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TOPOLOGICAL EQUIVALENCE OF NONAUTONOMOUS DIFFERENCE EQUATIONS WITH A FAMILY OF DICHOTOMIES ON THE HALF LINE

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Abstract. A linear system of difference equations and a nonlinear perturbation are considered, we obtain sufficient conditions to ensure the topological equivalence between them, namely, the linear part satisfies a property of dichotomy on the positive half–line while the nonlinearity has some boundedness and Lipschitz conditions. In addition, we provide new characterizations for the resulting homeomorphisms. When the linear system is asymptotically stable and the nonlinear system has a unique equilibrium, we deduce sharper results for the smoothness of the topological equivalence. Finally, we study the asymptotic stability and its preservation by topological equivalence.

1. Introduction.

1.1. Preliminaries.

The linearization of flows arising from autonomous ordinary differential equations and autonomous difference equations has a long history starting with the classical Hartman–Grobman Theorem [17, 16], which ensures the existence of a local homeomorphism between a nonlinear flow and its linearization around a fixed point, provided that a hyperbolicity condition on the corresponding linearized flow is verified. The reader is refered to [21, 29, 31] for an in depth look to the global case or an abstract setting.

The extension of the above results to the nonautonomous framework have dealed with the property of dichotomy [11, 12] which mimics some qualitative properties of the hyperbolicity condition, namely, the existence of stable and unstable directions of a linear system; this fact has been useful to develop some local [14, 20] and global linearization results.

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To the best of our knowledge, the global and nonautonomous linearization results started with the work of K.J. Palmer in [22], which considered two systems of ordinary differential equations: a linear one and a nonlinear perturbation. Under the assumption that the linear system satisfies a uniform exponential dichotomy property [12] and meanwhile the nonlinear perturbation verifies some Lipschitzness and boundedness assumptions, it is proven that both systems are topologically equivalent; property that will be explained in full later on in this paper.

In order to obtain a discrete version of the Palmer's result, let us consider the nonautonomous systems of difference equations

$$x_{k+1} = A(k)x_k, \qquad k \in \mathbb{Z}^+, \tag{1.1}$$

$$y_{k+1} = A(k)y_k + f(k, y_k), \quad k \in \mathbb{Z}^+, \tag{1.2}$$

where x_k and y_k are column vectors of $\mathbb{R}^d$ for any $k \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, the matrix function $k \mapsto A(k) \in M_d(\mathbb{R})$ is non singular and $f : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in $\mathbb{R}^d$.

The purpose of this article is to obtain a set of conditions ensuring that the above systems are topologically equivalent, this property was introduced in the continuous framework by K.J. Palmer in [22] and extended to the discrete case by several authors such as G. Papaschinopoulos and J. Schinas in [24, 34] who stated as follows:

Definition 1.1. Let $J \subseteq \mathbb{Z}$ be an integer interval, namely a set of consecutive integers. The systems (1.1) and (1.2) are J–topologically equivalent if there exists a function such as $H : J \times \mathbb{R}^d \to \mathbb{R}^d$ with the properties

(i) If $x(k)$ is a solution of (1.1), then $H[k, x(k)]$ is a solution of(1.2),

(ii) $H(k, u) - u$ is bounded in $J \times \mathbb{R}^d$,

(iii) For each fixed $k \in J$, $u \mapsto H(k, u)$ is a homeomorphism of $\mathbb{R}^d$.

In addition, the function $u \mapsto G(k, u) = H^{-1}(k, u)$ has properties (ii)–(iii) and maps solutions of (1.2) into solutions of (1.1).

The property of topological equivalence has several differences with the linearization arising from the classical Hartman–Grobman's theorem: i) it is inserted in a nonautonomous framework and there is not an univocal equivalent to the hyperbolicity condition, ii) it deals with a global linearization instead of a local one, iii) an explicit construction of the homeomorphisms is possible in some cases, iv) the smoothness properties are considerably less studied, v) a corresponding version of the resonance's condition is far from being completed.

The $\mathbb{Z}$–topological equivalence between (1.1) and (1.2) has been studied in several works inspired by Palmer's approach. First of all, A. Reinfelds in [32, 33] obtained a topological equivalence result by assuming that (1.1) has a dichotomy and by constructing two functions G and H based in the Green's function associated to the dichotomy combined with *ad hoc* conditions on the nonlinear part which are necessary to ensure that G is a bijection with H as its inverse. Secondly, we make the point of mentioning the work of G. Papaschinopoulos [26] which studied the topological equivalence in a continuous/discrete framework and the discrete case is studied as a technical step. We also mention the work [7], where the authors obtained a $\mathbb{Z}$–topological equivalence result by considering a generalized exponential dichotomy in the linear part combined with the Reinfelds's assumptions on the nonlinearities and the continuity of G and H is addressed in detail. We also highlight a related result from L. Barreira and C. Valls [3] which is not exactly a topological equivalence but considers a linear part with a nonuniform exponential dichotomy on $\mathbb{Z}$ and obtained properties of Hölder regularity on the corresponding homeomorphisms.

We point out that there exist other linearization results which follow ideas and methods different to the Palmer's construction. In particular, we highlight the approach based in the crossing times with the unit ball which has been employed with several variations in [5, 8, 19].

1.2. **Notation.** Throughout this paper, the symbols $|\cdot|$ and $||\cdot||$ will denote respectively a vector norm and its induced matrix norm. The Banach space of bounded sequences from $\mathbb{Z}^+$ to $\mathbb{R}^d$ will be denoted by $\ell^\infty(\mathbb{Z}^+,\mathbb{R}^d)$ with supremum norm $|\cdot|_\infty$.

Definition 1.2. A fundamental matrix of the system (1.1) is a matrix function $\Phi\colon \mathbb{Z}^+ \to M_d(\mathbb{R})$ such that its columns are a basis of solutions of (1.1) and satisfies the matrix difference equation

$$\Phi(n+1) = A(n)\Phi(n).$$

Definition 1.3. The transition matrix of (1.1) is defined by:

$$\Phi(k,n) = \begin{cases} A(k-1)A(k-2)\cdots A(n), & \text{if} \quad k > n, \\ I, & \text{if} \quad k = n, \\ A^{-1}(k)A^{-1}(k+1)\cdots A^{-1}(n-1), & \text{if} \quad k < n. \end{cases} \tag{1.3}$$

1.3. **Novelty of this work.** Our work is inscribed in the context of Palmer's approach considered previously in [7, 26, 32, 33] which construct the maps G and H by using the Green's function associated to the linear system (1.1) and two auxiliar difference systems. Nevertheless, this work has some differences that will be explained below.

First of all and contrarily to the previous references, we obtain a result of topological equivalence with $J = \mathbb{Z}^+$ instead of $J = \mathbb{Z}$. This fact induced technical differences and additional difficulties when constructing the maps G and H, mainly in the appropriate elaboration of the auxiliary systems above mentioned.

Secondly, we have obtained alternative characterizations of the maps G and H. In particular, we emphasize the remarkable simplicity for the map $u \mapsto G(k,u)$, this fact allow a simpler proof of its continuity and to deduce some nice and new identities for G in terms of fixed point properties.

Finally, when we restrict our attention to the case when the linear system (1.1) is asymptotically stable, we obtain sharper results of $\mathbb{Z}^+$–topological equivalence which allows a simpler and explicit study about the smoothness properties of G and H and to prove that the asymptotic stability is preserved by the equivalence when the nonlinear system has an equilibrium.

