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MULTIFIELD INFLATION CONSEQUENCES IN THE PRIMORDIAL BLACK HOLES GENERATION

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS, MENCIÓN FÍSICA

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RESUMEN PARA OPTAR AL GRADO DE
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POR: CRISTÓBAL TOMÁS ZENTENO GATICA
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En esta tesis estudiamos los efectos de una inflación generada por múltiples campos (multifield inflation) en el escenario donde la materia oscura puede ser descrita, totalmente o en una fracción, como agujeros negros primordiales (PBH por sus siglas en inglés). En el contexto en el que trabajamos, la producción de PBH, se puede relacionar directamente a una amplificación a pequeñas escalas en el espectro de potencias de las perturbaciones de curvatura primordial. Esto nos permite conectar la física inflacionaria con otro posible observable como lo serían estos agujeros negros, ayudando así a dilucidar de mejor forma los misterios detrás de la física fundamental detrás de este periodo.

Para el caso de una inflación con más de un campo, las perturbaciones de curvatura (o adiabáticas) interactúan con otros grados de libertad (modos de isocurvatura) describiendo una trayectoria no trivial en el espacio de campos. En la literatura, por lo general, se han considerado modelos donde los modos adiabáticos interactúan de manera débil con estos otros grados de libertad. Sin embargo, existen escenarios (gravedad cuántica por ejemplo) donde las perturbaciones de isocurvatura podrían interactuar fuertemente con las perturbaciones de curvatura.

En este trabajo presentamos un modelo exacto donde estos modos de isocurvatura producen grandes amplificaciones en los modos adiabáticos. Ocurriendo esto cuando la trayectoria inflacionaria experimenta giros abruptos en el espacio de los campos. Se resolvieron de forma analítica las ecuaciones de movimiento para los modos que representan las perturbaciones en el régimen de un acoplamiento fuerte (modelado por una función tipo top-hat). Los resultados obtenidos evidenciaron una dependencia exponencial entre el salto en el espectro de potencias de la perturbación de curvatura y el ángulo barrido por el giro en el camino inflacionario. Por lo que nuestra solución manifiesta el hecho de que para que tengamos un panorama cosmológico con una cantidad importante de PBH, es necesaria la existencia de términos cinéticos no canónicos en la acción de inflación con múltiples campos.

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In this thesis, we study the effects of multifield inflation on the scenario where dark matter can be described, in whole or in a fraction, as primordial black holes (PBH). In the context in which we work, the production of PBH can be directly related to a small-scale amplification in the power spectrum of primordial curvature perturbations. This fact allows us to connect inflationary physics with another possible observable one such as these black holes, thus helping to elucidate the mysteries behind the fundamental physics behind this period.

In the case of multifield inflation, curvature (or adiabatic) perturbations interact with other degrees of freedom (isocurvature modes), describing a non-trivial trajectory in the field space. Most of the models considered in the literature are the ones where adiabatic modes weakly interact with these other degrees of freedom. However, there are scenarios (quantum gravity, for example) where isocurvature perturbations could interact via a strong couple with curvature perturbations.

In this work, we present an exact model where these isocurvature modes produce a large amplification in the adiabatic modes. This occurs when the inflationary trajectory experiences sharp turns in the space of the fields. The equations of motion for the modes representing perturbations in a strong coupling regime (modeled by a top-hat-like function) were analytically solved. And the results obtained showed an exponential dependence between the jump in the power spectrum of the curvature perturbation and the angle swept by the rotation in the inflationary path. So our solution manifests the fact that to have a cosmological paradigm with a significant amount of PBH, the existence of non-canonical kinetic terms in the multifield action is necessary.

Dedicado a mi Nonita y a su amor infinito.

Agradecimientos

Realizar esta tesis en período de pandemia no fue para nada sencillo. La soledad y el aislamiento,características que parecieran ser muy típicas del trabajo científico, siempre dificultan el trabajo creativo natural en este tipo de trabajos. Por esto y más, le debo esta tesis principalmente a mi familia, que durante estos últimos dos años bastante complicados, siempre han estado a mi lado. A mi mamá, por tu apoyo, cariño, consejo y principalmente por permitirme crecer juntos de manera adulta y amorosa. A mi papá, por todo tu amor y cariño, por ser compañía en los largos días de escritura de esta tesis. A mi hermano Sebastián, por ser un gran amigo, compañero y ser tan diferente a mi, te quiero demasiado.

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Y por supuesto no puedo no agradecer a todas mis amistades que me han acompañado en todo este camino en los últimos años (me excuso por no hacer una lista con todos sus nombres, pero temo que se me olvide más de un@). Si bien en estos tiempos ha habido cierta distancia con algun@s, están y siempre estarán en mi corazón.

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Introduction

Questions about the Universe and our relationship as humans with it have existed for thousands of years since the origin of our civilized world. Questions related to its origin, our place in it, the movement of the elements that it contains, how it will end, and many others have tried to be answered by many cultures up to the present. In ancient times, the main tool used to be able to propose a cosmology (from Greek /kosmos/, cosmos or order and /logia/, study of) was the astronomical observation. So it is to be understood that, these times, a great part of the explanations have been based on intricate religious and mythological believings, where, in the center, there was a deity created by the human.

