vacuum. Now, with equations (1.141) and (1.139) we can fix the integration constants of (1.129) to be $C_1 = 1$ and $C_2 = 0$. Therefore we end up with the solution of the Mukhanov-Sasaki equation of the form

$$v_k(\tau) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}. \tag{1.142}$$

Due to the quantum nature of the fluctuations, we can only compute statistical quantities associated to it, for instance, we can compute its variance through the two-point correlation function,

$$\langle 0|\hat{v}(\tau, \mathbf{0})\hat{v}(\tau, \mathbf{0})|0\rangle = \int d\ln k \frac{k^3}{2\pi^2} |v_k(\tau)|^2, \tag{1.143}$$

where we can define the dimensionless power spectrum as

$$\mathcal{P}_v(k, \tau) \equiv \frac{k^3}{2\pi^2} P_v(\tau, k), \quad P_v(\tau, k) = |v_k(\tau)|^2. \tag{1.144}$$

In the following section we will return to this point.

For completeness, let us consider the super-horizon limit of the Mukhanov-Sasaki equation, this is when $k \ll aH$, then

$$v_k'' - \frac{z''}{z} v_k \approx 0, \tag{1.145}$$

which can be rewritten in terms of ζ as

$$\frac{d}{dt}\left(a^3 \epsilon \dot{\zeta}\right) = 0. \tag{1.146}$$

Additionally, if we consider the slow roll approximation in the previous equation, we find that

$$\dot{\zeta} = 0, \tag{1.147}$$

therefore, the comoving curvature perturbation is conserved on super-horizon scales. This allows us to relate predictions made at horizon crossing with observables on the horizon re-entry at late times of the cosmological evolution. Even more, independent of the constituents of the universe, the conservation of ζ on super-horizon scales holds. Then, ζ is an adiabatic mode as shown in [40, 43]. Additionally, this limit is particularly interesting because on super-horizon scales, it is possible to show that the commutator (1.132) vanishes, or equivalently $[\zeta, \dot{\zeta}] \to 0$ and therefore, quantum fluctuations become classical perturbations. Hence, quantum expectation values can be identified with the ensemble average of classical stochastic fields.

Now, let us move to the treatment of tensor modes described by the action (1.124). First, we can expand the tensorial perturbation γ_{ij} in Fourier space in terms of its polarization tensor ϵ^s_{ij} as

$$\gamma_{ij}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{s=\times,+} \epsilon^s_{ij} \gamma^s_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{1.148}$$

where, given the gauge choice, the polarization tensor satisfies $\epsilon^s_{ii} = 0$, $k^i \epsilon^s_{ij} = 0$ and $\epsilon^s_{ij}\epsilon^{s'}_{ij} = 2\delta_{ss'}$. In the same way as we did with scalar perturbations, we can introduce a Mukhanov variable $v^s_{\mathbf{k}} \equiv \frac{a}{2} h^s_{\mathbf{k}}$ and the resulting action is

$$S^{(2)}_{v_{\mathbf{k}}} = \frac{1}{2} \int d\tau \frac{d^3\mathbf{k}}{(2\pi)^3} \left[(v'^s_{\mathbf{k}})^2 - \left(k^2 - \frac{a''}{a} \right) (v^s_{\mathbf{k}})^2 \right]. \tag{1.149}$$

After varying the previous action, we end up with two copies of (1.125), therefore we can apply the same procedure that we did before to solve it.

It is possible to extend all the previous treatment to the perturbations arising from multi-field dynamics. However, we will return to that discussion in Chapter 5. For the moment, let us just mention that in multi-field inflation, the fluctuation of the scalar field $\delta\phi^a(t, \mathbf{x})$ can be projected along the basis vectors as

$$\delta\phi^a(t, \mathbf{x}) = \delta\phi_\parallel(t, \mathbf{x}) T^a + \sigma(t, \mathbf{x}) N^a, \tag{1.150}$$

where $\delta\phi_\parallel$ corresponds to the inflaton perturbation which is later related with the comoving curvature perturbation $\mathcal{R}$ and therefore with the adiabatic mode. On the other hand, $\sigma(t, \mathbf{x})$ is called the isocurvature perturbation which is characterized for being perpendicular to the inflationary trajectory, therefore it produces an amount of entropy and it is identified as a non-adiabatic perturbation.

1.4.2 The Power Spectrum

As we anticipated, correlation functions are one of the most important objects that we have to study the evolution of cosmological perturbations. In this context, symmetries play an important role constraining the shape of correlation functions [17]. Let us consider for a moment a perfect de Sitter space of the form

$$ds^2 = \frac{1}{H^2\tau^2} \left(-d\tau^2 + d\mathbf{x}^2 \right), \tag{1.151}$$

where it is direct to note that the rescaling of coordinates $\tau \to \lambda\tau$ and $\mathbf{x} \to \lambda\mathbf{x}$ with $\lambda =$ constant, leaves the line element invariant. The previous rescaling is known as a dilation and composes one of the 10 isometries of de Sitter space. Given this symmetry, the two-point correlation function of a field $f_{\mathbf{k}}(\tau)$ satisfies the following condition

$$\langle 0|f_{\mathbf{k}}(\tau)f_{\mathbf{k}'}(\tau)|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{1}{k^3}F(k\tau), \tag{1.152}$$

where $F(k\tau)$ is an unknown function. Nevertheless, without specifying any details of $f_{\mathbf{k}}(\tau)$ we can anticipate that the two-point function must be scale invariant.

Since we have solved the Mukhanov-Sasaki equation for the mode functions, we can compute its two-point correlation function yielding

$$\langle 0|\hat{v}_{\mathbf{k}}(\tau)\hat{v}_{\mathbf{k}'}(\tau)|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{a^2H^2}{2k^3}(1+k^2\tau^2). \tag{1.153}$$

Now, if we consider the late-time limit or equivalently, the super-horizon limit, *i.e.*, $|k\tau| \ll 1$ the two-point function in terms of ζ becomes

$$\langle 0|\hat{\zeta}_{\mathbf{k}}(\tau)\hat{\zeta}_{\mathbf{k}'}(\tau)|0\rangle = \langle\zeta_{\mathbf{k}}(\tau)\zeta_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}\frac{H^2}{\dot{\phi}_0^2}. \tag{1.154}$$

Since ζ is conserved on super-horizon scales, we can compute the two-point function at the moment of horizon crossing, t^*, such that $k = a(t^*)H(t^*)$, then

$$\langle\zeta_{\mathbf{k}}(t^*)\zeta_{\mathbf{k}'}(t^*)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')P_\zeta(k)|_{t^*}, \tag{1.155}$$

where the power spectrum $P_\zeta(k)$ is given by

$$P_\zeta(k) = \frac{H^2}{4\epsilon k^3}. \tag{1.156}$$

However, since inflation is not a perfectly de Sitter space, we expect to have deviations from exact scale-invariance, due to the changes in the potential during the slow roll phase. For that, we can parametrize the power spectrum as

$$P_\zeta(k) = \frac{2\pi^2}{k^3}A_s(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{1.157}$$

where A_s is a scalar amplitude at a given reference scale k_0 and the tilt $n_s - 1$ is known as the spectral index defined by

$$n_s - 1 \equiv \frac{d\ln k^3 P_\zeta(k)}{d\ln k}. \tag{1.158}$$

It is possible to show that, at first order in slow roll parameters, the spectral index is given by

$$n_s - 1 = -2\epsilon - \eta. \tag{1.159}$$

For a perfect de Sitter space, one expect to have $n_s = 1$. The latest constraint by Planck [31] in this parameter is reported to be

$$n_s = 0.9649 \pm 0.00042 \quad (68\%\mathrm{CL}), \tag{1.160}$$

which is 8σ away from scale-invariance.

Similarly, it is possible to compute the two-point function for tensorial perturbations as

$$\langle \gamma^s_{\mathbf{k}} \gamma^s_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_\gamma(k), \tag{1.161}$$

where the power spectrum is

$$P_\gamma(k) = \frac{4H^2}{k^3}. \tag{1.162}$$

Unlike the scalar power spectrum, the tensor power spectrum have not been observed yet. In the previous expression, the only variable is the Hubble expansion rate during inflation. Therefore, a detection of this tensorial spectrum or, which is the same, primordial gravitational waves, would tell us about the energy scale of inflation. Moreover, it is worth to emphasize that it is highly likely that gravitational waves from inflation arise from purely quantum gravitational effects. Thus, their observation might directly confirms the treatment of gravity as a quantum phenomenon. Another option, is that primordial gravitational waves might be sourced by another degrees of freedom, for instance SU(2) gauge fields [44, 45]. Although primordial gravitational waves have not been observed, it is possible to introduce a quantity in which we can put observational bounds regarding the tensorial power spectrum. Let us define the tensor-to-scalar ratio r as

$$r = \frac{P_\gamma(k)}{P_\zeta(k)} = 16\epsilon \tag{1.163}$$

which is currently constrained by Planck [46] as

$$r < 0.044 \quad (95\%\text{CL}). \tag{1.164}$$

Additionally, there exist a relation between the tensor-to-scalar ratio, the number of e-folds during inflation and the displacements of the inflaton field, know as the Lyth bound [47]. Using the background equations of motion, writing the time derivatives of the fields as derivatives respect to the number of e-folds, and using the definition of r it is possible to arrive to

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \Delta N \sqrt{\frac{r}{8}}, \tag{1.165}$$

where we have restored the Planck mass. Therefore a measure of primordial gravitational waves also would tell us about the amount of field displacement during inflation. In particular, if sizable primordial gravitational waves are detected, one may have super-Planckian field displacements $\Delta\phi > M_{\text{Pl}}$ (also known as *large-field inflation*) which is something problematic at the moment of completing inflation in a theory of quantum gravity [48]. However, in Chapter 5 we will present a mechanism to evade this issue.

So far, we have seen how the observables that inflation predicts, namely n_s and r, are written in terms of the two-point correlation functions of scalar and tensor modes. Nevertheless, there exist a third observable which would give us exquisite information about the mechanism that nature chose to produce the structures that we see today in the universe. Such observable is related with deviations from a pure Gaussian statistics.

1.4.3 Non-Gaussianities

Up to now, our analysis of perturbations have been done at the linear level. Within this approach, the statistical profile of the curvature perturbation ζ is exactly Gaussian. Then, the probability density function of ζ will be completely determined by its variance or what is the same, its two-point correlation function. Hence, all the odd correlations functions will be zero. Moreover, observations of the temperature fluctuations of the CMB [35] are consistent with a Gaussian profile for ζ, however consistency does not imply that the profile is exactly Gaussian. In fact, one may wonder, how Gaussian it is?

We know that gravity is non-linear; therefore, some amount of non-linearities should be expected. In principle, one can go beyond the linear perturbation theory and compute higher-order correlation functions to study deviations from an exactly Gaussian statistical profile. Moreover, these higher-order correlation functions carry information about the (self) interactions of the inflaton. We expect these deviations from Gaussianity to be small, of the order of slow-roll parameters. In the following, we will study the first contribution to the non-Gaussianity of ζ coming from single-field slow-roll inflation, through the three-point correlation function. Finally, we will see the power of non-Gaussianities constraining inflationary models.

To begin this discussion it is useful to introduce the bispectrum. As the two-point function defines the power spectrum, in analogy, the three-point function defines the bispectrum B_ζ as

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_\zeta(k_1,k_2,k_3). \tag{1.166}$$

Our main goal is to compute the prediction of single-field slow-roll inflation for the bispectrum. However, first, we can conveniently arrange the previous expression to introduce the observable parameter in which we are interested. One way to parametrize the non-Gaussianities of ζ phenomenologically, is to consider an expansion around its Gaussian part, namely ζ_g, in the form

$$\zeta = \zeta_g + \frac{3}{5}f_{\rm NL}^{\rm loc}\zeta_g^2. \tag{1.167}$$

This expansion is known as the local ansatz [49] and we have introduced the local $f_{\rm NL}^{\rm loc}$ parameter, which is an observable.[10] Since a Gaussian field is completely determined by its two-point function, the shape of the bispectrum with the previous expansion is

$$B_\zeta(k_1,k_2,k_3) = \frac{6}{5}f_{\rm NL}^{\rm loc}[P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_3)P_\zeta(k_1)]. \tag{1.168}$$

A particularly interesting limit of the previous expression, is when one of the momenta, for instance k_3 is very small, and the other two, by momentum conservation, are of the same order $k_1 \sim k_2$. This situation is known as the squeezed limit, and the bispectrum becomes

$$\lim_{k_3\to 0} B_\zeta(k_1,k_2,k_3) = \frac{12}{5}f_{\rm NL}^{\rm loc}P_\zeta(k_1)P_\zeta(k_3). \tag{1.169}$$

Note that in this limit, the fluctuation $\zeta_{\mathbf{k}_3}$ is super-horizon, therefore is frozen by the time the other two momenta cross the horizon. For phenomenological purposes, it is useful to

[10] There are various kinds of $f_{\rm NL}$ parameters. However, in this thesis, we will be working only with the local one. Therefore, when the superscript $^{\rm loc}$ is not specified, it is assumed that it is the local type.

isolate the $f_{\rm NL}^{\rm loc}$ parameter as

$$f_{\rm NL}^{\rm loc} = \frac{5}{12} \lim_{k_3 \to 0} \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1) P_\zeta(k_3)}, \tag{1.170}$$

which is currently constrained by Planck [50] as

$$f_{\rm NL}^{\rm loc} = -0.9 \pm 5.1 \quad (68\%\text{CL}). \tag{1.171}$$

Then, the natural question is, what is the prediction of single-field slow-roll inflation for $f_{\rm NL}^{\rm loc}$?

In order to compute higher-order correlation functions, it is necessary to find the action at higher-order in perturbations for ζ, which is not an easy task. Since we are interested in the bispectrum, we will need the cubic action for ζ. This is obtained by expanding at third order the action (1.112). In principle, one might think that for that it is necessary to expand the constraints N and N_i up to third order, fortunately this is not necessary. For example [51], if we consider the lapse function expanded as $N = N^{(0)} + N^{(1)} + \ldots N^{(n)}$ where n denotes the $\mathcal{O}(\zeta^n)$ terms, and the action (1.112) as $S = \int d^4x \mathcal{L}(N)$. In general we have

$$\delta S = \int d^4x \frac{\partial \mathcal{L}}{\partial N} \delta N = 0, \tag{1.172}$$

where it is possible to expand the constraint equation $\partial\mathcal{L}/\partial N = 0$ as

$$\left.\frac{\partial \mathcal{L}}{\partial N}\right|_{0,\zeta^0} = 0, \tag{1.173}$$

$$\left.\frac{\partial \mathcal{L}}{\partial N}\right|_{0,\zeta^1} + N^{(1)} \left.\frac{\partial^2 \mathcal{L}}{\partial N^2}\right|_{0,\zeta^0} = 0. \tag{1.174}$$

For the cubic order action we will have

$$S_\zeta^{(3)} \supset \int d^4x N^{(3)} \left.\frac{\partial \mathcal{L}}{\partial N}\right|_{0,\zeta^0} = 0, \tag{1.175}$$

$$S_\zeta^{(3)} \supset \int d^4x N^{(2)} \left(\left.\frac{\partial \mathcal{L}}{\partial N}\right|_{0,\zeta^1} + N^{(1)} \left.\frac{\partial^2 \mathcal{L}}{\partial N^2}\right|_{0,\zeta^0} \right) = 0, \tag{1.176}$$

where the same applies for N_i. Therefore, the third order terms in N and N_i would be multiplying the constraints at zeroth order on-shell, and the second order terms in N and N_i multiply the constraints evaluated at first order (1.118) which is zero. Hence, we only need to know N and N_i up to first order in order to compute the cubic-order action.

The computation of the cubic-order action is widely known for being a laborious task. It necessary to perform $\mathcal{O}(40)$ integration by parts in order to arrive to the following expression

$$\begin{aligned} S_\zeta^{(3)} = \int dt d^3x \big[& a^3 \epsilon^2 \zeta \dot{\zeta}^2 + a\epsilon^2 \zeta (\partial\zeta)^2 - 2a\epsilon \dot{\zeta} \partial\zeta \partial\chi + \frac{a^3\epsilon}{2} \dot{\eta} \zeta^2 \dot{\zeta} \\ & + \frac{\epsilon}{2a} \partial\zeta \partial\chi \partial^2\chi + \frac{\epsilon}{4a} \partial^2\zeta (\partial\chi)^2 + 2f(\zeta) \left.\frac{\delta \mathcal{L}}{\delta \zeta}\right|_1 + \mathcal{L}_b \big], \end{aligned} \tag{1.177}$$

where $\partial^2\chi = a^2\epsilon\dot{\zeta}$, $\delta\mathcal{L}/\delta\zeta|_1$ is a term proportional to the equations of motion, $f(\zeta)$ is

$$f(\zeta) = \frac{\eta}{4}\zeta^2 + \frac{\zeta\dot{\zeta}}{H} + \frac{1}{2a^2H}\left[-\frac{1}{2}\partial\zeta\partial\zeta + \frac{1}{2}\partial^{-2}[\partial^i\partial^j(\partial_i\zeta\partial_j\zeta)] + \partial\zeta\partial\chi - \partial^{-2}[\partial^i\partial^j(\partial_i\zeta\partial_j\chi)]\right], \tag{1.178}$$

and $\mathcal{L}_b$ is a boundary term of the form $\mathcal{L}_b = \partial_t\mathcal{A} + \partial_i\mathcal{B}$. If we keep in mind that we are interested in computing the three-point correlation function, then, the term $\delta\mathcal{L}/\delta\zeta|_1$ does not contribute, and the relevant contribution coming from the boundary term is

$$\mathcal{L}_b \simeq \partial_t\left(-\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta}\right). \tag{1.179}$$

Therefore, the cubic Lagrangian which contributes to the three point function is

$$\mathcal{L}_3 = a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}\partial\zeta\partial\chi - \partial_t\left(\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta}\right) + \frac{a^3\epsilon}{2}\dot{\eta}\zeta^2\dot{\zeta} + \frac{\epsilon}{2a}\partial\zeta\partial\chi\partial^2\chi + \frac{\epsilon}{4a}\partial^2\zeta(\partial\chi)^2, \tag{1.180}$$

where it is direct to see that $\mathcal{L}_3 \propto \epsilon^2$. As we expect, the dominant contribution for this Lagrangian, comes from gravitational interactions.

Now that we know the cubic order Lagrangian, we are in position to compute the three-point correlation function. To compute higher-order correlations functions of quantum fields in cosmology, it is necessary to make the distinction with the standard QFT transition amplitude computations. In QFT, one impose conditions at very early and late times over the particle states. In cosmology, we are interested in computing the expectation values of operators at a unique fixed time. In this case, the boundary conditions are not imposed on the fields at the asymptotic future and past. Instead, we impose boundary conditions only at very early times when the wavelengths of the operators are deep inside the horizon, and the spacetime is approximately Minkowskian. This procedure is known as the *in-in* formalism [52], because the expectation values are computed at two $|in\rangle$ states instead of at one $|in\rangle$ and a $|out\rangle$ states.

Let us consider an operator Q which will be a product in terms of ζ and its derivatives at the end of inflation. Then, the expectation value of the operator is given by

$$\langle Q\rangle \equiv \langle\Omega|Q(t)|\Omega\rangle, \tag{1.181}$$

where $|\Omega\rangle$ is the vacuum state of the interacting theory at some moment t_i in the far past and $t > t_i$ can be considered the end of inflation. To compute the expectation value, we evolve $Q(t)$ back to the initial time t_i, using the perturbed Hamiltonian $H = H_0 + H_{\rm int}$. Then we move to the interaction picture, where we compute the contribution coming from the interactions through a power of series in the interacting Hamiltonian yielding the master formula

$$\langle Q\rangle = \langle 0|\bar{\mathcal{T}}e^{i\int_{-\infty(1-i\varepsilon)}^{t}dt' H^I_{\rm int}(t')}Q^I(t)\mathcal{T}e^{-i\int_{-\infty(1+i\varepsilon)}^{t}dt' H^I_{\rm int}(t')}|0\rangle, \tag{1.182}$$

where $\bar{\mathcal{T}}$ and $\mathcal{T}$ are the anti- and time-ordering symbols respectively, the superscript $(\,)^I$ indicates operators evaluated in the interacting picture and $i\varepsilon$ is the prescription used to turn off the interaction in the far past and project the vacuum $|\Omega\rangle$ of the interacting theory onto the vacuum of the free theory $|0\rangle$. An operator $\mathcal{O}$ in the interacting picture is given by

$$\mathcal{O}^I(t,\mathbf{x}) = U_0^{-1}(t,t_i)\mathcal{O}(t_i,\mathbf{x})U_0(t,t_i), \tag{1.183}$$

where $U_0(t,t_i)$ is the time-evolution operator given by

$$U_0(t,t_i) = \mathcal{T} e^{-i\int_{t_i}^{t} dt' H_0(t')}. \tag{1.184}$$

Finally, by expanding the exponential in (1.182) it is possible to compute the correlations functions perturbatively in $H^I_{\rm int}$, where it is possible to organize the expansion with Feynman-like diagrams without time flow. For instance, at tree level, the leading order contribution is

$$\langle Q \rangle = -i\int_{-\infty}^{t} dt' \langle 0|[Q^I(t), H^I_{\rm int}(t')]|0\rangle. \tag{1.185}$$

For our purposes, the interacting Hamiltonian $H_{\rm int}$ is given by

$$H_{\rm int} = -\int d^3x \mathcal{L}_3, \tag{1.186}$$

therefore, the three-point correlation function is obtained as

$$\langle \zeta^3(t)\rangle = i\int_{-\infty}^{t} \langle 0|[\zeta^3(t), \int d^3x \mathcal{L}_3(t')]|0\rangle. \tag{1.187}$$

The computation although it is straightforward, may be a bit lenghty. A detailed review of the computation can be found in [53], however, the result in terms of the bispectrum in all its glory is

$$B_\zeta(k_1,k_2,k_3) = \frac{H^4}{4\epsilon^2 M_{\rm Pl}^4}\frac{1}{(k_1k_2k_3)^3}\left[\frac{1}{8}(\eta-\epsilon)\sum_i k_i^3 + \epsilon\sum_{i\neq j} k_i k_j^2 + \frac{8}{K}\sum_{i>j} k_i^2 k_j^2\right], \tag{1.188}$$

where $K = k_1+k_2+k_3$. If we take the squeezed limit in the previous expression, *i.e.*, $k_3 \to 0$, $k_1 \sim k_2$ we end up with

$$\lim_{k_3\to 0} B_\zeta(k_1,k_2,k_3) = \frac{H^4}{4\epsilon^2 M_{\rm Pl}^4}\frac{1}{(k_1k_2k_3)^3}\left[\frac{\eta+2\epsilon}{8}\sum_i k_i^3\right]. \tag{1.189}$$

Recall that the spectral index is $n_s - 1 = -2\epsilon - \eta$ and the scalar power-spectrum with the Planck mass recovered is $P_\zeta(k) = H^2/4\epsilon M_{\rm Pl}^2 k^3$. Therefore, the squeezed bispectrum finally is

$$\lim_{k_3\to 0} B_\zeta(k_1,k_2,k_3) = -(n_s-1)P_\zeta(k_3)P_\zeta(k_1), \tag{1.190}$$

which implies that

$$f_{\rm NL}^{\rm loc} = -\frac{5}{12}(n_s-1). \tag{1.191}$$

Equation (1.190) or (1.191) is known as the Maldacena's consistency relation [16, 42]. This relation can be understood as a soft-theorem regarding correlation functions, in the sense that, in the squeezed limit a n-point correlation function is completely determined by the $(n-1)$-point correlation function. In order to the theorem holds, it is only necessary to asume single-field slow-roll inflation, no matter what form the potential has. Therefore, the consistency relation it is one of the most powerful tools that we have to test inflation. Nevertheless, we have to be cautious, because although the consistency relation looks quite simple, relating two observables, it is plagued of subtleties. One could naively say that a detection of primordial local non-Gaussianity different from (1.191) ($\mathcal{O}(10^{-2})$) can rule out all models of single-field slow-roll inflation. However, if we study in detail the implications of the fact that the $\zeta_{\mathbf{k}_3}$ mode is already super-horizon when the other two-modes are sub-horizon, we will find that its only action is rescaling the background of the sub-horizon modes, and therefore, it is possible to demonstrate that the consistency relation is a gauge artifact [54–56]. Having a gauge artifact may sound terrible, but it is actually a good signal in this context. Since, we can make a most powerful assertion: *any* detection of local non-Gaussianity can rule out all single-field slow-roll models of inflation.

However, there are always ways to evade a theorem; it is only necessary to understand its premises well. And in fact, there is a way to violate the consistency relation within the context of single-field inflation. It is only necessary to relax the condition of attractive behavior of the background. When one asumes, slow-roll inflation, implicitly is saying that the background has an attractor behavior in the phase-space. Nevertheless, if we now consider non-attractor backgrounds, like for instance, ultra-slow-roll inflation the story is different. The first computation in single-field inflation that violates Maldacena's consistency relation was presented in [57].

In ultra-slow roll, the equation of motion for super-horizon modes (1.146) has solutions of the form

$$\zeta = c_1 + c_2 \int \frac{dt}{a^3 \epsilon}, \tag{1.192}$$

but, since the evolution of the slow-roll parameters is given by (1.28), now we cannot neglect the integral because it represents a growing mode conversely to the slow-roll case, therefore

$$\zeta \propto a^3, \tag{1.193}$$

on super-horizon scales. The previous behavior is important at the moment of computing the three-point function since terms that, in slow-roll, were neglected due to the super-horizon conservation, now becomes important. In fact, the computation of the three-point function for this models, predicts a local non-Gaussianity of

$$f_{\rm NL}^{\rm loc} = \frac{5}{2}. \tag{1.194}$$

In principle, one may be concerned because (1.192) is not constant on super-horizon scales, constituting a non-adiabatic mode. Nevertheless, it has been shown [58, 59] that this mode, even not being constant, is adiabatic in the thermodynamic sense.

Chapters 2, 3 and 4 are devoted to study the non-Gaussianity in ultra-slow roll inflation and its phenomenological consequences.

To conclude this chapter, let us mention that nowadays a novel technique to compute cosmological correlation functions is being developed. This technique relies in bootstrap methods where the computations are made without using Lagrangians nor Hamiltonians, just by using conformal symmetries, unitarity and locality principles. This ongoing research program is called the cosmological bootstrap. In particular, in [60] the three-point correlation function is computed by evaluating one of the legs of the de Sitter 4-point correlation function diagram on a time-dependent background.

Chapter 2

A Generalized Consistency Relation

Since the inflationary paradigm was introduced, a plethora of single-field models have been developed, see for instance Ref. [61] for an incomplete list of models. Therefore, if inflation really did happen, then we need to find a way to discriminate among different models. In this context, model independent result based on symmetries are one of the most powerful tools that we have to discriminate among different realizations of inflation. An example of such result, is Maldacena's consistency relation $f_{\rm NL} = 5(1-n_s)/12$ [42], linking together the amount of local (squeezed) non-Gaussianity $f_{\rm NL}$ with the spectral index $n_s - 1$, and valid for attractor models of single field inflation [16, 62–69]. It is by now well understood that this relation cannot be directly observed. A correct account of the observable amount of primordial local non-Gaussianity yields [54, 55, 70–72]

$$f_{\rm NL}^{\rm obs} = 0 + \mathcal{O}(k_L/k_S)^2, \tag{2.1}$$

where $\mathcal{O}(k_L/k_S)^2$ stands for non-Gaussianity produced by non-primordial phenomena such as gravitational lensing and redshift perturbations (the so called projection effects [73, 74]). This result may be understood as coming from a cancellation between the primordial value predicted in co-moving gauge $5(1-n_s)/12$, and a correction $-5(1-n_s)/12 + \mathcal{O}(k_L/k_S)^2$ that arises after considering a change of coordinates rendering gauge invariant observables. This coordinate change corresponds to a transformation from co-moving coordinates to the so called conformal Fermi coordinates [55, 75].

It appears to be entirely reasonable that the cancellation leading to (2.1) is only effective when the prediction of primordial non-Gaussianity corresponds to $f_{\rm NL} = 5(1-n_s)/12$. This is because Maldacena's consistency relation itself may be thought of as the consequence of a space-time reparametrization linking short- and long-wavelength curvature perturbations realized with the help of a symmetry of the system under a simultaneous spatial dilation and a field reparametrization [16]. Thus, any measurement of local non-Gaussianity would directly rule out single field models of slow-roll inflation [8–12] (attractor models of inflation), but it would not rule out other classes of inflation. In particular, one would be seriously motivated to consider more exotic models of inflation such as curvaton scenarios [76], multi-field models [77], or non-attractor models of inflation (that is, models for which the background depends on the initial conditions [57, 78–81]). For instance, in the case of ultra slow-roll inflation [78, 82], one finds $f_{\rm NL} = 5/2$, from where it seems unlikely that a cancellation could happen.

In this chapter, we show that there is a slightly more general class of non-Gaussian consistency relations, of which Maldacena's relation is an example. This generalization emerges from a space-time reparametrization (linking short- and long-wavelength curvature perturbations) that is realized with the help of a more general symmetry. This time, the symmetry transformation involves both a time dilation and a spatial dilation. We will show that this symmetry is approximate in the case of $\epsilon \ll 1$, but exact in the case of ultra slow-roll (independently of the value of ϵ).

The existence of a more general consistency relation (coming from space-time reparametrizations) suggests that the vanishing of Eq. (2.1) may be effective under more general conditions, valid beyond the attractor single field models of inflation. In particular, one could expect (2.1) to be valid in the extreme case of ultra slow-roll inflation. We will argue that this is indeed the case in Chapter 3, where the use of conformal Fermi coordinates is considered for the case of non-attractor models.

The outline of this chapter is the following: In Section 2.1 we offer a review of the derivation of the standard consistency relation for single field slow-roll inflation (attractor inflation). In Section 2.2 we derive the generalized version of the consistency relation. We do this first for the simple case $\epsilon \to 0$, and then extend this result to the more subtle case $\epsilon \neq 0$, where we pay some attention to the particular case of ultra slow-roll inflation. Then, in Sections 2.3 and 2.4 we briefly analyze and discuss our results, and ask how they could be modified by deviations from the canonical models of inflation for which our results are strictly valid.

2.1 Review of the Consistency Relation Derivation

Let us start by reviewing the derivation of the standard consistency relation for single field slow-roll attractor inflation, in which the curvature perturbation freezes on superhorizon scales. We will closely follow the discussion of Ref. [64], (see also the derivations in Refs. [16, 65]), but with a perspective that will show to be useful for generalizing the relation later on.

The metric line element describing a perturbed FLRW spacetime, in comoving gauge may be written as:

$$ds^2 = a^2(\tau)\bigg[- N^2 d\tau^2 + 2N_i d\tau dx^i + e^{2\zeta} dx^2 \bigg], \tag{2.2}$$

where a is the usual scale factor. We have adopted conformal time τ, which is related to cosmological time t via $d\tau = dt/a$. The lapse $\delta N = N - 1$ and shift N_i functions respect constraint equations that are found by varying the action of the perturbations. The linear solutions are given by:[1]

$$\delta N = \frac{1}{\mathcal{H}}\partial_0 \zeta, \qquad N_i = -\partial_i \frac{\zeta}{\mathcal{H}} + \epsilon \frac{\partial_i}{\partial^2}\partial_0 \zeta. \tag{2.3}$$

After replacing these solutions back into the action, one obtains a cubic action describing a single scalar degree of freedom ζ. Now, let us consider the following transformations of

[1]Here we are assuming regular Bunch-Davies initial conditions. For a discussion on the effect of considering different initial states, see [83].

coordinates and fields:

$$x = e^g x', \tag{2.4}$$
$$\tau = \tau', \tag{2.5}$$
$$\zeta = \zeta' + \Delta\zeta, \tag{2.6}$$

where g and $\Delta\zeta$ are functions of τ' only. We would like to know how these relations affect the form of the ζ-action for a certain choice of g and $\Delta\zeta$. Given that g and $\Delta\zeta$ are taken as perturbations, this would require us to consider the full initial action, Einstein-Hilbert plus scalar field, including the background contributions (this is because (2.4) implies that some background terms will be promoted to perturbations). Instead of examining this change by inserting (2.4)-(2.6) in the full action explicitly, we may analyze the way in which the metric (2.2) is affected. This will allow us to infer how the action itself is affected by the transformation. To proceed, first notice that (2.4) and (2.5) imply

$$dx^i = e^g dx'^i + e^g \partial_0 g x'^i d\tau', \tag{2.7}$$
$$d\tau = d\tau'. \tag{2.8}$$

In second place, recall that N and N_i were already fixed in terms of ζ, and so they must change according to (2.6). This is because we are examining how the transformations alter the form of the ζ-action after N and N_i were solved. One finds:

$$\delta N = \delta N' + \frac{1}{\mathcal{H}}\partial_0\Delta\zeta, \tag{2.9}$$
$$N_i = N_i' + \partial_i\Delta\psi, \tag{2.10}$$

where $\Delta\psi$ is such that

$$\partial^2\Delta\psi = -\partial^2\frac{\Delta\zeta}{\mathcal{H}} + \epsilon\partial_0\Delta\zeta. \tag{2.11}$$

Given that we are choosing $\Delta\zeta$ to be x'-independent, $\Delta\psi$ satisfies the simpler equation $\partial^2\Delta\psi = \epsilon\partial_0\Delta\zeta$. This equation is solved by $\Delta\psi = \frac{1}{6}x^i x_i \epsilon\partial_0\Delta\zeta$, and so we may write:

$$\partial^i\Delta\psi = \frac{1}{3}x'^i\epsilon\partial_0\Delta\zeta. \tag{2.12}$$

Then, replacing all of these results back into the metric (2.2), we obtain:

$$\begin{aligned} ds^2 &= a^2(\tau')\Big[- e^{2\delta N' + \frac{2}{\mathcal{H}}\partial_0\Delta\zeta} d\tau'^2 \\ &\quad + 2\Big(N_i' + \partial_0 g x_i' + \frac{1}{3}x_i'\epsilon\partial_0\Delta\zeta\Big) d\tau' dx'^i + e^{2\zeta' + 2\Delta\zeta + 2g} dx'^2 \Big]. \end{aligned} \tag{2.13}$$

It is important to keep the perturbations appearing in the term proportional to dx'^2 up to third order at least. In this case, we have kept $\Delta\zeta$ and g exactly as they appear from the definition of the transformations (2.4)-(2.6). On the other hand, in those terms proportional to $d\tau'^2$ and $d\tau' dx'^i$ we must keep the perturbations up to first order at least. The reason for doing this is that we want to understand how (2.4)-(2.6) change the form of the ζ-action up to third order. Given that the cubic action depends on the linear contributions to δN and

N_i, we do not need to worry about contributions coming from $\Delta\zeta$ and g beyond linear order in terms proportional to $d\tau'^2$ and $d\tau' dx'^i$.

Next, notice that if we choose both g and $\Delta\zeta$ constant, and demand them to satisfy $\Delta\zeta = -g$ we end up with

$$ds^2 \;\; = \;\; a^2(\tau')\Big[- N'^2 d\tau'^2 + 2N'_i d\tau' dx'^i + e^{2\zeta'} dx'^2 \Big]. \tag{2.14}$$

This metric has exactly the same form of (2.2), and therefore the action for ζ', obtained by using this metric, has the same form as the one for ζ. This in turn, implies that both ζ and ζ' are solutions of the same system of equations of motion. Moreover, these solutions are connected through the relation:

$$\zeta(\tau, x) = \zeta'(\tau', x') - g. \tag{2.15}$$

Since $\tau = \tau'$ and $x = e^g x'$, we may write instead:

$$\zeta(\tau, x) = \zeta'(\tau, e^{-g} x) - g. \tag{2.16}$$

This relation may be used to derive the squeezed limit of the bispectrum in terms of the power spectrum of the perturbations. First, let us consider a splitting of ζ into short- and long-wavelength contributions of the form:

$$\zeta = \zeta_S + \zeta_L. \tag{2.17}$$

This separation of scales is not directly related to the size of the horizon during inflation. Initially the wavelengths of both ζ_L and ζ_S fit inside the horizon, while at later times they are both of superhorizon size. The point here is that, independent of the size of the horizon, we want to understand the non-linear effect of the long mode on the short mode.

At length scales of order k_S^{-1}, the mode ζ_L is effectively x-independent. In addition, if we are interested in attractor models of single field inflation, ζ_L is also τ-independent. Then, if in Eq. (2.16) we choose $g = -\zeta_L$ (or, equivalently $\Delta\zeta = \zeta_L$), we end up with

$$\zeta_S(\tau, x) = \zeta'(\tau, e^{\zeta_L} x). \tag{2.18}$$

In other words, the long wavelength mode of ζ has been absorbed via a coordinate transformation.[2] Relation (2.18) tells us that $\zeta_S(\tau, x)$ may be expressed in terms of a fluctuation ζ' that is a solution of the same system of equations satisfied by ζ, but with $e^{\zeta_L} x$ instead of x in the spatial argument. In other words, we have non-linear information about how the long-wavelength mode ζ_L modulates the short wavelength mode ζ_S. Next, let us consider the 2-point correlation function $\langle \zeta_S(\tau, \mathbf{x}) \zeta_S(\tau, \mathbf{y}) \rangle \equiv \langle \zeta_S \zeta_S \rangle(\tau, |\mathbf{x} - \mathbf{y}|)$. Equation (2.18) tells us that

$$\langle \zeta_S \zeta_S \rangle(\tau, |\mathbf{x} - \mathbf{y}|) = \langle \zeta' \zeta' \rangle(\tau, e^{\zeta_L} |\mathbf{x} - \mathbf{y}|). \tag{2.19}$$

[2] This reveals that ζ corresponds to an adiabatic mode [20, 43], and that the evolution of the short wavelength contribution $\zeta_S(\tau, x)$ may be thought of as that of a perturbation ζ' on a new redefined background (obtained by the absorption of ζ_L).

Notice that $\langle\zeta'\zeta'\rangle(\tau,|\mathbf{x}-\mathbf{y}|)$ is nothing but the usual 2-point correlation function of the curvature perturbation in comoving gauge (because ζ' is a solution of the full system). Expanding the previous relation in powers of ζ_L, we obtain

$$\langle\zeta_S\zeta_S\rangle(\tau,|\mathbf{x}-\mathbf{y}|) = \langle\zeta'\zeta'\rangle(\tau,|\mathbf{x}-\mathbf{y}|) + \zeta_L\frac{d}{d\ln|\mathbf{x}-\mathbf{y}|}\langle\zeta'\zeta'\rangle(\tau,|\mathbf{x}-\mathbf{y}|) + \cdots . \tag{2.20}$$

Then, by writing the fields in Fourier space as

$$\zeta(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3k\zeta(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.21}$$

we end up with

$$\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2) = \langle\zeta'\zeta'\rangle(\mathbf{k}_1,\mathbf{k}_2) - \zeta_L(\mathbf{k}_L)\left[n_s(k_S,\tau)-1\right]P_\zeta(\tau,k_S), \tag{2.22}$$

where we have defined $\mathbf{k}_L = \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k}_S = (\mathbf{k}_1 - \mathbf{k}_2)/2$. In the previous expressions, the power spectrum $P_\zeta(\tau,k)$ and its spectral index $n_s(k)-1$ are defined as

$$P_\zeta(\tau,k) = \int d^3re^{-i\mathbf{k}\cdot\mathbf{r}}\left\langle\zeta\zeta\right\rangle(\tau,r), \tag{2.23}$$

$$n_s(k,\tau)-1 = \frac{\partial}{\partial\ln k}\ln\big(k^3P_\zeta(\tau,k)\big), \tag{2.24}$$

with $\mathbf{r}\equiv\mathbf{x}-\mathbf{y}$.