2. Main result.

2.1. **Statement.** In order to state the main result, we will assume that the linear system (1.1) satisfies the following properties:

(P1) The matrix function $k \mapsto A(k)$ is invertible and uniformly bounded, that is, there exists $M \geq 1$ such that

$$\max\left\{\sup_{k\in\mathbb{Z}^+} ||A(k)||, \sup_{k\in\mathbb{Z}^+} ||A^{-1}(k)||\right\} = M.$$

(P2) The linear system (1.1) has a nonuniform dichotomy. That is, there exists two invariant projectors $P(\cdot)$ and $Q(\cdot)$ such that $P(n) + Q(n) = I$ for any $n \in \mathbb{Z}^+$, a bounded sequence ρ and a nonincreasing sequence h convergent to zero with $h(0) = 1$ such that:

$$\begin{cases} ||\Phi(k,n)P(n)|| \leq \rho(n)\Big(\dfrac{h(k)}{h(n)}\Big), \forall k \geq n \geq 0 \\ ||\Phi(k,n)Q(n)|| \leq \rho(n)\Big(\dfrac{h(n)}{h(k)}\Big), \forall 0 \leq k \leq n. \end{cases} \tag{2.1}$$

Remark 2.1. By Lemma 2.2 from [1] it is known that the assumption (P1) is equivalent to the following properties:

a) The linear system (1.1) has bounded growth on $\mathbb{Z}^+$, that is

$$||\Phi(k,\ell)|| \leq M^{|k-\ell|} \quad \text{for any } k,\ell \in \mathbb{Z}^+,$$

b) For each $h \in \mathbb{N}$ and $\varepsilon > 0$ there is a corresponding $\delta : \delta(h,\varepsilon) > 0$ such that for $\xi,\eta \in \mathbb{R}^d$ with $|\xi - \eta| < \delta$, it follows that

$$|\Phi(k,\ell)(\xi - \eta)| < \varepsilon \quad \text{for all } k,\ell \in \mathbb{Z}^+ \text{ with } |k-\ell| < h,$$

and we refer the reader to [12] for details about the bounded growth property. Moreover, the invertibility of $A(n)$ is an essential property in the topological linearization literature. For example in [15, Example D.3] it is shown that the scalar equations

$$x(k+1) = a(k)x(k) \quad \text{and} \quad y(k+1) = a(k)y(k) + y(k)^2$$

cannot be conjugated when $a \equiv 0$. Indeed, otherwise the conjugacy H would satisfy $(H \circ a)(x) = H(x)^2$ for any x arbitrarily close to 0, this is $H(x)$ is constant near 0, contradicting H being one-to-one. Therefore, related results cannot be expected for noninvertible problems.

Remark 2.2. The assumption (P2) can be seen as a nonautonomous version of the hyperbolicity property of the autonomous case.

i) When $\rho(n) = K > 0$ and $h(n) = \theta^n$ with $\theta \in (0,1)$ for any $n \in \mathbb{Z}^+$, (P2) means that the system (1.1) has the property of uniform exponential dichotomy on $J = \mathbb{Z}^+$. This property and some of its consequences has been extensively studied in [19, 23, 24, 25].

ii) When $\rho(n) = K > 0$ and $h(n) = \exp(-\sum_{j=0}^{n} u_j)$, where the sequence u_j is positive and non summable, (P2) means that the system (1.1) has the property of generalized exponential dichotomy on $J = \mathbb{Z}^+$. This property and its applications in topological equivalence has been studied in [7] for the case $J = \mathbb{Z}$.

iii) When considering assumptions different and/or more general than those stated by (P2) we can obtain other dichotomies and we mention some as the (h,k)–dichotomies [13], the nonuniform exponential dichotomy [2, 5], the (μ,ν)– nonuniform dichotomies [6] and and the polynomial dichotomies [4].

Remark 2.3. The property (P2) restricted to the particular case $P(n) = I$ and $Q(n) = 0$ implies that the origin is an asymptotically stable equilibrium of (1.1) and a formal Definition will be stated in the Subsection 2.3. In addition, (P2) allows us to characterize several types of asymptotic stability. It is well known that the uniform asymptotic stability is verified if and only if (1.1) admits a uniform exponential dichotomy on $\mathbb{Z}^+$ with $P(n) = I$. The dichotomy property allows us to describe another type of asymptotic stability more general than the uniform one which is described in terms of $(\rho,h)-$contractions (see [10] for details).

Remark 2.4. The projectors P and Q are called invariant since they satisfy

$$P(k)\Phi(k,n)=\Phi(k,n)P(n) \quad \text{and} \quad Q(k)\Phi(k,n)=\Phi(k,n)Q(n) \quad \text{for any } k,n\in\mathbb{Z}^{+}$$

and finally, (P2) allows to define the Green's function $\mathcal{G}\colon \mathbb{Z}^{+}\times\mathbb{R}^{d}\to\mathbb{R}^{d}$ associated to the linear system (1.1) as follows

$$\mathcal{G}(k,n)=\begin{cases}\Phi(k,n)P(n) & \forall\, k\geq n\geq 0,\\ -\Phi(k,n)Q(n) & \forall\, 0\leq k<n.\end{cases} \tag{2.2}$$

It is easy to deduce that the above function verifies the following properties:

$$\mathcal{G}(k+1,n)=A(k)\mathcal{G}(k,n) \quad \text{and} \quad \mathcal{G}(n,n)=I+A(n-1)\mathcal{G}(n-1,n)$$

and it has been pointed out by Reinfelds in [32, p.10] that the property (P2) can be formulated in terms of any general Green's function satisfying the above properties, avoiding the use of projectors.

Moreover, we will assume that the nonlinear system (1.2) has a perturbation f that satisfies the following properties

(P3) For any $k\in\mathbb{Z}^{+}$ and any pair $(y,\tilde{y})\in\mathbb{R}^{d}\times\mathbb{R}^{d}$ it follows that:

$$|f(k,y)-f(k,\tilde{y})|\leq\gamma(k)|y-\tilde{y}| \quad \text{and} \quad |f(k,y)|\leq\mu(k).$$

(P4) The sequence μ defined above verifies

$$N(\ell,\mu)=\sum_{j=0}^{\infty}||\mathcal{G}(\ell,j+1)||\mu(j)=p<+\infty \quad \text{for any } \ell\in\mathbb{Z}^{+}.$$

(P5) The sequence γ defined above verifies

$$N(\ell,\gamma)=\sum_{j=0}^{\infty}||\mathcal{G}(\ell,j+1)||\gamma(j)=q<1 \quad \text{for any } \ell\in\mathbb{Z}^{+}.$$

(P6) The sequence $\gamma(\cdot)$ and $A(\cdot)$ are such that

$$||A^{-1}(\ell)\gamma(\ell)||<1 \quad \text{for any } \ell\in\mathbb{Z}^{+}.$$

Remark 2.5. The properties (P3)–(P5) have been used previously in the study of the topological equivalence problem in [33] and later on [7]. These properties allowed to generalize the construction of the homeomorphisms when the linear system (1.1) has dichotomies more general than the exponential one.

It is important to point out the existence of a trade off between the assumptions on the linear part and the nonlinear perturbation since milder properties on the dichotomies induce more restrictive assumptions on the sequences $\mu(\cdot)$ and $\gamma(\cdot)$. Indeed, if $j\mapsto||\mathcal{G}(\ell,j+1)||$ is summable for any $\ell\in\mathbb{Z}^{+}$ then we can choose constant sequences $\mu(\cdot)$ and $\gamma(\cdot)$ as in (i) of Remark 2.2. On the other hand, if $j\mapsto||\mathcal{G}(\ell,j+1)||$ is bounded but not summable, we have to assume that $\mu(\cdot)$ and $\gamma(\cdot)$ are summable sequences, and consequently convergent to zero. This last case could be observed in examples related to dichotomies described by (ii)–(iii) of Remark 2.2.

Remark 2.6. The property (P6) is a technical assumption and ensures that any solution $n\mapsto y(n,k,\eta)$ of the nonlinear system (1.2) passing through η at $n=k$ can be backward continued for any $n\in\{0,\ldots,k-1\}$. If fact, notice that $y(k-1,k,\eta)$ can be seen as the unique fixed point of the map $\Theta_{k-1}\colon\mathbb{R}^{d}\to\mathbb{R}^{d}$ defined by $\Theta_{k}(u)=A^{-1}(k-1)\eta-A^{-1}(k-1)f(k-1,u)$ and the terms $y(n,k,\eta)$ with $n\in$

$\{0,\ldots,k-2\}$ can be obtained in a similar way. We point out that this property has been considered previously in [33].

We must point out that (P5) implies that

$$||A^{-1}(\ell)Q(\ell)\gamma(\ell)|| < q - \sum_{j=0,j\neq\ell}^{\infty} ||\mathcal{G}(\ell,j+1)\gamma(j)|| < q < 1. \tag{2.3}$$

As we have set forth the premises now we are able to state our main result

Theorem 2.1. *If the assumptions (P1)-(P6) are satisfied then the systems* (1.1) *and* (1.2) *are* $\mathbb{Z}^+-$ *topologically equivalent.*

Proof. The proof of this result will be made in several steps.

Step 1: Preliminaries. Let $k\mapsto x(k,m,\xi)$ and $k\mapsto y(k,m,\eta)$ be the respective solutions of the systems (1.1) and (1.2) with initial conditions ξ and η at $k=m$.