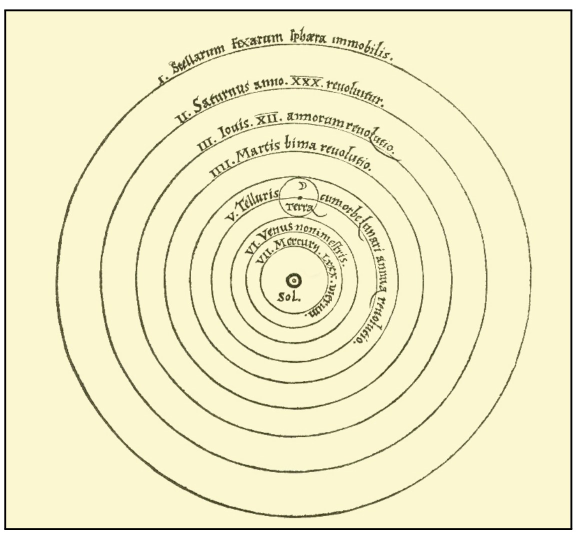


Figure 1.1: Heliocentric Model, **Source:** De revolutionibus orbium coelestium

The first breakthrough in this way of thinking was in the XVI-XVII centuries, with the origin of modern science. Contrary to religious beliefs, scientific truth must be based on observation and experimentation; besides, it must be justified in a common mathematical language. With this new way of thinking as a background, Copernicus presented the heliocentric model (figure: 1.1), where the Earth stopped to be the point from where the planets and stars orbit, and the sun becomes the center of our solar system. At the beginning of the seventeenth century, Kepler formulates his well-known laws, where with three simple postulates, he accurately describes the orbits of the planets around the sun. Later with Newton's laws, postulated in 1687, the world will be able, for the first time, to formulate a cosmology based on scientific proposals and the observations of the time. Although this understanding of the universe was quite simple and rudimentary, it was a great scientific leap of the time. The sky and the stars ceased moving at the whim of the deities and began to move according to the law of Universal Gravitation and describe orbits according to Newton's equations of motion.

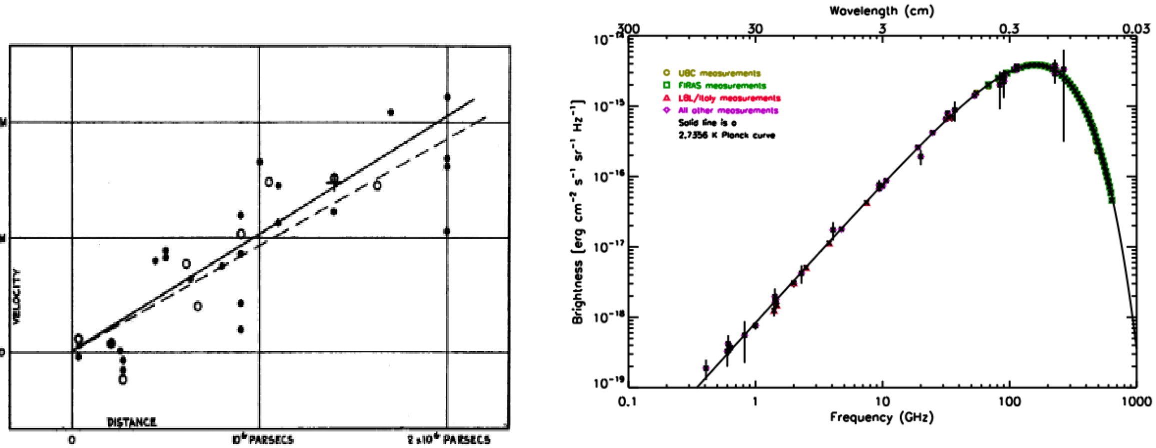
Of course, this early modern cosmology cannot explain all the questions about the universe and even generated more inquiries than answers. Questions such as; the origin of the universe and the stars, the action at distance of gravity, among others, still could not be understood in a good way. However, it laid the foundations for a conversation on which the study of the universe is based to this day: theory plus observation. In addition to these objective foundations, the universe is itself an infinitely complex system, so it is necessary to formulate a hypothesis about it that simplifies its study. This is why the cosmological principle was formulated. This principle proposes that our universe on large scales is homogeneous (matter is distributed equally in all space) and isotropic (there are no privileged directions). Although this paradigm helps significantly with the proposal of a physical model of the universe, when it was first thought, it did not have enough observational foundations. With advances in observations and technologies over time, the cosmological principle and its scope of applicability have been better justified. These elements mentioned above give us the bases and also the limitations of the cosmological model that we are going to have and on which we are going to work. This is why the next great leap in modern cosmology was in the 20th century.

In the first decades of the 20th century, two grand physical theories emerged and revolutionized the way of how we see the universe: General Relativity and Quantum Mechanics. The theory of General Relativity, proposed by Einstein in 1915 [1], models the universe as a 4-dimensional space (3 spatial and one temporal). The matter within it, changes its 4-D curvature so that gravity would be explained by straight paths in curved space-time. With general relativity, the fundamental understanding of the universe was expanded, several open questions of Newton's theory were solved (such as the precession in the orbit of Mercury and the interaction between light and gravity) in addition to adding new elements such as black holes and gravitational waves. This conversation between matter/energy and space-time dynamics is fundamental to understand the universe as a whole through General Relativity.