The first term at the rhs of Eq. (2.22) is independent of ζ_L, so by correlating Eq. (2.22) with $\zeta_L(\mathbf{k}_3)$, we obtain

$$\langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2)\rangle = -\langle\zeta_L(\mathbf{k}_3)\zeta_L(\mathbf{k}_L)\rangle\left[n_s(k_S,\tau)-1\right]P_\zeta(\tau,k_S). \tag{2.25}$$

The squeezed limit of the bispectrum appears as the formal limit:

$$\lim_{k_3\to0}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_\zeta(k_1,k_2,k_3) = \langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2)\rangle. \tag{2.26}$$

Thus, putting together Eqs. (2.25) and (2.26) we see that the squeezed limit acquires the form:

$$B_\zeta(k_1,k_2,k_3) \;=\; -\left[n_s(k_S,\tau)-1\right]P_\zeta(k_S)P_\zeta(k_L). \tag{2.27}$$

This corresponds to Maldacena's well known consistency relation. It was obtained with the help of transformation (2.18) linking short- and long-wavelength comoving curvature perturbations ζ_S and ζ_L through a "complete" curvature perturbation ζ' (that is, a curvature perturbation for which there has been no separation of scales). In other words, (2.27) gives us information on how the long wavelength mode ζ_L modulates the short wavelength mode ζ_S.

2.2 A generalized Consistency Relation

We would like to count with a consistency relation valid for cases in which the long mode ζ_L is time dependent, that is, when the curvature perturbation evolves on super-horizon scales. For simplicity, let us first attempt this in the formal limit $\epsilon\to0$. We will consider the case $\epsilon\neq0$ later.

2.2.1 Case with $\epsilon \to 0$

If $\epsilon = 0$, the Hubble parameter $H = \dot{a}/a$ is a constant, and the scale factor a is given by

$$a(\tau) = -\frac{1}{H\tau}, \tag{2.28}$$

Then, let us consider the following transformations:

$$x = e^{g} x', \tag{2.29}$$
$$\tau = e^{f} \tau', \tag{2.30}$$
$$\zeta = \zeta' + \Delta\zeta. \tag{2.31}$$

Here the quantities g, f and $\Delta\zeta$ are all functions of τ'. For concreteness, let us assume that $\tau = \tau'$ at a given reference time τ_*. This implies that $f = 0$ at $\tau' = \tau_*$. To make this explicit, one could write f as $f(\tau') = \int_{\tau_*}^{\tau'} d\tau h$ (this will not be important though). This choice is completely arbitrary, and one could certainly fix initial conditions for f and g in other ways, without modifying the main conclusions of this section. The change of coordinates implies:

$$dx^i = e^{g} dx'^i + e^{g} \partial_0 g x'^i d\tau', \tag{2.32}$$
$$d\tau = e^{f} d\tau'(1 + \tau' \partial_0 f). \tag{2.33}$$

Note that now $\partial_0 \equiv \partial_{\tau'}$. Replacing these relations back into the metric (2.2), we find:

$$\begin{aligned} ds^2 \;&=\; a^2(\tau')\bigg[- e^{2\delta N' - 2\tau' \partial_0 \Delta\zeta + 2\tau' \partial_0 f} d\tau'^2 \\ &\quad + 2\Big(N'_i + \partial_0 g x'_i\Big) d\tau' dx'^i + e^{2\zeta' + 2\Delta\zeta + 2g - 2f} dx'^2 \bigg]. \end{aligned} \tag{2.34}$$

As before, let us recall that the perturbations appearing together with δN and N_i may be treated up to linear order. On the other hand, those appearing together with ζ' must be treated up to cubic order. In this case, we are treating them exactly. Now, notice that if we demand that g is constant, and that

$$\Delta\zeta + g - f = 0, \tag{2.35}$$

the metric reduces back to (2.14). Then, we conclude that the ζ-action is invariant under the transformations (3.87)-(2.31). Therefore, we have two solutions ζ and ζ' related through the following relation

$$\zeta(\tau, x) = \zeta'(e^{-f}\tau, e^{-g}x) - g + f. \tag{2.36}$$

In order to deduce the squeezed limit of the bispectrum in this class of models, let us now again consider the splitting

$$\zeta = \zeta_S + \zeta_L. \tag{2.37}$$

Recall that this time we are assuming that ζ_L depends on time. As we did with (2.16), let us choose f and g in such a way that $\Delta\zeta = \zeta_L$:

$$-g + f = \zeta_L(\tau). \tag{2.38}$$

At this point we notice that the difference between f and g is a pure perturbation. At the background level, where perturbations are absent, f and g necessarily have to coincide, and we recover the well-known de Sitter isometry $\tau \to e^{\lambda}\tau$, $x \to e^{\lambda}x$, studied in, for example, [20,84].

Given that $f = 0$ for $\tau = \tau_*$, the previous relation sets the constant g as $g = -\zeta_L(\tau_*)$. Then we find that f is given by

$$f = \zeta_L(\tau) - \zeta_L(\tau_*). \tag{2.39}$$

This leads to a relation between ζ_S and ζ' given by:

$$\zeta_S(\tau, x) = \zeta'(e^{-[\zeta_L(\tau)-\zeta_L(\tau_*)]}\tau, e^{\zeta_L(\tau_*)}x). \tag{2.40}$$

If $\zeta_L(\tau)$ does not evolve, then $\zeta_L(\tau) = \zeta_L(\tau_*)$, and we recover Eq. (2.18). We may now compute the power spectrum of ζ_S. Up to first order in ζ_L, it is direct to find in Fourier space

$$\begin{aligned} \langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2) \quad &= \quad \langle\zeta'\zeta'\rangle(\mathbf{k}_1,\mathbf{k}_2) - [\zeta_L(\mathbf{k}_L) - \zeta_L^*(\mathbf{k}_L)]\,\frac{d}{d\ln\tau}P_\zeta(\tau,k_S) \\ &\quad -\zeta_L^*(\mathbf{k}_L)\,[n_s(k_S,\tau)-1]\,P_\zeta(\tau,k_S). \end{aligned} \tag{2.41}$$

Correlating this expression with $\zeta_L(\mathbf{k}_3)$, we end up with

$$\begin{aligned} \langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2)\rangle \quad &= \quad -\langle\zeta_L(\mathbf{k}_3)\,[\zeta_L(\mathbf{k}_L) - \zeta_L^*(\mathbf{k}_L)]\rangle\frac{d}{d\ln\tau}P_\zeta(\tau,k_S) \\ &\quad -\langle\zeta_L(\mathbf{k}_3)\zeta_L^*(\mathbf{k}_L)\rangle\,[n_s(k_S,\tau)-1]\,P_\zeta(\tau,k_S). \end{aligned} \tag{2.42}$$

This expression involves the correlation of $\zeta_L(\mathbf{k}_3)$ evaluated at a given time τ, with $\zeta_L^*(\mathbf{k}_3)$ which is evaluated at the reference time $\tau = \tau_*$. When super-horizon modes freeze, the first line cancels and there is no difference between $\zeta_L^*(\mathbf{k}_3)$ and $\zeta_L(\mathbf{k}_3)$, so we end up with Maldacena's standard attractor result. However, if ζ_L grows on super-horizon scales fast enough for ζ_L^* to become subdominant, and for the first line to dominate the second one, we end up with

$$B_\zeta(k_1,k_2,k_3) = -P_\zeta(k_L)\frac{d}{d\ln\tau}P_\zeta(k_S). \tag{2.43}$$

This is one of our main results. Equation (2.43) tells us that under a substantial super-horizon growth during inflation the squeezed limit is dominated by a time derivative of the power spectrum.

2.2.2 Non-Gaussianity in ultra slow-roll inflation

Before considering the more general case in which $\epsilon \neq 0$, let us briefly analyze (2.43) in the context of ultra slow-roll inflation, where the inflaton field moves over a constant potential and, as a consequence, the curvature perturbation evolves exponentially after horizon crossing. The salient feature of this model is the rapid decay of ϵ, which is found to be given by

$$\epsilon \propto \frac{1}{H^2a^6}. \tag{2.44}$$

Although $\epsilon \to 0$ very fast, the value of η is found to be large:

$$\eta = -6\left(1 - \frac{\epsilon}{3}\right). \tag{2.45}$$

The linear equation of motion respected by ζ on super-horizon scales is given by

$$\frac{d}{dt}\left(\epsilon a^3 \dot{\zeta}\right) = 0. \tag{2.46}$$

Then, neglecting terms subleading in ϵ, one finds that $\zeta \propto \tau^{-3}$. In other words, the power spectrum on superhorizon scales behaves as:

$$P_\zeta(k) \propto \tau^{-6}. \tag{2.47}$$

Using this result back into (2.43), we find that the bispectrum in ultra slow-roll is given by

$$B_\zeta(k_1, k_2, k_3) \;\; = \;\; 6P_\zeta(k_L)P_\zeta(k_S), \tag{2.48}$$

then

$$f_{\rm NL} = \frac{5}{2}, \tag{2.49}$$

which coincides with the well known expression previously found in the literature [57,81].

One should be careful with the result (2.48), even though it coincides with the known squeezed limit for ultra slow-roll inflation. Recall that we are judging the effect of the transformations (3.87)-(2.31) on the ζ-action from their effect on the metric. This implies that we are neglecting terms proportional to ϵ in the metric that could, according to Eq. (2.45), have a sizable impact on the action due to time derivatives of ϵ. Strictly speaking, at this point in our derivation the result of Eq. (2.43) is valid as long as $\epsilon \ll 1$ together with $\eta \ll 1$. But under these conditions it is hard (or impossible) to have a sizable super-horizon growth of ζ that could lead to an interesting situation where Eq. (2.43) could be used. To understand this issue more closely, let us analyze the case $\epsilon \neq 0$ in what follows.

2.2.3 Case with $\epsilon \neq 0$

Let us now analyze the more general case in which $\epsilon \neq 0$. Here, we may consider the following transformation of coordinates and fields:

$$x = e^g x', \tag{2.50}$$
$$a(\tau) = e^{-f} a(\tau'), \tag{2.51}$$
$$\zeta = \zeta' + \Delta\zeta. \tag{2.52}$$

Notice that we are defining the time reparametrization in terms of the scale factor a in order to keep the transformation in the spatial part of the metric (which involves $a(\tau)$) valid to all orders in the perturbation f. The effect of this transformation on the rest of the metric may be computed up to linear order. With this in mind, it is possible to derive that the time reparametrization to linear order is given by $\tau = \tau' - \frac{1}{\mathcal{H}} f$, where $\mathcal{H} = a^{-1}\partial_0 a$. Then, the transformations lead to:

$$dx^i = e^g dx'^i + e^g \partial_0 g x'^i d\tau', \tag{2.53}$$
$$d\tau = d\tau' + \left((1-\epsilon)f - \frac{1}{\mathcal{H}}\partial_0 f\right) d\tau', \tag{2.54}$$

where we used $\partial_0 \mathcal{H} = (1-\epsilon)\mathcal{H}^2$. Plugging these transformations back into the action (2.2), one finds:

$$\begin{aligned} ds^2 &= a^2(\tau')\Big[- e^{2\delta N' + 2\frac{1}{\mathcal{H}}\partial_0 \Delta\zeta - 2\epsilon f - 2\frac{1}{\mathcal{H}}\partial_0 f} d\tau'^2 \\ &\quad + 2\big(N_i' + \partial_0 g x_i' + \frac{1}{3} x_i' \epsilon \partial_0 \Delta\zeta\big) d\tau' dx'^i + e^{2\zeta' + 2\Delta\zeta + 2g - 2f} dx'^2 \Big]. \end{aligned} \tag{2.55}$$

Now, consider the following conditions on g and f:

$$\partial_0 \Delta\zeta - \epsilon \mathcal{H} f - \partial_0 f = 0, \tag{2.56}$$

$$\Delta\zeta + g - f = 0. \tag{2.57}$$

It is direct to see that these two equations imply:

$$\partial_0 g = -\epsilon \mathcal{H} f. \tag{2.58}$$

Then, the metric becomes

$$ds^2 = a^2(\tau')\Big[- e^{2\delta N'} d\tau'^2 + 2\big(N_i' + \Delta N_i\big) d\tau' dx'^i + e^{2\zeta'} dx'^2 \Big], \tag{2.59}$$

where we have defined ΔN_i as

$$\Delta N_i = -\epsilon \mathcal{H} f x_i' + \frac{1}{3} x_i' \epsilon (\epsilon \mathcal{H} f + \partial_0 f), \tag{2.60}$$

and where f is such that it is a solution of Eq. (2.56). Now, it is clear from this result that the ζ-action will not be invariant under the present transformation unless either $\Delta N_i = 0$, or ΔN_i leads to the appearance of a total derivative. This second option will not be true in general, and ΔN_i will imply terms in the action that are proportional to ϵ and η.

At this point, the metric of Eq. (2.59) differs from the original metric of Eq. (2.2) by the fact that ΔN_i does not vanish. The difference is of order ϵ, as expected from the analysis with $\epsilon \to 0$. In what follows, let us explore what would be required to satisfy the condition $\Delta N_i = 0$, independently of the size of ϵ (that is, we will not assume that ϵ is small). First, it is direct to see that $\Delta N_i = 0$ is equivalent to

$$\partial_0 (a^{-2} \mathcal{H}^{-1} f) = 0. \tag{2.61}$$

This implies that f must have the following dependence on time:

$$f = C a^2 \mathcal{H}, \tag{2.62}$$

where C is an integration constant that may be chosen conveniently. Note that here we cannot adopt the condition $f = 0$ at a given time $\tau = \tau_*$. This is because of the way in which f is introduced in Eq. (2.51). Now, according to Eq. (2.56), the solution for f given by Eq. (2.62) must be compatible with $\Delta\zeta$. In other words, it must be possible to choose C in such a way that

$$\partial_0 \Delta\zeta = 3 C H^2 a^4, \tag{2.63}$$

(where we have used $\mathcal{H} = Ha$). This corresponds to a strong restriction on $\Delta\zeta$, which has not been chosen yet. As in the previous subsections, we are interested in identifying $\Delta\zeta$ as:

$$\Delta\zeta = \zeta_L. \tag{2.64}$$

Inserting this back into (2.63), we see that $\Delta N_i = 0$ is only possible if (remember that in Eq. (2.63) $\partial_0 \equiv \partial_\tau$)

$$\dot{\zeta}_L = 3CH^2a^3. \tag{2.65}$$

Of course, this behavior is not guaranteed in general. However, in the particular case of ultra slow-roll inflation one has $\epsilon \propto 1/H^2a^6$, and so we may rewrite (2.65) as

$$\dot{\zeta}_L \propto \frac{1}{\epsilon a^3}, \tag{2.66}$$

which is nothing but (2.46). As a consequence, we see that in ultra slow-roll inflation one has $\Delta N_i = 0$ independently of the value of ϵ. Therefore, we have shown that the transformations (2.50)-(2.52) with f, g and $\Delta\zeta$ chosen as in (2.62), (2.58), and (2.64) respectively, correspond to an exact symmetry of the action for curvature perturbations in ultra slow-roll inflation (independent of the size of ϵ). This should not come as a surprise. Similar to exponential inflation, ultra slow-roll inflation never ends, and so the size of ϵ (which dilutes as $\sim a^{-6}$) cannot be regarded as a fundamental quantity describing the state of inflation.

The final step is to deduce an expression for ζ_S. This is found to be

$$\zeta_S(\tau, x) = \zeta'(e^{-\zeta_L - g}\tau, e^{-g}x), \tag{2.67}$$

with g the solution of Eq. (2.58). It is straightforward to see that g will contribute terms that are subleading in ϵ, and so we recover the expression (2.43) found where $\epsilon \to 0$. This in turn, leads to the well known result (2.48).

2.3 Analysis

Now that we know that (2.43) is valid for ultra slow-roll inflation, but not for general situations with $\epsilon \neq 0$, we would like to anticipate how this result could change once we consider models that depart from the exact ultra slow-roll behavior. First, if the action describing single-field inflation is canonical, then all of the couplings appearing in the ζ-action will consist of time derivatives of H, such as ϵ and η. Given that the action remains invariant under the set of transformations (2.50)-(2.52) in the case of ultra slow-roll, then models with a background close to ultra slow-roll have departures at most proportional to

$$6 + \eta.$$

However, in order to have a small spectral index in models close to ultra slow-roll it is necessary to have $|6 + \eta| \ll 1$, and so it would not be possible to have large departures from (2.43) unless the spectral index becomes incompatible with observations. Another possibility is to consider non-canonical models of inflation. In this class of models one has an additional parameter, the sound speed c_s, which is not directly related to variations of H. This time, the action for ζ could have terms (parametrizing departures from the ultra slow-roll case) proportional to:

$$\left(1 - \frac{1}{c_s^2}\right)\eta.$$

This type of departure would not be suppressed for small values of c_s, and one could expect sizable modifications to the result shown in (2.48). In fact, a direct computation shows that the modification to (2.48) due to c_s is given by [80]

$$B_\zeta(k_1, k_2, k_3) \;\simeq\; \frac{6}{c_s^2} P_\zeta(k_L) P_\zeta(k_S). \tag{2.68}$$

This result has also been obtained through symmetry arguments [81] pertaining the structure of the Lagrangian of $P(X)$-theories of inflation [85], but not through symmetry arguments related to space-time parametrizations, as considered here. Given that c_s appears as a consequence of non-gravitational interactions, it seems reasonable to assume that a space-time reparametrization leading to (2.68) does not exist.

2.4 Discussion

We have generalized the well known non-Gaussian consistency relation (2.25) to a broader class of relations that is able to cope with those classes of models where the curvature perturbation ζ evolves on super-horizon scales. This relation is given by Eq. (3.86), and in the case where the super-horizon growth dominates, it leads to:

$$B_\zeta(k_1, k_2, k_3) \;=\; -P_\zeta(k_L) \frac{d}{d\ln\tau} P_\zeta(k_S). \tag{2.69}$$

The standard non-Gaussian consistency relation (2.25) can be understood a symmetry involving a simultaneous spatial dilation and a reparametrization of the curvature perturbation. In the case of (2.69), the symmetry involves a time dilation together with a reparametrization of the curvature perturbation. In both cases, the reparametrization is induced by super-horizon evolution of the long-wavelength contributions of the curvature perturbation. While this symmetry is approximate in general when $\epsilon \ll 1$, it becomes exact in the case of ultra slow-roll, independently of the value of ϵ. (It is also exact when $\epsilon = 0$.)

Our result complements previous studies on consistency relations derived from symmetries of quasi-de Sitter spacetimes [18, 20, 21, 84, 86–88] applied to the context in which curvature perturbations freeze at horizon crossing. In addition, our result substantiates one more time the well known violation to the standard consistency relation found by the authors of Ref. [57]. However, our result raises the question how the non-Gaussianity expressed in (2.69) would survive the transition from a non-attractor phase —in which ultra slow-roll is dominant— to an attractor phase where standard slow-roll inflation is dominant (before inflation ends).

Given that the expression leading to (2.69) involves a time derivative of the power spectrum, one may suspect that once the non-attractor phase concludes, and the modes stop evolving on super-horizon scales, this contribution would become suppressed. In this case, the transition to the new phase would imply a leading contribution to the bispectrum dictated by the scale dependence of the power spectrum (through $n_s - 1$). Strictly speaking, our expression cannot describe this transition. This is because during such a transition the system is no longer invariant under the set of transformations (2.50)-(2.52). It is invariant during ultra slow-roll, and during slow-roll, but not in between.

One could speculate that in such a transition (from non-attractor to attractor, see also [89]) the amount of non-Gaussianity in the form of (2.69) could be transferred to a form of non-Gaussianity that is described by (2.25). But this would necessarily imply an unacceptably large value of the spectral index $n_s - 1$. Another possibility is that, instead of (2.69), the bispectrum produced during ultra slow-roll has to be read as

$$B_\zeta(k_1, k_2, k_3) \;\simeq\; 6P_\zeta(k_L)P_\zeta(k_S), \tag{2.70}$$

without taking into consideration the time derivative appearing in the expression preceding it. In other words, the factor 6 implied by the τ-derivative becomes engraved on the distribution of superhorizon modes, and survives until the modes re-enter the horizon much after inflation. Given that ultra slow-roll inflation has gained some prominence as a transient period of inflation that could explain certain phenomena associated to primordial physics [90,91], this seems to be a relevant issue to clarify.[3]

Ultimately, however, we are interested in the amount of squeezed non-Gaussianity available to a free-falling observer like us, rather than to a comoving observer. In Chapter 3 we will study this issue more closely, by introducing the use of conformal Fermi coordinates [55,70,71]. There, we will argue that the primordial squeezed non-Gaussianity produced in non-attractor models such as ultra slow-roll is non-observable ($f_{\rm NL}^{\rm obs} = 0$). That would render irrelevant the question to which extent the non-attractor corrections to $f_{\rm NL}$ in comoving coordinates, as computed in this paper, survive the end of inflation, or the transition to a slow-roll phase. Whatever comoving result one gets, we will conjecture that it will be cancelled by a similar term arising from the switch from comoving to Fermi coordinates.

[3] In [92] was found that, in comoving coordinates, the transition from non-attractor to attractor inflation is characterized by a reduction of non-Gaussianity. The amount of non-Gaussianity that survives depends crucially on the nature of this transition. Roughly speaking: a sharper transition leads to more surviving non-Gaussianity.

Chapter 3

Vanishing of the Local non-Gaussianity

Maldacena's consistency relation [42] has stood out as one of the key relations allowing us to test cosmic inflation [8–12]. It ties together two observables, the size of primordial local non-Gaussianity, $f_{\rm NL}$, and the power spectrum's spectral index, $n_s - 1$, in a simple relation given by:

$$f_{\rm NL} = \frac{5}{12}(1 - n_s). \tag{3.1}$$

This relation has been shown to remain valid for all single-field attractor models of inflation, characterized by the freezing of the curvature perturbation after horizon crossing (attractor models of single field inflation are models in which every background quantity during inflation is determined by a single parameter, for instance, the value of the Hubble expansion rate H, regardless of the initial conditions). Moreover, Eq. (3.1) is understood to be the consequence of how long wavelength modes of curvature perturbations modulate the amplitude of shorter wavelength modes [16, 62–69]. This modulation is found to be enforced by a symmetry of the action for curvature perturbations under a transformation simultaneously involving spatial dilations and a field reparametrization.

Relation (3.1) is not satisfied by non-attractor models of single field inflation, for which the background depends crucially on the initial conditions [57, 78, 79, 81, 82]. For instance, in the extreme case of ultra slow-roll inflation [78, 82], where the amplitude of comoving curvature perturbations grows exponentially fast outside the horizon,[1] it has been shown [22, 57, 79, 81, 94] that $f_{\rm NL}$ is given by:[2]

$$f_{\rm NL} = \frac{5}{2}. \tag{3.2}$$

This result has been extended to more general models of non-attractor inflation [80], for instance, realized with the help of nontrivial kinetic terms [59]. Given that local non-

[1] While the comoving curvature perturbation grows as a^3 on superhorizon scales, the non-adiabatic pressure still vanishes, providing a counterexample for the standard intuition that superhorizon freezing is caused by adiabaticity [58]. See also [93].

[2] Ultra slow-roll is not a realistic model in at least two ways: (1) The inflationary potential is exactly flat, so inflation does not have an end. (2) The value of the spectral index is essentially zero. In fact, Eq. (3.2) may be written as $f_{\rm NL} = 5/2 - (1 - n_s)5/4$, but $n_s - 1$ decays as e^{-6N}, where N is the number of e-folds after horizon crossing. For these reasons, we take ultra slow-roll as a proxy model allowing for large local non-Gaussianity.

Gaussianity is currently constrained by Planck as $f_{\rm NL} = -0.9 \pm 5.1$ (68%CL) [50], the result shown in Eq. (3.2) has strengthened the importance of reaching new observational targets as a way to rule out (or to confirm) exotic mechanisms underlying the origin of primordial perturbations. Other models known for predicting potentially large local non-Gaussianity include multi-field models of inflation [77] and curvaton models [76], but these scenarios require more than one scalar degree of freedom, so we will not consider them here.

Now, neither Eq. (3.1) nor (3.2) correspond to proper predictions for the amount of local non-Gaussianity available to inertial observers, such as us. In a cosmological setup, a physical local observer follows a geodesic path that does not necessarily coincide with that of an observer fixed at a given comoving coordinate. This is simply because inertial observers are themselves subject to the presence of perturbations. In the particular case of attractor models, when this fact is taken into account, one finds that the amount of local non-Gaussianity measured by an inertial observer is given by Eq. (3.1) plus a correction of the same order. More to the point, one finds [54, 55, 70–72]:

$$f_{\rm NL}^{\rm obs} = 0. \tag{3.3}$$

In other words, the true prediction for observable primordial local non-Gaussianity coming from attractor models of inflation is zero.[3] Because of this, one may wonder about the status of the observable local non-Gaussianity in more general situations in which the value for local non-Gaussianity computed in comoving gauge is predicted to be large, just like in non-attractor models of inflation such as ultra slow-roll.

The purpose of this chapter is to analyze the prediction of observable local non-Gaussianity in more general contexts, beyond those covered by attractor models of single field inflation. To achieve this, we will adopt the Conformal Fermi Coordinates (CFC) formalism introduced by Pajer, Schmidt and Zaldarriaga in Ref. [55] and further developed in refs. [70, 71]. These coordinates are the natural generalization of the so-called Fermi normal coordinates [75]. In short, the use of CFC in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime allows one to describe the local environment of freely falling inertial observers up to distances much longer than the Hubble radius. This allows one to follow the fate of primordial curvature perturbations within the CFC frame during the whole relevant period of inflation, and perform the computation of gauge invariant n-point correlation functions. As we shall see, the expressions for n-point correlation functions in the CFC frame differ from those computed in comoving gauge. To understand why this is so, one has to keep in mind that while an ordinary gauge transformation (diffeomorphism) does not change physical observables, the introduction of CFC's corresponds to a change of coordinates that relate global with local coordinates.

We shall see that the long wavelength perturbations do not only modulate the evolution of perturbations of shorter wavelengths, they also affect the geodesic path of inertial observers. These two effects combined imply that long wavelength modes cannot affect short wavelength modes in any observable way. This has been well understood in the case of attractor models. Here we extend this result to the entire family of canonical single field inflation, regardless

[3] More precisely, we should write $f_{\rm NL}^{\rm obs} = 0 + \mathcal{O}(k_L/k_S)^2$, where the $\mathcal{O}(k_L/k_S)^2$ terms are caused by non-primordial phenomena such as gravitational lensing and redshift perturbations (the so called projection effects [73, 74]). See Ref. [55] for more details on this.

of whether they are attractor or non-attractor. To show this, we will compute the squeezed limit of the bispectrum in CFC coordinates, and find that it is given by

$$B_{\rm CFC} = B_{\rm CM} + \Delta B, \tag{3.4}$$

where $B_{\rm CM}$ is the bispectrum computed in comoving gauge, and ΔB is the correction due to the change of coordinates. The term $B_{\rm CM}$ is dictated by the invariance of the action for curvature perturbations under a transformation involving both spatial and time dilations together with a field reparametrization [22]. On the other hand, we will show that the transformation dictating the form of ΔB is exactly the inverse of such a transformation. As a result, we find that during inflation, after all the modes have exited the horizon, $B_{\rm CFC}$ vanishes, regardless of whether inflation is attractor or non-attractor.

We begin this chapter by reviewing the construction of conformal Fermi coordinates in a perturbed FLRW spacetime in Section 3.1. There we also sketch the computation of the squeezed limit of the primordial non-Gaussian bispectrum leading to $f_{\rm NL}^{\rm obs}$. Then, in Section 3.2 we further develop and simplify some results found in refs. [70, 71]. In addition, we apply these results to show that the observable primordial local non-Gaussianity vanishes in both, attractor and non-attractor models of inflation. Finally, in Section 3.3 we offer some concluding remarks.

3.1 Conformal Fermi Coordinates

Here we offer a review on how Conformal Fermi Coordinates (CFC) are defined and used to compute correlation functions for inertial observers in terms of those valid for comoving observers. We will mostly base this discussion on refs. [70, 71], with some slight variations of the notation.

3.1.1 Central Geodesic

Let us start by considering an unperturbed cosmological background described by a FLRW metric in terms of conformal time τ:

$$ds_0^2 = a^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + d\mathbf{x}^2\right). \tag{3.5}$$

Here $a(\tau)$ is the scale factor and $\mathbf{x}$ is the position in comoving coordinates. The perturbed spacetime may be described with the help of the following metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left[\eta_{\mu\nu} + h_{\mu\nu}\right]dx^\mu dx^\nu, \tag{3.6}$$

where $h_{\mu\nu}$ parametrize deviations from the FLRW background. We will later use a specific form of $h_{\mu\nu}$ in which curvature perturbations are introduced. An inertial observer will follow a geodesic motion determined by $g_{\mu\nu}$, respecting the following equation of motion

$$\frac{d^2x^\mu}{d\eta^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\eta}\frac{dx^\sigma}{d\eta} = 0, \tag{3.7}$$

where $\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma})$ are the usual Christoffel symbols, and η is a given affine parameter. Let us call the resulting geodesic G, and take $\bar{t}$ to be the proper time employed by the inertial observer to parametrize time in his/her local environment. We will introduce a scale factor $a_F(\bar{t})$ that parametrizes the expansion felt by the observer in his/her vicinity. The precise definition of $a_F(\bar{t})$ is made in Eq. (3.25). In the meantime, notice that the introduction of $a_F(\bar{t})$ allows us to define a conformal time $\bar{\tau}$ through the following standard relation:

$$d\bar{\tau} = d\bar{t}/a_F(\bar{t}). \tag{3.8}$$

Because the inertial observer follows a geodesic motion that does not remain fixed to a comoving coordinate, it should be clear that τ and $\bar{\tau}$ will not coincide. Next, let us consider an arbitrary point P along G corresponding to conformal time $\bar{\tau}_P$. We wish to introduce a set of coordinates $x^{\bar{\alpha}} = \{\bar{\tau}, x^{\bar{\imath}}\}$ in such a way that $x^{\bar{\imath}}$ parametrizes the 3-dimensional slices of constant $\bar{\tau}_P$. In these coordinates, one has $x^{\bar{\alpha}}(P) = \{\bar{\tau}_P, 0\}$. At this point, one may introduce a set of tetrads $e^{\mu}_{\bar{\alpha}}$ such that:

$$g_{\mu\nu} e^{\mu}_{\bar{\alpha}} e^{\nu}_{\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}}, \tag{3.9}$$

in the vicinity of the entire geodesic. Equation (3.9) will be particularly true at the point P. We demand that the $\bar{0}$-component $U^\mu \equiv e^{\mu}_{\bar{0}}$ coincides with the normalized vector tangent to the geodesic. Then, the rest of the tetrads $e^{\mu}_{\bar{\imath}}$ correspond to the space-like vectors orthogonal to the geodesic.

In a perturbed FLRW spacetime parametrized by the metric (3.6), the tetrads $e^{\mu}_{\bar{\alpha}}$ may be conveniently written as:

$$e^{\mu}_{\bar{0}} = \frac{1}{a(\tau)} \left(1 + \frac{1}{2} h_{00} \, , \, V^i \right), \tag{3.10}$$

$$e^{\mu}_{\bar{\jmath}} = \frac{1}{a(\tau)} \left(V_j + h_{0j} \, , \, \delta^i_j - \frac{1}{2} h^i{}_j + \frac{1}{2} \varepsilon_j{}^{ik} \omega_k \right) \delta^j_{\bar{\jmath}}, \tag{3.11}$$

where V^i parametrizes the 3-component velocity of U^μ, and ω_k parametrizes the rotation of the spatial components of the tetrad induced by the perturbations. In these expressions, and in the rest of this chapter, spatial indices are raised and lowered with δ^{ij} and δ_{ij} respectively. Notice that the construction outlined in the previous section requires that the combination $e^{\mu}_{\bar{\alpha}}$ be parallel transported along the geodesic (recall that $U^\mu = e^{\mu}_{\bar{0}}$ and that $e^{\mu}_{\bar{\imath}}$ are normalized vectors orthogonal to U^μ). Then V^i and ω_k satisfy the following equations

$$\partial_0 V^i + \mathcal{H} V^i = \frac{1}{2} \partial^i h_{00} - \partial_0 h^i{}_0 - \mathcal{H} h^i{}_0, \tag{3.12}$$

$$\partial_0 \omega^k = -\frac{1}{2} \varepsilon^{kij} \left(\partial_i h_{0j} - \partial_j h_{0i} \right). \tag{3.13}$$

In Section 3.1.4 we will see that $h_{0i} = h_{i0}$ is given by a gradient of the curvature perturbation [see Eq. (3.31)]. This will imply that the right hand side of Eq. (3.13) vanishes, and ω^k may be chosen to vanish without loss of generality.

3.1.2 Construction of the CFC Map

Now we have the challenge to define the slice of constant $\bar{\tau}_P$. An arbitrary point Q in the vicinity of P, in the same slice of constant $\bar{\tau}_P$, will have coordinates $x^{\bar{\alpha}}(Q) = \{\bar{\tau}_P, x^{\bar{\imath}}_Q\}$. We may reach Q from P through certain classes of geodesics that are constructed as follows:

First, we introduce the conformally flat metric $\tilde{g}_{\mu\nu} \equiv a_F^{-2}(\bar{\tau}) g_{\mu\nu}$, and then solve the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0, \tag{3.14}$$

where $\tilde{\Gamma}^\mu_{\rho\sigma}$ are Christoffel symbols computed out of $\tilde{g}_{\mu\nu}$. To solve this equation, one chooses the following initial conditions:

$$\left. \frac{dx^\mu}{d\lambda} \right|_{\lambda=0} = a_F(\bar{\tau}_P) e^\mu_{\bar{\imath}} \Delta x^{\bar{\imath}}_Q, \tag{3.15}$$

where $\Delta x^{\bar{\imath}}_Q = x^{\bar{\imath}}_Q - x^{\bar{\imath}}_P$, and $x^{\bar{\imath}}_Q$ is the position of Q that is reached at $\lambda = 1$. One may solve Eq. (3.14) perturbatively by writing

$$x^\mu(\lambda) = \sum_{n=0}^{\infty} \alpha^\mu_n \lambda^n. \tag{3.16}$$

Since $\lambda = 0$ corresponds to the starting point P, one has $\alpha^\mu_0 = x^\mu(P)$. On the other hand, the initial condition (3.15) implies $\alpha^\mu_1 = e^\mu_{\bar{\imath}} \Delta x^{\bar{\imath}}_Q$. It is then possible to show that the solution to Eq. (3.14), up to cubic order, is given by

$$\begin{aligned} x^\mu(\lambda) &= x^\mu(P) + a_F(\bar{\tau}_P) e^\mu_{\bar{\imath}} \Big|_P \Delta x^{\bar{\imath}}_Q \, \lambda - \frac{1}{2} \tilde{\Gamma}^\mu_{\rho\sigma} a_F^2(\bar{\tau}_P) e^\rho_{\bar{\imath}} e^\sigma_{\bar{\jmath}} \Big|_P \Delta x^{\bar{\imath}}_Q \Delta x^{\bar{\jmath}}_Q \, \lambda^2 \\ &\quad - \frac{1}{6} \left(\partial_\nu \tilde{\Gamma}^\mu_{\rho\sigma} - 2 \tilde{\Gamma}^\mu_{\alpha\rho} \tilde{\Gamma}^\alpha_{\sigma\nu} \right) a_F^3(\bar{\tau}_P) e^\rho_{\bar{\imath}} e^\sigma_{\bar{\jmath}} e^\nu_{\bar{k}} \Big|_P \Delta x^{\bar{\imath}}_Q \Delta x^{\bar{\jmath}}_Q \Delta x^{\bar{k}}_Q \, \lambda^3 + \cdots . \end{aligned} \tag{3.17}$$

Evaluating this result at $\lambda = 1$ then gives us the position of an arbitrary point Q with respect to the position of P, and so, one may just drop the labels P and Q to obtain the coordinate transformation between the sets of coordinates x^μ and $x^{\bar{\alpha}}$ up to cubic order:

$$\begin{aligned} \Delta x^\mu &= a_F(\bar{\tau}_P) e^\mu_{\bar{\imath}} \Big|_P \Delta x^{\bar{\imath}} - \frac{1}{2} \tilde{\Gamma}^\mu_{\rho\sigma} a_F^2(\bar{\tau}_P) e^\rho_{\bar{\imath}} e^\sigma_{\bar{\jmath}} \Big|_P \Delta x^{\bar{\imath}} \Delta x^{\bar{\jmath}} \\ &\quad - \frac{1}{6} \left(\partial_\nu \tilde{\Gamma}^\mu_{\rho\sigma} - 2 \tilde{\Gamma}^\mu_{\alpha\rho} \tilde{\Gamma}^\alpha_{\sigma\nu} \right) a_F^3(\bar{\tau}_P) e^\rho_{\bar{\imath}} e^\sigma_{\bar{\jmath}} e^\nu_{\bar{k}} \Big|_P \Delta x^{\bar{\imath}} \Delta x^{\bar{\jmath}} \Delta x^{\bar{k}} + \cdots , \end{aligned} \tag{3.18}$$

where $\Delta x^\mu = x^\mu(Q) - x^\mu(P)$. From this change of coordinates, it is now possible to deduce the form of the metric in conformal Fermi coordinates:

$$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} \frac{\partial x^\nu}{\partial x^{\bar{\beta}}} g_{\mu\nu}. \tag{3.19}$$

From this expression, one explicitly finds:

$$g_{\bar{0}\bar{0}} = a_F^2(\bar{\tau}) \left[-1 - \left(\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}} \right)_P \Delta x^{\bar{k}} \Delta x^{\bar{l}} + \mathcal{O}(\Delta \bar{x}^3) \right], \tag{3.20}$$

$$g_{\bar{0}\bar{\jmath}} = a_F^2(\bar{\tau}) \left[-\frac{2}{3} \left(\tilde{R}_{\bar{0}\bar{k}\bar{\jmath}\bar{l}} \right)_P \Delta x^{\bar{k}} \Delta x^{\bar{l}} + \mathcal{O}(\Delta \bar{x}^3) \right], \tag{3.21}$$

$$g_{\bar{\imath}\bar{\jmath}} = a_F^2(\bar{\tau}) \left[\delta_{ij} - \frac{1}{3} \left(\tilde{R}_{\bar{\imath}\bar{k}\bar{\jmath}\bar{l}} \right)_P \Delta x^{\bar{k}} \Delta x^{\bar{l}} + \mathcal{O}(\Delta \bar{x}^3) \right], \tag{3.22}$$

where $\tilde{R}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$ are the components of the Riemann tensor constructed from $\tilde{g}_{\mu\nu}$ and projected along the CFC directions with the help of the tetrad introduced earlier:

$$\tilde{R}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = a_F^4(\bar{\tau}_P) e^\mu_{\bar{\alpha}} e^\nu_{\bar{\beta}} e^\rho_{\bar{\gamma}} e^\sigma_{\bar{\delta}} \tilde{R}_{\mu\nu\rho\sigma}. \tag{3.23}$$

In the previous expressions $\mathcal{O}(\Delta\bar{x}^3)$ stands for terms of order $\left(\partial_{\bar{\imath}}\tilde{R}_{\bar{\alpha}\bar{\jmath}\bar{\beta}\bar{k}}\right)_P \Delta x^{\bar{\imath}}\Delta x^{\bar{\jmath}}\Delta x^{\bar{k}}$. This is in fact one of the salient points of this construction: Higher order corrections to the metric $g_{\bar{\alpha}\bar{\beta}}$ are suppressed by both spatial derivatives of the Riemann tensor and powers of $x^{\bar{\imath}}$. We will see how this plays a role later on, when we examine the validity of the CFC to follow the evolution of perturbations during inflation.