Now, let us introduce the map:

$$\begin{aligned} w^*(k;(m,\eta)) &= -\sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,y(j,m,\eta)),\\ &= -\sum_{j=0}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,m,\eta))\\ &\quad +\sum_{j=k}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,m,\eta)), \end{aligned} \tag{2.4}$$

and for fixed $(m,\xi)\in\mathbb{Z}\times\mathbb{R}^d$ let us define the map $\Gamma\colon \ell^\infty(\mathbb{Z}^+,\mathbb{R}^d)\to\ell^\infty(\mathbb{Z}^+,\mathbb{R}^d)$ as follows

$$(\Gamma\phi)(k;(m,\xi)) = \sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,x(j,m,\xi)+\phi(j;(m,\xi))$$

Take notice that the backward continuation of the solutions of the nonlinear system (1.2) is necessary to ensure that the map $k\mapsto w^*(k;(m,\eta))$ is well defined; this is provided by (P6) and Remark 2.6.

Let $k\mapsto\phi(k;(m,\xi))$ and $k\mapsto\psi(k;(m,\xi))$ sequences in $\ell^\infty(\mathbb{Z}^+,\mathbb{R}^d)$. In addition, let us define

$$F(j,m,\xi) := f(j,x(j,m,\xi)+\phi(j;(m,\xi))) - f(j,x(j,m,\xi)+\psi(j;(m,\xi))),$$

thus by using (P2), (P3) and (P4) we can note that

$$\begin{aligned} &|(\Gamma\phi)(k;(m,\xi)) - (\Gamma\psi)(k;(m,\xi))| \\ \le& \sum_{j=0}^{+\infty}|\mathcal{G}(k,j+1)F(j,m,\xi)| \le \sum_{j=0}^{+\infty}\gamma(j)||\mathcal{G}(k,j+1)||\,|\phi-\psi|_\infty \le q\,|\phi-\psi|_\infty \end{aligned} \tag{2.5}$$

and by using the Banach contraction principle we have the existence of a unique fixed point

$$z^*(k;(m,\xi)) = \sum_{j=0}^{+\infty}\mathcal{G}(k,j+1)f(j,x(j,m,\xi)+z^*(j;(m,\xi))). \tag{2.6}$$

It is easy to verify that the maps $k \mapsto w^*(k;(m,\eta))$ and $k \mapsto z^*(k;(m,\xi))$ are solutions of the initial value problems:

$$\begin{cases} w_{k+1} = A(k)w_k - f(k, y(k,m,\eta)) \\ w_0 = -\displaystyle\sum_{j=0}^{\infty} \Phi(0,j+1)Q(j+1)f(j,y(j,m,\eta)). \end{cases}$$

and

$$\begin{cases} z_{k+1} = A(k)z_k + f(k, x(k,m,\xi) + z_k) \\ z_0 = \displaystyle\sum_{j=0}^{+\infty} \Phi(0,j+1)Q(j+1)f(j,x(j,m,\xi) + z^*(j;(m,\xi))) \end{cases} \tag{2.7}$$

Step 2: Constructing H and G. By the uniqueness of solutions we have that

$$x(k,m,\xi) = x(k,p,x(p,m,\xi)) \quad \text{for any } k,p,m \in \mathbb{Z}^+, \tag{2.8}$$

and the reader can verify that

$$z^*(k;(m,\xi)) = z^*(k;(p,x(p,m,\xi))) \quad \text{for any } k,p,m \in \mathbb{Z}^+. \tag{2.9}$$

For any fixed $k \in \mathbb{Z}^+$, let us construct the maps $H(k,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$ and $G(k,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$H(k,\xi) = \xi + \sum_{j=0}^{+\infty} \mathcal{G}(k,j+1)f(j,x(j,k,\xi) + z^*(j;(k,\xi))) = \xi + z^*(k;(k,\xi)), \tag{2.10}$$

and

$$G(k,\eta) = \eta - \sum_{j=0}^{+\infty} \mathcal{G}(k,j+1)f(j,y(j,k,\eta)) = \eta + w^*(k;(k,\eta)). \tag{2.11}$$

As $k \mapsto z^*(k;(k,\xi))$ and $k \mapsto w^*(k;(k,\eta))$ are uniformly bounded sequences, it follows that both H and G satisfy the statement (ii) from the Definition 1.1. Now, in order to study some additional properties of G, let us consider the initial value problem:

$$\begin{cases} y_{n+1} = A(n)y_n + f(n,y_n) \\ y_k = \eta. \end{cases} \tag{2.12}$$

If $n < k$ we have that:

$$y(n,k,\eta) = \Phi(n,k)\eta - \sum_{j=n}^{k-1} \Phi(n,j+1)f(j,y(j,k,\eta)), \tag{2.13}$$

which is equivalent to:

$$\begin{aligned} \Phi(k,n)y(n,k,\eta) =& \eta - \sum_{j=n}^{k-1} \Phi(k,j+1)f(j,y(j,k,\eta)), \\ =& \eta - \sum_{j=n}^{k-1} \Phi(k,j+1)\{P(j+1) + Q(j+1)\}f(j,y(j,k,\eta)), \\ =& \eta - \sum_{j=n}^{k-1} \Phi(k,j+1)P(j+1)f(j,y(j,k,\eta)) \end{aligned}$$

$$-\sum_{j=n}^{k-1}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta)).$$

In particular, if $n=0$ we have that

$$\begin{aligned}
&\Phi(k,0)y(0,k,\eta)\\
=&\eta-\sum_{j=0}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,k,\eta))-\sum_{j=0}^{k-1}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta)),\\
=&\eta-\sum_{j=0}^{k-1}\Phi(k,j+1)P(j+1)f(j,y(j,k,\eta))+\sum_{j=k}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta))\\
&-\sum_{j=0}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta)),\\
=&\eta-\sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,y(j,k,\eta))-\sum_{j=0}^{\infty}\Phi(k,j+1)Q(j+1)f(j,y(j,k,\eta)),\\
=&G(k,\eta)-\Phi(k,0)\sum_{j=0}^{\infty}\Phi(0,j+1)Q(j+1)f(j,y(j,k,\eta)).
\end{aligned}$$

Now, by using the definition of the map $n\mapsto w^*(0;(k,\eta))$ we can deduce that

$$\begin{aligned}
G(k,\eta)=&\Phi(k,0)\{y(0,k,\eta)+\sum_{j=0}^{\infty}\Phi(0,j+1)Q(j+1)f(j,y(j,k,\eta))\}\\
=&\Phi(k,0)\{y(0,k,\eta)+w^*(0;(k,\eta))\}
\end{aligned}\tag{2.14}$$

Step 3: H maps solutions of (1.1) into solutions of (1.2) and G maps solutions of (1.2) into solutions of (1.1). By using (2.8), (2.9) and (2.10) we can see that

$$\begin{aligned}
H[k,x(k,m,\xi)]=&x(k,m,\xi)+\sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,x(j,m,\xi)+z^*(j;(m,\xi))))\\
=&x(k,m,\xi)+z^*(k;(m,\xi)),
\end{aligned}$$

which has the alternative description of

$$H[k,x(k,m,\xi)]=x(k,m,\xi)+\sum_{j=0}^{\infty}\mathcal{G}(k,j+1)f(j,H[j,x(j,m,\xi)]).\tag{2.15}$$

The above identities combined with (1.1), (2.6) and $\mathcal{G}(k+1,j+1)=A(k)\mathcal{G}(k,j)$ allows us to prove that

$$\begin{aligned}
&H[k+1,x(k+1,m,\xi)]\\
=&x(k+1,m,\xi)+z^*(k+1;(m,\xi))\\
=&A(k)\left\{x(k,m,\xi)+z^*(k;(m,\xi)))\right\}+f(k,x(k,m,\xi)+z^*(k;(m,\xi)))\\
=&A(k)H[k,x(k,m,\xi)]+f(k,H[k,x(k,m,\xi)]),
\end{aligned}$$

at this point we conclude that $k\mapsto H[k,x(k,m,\xi)]$ is solution of (1.2) passing through $H(m,\xi)$ at $k=m$. In addition, as consequence of uniqueness of solution we obtain that

$$H[k,x(k,m,\xi)]=y(k,m,H(m,\xi)).$$

We can summarize several characterizations of $H[k, x(k, m, \eta)]$:

$$H[k,x(k,m,\xi)] = \begin{cases} x(k,m,\xi) + z^*(k;(m,\xi)) \\ x(k,m,\xi) + \sum\limits_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j, H[j,x(j,m,\xi)]) \\ y(k,m,H(m,\xi)) \end{cases} \tag{2.16}$$