Almost parallel to the development of the theory, Edwin Hubble, in 1929, observed that distant galaxies were moving away faster than those closer to the Earth (see Fig 1.2.a), thus proposing the famous Hubble law [2]:

$$v = H_0 d. \tag{1.1}$$

This observational law gave us the first intuition of the dynamics of the universe as a whole. It gave insights about an early universe where its components were much more compacted, which later expanded to give rise to the formation of structure that we see nowdays. This theory of the origin of the universe will finally be validated a couple of decades later with the observations of the cosmic microwave background radiation (CMB) [3], which corresponds to radiation emitted in early periods in the history of the universe where matter and radiation from the universe were coupled. When the universe had around 400,000 years, this radiation stopped interacting with the rest of the components of the universe and began to travel freely in space until it was observed. The spectrum of this radiation fit almost perfectly with a blackbody of about 2.7K (Fig 1.2.b), usually characterized as the temperature of the universe and has been declining due to expansion (it is estimated that the temperature of the radiation when it originated was approximately 3,000K).



(a) Hubble original Plot [2].

(b) CMB spectrum [4].

Figure 1.2: The Foundings of the Modern Cosmology

The relevant physical quantity in a model of space-time in general relativity is the metric, which can be understood in a simple way as a collection of functions that tell us how to measure distances in curved spaces. At the same time, the equations that communicate the metric with the matter content of space-time are Einstein's equations. It is impossible to solve a general Einstein equation that is valid for any metric at any point in space and with any source, so it is necessary to make assumptions that simplify our study problem. In this case, the assumption chosen is that our large-scale universe is governed by the cosmological principle. With this in mind, the most general metric that describes a homogeneous and isotropic universe is the so-called Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [5–8]. In this metric, the relevant dynamic quantity is called the scale factor, which is a function that depends on time and tells us how the distances measured in the universe change over time; if it expands, the scale factor will grow, while if the universe contracts, the scale factor will decrease over time. As in this first approximation, it is about understanding the universe on a large scale. The assumption that follows is to assume that all its content behaves like a perfect fluid. By rewriting Einstein's equation with all these ingredients, we have Friedmann's equation, which is a differential equation for the scale factor that depends on the content of the universe.

The content of the universe is another of the big questions that are being studied in modern cosmology. Our current model separates them into four parts: radiation, baryonic matter, dark matter, and dark energy. The first two are pretty simple to understand; radiation is mainly neutrinos and photons that travel through the universe whose origin is mainly from the CMB, while the baryonic matter is everything made up of atoms. On the other hand, dark matter and dark energy remain big questions in cosmology, even though together they correspond to more than 90% of the content of the universe today. Dark matter was proposed as a way to explain the dynamics of galaxies and the stars that compose them, while dark energy is necessary to explain the expansion of the universe and how it is accelerating.

The cosmological principle and the assumptions used to derive the Friedmann equation help us understand our universe at a first approximation. Unfortunately, this treatment does not predict the formation of structures such as galaxies and stars. In cosmology, the phenomena which occur on smaller scales than those prescribed by the cosmological principle are studied with a perturbation theory. Another justification for this treatment is the CMB observations, where small inhomogeneities of the order of $\pm 10^{-5} K$ of the background temperature are observed [9]. The theory of cosmological perturbations, in a few words, consists of perturbing the so-called "background" quantities governed by Friedmann's equations and (in principle) expanding the equations of motion to first order. With this theory, which complements the background cosmology, the anisotropies of the CMB radiation can be well understood theoretically, and the formation of structure is justified since, if we have small overdensities of matter in an early universe, these will begin to gather matter due to the effects of gravity and will generate stars and galaxies[1].

A fundamental quantity in the study of cosmological perturbations is the primordial curvature perturbation [10]. This perturbation acts as an initial condition (under certain assumptions) to all perturbations of the universe components. The primordial curvature perturbation did not have a clear physical explanation at first. It was modeled as a random field, where its behavior was characterized according to its statistics, which, to comply with the cosmological principle, must be gaussian as a first approximation. The two great successes of the cosmological perturbations theory are the computation of the angular power spectrum of the CMB [11] and the matter power spectrum [12], where, as is necessary for the cosmological proposal, the theoretical predictions correspond with the observations (Figures 1.3 and 1.4).

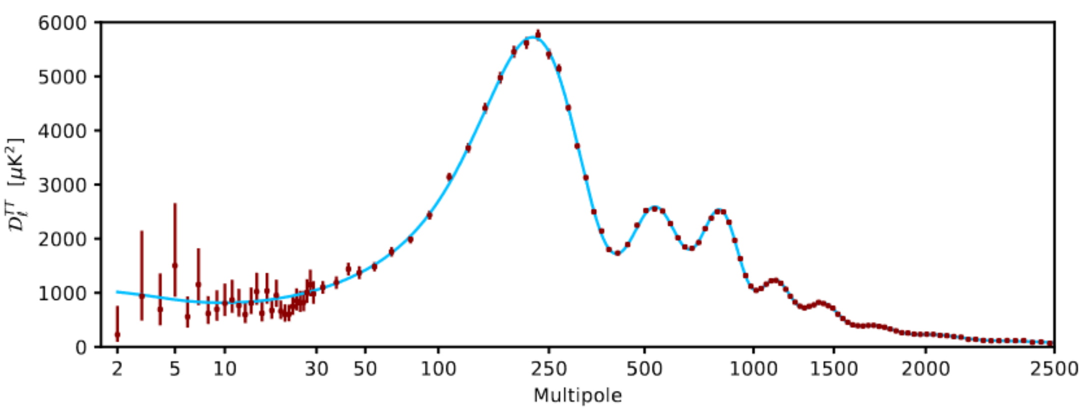


Figure 1.3: CMB angular power spectrum: the blue line is the theoretical prediction (according to the actual concordance model of cosmology), while the red points are the observations with their error bars. Source: Planck 2018 [13].