3.1.3 Choosing the Conformal Scale Factor a_F

Notice that up to now the scale factor a_F has not been properly defined. This may be done by first recalling that $U^\mu \equiv e^\mu_{\bar{0}}$ defined earlier satisfies the geodesic equation $U^\nu\nabla_\nu U^\mu = 0$. In order to study how two nearby parallel geodesics diverge, one may introduce the velocity divergence parameter ϑ as:

$$\vartheta \equiv \nabla_\mu U^\mu. \tag{3.24}$$

We then demand that the scale factor a_F satisfies the following equation:

$$H_F \equiv \frac{1}{a_F}\frac{da_F}{d\bar{t}} = \frac{1}{3}\vartheta. \tag{3.25}$$

One crucial reason behind this choice is that ϑ is a local observable, and so H_F is the true Hubble parameter describing the local expansion of the patch surrounding geodesic. One may read another consequence coming out from this choice. The velocity divergence ϑ satisfies the Raychaudhuri equation:

$$\frac{d\vartheta}{d\bar{t}} + \frac{1}{3}\vartheta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R^\mu{}_{\rho\mu\sigma}U^\rho U^\sigma, \tag{3.26}$$

where $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the trace-free symmetric and antisymmetric contributions to $\nabla_\mu U_\nu$ respectively. It is possible to work out $R^\mu{}_{\rho\mu\sigma}U^\rho U^\sigma = R^{\bar{\mu}}{}_{\bar{\rho}\bar{\mu}\bar{\sigma}}U^{\bar{\rho}}U^{\bar{\sigma}}$ to show that the Raychaudhuri equation reduces to

$$\frac{d\vartheta}{d\bar{t}} + \frac{1}{3}\vartheta^2 = 3(\dot{H}_F + H_F^2) - \sigma_{\bar{\imath}\bar{\jmath}}\sigma^{\bar{\imath}\bar{\jmath}} + \omega_{\bar{\imath}\bar{\jmath}}\omega^{\bar{\imath}\bar{\jmath}} - a_F^{-2}\tilde{R}^{\bar{\imath}}{}_{\bar{0}\bar{\imath}\bar{0}}. \tag{3.27}$$

But since a_F has been chosen to satisfy (3.25), this equation further reduces to

$$\sigma_{\bar{\imath}\bar{\jmath}}\sigma^{\bar{\imath}\bar{\jmath}} - \omega_{\bar{\imath}\bar{\jmath}}\omega^{\bar{\imath}\bar{\jmath}} = -a_F^{-2}\tilde{R}^{\bar{\imath}}{}_{\bar{0}\bar{\imath}\bar{0}}. \tag{3.28}$$

In a homogeneous background, both σ and ω vanish. Thus, we see that the choice of Eq. (3.25) implies that $\tilde{R}^{\bar{\imath}}{}_{\bar{0}\bar{\imath}\bar{0}}$ is necessarily of second order in perturbations.

3.1.4 CFC in Inflation

Now that we have in our hands the notion of conformal Fermi coordinates, we may examine their form in the specific case of a perturbed FLRW spacetime during inflation. First, it is convenient to consider the perturbed metric of Eq. (3.6) in comoving gauge, where the coordinates are such that the fluctuations of the fluid driving inflation vanish. In the case of single field canonical inflation, this gauge corresponds to the case in which the perturbations of the scalar field driving inflation satisfy $\delta\phi = 0$. In this gauge, it is customary to introduce the comoving curvature perturbation variable ζ through the relation:

$$h_{ij} = \left[e^{2\zeta} - 1\right]\delta_{ij}. \tag{3.29}$$

Then, Einstein's equations imply constraint equations for h_{00} and $h_{0i} = h_{i0}$. To linear order, the solutions of these equations are found to be given by

$$h_{00} = -\frac{2}{\mathcal{H}}\partial_0\zeta, \tag{3.30}$$

$$h_{0i} = -\partial_i\left[\frac{\zeta}{\mathcal{H}} - \epsilon\partial^{-2}\partial_0\zeta\right], \tag{3.31}$$

where $\mathcal{H} \equiv \partial_0 \ln a$ and $\partial_0 \equiv \partial/\partial\tau$. These solutions will change in non-canonical models of inflation. For instance, the fluid driving inflation could induce an effective sound speed c_s parametrizing the speed at which curvature perturbations propagate [64], as in the case of $P(X)$-models [85] or single field EFT's describing multi-field models with massive fields [95]. In these cases, c_s would modify the second constraint equation (3.31).

Now, we would like to integrate Eqs. (3.12) and (3.25) taking into account the introduction of the curvature perturbation ζ. This will allow us to extend the CFC map (3.18) at any time $\tau \geq \tau_P$. Let us start by considering the integration of Eq. (3.12) along the geodesic path. To do so, let us introduce the following combination involving V^i and h_{0i}:

$$\mathcal{F}^i = V^i + {h_0}^i. \tag{3.32}$$

Given that h_{0i} is a gradient, it should be clear that Eq. (3.12) implies that the non-trivial part of V^i is also given by a gradient. We therefore write $\mathcal{F}_i = \partial_i\mathcal{F}$. Then, direct integration of Eq. (3.12) gives

$$\mathcal{F}(\tau,\mathbf{x}) = e^{-\int_{\tau_*}^{\tau} ds\mathcal{H}(s)}\left[\frac{1}{\mathcal{H}_*}C_F(\tau_*,\mathbf{x}) + \frac{1}{2}\int_{\tau_*}^{\tau} ds\, e^{\int_{\tau_*}^{s} dw\mathcal{H}(w)}h_{00}(s,\mathbf{x})\right], \tag{3.33}$$

where $C_F(\tau_*,\mathbf{x}_c)$ is an integration constant defined on the geodesic path that must be taken to be linear in the perturbations. Given that we are interested in gradients of $\mathcal{F}$, we must allow for the existence of $\partial_i C_F(\tau_*,\mathbf{x})$ and $\partial^2 C_F(\tau_*,\mathbf{x})$. This result, together with h_{0i} found in (3.31) gives us back V^i. Let us now consider the integration of Eq. (3.25). By using Eq. (3.10) to write U^μ in terms of V^i and h_{00}, Eq. (3.25) gives a first order differential equation for the combination $a_F(\bar\tau)/a(\tau)$ which, to leading order in the perturbations yields

$$\frac{a_F(\bar\tau)}{a(\tau)} - 1 = C_a(\tau_*,\mathbf{x}_c(\tau_*)) + \int_{\tau_*}^{\tau} ds\left[\partial_0\zeta(s,\mathbf{x}_c(s)) + \frac{1}{3}\partial_i V^i(s,\mathbf{x}_c(s))\right], \tag{3.34}$$

where $\mathbf{x}_c(s)$ denotes the path of the geodesic in comoving coordinates. In the previous expression τ_* corresponds to a given initial time, and $C_a(\tau_*,\mathbf{x}_c)$ denotes an integration constant which should be considered to be of linear order in the perturbations.

We are now in a condition to deduce the form of the conformal Fermi coordinates valid at times $\tau > \tau_P$. To do so, let us consider an arbitrary point P_2 located on the central geodesic G at a given time $\tau > \tau_P$. It is clear that $x^\mu(P_2)$ will differ from the value $x_0^\mu(P_2)$ that would have been obtained in an unperturbed universe. The difference is accounted by a deviation $\rho^\mu(\tau)$ that is at least linear in the perturbations:

$$x^\mu(P_2) = x_0^\mu(P_2) + \rho^\mu(\tau). \tag{3.35}$$

Having this in mind, we may express the CFC map using the following ansatz for an arbitrary time $\tau > \tau_P$:

$$x^\mu(\bar\tau,\bar{\mathbf{x}}) = x_0^\mu(\bar\tau,\bar{\mathbf{x}}) + \rho^\mu(\tau) + A^\mu_{\bar\imath}(\tau)\Delta x^{\bar\imath} + B^\mu_{\bar\imath\bar\jmath}(\tau)\Delta x^{\bar\imath}\Delta x^{\bar\jmath} + C^\mu_{\bar\imath\bar\jmath\bar k}(\tau)\Delta x^{\bar\jmath}\Delta x^{\bar\jmath}\Delta x^{\bar k} + \cdots\,, \tag{3.36}$$

where $x_0^\mu(\bar\tau,\bar{\mathbf{x}})$ is the unperturbed map, for which the comoving coordinates and conformal Fermi coordinates coincide. More precisely, $x_0^\mu(\bar\tau,\bar{\mathbf{x}})$ is such that:

$$x_0^0(\bar\tau,\bar{\mathbf{x}}) = \bar\tau, \qquad x_0^i(\bar\tau,\bar{\mathbf{x}}) = x_c^i + \delta^i_{\bar\imath}\Delta x^{\bar\imath}. \tag{3.37}$$

The coefficients $\rho^\mu(\tau)$, $A^\mu_{\bar\imath}(\tau)$, $B^\mu_{\bar\imath\bar\jmath}(\tau)$ and $C^\mu_{\bar\imath\bar\jmath\bar k}(\tau)$ are all linear in the perturbations. In Appendix A.1 these are shown to be given by the following expressions:

$$\begin{aligned}
\rho^0(\tau) &= \int_{\tau_*}^{\tau} ds\left[\frac{a_F(\bar\tau)}{a(\tau)}(s,\mathbf{x}_c) - 1 + \frac{1}{2}h_{00}(s,\mathbf{x}_c)\right], && (3.38)\\
\rho^i(\tau) &= \int_{\tau_*}^{\tau} ds\, V^i(s,\mathbf{x}_c), && (3.39)\\
A^0_{\bar\imath}(\tau) &= \delta^i_{\bar\imath}F_i(\tau,\mathbf{x}_c), && (3.40)\\
A^i_{\bar\imath}(\tau) &= \left[\frac{a_F(\bar\tau)}{a(\tau)} - 1 - \zeta(\tau,\mathbf{x}_c)\right]\delta^i_{\bar\imath}, && (3.41)\\
B^\mu_{\bar\imath\bar\jmath}(\tau) &= -\frac{1}{2}\tilde\Gamma^\mu_{ij}(\tau,\mathbf{x}_c)\delta^i_{\bar\imath}\delta^j_{\bar\jmath}, && (3.42)\\
C^\mu_{\bar\imath\bar\jmath\bar k}(\tau) &= -\frac{1}{6}\partial_k\tilde\Gamma^\mu_{ij}(\tau,\mathbf{x}_c)\delta^i_{\bar\imath}\delta^j_{\bar\jmath}\delta^j_{\bar k}. && (3.43)
\end{aligned}$$

The right hand sides of the previous expressions are all expanded up to first order in the perturbations (recall that $a_F(\bar\tau)/a(\tau)(s,\mathbf{x}_c) - 1$ is a quantity of linear order in the perturbations). Along the geodesic one has $\Delta\bar{\mathbf{x}} = 0$ and the map reduces to $x^\mu(\bar\tau,\bar{\mathbf{x}}_c) = x_0^\mu(\bar\tau,\bar{\mathbf{x}}_c) + \rho^\mu(\tau)$. This implies that:

$$\tau = \bar\tau + \rho^0(\tau), \qquad x^i = x_c^i + \rho^i. \tag{3.44}$$

Then $\rho^0(\tau) = \tau - \bar\tau$ informs us how the perturbations shift the equal time slices parametrized by τ and $\bar\tau$. Similarly, ρ^i parametrizes the spatial shift of the geodesic from the unperturbed position x_c^i (that is, $x_c^i + \rho^i$ is the location of the geodesic at a time $\tau > \tau_*$).

3.1.5 Computation of Correlation Functions with CFC's

Here we consider the task of computing correlation functions using the CFC map of Eq. (3.36). We are particularly interested in the squeezed limit of the three-point function $\langle\zeta\zeta\zeta\rangle$. In this subsection we will sketch the procedure, that will be implemented in more detail in the next section, after we have considered some further simplifications of the map (3.36). The main idea is the following: We will split the curvature perturbation ζ into short and long wavelength contributions:

$$\zeta = \zeta_S + \zeta_L. \tag{3.45}$$

Then, we will use ζ_L to find the perturbed spacetime determining the map (3.36) deduced in the previous section. In other words, in global coordinates we study a FLRW metric perturbed by ζ_L. The map of Eq. (3.36) then shows that $\tau = \bar\tau + \mathcal{O}(\zeta_L)$, $x^i = \bar x^i + \mathcal{O}(\zeta_L)$. We then use the inverse of this map to see how local quantities (such as $\bar\zeta_S(\bar x)$ and its correlation functions) can be written in terms of well-known global quantities. This will allow us to derive an expression for the short wavelength curvature perturbation $\bar{\zeta}_S$ (defined in the inertial frame) as a function of both, the short and long wavelength curvature perturbations ζ_S and ζ_L (defined in the comoving frame):

$$\bar{\zeta}_S = \zeta_S + F_{\rm S}(\zeta_S, \zeta_L). \tag{3.46}$$

Here the function $F_{\rm S}(\zeta_S, \zeta_L)$ informs us about how the long wavelength mode ζ_L affects the behavior of $\bar{\zeta}_S$ due to the fact that this is a local quantity defined within the patch surrounding the geodesic. This function will be of the form (see next subsection)

$$F_{\rm S}(\zeta_S, \zeta_L) = \sum_a f_a(\zeta_L) g_a(\zeta_S), \tag{3.47}$$

where $f_a(\zeta_L)$ and $g_a(\zeta_S)$ are linear functions of ζ_L and ζ_S respectively (that could include space-times derivatives acting on the perturbations). Of course, the long mode ζ_L does not globally affect itself, in the sense that the CFC map corresponds to a local small scale coordinate transformation that can only affect the short scale contribution $\bar{\zeta}_S$ inside the patch around the central geodesic. Therefore we effectively write:

$$\bar{\zeta}_L = \zeta_L. \tag{3.48}$$

Having (3.46) and (3.48) then allows us to compute the squeezed limit of the three point correlation function $\langle \bar{\zeta}\bar{\zeta}\bar{\zeta} \rangle$. To this effect, one first considers the computation of the two point correlation function $\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2) \rangle$. Because $\bar{\zeta}_S$ is given by (3.46) with F_S given by (3.47), this two point correlation function may be expanded to linear order in ζ_L as:

$$\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2) \rangle = \langle \zeta_S(x_1)\zeta_S(x_2) \rangle + \sum_a f_a(\zeta_L) \left[\langle \zeta_S(\bar{x}_1) g_a(\zeta_S(\bar{x}_2)) \rangle + \langle g_a(\zeta_S(\bar{x}_1))\zeta_S(\bar{x}_2) \rangle \right]. \tag{3.49}$$

Then, by correlating this result with the long mode $\bar{\zeta}_L$ of Eq. (3.48), one obtains

$$\begin{aligned} \langle \bar{\zeta}_L(\bar{x}_3) \langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2) \rangle \rangle &= \langle \zeta_L(x_3) \langle \zeta_S(x_1)\zeta_S(x_2) \rangle \rangle \\ &\quad + \sum_a \langle \zeta_L(x_3) f_a(\zeta_L) \rangle \left[\langle \zeta_S(\bar{x}_1) g_a(\zeta_S(\bar{x}_2)) \rangle + \langle g_a(\zeta_S(\bar{x}_1))\zeta_S(\bar{x}_2) \rangle \right]. \end{aligned} \tag{3.50}$$

The squeezed limit in momentum space of this $\langle \bar{\zeta}_L(\bar{x}_3) \langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2) \rangle \rangle$ directly gives $f_{\rm NL}^{\rm obs}$, whereas the squeezed limit of the first term of the right hand side gives the usual $f_{\rm NL}$-parameter computed in comoving coordinates.[4] This means that one finally arrives to an expression of the form

$$f_{\rm NL}^{\rm obs} = f_{\rm NL} + \Delta f_{\rm NL}, \tag{3.51}$$

where $\Delta f_{\rm NL}$ arises from those terms entering the second line of Eq. (3.50), due to the CFC transformation. We will see how to perform all these steps in detail in the following section.

[4] To show that $\langle \zeta_L(x_3) \langle \zeta_S(x_1)\zeta_S(x_2) \rangle \rangle$ gives the squeezed limit of the bispectrum, one may start from $\langle \zeta(x_3)\zeta(x_1)\zeta(x_2) \rangle$ and split both the curvature perturbation and the vacuum state as $\zeta = \zeta_L + \zeta_S$ and $|0\rangle = |0\rangle_L \otimes |0\rangle_S$ respectively. Then one only needs to recall that, because of non-linearities, ζ_S will in general depend on ζ_L.

3.1.6 Short Wavelength Modes in CFC

To finish this section, we deduce how long wavelength modes affect short wavelength modes through the CFC transformation of Eq. (3.36). We will only consider the effects of the first three terms in (3.36), involving the perturbations $\rho^\mu(\tau)$ and $A^\mu_{\bar\imath}(\tau)$. The remaining pieces involving $B^\mu_{\bar\imath\bar\jmath}(\tau)$ and $C^\mu_{\bar\imath\bar\jmath\bar k}(\tau)$ introduce the so called projection effects, which are suppressed in the squeezed limit. We may organize the transformation as follows:

$$x^\mu(\tau,\bar{\mathbf{x}}) = x_0^\mu + \xi^\mu(\tau,\bar{\mathbf{x}}), \tag{3.52}$$

where

$$\xi^\mu(\tau,\bar{\mathbf{x}}) \equiv \rho^\mu(\tau) + A^\mu_{\bar\imath}(\tau)\Delta x^{\bar\imath}. \tag{3.53}$$

Now, notice that the proper definition of the curvature perturbation in the CFC frame may be written as [96, 97]:

$$\bar\zeta(\bar x) = \frac{1}{6}\log\det\big(g_{\bar\imath\bar\jmath}/a_F^2(\bar\tau)\big), \tag{3.54}$$

where the elements $g_{\bar\imath\bar\jmath}$ are given by the spatial components of Eq. (3.19) as:

$$g_{\bar\imath\bar\jmath} = a^2(\tau)\left[\frac{\partial\tau}{\partial x^{\bar\imath}}\frac{\partial\tau}{\partial x^{\bar\jmath}}(1+h_{00}) + \frac{\partial x^i}{\partial x^{\bar\imath}}\frac{\partial\tau}{\partial x^{\bar\jmath}}h_{i0} + \frac{\partial\tau}{\partial x^{\bar\imath}}\frac{\partial x^j}{\partial x^{\bar\jmath}}h_{0j} + \frac{\partial x^i}{\partial x^{\bar\imath}}\frac{\partial x^j}{\partial x^{\bar\jmath}}h_{ij}\right]. \tag{3.55}$$

We now consider the role of the long and short modes in the splitting $\zeta = \zeta_L + \zeta_S$ in order to compute $\bar\zeta(\bar x)$ from (3.54). First, notice that in the previous expression, the partial derivatives $\partial\tau/\partial x^{\bar\imath}$ and $\partial x^i/\partial x^{\bar\imath}$ are determined by the CFC map of Eq. (3.36), and therefore depend on ζ_L alone. On the other hand, h_{00}, $h_{0i} = h_{i0}$ and $h_{ij} = \delta_{ij}e^{2\zeta}$ depend on both ζ_L and ζ_S. In addition, one has to keep in mind that ζ_S is evaluated at $x = x(\bar x)$, which is determined by the CFC map. Putting together all of these factors, it is possible to deduce

$$\bar\zeta_S(\bar x) = \zeta_S(x(\bar x)) + \frac{1}{3}[h_S]_0{}^i(x(\bar x))\partial_i\xi^0_L(\bar x), \tag{3.56}$$

where the arguments of the short scale perturbations are evaluated at $x(\bar x)$. For example, in the case of the first term $\zeta_S(x(\bar x))$, since $x = x_0 + \xi$ (as in Eq. (3.52)), one has

$$\zeta_S(x(\bar x)) = \zeta_S(x_0) + \xi^\mu(\bar x)\partial_\mu\zeta_S(x_0). \tag{3.57}$$

Recall that x_0^μ is given by Eq. (3.37), and therefore $\zeta(x_0)$ is nothing but the comoving curvature perturbation evaluated with unperturbed comoving coordinates. For this reason, we could simply write $\zeta_S(x_0) = \zeta_S(\bar x)$.

3.2 Local non-Gaussianity in Single-field Inflation

We now put together the results of the previous sections to compute the squeezed limit of the bispectrum in the two regimes of interest: attractor and non-attractor. To simplify matters, we will work to leading order in the slow-roll parameter ϵ and neglect any corrections that would modify the final results by terms suppressed in ϵ.

3.2.1 Further Developments

Let us start by obtaining explicit expressions for the integrals of Eqs. (3.33) and (3.34), which in turn lead to simple and manageable expressions for the coefficients $\rho^\mu(\tau)$ and $A^\mu_{\bar{i}}(\tau)$ appearing in the map (3.52). First, notice that $\int_{\tau_*}^{\tau} ds\, \mathcal{H}(s)$ appearing in (3.33) may be directly integrated as:

$$\int_{\tau_*}^{\tau} ds\, \mathcal{H}(s) = \ln \frac{a(\tau)}{a(\tau_*)}. \tag{3.58}$$

Then, one may re-express the integral of Eq. (3.33) as

$$\mathcal{F} = \frac{a(\tau_*)}{a(\tau)} \left(\frac{1}{\mathcal{H}_*} C_F - \left[\frac{1}{a(\tau_*)} \int_{\tau_*}^{\tau} d\tau \frac{a(\tau)}{\mathcal{H}(\tau)} \frac{\partial \zeta}{\partial \tau} \right] \right). \tag{3.59}$$

Now, notice that $\mathcal{H} = -1/\tau + \mathcal{O}(\epsilon)$ and $a(\tau)/a(\tau_*) = \tau_*/\tau + \mathcal{O}(\epsilon)$. This allows one to integrate Eq. (3.59) to obtain the following expression for V_i:

$$V_i \;=\; \frac{1}{\mathcal{H}_*} \frac{a(\tau_*)}{a(\tau)} \partial_i \Big(C_F + \zeta_* \Big) - \partial_i \left[\epsilon \partial^{-2} \frac{\partial \zeta}{\partial \tau} \right] + \mathcal{O}(\epsilon), \tag{3.60}$$

where $\mathcal{O}(\epsilon)$ stands for a function of order ϵ that decays quickly on superhorizon scales. We will soon argue how to choose the integration constant C_F. Our choice will imply that V_i is a function of order ϵ. Irrespective of this, V_i will contribute terms that quickly decay on superhorizon scales (for both regimes, attractor and non-attractor), and that become negligible in the computation of the bispectrum squeezed limit. Next, we move on to compute the integral of Eq. (3.34). Given that V^i is sub-leading, Eq. (3.34) may be directly integrated, giving us back

$$\frac{a_F(\bar{\tau})}{a(\tau)} - 1 = C_a + \zeta - \zeta_* + \mathcal{O}(\epsilon), \tag{3.61}$$

where $\mathcal{O}(\epsilon)$ stands for those decaying terms of order ϵ. Finally, we may use these results to rewrite the map coefficients (3.38)-(3.41) that will be used to compute the squeezed limit. Using the fact that $\mathcal{H} = -1/\tau$ up to corrections of order ϵ, these are found to be:

$$\rho^0(\tau) \;=\; C_a(\tau - \tau_*) + \tau(\zeta - \zeta_*) + \cdots, \tag{3.62}$$
$$\rho^i(\tau) \;=\; 0 + \cdots, \tag{3.63}$$
$$A^0_{\bar{i}}(\tau) \;=\; \delta^i_{\bar{i}} \tau \partial_i \left[\zeta - \zeta_* - C_F \right] + \cdots, \tag{3.64}$$
$$A^i_{\bar{i}}(\tau) \;=\; \left[C_a - \zeta_* \right] \delta^i_{\bar{i}} + \cdots, \tag{3.65}$$

where the ellipses $\cdots$ denote those decaying terms of order ϵ.

3.2.2 Initial Conditions

As discussed in Section 3.1.5, we are interested in understanding how long wavelength modes affect the geodesic motion of an inertial observer that has access to short wavelength perturbations. This means that the perturbed FLRW spacetime considered in the previous section deviates from its unperturbed version due to long wavelength modes ζ_L. We will choose τ_* at a time when all the relevant modes of ζ_L have exited the horizon. In practice, we are interested in computing the effects due to a single mode (or a small range of modes) appearing in ζ_L, that will later be selected in the squeezed limit in momentum space. So we could simply say that our condition is that τ_* corresponds to a moment in time at which ζ_L has just crossed the horizon.

Given that at τ_* the perturbation ζ_L has just crossed the horizon, deviations to the geodesic path are just starting to take over. In particular, any effect of ζ_L on the velocity field V_i must be negligible. We therefore choose C_F in such a way that $V_*^i = 0$. To leading order this corresponds to

$$C_F = -\zeta_L^*. \tag{3.66}$$

As explained, this implies that V_i may be neglected, leading to $\rho^i(\tau) = 0$, which was already stated in Eq. (3.63). Next, a similar argument may be used to state that since ζ_L has just crossed the horizon, we require that $a(\tau_*) = a_F(\bar{\tau}_*)$, this corresponds to a synchronized map choice. This leads to $C_a = 0$ as evident from Eq. (3.61). Then, we finally arrive at a simple version of the map coefficients needed to connect the coordinates of inertial and comoving observers, that may be summarized as follows:

$$\rho^0(\tau) = \tau(\zeta_L - \zeta_L^*) + \cdots , \tag{3.67}$$
$$\rho^i(\tau) = 0 + \cdots , \tag{3.68}$$
$$A^0_{\bar{i}}(\tau) = \tau\partial_{\bar{i}}\zeta_L + \cdots , \tag{3.69}$$
$$A^i_{\bar{i}}(\tau) = -\zeta_L^*\delta^i_{\bar{i}} + \cdots . \tag{3.70}$$

From Eq. (3.67) it is clear that in attractor inflation (where ζ_L freezes) time remains synchronized, while in the non-attractor case (where ζ_L evolves) the watches of a comoving and a free-falling observer do not run at the same rate.

We notice that the authors of Ref. [71] discuss alternative choices to fix τ_* and the associated initial conditions.

3.2.3 Computation of the Squeezed Limit

We are finally ready to compute the observed bispectrum's squeezed limit. This computation was already outlined in Section 3.1.5. We start by explicitly computing the two point correlation function $\langle\bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle$ using (3.56) to express $\bar{\zeta}_S$ in terms of ζ_S:

$$\langle\bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle = \left\langle \left(\zeta_S(x(\bar{x}_1)) + \frac{1}{3}[h_S]_0{}^i\partial_i\xi_L^0\right)\left(\zeta_S(x(\bar{x}_2)) + \frac{1}{3}[h_S]_0{}^i\partial_i\xi_L^0\right)\right\rangle . \tag{3.71}$$

Notice that we have kept $x(\bar{x})$ in the argument of ζ_S at the right hand side, which also depends on ζ_L. We shall deal with this dependence in a moment. Expanding the previous expression up to linear order in ξ_L^0, we have

$$\begin{aligned}\langle\bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle &= \langle\zeta_S(x(\bar{x}_1))\zeta_S(x(\bar{x}_2))\rangle \\ &+\frac{1}{3}\left[\left\langle\zeta_S(x(\bar{x}_1))[h_S]_0{}^i(\bar{x}_2)\right\rangle + \left\langle[h_S]_0{}^i(\bar{x}_1)\zeta_S(x(\bar{x}_2))\right\rangle\right]\partial_i\xi_L^0. \end{aligned}\tag{3.72}$$

It is not difficult to show that, because $[h_S]_{0i}$ consists of a gradient, the two last terms of this expression cancel each other. Then, we are left with:

$$\langle\bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2))\rangle = \langle\zeta_S(x(\bar{x}_1))\zeta_S(x(\bar{x}_2))\rangle . \tag{3.73}$$

Next, we may expand $x(\bar{x})$ appearing in the argument of ζ_S in terms of ζ_L. This gives:

$$\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle = \left[1 + \xi^\mu_L(\tau, \bar{\mathbf{x}}_1)\partial^{(1)}_\mu + \xi^\mu_L(\tau, \bar{\mathbf{x}}_2)\partial^{(2)}_\mu\right] \langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle \,, \tag{3.74}$$

where $\xi^\mu(\tau, \bar{\mathbf{x}})$ is given in Eq. (3.53), and where $\partial^{(1)}_\mu$ and $\partial^{(2)}_\mu$ are partial derivatives with respect to $\bar{x}_1$ and $\bar{x}_2$ respectively. As already explained in Section 3.1.5, the two point correlation function $\langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle$ depends on ζ_L. But given that ξ^μ in (3.74) already depends linearly on ζ_L, in order to keep the leading terms, we may re-write the previous expression as

$$\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle = \langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle + \left[\xi^\mu_L(\tau, \bar{\mathbf{x}}_1)\partial^{(1)}_\mu + \xi^\mu_L(\tau, \bar{\mathbf{x}}_2)\partial^{(2)}_\mu\right] \langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0 \,, \tag{3.75}$$

where $\langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0$ is the two point correlation function in comoving coordinates with $\zeta_L \to 0$ (that is, without a modulation coming from the long wavelength mode). Notice that $\langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0$ is nothing but the two point correlation function of ζ_S with spatial arguments given by $\bar{x}_1$ and $\bar{x}_2$ as if it was computed in comoving coordinates. The result is a function of time τ, and the difference $|\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2|$:

$$\langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0 = \langle \zeta_S\zeta_S\rangle(\bar{\tau}, \bar{r}), \qquad \bar{r} \equiv |\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1|. \tag{3.76}$$

Using this result back into Eq. (3.75) together with the map coefficients of Eqs. (3.67)-(3.67), we find (see Appendix A.2 for details):

$$\begin{aligned}\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle &= \langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle \\ &+ \left[(\zeta_L - \zeta^*_L)\frac{\partial}{\partial \ln \tau} + \frac{1}{2}(x^{\bar{\imath}}_1 + x^{\bar{\imath}}_2)\partial_{\bar{\imath}}\zeta_L\frac{\partial}{\partial \ln \tau} - \zeta^*_L\frac{\partial}{\partial \ln r}\right]\langle \zeta_S\zeta_S\rangle(\tau, r)\end{aligned} \tag{3.77}$$

Recall that ζ_L is evaluated at $\mathbf{x}_c$. However, given that it is a long wavelength mode, we may as well consider it to be evaluated at $\bar{\mathbf{x}}_L = (\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)/2$ without modifying any conclusion. Note that the second term inside the square brackets is necessarily subleading since it involves a spatial derivative of the long-wavelength mode ζ_L. For this reason, we disregard it. To continue, we may now Fourier transform this expression. First, we introduce

$$\zeta(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3k\zeta(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.78}$$

which implies that

$$\begin{aligned}\langle \bar{\zeta}_S(\bar{x}_1)\bar{\zeta}_S(\bar{x}_2)\rangle &= \langle \zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle \\ &+ \int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}_L}\left([\zeta(\mathbf{k}) - \zeta_*(\mathbf{k})]\frac{\partial}{\partial \ln \tau} - \zeta_*(\mathbf{k})\frac{\partial}{\partial \ln r}\right)\langle \zeta_S\zeta_S\rangle(\tau, r).\end{aligned} \tag{3.79}$$

Then, Fourier transforming the fields $\bar{\zeta}_S(\bar{x}_1)$ and $\bar{\zeta}_S(\bar{x}_2)$, we arrive to (see Appendix A.2 for details)

$$\begin{aligned}\langle\bar{\zeta}_S\bar{\zeta}_S\rangle(\mathbf{k}_1, \mathbf{k}_2) &= \langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2) \\ &+ \left([\zeta(\mathbf{k}_L) - \zeta_*(\mathbf{k}_L)]\frac{\partial}{\partial \ln \tau} + \zeta_*(\mathbf{k}_L)\left[n_s(k_S, \tau) - 1\right]\right)P_\zeta(\tau, k_S),\end{aligned} \tag{3.80}$$

where $\mathbf{k}_L = \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k}_S = (\mathbf{k}_1 - \mathbf{k}_2)/2$. In the previous expressions, the power spectrum $P_\zeta(\tau, k)$ of $\zeta(\mathbf{k})$ and its spectral index $n_s(k) - 1$ are defined as

$$P_\zeta(\tau, k) = \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \langle\zeta\zeta\rangle(\tau, r), \tag{3.81}$$

$$n_s(k, \tau) - 1 = \frac{\partial}{\partial \ln k} \ln\big(k^3 P_\zeta(\tau, k)\big). \tag{3.82}$$

Equation (3.80) gives the power spectrum in conformal Fermi coordinates expressed in terms of the curvature perturbations defined in comoving coordinates.

To continue, notice that since we have split the curvature perturbation in short and long wavelength modes, the two point correlation function $\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2)$ will be modulated by the long wavelength mode ζ_L in comoving coordinates. The squeezed limit of the bispectrum in comoving coordinate appears as the formal limit:

$$\lim_{k_3\to 0}(2\pi)^3\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)B_\zeta(k_1, k_2, k_3) = \langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2)\rangle. \tag{3.83}$$

Thus, we see that if we correlate the expression of Eq. (3.80) with $\zeta_L(\mathbf{k}_3)$ we obtain (after using Eq. (3.48))

$$\begin{aligned} \langle\bar{\zeta}_L(\mathbf{k}_3)\langle\bar{\zeta}_S\bar{\zeta}_S\rangle(\mathbf{k}_1, \mathbf{k}_2)\rangle \;=\;& (2\pi)^3\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ &+(2\pi)^3\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)P_\zeta(\tau, k_3)\frac{\partial}{\partial \ln\tau}P_\zeta(\tau, k_S) \\ &-\langle\zeta_L(\mathbf{k}_3)\zeta_*(\mathbf{k}_L)\rangle\left[\frac{\partial}{\partial\ln\tau} - [n_s(k_S, \tau) - 1]\right]P_\zeta(\tau, k_S). \end{aligned} \tag{3.84}$$

Here we still work with the notation $\mathbf{k}_L = \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k}_S = (\mathbf{k}_1 - \mathbf{k}_2)/2$. Notice that this expression contains the quantity $\langle\zeta_L(\mathbf{k}_3)\zeta_*(\mathbf{k}_L)\rangle$, which correlates two ζ_L at two different times. In what follows, we show that

$$\langle\bar{\zeta}_L(\mathbf{k}_3)\langle\bar{\zeta}_S\bar{\zeta}_S\rangle(\mathbf{k}_1, \mathbf{k}_2)\rangle = 0, \tag{3.85}$$

for both attractor and non-attractor models of inflation.

3.2.4 Vanishing of Local non-Gaussianity

In [22] (see Chapter 2 for the details) we have derived a generalized version of the non-Gaussian consistency relation valid for the two regimes of interest: attractor and non-attractor models. The squeezed limit of the 3-point correlation function for ζ was found to be given by:[5]

$$\begin{aligned} \langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2)\rangle \;=\;& -\langle\zeta_L(\mathbf{k}_3)\left[\zeta_L(\mathbf{k}_L) - \zeta_L^*(\mathbf{k}_L)\right]\rangle\frac{d}{d\ln\tau}P_\zeta(\tau, k_S) \\ &-\langle\zeta_L(\mathbf{k}_3)\zeta_L^*(\mathbf{k}_L)\rangle\left[n_s(k_S, \tau) - 1\right]P_\zeta(\tau, k_S), \end{aligned} \tag{3.86}$$

[5] In [94] a different expression for the generalization of the squeezed limit was obtained. The derivation of [94] is based on the use of the operator product expansion to find the squeezed limit of the three point functions for ζ. The main difference with the result (3.86) found in [22] consists of the presence of terms with two $\ln\tau$-derivatives, instead of one.

where the fields ζ_L and ζ_S are evaluated at a time τ, and ζ_L^* is evaluated at a reference initial time τ_*. In order to derive this expression, we used the fact that the cubic action for the curvature perturbation ζ is approximately invariant under space-time reparametrizations given by

$$x \to x' = e^{\zeta_L(\tau_*)}x, \tag{3.87}$$
$$\tau \to \tau' = e^{-\zeta_L(\tau)+\zeta_L(\tau_*)}\tau. \tag{3.88}$$

It may be seen how these relations resemble the coordinate transformation implied by Eqs. (3.67)-(3.70), except for the signs of $\zeta_L(\tau)$ and $\zeta_L(\tau_*)$.

In the case of attractor models, one has that the comoving curvature perturbation ζ becomes constant on superhorizon scales. This implies that $\zeta_*(\mathbf{k}_L) = \zeta_L(\mathbf{k}_3)$ and that $\ln\tau$-derivatives of the power spectrum vanish. Then Eq. (3.86) reduces to

$$B_\zeta(k_1,k_2,k_3) \quad = \quad -\left[n_s(k_S,\tau)-1\right]P_\zeta(\tau,k_L)P_\zeta(\tau,k_S). \tag{3.89}$$

This is the well known Maldacena's consistency condition. On the other hand, in non-attractor models of inflation (ultra slow-roll) the modes grow exponentially fast on superhorizon scales, and one finds a leading contribution given by (one finds that $n_s(k_3,\tau)-1 \propto \epsilon_*(\tau/\tau_*)^6$ and so it may be regarded as formally zero in the long wavelength limit):

$$B_\zeta(k_1,k_2,k_3) \quad = \quad -P_\zeta(\tau,k_L)\frac{d}{d\ln\tau}P_\zeta(\tau,k_S). \tag{3.90}$$

More precisely, the superhorizon modes evolve like $\zeta(\mathbf{k},\tau) = (\tau_*/\tau)^3\zeta(\mathbf{k},\tau_*)$ [57, 78, 79, 81]. For this reason, the power spectrum scales as τ^{-6}:

$$P_\zeta(\tau,k) = \left(\frac{\tau_*}{\tau}\right)^6 P_\zeta(\tau_*,k). \tag{3.91}$$

In this case, Eq. (3.90) finally gives

$$B_\zeta(k_1,k_2,k_3) = 6P_\zeta(\tau,k_L)P_\zeta(\tau,k_S), \tag{3.92}$$

which is the well known ultra slow-roll bispectrum [57].

Now, independently of the specific form of the bispectrum in these two regimes, we see that Eq. (3.86) implies a cancellation between $B_\zeta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ and the rest of the terms in Eq. (3.84):

$$B_\zeta(k_1,k_2,k_3) + \Delta B_\zeta(k_1,k_2,k_3) = 0 \tag{3.93}$$

Thus, we conclude that in the CFC frame the bispectrum vanishes during both the attractor and non-attractor regimes. In the next section we will argue why we expect this result to survive after inflation ends.

3.2.5 On the Validity of CFC for non-Attractor Models

Let us briefly come back to the issue raised in Section 3.1.2, regarding the validity of the CFC map in the case of non-attractor models. We note that even if ζ grows as a^3 on superhorizon scales, during inflation it is always small, since it should reach its value observed in the CMB $\zeta_{\rm CMB} \simeq 10^{-5}$ when it stops to evolve, *i.e.,* at the end of inflation.

Therefore, the validity of the CFC transformation does not rely on the size of ζ but on the possibility that the map depends on time derivatives of ζ, which are of order $\mathcal{H}\zeta$. To be more precise, notice from Eqs. (3.20)-(3.22) that the size of the patch surrounding the central geodesic G has to be such that

$$\left| \left(\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}} \right)_P \Delta x^{\bar{k}} \Delta x^{\bar{l}} \right| \ll 1, \tag{3.94}$$

where $\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}}$ are components of the Riemann tensor constructed from the conformally flat metric $\tilde{g}_{\mu\nu} \equiv a_F^{-2}(\bar{\tau}) g_{\mu\nu}$. This means that $\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}}$ is at least linear in the perturbations. Then, Eq. (3.94) simply translates into the condition that $|\Delta x|^2 \mathcal{H}^2 \zeta_L \ll 1$, where we have used that time derivatives of ζ are of order $\mathcal{H}\zeta$. This last inequality is guaranteed by our previous remark on the size of ζ.