Similarly, the uniqueness of solutions implies the identity

$$y(k,m,\eta) = y(k,p,y(p,m,\eta)) \quad \text{for any } k,p,m \in \mathbb{Z}^+, \tag{2.17}$$

which allows us to deduce that

$$w^*(k;(m,\eta)) = w^*(k;(p,y(p,m,\eta))) \quad \text{for any } k,p,m \in \mathbb{Z}^+. \tag{2.18}$$

Thus from the previous expression it follows that

$$G[k,y(k,m,\eta)] = y(k,m,\eta) - \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,y(j,m,\eta)) = y(k,m,\eta) + w^*(k;(m,\eta)),$$

now let us note that

$$\begin{aligned} &G[k+1,y(k+1,m,\eta)] \\ =&y(k+1,m,\eta) + w^*(k+1;(m,\eta)) \\ =&A(k)\{y(k,m,\eta) + w^*(k;(m,\eta))\} + f(k,y(k,m,\eta)) - f(k,y(k,m,\eta)) \\ =&A(k)G[k,y(k,m,\eta)] \end{aligned}$$

then $k \mapsto G[k, y(k, m, \eta)]$ is solution of (1.1) passing through $G(m, \eta)$ at $k = m$.

In addition, since $k \mapsto G[k, y(k, m, \eta)]$ is solution of (1.1) passing through $G(m, \eta)$ at $k = m$, then we have that

$$G[k,y(k,m,\eta)] = x(k,m,G(m,\eta)) = \Phi(k,m)G(m,\eta),$$

which also has an alternative formulation by using (2.14) and (2.18):

$$G[k,y(k,m,\eta)] = \Phi(k,0)\{y(0,m,\eta) + w^*(0;(m,\eta))\}.$$

We are also available to summarize several characterizations of $G[k, y(k, m, \eta)]$:

$$G[k,y(k,m,\eta)] = \begin{cases} y(k,m,\eta) + w^*(k;(m,\eta)) \\ x(k,m,G(m,\eta)) = \Phi(k,m)G(m,\eta) \\ \Phi(k,0)\{y(0,m,\eta) + w^*(0;(m,\eta))\}. \end{cases} \tag{2.19}$$

Step 4: $u \mapsto G(k, u)$ and $u \mapsto H(k, u)$ are bijective for any fixed $k \in \mathbb{Z}^+$.

By using the description of $H[k, x(k, m, \xi)]$ combined with identities (2.17) and (2.16) we can deduce that

$$\begin{aligned} &G[k,H[k,x(k,m,\xi)]] \\ =&H[k,x(k,m,\xi)] - \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,y(j,k,H[k,x(k,m,\xi)])) \\ =&x(k,m,\xi) + \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,H[j,x(j,m,\xi)]) \\ &- \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,y(j,k,H[k,x(k,m,\xi)])) = x(k,m,\xi). \end{aligned}$$

In order to study $H[k,G[k,y(k,m,\eta)]]$, the first identity of (2.16) allows us to verify that

$$\begin{aligned} H[j,x(j,k,G[k,y(k,m,\eta)])] =& x(j,k,G[k,y(k,m,\eta)]) + z^*(j;(k,G[k,y(k,m,\eta)])) \\ =:& L[j,y(k,m,\eta)]. \end{aligned}$$

On the other hand, by using (2.8), (2.9), (2.16) and (2.19) it can be proved that

$$\begin{aligned} H[j,x(j,k,G[k,y(k,m,\eta)])] =& x(j,m,G(m,\eta)) + z^*(j;(m,G(m,\eta))) \\ =& H[j,x(j,m,G(m,\eta))] = H[j,G[j,y(j,m,\eta)]]. \end{aligned} \quad (2.20)$$

At this juncture, we have that:

$$\begin{aligned} & H[k,G[k,y(k,m,\eta)]] \\ =& G[k,y(k,m,\eta)] + \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,L[j,u(k,m,\eta)]) \\ =& y(k,m,\eta) - \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,y(j,k,y(k,m,\eta))) + \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) f(j,L[j,y(k,m,\eta)]) \\ =& y(k,m,\eta) - \sum_{j=0}^{\infty} \mathcal{G}(k,j+1) \{ f(j,y(j,m,\eta)) - f(j,L[j,y(k,m,\eta)]) \}. \end{aligned}$$

Now, let us define

$$w(k) = \left| H[k,G[k,y(k,m,\eta)]] - y(k,m,\eta) \right|.$$

Then by using the above inequalities combined with (2.20) we have that

$$\begin{aligned} w(k) \leq& \sum_{j=0}^{\infty} ||\mathcal{G}(k,j+1)|| \, |f(j,x(j,L[j,y(k,m,\eta)]) - f(j,y(j,m,\eta))|, \\ \leq& \sum_{j=0}^{\infty} \gamma(j) ||\mathcal{G}(k,j+1)|| \, |H[j,G(j,y(j,m,\eta))] - y(j,m,\eta)|, \\ \leq& \sum_{j=0}^{\infty} ||\mathcal{G}(k,j+1)|| \, \gamma(j) w(j). \end{aligned}$$

By (P4) we know that $w \in \ell^\infty(\mathbb{Z}^+,\mathbb{R}^d)$ and by (P5) it follows that $|w|_\infty \leq q\,|w|_\infty$. Therefore if $w > 0$ then $1 \leq q$, obtaining a contradiction. Hence $w(k) = 0$ for any $k \in \mathbb{Z}^+$ and therefore we have

$$H[k,G[k,y(k,m,\eta)]] = y(k,m,\eta), \quad \forall k \in \mathbb{Z}^+.$$

In particular, if $k = m$ then

$$H(m,G(m,\eta)) = \eta,$$

and hence we conclude that $u \mapsto H(k,u)$ is a bijection for any $k \in \mathbb{Z}^+$ and $u \mapsto G(k,u)$ is its inverse.

Step 5: G is a continuous map. By (2.14), we have to prove that $\eta \mapsto y(0,k,\eta)$ and $\eta \mapsto w^*(0;(k,\eta))$ are continuous functions for any $k \in \mathbb{Z}^+$.

Firstly, we can see that if $n < k$ then

$$y(n,k,\eta) = \Phi(n,k)\eta - \sum_{j=n}^{k-1} \Phi(n,j+1) f(j,y(j,k,\eta)), \quad (2.21)$$

thus (see Remark 2.6 for details) we have that

$$y(k-1,k,\eta) = A^{-1}(k-1)\eta - A^{-1}(k-1)f(k-1,y(k-1,k,\eta)),$$

then

$$\begin{aligned}&|y(k-1,k,\eta)-y(k-1,k,\tilde{\eta})|\\ \leq&||A^{-1}(k-1)||\,|\eta-\tilde{\eta}| + ||A^{-1}(k-1)\gamma(k-1)||\,|y(k-1,k,\eta)-y(k-1,k,\tilde{\eta})|,\end{aligned}$$

which implies by (P6) that

$$|y(k-1,k,\eta)-y(k-1,k,\tilde{\eta})| \leq \frac{||A^{-1}(k-1)||}{(1-||A^{-1}(k-1)\gamma(k-1)||)}|\eta-\tilde{\eta}|. \tag{2.22}$$

Similarly, it follows that

$$|y(k-2,k,\eta)-y(k-2,k,\tilde{\eta})| \leq \frac{||A^{-1}(k-2)||}{(1-||A^{-1}(k-2)\gamma(k-2)||)}|y(k-1,k,\eta)-y(k-1,k,\tilde{\eta})|,$$

and from (2.22) we have that

$$|y(k-2,k,\eta)-y(k-2,k,\tilde{\eta})| \leq \prod_{p=k-2}^{k-1}\frac{||A^{-1}(p)||}{(1-||A^{-1}(p)\gamma(p)||)}|\eta-\tilde{\eta}|.$$

Hence, inductively we can deduce that

$$|y(k-j,k,\eta)-y(k-j,k,\tilde{\eta})| \leq \prod_{p=k-j}^{k-1}\frac{||A^{-1}(p)||}{(1-||A^{-1}(p)\gamma(p)||)}|\eta-\tilde{\eta}|.$$