[1] For the full study of the gravitational collapse, a non-linear treatment of general relativity is used.

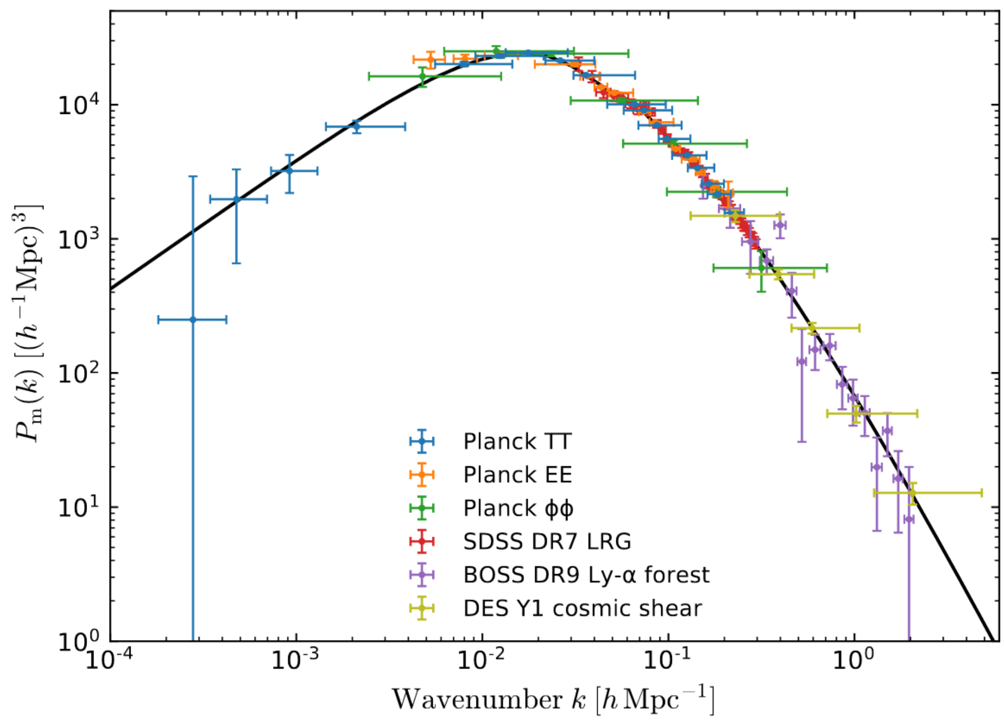


Figure 1.4: Matter Power Spectrum, source: Planck 2018 [13].

The Big Bang model affirm that the universe was originally in a state of infinite density and temperature and then began to expand. In this model, the first fractions of a second cannot be explained by known physical theories, this epoch, it is known as the Planck era. As the universe expands, fundamental forces, which are unified at high energies, begin to decouple and the standard model particles begin to appear, giving rise to hadrons, of which protons and neutrons stand out, which would form atomic nuclei. From what has been described above and in a very superficial way, it seems that, except for what happened in the Planck era and for the unknown regarding our standard model of particles, the Big Bang theory manages to satisfactorily explain the origin of our universe and how it evolves over time. However, the latter is very far from the reality since the Big Bang theory presents a couple of fundamental problems. The most obvious are: the horizon problem, the flatness problem, and the initial conditions [14, 15, 10].

To understand the horizon problem, we can look at two diametrically opposite areas of the CMB radiation. To correspond to the homogeneity of the universe, we must find that these two zones, at the moment of origin of said radiation, must be in thermodynamic equilibrium. But the reality is that these two zones, according to the usual Big Bang theory, were never in causal contact, so it is impossible for them to have the same temperature (unless the initial conditions were very fine-tuned). So, the big bang theory fails to explain that our night sky is so homogeneous if it comes from non-causally connected patches. On the other hand, current observations show us an approximately flat universe today (curvature in 4 dimensions). Still, one of the dynamic effects of the expansion of the universe is to increase that curvature, which implies that the early universe must still be much flatter than today. This is the flatness problem that the classical Big Bang theory cannot explain. Finally, as we had previously proposed, the Big Bang also fails to explain that the primordial perturbations are adiabatic and it statistic is mainly gaussian [16].

The most elegant way to solve these problems of the Big Bang theory is by considering a period during the first moments in the history of the universe where it has an accelerated expansion. This period is known as inflation [17], and it will be the main focus of the work in this Thesis. Assuming a period of accelerated growth in the primordial universe instantly solves the horizon and flatness problems. The horizon problem is solved since prior to this accelerated expansion, these points that we considered causally disconnected were in contact, to later be separated by the effect of the great expansion of the universe. On the other hand, with an accelerated expansion, a flat universe is an equilibrium solution, so an inflationary epoch long enough gives us a universe that is flat enough to correspond to current predictions. To explain the problem of the initial conditions is necessary to study perturbations in inflation.