Next, one could be worried about higher order time derivatives of ζ emerging in terms of order, for instance those of order $\mathcal{O}(\Delta \bar{x}^3)$ that appear in Eqs. (3.20)-(3.22). However, as noted before, higher order contributions that are linear in $\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}}$ (and therefore linear in ζ) only bring in spatial derivatives with respect to $x^{\bar{\imath}}$. Higher order derivatives with respect to time will only come about from terms which are quadratic in $\tilde{R}_{\bar{0}\bar{k}\bar{0}\bar{l}}$, and are thus quadratic in ζ_L^2. Therefore, even if in ultra slow-roll ζ does grow exponentially on superhorizon scales, the convergence of both these expansions is still guaranteed.

3.3 Discussion

We have studied the computation of local non-Gaussianity accessible to inertial observers in canonical models of single field inflation. It was already known [54,55,71,72] that observable local non-Gaussianity vanishes in the case of single field attractor models ($f_{\rm NL}^{\rm obs} = 0$) modulo projection effects. In this work, we have extended this result to the case of non-attractor models (ultra slow-roll) in which the standard derivation gives a sizable value $f_{\rm NL} = 5/2$. This result (the standard result) was thought to represent a gross violation of Maldacena's consistency relation. We have instead shown that for both classes of models, the consistency relation is simply:

$$f_{\rm NL}^{\rm obs} = 0. \tag{3.95}$$

This result is noteworthy: In ultra slow-roll comoving curvature perturbations experience an exponential superhorizon growth, and this growth was taken to be the natural explanation underlying large local non-Gaussianity. But this is certainly not the case.

Our results shed new light on our understanding of the role of the bispectrum squeezed limit in inflation to test primordial cosmology. We now know that non-Gaussianity cannot discriminate between two drastically different regimes of inflation. Instead, we are forced to think of new ways of testing the evolution of curvature perturbations in non-attractor backgrounds. This is particularly important once we face the possibility that ultra slow-roll could be representative of a momentary phase within a conventional slow-roll regime [90, 91].

In order to derive (3.95), we have re-examined the use of conformal Fermi coordinates introduced in Ref. [55] and perfected in Refs. [70, 71]. Our results complement these works. For instance, the vanishing of $f_{\rm NL}^{\rm obs}$ in the case of non-attractor models required us to consider in detail the contribution of time-displacements of the CFC map that are irrelevant in the case of attractor models.

The previous remark offers a way to understand the vanishing of local $f_{\rm NL}^{\rm obs}$ for the case of non-attractor models. To appreciate this, let us first focus on the case of attractor models. Notice that in the case of attractor models the freezing of the curvature perturbation can be absorbed at superhorizon scales through a re-scaling of the spatial coordinates, which, to linear order in the perturbations, looks like $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \zeta_* \mathbf{x}$, where ζ_* is the value of the mode at horizon crossing. It is precisely this scaling that gives rise to the modulation of small scale perturbations by long scale perturbations in comoving gauge. The map coefficients of Eq. (3.67) show that in attractor inflation the local transformation corresponds to $\mathbf{x} \to \bar{\mathbf{x}} = \mathbf{x} - \zeta_* \mathbf{x}$. This transformation is opposite to the previous re-scaling, and therefore it cancels the effect of the modulation in comoving coordinates. Note that all these transformations act only on spatial coordinates, the time coordinate remains untouched in the attractor case.

Now, something similar happens in the case of non-attractor models. Here, the curvature perturbation does not freeze on superhorizon scales. Instead, on superhorizon scales the mode acquires a time dependence that may be absorbed by a re-scaling of time $\tau \to \tau' = \tau - \zeta(\tau)\tau$ in the argument of the scale factor (in comoving coordinates). Similar to the case of attractor models, the map coefficients of Eq. (3.67) show that in the non-attractor regime the local CFC transformation corresponds to $\tau \to \bar{\tau} = \tau + \zeta(\tau)\tau$, which is opposite to the previous re-scaling, and so it cancels the whole modulation effect.

More generally, and independently of whether we are looking into the attractor regime, or the non-attractor regime, the cancellation may be understood as follows: The squeezed limit of the 3-point correlation function of canonical models of inflation is the consequence of a symmetry of the action for ζ under the special class of space-time reparametrization shown in Eqs. (3.87)-(3.88). This symmetry is exact in the two regimes that we have studied, but approximate in intermediate regimes. In addition, this symmetry dictates the way in which long-wavelength ζ-modes modulate their short wavelength counterparts. The CFC transformation is exactly the inverse of the symmetry transformation, and so the modulation deduced with the help of the symmetry is cancelled by moving into the CFC frame.

At this point, it is important to emphasize that our computation was performed during inflation. That is, we have performed the CFC transformation while inflation takes place, and the result $B + \Delta B = 0$ found in Section 3.2.4 is strictly valid during inflation. The claim that the primordial contribution to $f_{\rm NL}^{\rm obs}$ vanishes for a late time observer must be a consequence of the CFC transformation, taking into account the entire cosmic history. This would require studying the transition from the non-attractor phase to the next phase, which presumably could be of the attractor class, a study already begun in [89]. Note that the non-attractor nature of USR inflation leads to many different ways to end this phase. However, given that in both regimes (pure ultra slow-roll and pure slow-roll inflation), we have seen that both B and ΔB found in Section 3.2.4 are exactly the same (but of opposite signs) and determined by τ-derivatives and k-derivatives of the power spectrum, we expect that the end of non-attractor inflation (which could be a transition to a slow-roll phase) will affect equally B and ΔB, in such a way that the net result will continue to be $B + \Delta B = 0$. We will present an argument in favor of this claim in the next chapter.[6]

Our main reason for expecting that local non-Gaussianity when expressed in free-falling Fermi vanishes in both attractor and non-attractor models of single field inflation is that in both cases, after the inflaton scalar degree of freedom is swallowed by the gravitational field, the only dynamical scalar degree of freedom corresponds to the curvature perturbation. As a consequence, the interaction coupling together long and short wavelength modes is purely gravitational, and therefore the equivalence principle dictates that long wavelength physics cannot dictate the evolution of short wavelength dynamics, implying that any observable effect must be suppressed by a ratio of scales $\mathcal{O}(k_L/k_S)^2$. All of this calls for a better examination of the relation between the local ansatz and the squeezed limit of the bispectrum [98].

Our work leaves several open challenges ahead. First, we have focussed our interest in canonical models of inflation, namely, those in which the inflaton field is parametrized by a Lagrangian containing a canonical kinetic term. In this category, the ultra slow-roll regime is not fully realistic, and at best should be considered as a toy model allowing the study of perturbations under the extreme conditions of a non-attractor background. However, it has been shown that non-attractor regimes may appear more realistically within non-canonical models of inflation such as $P(X)$ models. In these models one has non-gravitational interactions inducing a sound speed $c_s \neq 1$, and so we suspect that our result (3.95) will not hold in those cases. This intuition is mainly based on the fact that in non-attractor models, the squeezed limit gets an enhancement when $c_s^2 \neq 1$ as shown in Ref. [80]. At any rate, our results (together with Ref. [22]) calls for a better understanding of the non-Gaussianity predicted by non-attractor models in general.

Second, given that observable local non-Gaussianity vanishes in ultra slow-roll, in which curvature perturbations grow exponentially on superhorizon scales, one should revisit the status of other classes of inflation, such as multi-field inflation, where local non-Gaussianity may be large (a first look into this issue has already been undertaken in Ref. [72]). It is quite feasible that in some models of multi-field inflation the amount of local non-Gaussianity may be understood as the consequence of a space-time symmetry dictating the way in which long-wavelength modes module short modes.

Third, a deeper understanding of our present result is in order. In the case of attractor models, Maldacena's consistency relation (and its vanishing) may be understood as a consequence of soft limit identities linking the non-linear interaction of long wavelength perturbations with shorter ones [18, 20, 21, 84, 86–88]. However, there were good reasons to suspect that these relations would not hold anymore in the case of non-attractor models [55].

[6]In [92], Cai *et al.* studied the effects on the bispectrum B of a transition from a non-attractor phase to an attractor phase. They discovered that the transition can drastically change the comoving value of $f_{\rm NL}$, suppressing its value if the transition is smooth. Then, the question would be: what happens with ΔB during such transitions? Does the transition from comoving to Fermi coordinates continue to cancel the squeezed bispectrum computed in comoving coordinates? As we explain above, we expect the answer to be affirmative.

Our results suggest that, regardless of the background, these identities continue to be valid, and in an inertial frame the gravitational interaction cannot be responsible of making long wavelength modes affect the local behavior of short wavelength modes.

Chapter 4

A New Soft Theorem for Single-field Inflation

In Chapter 2, we have seen that although the symmetries that lead to the generalized consistency relation are exact, in principle, one needs to know the explicit form of g from (2.58) in order to do an expansion in (2.67) to obtain a consistency relation valid in any moment at any order in slow-roll parameters. In this chapter, we will adopt a slightly different strategy, and we will derive a robust version of the consistency relation.

As discussed, during cosmic inflation [8–12], the universe is approximately a de Sitter spacetime. This fact helps to constrain the expected shape of n-point correlation functions of the primordial curvature fluctuations ζ, responsible for the existence of structure in our universe. In particular, the de Sitter dilation symmetry severely restricts the momentum dependence of ζ's n-point correlation functions [17, 20, 67, 68].

But inflationary backgrounds are not de Sitter. The existence of an evolving scalar field $\phi(t)$ breaks the de Sitter isometries with departures of order $\epsilon \equiv -\dot{H}/H^2$. Thus, some statements based on symmetries are not exact (*e.g.* the power spectrum is scale invariant up to corrections of order ϵ). However, other statements remain valid to all orders in slow-roll. For example, in single-field inflation, the squeezed limit of the bispectrum respects Maldacena's consistency relation [42]

$$\lim_{k_3 \to 0} B_\zeta(k_1, k_2, k_3) = -(n_s - 1) P_\zeta(k_3) P_\zeta(k_1). \tag{4.1}$$

This result is valid to all orders in the slow-roll parameters as long as the background is attractor [16, 64]. This can be understood as a consequence of the invariance of Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes under a special class of (residual) spatial diffeomorphisms [18,19,84,87]. However, this understanding is restricted to attractor models, where ζ becomes constant for wavelengths greater than the Hubble horizon H^{-1} [43].

General statements valid for non-attractor models have remained more elusive. In non-attractor models, such as ultra slow-roll inflation [78,82,91,99,100], ζ's amplitude experiences a rapid growth for wavelengths larger than H^{-1}. This property has propelled considerable interest in the study of non-attractor phases during inflation as a way of generating primordial black-holes [90, 101–113]. It is well understood that during a non-attractor phase the bispectrum is amplified, leading to a violation of (4.1) which, in the particular case of ultra slow-roll, takes the form [22, 57, 59, 79–81, 94, 114]

$$\lim_{k_3\to 0} B_\zeta(k_1, k_2, k_3) = 6P_\zeta(k_3)P_\zeta(k_1). \tag{4.2}$$

But, it has been pointed out that this result does not necessarily stay imprinted in the primordial spectra after the non-attractor phase is over, unless the background dynamics experiences a sharp transition from ultra slow- roll to the attractor phase [22, 23, 92, 115]. The status of local non-Gaussianity in non-attractor phases of inflation has become a relevant subject with important consequences to our understanding of the early universe [116–118].

In fact, both (4.1) and (4.2) are expressions strictly valid in comoving coordinates. As discussed in [54] (see also [56]) comoving coordinates contain spurious couplings between short- and long-wavelength perturbations, altering the derivation of the observable squeezed limit of the bispectrum. To obtain the observable squeezed limit, one may employ a special class of coordinates, the so called Conformal Fermi Coordinates [23, 55, 70, 71] (CFC's), allowing the computation of physical quantities observed by inertial observers. As emphasized in [70], CFC's remove any diffeomorphism invariance to isolate all locally observable effects, from inflation all the way up to our present epoch. The use of CFC's has been well understood in the case of attractor models of inflation, where the observable bispectrum has been shown to consists of (4.1) corrected by a term $\Delta B = (n_s - 1)P_\zeta(k_3)P_\zeta(k_1) + \mathcal{O}(k_3^2/k_1^2)$ giving us back

$$\lim_{k_3\to 0} B_\zeta^{\rm obs}(k_1, k_2, k_3) = 0 + \mathcal{O}(k_3^2/k_1^2), \tag{4.3}$$

where $\mathcal{O}(k_3^2/k_1^2)$ stands for projection effects (non-Gaussian contributions due to post inflationary cosmological evolution). Nevertheless, the use of CFC's to compute the bispectrum resulting from non-attractor phases has remained a challenge. CFC's were implemented in the particular case of ultra slow-roll in Ref. [23], where (4.3) was indeed recovered. But the computation offered in [23] was limited to the assumption that the universe never abandons the ultra slow-roll phase. Despite of this shortcoming, the result was indicative of a non-trivial cancellation potentially present in more general regimes.

The purpose of this chapter is to understand the status of the bispectrum's squeezed limit in its three incarnations (4.1), (4.2) and (4.3) under the scope offered by the diffeomorphism invariance of canoncial single field inflation. In particular, we are interested in the status of (4.3) in single field canonical inflation independently of whether the inflationary background is attractor or non-attractor). We find that attractor backgrounds of single field inflation are characterized for having a vanishing primordial local non-Gaussianity to all orders in slow-roll parameters (a result previously understood up to first order in slow-roll).

Outstandingly, our approach makes use of time diffeomorphisms (in addition to spatial diffeomorphisms) in order to unify both (4.1) and (4.2) under a single soft theorem valid as long as the third slow-roll parameter does not experience sudden changes. It is well known that time diffeomorphisms break the comoving gauge chose to study primordial perturbations [18, 19, 84, 87] (see also [119, 120]). In Ref. [94], Finelli *et al.* sorted this difficulty out by restricting the computation of n-point functions to models with exact shift symmetries.

This shift symmetry allows an additional transformation that restitutes back the initial gauge choice. As we shall see, one can perform time diffeomorphisms without breaking the choice of comoving gauge provided that long wavelength perturbations are renormalized away into the background. This is a key aspect that we exploit, allowing us to interpret the role of CFC's in the computation of the bispectrum.

4.1 The General Picture

Before we commence, let us describe the main idea underlying our approach. The dynamics of ζ is governed by a non-linear equation of motion (EOM) derived from an action $S[\zeta, B]$, where $B = B(t)$ represents time-dependent background quantities, including the scale factor $a(t)$, the Hubble parameter $H = \dot{a}/a$ and further time derivatives of H. The solution to this EOM is a function $\zeta(t, \mathbf{x})$ that may be split into short- and long-wavelength contributions $\zeta(t, \mathbf{x}) = \zeta_S(t, \mathbf{x}) + \zeta_L(t, \mathbf{x})$. If the lengthscale λ_* determining this splitting is larger than H^{-1}, then spatial gradients of ζ_L will have a suppression of order $1/H\lambda_*$. Then, at length-scales smaller than λ_*, $\zeta_L = \zeta_L(t)$ is effectively a function of time only (a background perturbation). This implies that $\zeta_L(t)$ can be absorbed in the background in such a way that $S[\zeta, B] = S[\zeta_S, \bar{B}]$, where $\bar{B}$ represents background quantities containing $\zeta_L(t)$ as part of them. For this to be consistent, the background EOM's respected by $\bar{B}$ must have the same form as those of B. Similarly, the EOM for $\zeta_S(t, \mathbf{x})$ must be the same as that of $\zeta(t, \mathbf{x})$, but with background coefficients given by $\bar{B}$ instead of B.

As we shall see, ζ_S and ζ are related by a space-time diffeomorphism (a change of coordinates) that gives us $\zeta_S(t, \mathbf{x}) = \zeta(t + \Delta_1, \mathbf{x}e^{\Delta_2})$, where $\Delta_{1,2}$ are functions of ζ_L and $\dot{\zeta}_L$. The concrete form of this relation is found to be given by (4.48). In what follows, we show how to use space-time diffeomorphisms to relate $S[\zeta, B]$ and $S[\zeta_S, \bar{B}]$, and derive (4.48). This in turn, will directly lead to (4.51).

4.2 Time Diffeomorphisms and FLRW Backgrounds

We start by analyzing the effects of small changes of the time coordinate on background fields. The FLRW metric describing a flat expanding universe is

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{4.4}$$

where $a(t)$ is the scale factor, and $d\mathbf{x}^2 = \delta_{ij}dx^i dx^j$. In single-field inflation the two quantities determining the background configuration are the scalar field $\phi(t)$ and the Hubble parameter $H = \dot{a}/a$. They satisfy

$$\frac{\partial^2\phi}{\partial t^2} + 3H\frac{\partial\phi}{\partial t} + V_\phi(\phi) = 0, \quad 3H^2 = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + V(\phi), \tag{4.5}$$

where $V_\phi \equiv \partial V/\partial\phi$. Let us consider the effect of a time diffeomorphism of the form:

$$t \to \bar{t} = t + \xi^0(t). \tag{4.6}$$

Given that ϕ is a scalar field, this change of coordinates leads to $\phi(t) \to \bar{\phi}(\bar{t}) = \phi(t)$. Normally, one would use this relation to write $\bar{\phi}(t) = \phi(t - \xi^0) = \phi(t) - \dot{\phi}(t)\xi^0$. Here, instead, we are happy to keep $\bar{\phi}(\bar{t})$ intact, and ask whether it is able to satisfy the same EOM as $\phi(t)$, but with $\bar{t}$ instead of t. On the other hand $a(t)$, which belongs to the spatial part of the metric, does not transform at all under (4.6). That is, in terms of the new time coordinate $\bar{t}$, the metric (4.4) is given by

$$ds^2 = -(1-2\dot{\xi}^0)d\bar{t}^2 + a^2(t)d\mathbf{x}^2, \quad a(t) = a(\bar{t}-\xi^0). \tag{4.7}$$

Let us now define a new scale factor $\bar{a}(\bar{t})$ as

$$\bar{a}(\bar{t}) = a(t)e^{\alpha}, \tag{4.8}$$

where $\alpha = \alpha(t)$ is a function of time to be determined. So far, α and ξ^0 are arbitrary functions. We will fix them in such a way that $\bar{\phi}(\bar{t})$ and $\bar{a}(\bar{t})$ satisfy the same equations of motion (4.5) obeyed by $\phi(t)$ and $a(t)$, but with $\bar{t}$ instead of t. That is:

$$\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} + 3\bar{H}\frac{\partial\bar{\phi}}{\partial\bar{t}} + V_{\bar{\phi}}(\bar{\phi}) = 0, \quad 3\bar{H}^2 = \frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + V(\bar{\phi}). \tag{4.9}$$

Using $\partial t/\partial\bar{t} = 1-\dot{\xi}^0$, we see that $\bar{H}(\bar{t}) = (1+\dot{\alpha}/H - \dot{\xi}^0)H(t)$. Then, these equations are satisfied as long as:

$$\frac{d}{dt}\left(a^3\epsilon H\dot{\xi}^0\right) = 0, \tag{4.10}$$
$$3\dot{\alpha} = (3-\epsilon)H\dot{\xi}^0. \tag{4.11}$$

Thus, there exists a non-trivial time diffeomorphism $t \to \bar{t} = t + \xi^0$ for which the scalar field $\bar{\phi}(\bar{t}) = \phi(t)$ satisfies the same original background equations, but with $\bar{a}(\bar{t})$ instead of $a(t)$. However, notice that (4.4) and (4.7) don't share the same form. We address this in the following by including perturbations.

4.3 Time Diffeomorphisms and Perturbations

As a next step, let us consider perturbing the system in two different ways. First, we define perturbations ζ, $\delta\mathcal{N}$, $\boldsymbol{\mathcal{N}}$ and $\delta\phi$ in such a way that

$$ds^2 = -e^{2\delta\mathcal{N}}dt^2 + a^2(t)e^{2\zeta}(d\mathbf{x}+\boldsymbol{\mathcal{N}}dt)^2, \tag{4.12}$$
$$\phi = \phi(t) + \delta\phi(t,\mathbf{x}). \tag{4.13}$$

where $(d\mathbf{x}+\boldsymbol{\mathcal{N}}dt)^2 = \delta_{ij}(dx^i+\mathcal{N}^i dt)(dx^j+\mathcal{N}^j dt)$. In the previous expression ζ is the spatial curvature perturbation, and $\delta\mathcal{N}$ and $\boldsymbol{\mathcal{N}}$ are the usual lapse and shift functions. We also consider a second way of perturbing the same metric through perturbations $\bar{\zeta}$, $\delta\bar{\mathcal{N}}$, $\bar{\boldsymbol{\mathcal{N}}}$ and $\delta\bar{\phi}$ in such a way that

$$ds^2 = -e^{2\delta\bar{\mathcal{N}}}d\bar{t}^2 + \bar{a}^2(\bar{t})e^{2\bar{\zeta}}(d\bar{\mathbf{x}}+\bar{\boldsymbol{\mathcal{N}}}d\bar{t})^2, \tag{4.14}$$
$$\phi = \bar{\phi}(\bar{t}) + \delta\bar{\phi}(\bar{t},\bar{\mathbf{x}}), \tag{4.15}$$

where $\bar{\phi}(\bar{t}) = \phi(t)$. Here $\bar{t}$ is the same time coordinate defined in (4.6), and $\bar{a}(\bar{t})$ is the same scale factor defined in (4.8), with ξ^0 and α satisfying (4.10) and (4.11). We have also introduced the spatial coordinate $\bar{x}^i$ as

$$\bar{\mathbf{x}} = e^{\beta/3}\mathbf{x}, \tag{4.16}$$

where $\beta = \beta(t)$ is a function of time to be determined [a more standard approach would have consisted in writing $\bar{x}^i = x^i + \xi^i_s$, with $\xi^i_s \equiv \frac{x^i}{3}\beta(t)$ so that $\partial_i \xi^i_s = \beta(t)$]. It should be clear that both (4.12) and (4.14) are just different ways of expressing the same metric. However, the background fields and perturbations differ. This is the key aspect that we exploit in what follows.

We now fix the gauge. We choose to work in co-moving gauge, whereby $\delta\phi$ of Eq. (4.13) satisfies $\delta\phi(t, \mathbf{x}) = 0$. Given that $\bar{\phi}(\bar{t}) = \phi(t)$, this condition implies that $\delta\bar{\phi}$ in Eq. (4.15) satisfies $\delta\bar{\phi}(\bar{t}, \bar{\mathbf{x}}) = 0$. Thus, both ζ and $\bar{\zeta}$ are comoving curvature perturbations. Is this even possible? As we shall see, (4.10) and (4.11) play an important role in allowing this. Indeed, notice that (4.16) implies that $d\bar{\mathbf{x}} = e^{\beta/3} d\mathbf{x} + \frac{1}{3}\mathbf{x}\dot{\beta} e^{\beta/3} dt$, which in (4.14) leads to

$$\begin{aligned} ds^2 &= -e^{2(\delta\bar{\mathcal{N}} + \dot{\xi}^0)} dt^2 \\ &\quad + a^2(t) e^{2(\bar{\zeta} + \alpha + \beta/3)} \left[d\mathbf{x} + \bar{\boldsymbol{\mathcal{N}}} dt + \frac{1}{3}\mathbf{x}\dot{\beta} dt \right]^2, \end{aligned} \tag{4.17}$$

where we also made use of (4.6) and (4.8). Comparing (4.17) with (4.12) gives us the following relations:

$$\zeta = \bar{\zeta} + \alpha + \frac{1}{3}\beta, \tag{4.18}$$

$$\delta\mathcal{N} = \delta\bar{\mathcal{N}} + \dot{\xi}^0, \tag{4.19}$$

$$\boldsymbol{\mathcal{N}} = \bar{\boldsymbol{\mathcal{N}}} + \frac{1}{3}\mathbf{x}\dot{\beta}. \tag{4.20}$$

Now, let us split ζ, $\delta\mathcal{N}$ and $\boldsymbol{\mathcal{N}}$ appearing in the left hand side of the previous equations into short- and long-wavelength modes: $\zeta(t, \mathbf{x}) = \zeta_S(t, \mathbf{x}) + \zeta_L(t)$, $\delta\mathcal{N}(t, \mathbf{x}) = \delta\mathcal{N}_S(t, \mathbf{x}) + \delta\mathcal{N}_L(t)$ and $\boldsymbol{\mathcal{N}}(t, \mathbf{x}) = \boldsymbol{\mathcal{N}}_S(t, \mathbf{x}) + \boldsymbol{\mathcal{N}}_L(t, \mathbf{x})$. The long-wavelength part $\zeta_L(t)$ corresponds to the zeroth order term of the Taylor expansion $\zeta_L(t, \mathbf{x}) = \zeta_L(t) + \partial_i \zeta_L(t)\mathbf{x}^i + \frac{1}{2}\partial_i\partial_j\zeta_L(t)\mathbf{x}^i\mathbf{x}^j$, and must satisfy the e.o.m. for ζ in the long-wavelength limit:

$$\frac{d}{dt}\left(\epsilon a^3 \dot{\zeta}_L\right) = 0. \tag{4.21}$$

On the other hand $\delta\mathcal{N}_L(t)$ and $\boldsymbol{\mathcal{N}}_L(t, \mathbf{x})$ satisfy the constraint equations:

$$\delta\mathcal{N}_L(t) = \frac{1}{H}\dot{\zeta}_L(t), \qquad \boldsymbol{\mathcal{N}}_L(t, \mathbf{x}) = \frac{1}{3}\epsilon\, \mathbf{x}\, \dot{\zeta}_L(t). \tag{4.22}$$

Given that we are interested in making statements about the action of ζ valid up to cubic order, it is enough to express the lapse and shift function linearly with respect to ζ (see Ref. [42]). But notice that $\dot{\xi}^0$, α and β in the right hand side of (4.18) can be adjusted to absorb the long wavelength perturbations $\zeta_L(t)$, $\delta\mathcal{N}_L(t)$ and $\boldsymbol{\mathcal{N}}_L(t, \mathbf{x})$ appearing at the left hand side of the same equations. That is:

$$\dot{\xi}^0 = \frac{1}{H}\dot{\zeta}_L, \tag{4.23}$$

$$\alpha + \frac{1}{3}\beta = \zeta_L, \tag{4.24}$$

$$\dot{\beta} = \epsilon\dot{\zeta}_L. \tag{4.25}$$

Let us recall that ξ^0 and α are restricted to satisfy (4.10) and (4.11), so it is not obvious that (4.23)-(4.25) can hold (even if β is a function to be fixed freely). However, ζ_L respects the e.o.m. (4.21) which, thanks to (4.23), coincides with (4.10). Similarly by taking a time derivative of (4.24), and combining the result with (4.23) and (4.25) we obtain $3\dot{\alpha} = (3-\epsilon)H\dot{\xi}^0$, which is precisely (4.11).

Thus, we conclude that (4.23)-(4.25) [together with (4.21)] are consistent with (4.10) and (4.11). A direct consequence of this, is that the remaining spatially dependent functions appearing in (4.18)-(4.20) satisfy:

$$\bar{\zeta} = \zeta_S, \qquad \delta\bar{\mathcal{N}} = \delta\mathcal{N}_S, \qquad \bar{\boldsymbol{\mathcal{N}}} = \boldsymbol{\mathcal{N}}_S. \tag{4.26}$$

This means that the short wavelength perturbations of the metric (4.12) can be identified as the full perturbations of the metric (4.14). In other words, $\bar{\phi}(\bar{t})$ and $\bar{a}(\bar{t})$ are the background fields felt by ζ_S, $\delta\mathcal{N}_S$ and $\boldsymbol{\mathcal{N}}_S$, once the long wavelength modes have become part of the background. Given that the full action describing single-field inflation is invariant under space-time diffeomorphisms, the action for ζ derived using (4.12) and the action for $\bar{\zeta} = \zeta_S$ derived using (4.14) have the same form (at least at cubic order). However, the background quantities in the action for ζ_S contain ζ_L as part of it.

Setting Initial Conditions

Having determined the equations that gives us ξ^0, α and β in terms of ζ_L, we can proceed to solve them. This requires knowledge of the initial conditions for ξ^0, α and β. From its definition in (4.8), we may set initial conditions in such a way that $\alpha = 0$ for a given choice of time t_*:

$$\alpha(t_*) = 0. \tag{4.27}$$

This simply ensures that the scale factor $\bar{a}$ coincides with a at a given time t_*. That is:

$$\bar{a}(\bar{t}_*) = a(t_*). \tag{4.28}$$

Given that a cannot be measured directly, we can choose t_* arbitrarily. However, in order to weigh the amount of expansion experienced by short wavelength modes with ζ_L absorbed in the background, it is convenient to choose t_* to be the time at which the wavelength $\lambda_L = 2\pi a(t)/k_L$ separating long- and short-wavelengths crosses the horizon. This automatically sets

$$\beta(t_*) = 3\zeta_L(t_*). \tag{4.29}$$

Furthermore, we must decide how to fix the initial conditions for ξ^0. In fact, we can do this by directly solving (4.23), which is the subject of the next discussion.

An Explicit Expression for ξ^0

To round up this discussion, we derive a useful expression for the time diffeomorphism ξ^0. Note that it is possible to integrate $\dot{\xi}^0 = \dot{\zeta}_L/H$ to obtain an analytical expression with slow-roll parameters to all orders. To do so, we try the ansatz

$$\xi^0 = C_1 + F\,\dot{\zeta}_L, \tag{4.30}$$

where C_1 is a constant of integration and $F = F(t)$ is a function of time to be determined. By taking a derivative of this ansatz, and matching it with $\dot{\xi}^0 = \dot{\zeta}_L/H$ we obtain $H\dot{F}\dot{\zeta}_L + FH\ddot{\zeta}_L = \dot{\zeta}_L$. But recall from (4.21) that $\ddot{\zeta}_L = -3H\dot{\zeta}_L - H\eta\dot{\zeta}_L$. Then, F must respect the following equation $H\dot{F} - (3+\eta)FH^2 - 1 = 0$. The solution (See Appendix B for the details) to this equation is the integral

$$F = a^3\epsilon\, C_2 + 2A(t), \qquad A(t) \equiv \frac{a^3\epsilon}{2}\int^t \frac{dt}{a^3\epsilon H}, \tag{4.31}$$

where C_2 is a constant of integration. In the previous expression, A contains the indefinite part of the integral, without the constant part already accounted in C_2. The integral can be solved by iterating partial integrations infinite times. We arrive to the formal result:

$$A(t) = -\frac{\epsilon}{3}\sum_{n=0}^{\infty}\frac{e^{3N/2}}{(n+1)!}\left[\frac{2}{3}\frac{d}{dN}\right]^n\left(\frac{e^{-3N/2}}{H^2\epsilon}\right), \tag{4.32}$$

where $N = \ln a(t)$ is the usual e-fold number. It is easy to appreciate that A is a function of slow-roll parameters. The first few terms of the previous expression are:

$$\begin{aligned} H^2 A \;=\; & -\frac{1}{3} + \frac{1}{6}\left(1 + \frac{4}{3}\epsilon + \frac{2}{3}\eta\right) \\ & -\frac{1}{27}\left[\left(1 + \frac{4}{3}\epsilon + \frac{2}{3}\eta\right)(2\epsilon+\eta) - \left(\frac{4}{3}\epsilon\eta + \frac{2}{3}\eta\xi\right)\right] \\ & + \cdots, \end{aligned} \tag{4.33}$$

where $\eta = \frac{\dot{\epsilon}}{\epsilon H}$ and $\xi = \frac{\dot{\eta}}{\eta H}$ are the second and third slow-roll parameters. In the particular case of models for which every slow-roll parameter is small except for η (of which, ultra-slow-roll inflation is an example), the result (4.32) can be resumed back to

$$A \simeq -\frac{1}{2(3+\eta)H^2}, \tag{4.34}$$

It would seem that for $\eta = -3$ the function A becomes ill defined. However, in that case one has to resume back the neglected slow-roll parameters.

To continue, from (4.30) we now have $\xi^0 = C_1 + a^3\epsilon\, C_2\dot{\zeta}_L + 2A(t)\dot{\zeta}_L$. But recall from (4.21) that $a^3\epsilon\dot{\zeta}_L$ is a constant, and so we can simplify ξ^0 as

$$\xi^0 = C + 2A(t)\dot{\zeta}_L. \tag{4.35}$$

What value should we choose for C? Notice that C plays no role what so ever in our arguments relating the long wavelength mode ζ_L with the background. In our analysis of the background leading to Eqs. (4.10) and (4.11) the physically relevant quantity is $\dot{\xi}^0$. The same was true in our analysis leading to (4.23), (4.24), and (4.25). In addition, we already know that if $\dot{\zeta}_L = 0$, we ought to not consider any time diffeomoerphism to incorporate the effects of ζ_L on the background.

4.4 Modulation of Short Wavelengths in Comoving Coordinates

Before we show the validity of (4.3), let us examine how the compelling result of the previous discussion can be used to derive both (4.1) and (4.2). Let us denote

$$\zeta(t,\mathbf{x}) = \zeta[t,\mathbf{x},a(t)] \tag{4.36}$$

as the solution of the perturbed system (4.14), where the notation emphasizes that $\zeta(t,\mathbf{x})$ is a solution to a system with a scale factor $a(t)$. This definition includes a prescription to fix the initial conditions of $\zeta(t,\mathbf{x})$. Now, notice that $\bar{\zeta}$ of (4.14) must be a solution of the same equations of motion governing the system (4.12) but with coordinates $(\bar{t},\bar{\mathbf{x}})$ and a scale factor $\bar{a}(\bar{t})$, instead of $(t,\mathbf{x})$ and $a(t)$. That is

$$\bar{\zeta}(\bar{t},\bar{\mathbf{x}}) = \zeta[\bar{t},\bar{\mathbf{x}},\bar{a}(\bar{t})]. \tag{4.37}$$

But thanks to (4.26) we have

$$\zeta_S(t,\mathbf{x}) = \bar{\zeta}(\bar{t},\bar{\mathbf{x}}) = \zeta[\bar{t},\bar{\mathbf{x}},\bar{a}(\bar{t})], \tag{4.38}$$

where it is understood that $(\bar{t},\bar{\mathbf{x}})$ appearing in $\bar{\zeta}$ can be expressed in terms of $(t,\mathbf{x})$ thanks to the change of coordinates determined by ξ^0 and β. Therefore, it follows that

$$\begin{aligned}\zeta_S(t,\mathbf{x}) &= \zeta[\bar{t},\bar{\mathbf{x}},\bar{a}(\bar{t})]\\ &= \zeta[t+\xi^0, e^{\beta/3}\mathbf{x}, a(t)e^{\alpha}]\\ &= \zeta[t+\xi^0, e^{\beta/3}\mathbf{x}, a(t+\xi^0)e^{D_L}],\end{aligned} \tag{4.39}$$

where in the last line we have defined D_L as

$$D_L \equiv \alpha - H\xi^0. \tag{4.40}$$

To continue, we must notice that under very general conditions D_L asymptotes quickly to a constant. Indeed, a time derivative of D_L is given by:

$$\dot{D}_L = \dot{\alpha} + \epsilon H^2\xi^0 - H\dot{\xi}^0, \tag{4.41}$$

$$= -\frac{1}{3}\epsilon H\dot{\xi}^0 + \epsilon H^2\xi^0, \tag{4.42}$$

$$= -\frac{1}{3}a^3\epsilon H\frac{d}{dt}\left(a^{-3}\xi^0\right), \tag{4.43}$$

where we used (4.24) together with (4.25) to obtain the second equality. This result already shows that $\dot{D}_L$ is of order ϵ. We can go one step further and show that it dilutes as a^{-3}. First, using the result $\xi^0 = 2A\dot{\zeta}_L$ obtained in Section 4.3, we get

$$\dot{D}_L = -\frac{2}{3}a^3\epsilon H\frac{d}{dt}\left(a^{-3}A\dot{\zeta}_L\right). \tag{4.44}$$

Then, by using the equation of motion (4.21) for the long wavelength mode, and the definition (4.31) for A, we finally obtain:

$$\dot{D}_L = \left(2H^2A - \frac{1}{3}\right)\epsilon\dot{\zeta}_L. \tag{4.45}$$

But given that Eq. (4.21) ensures that $\epsilon\dot{\zeta}_L \propto a^{-3}$, we see that

$$\dot{D}_L \propto a^{-3}, \tag{4.46}$$

and so D_L evolves quickly to a constant after the long-wavelength mode has crossed the horizon. The caveat to this statement is the fact that $H^2 A$ could evolve in such a way as to compensate for the dilution factor $\propto a^{-3}$. To continue our analysis we shall assume that D_L is a constant and come back later to the case in which this assumption is violated.

Now, in the equation of motion for ζ the quantities a and $\mathbf{x}$ appear together through the combination $a^{-2}\partial_{\mathbf{x}}^2$. Then, under the assumption that D_L is constant, it follows that:

$$\zeta[t, \mathbf{x}, a(t)] = \zeta[t, \mathbf{x}e^{D_L}, a(t)e^{-D_L}]. \tag{4.47}$$

This result then allows us to go back to Eq. (4.39) and write

$$\begin{aligned} \zeta_S(t, \mathbf{x}) &= \zeta[t + \xi^0, e^{\beta/3+\alpha-H\xi^0}\mathbf{x}, a(t+\xi^0)], \\ &= \zeta(t + \xi^0, e^{\beta/3+\alpha-H\xi^0}\mathbf{x}), \\ &= \zeta(t + 2A\dot{\zeta}_L, e^{\zeta_L - 2AH\dot{\zeta}_L}\mathbf{x}), \end{aligned} \tag{4.48}$$

where in the last step we used $\beta/3 + \alpha = \zeta_L$ and the definition (4.31) with $C = 0$ (given that C has units of time, it would be impossible to fix it in terms of background quantities unless it consists of a quantity evaluated at a specific time, which in the present formalism, cannot be chosen). This result shows how ζ_L modulates ζ_S. From here, it is direct to derive the following expression for the bispectrum in co-moving coordinates (see for instance [64]). To proceed, let us use the notation $\langle\zeta(\tau, \mathbf{x})\zeta(\tau, \mathbf{y})\rangle = \langle\zeta\zeta\rangle(\tau, |\mathbf{x} - \mathbf{y}|)$. Then, from (4.48) the two-point correlation function of $\zeta_S(t, \mathbf{x})$ is given by

$$\langle\zeta_S\zeta_S\rangle(t, |\mathbf{x} - \mathbf{y}|) = \langle\zeta\zeta\rangle(t + 2A\dot{\zeta}_L, e^{\zeta_L - 2HA\dot{\zeta}_L}|\mathbf{x} - \mathbf{y}|). \tag{4.49}$$

Expanding this expression, and writing it in Fourier space, we obtain

$$\begin{aligned} \langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2) = \langle\zeta\zeta\rangle(\mathbf{k}_1, \mathbf{k}_2) + 2A\dot{\zeta}_L(\mathbf{k}_L)\dot{P}_\zeta(k_S, t) \\ - \left[\zeta_L(\mathbf{k}_L) - 2HA\dot{\zeta}_L(\mathbf{k}_L)\right](n_s - 1)P_\zeta(k_S, t), \end{aligned} \tag{4.50}$$

where $\mathbf{k}_S \equiv (\mathbf{k}_1 - \mathbf{k}_2)/2$ and $\mathbf{k}_L \equiv \mathbf{k}_1 + \mathbf{k}_2$. In the previous expression, $n_s - 1 \equiv \frac{\partial}{\partial \ln k} \ln\left[k^3 P_\zeta(k, t)\right]$ is the spectral index of the power spectrum $P_\zeta(k_S, t)$. Then, correlating (4.50) with a long mode $\zeta_L(\mathbf{k}_3)$, and using $\langle\zeta_L(\mathbf{k}_3)\langle\zeta_S\zeta_S\rangle(\mathbf{k}_1, \mathbf{k}_2)\rangle = \lim_{k_3\to 0}(2\pi)^3\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)B_\zeta(k_1, k_2, k_3)$, to identify the squeezed limit of the bispectrum, we finally obtain

$$\begin{aligned} \lim_{k_3\to 0} B_\zeta(k_1, k_2, k_3) &= -(n_s - 1)P_\zeta(k_L)P_\zeta(k_S) \\ &\quad + A\dot{P}_\zeta(k_L)\left[\dot{P}_\zeta(k_S) + H(n_s - 1)P_\zeta(k_S)\right]. \end{aligned} \tag{4.51}$$

This is one of our main result. It gives the consistency relation for canonical single field inflation valid to all orders in slow-roll parameters in co-moving coordinates. Noteworthily, this result displays the same form of the consistency relation found in Ref. [94], valid for scalar field theories with an exact shift symmetry (see also [121, 122]). There, the factor A takes the particular form $A = -\dot{\phi}/2\ddot{\phi}\,\Theta$, where Θ is a time dependent function that generalizes the constraint equation (4.22) as $\delta N_L = \dot{\zeta}_L/\Theta$ to incorporate non-canonical theories.