Now, if $n<k$ then there exists $j\in\mathbb{Z}^+$ such that $n+j=k$. Thus

$$|y(n,k,\eta)-y(n,k,\tilde{\eta})| \leq \mathcal{C}_k(n)|\eta-\tilde{\eta}|, \quad \text{for all } n<k, \tag{2.23}$$

where

$$\mathcal{C}_k(n) = \prod_{j=n}^{k-1}\frac{||A^{-1}(j)||}{(1-||A^{-1}(j)\gamma(j)||)}.$$

In particular

$$|y(0,k,\eta)-y(0,k,\tilde{\eta})| \leq \mathcal{C}_k(0)|\eta-\tilde{\eta}|. \tag{2.24}$$

Hence, for each $k\in\mathbb{Z}^+$ we have that $\eta\mapsto y(0,k,\eta)$ is a continuos map. Moreover, it is easy to see that if $n>k$, the discrete Gronwall's inequality (see for example [15, 28]) implies

$$|y(n,k,\eta)-y(n,k,\tilde{\eta})| \leq |\eta-\tilde{\eta}|\prod_{p=k}^{n-1}(||A(p)-I||+\gamma(p)). \tag{2.25}$$

In particular, for each $j\in\mathbb{Z}^+$ we have that $\eta\mapsto y(j,0,\eta)$ is a continuous function. Secondly, let us see that

$$\eta \to w^*(0;(k,\eta)),$$

is a continuous map for any fixed $k\in\mathbb{Z}^+$. For this, let us consider $\eta\in\mathbb{R}^d$ and a sequence $\{\eta_n\}_{n\in\mathbb{Z}^+}\subseteq\mathbb{R}^d$ such that $\lim\limits_{n\to\infty}\eta_n=\eta$.

In addition if

$$a_n(j) = \mathcal{G}(0,j+1)f(j,y(j,0,\eta_n)), \quad \forall n\in\mathbb{Z}^+$$

then by (P3) we have that

$$|a_n(j)| \leq ||\mathcal{G}(0,j+1)||\mu(j), \quad \forall n,j\in\mathbb{Z}^+,$$

where

$$\sum_{j=0}^{\infty} ||\mathcal{G}(0,j+1)||\mu(j) < +\infty,$$

by (P4) On the other hand, from the fact that $v \mapsto f(j,v)$ and $\eta \mapsto y(j,0,\eta)$ are a continuous functions for all $j \in \mathbb{Z}^+$ it follows that

$$\lim_{n\to\infty} a_n(j) = \lim_{n\to\infty} \mathcal{G}(0,j+1)f(j,y(j,0,\eta_n)) = \mathcal{G}(0,j+1)f(j,y(j,0,\eta)), \quad \forall j \in \mathbb{Z}^+.$$

Therefore, by the dominated convergence Theorem together with the expression (2.4) we have

$$\begin{aligned}\lim_{n\to\infty} w^*(0;(k,\eta_n)) &= \lim_{n\to\infty} -\sum_{j=0}^{\infty} \mathcal{G}(0,j+1)f(j,y(j,0,\eta_n)) = \lim_{n\to\infty} -\sum_{j=0}^{\infty} a_n(j)\\ &= -\sum_{j=0}^{\infty} \mathcal{G}(0,j+1)f(j,y(j,0,\eta)) = w^*(0;(k,\eta))\end{aligned}$$

and the continuity of $\eta \mapsto w^*(0;(k,\eta))$ for any $k \in \mathbb{Z}^+$ follows. Finally, since $\eta \mapsto y(0,k,\eta)$ is a continuous function for any $k \in \mathbb{Z}^+$ we can conclude that G is a continuous map.

Step 6: H is a continuous map. In order to study the continuity of (2.10) for any fixed $k \in \mathbb{Z}^+$, we will prove that

$$\xi \mapsto z^*(k;(k,\xi))$$

is continuous for any $k \in \mathbb{Z}^+$ since $H(k,\xi) = \xi + z^*(k;(k,\xi))$. For this, let us consider $\xi \in \mathbb{R}^d$ and a sequence $\{\xi_n\}_{n\in\mathbb{Z}^+} \subseteq \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \xi_n = \xi$$

and $\xi \mapsto \phi(j;(m,\xi))$ a continuous function for each $m \in \mathbb{Z}^+$ fixed. Furthermore, for each $m \in \mathbb{Z}^+$ fixed let us define

$$b_n(j,\phi(j;(m,\xi_n))) := b_n(j) = \mathcal{G}(k,j+1)f(j,x(j,m,\xi_n) + \phi(j;(m,\xi_n)), \quad \forall n \in \mathbb{Z}^+$$

Then by (P3) we have

$$|b_n(j)| \leq ||\mathcal{G}(k,j+1)||\mu(j), \quad \forall n,j \in \mathbb{Z}^+,$$

where

$$\sum_{j=0}^{\infty} ||\mathcal{G}(k,j+1)||\mu(j) < +\infty,$$

by (P4). Moreover, from Remark 1 we have that $\xi \mapsto x(k,m,\xi)$ is a continuous map for any $k \in \mathbb{Z}^+$ fixed. Thus, from the above combined with the fact that $\xi \mapsto \phi(k;(m,\xi))$ and $v \mapsto f(j,v)$ are continuous functions it is follows that

$$\begin{aligned}\lim_{n\to\infty} b_n(j) &= \lim_{n\to\infty} \mathcal{G}(k,j+1)f(j,x(j,m,\xi_n) + \phi(j;(m,\xi_n)),\\ &= \mathcal{G}(k,j+1)f(j,x(j,m,\xi) + \phi(j;(m,\xi)),\end{aligned}$$

for each $j \in \mathbb{Z}^+$. Therefore, by the dominated convergence Theorem we have that

$$\begin{aligned}\lim_{n\to\infty} (\Gamma\phi)(k;(m,\xi_n)) &= \lim_{n\to\infty} \sum_{j=0}^{\infty} \mathcal{G}(k,j+1)f(j,x(j,m,\xi_n) + \phi(j;(m,\xi_n))\\ &= \lim_{n\to\infty} \sum_{j=0}^{\infty} b_n(j) = \sum_{j=0}^{\infty} \mathcal{G}(k,j+1)f(j,x(j,m,\xi) + \phi(j;(m,\xi))\end{aligned}$$

$$=(\Gamma\phi)(k;(m,\xi)).$$

Thus, we have that $\xi \mapsto (\Gamma\phi)(k;(m,\xi))$ is a continuous function and therefore its fixed point

$$\xi \mapsto z^*(k;(m,\xi))$$

is a continuous function too. Hence, in particular, the map $\xi \mapsto z^*(k;(k,\xi))$ is continuous. □

Remark 2.7. We emphasize the remarkable simplicity of the characterization of the map $\eta \mapsto G(k,\eta)$ provided by the identity (2.14) rather than the classical one (2.11). To the best of our knowledge, this characterization is a novelty in the discrete framework.

2.2. **Some consequences.** The intermediate computations proving Theorem 2.1 allow us to verify some interesting identities:

Remark 2.8. The maps w^* and z^* defined respectively by (2.4) and (2.6) verify the following identities:

$$\begin{aligned} w^*(k;(m,\eta)) + z^*(k;(m,G(m,\eta))) &= 0, \\ z^*(k;(m,\xi)) + w^*(k;(m,H(m,\xi))) &= 0. \end{aligned}$$

In fact, by using the identity $H[k,G[k,y(k,m,\eta)]] = y(k,m,\eta)$ combined with (2.10), the first and second identities of (2.19) and (2.9) we have that:

$$\begin{aligned} w^*(k;(m,\eta)) + z^*(k;(k,G[k,y(k,m,\eta)])) &= 0 \\ w^*(k;(m,\eta)) + z^*(k;(k,x(k,m,G(m,\eta)])) &= 0 \\ w^*(k;(m,\eta)) + z^*(k;(m,G(m,\eta))) &= 0, \end{aligned}$$

and the first identity follows. The second identity can be deduced by replacing η by $H(m,\xi)$ in the first identity and using $G(m,H(m,\xi)) = \xi$.

By considering $m = k$ in the above identities, we have the following consequence

Corollary 1. *The maps G and H of the $\mathbb{Z}^+$–topological equivalence satisfies the following fixed point properties:*

$$\begin{aligned} G(k,\eta) &= \eta - z^*(k;(k,G(k,\eta))), \\ H(k,\xi) &= \xi - w^*(k;(k,H(k,\xi))). \end{aligned}$$

Remark 2.9. As mentioned in the introduction, the above Corollary provides a characterization of the maps $\eta \to G(k,\eta)$ and $\xi \mapsto H(k,\xi)$ in terms of fixed points, which seems not be noticed previously.