The simplest way to model the inflationary period is through a scalar field, called inflaton, coupled with gravity. The conditions to generate an accelerated expansion through a field is not unique, and there is no fundamental theory (like the standard model of particles or general relativity) where the existence of the inflaton is contemplated, which makes the inflationary theory intrinsically phenomenological. However, physically speaking, this period is especially interesting because, in inflation, we can connect quantum phenomena with classical scales. In fact, the explanation for the initial conditions is through considering the perturbations in the inflaton as quantum fields. As the universe expands rapidly, these quantum fluctuations are frozen and then give rise to primordial curvature perturbations.

The most common inflation model that gives us an accelerated expansion and satisfies current observations is called slow-roll and is characterized by an approximately flat potential. In addition, the perturbations in slow-roll inflation give us primordial perturbations that follows gaussian statistics. The gaussianity of these perturbations has been observed with a certain degree of accuracy in the angular spectrum of the CMB and measurements of the large-scale structure, characterized mainly by an almost scale-invariant power spectrum (in Fourier space).

The gaussianity of primordial perturbations is not the end of the story since, as we stated earlier, the physical nature that leads to inflation is unknown. Many theories that try to explain inflation go beyond conventional ones, such as string theory and supergravity, and many others. For this reason, if it is possible to observe and understand the departures (or any exotic difference) in the gaussian statistics (non-gaussianities) or the scale-invariance of the primordial perturbations, we would have valuable information on the physics behind inflation [18–21]. This idea is the great motivation behind this work.

We had previously mentioned that the observations of the CMB and the large-scale structure have corresponded to a gaussian statistic and scale-invariant in the primordial perturbations, so how do we justify going further? First of all, it is necessary to emphasize that the observations of the angular spectrum of the CMB restrict the power spectrum in a specific range of scales (large ones), but it is not general. There are many reasons, but one of the recent events that make us suspect going beyond a gaussian model comes from the detections of gravitational waves due (mainly) to collisions between black holes.

In 2016, the first measured observation of a gravitational wave was reported by the LIGO observatory in the United States [22]. This event was a great revolution in science, not only because it once again corroborated Einstein's theory of General Relativity, which predicted its existence, but also because the first detection was not expected so soon, let alone that the following were so frequent. If gravitational-wave emissions were considered to be due to astrophysical black hole collisions, these phenomena are highly unlikely. In addition, some of the masses reported for these black holes that originate gravitational waves were within the range where it is not possible for them to exist, called the "mass gap" [23]. All these inconsistencies made the idea of the existence of Primordial Black Holes (PBH) resurface and that the LIGO detections come from collisions of this type of black hole.

Primordial Black Holes were first theorized in the 1960s [24, 25] and differ from astrophysical black holes as the former comes from large curvature perturbations in the early universe. At the time of these first ideas about PBH, the initial conditions were worked as a random field, without taking much relevance to its physical origin, so these large perturbations in the curvature were possible but unlikely, so the theorized amount of PBH should be negligible. What is relevant to our discussion on non-gaussianities and scale-invariance departures is the fact that these large perturbations are characterized by tails in the probability distributions for curvature perturbations so that a scenario with an odd number of primordial black holes is necessarily generated by deviations in the gaussianity or the scale-invariance of the primordial statistics (mainly with a power spectrum that grows several orders of magnitude on small scales [26]). Furthermore, due to the nature of the perturbations that would generate the PBHs (smaller in relation to those of the CMB), a scenario with PBH is compatible, under certain restrictions, with the observations that are available to date [27]. Additionally, in recent years, primordial black holes have also received more attention since they are a very good candidate to be dark matter [28]. This is mainly due to their properties, characteristic size, and that they are well explained by a well-known theory such as general relativity.

Trying to understand inflationary models that allow changes in the type of primordial statistic to a non-gaussian/non scale-invariant one is the focus of this thesis. One of the inflationary models that enable this is those with more than one field, called multifield inflation [29–31]. In this type of inflation, we have multiple fields which drive inflation; with this paradigm, the usual single-field slow-roll inflation can be understood as a trajectory in the space defined by the fields. The non-gaussianities in this model can be understood through the coupling between the perturbations in the curvature and those associated with the additional fields, called isocurvature perturbations. This coupling is described as changes in the trajectory in the field's space and has generally been treated as a weak coupling (to use a perturbative treatment in the equations). The main objective of this thesis will be to study the effects of a strong coupling between these fields, characterized by large turns in the space of the fields, and to relate characteristics of the turn with the conditions necessary to generate a considerable amount of PBH.

This thesis will be structured as follows: in chapter 2, we will introduce the general concepts of cosmology for a homogeneous and isotropic universe, and we will present the inflationary theory in its most general version; in chapter 3, we will present the theory of cosmological perturbations, and we will develop the tools needed to understand perturbations in inflation; in chapter 4 we will discuss going beyond the general paradigm on primordial perturbations (adiabaticity, gaussianity, and scale invariance) to explore the fundamental physics behind inflation; in chapter 5, we will make a brief review on the theory of primordial black holes as dark matter; in chapter 6 we will present our model of multifield inflation, and we will derive an analytical solution for the curvature perturbation that allows the generation of PBH-DM, in chapter 7 we will terminate our work with the conclusions.

Chapter 2

Modern Cosmology in a Homogeneus universe

2.1. General Relativity and Homogeneous Cosmology Overview

In this section, we will show the main results and equations that are useful in the context of the study of modern cosmology. For more details, we recommend the usual literature: [32–35].