Equation (4.51) implies that non-Gaussianity can be large as long as $\dot{P}_\zeta(k_L)$ is sizable. In the particular case of ultra-slow roll one has $\eta = -6$, and ζ_L grows as a^3, implying that $\dot{P}_\zeta(k) = 6HP_\zeta(k)$. Also, thanks to (4.34) we see that this corresponds to the case $A = 1/H^2 6$, from where it follows that (4.2) is a particular case of (4.51). However, as soon as the non-attractor phase finishes, and inflation goes back to a more standard attractor phase, one has $\dot{P}_\zeta(k_L) = 0$ and the standard Maldacena's consistency relation (4.1) is recovered.

Nevertheless, recall that (4.51) was derived assuming that D_L defined in (4.40) evolves quickly to a constant after horizon crossing. This is valid as long as the time evolution of $H^2 A$ does not compete with the decaying factor a^{-3} in (4.45). Thus, our result (4.51) remains valid for general backgrounds (attractor and/or non-attractor) where $H^2 A$ evolve slowly. Of course, this is not automatically ensured by the general dynamics of ζ_L. For instance, during a sharp transition between ultra-slow roll and slow-roll phases can affect the evolution of D_L. In fact, in [92] it was found that the bispectrum can pick a large value if such transitions happen, but it becomes suppressed if the transitions are soft, in agreement with our result (4.51). Moreover, in the recent work [123], the obstruction of taking D_L as a constant was surpassed, and a more general result for the bispectrum in co-moving coordinates was obtained, in agreement with that of [92].

4.5 Conformal Fermi Coordinates

We now consider the task of moving from co-moving coordinates to conformal Fermi coordinates. As explained in detail in Ref. [55] (see also [70, 71]), these are coordinates that describe the local environment of inertial observers. To make this discussion easy to compare with the existing literature, here we adopt conformal time τ, defined through the relation $dt = a(\tau)d\tau$, where the scale factor $a(\tau)$ (with τ as an argument) should be understood as the composition function $a(\tau) \equiv a(t(\tau))$. With this, using (4.26), the metric (4.17) takes the form

$$\begin{aligned} ds^2 &= a^2(\tau)\Bigg[- e^{2(\delta\mathcal{N}_S + \xi^{0\prime}/a(\tau))} d\tau^2 \\ &\quad + e^{2(\zeta_S + \alpha + \beta/3)} \left(d\mathbf{x} + a(\tau)\boldsymbol{\mathcal{N}}_S d\tau + \frac{1}{3}\mathbf{x}\beta' d\tau \right)^2 \Bigg], \end{aligned} \tag{4.52}$$

where primes ($'$) denote derivatives with respect to conformal time. To re-express this metric in CFC's we need to consider the following change of coordinates from the conformal co-moving coordinates $(\tau, \mathbf{x})$ to the conformal Fermi coordinates $(\tau_F, \mathbf{x}_F)$

$$\tau = \tau_F + \xi_F^0, \qquad \mathbf{x} = \mathbf{x}_F + \boldsymbol{\xi}_F, \tag{4.53}$$

with [23, 71]

$$\xi^0_F = \int^\tau ds \left[\frac{a_F(s)}{a(\tau(s))} - 1 - \frac{1}{\mathcal{H}}\zeta_L'(s)\right] + \mathcal{O}(\mathbf{x}_F), \tag{4.54}$$

$$\boldsymbol{\xi}_F = \left[\frac{a_F(\tau_F)}{a(\tau)} - 1 - \zeta_L(\tau)\right]\mathbf{x}_F + \mathcal{O}(\mathbf{x}_F^2), \tag{4.55}$$

where we have only included the scalar contributions to ξ^0_F and ξ^i_F. In the previous expressions, $\mathcal{H} \equiv a'/a = aH$. In addition, $\mathcal{O}(\mathbf{x}_F)$ and $\mathcal{O}(\mathbf{x}_F^2)$ stand for terms linear in ζ_L that contribute to the appearance of the projection effects in (4.3). On the other hand, the function $a_F(\tau_F)$ is the scale factor experienced by the inertial observer, given by

$$a_F(\tau_F) = a(\tau)\left[1 + \zeta_L + \frac{1}{3}\int^\tau ds\, \partial_i V^i\right], \tag{4.56}$$

where V^i are the spatial components of the 4-velocity of the inertial observer in co-moving coordinates, given in terms of ζ_L as

$$V^i = -\frac{1}{3}\epsilon\zeta_L' x^i_F + \mathcal{O}(\mathbf{x}_F^2). \tag{4.57}$$

For comparison, Eqs. (4.54), (4.55), (4.56) and (4.57) correspond to the respective Eqs. (2.19a), (2.19b), (2.15) and (2.14) of Ref. [71] (or Eqs. (2.34), (2.37), (2.30) and (2.8) of Ref. [23]).[1] Now, from (4.25) notice that $\epsilon\zeta_L'$ is nothing but β'. Thus, we can integrate $\int^\tau ds\, \partial_i V^i$ in (4.56) to obtain $a_F(\tau_F) = a(\tau)\left[1 + \zeta_L - \beta/3\right]$. Then, using (4.24), it follows that $a_F(\tau_F)$ coincides with $\bar{a}(\bar{\tau})$:

$$a_F(\tau_F) = \bar{a}(\bar{\tau}). \tag{4.58}$$

Moreover, the diffeomorphism leading to the CFC's takes the form

$$\xi^0_F = \int^\tau ds \left[\alpha - \frac{1}{\mathcal{H}}\zeta_L'(s)\right] + \mathcal{O}(\mathbf{x}_F), \tag{4.59}$$

$$\boldsymbol{\xi}_F = -\frac{1}{3}\beta\mathbf{x}_F + \mathcal{O}(\mathbf{x}_F^2). \tag{4.60}$$

Using these results back into (4.52) we see that the diffeomorphisms ξ^0_F and $\boldsymbol{\xi}_F$ cancel out with the various functions of ζ_L (such as ${\xi^0}'$, α, β, and β') and we finally obtain

$$\begin{aligned} ds^2 = a_F^2(\tau_F)\Big[& - e^{2\delta\mathcal{N}_S} d\tau_F^2 \\ & + e^{2\zeta_S}\left(d\mathbf{x}_F + a_F(\tau_F)\boldsymbol{\mathcal{N}}_S d\tau_F\right)^2\Big] + \cdots, \end{aligned} \tag{4.61}$$

where the elipses stand for terms that give rise to projection effects. This result shows that the CFC frame coincides with the frame studied in Section 4.3, in which the long wavelength perturbation is completely absorbed in the background.

[1]A technical note for readers interested in matching our expressions with those of Refs. [71] and [23]: Notice that both in [71] and [23] the velocity field V^i appears in the definition of ξ^i_F (c.f. Eq. (2.19a) of Ref. [71] and Eq. (2.35) in Ref. [23]). However, it is the vector part V^i_v of V^i that appears in those definitions (with $\partial_i V^i_v = 0$), and not the full velocity V^i. This is because the scalar part V^i_s of V^i is already accounted in (4.55) as the coefficients multiplying x^i_F, thanks to Eq. (4.56). Given that here we are only interested in scalar modes, V^i does not appear explicitly in Eqs. (4.54) and (4.55).

Now, it is useful to compare this form of the metric with the one obtained by directly performing the change of coordinates from co-moving coordinates to conformal Fermi coordinates. The metric components in the CFC frame can be obtained as

$$g^F_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}\frac{\partial x^\nu}{\partial x^{\bar{\nu}}} g_{\mu\nu}. \tag{4.62}$$

One then obtains that the short- and long-wavelength contributions of the curvature perturbation ζ^F in the CFC frame are related to the short- and long-wavelength contributions of the curvature perturbation ζ in co-moving coordinates as [71]:

$$\zeta^F_S(\tau_F, \mathbf{x}_F) = \zeta_S(\tau, \mathbf{x}), \tag{4.63}$$

$$\zeta^F_L(\tau_F) = \zeta_L - \frac{1}{3}\beta - H\xi^0. \tag{4.64}$$

Because of (4.24), notice that the second equation is equivalent to

$$\zeta^F_L(\tau_F) = \alpha - H\xi^0, \tag{4.65}$$

from where it follows that

$$a_F(\tau_F) = a(\tau_F) e^{\zeta^F_L}. \tag{4.66}$$

This allows one to re-write the metric line element (4.61) as

$$\begin{aligned} ds^2 = a^2(\tau_F) e^{2\zeta^F_L} \Big[& - e^{2\delta \mathcal{N}_S} d\tau_F^2 \\ & + e^{2\zeta^F_S} \left(d\mathbf{x}_F + a(\tau_F) \boldsymbol{\mathcal{N}}_S d\tau_F \right)^2 \Big] + \cdots . \end{aligned} \tag{4.67}$$

The result (4.66) shows how explicitly how the background quantity $a_F(\tau_F)$ depends on long wavelength perturbations. The scale factor $a(\tau_F)$ contains no dependence on perturbations, and all long wavelength perturbations are contained are in the single variable ζ^F_L.

Notice that that $\zeta^F_L = \alpha - H\xi^0$ is nothing but the long wavelength quantity D_L defined in Section 4.4. This result shows that $\dot{\zeta}^F_L$ is suppressed by by ϵ, from where it follows that ζ^F_L does not evolve significantly as long as $\epsilon \ll 1$. This is in contrast with the evolution of ζ_L in non-attractor backgrounds, which is dominated by a growing mode. For instance, in the particular case of ultra slow-roll one has $A = 1/6H^2$, and so $\dot{\zeta}^F_L = 0$. Moreover, recall that we showed that D_L is constant unless $H^2 A$ changes quickly over time (that is, its time evolution competes with a^{-3}). This result is completely general, and independent of whether single field inflation is attractor or non-attractor.

4.6 Observable bispectrum's squeezed limit

Finally, we turn to the computation of the observable bispectrum's squeezed limit. First, let us notice that according to (4.63) the two point function of the short-wavelength modes of ζ in the CFC frame is given by

$$\langle \zeta^F_S \zeta^F_S \rangle (\tau_F, |\mathbf{x}_F - \mathbf{x}'_F|) = \langle \zeta_S \zeta_S \rangle (\tau, |\mathbf{x} - \mathbf{x}'|). \tag{4.68}$$

But recall from (4.38) that $\zeta_S(t,\mathbf{x}) = \zeta[\bar{t},\bar{\mathbf{x}},\bar{a}(\bar{t})]$. In terms of conformal Fermi coordinates, this menas

$$\zeta_S(\tau_F,\mathbf{x}_F) = \zeta[\tau_F,\mathbf{x}_F,a_F(\tau_F)]. \tag{4.69}$$

Let us see now how this result can be used to compute the observable squeezed limit of the bispectrum. We start by examining the standard case of single field slow-roll inflation. We then move on to examine more general backgrounds.

4.6.1 Attractor backgrounds

In canonical single field inflation, attractor models are characterized for having their long wavelength perturbations ζ_L constant. From the discussion of Section 3.2.2, this implies the following constant values for α, β and ξ^0:

$$\alpha = 0, \tag{4.70}$$
$$\beta = 3\zeta_L, \tag{4.71}$$
$$\xi^0 = 0. \tag{4.72}$$

Thus, Eq. (4.64) tells us that the long-wavelength mode ζ_L^F in the CFC frame vanishes and, thanks to (4.66), the scale factor $a_F(\tau_F)$ of the CFC frame acquires no dependence on long wavelength perturbations. Using the notation of Section 4.4, we can write

$$\zeta_S(\tau_F,\mathbf{x}_F) = \zeta[\tau_F,\mathbf{x}_F,a(\tau_F)] = \zeta(\tau_F,\mathbf{x}_F), \tag{4.73}$$

which shows that the solution $\zeta_S(\tau_F,\mathbf{x}_F)$ is simply the solution $\zeta(\tau,\mathbf{x})$ with the coordinates $(\tau,\mathbf{x})$ replaced by $(\tau_F,\mathbf{x}_F)$, unaffected by the long wavelength perturbation ζ_L. As a result, the two point function of ζ_S^F is simply given as

$$\langle\zeta_S^F\zeta_S^F\rangle(\tau_F,|\mathbf{x}_F-\mathbf{x}'_F|) = \langle\zeta\zeta\rangle(\tau_F,|\mathbf{x}_F-\mathbf{x}'_F|). \tag{4.74}$$

We therefore re-obtain the well known result that in single field inflation the two point function $\langle\zeta_S^F\zeta_S^F\rangle(\tau_F,|\mathbf{x}_F-\mathbf{x}'_F|)$ is not modulated by ζ_L. This, in turn, leads to (4.3).

4.6.2 Non-attractor backgrounds

We can use Eq. (4.45) to relate $\zeta_L^F(t)$ with $\zeta_L(t)$. First, notice that we can write $\zeta_L^F(t)$ in terms of $\dot{\zeta}_L(t)$ as

$$\zeta_L^F(t) = \epsilon a^3\dot{\zeta}_L\left[-2\frac{H_*A_*}{\epsilon_* a_*^3} + \int_{t_*}^t dt\frac{1}{a^3}\left(2H^2A - \frac{1}{3}\right)\right], \tag{4.75}$$

where H_*, A_*, ϵ_* and a_* represent background quantities evaluated at a given time t_*. To obtain this result, we used the fact that $\epsilon a^3\dot{\zeta}_L$ is constant, together with $\zeta_L^F(t_*) = \alpha(t_*) - H_*\xi^0(t_*) = -2H_*A_*\dot{\zeta}_L(t_*)$ where t_* is the initial time introduced in Section 3.2.2 whereby $\alpha_* = 0$. On the other hand, we can write ζ_L as

$$\zeta_L(t) = \zeta_L(t_*) + \epsilon a^3\dot{\zeta}_L\int_{t_*}^t\frac{dt}{\epsilon a^3}. \tag{4.76}$$

Here, the second term is a decaying solution if the background is attractor, or a growing solution in the case the background is non-attractor. Thus we obtain the following general expression relating $\zeta_L^F(t)$ and $\zeta_L(t)$

$$\zeta_L^F(t) = [\zeta_L(t) - \zeta_L(t_*)]\, G(t,t_*), \tag{4.77}$$

where

$$G(t,t_*) \equiv \frac{1}{\int_{t_*}^t dt \frac{1}{\epsilon a^3}} \left[\int_{t_*}^t dt \frac{1}{a^3} \left(2H^2 A - \frac{1}{3} \right) - 2\frac{H_* A_*}{\epsilon_* a_*^3} \right]. \tag{4.78}$$

If inflation experiences a long enough period of non-attractor evolution, we expect ζ_L to become dominated by the growing solution. Thus we can simply write:

$$\zeta_L^F(t) \simeq \zeta_L(t) G(t,t_*). \tag{4.79}$$

Now, given that $\zeta_L^F(t)$ stays almost constant, the rapid growth of $\zeta_L(t)$ during a non-attractor phase is characterized by the fact that $G(t,t_*)$ dilutes quickly within a couple of e-folds:

$$G(t,t_*) \to 0, \tag{4.80}$$

which is due to the rapid growth of the denominator $\int_{t_*}^t dt \frac{1}{\epsilon a^3}$ during the non-attractor phase. Recall that t_* is chosen in such a way that the wavelength scale splitting short- and long-wavelength modes is crossing the horizon at a time t_*. For instance, if we are dealing with the Fourier modes of ζ, we may choose to work with the mode $\zeta_L^F(t,\mathbf{k})$ of smallest co-moving wavelength $\mathbf{k}_L^{-1}$. In this case, we would find

$$\zeta_L^F(t,\mathbf{k}_L) \simeq \zeta_L(t,\mathbf{k}_L) G(t,t_*(\mathbf{k}_L)) \tag{4.81}$$

where $t_*(\mathbf{k}_L)$ is the time where $\mathbf{k}_L = a(t_*)H(t_*)$.

To continue, let us first assume that the background quantity A does not experiences sudden changes. In this case, we can treat ζ_L^F to be nearly constant. Then, starting with (4.69) we can write

$$\begin{aligned} \zeta_S(\tau_F,\mathbf{x}_F) &= \zeta[\tau_F,\mathbf{x}_F, a(\tau_F)e^{\zeta_L^F}], \\ &\simeq \zeta[\tau_F,\mathbf{x}_F e^{\zeta_L^F}, a(\tau_F)], \\ &\simeq \zeta(\tau_F,\mathbf{x}_F e^{\zeta_L^F}), \end{aligned} \tag{4.82}$$

where, just as in we did in Section 4.4, we used the fact that $\zeta[t,\mathbf{x},a(t)] = \zeta[t,\mathbf{x}e^D, a(t)e^{-D}]$ for a constant D (we shall soon address the size of the corrections implied by the fact that ζ_L^F does not stay exactly constant). It therefore follows that the two point function $\langle \zeta_S^F \zeta_S^F \rangle$ is modulated by ζ_L^F. Expanding it to first order in ζ_L^F we obtain:

$$\begin{aligned} \langle \zeta_S^F \zeta_S^F \rangle(\tau_F, r_F) &= \langle \zeta\zeta \rangle(\tau_F, r_F e^{\zeta_L^F}) \\ &= \langle \zeta\zeta \rangle(\tau_F, r_F) + \zeta_L^F \frac{\partial}{\partial r_F} \langle \zeta\zeta \rangle(\tau_F, r_F) + \cdots . \end{aligned} \tag{4.83}$$

Now, in [55] (see also [70]) it is argued that the squeezed limit of the bispectrum in the CFC frame may be obtained by correlating $\langle \zeta_S^F \zeta_S^F \rangle$ with ζ_L (not ζ_L^F). That is, the observable squeezed limit is related to the correlation between ζ_L and $\langle \zeta_S^F \zeta_S^F \rangle$ as:

$$\langle \zeta_L(\mathbf{k}_3) \langle \zeta_S^F(\mathbf{k}_3) \zeta_S^F(\mathbf{k}_3) \rangle \rangle \equiv (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_\zeta^{\rm obs}(k_1,k_2,k_3). \tag{4.84}$$

Inserting (4.83) in (4.84) and using our previous result (4.81) we are then finally led to

$$\lim_{k_3 \to 0} B_\zeta^{\rm obs}(k_1, k_2, k_3) = -(n_s - 1) P_\zeta(k_L) P_\zeta(k_S) G(t, t_*(\mathbf{k}_L)). \tag{4.85}$$

Thus, thanks to the fact that G decays quickly during a non-attractor phase, we finally re-obtain (4.3). This result shows that non-attractor phases do not, per se, imply large local non-Gaussianity (a point already stressed in [92]). However, even though ζ_L^F remains almost constant during the whole period of inflation (as long as ϵ stays small), its evolution does not need to be smooth, implying that its role as a background quantity for the evolution of ζ_S^F may have a significant impact in the generation of local non-Gaussianity.

4.6.3 Large observable non-Gaussianity?

Let us now consider those situations in which the steps followed in (4.82) cannot be performed. This could happen, for instance, in the case where higher time derivatives of ζ_F experience sudden variations. That is, even though ζ_L^F stays almost constant during inflation, higher derivatives of ζ_L^F can have a relevant role in the equations of motion of ζ_S^F. To analyze this case we may resort to the in-in formalism: We must use the metric (4.67) to obtain the full action S of ζ_S^F. In this action, ζ_L^F appears as a background quantity. We can then identify the cubic term in S containing ζ_S^F at quadratic order, and ζ_L^F at linear order. This gives the non-linear interaction sourcing the squeezed limit of the bispectrum. The action to consider is just the free field action for ζ_S^F written in the CFC frame:

$$S = \int d^4x_F a_F^2(\tau_F) \epsilon_F(\tau_F) \left[(\zeta_S^{F\prime})^2 - (\nabla \zeta_S^F)^2 \right] + \cdots . \tag{4.86}$$

Here $a_F(\tau_F) = a(\tau_F) e^{\zeta_L^F}$ and $\epsilon_F(\tau_F)$ is computed from $a_F(\tau_F)$. One finds

$$\epsilon_F(\tau_F) = \epsilon(\tau_F)(1 - 2\dot{\zeta}_L^F/H) - \ddot{\zeta}_L^F/H^2. \tag{4.87}$$

This implies that the part of the action quadratic in ζ_S^F and linear in ζ_L^F is given by

$$S_{\rm int} = \int d^4x_F a^2(\tau_F) \epsilon(\tau_F) \Delta_L \left[(\zeta_S^{F\prime})^2 - (\nabla \zeta_S^F)^2 \right] + \cdots , \tag{4.88}$$

where

$$\Delta_L \equiv 2\zeta_L^F - \frac{2}{H}\dot{\zeta}_L^F - \frac{1}{\epsilon H^2}\ddot{\zeta}_L^F. \tag{4.89}$$

We can now isolate the relevant part by focussing on those terms that would have the largest impact on the squeezed limit of the bispectrum given a time varying ζ_L^F. This will necessarily come from the third term in (4.89), which is not suppressed by ϵ. To proceed, let us assume that ϵ stays small throughout the full period of inflation, and that η is most of order 1 (so as to keep ϵ small). Then, the largest contribution to the computation of the bispectrum comes from

$$S_{\rm int} = \int d^4x_F a^2 \epsilon \, \eta' \zeta_L' (\zeta_S^F)^2 + \cdots , \tag{4.90}$$

which is obtained from (4.88) after performing partial integrations. This term is precisely what gives rise to large local non-Gaussianity in co-moving coordinates (with ζ_S instead of ζ_S^F) provided that the background transit from a non-attractor phase to an attractor phase abruptly, as studied in [92]. In other words, with abrupt transitions the observed bispectrum can be as large as the bispectrum computed in the co-moving frame:

$$\lim_{k_3\to 0} B_\zeta^{\rm obs}(k_1,k_2,k_3) \simeq \lim_{k_3\to 0} B_\zeta^{\rm com}(k_1,k_2,k_3). \tag{4.91}$$

For instance, as shown in [92], in the particular case where η' is negligible for all times except for a small period of time, the term in Eq. (4.90) leads to

$$\lim_{k_3\to 0} B_\zeta^{\rm obs}(k_1,k_2,k_3) \simeq P_\zeta(k_3)P_\zeta(k_2)\int d\tau\,\eta', \tag{4.92}$$

which, in the case of a sudden transition from ultra slow roll ($\eta=-6$) to slow roll ($|\eta|\ll 1$) gives back (4.2). However, we emphasize that what is at play here is not the fact that the background is non-attractor (which as we saw in Section 4.6.2 does not automatically lead to large non-Gaussianity) but a momentary large value of η'.

4.7 Discussion

We have analyzed the squeezed limit of the non-Gaussian bispectrum in canonical single field inflation using tools that allow certain statements to all orders in slow roll parameters. In particular, we have derived a consistency relation for the squeezed limit [c.f. Eq. (4.51)] in co-moving coordinates for both, attractor and non-attractor backgrounds, valid as long as the third slow roll parameter η' does not experiences sudden changes. We have also analyzed the computation of the bispectrum's squeezed limit in the conformal Fermi coordinate frame, which gives access to the observable squeezed limit. In general, both attractor and non-attractor backgrounds leads to a suppressed amount of local non-Gaussianiy except for those cases in which η' is momentarily large. As we saw, large local non-Gaussianity is not a direct consequence of being in a non-attractor background.

Our approach highlighted the important role of time diffeomorphisms in order to understand single field inflation in more general terms. As we saw, it is possible to organize perturbation theory in such a way that a time diffeomorphism does not take us away from co-moving gauge, allowing us to evade a well known obstruction of using time-diffeomorphism to study the consequences of residual diffeomorphisms [18, 19, 84, 87]. This diffeomorphism coincides with the change of variables from co-moving coordinates to conformal Fermi coordinates, which allows the computation of observable n-point correlation functions.

To finish, let us notice that in order to compute the observable bispectrum we have used the approach followed in [55] whereby the observable squeezed limit of the bispectrum corresponds to (4.84). However, we may alternatively consider the computation of the three point function involving only perturbations within the CFC frame:

$$\langle\zeta_L^F(\mathbf{k}_3)\langle\zeta_S^F(\mathbf{k}_3)\zeta_S^F(\mathbf{k}_3)\rangle\rangle \equiv (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B_\zeta^{\rm obs}(k_1,k_2,k_3). \tag{4.93}$$

In this case, instead of obtaining (4.91) we would obtain

$$B_\zeta^{\rm obs}(k_1,k_2,k_3) \simeq G(t,t_*(\mathbf{k}_3))B_\zeta^{\rm com}(k_1,k_2,k_3), \tag{4.94}$$

and so, the bispectrum's squeezed limit is found to be suppressed with respect to the value computed in co-moving gauge.

Chapter 5

Inflation and Quantum Gravity

Inflation is an effective field theory (EFT) that, in principle, provides a phenomenological window to study some quantum gravitational effects [48]. Indeed, we need to be sure that inflation can be UV-completed in a quantum theory of gravitation. In order to do so it is necessary to incorporate into the EFT description, universal features that any theory of quantum gravity must have, such as, having gravity as the weakest gauge force [125].

It is commonly stated that string theory is far from being fully understood yet still the most promising, mathematically consistent, unified framework, which allows us to make sense of gravity in the quantum realm beyond the Planck scale.[1] In the quest of trying to recover our 4-dimensional physical world, string theorists realized that the procedure of doing so was not unique but actually quite degenerate. Roughly speaking, they have found that the number of metastable vacua of string theory, the so-called *landscape* [128], is $\mathcal{O}\left(10^{500}\right)$ [129, 130]. Such a number, while huge, and maybe disappointing for those who expected a unique fundamental prediction of how a consistent universe should look like, is still far smaller than the number of seemingly consistent EFT's that, however, do not accept an ultraviolet (UV) completion within quantum gravity. The latter are said to belong to the *swampland*, a term originally coined by Vafa and collaborators in [131, 132]. Since the inception of this seminal idea, different so-called swampland conjectures, such as the weak gravity conjecture[2] (WGC) [125], have been devised in order to ascertain whether an EFT may or may not arise as a low-energy approximation stemming from a fundamental quantum gravity theory like string theory.

In the following, we weigh the constraining power of the so-called swampland distance conjecture (SDC) [132] taken together with the famous Lyth bound [47] on the dynamics of cosmic inflation. As we shall quickly review, a non-trivial consequence of the SDC is that the *geodesic* field excursion $\Delta\phi$ of *any* scalar field ϕ weakly coupled with Einstein's

[1]The fact that "quantum mechanics and General Relativity are irreconcilable theories associated with extremely different length scales" is not only an *old-fashioned* but actually *wrong* statement. To illustrate, quantum gravity well below the Planck scale is a well-developed effective field theory that leads to definite predictions such as quantum corrections to Newton's gravitation law. See for instance [126, 127].

[2]In short the WGC states that, in suitable units, any conceivable consistent universe has gravity as the *weakest* gauge force.

gravity, should always remain sub-Planckian, $\Delta\phi/M_{\rm Pl} < \mathcal{O}(1)$, in order to be consistent with quantum gravity. On the other hand, the Lyth bound establishes that the observation of primordial tensor perturbations sets a minimum amount of field excursion $[\Delta\phi]_r$ which, in the case of canonical single-field inflation, is given by

$$\frac{[\Delta\phi]_r}{M_{\rm Pl}} \equiv \Delta N \sqrt{\frac{r}{8}}, \tag{5.1}$$

where r is the tensor-to-scalar ratio (currently constrained as $r < 0.044$ [46]), and ΔN is the number of e-folds elapsed from the time when the largest observable scales crossed the horizon to the end of inflation. Given that $\Delta N \sim 60$, the observation of r within the range accessible by current surveys ($r \sim 0.01$-0.07)[3], would imply that the inflaton field necessarily had a super-Planckian field excursion $\Delta\phi > [\Delta\phi]_r \sim \mathcal{O}(1)\, M_{\rm Pl}$. Considering that in a single-field context $[\Delta\phi]_r$ is geodesic by default, this last observation effectively leaves canonical single-field inflation in the swampland of inconsistent EFT's. However, when considering the very well-motivated scenario of multi-field inflation, one needs to be more cautious, as there is an emergent non-trivial geometrical structure in the field space spanned by the set of scalar fields which may change quite drastically the conclusion that inflation, as a framework, is doomed by the aforementioned considerations [39]. In particular, multi-field scenarios allow for the possibility of *non-geodesic* field excursions, which are not directly subjected to satisfy the distance conjecture [134].

Indeed, an important aspect of multi-field models of inflation, completely absent in single-field scenarios, is the distinction between geodesic and non-geodesic trajectories. Non-geodesic inflationary trajectories (in field space) are those for which the background solution follows a path that locally bends at a non-vanishing rate. Crucially, at the perturbation theory level, these bends generate non-trivial interactions between the primordial curvature perturbations (that seeded the observed inhomogeneities of our universe) and isocurvature fluctuations, defined as field fluctuations orthogonal to the inflationary trajectory. These interactions have a series of important consequences for the statistics of primordial curvature perturbations which, in addition to the tensor-to-scalar ratio, will be probed by future cosmological surveys.

The summary of this chapter may be condensed as follows: The very same mechanism that generates non-geodesic trajectories in multi-field space induces an enhancement of the Lyth bound. In other words, non-geodesic trajectories come together with two competing effects: (1) an attenuation of the SDC bound and (2) an amplification of the Lyth bound. These two competing effects, combined together, imply novel bounds on the parameter space of multi-field models. To anticipate how this happens, we should start by noticing that the first effect (the attenuation of the SDC bound) is simply a consequence of the fact that non-geodesic field excursions are always greater (or equal) than their geodesic counterpart. This entails the existence of a concrete relation connecting the geodesic and non-geodesic distances (denoted as $[\Delta\phi]_{\rm G}$ and $[\Delta\phi]_{\rm NG}$, respectively) between any two points laying over

[3]Let us just mention that r may have a significantly lesser value ($r \sim 0.003$) in single-field models like Starobinsky's [9] and Higgs Inflation [133]. In this work, however, we focus in the scenario where the measurement of r is just around the corner.

the inflationary trajectory. The relation takes the general form

$$\frac{[\Delta\phi]_{\rm G}}{\Lambda_g} = f\left(\frac{[\Delta\phi]_{\rm NG}}{\Lambda_g}\right), \tag{5.2}$$

where Λ_g is a characteristic mass scale, and f is a function that satisfies $f(x) \leq x$. As we shall see with the help of concrete examples, this function is determined by the specific model under study, and it parametrizes the extent to which $[\Delta\phi]_{\rm G}$ and $[\Delta\phi]_{\rm NG}$ differ as a result of the bending inflationary trajectory. On the other hand, we will show that the second effect (the amplification of the Lyth bound) comes down to the expression

$$[\Delta\phi]_{\rm NG} = \frac{[\Delta\phi]_r}{\sqrt{\beta}}, \tag{5.3}$$

where $[\Delta\phi]_r$ is the same quantity defined in (5.1), and β (with $0 < \beta \leq 1$) is a function of local properties of the trajectory (such as the bending rate and the mass of the field fluctuations normal to the trajectory), which is implicitly defined through a modified version of the well-known power spectrum of primordial curvature perturbations $\mathcal{R}$

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{H^2}{8\pi^2 M_{\rm Pl}^2\, \epsilon\, \beta}, \tag{5.4}$$

where H and $\epsilon \equiv -\frac{\dot{H}}{H^2}$ are the usual Hubble scale and first slow-roll parameter of inflation, respectively. As we shall see in more details, $\beta = 1$ is achieved only for geodesic trajectories, so non-geodesic trajectories necessarily lead to an amplification of the Lyth bound, and one may even attain situations where $\beta \ll 1$.[4] As a consequence, putting together equations (5.2) and (5.3), and using the fact that the SDC bound acts on $[\Delta\phi]_{\rm G}$, we arrive at an alternative version of the bound of the form

$$\frac{\Lambda_g}{M_{\rm Pl}}\, f\left(\frac{[\Delta\phi]_r}{\Lambda_g\sqrt{\beta}}\right) < \mathcal{O}(1). \tag{5.5}$$

This relation combines information pertaining the background solution of the theory, and quantities parametrizing the dynamics of fluctuations. Given that both $f(x) < x$ and $\beta < 1$ are consequences of non-geodesic trajectories, equation (5.5) gives us a non-trivial restriction on the local characteristics of the inflationary path in multi-field space. The bound of equation (5.5) can be satisfied in simple and well motivated multi-field setups where the geometry of the field space plays a decisive role. For instance, in two-field models with a *hyperbolic* field space geometry[5] (*i.e.* where the Ricci curvature is given by $\mathbb{R} = -2/R_0^2$, with R_0 a constant parameter with mass dimension 1), if the non-geodesic trajectory bends at a constant rate, one finds that the function f and the scale Λ_g appearing in (5.2) are respectively given by

$$f(x) = \text{arcsinh}(x) \qquad \text{and} \qquad \Lambda_g = 2R_0. \tag{5.6}$$

This form of the function f turns Eq. (5.5) into a constraint on the minimal amount of bending rate necessary to satisfy the SDC, and on the possible values of masses for the

[4]The fact that for multi-field models of inflation β may be significantly smaller than unity was already noted in [135] while considering the simple case of inflation driven by two scalar fields, and then emphasized again in [136].

[5]Current work related to this subject may be found in [137–146].

isocurvature fluctuations interacting with the inflaton. This simple example highlights the constraining power of future observations at restricting the parameter space of stringy models characterized by nontrivial geometries, resulting from compactifications.

Arriving to (5.5) and analyzing its non-trivial consequences is the aim of the rest of this chapter. The plan is the following: In Section 5.1 we deepen within the arguments already exposed in this introduction, first by offering a brief stringy-like derivation of the SDC as well as giving precise statements between the later and the Lyth bound, while acknowledging the expected power (and limitations) of multi-field EFT's when trying to address the tension of our plot. In Section 5.2 we announce the caveat that will enable us to relax such a tension, then we quickly review the multi-field formalism, introducing the main equations that are relevant for our subsequent calculations. Then in Section 5.2.2 we consider the general case of two-field models of inflation with constant turning rates, at both the background and perturbation levels. As an example, a particular well-motivated model in which the geometry of the field space is hyperbolic is further explored. In Section 5.3 we will show that the new scale in the problem (the negative curvature in field space) and the constant turning rate condition, allow us to find a non-trivial relation between the geodesic distance $[\Delta\phi]_{\rm G}$ and the non-geodesic distance $[\Delta\phi]_{\rm NG}$. Such a relation is indeed the incarnation of the aforementioned non-geodesic motion caveat. Armed with this relation and a couple of other well-defined phenomenological considerations, in Section 5.4 we derive what is probably the main result of this paper: the naive parameter space and the geometrical scales of multi-field inflation models are highly constrained in order to be swampland-safe. While current bounds on non-Gaussianities [50] are not useful to constrain the aforesaid parameter space, in Section 5.5 we briefly address how futuristic observations of non-Gaussian signals may indeed drastically affect our findings. Finally, we give a general summary of our findings in Section 5.6, leaving the discussion of other coordinate systems for the hyperbolic geometry, and of the other maximally symmetric 2d field space geometries (and why they are not useful backgrounds for our purposes) for Appendices C and D, respectively.

5.1 Super-Planckian Displacements in String Theory

We may naively worry that super-Planckian field displacements will lead to super-Planckian energy densities and a correspondingly large gravitational backreaction.[6] However, it so happens that large field displacements along flat directions of the inflaton potential will not induce large variations of the energy density ρ of the universe during inflation, and $\rho \sim V \ll M_{\rm Pl}^4$ can be kept valid as long as the slow-roll parameters remain small. The real issue is that gravity needs a UV-completion, and the couplings between the inflaton and the new degrees of freedom of such a UV-completion are not necessarily constrained to respect the symmetries that one may naively impose to render a flat inflaton potential. EFT reasoning leads us to expect that when integrating out the heavy modes of the full theory we are left with an effective action with a structure of the form [48]

$$\mathcal{L}_{\rm eff}[\phi] = \mathcal{L}_0[\phi] + \sum_{i=1}^{\infty}\left(\frac{c_i}{\Lambda^{2i}}\phi^{4+2i} + \frac{d_i}{\Lambda^{2i}}(\partial\phi)^2\phi^{2i} + \frac{e_i}{\Lambda^{4i}}(\partial\phi)^{2(i+1)} + \dots\right), \tag{5.7}$$

[6] A nice discussion about super-Planckian field displacements occurring at sub-Planckian energies may be found, for instance, in [147].

where $\mathcal{L}_0[\phi]$ is the Lagrangian describing the light degrees of freedom, the ellipsis represents higher-order (in derivatives) operators, $\{c_i, d_i, e_i, \ldots\}$ are dimensionless Wilson coefficients which are expected to be $\mathcal{O}(1)$, and Λ is the mass of the heavy modes which is at least Planckian. Unless one finely-tunes all the Wilson coefficients to be much smaller than 1, dangerous corrections to the two-derivative kinetic term as well as to the potential are expected for super-Planckian displacements.[7]

5.1.1 The Swampland Distance Conjecture

The swampland distance conjecture may be considered as a particular instance of the previous statement regarding EFT's, placed in the well defined context of string theory. Within this scheme, it is widely known that the bosonic string and the superstring live in 26 and 10 dimensions respectively. These both seem to be incompatible with the 4-dimensional universe we live in. However, this may be a hasty claim, since most of these dimensions may be compact and small, so that they have not been observed yet. Therefore, this lead us to think about string theory in a spacetime which has compactified dimensions. In general, these last are wrapped up on themselves on a special class of manifolds, such as Calabi-Yau spaces or orbifolds [148]. In the following, in order to make the appearance of the swampland distance conjecture manifest in the simplest way possible, we will consider compactifications on a circle.

Compactifications on a Circle

Let us consider a $D = d+1$ spacetime in which one of the spatial dimensions, let us say, X^d is taken to be compact in the shape of a circle, *i.e.*, we can always do the following identification

$$X^d \simeq X^d + 1. \tag{5.8}$$

We are interested in address, how an EFT looks in d non-compact dimensions. To do so, we can write the D-dimensional metric as

$$ds^2 \equiv G_{MN} dX^M dX^N = e^{2\alpha\phi} g_{\mu\nu} dX^\mu dX^\nu + e^{2\beta\phi} dX^d dX^d. \tag{5.9}$$

Several things have been introduced; the D-dimensional metric is G_{MN} with coordinates X^M, where $M, N = 0, \ldots, d$. The d-dimensional metric is $g_{\mu\nu}$, where $\mu, \nu = 0, \ldots, d-1$. Additionally, the full metric has a parameter ϕ which can be regarded as a d-dimensional scalar field.[8] The constants α and β are chosen to be

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}, \quad \beta = -(d-2)\alpha. \tag{5.10}$$

It is possible to compute the circumference of the circle ($2\pi R$) as

$$2\pi R = \int_0^1 dX^d \sqrt{G_{dd}} = e^{\beta\phi}, \tag{5.11}$$

[7]The seminal idea of implementing a weakly broken shift symmetry $\phi \to \phi + c$, is useful for building radiatively stable models of large-field inflation. However, whether this symmetry is actually compatible with a UV-completion of gravity, like string theory, remains a question of debate [48].