Moreover, as the topological equivalence is an equivalence relation, the Theorem 2.1 has a direct byproduct:

Corollary 2. *If the linear system (1.1) satisfies the assumptions (P1)-(P2), then for any function $g\colon \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the assumptions (P3)-(P6) it follows that the nonlinear systems*

$$y_{k+1} = A(k)y_k + g(k,y_k)$$

and (1.2) are $\mathbb{Z}^+-$ topologically equivalent.

2.3. **The case $P = I$ and $Q = 0$.** A second byproduct of Theorem 2.1 considers the case where the projectors $P(n) = I$ and $Q(n) = 0$ for any $\mathbb{Z}^+$ are considered, that is, the system (1.1) is a nonuniform contraction. Nevertheless this result has interest of itself and should be treated separately.

Under these specific projectors, the assumptions (P2), (P4) and (P5) become:

(S1) There exists a bounded sequence ρ and a decreasing sequence h convergent to zero with $h(0) = 1$ such that:

$$||\Phi(k,n)|| \leq \rho(n)\frac{h(k)}{h(n)}, \quad \forall k \geq n \geq 0.$$

(S2) The sequences ρ and h defined above verify

$$\sum_{j=0}^{k-1} \mu(j)\rho(j+1)\frac{h(k)}{h(j+1)} < \infty \quad \text{for any } k \in \mathbb{Z}^+.$$

(S3) The sequences ρ and h defined above verify

$$\sum_{j=0}^{k-1} \gamma(j)\rho(j+1)\frac{h(k)}{h(j+1)} := q < 1 \quad \text{for any } k \in \mathbb{Z}^+.$$

Remark 2.10. In order to highlight the relevance of the property **(S1)**, we recall the following definition:

Definition 2.2. [18, p.255] The linear system (1.1) is asymptotically stable at $k_0 \geq 0$ if

$$\lim_{k\to\infty} ||\Phi(k,k_0)|| = 0.$$

In addition, we will say that the system (1.1) is asymptotically stable if it is asymptotically stable at each $k_0 \geq 0$. As we stated in Remark 2.3, the property (P2) with $P = I$ says that the asymptotic stability of (1.1) can be characterized more specifically as a non uniform contraction or as a (ρ, h)–contraction in the sense of [10, Def. 2.2], and a suitable choice of ρ and h allows to define sharper types of asymptotic stability for example:

- If $\rho(n) = K$ and $h(k) = \theta^k$ for any $k \geq n \geq 0$ with $k > 0$ and $\theta \in (0,1)$, then the system (1.1) is uniformly asymptotically stable or uniformly exponentially stable. We refer the reader to [15] and [18, p.257] for details.
- If $\rho(n) = K\theta_0^n$ and $h(k) = \theta^k$ for any $k \geq n \geq 0$ with $K > 0$, $\theta \in (0,1)$ and $\theta_0 \geq 1$, then the system (1.1) is non uniformly asymptotically stable or non uniformly exponentially stable. We refer the reader to [35] where $\theta = e^{-\alpha}$ and $\theta_0 = e^{\delta}$ with $\alpha < 0 \leq \delta$ are considered.

Now, we revisit Theorem 2.1 under the above mentioned restrictions:

Corollary 3. *If the properties* **(P1)***,* (P3)*,* (P6) *and* (S1)–(S3) *are satisfied then the systems (1.1) and (1.2) are $\mathbb{Z}^+$–topologically equivalent with*

$$\begin{cases} G(k,\xi) = x(k,0,y(0,k,\xi)) = \Phi(k,0)y(0,k,\xi), \\ H(k,\xi) = y(k,0,x(0,k,\xi)). \end{cases} \tag{2.26}$$

Proof. The topological equivalence is immediate since (S1), (S2) and (S3) are particular cases of (P2), (P4) and (P5) respectively. Nevertheless, we can gain more insight about G and H. In fact, as $Q(n) = 0$ for any $n \in \mathbb{Z}^+$ we have the identity (2.14). Moreover, we can easily prove that $\xi \mapsto G(k,\xi)$ is a bijection for any

$k \in \mathbb{Z}^+$. In fact, the injectiveness is a straightforward consequence of the uniqueness of solutions. On the other hand, given an arbitrary $z \in \mathbb{R}^d$ it is easy to see that $G(k,\xi) = z$ with $\xi = y(k,0,\Phi(0,k)z)$ and the surjectivity follows.

As we know that

$$H(k,\xi) = G^{-1}(k,\xi) \tag{2.27}$$

for any $k \in \mathbb{Z}^+$, from (2.11) we have that

$$G(k,H(k,\xi)) = \Phi(k,0)y(0,k,H(k,\xi)) = \xi$$

or equivalently

$$y(0,k,H(k,\xi)) = \Phi(0,k)\xi = x(0,k,\xi).$$

In addition, from (2.16) combined with the identity above we have that

$$H(k,\xi) = y(k,0,y(0,k,H(k,\xi)) = y(k,0,x(0,k,\xi))$$

and the result follows. □

Remark 2.11. We point out the remarkable simplicity of the identities (2.26) compared with previous references [7, 32, 33]. In addition, we emphasize its novelty for the discrete case.

Last but not least, the restriction to the projectors $P = I$ and $Q = 0$, combined with additional boundeness and smoothness properties on $x \mapsto f(k,x)$ for any $k \in \mathbb{Z}^+$, allow us a simple study of the smoothness properties of the topological equivalence as stated in the following result:

Lemma 2.3. *If the properties* (P1), (P3),(P6), (S1)–(S3) *are satisfied and the following properties are satisfied for any fixed $k \in \mathbb{Z}^+$:*

a) The map $x \mapsto f(k,x)$ and its derivatives up to order r–th are continuous with $r \geq 1$,

b) $\sup\limits_{x\in\mathbb{R}^d} ||\frac{\partial f}{\partial x}(k,x)|| < \infty$ is bounded.

Then the map $\xi \mapsto G(k,\xi)$ is a diffeomorphism of class C^r for any fixed $k \in \mathbb{Z}^+$.

Proof. By using (2.21), it can be proved that $\xi \mapsto \frac{\partial y}{\partial \xi}(n,k,\xi)$ is well defined for any $0 \leq n < k-1$ and is solution of the matrix difference equation:

$$\begin{cases} z_{n+1} = [A(n) + \dfrac{\partial f}{\partial x}(n,y(n,k,\xi))]z_n, \\ z_k = I. \end{cases}$$

provided that the matrix $A(n)+\frac{\partial f}{\partial x}(n,y(n,k,\xi))$ is invertible for any $n \in \{0,\dots,k\}$.

In order to prove the invertibility, notice that

$$A(n) + \frac{\partial f}{\partial x}(n,y(n,k,\xi)) = A(n)[I + A^{-1}(n)\frac{\partial f}{\partial x}(n,y(n,k,\xi))],$$

and as $\sup_{x\in\mathbb{R}^d} ||\frac{\partial f}{\partial x}(n,x)||$ is bounded for any fixed n, the Lipschitz constant $\gamma(n)$ is such that

$$\gamma(n) = \sup_{x\in\mathbb{R}^n} |||\frac{\partial f}{\partial x}(n,x)||.$$

By using (P6), we can see that

$$||A^{-1}(n)\frac{\partial f}{\partial x}(n,y(n,k,\xi))|| \leq ||A^{-1}(n)||||\frac{\partial f}{\partial x}(n,y(n,k,\xi))|| = ||A^{-1}(n)||\gamma(n) < 1,$$

for any $n \in \{0,\dots,k\}$, which implies the invertibility of $A(n)[I + A^{-1}(n)Df(n,y(n,k,\xi))]$ for any $n \in \{0,\dots,k\}$. This property implies the backward continuation of the above matrix difference system, and particular $\xi \mapsto \frac{\partial y}{\partial \xi}(0,k,\xi)$ is well defined and is non singular. Moreover, it can be proved that the invertibility of the above matrix combined with the fact that $x \mapsto f(n,x)$ is of class C^r also implies that $\xi \mapsto y(n,k,\xi)$ is of class C^r for any fixed k and $n \in \{0,\ldots,k\}$ and the partial derivatives

$$\xi \mapsto \frac{\partial^{|m|} y(0,k,\xi)}{\partial \xi_1^{m_1} \cdots \partial \xi_n^{m_n}} \quad \text{with } |m| = m_1 + \cdots + m_n \leq r$$

are continuous. In addition, by using the first identity of (2.26) we have that

$$\frac{\partial G}{\partial \xi}(k,\xi) = \Phi(k,0)\frac{\partial y}{\partial \xi}(0,k,\xi) \quad \text{for any } k \in \mathbb{Z}^+.$$

Moreover, the invertibility of $\frac{\partial y}{\partial \xi}(0,k,\xi)$ implies that

$$\det \frac{\partial G}{\partial \xi}(k,\xi) \neq 0.$$

The final part of the proof will follow the lines of [9]: as G satisfies the properties of a $\mathbb{Z}^+$–topological equivalence, we have that $\xi \mapsto G(k,\xi) - \xi$ is bounded for any k, which implies that $G(k,\xi) \to \infty$ when $|\xi| \to \infty$. This last property combined with $\det \frac{\partial G}{\partial \xi}(k,\xi) \neq 0$ allow us to use the Corollary 2.1 from Plastock [27] and to obtain that $\xi \mapsto G(k,\xi)$ is a diffeomorphism.