The theory that rises to modern cosmology is mainly the theory of General Relativity (GR) (excluding the fact that some modern cosmology equations can be derived from a purely Newtonian treatment [36]), which Einstein proposed at the beginning of the 20th century [1]. Briefly, the theory of general relativity explains how space-time behaves under the effects of matter and how objects move in these curved space-times. A remarkable fact of this theory is that gravity is no longer a force and can be understood as an "effect" due to space-time curvature. In contrast to classical physics, where the dynamic object to be studied through the equations of motion are the particles' trajectories, in GR, the dynamic object is the metric. Without going into much detail, the metric is the object that characterizes distances in curved space-times, that is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

When talking about curved space-times, the intuition of derivative changes since it will depend on the direction (and the coordinates) of the point where we are applying it. To preserve an invariant definition of the reference frame of the derivative, we introduce the covariant derivative:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho, \tag{2.2}$$

where $\Gamma^\mu_{\nu\rho}$ are known as Christoffel symbols, defined from the metric as:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{2.3}$$

It is important to note that the definition written in equation (2.2) is valid only for tensors of rank (0,1), but it can be simply extended to any tensor.

With the covariant derivative, we can derive an equation for the paths that minimize the distance between two points, called geodesics, which in GR are defined from an affine parameter along the path $x^\mu(\lambda)$.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{2.4}$$

This equation is of utmost relevance since point particles move along geodesics.

The equation of motion for the metric in GR is Einstein's equation, which has the form of:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{2.5}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, defined by the following contraction: $R = R^\sigma_\sigma$ and G is the Newton gravitational constant. The Ricci tensor is a contraction of the Riemann tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$. The Riemann tensor can be understood as a quantification of the curvature of space-time at a certain place. Written in terms of Christoffel's symbols, it is:

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}. \tag{2.6}$$

The Einstein equation's (2.5) left side contains the kinematic (or curvature) information associated with the metric and its derivatives, while the right side is the dynamic information contained in the energy-momentum tensor of the space-time that we are studying.

Knowing the metric of the universe would be the ultimate goal of modern cosmology, where we could predict the movement of its components as well as knowing the origin and end of it exactly. Unfortunately, as we mentioned earlier, this is practically impossible. Therefore, it is necessary to apply various assumptions to the type of metric that we imagine the universe must have and how we model its components.

On the metric side, we must impose that the space-time that describes the universe must comply with the cosmological principle. In this way, the only metric representing a homogeneous and isotropic space-time corresponds to the metric known as Friedmann-Lemaitre-Robertson-Walker (FLRW), which in spherical coordinates has the form:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right], \tag{2.7}$$

where K corresponds to the curvature of this space in 4 dimensions[2]. If $K = 0$, we have a flat space, if $K < 0$, we have an open (hyperbolic) space, and $K < 0$ a closed space (if this is the case, our universe could be characterized as the surface of a 3-sphere). On the other hand, the function a(t) is the scale factor that tells us how spatial distances increase (or decrease) with time, this dependence is due to the homogeneity and isotropy of space-time.

To continue, we must make assumptions about the content of the universe. As we are interested in knowing the universe on a large scale (since we want our metric to obey the

[2] In this metric, and for all of this thesis, we work with units that make the speed of light equal to one.

cosmological principle), we can approximate the galaxies to points and the content of the universe as a composition of perfect fluids. The energy-momentum tensor of a perfect fluid is written as:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.8}$$

where u^μ is the 4-speed of the fluid and ρ, p are the energy density and pressure of the fluid respectively. As an intrinsic property of the energy-momentum tensors that are valid as sources for the Einstein equation, they must obey a continuity equation $\nabla_\mu T^\mu_\nu$. If we place ourselves in the reference system where the universe is homogeneous and isotropic, the fluid is at relative rest, so we have $u^0 = 1$ and $u^i = 0$, in this way, the continuity equation is:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{2.9}$$

where H is the Hubble expansion rate, defined by $H(t) \equiv \frac{\dot{a}}{a}$.

Before writing Einstein's equation for the FLRW metric, in the last definition, we write H as a function of the scale factor and its derivative. Still, in the equation (1.1) we present H_0 as the proportionality constant between the redshift and the distance to state Hubble's law. This is not a coincidence since we can show that Hubble's experimental law in 1929 is directly related to the evolution of the universe and the scale factor. If we do a Taylor expansion to the scale factor around the current time t_0 and keep up at first order, we have

$$a(t) \approx a(t_0) + \dot{a}(t_0)(t - t_0) = a(t_0)(1 + H(t_0)(t - t_0)). \tag{2.10}$$

On the other hand, we can relate the scale factor and the redshift, defining the latter as the difference between the wavelengths of a signal due to the expansion of the universe

$$z \equiv \frac{\lambda - \lambda_e}{\lambda_e}, \tag{2.11}$$

where, λ_e is the emitted wavelength of the signal. As the signal travel through the universe, his wavelengths are affected by its expansion as $\lambda(t) = \lambda_e/a(t)$, and the redshift is:

$$z = \frac{1}{1 + a}. \tag{2.12}$$

Then if $H(t_0)(t - t_0) \ll 1$, we have a relationship between the redshift and the distances (since $c = 1$) where $H_0 \equiv H(t_0)$

$$z \approx H(t_0)(t - t_0) = H_0 d. \tag{2.13}$$

Which is the same law that Hubble gets from his observations. This is because he only could observe some "near" galaxies at that time, so the travel time of the signals can be considered small.