[8]Actually this field is called the dilaton.

then, the radius of the circle is a dynamical field in d-dimensions. The Hilbert-Einstein action in D-dimensions, according to (5.9) is

$$S = \int d^D X \sqrt{-G} R^D = \int d^d X \sqrt{-G} \left[R^d - \frac{1}{2} (\partial\phi)^2 \right]. \tag{5.12}$$

Note that, in the previous expression is manifestly the fact that ϕ is a dynamical field. Now, let us introduce a massless D-dimensional scalar field Ψ. Since the d^{th} dimension is periodic so must Ψ be, therefore we can decompose it as

$$\Psi(X^M) = \sum_{n=-\infty}^{\infty} \psi_n(X^\mu)\, e^{2\pi i n X^d} \tag{5.13}$$

the modes ψ_n are d-dimensional scalar fields, and they receive the names

$$\psi_0 \to \text{Zero-mode of } \Psi, \tag{5.14}$$
$$\psi_n (n \neq 0) \to \text{Kaluza-Klein (KK) modes of } \Psi. \tag{5.15}$$

Note that due to (5.13), the momentum is quantized along the compact direction as

$$- i \frac{\partial}{\partial X^d} \Psi = 2\pi n \Psi. \tag{5.16}$$

Just for simplicity, let us consider $g_{\mu\nu} = \eta_{\mu\nu}$. Since Ψ is massless, in D-dimensions its Klein-Gordon equations is

$$\partial^M \partial_M \Psi = (e^{-2\alpha\phi} \partial^\mu \partial_\nu + e^{-2\beta\phi} \partial^2_{X^d}) \Psi = 0, \tag{5.17}$$

which gives the following equation of motion for the ψ_n modes

$$\left[\partial^\mu \partial_\mu - \left(\frac{n}{R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}} \right] \psi_n = 0. \tag{5.18}$$

Therefore, we can identify the mass of the KK modes as

$$M_n^2 = \left(\frac{n}{R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}}, \tag{5.19}$$

hence, in the d-dimensional theory the KK modes are a massive tower of states with increasing masses as is shown in the previous equation.

Compactifications of string theory on a circle matches the simple field theory calculation for the KK masses

$$(M_{n,\omega})^2 = \left(\frac{n}{R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}} + (2\pi R)^{\frac{2}{d-2}} \left(\frac{\omega R}{\alpha'}\right), \tag{5.20}$$

where ω ($\omega \in \mathbb{Z}$) is the winding number which inform us about how many times the string is wrapping around the circle and α' is the inverse of the string tension. The natural question is, how this spectrum behaves under variations of the expectation value of the field ϕ? We can address this question by looking the relation (5.11). The possible expectation values of the field ϕ define a moduli space $\mathcal{M}_\phi$, which in this case has one infinite real dimension, *i.e.,* $\mathcal{M}_\phi : -\infty < \phi < \infty$, $\mathcal{M}_\phi \simeq \mathbb{R}$. Additionally, it is useful to define the variations of ϕ between fixed initial and final values as

$$\Delta\phi = \phi_f - \phi_i. \tag{5.21}$$

By looking at (5.20) it is possible to note that there are two infinite towers of massive states in this EFT; the tower of KK modes with masses given by $M_{n,0}$ and a tower of winding modes given by $M_{0,\omega}$.[9] We can assign to each tower a characteristic mass scale, which is the universal factor multiplying the integers n and ω. Using (5.11) we can write these mass scales as

$$M_{KK} \sim e^{\gamma\phi}, \qquad M_\omega \sim e^{-\gamma\phi}, \tag{5.22}$$

where

$$\gamma = \sqrt{2\left(\frac{d-1}{d-2}\right)} > 0. \tag{5.23}$$

Consequently, the following observation can be made. For any $\Delta\phi$ there exists an infinite tower of states, with some associated scale M, which becomes light at an exponential rate in $\Delta\phi$,

$$M(\phi_i + \Delta\phi) \sim M(\phi_i)e^{-\gamma|\Delta\phi|}. \tag{5.24}$$

There are a couple of important implications of this observation: the exponent γ is typically a constant of order one (in reduced Planck units) and, if $|\Delta\phi| \to \infty$, then a infinite tower of states becomes massless, which means that there is no description of this region of the Moduli in a d-dimensional quantum field theory. In other words, if the EFT has a cutoff Λ below the mass scale of an infinite tower of states, then this field theory can only hold for a finite range of expectations values of ϕ.

The previous kind of reasoning motivated by different compactifications in string theory, let to proposal the following conjecture [132]:

Conjecture 1: *A theory like (5.12) with a moduli space $\mathcal{M}_\phi$ always have two points P_0, $P \in \mathcal{M}_\phi$, such that, their geodesic distance $d(P_0, P)$ is infinite.*
There exist an infinite tower of states, with an associated mass scale M, such that

$$M(P) \sim M(P)e^{-\gamma d(P_0,P)}, \tag{5.25}$$

with $\gamma \sim \mathcal{O}(1) > 0$.

Nevertheless, in realistic string compactifications, typically the moduli space $\mathcal{M}_\phi$ is a non-trivial manifold, and instead of (5.12) we should consider the effective d-dimensional action

$$S = \int d^dx\sqrt{-g}\left[\frac{1}{2}R - \gamma_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + \dots\right], \tag{5.26}$$

where the scalar fields ϕ^a are the coordinates on $\mathcal{M}_\phi$ and their kinetic terms define the metric γ_{ab} of this space. The geodesic distance between two points $P_0, P_1 \in \mathcal{M}_\phi$ is defined as

$$d(P_0, P_1) \equiv \int_{\mathcal{C}} d\lambda\sqrt{\gamma_{ab}(\phi)\frac{d\phi^a}{d\lambda}\frac{d\phi^b}{d\lambda}}, \tag{5.27}$$

[9] These two towers of states are related between them through a $\mathbb{Z}_2$ symmetry known as T-duality.

where $\mathcal{C}$ is the geodesic connecting the points P_0 and P_1, and λ is some parameter along this geodesic. In this scenario, the first part of the Conjecture 1 can be easily violated due to the possibility of having a moduli with non-trivial topology. For instance, a certain compactification can lead to $\mathcal{M}_\phi \simeq S^1$,[10] in such a space, the maximum distance that can exist between two points is 2π. Therefore, we cannot expect that the infinite geodesic distance to be a general property of moduli spaces in quantum gravity, so we may consider then how large does $d(P_0, P_1)$ need to be in order that the asymptotic exponential behaviour gives a good approximation for any starting point P_0.

In Ref. [149], refinements have been made to Conjecture 1, where it is claimed that the tower of states quickly flows to exponential behaviour for any $d(P_0, P_1) > \mathcal{O}(1)M_{\text{Pl}}$. Moreover, in [149] a conjecture based on aspects of a general theory of quantum gravity is formulated, not only restricted to be a string theory. The conjecture states the following:

Conjecture 2: *Consider a theory like (5.26) with a Moduli space $\mathcal{M}_\phi$. Let the geodesic distance between any two points $P_0, P \in \mathcal{M}_\phi$ be denoted $d(P_0, P_1)$.*
There exist an infinite tower of states, with an associated mass scale M, such that

$$M(P) < M(P_0) e^{-\gamma \frac{d(P_0,P_1)}{M_{Pl}}}, \tag{5.28}$$

with $\gamma \sim \mathcal{O}(1) > 0$, if $d(P_0, P_1) \gtrsim M_{Pl}$.

The previous statement holds even for fields with a potential, not just for moduli, where the moduli space is replaced with the field space in the effective theory.

Conjectures 1 and 2 are named the distance conjecture and the refined distance conjecture, respectively. However, both are simply referred to, as the swampland distance conjecture (SDC). To sum up, the SDC states that traversing super-Planckian field distances in EFT's derived from quantum gravity will always imply the appearance of an *infinite* tower of light modes, which openly undermines the initial effective description. This means that any EFT has a proper geodesic field range $\Delta\phi$ in which the theory is valid, and is set to be sub-Planckian.

The SDC gives rise to the so-called "first swampland criterion" which establishes that field distances $\Delta\phi$ involved in phenomenologically successful EFT's —consistent with quantum gravity— must be bounded from above [132], meaning

$$\Delta\phi < \vartheta \cdot M_{\text{Pl}}, \tag{5.29}$$

where ϑ is an $\mathcal{O}(1)$ number that depends on the details of the UV-completion. The authors of [150] have also proposed a second swampland criterion, which rules out the existence of stable de Sitter vacua in consistent EFT's, by establishing the inequality

$$\frac{|V_\phi|}{V} \geq \frac{\varsigma}{M_{\text{Pl}}}, \tag{5.30}$$

where V is the scalar field potential, $V_\phi \equiv \partial_\phi V$, and ς is another $\mathcal{O}(1)$ number. Furthermore, in [151] it has been argued that single-field slow-roll inflationary models may, in general,

[10]This kind of compactification can be generated by periodic scalars $\phi \sim \phi + 2\pi$, for example, axions.

be in conflict with these two bounds. Consequently, the authors of [152] have studied the real impact of the swampland conjectures in light of data. Nevertheless, it is likely that the second criteria, seemingly dubbed the “de Sitter conjecture”, will be abandoned as it does not have strong theoretical support (see however [153–155] and references therein). Instead, the first criteria is based in sound theoretical arguments such as the WGC [125], so it will not so easily fade away.

5.1.2 A Geometrical Scalar Cutt-off

Given that the SDC is formulated in terms of geodesic distances, it is only logical to study its effects for inflation within setups with many fields or, at least, two fields. In the following we will consider effective field theories of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\gamma_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi)\right] + \Delta S_\Lambda, \tag{5.31}$$

where R is the Ricci scalar determined by the spacetime metric $g_{\mu\nu}$, and ϕ^a, with $a = \{1,\ldots,N\}$, are scalar fields spanning a field space which is itself endowed with its own sigma model metric $\gamma_{ab}(\phi)$. On the other hand, $V(\phi)$ stands for the scalar potential of the system. Because the naive action in equation (5.31) must be understood as an effective description valid up to a given cut-off energy scale Λ, we have included a term ΔS_Λ standing for corrections that emerge from unknown physics which takes place at energies above Λ (*e.g.* loop corrections, or the integration of degrees of freedom kinematically suppressed at energies below Λ). Among these corrections, there will necessarily be an operator of the form

$$\Delta S_\Lambda \supset -\frac{1}{4}\int d^4x\sqrt{-g}\, g^{\mu\nu}\frac{f_{abcd}}{\Lambda^2}\,\Delta\phi^c\Delta\phi^d\partial_\mu\phi^a\partial_\nu\phi^b, \tag{5.32}$$

where f_{abcd} represents a collection of order one Wilson coefficients. In the previous expression $\Delta\phi^a \equiv \phi^a - \phi^a_\star$, where $\phi^a_\star$ denotes a given field value around which S is taken to be valid. It should be clear that the presence of (5.32) sets a maximum field range centered at $\phi^a_\star$ beyond which one needs to become skeptical about the first term in (5.31). Indeed, as soon as we depart from $\phi^a_\star$ a field distance $\Delta\phi \sim \Lambda$, we are forced to resume every operator (suppressed by powers of Λ^{-2}) comprising ΔS_Λ. Actually, the presence of corrections like the one outlined in (5.32) has some consequences on our attitude towards the field geometry parametrized by γ_{ab}. If we allow (5.32) back into the first term of (5.31), so as to track the small corrections implied by Λ^2 in our computations, we may define an effective metric given by

$$\gamma^\Lambda_{ab}(\phi) \equiv \gamma_{ab}(\phi) + \frac{1}{2\Lambda^2}f_{abcd}\Delta\phi^c\Delta\phi^d + \ldots, \tag{5.33}$$

where the ellipsis stands for higher order terms in the fields, suppressed by higher powers of Λ^{-2}. On the other hand, without loss of generality, we may always choose $\phi^a_\star = 0$ and adopt a field parametrization by which

$$\gamma^\Lambda_{ab}(\phi) = \delta_{ab} - \frac{1}{3}\mathbb{R}^\Lambda_{acbd}(\phi_\star)\,\phi^c\phi^d + \ldots, \tag{5.34}$$

where $\mathbb{R}^\Lambda_{acbd}(\phi_\star)$ is the Riemann tensor, constructed out of γ^Λ_{ab} in (5.33), and evaluated at $\phi^a = \phi^a_\star = 0$.[11] We then see that, in these coordinates, the presence of the $\frac{1}{\Lambda^2}f_{abcd}$ operator

[11] These are nothing but the well-known Riemann Normal Coordinates.

may be understood as a correction to the Riemann tensor. That is, the "true" Riemann tensor of the EFT, at $\phi^a = 0$, should be read as

$$\mathbb{R}^{\Lambda}_{abcd} = \mathbb{R}_{abcd} + \frac{1}{\Lambda^2}\, g_{abcd}\,(f) + \ldots, \tag{5.35}$$

where (again) the ellipsis denotes terms suppressed by higher powers of Λ^{-2} and $g_{abcd}(f)$ is a "Riemann-symmetrized"[12] set of linear combinations among the Wilson coefficients introduced in (5.32). Now, let us assume that the field space has a characteristic curvature determined by a mass scale R_0, meaning $\mathbb{R} \sim R_0^{-2}$. Then if $R_0 > \Lambda$, we should consider, for all practical purposes, the theory to be indistinguishable from a theory with a flat field geometry $\gamma_{ab} = \delta_{ab}$, which is indeed attained as $R_0 \to \infty$. This is simply because the physical effects from such geometries would be suppressed against corrections of order Λ^{-2}, which are already assumed to be sub-leading. Hence, if we are interested in studying genuine non-trivial effects from γ_{ab} due to the field space geometry, we are forced to consider geometries for which $R_0 < \Lambda$.[13]

We may connect the present discussion with that of the previous section. For example, the scale Λ appearing in (5.32) may be identified with the scale $1/\gamma$ of equation (5.28). That is, the SDC may be taken as a specific realization within string theory, whereby the low-energy EFT cannot be probed beyond a field range specified by the string compactifications where it descents from. For our porpuses, we will take $\Lambda = M_{\rm Pl}$, in line with equation (5.29).

5.1.3 A Multi-Field Lyth Bound

Canonical single-field slow-roll inflation generically predicts that the overall field displacement $\Delta\phi$ experienced by the inflaton during the quasi-de Sitter phase must satisfy a lower bound. To derive it, it is enough to plug the background equation $\dot{H} = -\dot{\phi}^2/2M_{\rm Pl}^2$ back into the defining relation of the first slow-roll parameter, namely $\epsilon \equiv -\dot{H}/H^2$. By doing so we get $\epsilon = \dot{\phi}^2/2H^2M_{\rm Pl}^2$ which, assuming a nearly constant value of ϵ, allows us to write

$$\frac{\Delta\phi}{M_{\rm Pl}} \simeq \sqrt{2\,\epsilon}\,\Delta N, \tag{5.36}$$

where ΔN is the effective number of e-folds during inflation. In canonical single-field slow-roll inflation the amplitudes of the dimensionless power spectra of scalar and tensor modes are respectively given by

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{H^2}{8\pi^2 M_{\rm Pl}^2\,\epsilon} \quad \text{and} \quad \mathcal{P}_h(k) = \frac{2H^2}{\pi^2 M_{\rm Pl}^2}, \tag{5.37}$$

implying that the tensor-to-scalar ratio is uniquely determined by ϵ through $r = 16\,\epsilon$, which immediately leads to the well-known relation

$$\frac{\Delta\phi}{M_{\rm Pl}} = \Delta N\sqrt{\frac{r}{8}}. \tag{5.38}$$

[12]Explicitly, $g_{abcd}(f) \equiv \frac{1}{2}\,(f_{adbc} - f_{dbac} - f_{acbd} + f_{cbad})$. It is then easy to check, using the fact that $f_{abcd} = f_{(ab)(cd)}$, that $g_{abcd} = -g_{bacd} = -g_{abdc}$, $g_{abcd} = g_{cdab}$, and $g_{a[bcd]} = 0$, where the brackets ( ) and [ ] denote the symmetric and antisymmetric part of the indicated indices, respectively. It can be shown that these last three identities g_{abcd} satisfies form a *complete* list of symmetries of the curvature tensor.

[13]Note that we are *not* claiming that one cannot study the dynamics of theories with $R_0 > \Lambda$; we are simply emphasizing the fact that any conclusion from such a theory, where R_0 plays an essential role, should not be trusted from an EFT point of view.

Given that the minimal amount of e-folds necessary to account for the CMB anisotropies is about $\Delta N \sim 60$, one infers a lower bound on the field displacement given by

$$\frac{\Delta\phi}{M_{\rm Pl}} \gtrsim \mathcal{O}(1)\sqrt{\frac{r}{0.01}}. \tag{5.39}$$

In words, (5.39) implies that *if* we ever measure primordial gravitational waves, meaning that r happens to be around ~ 0.01, *then* the field distance $\Delta\phi$ traversed by the inflaton is necessarily super-Planckian, in clear conflict with the bound in (5.29).

Now, in multi-field models of inflation, the background equations of motion determined by an action of the form (5.31) leads to the same relation $\epsilon = \dot\phi^2/2H^2M_{\rm Pl}^2$ connecting ϵ with the scalar field rapidity, though (importantly) $\dot\phi^2 \equiv \gamma_{ab}\,\dot\phi^a\dot\phi^b$ in this context. This leads to the same relation (5.36), but this time with $\Delta\phi$ given by

$$\Delta\phi(t') = \int^{t'} dt\,\sqrt{\gamma_{ab}\,\dot\phi^a\dot\phi^b}. \tag{5.40}$$

This is the *non-geodesic* field distance traversed by the fields in multi-field space. A crucial difference when contrasted with the single-field case is that, in the multi-field context, the bends experienced by the *non-geodesic* inflationary trajectory will turn on interactions between the curvature perturbation $\mathcal{R}$ and field fluctuations normal to the trajectory. As a result, the power spectrum of scalar fluctuations will pick up a dependence on new background parameters in addition to ϵ. For instance, in the particular case of two-field models, the power spectrum becomes

$$\mathcal{P}_{\mathcal{R}} = \frac{H^2}{8\pi^2 M_{\rm Pl}^2\,\epsilon\,\beta}, \tag{5.41}$$

where $\beta = \beta\,(\lambda, \tilde\mu)$ is a function of $\lambda \equiv -2\,\Omega/H$ (where Ω is the local bending rate of the trajectory), and $\tilde\mu \equiv \mu/H$ is, up to the normalization by H, the so-called entropy mass of the fluctuation normal to the path [156]. Thus, the Lyth bound that will be relevant for us, let us just announce it for the time being, is of the form

$$\frac{\Delta\phi}{M_{\rm Pl}} = \Delta N\sqrt{\frac{r}{8\,\beta}} \gtrsim \frac{\mathcal{O}(1)}{\sqrt{\beta}}\sqrt{\frac{r}{0.01}}. \tag{5.42}$$

Since $\beta\,(\lambda, \tilde\mu)$ is, as it turns out, less or equal to unity, this version of the Lyth bound for multi-field models is more stringent than the original one.

For completeness, let us mention that Ref. [157] offers a "generalized Lyth bound" based on the EFT of inflation [13], a framework that captures many classes of single-field models of inflation. Denoting $\Delta\varphi$ "the physically relevant field range" they have found that

$$\frac{\Delta\varphi}{M_{\rm Pl}} = c_p^{-3/2}\,\Delta N\sqrt{\frac{r}{8}}, \tag{5.43}$$

where $c_p \equiv \frac{\omega}{k}\big|_{\omega=H}$ is the *phase velocity* at horizon crossing.[14] Equation (5.43) recovers the usual slow-roll Lyth bound when $c_p = 1$. On the other hand, if $c_p < 1$, this generalized

[14]For example, for $P(X)$ theories $c_p = c_s$, where c_s is the usual speed of sound [158].

bound is stronger than the original one. At this point, it is interesting to note that multi-field models have a well known single-field limit where the non-vanishing bending rate $\Omega \neq 0$ induces the appearance of a nontrivial speed of sound $c_s < 1$ for the primordial curvature perturbation [95,159]. In that limit, which is only possible if the isocurvature mode is massive enough, one ends up finding that $\beta = c_s$. Given that the phase velocity at horizon crossing coincides with c_s in this limit, it might seem intriguing to find out that (5.43) and (5.42) do not coincide by a factor of c_p. However, as the authors of [157] point out, in the case of effective field theories descending from multi-field models, there is more than one scale at play, and the rule determining how to identify the field range in terms of EFT quantities gets modified.[15] Taking that into account, they find

$$\frac{\Delta\varphi}{M_{\rm Pl}} = c_p^{-1/2}\,\Delta N\,\sqrt{\frac{r}{8}}, \tag{5.44}$$

which indeed coincides with our version of the non-geodesic field range (5.42) in the appropriate limit.

5.2 Multi-field Inflation Overcomes the SDC

So far, we have seen if gravitational waves with a sizable r are detected in the near future, the Lyth bound (in any of its forms) would imply super-Planckian displacements of the inflaton in field space, in open tension with the swampland distance conjecture. However, both the Lyth bound and the SDC refer to different classes of field distances. More to the point, the displacement upon which the Lyth bound is operative is non-geodesic, whereas the SDC applies on field distances measured with the help of geodesic paths. Thus, as long as the Lyth bounds apply to non-geodesic inflationary trajectories of multi-field scenarios, and the swampland criterion applies only to the geodesic trajectories, there is a chance that observable gravitational waves may only rule out single-field inflation, while keeping multi-field inflation as a consistent low-energy EFT. In fact, this opens a window of opportunity: non-geodesic trajectories turn on non-trivial interactions between the curvature perturbation and fluctuations representing fields orthogonal to the non-geodesic path. As a consequence, a measurement of tensor modes should imply, within the context of string theory compactifications (or more generally, quantum gravity consistent UV-completions), other observable effects related to bending trajectories in multi-field models.

5.2.1 Multi-Field Inflation

As previously discussed, EFT reasoning stemming from string theory compactifications can easily justify a 4d theory defined by an action of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\,R - \frac{1}{2}g^{\mu\nu}\gamma_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi)\right]. \tag{5.45}$$

In a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, defined through the background metric $ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2$, it is useful to write all the fields in the system (including

[15]This, in turn, signals that a proper notion of field range within the EFT requires information from the UV theory that it describes at low energies.

the metric), generically denoted by $\Psi(\boldsymbol{x},t)$, as the sum of a background and perturbations $\Psi(\boldsymbol{x},t) = \Psi_0(t) + \delta\Psi(\boldsymbol{x},t)$. The equations of motion (EOM) for the background system defined by (5.45) then read

$$3M_{\rm Pl}^2H^2 = \frac{1}{2}\dot{\phi}_0^2 + V, \tag{5.46}$$

$$D_t\dot{\phi}_0^a + 3H\dot{\phi}_0^a + V^a = 0, \tag{5.47}$$

where $H \equiv \dot{a}/a$ is the Hubble expansion rate, $\dot{\phi}_0^2 \equiv \gamma_{ab}\,\dot{\phi}_0^a\dot{\phi}_0^b$, and $V^a \equiv \gamma^{ab}V_b \equiv \gamma^{ab}\partial_b V$. In the previous expression D_t stands for a "time covariant derivative" defined to act on a given field space vector X^a as $D_tX^a \equiv \dot{X}^a + \Gamma^a_{bc}X^b\dot{\phi}_0^c$, where Γ^a_{bc} are the usual Christoffel symbols compatible with the field space metric γ_{ab}. Moreover, as usual, the EOM may be used to derive a "conservation law" of the form

$$\dot{H} = -\frac{\dot{\phi}_0^2}{2M_{\rm Pl}^2}. \tag{5.48}$$

A given background solution $\phi_0^a(t)$ defines a path in field space parametrized by time t. Therefore, it is natural to define a unit-norm vector which is tangent to the inflationary trajectory, namely [160]

$$T^a \equiv \frac{\dot{\phi}_0^a}{\dot{\phi}_0}. \tag{5.49}$$

A time covariant derivative of this tangent vector defines an orthonormal vector N^a together with an angular velocity Ω parametrizing the rate of bending of the trajectory through the equation

$$D_tT^a \equiv -\Omega\,N^a. \tag{5.50}$$

By projecting (5.47) along the two directions T^a and N^a one obtains two equations:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V_\phi = 0, \tag{5.51}$$

$$\Omega = \frac{V_N}{\dot{\phi}_0}, \tag{5.52}$$

where $V_\phi \equiv T^aV_a$ and $V_N \equiv N^aV_a$. The first one of these equations describes the displacement of the field along the trajectory, whereas the second gives us back a relation tying Ω with the slope of the potential V_N away from the trajectory.

In order to study the dynamics of the perturbations, it is convenient to write the metric using the Arnowitt-Deser-Misner (ADM) formalism [42, 161] as

$$ds^2 = -N^2dt^2 + a^2(t)\,e^{2\mathcal{R}(\boldsymbol{x},t)}\delta_{ij}\left(dx^i + N^idt\right)\left(dx^j + N^jdt\right), \tag{5.53}$$

where N and N^i are the lapse and the shift functions, respectively, and $\mathcal{R}(\boldsymbol{x},t)$ is the spatial curvature perturbation. In the two-field case where $\phi^a = \{\phi^1,\phi^2\}$, it is possible to project the perturbations $\delta\phi^a(\boldsymbol{x},t)$ along the tangent and normal vectors in such a way that

$$\delta\phi^a(\boldsymbol{x},t) = T^a(t)\,\delta\phi_\parallel(\boldsymbol{x},t) + N^a(t)\,\sigma(\boldsymbol{x},t), \tag{5.54}$$

where $\delta\phi_{\parallel}(\boldsymbol{x},t)$ corresponds to the inflaton perturbation and $\sigma(\boldsymbol{x},t)$ is the so-called isocurvature perturbation [162]. Moreover, it is useful to adopt the co-moving gauge, defined through $\delta\phi_{\parallel}(\boldsymbol{x},t) = 0$, so that the variable $\mathcal{R}(\boldsymbol{x},t)$ truly represents the adiabatic mode of perturbations. After writing the action (5.45) in terms of (5.53) one may solve the constraint equations[16], which to linear order yield

$$N = 1 + \dot{\mathcal{R}}/H, \tag{5.55}$$

$$N_i = \partial_i \left(\chi - \frac{\mathcal{R}}{H} \right), \tag{5.56}$$

where χ is a function that satisfies $a^{-2}\nabla^2\chi = \epsilon\,\dot{\mathcal{R}} + \sqrt{2\,\epsilon}\,\Omega\,\sigma$. Plugging these expressions for N and N_i back into the action (5.45), it is possible to find a quadratic action for the perturbations $\mathcal{R}$ and σ given by[17]

$$S^{(2)} = \int d^4x\, a^3 \left[\epsilon \left(\dot{\mathcal{R}} - \lambda \frac{H}{\sqrt{2\epsilon}}\,\sigma \right)^2 - \epsilon\,\frac{(\nabla\mathcal{R})^2}{a^2} + \frac{1}{2}\left(\dot{\sigma}^2 - \frac{(\nabla\sigma)^2}{a^2} \right) - \frac{1}{2}\mu^2\sigma^2 \right], \tag{5.57}$$

where we have defined

$$\lambda \equiv -\frac{2\,\Omega}{H}, \tag{5.58}$$

$$\mu^2 \equiv N^a N^b \left(V_{ab} - \Gamma^c_{ab} V_c \right) + \epsilon\, H^2 \mathbb{R} + 3\,\Omega^2. \tag{5.59}$$

Here μ is the entropy mass of σ, defined in terms of the projection of the second derivative of the potential along the normal direction, the Ricci scalar $\mathbb{R}$ determined by the field space metric γ_{ab}, and the angular velocity Ω. In Subsection 5.2.3 we will deal with a particular realization of the previous action.

5.2.2 Two-Field Inflation with Constant Turning Rates

In this section we study, in some detail, general two-field models characterized for having a constant turning rate Ω during a long period of inflation. To start with, notice that in the particular case of two-dimensional field spaces, given a metric γ_{ab}, we may always express its inverse as

$$\gamma^{ab} = \frac{1}{\gamma} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix}, \tag{5.60}$$

where $\gamma \equiv \det \gamma_{ab}$. Then, given a tangent vector T^a parametrizing an arbitrary trajectory, the normal vector may be conveniently fixed as

$$N_a \equiv -\sqrt{\gamma}\,\epsilon_{ab}\,T^b. \tag{5.61}$$

Moreover, it is always possible to adapt the field coordinate system around a given inflationary trajectory such that one of the field coordinates remains constant along it. That is, during inflation, one of the fields evolves and the second field remains fixed to a nearly constant

[16] Recall that N and N^i are, ultimately, Lagrange multipliers that enforce the diffeomorphism constraints of gravity. See for instance [163].

[17] From here on we work in units where the reduced Planck mass is set to unity, $M_{\rm Pl} = 1$, unless explicitly written for convenience and clarity.

value. Notice that in practice this strategy is commonly adopted by model builders, which assign the role of the evolving "inflaton" to one of the fields of their systems. However, in the present approach, this is just the consequence of adopting the field parametrization to a trajectory characterized for having a constant turning rate Ω. For definiteness, let us consider a system with two fields $\mathcal{X}$ and $\mathcal{Y}$, in such a way that the inflationary trajectory keeps the $\mathcal{Y}$ field nearly constant, *i.e.* $\mathcal{Y} = \mathcal{Y}_0$. In this case $\dot{\mathcal{Y}} = 0$, and one immediately obtains

$$T^a = \frac{1}{\sqrt{\gamma_{\mathcal{X}\mathcal{X}}}}\,(1,0)\,, \qquad T_a = \frac{1}{\sqrt{\gamma_{\mathcal{X}\mathcal{X}}}}\,\big(\gamma_{\mathcal{X}\mathcal{X}}, \gamma_{\mathcal{X}\mathcal{Y}}\big) \tag{5.62}$$

$$N^a = \frac{1}{\sqrt{\gamma_{\mathcal{X}\mathcal{X}}\,\gamma}}\,\big(-\gamma_{\mathcal{X}\mathcal{Y}}, \gamma_{\mathcal{X}\mathcal{X}}\big)\,, \qquad N_a = \sqrt{\frac{\gamma}{\gamma_{\mathcal{X}\mathcal{X}}}}\,(0,1)\,. \tag{5.63}$$

These expressions may be used in Eq. (5.50) to obtain a simple relation between $\Omega \neq 0$ and $\dot{\mathcal{X}}$, determining the first-order system

$$\dot{\mathcal{X}} = -\frac{\gamma_{\mathcal{X}\mathcal{X}}}{\sqrt{\gamma}}\frac{\Omega}{\Gamma^{\mathcal{Y}}_{\mathcal{X}\mathcal{X}}} \quad \text{while} \quad \dot{\mathcal{Y}} = 0. \tag{5.64}$$

On the other hand, given the assumed constancy of $\mathcal{Y}$, one directly obtains a relation between the rapidity of the field displacement along the non-geodesic motion $[\Delta\phi]_{\rm NG}$ and $\dot{\mathcal{X}}$, namely

$$[\dot{\phi}]_{\rm NG} = \sqrt{\gamma_{\mathcal{X}\mathcal{X}}}\,\dot{\mathcal{X}}. \tag{5.65}$$

Then, since both $[\dot{\phi}]_{\rm NG}$ and Ω must evolve slowly in order to keep the scale invariance of the primordial spectra, we finally arrive at the simple condition

$$\left.\frac{\sqrt{\gamma}}{\gamma^{3/2}_{\mathcal{X}\mathcal{X}}}\,\Gamma^{\mathcal{Y}}_{\mathcal{X}\mathcal{X}}\right|_{\mathcal{Y}=\mathcal{Y}_0} = -\frac{1}{\rho_{\rm NG}} \simeq \text{constant}, \qquad \rho_{\rm NG} \equiv \frac{[\dot{\phi}]_{\rm NG}}{\Omega}, \tag{5.66}$$

where we have introduced the radius of curvature $\rho_{\rm NG}$ of the bending trajectory. The previous condition, which is *independent* of the potential responsible for the inflationary dynamics, informs us that *not* any geometry will be able to accommodate a constant turning rate. Indeed, at this stage it is quite fair to ask: Why the field potential V and its derivative have somehow "dissapeared"? In short, the reason behind this fact is that we are *assuming* that the potential in (5.31) must be such that it ensures a trajectory with a nearly constant rate. By now, there are several working examples in the literature of such potentials[18], which is more than enough for our purposes. At any rate, it is immediately clear that a two-field metric that is independent of $\mathcal{X}$ will allow for constant turns. Having this result in mind, in Section 5.3 we will consider models where the metric is independent of such a field.

5.2.3 Perturbations

Let us now turn our attention to the dynamics of perturbations in a multi-field system with a constant turning rate. The action of such a system is given by (5.57) with a constant λ coupling. For the purposes of the present discussion, it is useful to consider a canonical version of $\mathcal{R}$ given by

$$\mathcal{R}_c \equiv \sqrt{2\,\epsilon}\,\mathcal{R}. \tag{5.67}$$

[18]See for instance [159, 164–168]

After this reparametrization, it is easy to obtain the following linear equations of motion:

$$(\ddot{\mathcal{R}}_c - \lambda H\dot{\sigma}) + 3H(\dot{\mathcal{R}}_c - \lambda H\sigma) + \frac{\nabla^2}{a^2}\mathcal{R}_c = 0, \tag{5.68}$$

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{\nabla^2}{a^2}\sigma + H^2\tilde{\mu}^2\sigma + \lambda H(\dot{\mathcal{R}}_c - \lambda H\sigma) = 0, \tag{5.69}$$

where we have defined

$$\tilde{\mu} \equiv \frac{\mu}{H}. \tag{5.70}$$

Notice that in order to derive these equations we have assumed that $\eta = \dot{\epsilon}/H\epsilon$ and $\xi = \dot{\eta}/H\eta$ remain suppressed for a sufficiently long time. To keep the scale invariance of the system, we do not only require small η and ξ, but also that λ and μ remain almost constant. This means that we must assume that $|\dot{\lambda}| \ll |H\lambda|$ and $|\dot{\mu}| \ll |H\mu|$, so that λ and μ may be effectively taken to be constants.

The main problem that we wish to tackle here is the computation of the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ of $\mathcal{R}$ as affected by the isocurvature perturbation σ when the rate of turn remains constant. This problem has been previously studied in different regimes of the parameter space $\{\lambda, \tilde{\mu}\}$ in model dependent setups (see for instance [136,162,169–173]) and it was dealt with in a model independent manner in [174]. From dimensional analysis, $\mathcal{P}_{\mathcal{R}}(k)$ is necessarily proportional to the Hubble expansion rate (which sets the size of the horizon H^{-1} during horizon crossing) squared, H^2. In the absence of mixing between $\mathcal{R}$ and σ, that is to say when $\lambda = 0$, we would obtain that the power spectrum of the canonical curvature perturbation $\mathcal{R}_c$ is given by $\mathcal{P}_{\mathcal{R}_c} = H^2/4\pi^2$, from where it follows, by using (5.67), that $\mathcal{P}_{\mathcal{R}} = H^2/8\pi^2\epsilon$. Therefore, given that the only parameters present in the equations of motion (5.68) and (5.69) consist of λ and $\tilde{\mu}$, it follows that for $\lambda \neq 0$ the power spectrum of $\mathcal{R}_c$ must be of the form $\mathcal{P}_{\mathcal{R}_c} = H^2/4\pi^2\beta(\lambda,\tilde{\mu})$, where $\beta(\lambda,\tilde{\mu})$ is a dimensionless function of λ and $\tilde{\mu}$. As a result we conclude that the power spectrum for the curvature perturbation $\mathcal{R}$ is necessarily of the form

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{H^2}{8\pi^2\,\epsilon\,\beta\,(\lambda,\tilde{\mu})}. \tag{5.71}$$

In order to determine the shape of $\beta(\lambda,\tilde{\mu})$ we must solve the equations of motion (5.68) and (5.69) for quantum fields $\mathcal{R}_c(\boldsymbol{x},t)$ and $\sigma(\boldsymbol{x},t)$ satisfying standard commutation relations with their respective canonical momenta. To proceed, it is useful to write the perturbations in Fourier space

$$\mathcal{R}_c(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\hat{\mathcal{R}}_c(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \sigma(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\hat{\sigma}(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.72}$$

where $\hat{\mathcal{R}}_c(\boldsymbol{k},t)$ and $\hat{\sigma}(\boldsymbol{k},t)$ may be expressed in terms of two mode-functions $u_\alpha(k,t)$ and $v_\alpha(k,t)$ (actually, Mukhanov-Sasaki variables) as

$$\hat{\mathcal{R}}_c(\boldsymbol{k},t) = \frac{1}{a}\sum_{\alpha=1}^{2}\left[\hat{a}_\alpha(\boldsymbol{k})u_\alpha(k,t) + \hat{a}_\alpha^\dagger(-\boldsymbol{k})u_\alpha^*(k,t)\right], \tag{5.73}$$

$$\hat{\sigma}(\boldsymbol{k},t) = \frac{1}{a}\sum_{\alpha=1}^{2}\left[\hat{a}_\alpha(\boldsymbol{k})v_\alpha(k,t) + \hat{a}_\alpha^\dagger(-\boldsymbol{k})v_\alpha^*(k,t)\right], \tag{5.74}$$

such that the annihilation and creation operators $\hat{a}_\alpha(\boldsymbol{k})$ and $\hat{a}^\dagger_\alpha(\boldsymbol{k})$ satisfy the usual commutation relations, meaning the only non-trivial commutators read

$$\left[\hat{a}_\alpha(\boldsymbol{k}), \hat{a}^\dagger_\beta(\boldsymbol{k'})\right] = (2\pi)^3 \delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}), \quad \alpha = 1,2. \tag{5.75}$$

As usual, the vacuum state of the theory $|0\rangle$ is such that $\hat{a}_{1,2}(\boldsymbol{k})\,|0\rangle = 0$. After plugging (5.72) back into the equations of motion (5.68) and (5.69), one finds new equations of motion for the mode functions $u_\alpha(k,t)$ and $v_\alpha(k,t)$. By using conformal time τ (introduced through the relation $d\tau = dt/a$), one ends up with

$$u''_\alpha - \frac{2}{\tau^2}\,u_\alpha + k^2\,u_\alpha + \frac{\lambda}{\tau}\,v'_\alpha - \frac{2\lambda}{\tau^2}\,v_\alpha = 0, \tag{5.76}$$

$$v''_\alpha - \frac{2}{\tau^2}\,v_\alpha + k^2\,v_\alpha + \frac{\tilde{\mu}}{\tau^2}\,v_\alpha - \frac{\lambda}{\tau}\left(u'_\alpha + \frac{1}{\tau}\,u_\alpha + \frac{\lambda}{\tau}\,v_\alpha\right) = 0. \tag{5.77}$$

In the previous expression, $(\ )' \equiv \frac{d}{d\tau}(\ ) \equiv a\,\frac{d}{dt}(\ )$ denotes a derivative with respect to conformal time. Imposing the Bunch-Davies initial conditions on subhorizon scales

$$u_1 = 0, \quad u_2 = \frac{1}{\sqrt{2k}}e^{-ik\tau}, \quad v_1 = \frac{1}{\sqrt{2k}}e^{-ik\tau}, \quad v_2 = 0, \tag{5.78}$$

the system of coupled differential equations (5.76) and (5.77) with initial conditions (5.78) is suitable for numerical methods. This way, we may obtain the curvature perturbation power spectrum (5.71) using the definition

$$\mathcal{P}_\mathcal{R}(k) \equiv \frac{k^3}{2\pi^2}\left(|\mathcal{R}_1|^2 + |\mathcal{R}_2|^2\right), \tag{5.79}$$

so we can then isolate $\beta\,(\lambda,\tilde{\mu}) = \frac{H^2}{8\pi^2\,\epsilon\,\mathcal{P}_\mathcal{R}(k)}$, and plot it as a function of its arguments. Proceeding so delivers Figure 5.1 as an output.