In order to prove that $\xi \mapsto G(k,\xi)$ is a homeomorphism of class C^r for any fixed $k \in \mathbb{Z}^+$, we use again the first identity of (2.26) to verify that

$$\frac{\partial^{|m|} G(k,\xi)}{\partial \xi_1^{m_1} \cdots \xi_n^{m_n}}(k,\xi) = \Phi(k,0)\frac{\partial^{|m|} y(0,k,\xi)}{\partial \xi_1^{m_1} \cdots \xi_n^{m_n}}(0,k,\xi) \quad \text{where } m = m_1 + \cdots + m_n \leq r.$$

On the other hand, as we know that $G(k,H(k,\xi)) = \xi$ implies

$$\frac{\partial G}{\partial x}(k,H(k,\xi))\frac{\partial H}{\partial \xi}(k,\xi) = I$$

and we have that $\frac{\partial H}{\partial \xi}(k,\xi) = [\frac{\partial G}{\partial x}(k,H(k,\xi))]^{-1}$. Finally, the higher formal derivatives of $\xi \mapsto H(k,\xi)$ up to order r–th and its continuity can be deduced recursively. □

Remark 2.12. In the context of topological equivalence, a challenging problem is to determine if the homeomorphism in Theorem 2.1 is Lipschitz: As a first illustration for the autonomous framework, we recall that in [17] Hartman constructed an example of hyperbolic system that not admit C^1 linearization. Later, Rayskin in [30] showed that for any $\alpha \in (0,1)$ there exists an $\alpha-$ Hölder linearization in a neighborhood of the origin for the example of Hartman. Our previous result also illustrates this problem in a nonautonomous framework, in particular in the stable and global case, by using (2.24) combined with the first identity of (2.26) and the asymptotic stability of (1.1) we can deduce that

$$|G(k,\eta) - G(k,\tilde{\eta})| \leq \mathcal{L}(k)|\eta - \tilde{\eta}| \quad \text{where} \quad \mathcal{L}(k) = ||\Phi(k,0)||\,\mathcal{C}_k(0),$$

then a Lipschitz linearization could be obtained if $k \mapsto ||\Phi(k,0)||\mathcal{C}_k(0)$ is bounded for any $k \in \mathbb{Z}^+$ or equivalently if $\xi \mapsto \frac{\partial y}{\partial \xi}(0,k,\xi)$ is bounded for any $k \in \mathbb{Z}^+$. Nevertheless, finding conditions ensuring boundedness for the above cases will impose additional restrictions to $A^{-1}(\cdot)$ and $\gamma(\cdot)$.

3. **Topological equivalence and asymptotic stability.** This section is focused in the special case that the linear system (1.1) is asymptotically stable in the sense of the Definition 2.2. We recall the Definition of equilibrium for the nonlinear system (1.2) and its stability properties. Moreover we prove that, in case of existence of an equilibrium point y^* of the nonlinear system, the conditions ensuring the topological equivalence between (1.1) and (1.2) also implies its uniqueness. Finally, we study if the conditions ensuring the topological equivalence also preserves the asymptotic stability of y^*.

Definition 3.1. The solution y^* is an equilibrium of the nonlinear system (1.2) if

$$y^* = A(n)y^* + f(n,y^*) \quad \text{for any } n \in \mathbb{Z}^+. \tag{3.1}$$

If y^* is an equilibrium of the nonlinear system (1.2) it follows that $y(n,m,y^*) = y^*$ for any $(n,m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, which is equivalent to

$$y^* = \Phi(n,m)y^* + \sum_{j=m}^{n-1} \Phi(n,j+1)f(j,y^*)$$

Definition 3.2. [15, Def. 4.2] The equilibrium y^* of the nonlinear system (1.2) is:

i) Stable if given $\varepsilon > 0$ and $m \in \mathbb{Z}^+$ there exists $\delta := \delta(\varepsilon,m) > 0$ such that $|\eta - y^*| < \delta$ implies $|y(n,m,\eta) - y^*| < \varepsilon$ for any $n \geq m$, uniformly stable if δ can be chosen independently of m.
ii) Attracting if there exists $\mu := \mu(m) > 0$ such that $|\eta - y^*| < \mu$ implies $\lim\limits_{n\to+\infty} y(n,m,\eta) = y^*$, uniformly attracting if μ can be chosen independently of m.
iii) Asymptotically stable if is stable and attracting, and uniformly asymptotically stable if it is uniformly stable and uniformly attracting.
iv) Uniformly exponentially stable if there exists $\mu : \mu(\varepsilon) > 0$, $\sigma \in (0,1)$ and $K > 0$ such that $|y(n,m,\eta) - y^*| \leq K|\eta - y^*|\sigma^{n-m}$ whenever $|\eta - y^*| < \mu$.

As stated in [15], the condition of uniform attractivity is equivalent to the existence of $\mu > 0$ such that for any $\varepsilon > 0$ and $m \geq 0$ there exist $N(\varepsilon)$ independent of m such that $|y(n,m,\xi) - y^*| < \varepsilon$ for any $n > m + N$ whenever $|\eta - y^*| < \mu$. The uniform exponential stability is a particular case of uniform asymptotic stability.

By the above Definition, we have that a unique equilibrium y^* of (1.2) is globally asymptotically stable if it is stable and globally attracting, namely, there are no restriction for μ. Similarly, the global uniform exponential stability of y^* ignores the boundedness for $|\eta - y^*|$.

The next result shows that if (1.2) has an equilibrium then it is unique and provides conditions ensuring its global asymptotic stability. Before to state it, we will introduce an additional assumption:

(S4) The sequences ρ and h defined previously verify

$$\lim_{k\to+\infty} h(k) \prod_{j=0}^{k-1} \left(1 + \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)}\right) = 0$$

Theorem 3.3. *If the properties* (P1),(P3),(P6),(S1)–(S3) *are satisfied then*

(i) If the system (1.2) has an equilibrium y^*, then it is unique,
(ii) If $y^* = 0$, that is $f(n,0) = 0$ for any $n \in \mathbb{Z}^+$. Then:

$$H(k,0) = G(k,0) = 0 \quad \text{for any } k \in \mathbb{Z}^+,$$

(iii) If $y^*\neq 0$, then $\lim\limits_{k\to+\infty} G(k,y^*)=0$.

(iv) If $y^*\neq 0$ and (S4) *is verified then* $\lim\limits_{k\to+\infty} H(k,0)=y^*$.

Proof. In order to prove (i) notice that if y^* and $\bar{y}$ are equilibria of (1.2) then it follows that $y(n,0,y^*)=y^*$ and $y(n,0,\bar{y})=\bar{y}$ for any $n\in\mathbb{Z}^+$, this is equivalent to:

$$y^*-\bar{y}=\Phi(n,0)(y^*-\bar{y})+\sum_{k=0}^{n-1}\Phi(n,k+1)\{f(k,y^*)-f(k,\bar{y})\}\quad \text{for any } n\in\mathbb{Z}^+.$$

Then, it follows by (S1) and (S3) that

$$\begin{aligned}|y^*-\bar{y}| \leq & ||\Phi(n,0)|||y^*-\bar{y}|+\sum_{k=0}^{n-1}||\Phi(n,k+1)|||f(k,y^*)-f(k,\bar{y})|\\ \leq & \rho(0)h(n)|y^*-\bar{y}|+\sum_{k=0}^{n-1}\gamma(k)\rho(k+1)\frac{h(n)}{h(k+1)}|y^*-\bar{y}|\\ \leq & \rho(0)h(n)|y^*-\bar{y}|+q|y^*-\bar{y}|.\end{aligned}$$

We will see that $y^*=\bar{y}$. Indeed otherwise $|y^*-\bar{y}|\neq 0$ which combined with the above inequality leads to

$$1\leq \rho(0)h(n)+q\quad \text{for any } n\in\mathbb{Z}^+.$$

Now, letting $n\to+\infty$ and using (S1) we obtain a contradiction with (S3) and the uniqueness of the equilibrium follows. Subsequently from (2.14) we have that

$$G(k,y^*)=\Phi(k,0)y(0,k,y^*)=\Phi(k,0)y^*.$$

Thus if $y^*=0$ we have that $G(k,0)=0$. On the other hand, from the fact that $H(k,G(k,0))=0$ we can deduce that $H(k,0)=0$ for any $k\in\mathbb{Z}^+$ and thus (ii) has been proved.