If we go back to studying the equation (2.9). First of all, we would like to know the relation between the energy density and the different components' pressure. It can be demonstrated, using mainly the kinetic theory of the components, that for the fluids that we are interested in studying, the relation between the density and the pressure is through a simple equation of state:

$$p = \omega\rho, \tag{2.14}$$

where ω is a constant known as the state parameter, and it will be different according to each component of the universe. With this, we can integrate the equation (2.9) and obtain the energy density dependence with respect to the scale factor:

$$\rho_a \propto a^{-3(1+\omega_a)}. \tag{2.15}$$

Where the subscript a represents each element of the universe differentiated by its state parameter. For the case of our universe, we can characterize three different components. Each component will have a different state parameter:

- Relativistic matter ($\omega_r = 1/3$), are mainly the photons and neutrinos that are in the universe, we can observe part of the radiation of the universe with the CMB (mainly photons). The evolution of the energy density of the radiation as a function of the scale factor will be:
$$\rho_r \propto a^{-4}. \tag{2.16}$$
- Non-relativistic matter ($\omega_m \approx 0$). This type of matter is characterized as non-interacting with each other. It mainly describes baryonic matter and dark matter. In this case, the energy density associated with non-relativistic matter will have the form:
$$\rho_m \propto a^{-3}. \tag{2.17}$$
- Dark energy (or vacuum energy) ($\omega_\Lambda \approx -1$); is an element that characterizes a type of energy with negative pressure, generating the expansion of the universe. In general, this element is introduced into space-time dynamics from a cosmological constant Λ in Einstein's equations.
$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}. \tag{2.18}$$
Replacing the state parameter for this case, we arrive at a fairly intuitive result, the energy density associated with the cosmological constant does not depend on time (through the scale factor).
$$\rho_\Lambda \propto \text{const.} \tag{2.19}$$

Considering all the components of the universe (i.e $T_{\mu\nu} = \sum\limits_a T^a_{\mu\nu}$), Einstein's equation (2.5) translates into two equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_a \rho_a + \frac{K}{a^2} \tag{2.20}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_a (\rho_a + 3p_a) \tag{2.21}$$

The first equation is known as the Friedmann equation, while the second is interpreted as the acceleration equation. From the second equation, it is straightforward to see that we will have an accelerated expanding universe for the dark energy component ($p = -\rho$). In any case, the first equation will be the relevant one to relate the density of matter of the universe in its history with the scale factor.

The goal will be to solve the Friedmann equation for all times. Being a first order differential equation, we need an initial condition. This initial time will be the current time since we can only access the observations measured today. So we set the scale factor today to $a(t_0) = 1$. We can make the last statement because the scale factor does not have a physical meaning by itself, we only see its effects when we measure proper quantities in the universe. Another way to understand it is that the observable and measurable physical quantity is the scale factor variation, characterized by the Hubble parameter $H(t)$.

With this in mind, we are going to rewrite the Friedmann equation by making two crucial definitions: the critical density

$$\rho_{\text{crit}}(t) \equiv \frac{3H^2}{8\pi G}, \tag{2.22}$$

and the density parameter of each component of the universe

$$\Omega_a \equiv \frac{\rho_a}{\rho_{\text{crit}}}. \tag{2.23}$$

With this, we are left with a more compact version of the Friedmann equation.

$$H^2(t) = H_0^2 \left[\frac{\Omega_{0,r}}{a^4} + \frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,k}}{a^2} + \Omega_{0,\Lambda} \right], \tag{2.24}$$

in this version of the equation, we introduce the curvature energy density as:

$$\rho_k \equiv \frac{-3K}{8\pi G a^2}. \tag{2.25}$$

In general, to describe the various epochs of the universe, we use the scale factor (which goes from 0 to 1 today) or the redshift as a temporal quantity. The latter is 0 today and grows as that we are going backward in time (reasonable since the oldest signals have more time for their wavelengths to be stretched due to the expansion of the universe).

Another important element is that although we include the effects of the intrinsic curvature of space-time in this way of writing the Friedmann equation, it must be separated from the contributions of the other sources of energy density $\sum\limits_a \Omega_a = 1 - \Omega_k$. This separation will help us to deduce the curvature of the universe only by measuring the densities of the other components.

2.2. ΛCDM model

The model that best fits the observations is the so-called Lambda Cold Dark Matter (ΛCDM), where our universe contains various types of components, all of which behave with one of the state parameters mentioned above. We will go on to explain in more detail what each type of component corresponds to and its values observed today.

- Photons: They are photons that travel through the universe freely. The vast majority of these come from the microwave background radiation (CMB). In the early universe, the temperature was so high that the photons were coupled with the matter, but as the universe was cooling, the photons had one last interaction with matter and began to travel freely until we observed them. The CMB is a photo of that moment. Measuring the temperature of the CMB allows us to deduce the density parameter for photons today:

$$\Omega_{0,\gamma} = 4.48 \times 10^{-5}, \tag{2.26}$$

we can notice that photons correspond to a very small fraction of the energy density today.