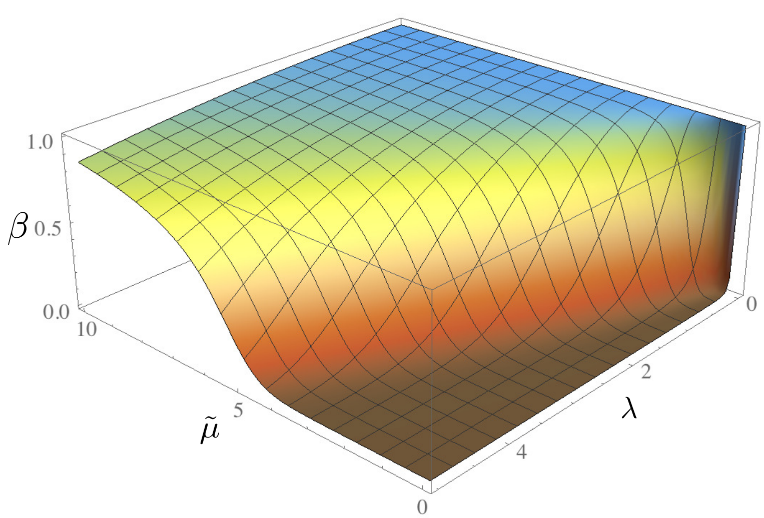

Figure 5.1: Numerical solution for $\beta\,(\lambda,\tilde{\mu})$.

The result shown in the figure agrees with that of Ref. [174] and it is consistent with previous analytical results found in the literature. For instance, it is well known that isocurvature fields with large entropy masses can be integrated out to yield a single-field effective field theory for the curvature perturbation [95, 159, 167, 175–183]. In this effective theory, the final form of the power spectrum will depend on whether the modes crossed the horizon while their dispersion relation was linear ($\omega \propto k$) or quadratic ($\omega \propto k^2$) [166, 179, 184]. If the mode crossed the horizon in the linear regime (which happens as long as $(1-c_s^2)H < c_s M$), then the function β is well approximated by

$$\beta \simeq c_s, \tag{5.80}$$

where c_s is the speed of sound of the curvature perturbation, given by the well known result [95]

$$c_s = \left(1 + \frac{\lambda^2 H^2}{M^2}\right)^{-1/2}, \tag{5.81}$$

where $M \equiv H\sqrt{\tilde{\mu}^2 - \lambda^2}$ (recalling that $\lambda \equiv -2\,\Omega/H$). On the other hand, if horizon crossing takes place in the non-linear regime (which happens if $(1-c_s^2)H > c_s M$), then β is well approximated by [179, 185]

$$\beta \simeq \frac{\pi}{8\,\Gamma^2(5/4)}\left(\frac{Hc_s}{M}\right)^{1/2}, \tag{5.82}$$

where $\Gamma(5/4) \simeq 0.91$. Moreover, the result is also consistent with the so-called ultralight limit, where $\tilde{\mu}^2 = 0$ and $\lambda \neq 0$, for which β has been computed perturbatively in the case $\lambda \ll 1$ [186]. In this ultralight limit, the value of β becomes suppressed by the amount of e-folds elapsed from the moment in which the mode crossed the horizon and the end of inflation (similarly, in Ref. [142] an analytical expression for the power spectrum was obtained for large values of λ in which a superhorizon growth is reported, in agreement with a suppressed value of β for large values of λ). The numerical result shown in Figure 5.1 allows us to see how β behaves for intermediate regimes that have not been solved analytically. For instance, one can appreciate that the ultralight behavior (whereby β becomes suppressed by the number of e-folds) extends to values of $\tilde{\mu}$ and λ greater than one.

Notice that the modified power spectrum (5.71) gives rise to a modified tensor-to-scalar ratio given by

$$r = 16\,\epsilon\,\beta\,(\lambda, \tilde{\mu})\,. \tag{5.83}$$

As usual, using the Friedmann equation in its conservation law form, Eq. (5.48), and the definition of the first slow-roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$, along with $\Delta N \equiv H\Delta t$, we may straightforwardly relate the field displacement with the tensor-to-scalar ratio, finding a Lyth bound of the form

$$\frac{\Delta\phi}{M_{\rm Pl}} = \Delta N\sqrt{\frac{r}{8\,\beta\,(\lambda,\tilde{\mu})}} \gtrsim \frac{\mathcal{O}(1)}{\sqrt{\beta\,(\lambda,\tilde{\mu})}}\sqrt{\frac{r}{0.01}}. \tag{5.84}$$

Equation (5.84) is the relevant Lyth bound we will consider when analyzing the viability of our evading mechanism. Compared with the one pertaining the single-field scenario, besides the fact that here $\Delta\phi \equiv \sqrt{\gamma_{ab}\,\Delta\phi^a\Delta\phi^b}$, it is clear that the ratio $\Delta\phi/M_{\rm Pl}$ is rescaled by a factor of $[\beta\,(\lambda,\tilde{\mu})]^{-1/2}$. Unfortunately, it has not been possible to find an analytic (closed) expression for $\beta\,(\lambda,\tilde{\mu})$ in the *general* case (meaning for *arbitrary* values of λ and $\tilde{\mu}$); this lies beyond the scope of this thesis and remains to be a quite challenging open problem.

5.2.4 Example: Inflation in Hyperbolic Spaces

Let us now review the previous results of this section by focusing our attention in the case of inflationary models where the field geometry is hyperbolic. Consider a set of fields $\phi_0^1 = \mathcal{X}$, $\phi_0^2 = \mathcal{Y}$, and a field space metric given by

$$\gamma_{ab} = \begin{pmatrix} e^{2\mathcal{Y}/R_0} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.85}$$

where R_0 is a constant of mass dimension 1. Given the non-vanishing Christoffel symbols $\Gamma^{\mathcal{X}}_{\mathcal{Y}\mathcal{X}} = \Gamma^{\mathcal{X}}_{\mathcal{X}\mathcal{Y}} = \frac{1}{R_0}$ and $\Gamma^{\mathcal{Y}}_{\mathcal{X}\mathcal{X}} = -\frac{e^{2\mathcal{Y}/R_0}}{R_0}$, it is straightforward to check that the field space Ricci scalar $\mathbb{R}$ is then given by

$$\mathbb{R} = -\frac{2}{R_0^2}, \tag{5.86}$$

so the model indeed represents a *negative* curvature field space.[19] The equations of motion as given by (5.46) and (5.47) read

$$3M_{\rm Pl}^2 H^2 - \frac{1}{2} e^{2\mathcal{Y}/R_0} \dot{\mathcal{X}}^2 - \frac{1}{2}\dot{\mathcal{Y}}^2 - V = 0, \tag{5.87}$$

$$\ddot{\mathcal{X}} + 3H\dot{\mathcal{X}} + \frac{2}{R_0}\dot{\mathcal{Y}}\dot{\mathcal{X}} + e^{-2\mathcal{Y}/R_0} V_{\mathcal{X}} = 0, \tag{5.88}$$

$$\ddot{\mathcal{Y}} + 3H\dot{\mathcal{Y}} - \frac{1}{R_0}\, e^{2\mathcal{Y}/R_0}\dot{\mathcal{X}}^2 + V_{\mathcal{Y}} = 0. \tag{5.89}$$

It is clear that $\dot{\mathcal{Y}} = 0$ is allowed by the equations of motion as long as the potential is suitably chosen. However, notice that our present approach does not care about the precise form of the potential. Instead, we just need to make sure that the geometry allows for a trajectory with a constant turning rate.

In the present case, we see that (5.66) takes the form

$$\rho_{\rm NG} = R_0, \tag{5.90}$$

and so we conclude that trajectories with nearly constant rates are indeed possible.

Next, using the general first-order form of the EOM given in (5.64), we get that

$$\dot{\mathcal{X}} = R_0\, e^{-\mathcal{Y}/R_0}\,\Omega \Rightarrow \mathcal{X}(t) = R_0\, e^{-\mathcal{Y}/R_0}\,\Omega\, t + C_1, \tag{5.91}$$

$$\dot{\mathcal{Y}} = 0 \Rightarrow \mathcal{Y}(t) = C_2, \tag{5.92}$$

where the $\{C_i\}$ are integration constants. Moreover, using (5.48), (5.91), and (5.92) we find, for later use, that the first slow-roll parameter becomes

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{R_0^2\,\Omega^2}{2M_{\rm Pl}^2 H^2}. \tag{5.93}$$

[19]See Appendix C for a discussion of other well-known coordinatizations of two-dimensional hyperbolic field space.

Let us now calculate the non-geodesic field distance defined through

$$[\Delta\phi]_{\rm NG} \equiv \int_{\mathcal{C}_1} \sqrt{\gamma_{ab}\,\dot\phi_0^a\dot\phi_0^b}\,dt, \tag{5.94}$$

where $\mathcal{C}_1$ denotes the specific non-geodesic path characterized for the condition $\mathcal{Y} = \mathcal{Y}_0$. The integration constants C_1 and C_2 are easily solved by imposing the following initial ($t = 0$) and final ($t = T$) conditions

$$\mathcal{Y}(0) = \mathcal{Y}(T) = \mathcal{Y}_0, \quad \mathcal{X}(0) = \mathcal{X}_0, \quad \text{and} \quad \mathcal{X}(T) = \mathcal{X}_0 + \Delta\mathcal{X}. \tag{5.95}$$

One then finds that

$$C_1 = \mathcal{X}_0, \quad C_2 = \mathcal{Y}_0, \quad \text{and} \quad \Omega = \frac{\Delta\mathcal{X}}{R_0\,T}\,e^{\mathcal{Y}_0/R_0}. \tag{5.96}$$

Finally, using (5.65) we arrive at

$$[\Delta\phi]_{\rm NG} = e^{\mathcal{Y}_0/R_0}\,|\Delta\mathcal{X}|\,. \tag{5.97}$$

Another useful expression for the above quantity is given by

$$[\Delta\phi]_{\rm NG} = R_0\,\Delta N\,\frac{|\Omega|}{H}, \tag{5.98}$$

where use has been made of (5.96), and the defining equation for e-folds $dN \equiv H dt$, which assuming $\dot H = 0$, implies $\Delta N = H\,T$ upon integration. Last but not least, given that Ω determines the λ coupling via (5.58), we may rewrite (5.98) as

$$[\Delta\phi]_{\rm NG} = \frac{1}{2}\,R_0\,\Delta N\,|\lambda|\,. \tag{5.99}$$

We will come back to this result in Section 5.4.

5.3 Geodesic Distances in Two-Field Models

We now move on to consider the computation of geodesic field distances in situations where the inflationary trajectory is non-geodesic. We will keep the field coordinate system employed in Section 5.2.2, whereby one of the fields, say $\mathcal{Y}$, is kept constant. To obtain the geodesic field distance separating any two points in field space, we may adopt any parametrization of the fields ϕ^a along the path. In particular, if we take time t as the parameter, the field distance functional along a path $\mathcal{C}$ takes the form

$$\Delta\phi \equiv \int_{\mathcal{C}} \sqrt{\gamma_{ab}(\phi)\dot\phi_0^a\dot\phi_0^b}\,dt, \tag{5.100}$$

which under extremization with respect to ϕ_0^a yields the geodesic equations

$$D_t\dot\phi_0^a \equiv \partial_t\dot\phi_0^a + \Gamma^a_{bc}\,\dot\phi_0^b\,\dot\phi_0^c = 0. \tag{5.101}$$

This is a coupled system of second order differential equations. Solving it, a task that may be quite non-trivial, will yield solutions of the form $\mathcal{X} = \mathcal{X}(t)$ and $\mathcal{Y} = \mathcal{Y}(t)$, which depend on four integration constants. Given that the non-geodesic motion is characterized for the condition $\mathcal{Y} = \mathcal{Y}_0$, then the geodesic path must be such that the initial and final values of $\mathcal{Y}$ coincides with $\mathcal{Y}_0$. This is simply achieved by imposing the following initial ($t = 0$) and final ($t = T$) conditions

$$\mathcal{Y}(0) = \mathcal{Y}(T) = \mathcal{Y}_0, \quad \mathcal{X}(0) = \mathcal{X}_0, \quad \text{and} \quad \mathcal{X}(T) = \mathcal{X}_0 + \Delta\mathcal{X}, \tag{5.102}$$

for the geodesic path. These conditions will then allow us to find a non-trivial relation between $[\Delta\phi]_{\rm G}$ and $[\Delta\phi]_{\rm NG}$ by using the crucial general result of equation (5.65). The general form of this relation will necessarily be of the form

$$\frac{[\Delta\phi]_{\rm G}}{\Lambda_g} = f\left(\frac{[\Delta\phi]_{\rm NG}}{\Lambda_g}\right), \tag{5.103}$$

where f is a function satisfying $f(x) \leq x$ and Λ_g is a mass scale determined by the specifics of the system under consideration. The condition that $f(x) \leq x$ (or $[\Delta\phi]_{\rm G} \leq [\Delta\phi]_{\rm NG}$) simply reminds us that a geodesic, by definition, is the shortest path between any two points in a given geometry. We now specialize to 2d hyperbolic geometry, simply because it is a minimal setup which enjoys all the desirable features we are looking for. For completeness, the other two maximally symmetric 2d spaces are discussed in Appendix D.

5.3.1 Example: Inflation in Hyperbolic Spaces

Let us again consider the example of inflation taking place in a field space with a hyperbolic geometry. The geodesic motion is determined by the equations (5.101), which in this case read

$$\ddot{\mathcal{X}} + \frac{2}{R_0}\dot{\mathcal{X}}\dot{\mathcal{Y}} = 0, \tag{5.104}$$

$$\ddot{\mathcal{Y}} - \frac{1}{R_0}\, e^{2\mathcal{Y}/R_0}\dot{\mathcal{X}}^2 = 0. \tag{5.105}$$

The solutions to the set of differential equations (5.104) and (5.105) are given by

$$\mathcal{X}(t) = c_1 + c_2 \tanh\left(c_3\left(t + c_4\right)\right), \tag{5.106}$$

$$\mathcal{Y}(t) = R_0 \ln\left(\frac{R_0}{c_2}\cosh\left(c_3\left(t + c_4\right)\right)\right), \tag{5.107}$$

where the $\{c_i\}$ are integration constants. We may now calculate the geodesic distance

$$[\Delta\phi]_{\rm G} \equiv \int_{\mathcal{C}_2} \sqrt{\gamma_{ab}(\phi)\dot{\phi}_0^a\dot{\phi}_0^b}\, dt, \tag{5.108}$$

where $\mathcal{C}_2$ is the specific geodesic path depicted in Figure 5.2 and the $\dot{\phi}_0^a$'s are derived using (5.106) and (5.107). It is then straightforward to show that under these circumstances

$$[\Delta\phi]_{\rm G} = c_3 R_0 T, \tag{5.109}$$

where $T \equiv \int_{\mathcal{C}_2} dt$. Imposing the boundary conditions (5.102), one finds

$$c_1 = \mathcal{X}_0 + \frac{\Delta\mathcal{X}}{2}, \quad c_2 = \frac{1}{2}\sqrt{(\Delta\mathcal{X})^2 + 4R_0^2\, e^{-2\mathcal{Y}_0/R_0}},$$
$$c_3 = \frac{2}{T}\,\mathrm{arcsinh}\left(e^{\mathcal{Y}_0/R_0}\frac{\Delta\mathcal{X}}{2R_0}\right), \quad c_4 = -\frac{T}{2}. \tag{5.110}$$

This finally leads, using (5.109), to the following geodesic field distance

$$[\Delta\phi]_{\mathrm{G}} = 2R_0\,\mathrm{arcsinh}\left(e^{\mathcal{Y}_0/R_0}\,\frac{\Delta\mathcal{X}}{2R_0}\right). \tag{5.111}$$

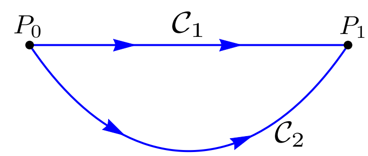

Figure 5.2: Sketch of the trajectories in hyperbolic field space. The curve $\mathcal{C}_1$ corresponds to a non-geodesic path (satisfying the EOM), while the curve $\mathcal{C}_2$ corresponds to a geodesic path. The boundary conditions were chosen in such a way that these trajectories share their initial (P_0) and final (P_1) points in field space.

5.3.2 Mixing Geodesic and Non-Geodesic Field Distances

We are now in a position to find a non-trivial relation between $[\Delta\phi]_{\mathrm{G}}$ and $[\Delta\phi]_{\mathrm{NG}}$. Using (5.111) and (5.97), we finally find that

$$[\Delta\phi]_{\mathrm{G}} = 2\sqrt{\frac{2}{|\mathbb{R}|}}\,\mathrm{arcsinh}\left(\frac{1}{2}\sqrt{\frac{|\mathbb{R}|}{2}}\,[\Delta\phi]_{\mathrm{NG}}\right). \tag{5.112}$$

Equation (5.112) is exactly the map between the geodesic and non-geodesic field distances we were looking for; it is 1-to-1 and only depends on the geometrical invariant of the field space. Indeed, this relation may allow us to simultaneously satisfy both the swampland criterion for $[\Delta\phi]_{\mathrm{G}}$ and the Lyth bound for $[\Delta\phi]_{\mathrm{NG}}$. To achieve this, it is crucial that the argument in the inverse hyperbolic function is bigger than 1; otherwise $[\Delta\phi]_{\mathrm{G}} \approx [\Delta\phi]_{\mathrm{NG}}$. Happily, this is exactly what occurs. Recalling the EFT arguments exposed in section 5.1.2, we will demand the sub-Planckian condition on the field geometry

$$R_0 < M_{\mathrm{Pl}}. \tag{5.113}$$

Moreover, it is a numerical (and theoretically appealing) result that a "subluminality" condition

$$\beta\,(\lambda, \tilde{\mu}) \leq 1 \quad \text{holds} \quad \forall\ \{\lambda, \tilde{\mu}\}\,. \tag{5.114}$$

Then using the Lyth bound in (5.84) we find that

$$\frac{[\Delta\phi]_{\rm NG}}{2R_0} = \frac{M_{\rm Pl}\,\Delta N}{2R_0}\sqrt{\frac{r}{8\,\beta\left(\lambda,\tilde{\mu}\right)}}. \tag{5.115}$$

It is easy to check that the above ratio is generically bigger than 1, so that it is indeed possible via (5.112), to achieve the hierarchy[20]

$$[\Delta\phi]_{\rm G} < M_{\rm Pl} < [\Delta\phi]_{\rm NG}, \tag{5.116}$$

which, as previously argued, is necessary in order to produce measurable primordial gravitational waves without the need for a geodesic super-Planckian field displacement. It is now important to understand what are the non-trivial consequences on the naive EFT we started with when all moving parts conspire to reach (5.116). This is what we do in the next section.

5.4 SDC, The Lyth Bound, and Non-Geodesic Motion

Equation (5.112) neatly shows how geodesic and non-geodesic field distances relate in two-field inflation with constant turns within a hyperbolic field space. In this section we will study some of the consequences emerging from having such a relation. To start with, let us consider the result of imposing the SDC, given by Eq. (5.29), over $[\Delta\phi]_{\rm G}$ in the left-hand side of (5.112), while the right-hand side is written in terms of the Lyth bound, Eq. (5.84). Doing so leads to the following inequality

$$\frac{1}{2R_0}\frac{\Delta N}{\sqrt{\beta\left(\lambda,\tilde{\mu}\right)}}\sqrt{\frac{r}{8}} < \sinh\left(\frac{\vartheta}{2R_0}\right). \tag{5.117}$$

A more enlightening expression may be reached by replacing the EFT parameter R_0 by

$$R_0 = \frac{1}{\lambda}\sqrt{\frac{r}{2\,\beta}}, \tag{5.118}$$

where use has been made of (5.93), (5.58), and (5.83). Writing (5.117) in terms of (5.118) we get

$$|\lambda| < \frac{4}{\Delta N}\sinh\left(\frac{\vartheta}{4}\sqrt{\frac{8\,\beta\left(\lambda,\tilde{\mu}\right)}{r}}\,|\lambda|\right). \tag{5.119}$$

The above inequality may be inverted to get a theoretical bound of the form

$$r < \frac{\vartheta^2\lambda^2}{2}\,\mathrm{arcsinh}^{-2}\left(\frac{\Delta N|\lambda|}{4}\right)\beta\left(\lambda,\tilde{\mu}\right). \tag{5.120}$$

This bound implies a non-trivial constraint on the parameter space $\{\lambda,\tilde{\mu}\}$; *in short, for a given value of r only certain values of such parameters are allowed in order to simultaneously satisfy both the SDC and the Lyth Bound.* We plot this in Figure 5.3, where we see how the allowed parameter space regions, for fixed values of $\{\vartheta,\Delta N\}$ and different values of r,

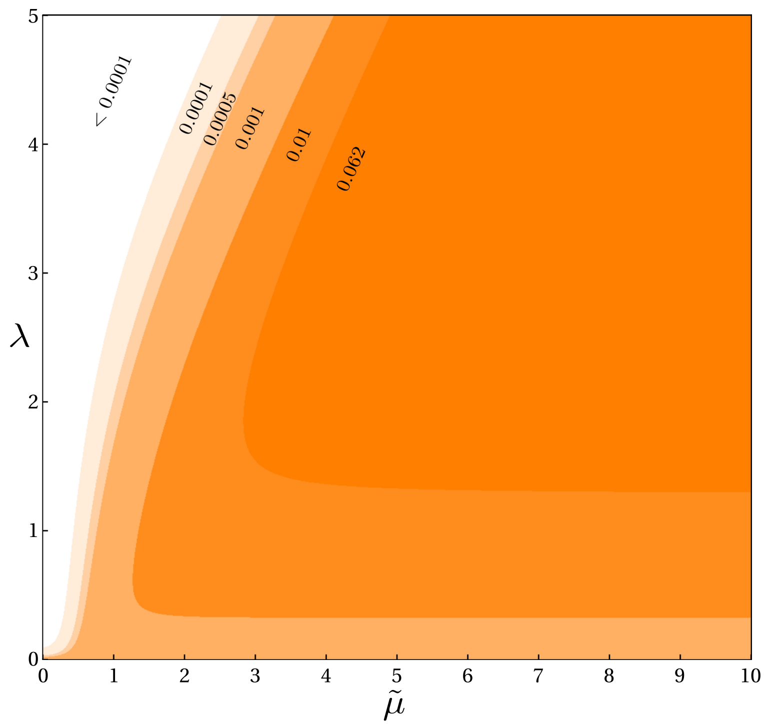

Figure 5.3: Leftover parameter space $\{\lambda, \tilde{\mu}\}$ when constrained by (5.120), for different values of r, while fixing $\vartheta = 1$ and $\Delta N = 60$. Here we can appreciate, in different shades of orange, the portions of parameter space that are still compatible, under these conditions, with the demands of the SDC and the Lyth bound.

are constrained by this requirement. As expected, larger values of r imply more restrictions for the possible combinations of λ and $\tilde{\mu}$. In particular, we see that a value of $r = 0.01$ implies a lower bound on $\tilde{\mu}$ of about ~ 1.3. This result is particularly interesting for the case of multi-field models within the framework of the Hamilton-Jacobi formalism (of which holographic inflation is an example). It has been recently shown that two-field models of inflation satisfying Hamilton-Jacobi equations must satisfy $\tilde{\mu} \leq 1.5$ [190]. Thus, a future measurement of r together with the swampland distance conjecture would severely constrain models based on the Hamilton-Jacobi formalism.

This, however, is not the end of the story. Besides applying the constraint (5.120) on $\beta(\lambda, \tilde{\mu})$, there are some other considerations to be taken into account. Using (5.118) and the sub-Planckian condition (5.113), we may express β as

$$\beta(\lambda, \tilde{\mu}) = \frac{r}{2\,\lambda^2\,R_0^2} \gg \frac{r}{2\,\lambda^2}, \tag{5.121}$$

where the strong inequality follows from the fact that $R_0 < 1$. With this at hand, the subluminality condition in (5.114) translates into a lower bound on R_0,

$$R_0 \geq \frac{1}{\lambda}\sqrt{\frac{r}{2}}, \tag{5.122}$$

[20]Multi-field models enjoying this feature do exist in the literature. See for instance [187–189].

while the SDC bound (5.120) translates into an upper one,

$$R_0 < \frac{\vartheta}{2\,\mathrm{arcsinh}\left(\frac{\Delta N|\lambda|}{4}\right)}. \tag{5.123}$$

Finally, it is crucial that our solution actually inflates. Using $\epsilon \ll 1$ we find a further constraint on β and R_0, namely

$$\left\{\epsilon = \frac{r}{16\,\beta\left(\lambda,\tilde{\mu}\right)} = \frac{\lambda^2 R_0^2}{8}\right\} \ll 1 \iff \left\{\beta\left(\lambda,\tilde{\mu}\right) \gg \frac{r}{16},\ R_0 < \frac{2\sqrt{2}}{\lambda}\right\}. \tag{5.124}$$

To sum up, we actually have to consider not one but three bounds over $\beta\left(\lambda,\tilde{\mu}\right)$; in addition to the swampland condition in (5.120), we have to impose both the sub-Planckian condition in (5.121) and the inflating-solution condition in (5.124). In Figure 5.4 we plot the portions of parameter space which are still allowed when considering all such bounds, while taking the benchmark point[21]

$$\Delta N = 60, \quad \vartheta = 1, \quad \text{and} \quad r = 0.01, \tag{5.125}$$

in order to assess which is the most constraining one. For this particular benchmark point we observe that the sub-Planckian bound (in blue) is subdominant with respect to the swampland bound (in orange), though increasing the $\mathcal{O}(1)$ constant ϑ eventually inverts this hierarchy, as suggested by the dashed orange line labeled by $\vartheta = 2$. Moreover, we appreciate that demanding inflationary solutions does enforce further restrictions on the allowed parameter space, depending on how small we expect the slow-roll parameter ϵ to be, as illustrated by the green lines labeled by $\epsilon = \{10^{-2}, 4\times10^{-3}, 10^{-3}\}$[22], respectively.

On the other hand, we have found that R_0 is bounded so that it is effectively allowed to lie only in the range

$$\frac{1}{\lambda}\sqrt{\frac{r}{2}} \leq R_0 < \min\left\{1, \frac{2\sqrt{2}}{\lambda}, \frac{\vartheta}{2\,\mathrm{arcsinh}\left(\frac{\Delta N|\lambda|}{4}\right)}\right\}. \tag{5.126}$$

This is, to our eyes, a very interesting result. Indeed, we see that by imposing very sensible conditions, one can heavily constrain the field geometry parameter R_0 of the naive two-field EFT. In Figure 5.5, we plot the allowed values of R_0, compatible with the bounds in (5.126), for the benchmark point in (5.125). We observe a non-trivial stripe bounded from below by the subluminality condition (5.114), and bounded from above by the swampland bound (5.123) or the slow-roll bound (5.124), whichever gives the lesser value. As bigger values of ϑ and ϵ and smaller of ΔN and r, are considered, the area of the stripe increases allowing more values of R_0. Incidentally, for perturbative (< 1) values of λ, the relevant bound, in order to get consistent inflation, is the one stemming from the swampland criterion, while for non-perturbative ($\gtrsim 2$) values of λ, satisfying the swampland bound is not enough to ensure a successful inflationary period. Moreover, the only possible way to obtain allowed values of R_0 in the perturbative regime, is decreasing the value of r.

[21]The value $r = 0.01$ is *not* a fanciful one, but actually the *smallest* value of r which will be experimentally accessible for next generation CMB surveys [191–194]. Nevertheless, values of order $r \sim \mathcal{O}\left(10^{-4}\right)$ may be achieved by futuristic observations [195].

[22]Here we consider ϵ values which are compatible with the latest Planck Collaboration release [35].

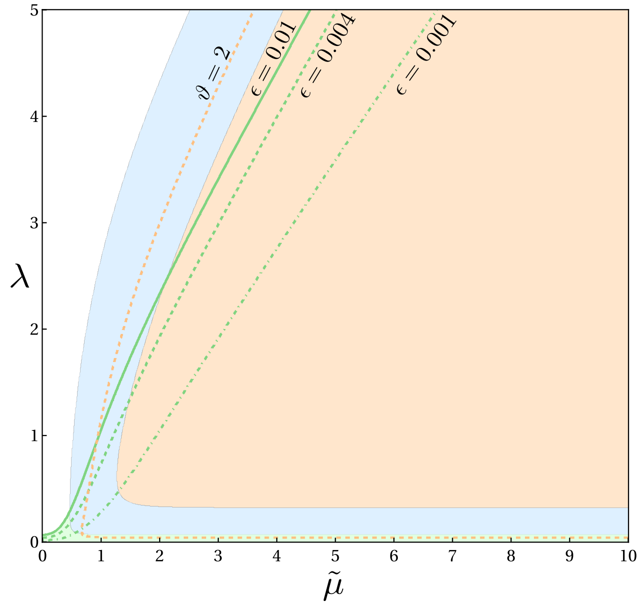

Figure 5.4: Green: Slow-roll bound $r \ll 16\,\beta(\lambda,\tilde{\mu})$, Blue: Sub-Planckian bound $r \ll 2\lambda^2\beta(\lambda,\tilde{\mu})$, and Orange: Swampland bound $r < \frac{\vartheta^2\lambda^2}{2}\left[\text{arcsinh}\left(\frac{\Delta N|\lambda|}{4}\right)\right]^{-2}\beta\left(\lambda,\tilde{\mu}\right)$, when using the benchmark point in (5.125). For this particular benchmark point, we observe that the swampland bound is more confining than the sub-Planckian bound. One can check that for $\vartheta \approx 2$, the sub-Planckian bound starts to compete with the swampland bound. When $\vartheta > 3$ the swampland bound becomes sub-dominant with respect to the sub-Planckian bound. On the other hand, the constraining power of the slow-roll bound also depends on how small ϵ is taken to be; for a standard value $\epsilon = 10^{-2}$ it is almost fully compatible with the swampland bound, while decreasing its value towards $\epsilon = 10^{-3}$ does invalidate non-neglilible portions of the otherwise swampland-safe parameter space.

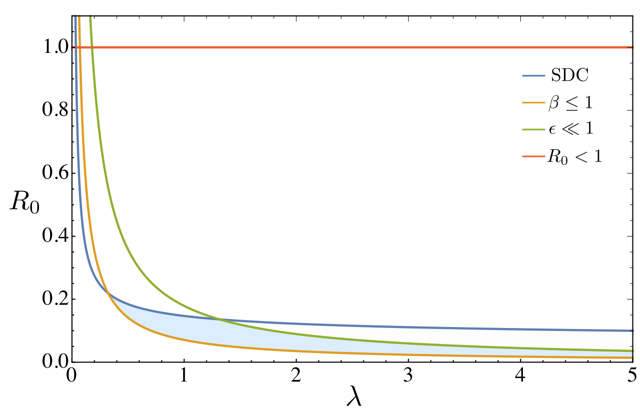

Figure 5.5: The different bounds for R_0 are plotted as a function of λ. The shaded region corresponds to the allowed values of R_0, when taking the benchmark point (5.125) and $\epsilon = 4 \times 10^{-3}$. The area of the stripe depends on the choice of the parameters, see text for further details.

5.5 Non-Gaussianity

Non-geodesic trajectories in multi-field spaces also induce the transfer of non-Gaussianity from the isocurvature field to the curvature perturbation, at a rate that depends on the values of $\tilde{\mu}$ and λ [165,168,196–198]. This means that non-Gaussianity observations would allow us to place additional constraints on the parameter space studied in the previous section. For instance, it is well understood that a non-unit speed of sound generated by a non-vanishing turning rate (recall Eq. (5.81)) will generate a non-negligible amount of equilateral and folded non-Gaussianity [95,159], which future surveys will be able to constrain.

To get an idea about how future observations may contribute to further constrain the parameter space $\{\lambda, \tilde{\mu}\}$, let us consider the regime in which the isocurvature field can be integrated out [95,159]. Necessarily, there will exist a region within the parameter space for which the two-field system is accurately described by a single-field EFT with a sound speed c_s given by (5.81). This region is characterized by $\beta \simeq c_s$. Figure 5.6 gives an account of this region by plotting the difference $|\beta - c_s|$.

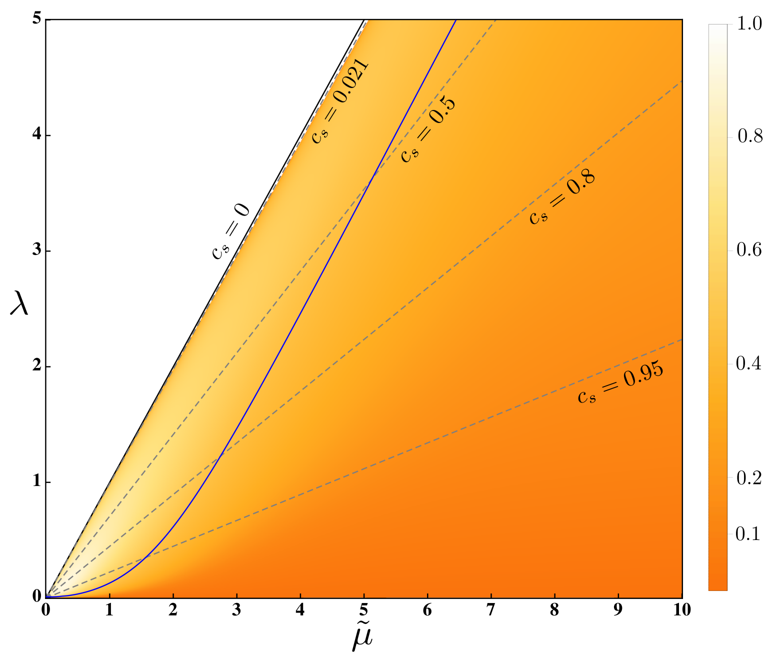

Figure 5.6: The color scale indicates the different values of $|\beta - c_s|$. The white area corresponds to imaginary values of c_s. The dashed lines denote different values of the sound speed. The value $c_s = 0.021$ corresponds to the current lower bound coming from non-Gaussianity constraints [50]. The solid blue line denotes points where $|\beta - c_s|/\beta = 0.1$.

The dashed lines show different fixed values of the sound speed, whereas the solid blue line shows the boundary beyond which $|\beta - c_s|/\beta$ becomes larger than 0.1. In other words, points to the right-hand side of the solid blue line correspond to values of $\tilde{\mu}$ and λ for which the power spectrum of the EFT is at least 10% off from the full two-field prediction. The line $c_s = 0$ and points to its left correspond to cases in which the EFT miserably fails.

Now, in order to constrain the parameter space $\{\lambda, \tilde{\mu}\}$ we can only trust bounds on c_s as long as they affect the region to the right of the solid blue line (assuming that we want an accuracy of 10%). For instance, current constraints on primordial non-Gaussianity imply $c_s \geq 0.021$ (95%CL) [50], but this is outside the region of validity of the EFT, and so we cannot use such a bound to infer constraints on both $\tilde{\mu}$ and λ. However, the plot shows how future observations may contribute to constrain $\tilde{\mu}$ and λ, and only very restrictive bounds on c_s could restrict the parameter space using the EFT. Comparing Figure 5.6 with the plots of the previous region, we see that a detection of primordial non-Gaussianity compatible with a single-field EFT would indeed dramatically restrict the parameter space in addition to the bounds required by the distance conjecture.

5.6 Discussion

It has been recently claimed [151] that the inflationary paradigm, at least in its single-field incarnation, is doomed as a consistent EFT when considering "UV-lessons" stemming from quantum gravity in light of eventual measurable primordial tensor perturbations. Reference [151] takes into account a couple of swampland conjectures to draw its conclusions. In this work we have only considered the one that, to our eyes, has much stronger theoretical support; the swampland distance conjecture. It is important to emphasize though, that the SDC still puts significant pressure on models of inflation once we also take into account the far-reaching observation made by Lyth [47] more than twenty years ago.

As already emphasized elsewhere [39, 134, 199], the conclusions on inflation derived from the SDC change dramatically if one considers multi-field inflation instead of single-field inflation. As we have shown, the SDC implies strong constraints on parameters related to the dynamics of perturbations. To understand how this happens, we have contemplated with some attention the particular case of multi-field inflation in a hyperbolic field space, which is a well-motivated background model that allows for simple analytical expressions. In particular, we found that for inflationary trajectories with constant turning rates in hyperbolic field spaces, the geodesic and non-geodesic distances $[\Delta\phi]_{\mathrm{G}}$ and $[\Delta\phi]_{\mathrm{NG}}$ are related through

$$[\Delta\phi]_{\mathrm{G}} = 2R_0 \,\mathrm{arcsinh}\left(\frac{[\Delta\phi]_{\mathrm{NG}}}{2R_0}\right), \tag{5.127}$$

where R_0 is the radius of curvature of the field space geometry. We found that this relation, together with the SDC and the Lyth bound, leads to powerful constraints on the entropy mass μ and turning rate λ parameters characterizing the dynamics of perturbations. Our main results are summarized in Figure 5.3, where we have plotted the allowed contour regions on the $\{\lambda, \tilde{\mu}\}$ space for different values of the tensor-to-scalar ratio. Our work provides an example where UV-physics constrains the possible values of the naively free parameters of the EFT describing the low-energy theory.

Clearly, the results of this work can be extended to any desired multi-field model. Multi-field models of inflation will necessarily lead to relations analogous to (5.127) tying together $[\Delta\phi]_{\mathrm{G}}$ and $[\Delta\phi]_{\mathrm{NG}}$. Then, by means of the SDC and the Lyth bound, it will always be possible to derive a bound on quantities parametrizing the dynamics of perturbations. Our results show, once more, the importance of the tensor-to-scalar ratio to characterize the early universe. An observation of the tensor-to-scalar ratio within the range targeted by future observatories ($r \sim 0.01$) will severely restrict the building of inflationary models. To say the least, as long as the SDC is taken seriously, it would provide a strong argument in favour of multi-field models of inflation.

Last but not least, non-geodesic trajectories in multi-field spaces also induce the transfer of non-Gaussianity from the isocurvature field to the curvature perturbation, at a rate that depends on the values of $\tilde{\mu}$ and λ [165, 168, 196–198]. This means that non-Gaussianity observations would allow us to place additional constraints on the parameter space studied in this work. For instance, it is well understood that a non-unit speed of sound generated by a non-vanishing turning rate (recall Eq. (5.81)) will generate a non-negligible amount of equilateral and folded non-Gaussianity [95,159], which future surveys will be able to constrain.

Conclusions

Throughout this work, we have addressed problems related to the UV sensitivity of inflation. We derived a novel soft-theorem for inflation that generalizes the well known Maldacena's consistency relation, which allowed us to claim that the observable local non-Gaussianity in canonical single-field inflation should vanish. Also, we evade an obstruction from low energy implications of quantum gravity on super-Planckian displacements in inflation, which allowed us to built a mechanism to generate sizable primordial gravitational waves while maintaining the inflaton's geodesic field-displacement sub-Planckian.

The derivation of the soft-theorem is based on a new class of symmetries of a perturbed FLRW spacetime. Unlike what is found in literature, where soft-theorems for inflation are constructed by invoking additional symmetries; either de Sitter isometries or shift-symmetries, in our case, we only required a FLRW spacetime. The symmetries that we found, are constituted by spacetime diffeomorphisms as well as field redefinitions that together are part of what is known as residual diffeomorphisms. While searching for these symmetries, we were able to evade a well-known obstruction regarding temporal diffeomorphisms; it was thought that a temporal diffeomorphism destroys the comoving gauge, and therefore they are prohibited if one desires to keep the gauge intact. Nevertheless, as shown in Chapter 4, a non-trivial modification of the scale factor allows the existence of a temporal diffeomorphism which keeps the gauge-choice intact. Another important ingredient in our construction, was the fact that no matter the inflationary background is attractor or not, a long-wavelength mode ζ_L is always adiabatic, and consequently can be absorbed in a redefinition of coordinates. This long-mode act as a background for short-modes, and therefore we computed the two-point function of the short-modes in this new background. Since the two-point function of this short-modes contains information about the background long-mode, we then correlated this result with a long-mode, a procedure that is equivalent to compute the squeezed limit of the three-point function. The result that we obtained, was a consistency relation valid for canonical single-field models of inflation whether the background is attractor or non-attractor, at all order in slow-roll parameters. As a consistency check, our consistency relation was able to recover the bispectrum for the well known cases for slow-roll and ultra-slow-roll inflation.