Finally if $y^*\neq 0$ then

$$|G(k,y^*)|=|\Phi(k,0)y^*|\leq ||\Phi(k,0)||\,|y^*|\leq \rho(0)h(k)|y^*|,\quad \forall k\in\mathbb{Z}^+$$

Thus, letting $k\to+\infty$ we conclude that $\lim\limits_{k\to+\infty} G(k,y^*)=0$ and (iii) follows.

Now from (2.10) combined with the fact that y^* is an equilibrium we have that

$$\begin{aligned}|H(k,0)-y^*| \leq & |\Phi(k,0)y^*|+\sum_{j=0}^{k-1}|\Phi(k,j+1)||f(j,z^*(j;(k,0))+x(j,k,0))-f(j,y^*)|\\ \leq & |\Phi(k,0)y^*|+\sum_{j=0}^{k-1}\gamma(j)|\Phi(k,j+1)|\,|z^*(j;(k,0))+x(j,k,0)-y^*|\\ \leq & ||\Phi(k,0)||\,|y^*|+\sum_{j=0}^{k-1}\gamma(j)|\Phi(k,j+1)|\,|H(j,0)-y^*|\\ \leq & \rho(0)h(k)|y^*|+\sum_{j=0}^{k-1}\gamma(j)\rho(j+1)\frac{h(k)}{h(j+1)}|H(j,0)-y^*|\end{aligned}$$

which is equivalent to

$$\frac{1}{h(k)}|H(k,0)-y^*| \leq \rho(0)|y^*|+\sum_{j=0}^{k-1}\gamma(j)\rho(j+1)\frac{1}{h(j+1)}|H(j,0)-y^*|$$

$$\leq \rho(0)|y^*| + \sum_{j=0}^{k-1} \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)}\frac{1}{h(j)}|H(j,0) - y^*|,$$

Later if $W(k) = \frac{1}{h(k)}|H(k,0) - y^*|$ then

$$W(k) \leq \rho(0)|y^*| + \sum_{j=0}^{k-1} \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)}W(j).$$

Now from the discrete Gronwall's inequality (see for example [15, 28]) we have that

$$W(k) \leq \rho(0)|y^*| \prod_{j=0}^{k-1} \left(1 + \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)}\right).$$

Thus

$$|H(k,0) - y^*| \leq \rho(0)|y^*|\, h(k) \prod_{j=0}^{k-1} \left(1 + \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)}\right).$$

Therefore, the property (iv) follows from (S4). □

Theorem 3.4. *If the properties* (P1),(P3),(P6),(S1)*–(S4) hold and the nonlinear system (1.2) has an equilibrium y^*, then it is globally asymptotically stable.*

Proof. Given a fixed $m \geq 0$, for any $k \geq m$ it follows that

$$y(k,m,\eta) - y^* = \Phi(k,m)(\eta - y^*) + \sum_{j=m}^{k-1} \Phi(k,j+1)\{f(j,y(j,m,\eta)) - f(j,y^*)\}$$

By using (S1)–(S3), we have that

$$|y(k,m,\eta) - y^*| \leq \rho(m)\frac{h(k)}{h(m)}|\eta - y^*| + \sum_{j=m}^{k-1} \rho(j+1)\frac{h(k)}{h(j+1)}\gamma(j)\,|y(j,m,\eta)) - y^*|$$

which implies that

$$\begin{aligned}\frac{1}{h(k)}|y(k,m,\eta) - y^*| \leq& \rho(m)\frac{1}{h(m)}|\eta - y^*| \\ &+ \sum_{j=m}^{k-1} \rho(j+1)\frac{h(j)}{h(j+1)}\gamma(j)\,\frac{1}{h(j)}|y(j,m,\eta)) - y^*|,\end{aligned}$$

and by Gronwall's inequality it follows that

$$|y(k,m,\eta) - y^*| \leq C(m)D(k)|\eta - y^*| \tag{3.2}$$

where

$$C(m) = \frac{\rho(m)}{h(m)} \quad \text{and} \quad D(k) = h(k)\prod_{j=m}^{k-1}\left(1 + \rho(j+1)\frac{h(j)}{h(j+1)}\gamma(j)\right).$$

Now, we will verify that the equilibrium y^* is globally asymptotically stable in the sense of Definition 3.2, namely, y^* is stable and globally attractive: firstly, by (S4) we know that $D(k)$ is a convergent positive sequence and consequently is bounded. Then, given a fixed $\varepsilon > 0$, the stability of y^* follows by considering

$$\delta(\varepsilon, m) = \frac{\varepsilon}{C(m)}\left(\sup_{k\geq m} D(k)\right)^{-1}.$$

Secondly, by using again (S4) we have that $y(k,m,\eta) \to y^*$ when $k \to +\infty$ and the global attractiveness property follows since (3.2) shows that there are no restriction for $|\eta - y^*|$. □

Let us note that the above result states that the asymptotic stability is preserved by the same properties ensuring the topological equivalence provided that (S4) is verified. Moreover, we have to recall that the asymptotic stability of y^* is not uniform. An open question is to determine that if (S4) can be replaced by less restrictive properties.

In the particular case that the linear system (1.1) is uniformly exponentially stable, namely, it has an exponential dichotomy on $\mathbb{Z}^+$ with projectors $P(n) = I$ and $Q(n) = 0$ (see Remark 2.3), the assumption (S4) can be dropped and the above result becomes sharper:

Corollary 4. *Assume that the property* (P1) *holds, the property* (S2) *is satisfied with* $\rho(n) := K > 0$ *and* $h(n) = \theta^n$ *with* $\theta \in (0,1)$ *and the property* (P3) *is verified with constants sequences* $\gamma(n) := \gamma$ *and* $\mu(n) := \mu$ *such that*

$$\frac{\gamma K}{1-\theta} < q < 1 \quad \text{and} \quad ||A^{-1}(\ell)\gamma|| < 1, \quad \forall \ell \in \mathbb{Z}^+, \tag{3.3}$$

then if the nonlinear system (1.2) has an equilibrium y^**, then it is unique and globally uniformly exponentially stable.*

Proof. It is straightforward to see that the inequalities (3.3) implies (S3) and (P6). The property (S2) is verified since

$$\sum_{j=0}^{\infty} \mu(j)\rho(j+1)\frac{h(k)}{h(j+1)} < \frac{\mu K}{1-\theta}.$$

Now, we will see that (3.3) also implies (S4). In fact, notice that the left inequality of (3.3) implies that $0 < \theta + \gamma K < 1$. Moreover we can see that

$$\begin{aligned} h(k)\prod_{j=0}^{k-1} 1 + \gamma(j)\rho(j+1)\frac{h(j)}{h(j+1)} \\ &= \theta^k \prod_{j=0}^{k-1}\left(1 + \frac{K\gamma}{\theta}\right) = \theta^k \prod_{j=0}^{k-1} \frac{\theta + \gamma K}{\theta} = (\theta + \gamma K)^k \end{aligned}$$

and (S4) follows since $0 < \theta + \gamma K < 1$.

The uniqueness and global asymptotic stability of the equilibrium y^* is a consequence of the Theorems 3.3 and 3.4. Nevertheless, by following the lines of the proof of Theorem 3.4 we can deduce that

$$|y(k,m,\eta) - y^*| \leq K|\eta - y^*|\theta^{k-m} \prod_{j=m}^{k-1}\left(1 + \frac{\gamma K}{\theta}\right) = K|\eta - y^*|(\theta + \gamma K)^{k-m}$$

and the global uniform exponential stability is verified. □

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