The development of the consistency relation through a transformation of coordinates, as we did in Chapters 2 and 4, suggested us the presence of a gauge-artifact in the computation. It is known that, at the moment of connecting theoretical predictions with cosmological observables, it is necessary to use a set of coordinates free of any redundancies. That is what we did in Chapter 3, were we demonstrated with the use of Conformal Fermi Coordinates, that it was possible to eliminate the presence of spurious background long-modes at the moment of computing correlation functions. There, we showed that the observable bispectrum during either, slow-roll or ultra-slow-roll inflation vanishes. At the moment of developing those results, an open question was if the computation holds after an ultra-slow roll phase. With the further understanding of the symmetries of FLRW and the development of the new soft-theorem in Chapter 4, we concluded that the vanishing of the local non-Gaussianity holds after an USR phase. This claim relies in the fact that the symmetries that we found are always present, no matter the phase of inflation, therefore the long-mode it will *always* be able to be removed.

Apart from above, we were able to evade the obstruction of super-Planckian displacements in inflation, by using the fact that those restrictions apply to *geodesic* distances. As shown in Chapter 5, we exploited the fact that the inflationary trajectory in multi-field inflation, follows a *non-geodesic* trajectory in field-space, a concept that is completely absent when considering single-field inflation. Since the upper bounds on the field displacement applies exclusively to geodesic distances, we established a geometrical relation between geodesic and non-geodesic distances in a two-field hyperbolic model of multi-field inflation. There, we applied the upper bound on the correspondent geodesic distance, and with the help of a modified Lyth bound, we showed that it is possible to have super-Planckian non-geodesic distances capable of generating a tensor-to-scalar ratio r as large as we want, while the corresponding geodesic distance remains as small as we want. In order to achieve this, the curvature of the field space played an important role. Moreover, we were able to apply these restrictions, which in principle are at the background level, to the perturbation theory of multi-field inflation. We found novel bounds on the parameter space of multi-field inflation, namely the mass of the entropy isocurvature fluctuation μ, as well as the rate of turn Ω of the inflationary trajectory. Therefore, if future observations confirm the existence of primordial gravitational waves, they would help us to constrain the parameter space of multi-field inflation.

Our studies and results allows us to put in test the current notions of inflationary theories. On the one hand, if a non-negligible amount of primordial local non-Gaussianity is detected in the future, now we are sure that it does not come from single-field inflation, leaving room to consider new mechanisms capable of producing such amount of local non-Gaussianity. If, on the contrary, a vanishing of the local non-Gaussianity is measured, this would also not confirm that single-field inflation is the underlying mechanism of such measurement since it is possible to engineer models (for example, in multi-field or non-canonical theories) capable of producing the same output. On the other hand, now we have the certainty that a measurement of primordial gravitational waves with a significant value of r does not necessarily imply super-Planckian excursions of the inflaton field. The above was one of the concerns that arose when BICEP announced the detection of B-modes back in 2014 [200]. Although these ended up being dust, they gave rise to the discussion about super-Planckian displacements. To sum up, detections of $f_{\rm NL}^{\rm loc}$ and r in the future are going to surprise us with information that goes beyond the canonical models of inflation, probably giving us access to UV physics.

Let us finish by mentioning that our work leaves room for new research directions that remains to be studied. For instance, a symmetry-based treatment of the non-Gaussianities in multi-field inflation needs to be done. We suspect that the arguments we find in developing consistency relations may be directly extended to multi-field setups.

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Appendices

Appendix A

Details of computations in CFC

A.1 Map coefficients

Here we show how to compute the map coefficients appearing in Eqs. (3.38)-(3.43). To start with, we notice that the map of Eq. (3.36) implies that along the geodesic (where $\Delta x^{\bar{\imath}} = 0$) the following relations must be satisfied:

$$\frac{\partial \tau}{\partial x^{\bar{\imath}}} = A^0_{\bar{\imath}}, \tag{A.1}$$

$$\frac{\partial x^i}{\partial x^{\bar{\imath}}} = A^i_{\bar{\imath}}, \tag{A.2}$$

$$\frac{\partial x^i}{\partial \bar{\tau}} = \frac{\partial \rho^i}{\partial \bar{\tau}}. \tag{A.3}$$

In addition, it is useful to recall the transformation rule of Eq. (3.19) connecting the metric in comoving coordinates and conformal Fermi coordinates:

$$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} \frac{\partial x^\nu}{\partial x^{\bar{\beta}}} g_{\mu\nu}. \tag{A.4}$$

At any point on the geodesic we have that $g_{\bar{\alpha}\bar{\beta}} = a_F^2(\bar{\tau})\eta_{\bar{\alpha}\bar{\beta}}$. Then, given that the tetrads satisfy $g_{\mu\nu} e^\mu_{\bar{\alpha}} e^\nu_{\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}}$, we see that Eqs. (A.1) and (A.2) imply

$$A^0_{\bar{\imath}} = a_F(\bar{\tau}) e^0_{\bar{\imath}}, \tag{A.5}$$

$$A^i_{\bar{\imath}} = a_F(\bar{\tau}) e^i_{\bar{\imath}}. \tag{A.6}$$

Then, using the expression of Eq. (3.11) for the tetrad $e^\mu_{\bar{\imath}}$, and keeping the leading contributions in terms of the perturbations, we find

$$A^0_{\bar{\imath}} = \delta^i_{\bar{\imath}} \partial_i \mathcal{F}, \tag{A.7}$$

$$A^i_{\bar{\imath}} = \left[\frac{a_F(\bar{\tau})}{a(\tau)} - 1 - \zeta \right] \delta^i_{\bar{\imath}}, \tag{A.8}$$

where we used the fact that $a_F(\bar{\tau})/a(\tau) - 1$ is of linear order. These correspond to the desired expressions for the map coefficients (3.40) and (3.41).

Next, let us consider Eq. (A.4) with the choice $(\bar{\alpha},\bar{\beta})=(\bar{0},\bar{0})$. Because along the geodesic one has $g_{\bar{0}\bar{0}}=-a_F^2(\bar{\tau})$, using the form of the metric $g_{\mu\nu}$ introduced in Eq. (3.6) one arrives at

$$\frac{a_F^2(\bar{\tau})}{a^2(\tau)}=\left(\frac{\partial\tau}{\partial\bar{\tau}}\right)^2[1-h_{00}]-2\frac{\partial\tau}{\partial\bar{\tau}}\frac{\partial x^i}{\partial\bar{\tau}}h_{0i}-\frac{\partial x^i}{\partial\bar{\tau}}\frac{\partial x^j}{\partial\bar{\tau}}\delta_{ij}e^{2\zeta}. \tag{A.9}$$

Keeping the leading terms, the previous expression may be rewritten as:

$$\frac{\partial\tau}{\partial\bar{\tau}}=\frac{a_F(\bar{\tau})}{a(\tau)}+\frac{1}{2}h_{00}. \tag{A.10}$$

Then, integrating with respect to time we arrive to:

$$\tau-\tau_*=\int_{\tau_*}^{\tau}ds\left[\frac{a_F(\bar{\tau})}{a(\tau)}+\frac{1}{2}h_{00}\right]. \tag{A.11}$$

As a last step, note that $\rho^0=\tau-\bar{\tau}$ by definition. This implies that:

$$\rho^0=\int_{\tau_*}^{\tau}ds\left[\frac{a_F(\bar{\tau})}{a(\tau)}-1+\frac{1}{2}h_{00}\right]. \tag{A.12}$$

which is the desired expression giving us the map coefficient of Eq. (3.38). Notice that we have imposed the condition $\rho^0=0$ at $\tau=\tau_*$ to synchronize the map.

To conclude, let us consider Eq. (A.4) one more time for the case $(\bar{\alpha},\bar{\beta})=(\bar{0},\bar{\imath})$:

$$0=\frac{\partial\tau}{\partial\bar{\tau}}\frac{\partial\tau}{\partial x^{\bar{\imath}}}[-1+h_{00}]+\frac{\partial x^i}{\partial\bar{\tau}}\frac{\partial\tau}{\partial x^{\bar{\imath}}}h_{i0}+\frac{\partial\tau}{\partial\bar{\tau}}\frac{\partial x^j}{\partial x^{\bar{\imath}}}h_{0j}+\frac{\partial x^i}{\partial\bar{\tau}}\frac{\partial x^j}{\partial x^{\bar{\imath}}}\delta_{ij}e^{2\zeta}. \tag{A.13}$$

Keeping the leading terms in the perturbations, we obtain

$$\frac{\partial\rho^i}{\partial\bar{\tau}}=A^0_{\bar{\imath}}\delta^{i\bar{\imath}}-\delta^{ij}h_{0j}. \tag{A.14}$$

Then, inserting our previous result of Eq. (A.7) and integrating with respect to time, we finally arrive to:

$$\rho^i=\int_{\tau_*}^{\tau}ds\,V^i, \tag{A.15}$$

which corresponds to the map coefficient of Eq. (3.39).

Finally, the map coefficients of Eqs. (3.42) and (3.43) directly follow directly from Eq. (3.18) after keeping the leading therms in the perturbations.

A.2 The 2-point correlation function for short modes

Here we give details on how to derive Eqs. (3.77) and (3.80). Let us start with Eq. (3.77). First, notice that the second term at the right hand side of Eq. (3.75) may be rewritten as

$$\begin{aligned}\left[\xi^\mu_L(\tau,\bar{\mathbf{x}}_1)\partial^{(1)}_\mu+\xi^\mu_L(\tau,\bar{\mathbf{x}}_2)\partial^{(2)}_\mu\right]\langle\zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0&=\left[\frac{1}{2}(\xi^0_L(\tau,\bar{\mathbf{x}}_2)-\xi^0_L(\tau,\bar{\mathbf{x}}_1))(\partial^{(2)}_0-\partial^{(1)}_0)\right.\\&\left.+\frac{1}{2}\Big(\xi^0_L(\tau,\bar{\mathbf{x}}_1)+\xi^0_L(\tau,\bar{\mathbf{x}}_2)\Big)\partial_0+\xi^i_L(\tau,\bar{\mathbf{x}}_1)\partial^{(1)}_i+\xi^i_L(\tau,\bar{\mathbf{x}}_2)\partial^{(2)}_i\right]\langle\zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0.\end{aligned} \tag{A.16}$$

Let us focus for a moment on the first contribution of the right hand side, which is given by

$$\frac{1}{2}A^0_{\bar{i}}(x^{\bar{i}}_2 - x^{\bar{i}}_1)\,\langle(\zeta_S(\bar{x}_1)\zeta'_S(\bar{x}_2) - \zeta'_S(\bar{x}_1)\zeta_S(\bar{x}_2))\rangle_0\,, \tag{A.17}$$

where we have used $\xi^0_L(\tau,\bar{\mathbf{x}}_2)-\xi^0_L(\tau,\bar{\mathbf{x}}_1) = A^0_{\bar{i}}(x^{\bar{i}}_2-x^{\bar{i}}_1)$. Now recall that the label 0 reminds us that we are dealing with a comoving curvature perturbation in the absence of non-linearities. We may therefore expand it as

$$\zeta_S(\bar{x}) = \frac{1}{(2\pi)^3}\int dk e^{-i\mathbf{k}\cdot\mathbf{x}}\zeta(\tau,\mathbf{k}), \qquad \zeta(\tau,\mathbf{k}) = \zeta_k(\tau)a_{\mathbf{k}} + \zeta^*_k(\tau)a^\dagger_{-\mathbf{k}}, \tag{A.18}$$

where $\zeta_k(\tau)$ are is the amplitude of the mode, and $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ are creation and annihilation operators. I is then easy to show that

$$\left\langle\zeta(\tau,\mathbf{k}_1)\zeta^{*\prime}(\tau,\mathbf{k}_2) - \zeta'(\tau,\mathbf{k}_1)\zeta^*(\tau,\mathbf{k}_2)\right\rangle_0 = (2\pi)^3\delta^{(3)}(\mathbf{k}_2-\mathbf{k}_1)[\zeta'_{k_1}(\tau)\zeta^*_{k_1}(\tau) - \zeta_{k_1}(\tau)\zeta^*_{k_1}{}'(\tau)]. \tag{A.19}$$

Now, $[\zeta'_{k_1}(\tau)\zeta^*_{k_1}(\tau) - \zeta_{k_1}(\tau)\zeta^*_{k_1}{}'(\tau)]$ must be such that the canonical commutation condition for ζ is satisfied. This implies that (A.17) is given by

$$\frac{1}{2}A^0_{\bar{i}}(x^{\bar{i}}_2 - x^{\bar{i}}_1)\,\langle(\zeta_S(\bar{x}_1)\zeta'_S(\bar{x}_2) - \zeta'_S(\bar{x}_1)\zeta_S(\bar{x}_2))\rangle_0 = \frac{1}{4\epsilon a^2}A^0_{\bar{i}}(x^{\bar{i}}_2 - x^{\bar{i}}_1)\delta^{(3)}(\mathbf{x_2}-\mathbf{x_1}), \tag{A.20}$$

which gives vanishing contribution to (A.16). Next, using (3.76) and replacing in ξ^μ_L the map coefficients of Eqs. (3.38)-(3.41), we find that the remaining terms in Eq. (A.16) give:

$$\begin{aligned}&\left[\xi^\mu_L(\tau,\bar{\mathbf{x}}_1)\partial^{(1)}_\mu + \xi^\mu_L(\tau,\bar{\mathbf{x}}_2)\partial^{(2)}_\mu\right]\langle\zeta_S(\bar{x}_1)\zeta_S(\bar{x}_2)\rangle_0\\ &= \left[\frac{1}{\tau}\left(\rho^0 + \frac{1}{2}(x^i_1+x^i_2)A^0_i\right)\frac{\partial}{\partial\ln\tau} + A^i_{\bar{i}}(x^{\bar{i}}_2 - x^{\bar{i}}_1)(x^i_2-x^i_1)\frac{1}{r^2}\frac{\partial}{\partial\ln r}\right]\langle\zeta_S\zeta_S\rangle(\tau,r)\end{aligned} \tag{A.21}$$

This result then leads directly to Eq. (3.77).

Next, we wish to derive Eq. (3.80) out from Eq. (3.79). The first step is to simply multiply Eq. (3.79) by $e^{-i\mathbf{k}_1\cdot\bar{\mathbf{x}}_1-i\mathbf{k}_2\cdot\bar{\mathbf{x}}_2}$ and integrate the two spatial coordinates $\bar{x}_1$ and $\bar{x}_2$. This directly gives

$$\begin{aligned}\langle\bar{\zeta}_S\bar{\zeta}_S\rangle(\mathbf{k}_1,\mathbf{k}_2) &= \langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2) + \int d^3\bar{x}_1d^3\bar{x}_2e^{-i\mathbf{k}_1\cdot\bar{\mathbf{x}}_1-i\mathbf{k}_2\cdot\bar{\mathbf{x}}_2}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}_L}\\ &\quad\times\left([\zeta(\mathbf{k})-\zeta_*(\mathbf{k})+\zeta(\mathbf{k})i\mathbf{k}\cdot\mathbf{x}_L]\frac{\partial}{\partial\ln\tau} - \zeta_*(\mathbf{k})\frac{\partial}{\partial\ln r}\right)\langle\zeta_S\zeta_S\rangle(\tau,r).\end{aligned} \tag{A.22}$$

We may re-express the integral in terms of $\mathbf{r}=\mathbf{x_1}-\mathbf{x_2}$ and $\mathbf{x}_L = (\mathbf{x_1}+\mathbf{x_2})/\mathbf{2}$ instead of $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$. One has that $d^3\bar{x}_1d^3\bar{x}_2 = d^3rd^3x_L$, and so one finds

$$\begin{aligned}\langle\bar{\zeta}_S\bar{\zeta}_S\rangle(\mathbf{k}_1,\mathbf{k}_2) &= \langle\zeta_S\zeta_S\rangle(\mathbf{k}_1,\mathbf{k}_2) + \int d^3rd^3x_Le^{-i\mathbf{k}_S\cdot\mathbf{r}/2}\int\frac{d^3k}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{k}_L)\cdot\mathbf{x}_L}\\ &\quad\times\left([\zeta(\mathbf{k})-\zeta_*(\mathbf{k})+\zeta(\mathbf{k})i\mathbf{k}\cdot\mathbf{x}_L]\frac{\partial}{\partial\ln\tau} - \zeta_*(\mathbf{k})\frac{\partial}{\partial\ln r}\right)\langle\zeta_S\zeta_S\rangle(\tau,r),\end{aligned} \tag{A.23}$$

where we have defined $\mathbf{k}_L = \mathbf{k}_1+\mathbf{k}_2$ and $\mathbf{k}_S = (\mathbf{k}_1-\mathbf{k}_2)/2$. The final step consists of explicitly integrating the coordinates $\mathbf{x}_L$ and $\mathbf{r}$. The r-integral gives us the power spectrum $P_\zeta(\tau,k_S)$. On the other hand, the x_L-integral gives us a $\delta^{(3)}(\mathbf{k}-\mathbf{k}_L)$ that can be used to get rid of the k-integral (and produces the appearance of $\nabla_{\mathbf{k}_L}$). All of this together gives us the final result expressed in Eq. (3.80), after we throw away the term suppressed by $\mathbf{k}_L$.

Appendix B

Temporal Diffeomorphism Generator

Here we show the details to arrive at equation (4.32). First, we note that the differential equation that determines F

$$H\dot{F} - (3+\eta)FH^2 = 1, \tag{B.1}$$

can be rewritten as

$$a^3\epsilon \frac{d}{dt}\left(\frac{F}{a^3\epsilon}\right) = \frac{1}{H}. \tag{B.2}$$

Then

$$F = a^3\epsilon C_2 + 2A(t), \tag{B.3}$$

where $A(t)$ is given by

$$A(t) = \frac{a^3\epsilon}{2}\int^t \frac{dt'}{a^3\epsilon H}. \tag{B.4}$$

To solve the previous integral, it is convenient to use the number of e-folds, N, as variable. We have $a(t) = e^N$, $\frac{dN}{dt} = H$ and $\epsilon = -\frac{1}{H}\frac{dH}{dN} = -\frac{H'}{H}$, then

$$\begin{aligned}
2A(t) &= -e^{3N}\epsilon \int^N \frac{e^{-3N}dN}{HH'} && \text{(B.5)}\\
&= -e^{3N}\epsilon \int^H \frac{e^{-3N}}{H^2}\left(\frac{dN}{dH}\right)^2 HdH && \text{(B.6)}\\
&= -\frac{8}{9}e^{3N}\epsilon \int^{H^2}\left(\frac{de^{-3N/2}}{dH^2}\right)^2 dH^2. && \text{(B.7)}
\end{aligned}$$

Now if we do the change of variables $y = e^{-3N/2}$ and $x = H^2$, the integral acquires the form

$$I = \int^x \left(\frac{dy}{dx}\right)^2 dx. \tag{B.8}$$

Which can be solved in the following way. First note that

$$I = \int^x \left(\frac{dy}{dx}\right)^2 dx = \int^x \frac{(dy)^2}{dx} = \int^y y'dy = y'y - \int^y yd(y') \tag{B.9}$$

The previous step can be repeated infinitely many times as

$$I = y'y - \int^{y} y\frac{dy'}{dy}dy = y'y - \frac{1}{2}\int^{y^2}\frac{dy'}{dy}dy^2 = y'y - \frac{1}{2}\left(\frac{dy'}{dy}y^2 - \int y^2 d\left(\frac{dy'}{dy}\right)\right) \tag{B.10}$$

$$= y'y - \frac{1}{2}\left(\frac{dy'}{dy}y^2 - \frac{1}{3}\int\frac{dy'}{dy^2}dy^3\right) = \dots \tag{B.11}$$

$$= y'y - \frac{1}{2!}y^2\frac{dy'}{dy} + \frac{1}{3!}y^3\frac{d^2y'}{dy^2} - \frac{1}{4!}y^4\frac{d^3y'}{dy^3} + \dots \tag{B.12}$$

$$= y\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}y^n\frac{d^n}{y^n}y' = y\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}y^n\frac{d^n}{y^n}\left(\frac{dy}{dx}\right). \tag{B.13}$$

Returning back to the original variables, we end up with

$$I = \frac{3}{4}e^{-3N/2}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left[\frac{2}{3}\frac{d}{dN}\right]^n\left(\frac{e^{-3N/2}}{H^2\epsilon}\right), \tag{B.14}$$

thus

$$A(t) = -\frac{\epsilon}{3}\sum_{n=0}^{\infty}\frac{e^{3N/2}}{(n+1)!}\left[\frac{2}{3}\frac{d}{dN}\right]^n\left(\frac{e^{-3N/2}}{H^2\epsilon}\right). \tag{B.15}$$

Appendix C

Hyperbolic Coordinate Systems

C.1 Upper Half-Plane

We start considering the line element associated with the metric (5.85), that is to say

$$ds^2 = e^{2\mathcal{Y}/R_0} d\mathcal{X}^2 + d\mathcal{Y}^2. \tag{C.1}$$

Subsequently, we perform the change of coordinates defined by $d\mathcal{Y} = \mathfrak{a}\left(\mathscr{Y}\right) d\mathscr{Y}$, which upon integration when using $\mathfrak{a}\left(\mathcal{Y}\right) = e^{\mathcal{Y}/R_0}$, implies that $\mathfrak{a}\left(\mathscr{Y}\right) = -\frac{1}{R_0 \mathscr{Y}}$, so the line element becomes

$$ds^2 = R_0^2 \frac{d\mathcal{X}^2 + d\mathscr{Y}^2}{\mathscr{Y}^2}. \tag{C.2}$$

Equation (C.2) defines the so-called "upper half-plane" coordinate system for the hyperbolic geometry. It is straightforward to find that $\Gamma^{\mathcal{X}}_{\mathscr{Y}\mathcal{X}} = -\frac{1}{\mathscr{Y}}$, $\Gamma^{\mathscr{Y}}_{\mathcal{X}\mathcal{X}} = \frac{1}{\mathscr{Y}}$, and $\Gamma^{\mathscr{Y}}_{\mathscr{Y}\mathscr{Y}} = -\frac{1}{\mathscr{Y}}$, and to check that the Ricci scalar is (still) given by $\mathbb{R} = -\frac{2}{R_0^2}$.

Geodesic Motion

The geodesic equations read

$$\ddot{\mathcal{X}} - \frac{2\dot{\mathcal{X}}\dot{\mathscr{Y}}}{\mathscr{Y}} = 0, \tag{C.3}$$

$$\ddot{\mathscr{Y}} + \frac{\dot{\mathcal{X}}^2 - \dot{\mathscr{Y}}^2}{\mathscr{Y}} = 0, \tag{C.4}$$

whose solutions can be found to be

$$\mathscr{Y}\left(t\right) = \frac{\sqrt{\mathfrak{C}}}{\ell} \operatorname{sech}\left(\sqrt{\mathfrak{C}}\left(t - \mathfrak{D}\right)\right), \tag{C.5}$$

$$\mathcal{X}\left(t\right) = \frac{\sqrt{\mathfrak{C}}}{\ell} \tanh\left(\sqrt{\mathfrak{C}}\left(t - \mathfrak{D}\right)\right) + \mathfrak{E}, \tag{C.6}$$

where $\{\mathfrak{C}, \ell, \mathfrak{D}, \mathfrak{E}\}$ are integration constants. The geodesic field distance is given by

$$[\Delta\phi]_{\mathrm{G}} = \int \sqrt{\gamma_{ab}\dot{\phi}_0^a \dot{\phi}_0^b}\, dt = \sqrt{\mathfrak{C}}\, R_0\, T, \tag{C.7}$$

where $T \equiv \int dt$. Imposing the boundary conditions

$$\mathscr{Y}(0) = \mathscr{Y}(T) = \mathscr{Y}_0, \quad \mathcal{X}(0) = \mathcal{X}_0, \quad \text{and} \quad \mathcal{X}(T) = \mathcal{X}_0 + \Delta\mathcal{X}, \tag{C.8}$$

one finds that

$$\mathfrak{D} = \frac{T}{2}, \;\; \mathfrak{E} = \mathcal{X}_0 + \frac{\Delta\mathcal{X}}{2}, \;\; \mathfrak{C} = \frac{4}{T^2}\left(\operatorname{arcsinh}\left(\frac{\Delta\mathcal{X}}{2\mathscr{Y}_0}\right)\right)^2, \text{and } \ell = \frac{4\operatorname{arcsinh}\left(\frac{\Delta\mathcal{X}}{2\mathscr{Y}_0}\right)}{T\sqrt{4\mathscr{Y}_0^2 + (\Delta\mathcal{X})^2}}. \tag{C.9}$$

Moreover,

$$\Delta\mathcal{X} = \frac{2\sqrt{\mathfrak{C}}}{\ell}\tanh\left(\frac{\sqrt{\mathfrak{C}}\,T}{2}\right) = 2\,\mathscr{Y}_0 \sinh\left(\frac{\sqrt{\mathfrak{C}}\,T}{2}\right). \tag{C.10}$$

Non-Geodesic Motion

The non-geodesic motion we care about is determined by the first-order system defined by equation (5.64), which in this case becomes

$$\dot{\mathcal{X}} = -\mathscr{Y}\Omega, \quad \dot{\mathscr{Y}} = 0 \quad \Rightarrow \quad \mathscr{Y}(t) = \mathfrak{Y}, \quad \mathcal{X}(t) = -\mathfrak{Y}\,\Omega\, t + \mathfrak{X}, \tag{C.11}$$

with $\{\mathfrak{Y}, \mathfrak{X}\}$ integration constants. Using the same boundary conditions as in the geodesic case, equations (C.8), we find that

$$\Omega = -\frac{\Delta\mathcal{X}}{\mathscr{Y}_0\, T}. \tag{C.12}$$

Moreover, applying (5.65) the non-geodesic field distance is found to be given by

$$[\Delta\phi]_{\mathrm{NG}} = \frac{R_0\,\Delta\mathcal{X}}{\mathscr{Y}_0}. \tag{C.13}$$

Finally, using (C.7), (C.10), and (C.13), it is easy to show that

$$[\Delta\phi]_{\mathrm{G}} = 2R_0 \operatorname{arcsinh}\left(\frac{1}{2R_0}\,[\Delta\phi]_{\mathrm{NG}}\right) = 2\sqrt{\frac{2}{|\mathbb{R}|}}\operatorname{arcsinh}\left(\frac{1}{2}\sqrt{\frac{|\mathbb{R}|}{2}}\,[\Delta\phi]_{\mathrm{NG}}\right), \tag{C.14}$$

where we have used $R_0 = \sqrt{\frac{2}{|\mathbb{R}|}}$ in the second equality. This expression of course coincides with (5.112), as the half-plane is just another coordinatization of the hyperbolic geometry discussed in this paper.

C.2 Poincaré Disk

Consider the hyperbolic metric as written in the half-plane coordinate system

$$ds^2 = R_0^2 \, \frac{d\mathrm{u}^2 + d\mathrm{v}^2}{\mathrm{v}^2}. \tag{C.15}$$

Let us now perform a Möbius transformation defined through

$$z = \frac{i - w}{i + w}, \quad \text{where} \quad w \equiv \mathrm{u} + i\mathrm{v}. \tag{C.16}$$

It is easy to check that $\frac{dz}{dw} = -\frac{2\,i}{(w+i)^2}$ and $1 - |z|^2 = \frac{4\,\mathrm{v}}{|w+i|^2}$, which allows us to rewrite (C.15) as

$$ds^2 = 4R_0^2 \, \frac{|dz|^2}{\left(1 - |z|^2\right)^2}. \tag{C.17}$$

Equation (C.17) is known as the Poincaré Disk line element. Introducing polar coordinates $z = \frac{1}{\sqrt{3\alpha}}\, r e^{i\theta}$ where $\alpha \equiv \frac{R_0^2}{3}$, leads to

$$ds^2 = 4\, \frac{dr^2 + r^2 d\theta^2}{\left(1 - \frac{r^2}{3\alpha}\right)^2}. \tag{C.18}$$

The metric in (C.18) reduces to that of the so-called α-attractor models of inflation [201], whose characteristic kinetic term is of the form

$$\mathscr{L} \supset -\frac{1}{2} \frac{(\partial\phi)^2}{\left(1 - \frac{\phi^2}{6\alpha}\right)^2}, \tag{C.19}$$

which is achieved by taking into account a suitable normalization factor, defining $r \equiv \frac{1}{\sqrt{2}}\, \phi$, and fixing $\theta =$ constant. The Ricci curvature scalar stemming from (C.18) is (again) given by

$$\mathbb{R} = -\frac{2}{R_0^2} = -\frac{2}{3\alpha}, \tag{C.20}$$

since (again) this is just another coordinatization of the hyperbolic geometry. Moreover, applying identical reasoning as in Sections 5.3.1 and C.1, it is possible to show that the relation for the geodesic and non-geodesic trajectories is given by

$$[\Delta\phi]_{\mathrm{G}} = 2\sqrt{\frac{2}{|\mathbb{R}|}}\, \mathrm{arcsinh}\left(\frac{1}{2}\sqrt{\frac{|\mathbb{R}|}{2}}\, [\Delta\phi]_{\mathrm{NG}}\right), \tag{C.21}$$

where care must be taken when comparing “angular” vs. “radial” motion, because θ is *not* canonically normalized, as can be seen from the form of the metric in (C.18).

Appendix D

Maximally Symmetric Geometries

D.1 Planar Geometry

Geodesic Motion

Consider the system defined by taking $\phi^1 = r$ and $\phi^2 = \theta$, and the *planar* field metric

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{D.1}$$

with corresponding non-trivial Christoffel symbols $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{\theta r} = \frac{1}{r}$, and a trivial field space Riemann tensor $\mathbb{R}^a{}_{bcd} = 0$. The geodesic equations for this geometry are then

$$\ddot{r} - r^2\dot{\theta}^2 = 0, \tag{D.2}$$

$$\ddot{\theta} + \frac{2\,\dot{r}\,\dot{\theta}}{r} = 0. \tag{D.3}$$

Moreover, (D.3) may be casted as

$$r^2\dot{\theta} = L, \tag{D.4}$$

where L is an integration constant, a.k.a. nothing but good old angular momentum. The solutions to the system determined by (D.2) and (D.3) are given by

$$r(t) = \sqrt{\mathrm{c}_1\,(t + \mathrm{c}_2)^2 + \frac{L^2}{\mathrm{c}_1}}, \tag{D.5}$$

$$\theta(t) = \tan^{-1}\left(\frac{\mathrm{c}_1}{L}\,(t + \mathrm{c}_2)\right) + \mathrm{c}_3, \tag{D.6}$$

where the $\{\mathrm{c}_i\}$ are integration constants. The geodesic field distance is then given by

$$[\Delta\phi]_{\mathrm{G}} = \int \sqrt{\gamma_{ab}\dot{\phi}^a_0\dot{\phi}^b_0}\,dt = \sqrt{\mathrm{c}_1}\,T, \tag{D.7}$$

where $T \equiv \int dt$. Imposing the boundary conditions

$$r(0) = r(T) = r_0, \quad \theta(0) = \theta_0, \quad \text{and} \quad \theta(T) = \theta_0 + \Delta\theta, \tag{D.8}$$

one finds that

$$\mathrm{c}_2 = -\frac{T}{2}, \quad \mathrm{c}_3 = \theta_0 + \frac{\Delta\theta}{2}, \quad \mathrm{c}_1 = \frac{2\,r_0^2}{T^2}\,(1-\cos\Delta\theta)\,, \quad L = \frac{r_0^2\,\sin\Delta\theta}{T}, \tag{D.9}$$

so that

$$[\Delta\phi]_{\mathrm{G}} = 2\,r_0 \sin\left(\frac{\Delta\theta}{2}\right). \tag{D.10}$$

Non-Geodesic Motion

Using the first-order system of equations (5.64), we get

$$\dot\theta = \Omega \Rightarrow \theta(t) = \Omega\,t + \theta_c, \quad \text{and} \quad \dot r = 0 \Rightarrow r = r_c, \tag{D.11}$$

with $\{\theta_c, r_c\}$ integration constants. Using the same boundary conditions as in the geodesic case given in (D.8), one finds that

$$r_c = r_0, \quad \theta_c = \theta_0, \quad \text{and} \quad \Omega = \frac{\Delta\theta}{T}. \tag{D.12}$$

Moreover, the non-geodesic field distance is then given by

$$[\Delta\phi]_{\mathrm{NG}} = r_0\,|\Omega|\,T = r_0\,\Delta\theta. \tag{D.13}$$

Using (D.10) and (D.13) we finally find that

$$[\Delta\phi]_{\mathrm{G}} = 2\,r_0 \sin\left(\frac{1}{2\,r_0}\,[\Delta\phi]_{\mathrm{NG}}\right), \tag{D.14}$$

which may be casted as

$$\frac{[\Delta\phi]_{\mathrm{G}}}{\Lambda_g} = \mathcal{F}\left(\frac{[\Delta\phi]_{\mathrm{NG}}}{\Lambda_g}\right), \quad \text{where} \quad \mathcal{F}(x) = \sin x \quad \text{and} \quad \Lambda_g = 2\,r_0. \tag{D.15}$$

The previous relation depends explicitly on the initial condition r_0, which being dimensionful, is enforced to play the role of the scale Λ_g in this curvatureless space. Moreover, the periodicity of the sine function is clearly not useful for our purposes. That should suffice the discussion of the planar geometry.

D.2 Spherical Geometry

Geodesic Motion

One could try to do better than in the planar geometry case, and consider the system $\phi^1 = \theta$ and $\phi^2 = \varphi$ with a *spherical* field space metric given by

$$\gamma_{ab} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{D.16}$$

with corresponding non-trivial Christoffel symbols given by $\Gamma^{\theta}_{\varphi\varphi} = -\cos\theta\sin\theta$ and $\Gamma^{\varphi}_{\varphi\theta} = \cot\theta$. In this case, the Ricci scalar is given by $\mathbb{R} = +\frac{2}{R^2}$. The geodesic equations for this system then read

$$\ddot{\theta} - \cos\theta\sin\theta\,\dot{\varphi}^2 = 0, \tag{D.17}$$

$$\ddot{\varphi} + 2\cot\theta\,\dot{\theta}\,\dot{\varphi} = 0. \tag{D.18}$$

The general solutions to the system determined by (D.17) and (D.18) are found to be

$$\theta(t) = \cos^{-1}\left(\sqrt{\frac{\mathsf{c}_2 - \mathsf{c}_1^2}{\mathsf{c}_2}}\cos\left(\sqrt{\mathsf{c}_2}\,(t+\mathsf{c}_3)\right)\right), \tag{D.19}$$

$$\varphi(t) = \tan^{-1}\left(\frac{\sqrt{\mathsf{c}_2}}{\mathsf{c}_1}\tan\left(\sqrt{\mathsf{c}_2}\,(t+\mathsf{c}_3)\right)\right) + \mathsf{c}_4, \tag{D.20}$$

where the $\{\mathsf{c}_i\}$ are integration constants. The geodesic field distance is then given by

$$[\Delta\phi]_{\mathrm{G}} = \int\sqrt{\gamma_{ab}\dot{\phi}^a_0\dot{\phi}^b_0}\,dt = \sqrt{\mathsf{c}_2}R\,T, \tag{D.21}$$

where $T \equiv \int dt$. We now impose the following boundary conditions

$$\theta(0) = \theta(T) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \varphi(T) = \varphi_0 + \Delta\varphi. \tag{D.22}$$

It can be shown that this picking implies

$$\mathsf{c}_3 = -\frac{T}{2}, \quad \mathsf{c}_4 = \varphi_0 + \frac{\Delta\varphi}{2}, \quad \mathsf{c}_2 = \frac{4}{T^2}\left(\sin^{-1}\left(\sin\left(\frac{\Delta\varphi}{2}\right)\sin\theta_0\right)\right)^2, \tag{D.23}$$
$$\mathsf{c}_1 = \sqrt{\mathsf{c}_2}\,\frac{\tan\left(\frac{\sqrt{\mathsf{c}_2}\,T}{2}\right)}{\tan\left(\frac{\Delta\varphi}{2}\right)}.$$

Moreover, under these circumstances, the following somewhat non-trivial relation holds

$$\sin\left(\frac{\sqrt{\mathsf{c}_2}\,T}{2}\right) = \sin\left(\frac{\Delta\varphi}{2}\right)\sin\theta_0. \tag{D.24}$$

Non-Geodesic Motion

For the non-geodesic case we use the general result of (5.64) to get that

$$\dot{\varphi} = \sec\theta\,\Omega \Rightarrow \varphi\,(t) = \sec\theta\,\Omega\,t + \varphi_* \quad \text{and} \quad \dot{\theta} = 0 \Rightarrow \theta\,(t) = \theta_* \tag{D.25}$$

where $\{\varphi_*, \theta_*\}$ are integration constants. We now impose the following boundary conditions

$$\theta(0) = \theta(T) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \varphi(T) = \varphi_0 + \Delta\varphi, \tag{D.26}$$

which are the same as in the geodesic case. This picking then yields

$$\theta_* = \theta_0, \quad \varphi_* = \varphi_0, \quad \text{and} \quad \Omega = \frac{\cos\theta_0\,\Delta\varphi}{T}. \tag{D.27}$$

Furthermore, using (5.65) the non-geodesic field distance becomes

$$[\Delta\phi]_{\rm NG} = R \sin\theta_0\, \Delta\varphi. \tag{D.28}$$

Finally, using (D.21), (D.28), and the relation (D.24) we may finally state that

$$[\Delta\phi]_{\rm G} = 2\sqrt{\frac{2}{|\mathbb{R}|}} \sin^{-1}\left[\sin\left(\frac{1}{2\sin\theta_0}\sqrt{\frac{|\mathbb{R}|}{2}}\,[\Delta\phi]_{\rm NG}\right)\sin\theta_0\right], \tag{D.29}$$

where we have used that $R = \sqrt{\frac{2}{|\mathbb{R}|}}$. Equation (D.29) may be casted as

$$\frac{[\Delta\phi]_{\rm G}}{\Lambda_g} = \mathcal{F}\left(\frac{[\Delta\phi]_{\rm NG}}{\Lambda_g}, \theta_0\right) \quad \text{where} \quad \mathcal{F}\,(x, \theta_0) \equiv \sin^{-1}\left[\sin\left(\frac{x}{\sin\theta_0}\right)\sin\theta_0\right] \quad \text{and} \quad \Lambda_g = 2R. \tag{D.30}$$

We observe, for example, that when $\theta_0 = \frac{\pi}{2}$,

$$[\Delta\phi]_{\rm NG} = [\Delta\phi]_{\rm G} + 2\,\mathsf{n}\,\pi\,\Lambda_g \quad \text{where} \quad \mathsf{n} \in \mathbb{N}_0, \tag{D.31}$$

which indeed makes sense, since when confined to the *Equator* the two distances necessarily coincide, up to “windings”, which in the context of inflation, are unphysical.[1] Again, though we have found a relation between the geodesic and non-geodesic trajectories, it is *not* monotonically growing, 1-to-1, and independent of initial conditions, features only enjoyed by the hyperbolic geometry.

[1]After 60 e-folds of inflationary evolution it is highly unlikely that the background comes back to its initial value in field space.