- Neutrinos: they also correspond to relativistic particles. Unlike photons, they were uncoupled from the rest of the universe elements much earlier. The energy density of neutrinos can be related to the energy density of photons by:

$$\rho_\nu = 3 \times \frac{7}{8} \times \left(\frac{4}{3}\right)^{4/3} \rho_\gamma. \tag{2.27}$$

In this way, the neutrino density will also correspond to a very small contribution to the energy density of the universe today:

$$\Omega_{0,\nu} = 3.4 \times 10^{-5}, \tag{2.28}$$

- Baryonic Matter: Corresponds to everything made up of baryons in the universe. It is the component of which galaxies, stars, planets, we, etc., are composed. Baryons interact with light so that we can observe them. Currently, the observations of the CMB temperature perturbations allow us to approximate the density parameter of baryonic matter, where:

$$\Omega_{0,b} = 0.048 \pm 0.003, \tag{2.29}$$

- Dark matter: It is another type of non-relativistic matter whose main characteristic is that it interacts only gravitationally with the baryonic matter. The dark part of its name came since it does not interact with light. Its existence is deduced from observations in the rotations of galaxies and galaxy clusters, among others. You can approximate the density parameter of dark matter from studying the peaks in the angular spectrum of the CMB, where it currently has to be:

$$\Omega_{0,DM} = 0.258 \pm 0.015. \tag{2.30}$$

Dark matter is an essential part of this thesis, so we will discuss it later in chapter 5.

- Curvature: Observations of the large-scale structure of the universe, besides the CMB analysis, can constrain the effects of the curvature of the universe. The density parameter associated with the curvature has the value:

$$\Omega_{0,K} = 0.001 \pm 0.002. \tag{2.31}$$

This is why it is generally said that our universe is approximately flat.

- Dark energy: Observations tell us that our universe is expanding, and this can be explained mainly by a type of energy density with negative pressure as we present in the equation (2.18), if we complement all the values of the other density parameters mentioned above, we get:

$$\Omega_{0,\Lambda} = 0.6889 \pm 0.0056. \tag{2.32}$$

In this way, we will group neutrinos and photons as a single fluid called radiation and baryonic and dark matter as a fluid that we simply call matter. In the figure 2.1, we summarize the values for each component's energy density. Noting that at present, the dynamics of the universe is dominated by dark energy.

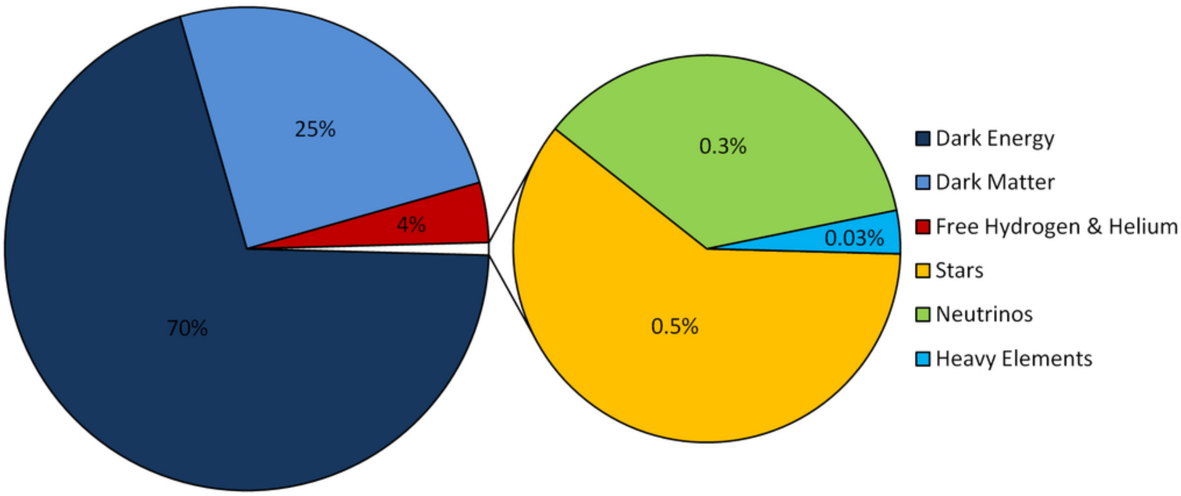


Figure 2.1: Components of the universe, source.

The fact that Λ is the majority component today does not necessarily mean that it always was. If we look at the relations (2.16) and (2.17), as we go further back in time, the scale factor gets smaller, so the other components of the universe (radiation and matter) were the ones that dominated the dynamics of the universe in early times. As we see in the relation (2.17), we can determine the moment in which the energy density associated with the matter was equal to the energy density associated with dark energy. This is for $a_\Lambda = 0.76$, or $z = 0.31$. While from the equation (2.16) we can also deduce the moment when the energy density associated with radiation was equal to the density of matter. This is $a_{rad} = 2.7 \times 10^{-4}$ or $z = 3676$.

With the above, we can define (a priori) 3 eras in the universe that occurred in the following order: the radiation dominated era, the matter dominated era, and the dark energy dominated era (i which this thesis is written). We have left out inflation since its origin is to solve certain problems associated with the Big Bang paradigm, and we will explain it later in this chapter.

The values proposed above are essential since we can solve the Friedmann equation (2.24) at different times, considering only the element whose density is greater than the rest and thus better understand the dynamics in those eras. If we solve the Friedmann equation for the different eras, we have the following time dependencies for the scale factor:

$$a_r(t) \propto \sqrt{t}, \qquad a_m(t) \propto t^{2/3}, \qquad a_\Lambda \propto e^{H_0 t}. \tag{2.33}$$

2.3. A Quick Review of the universe's Thermal History

Although understanding the components of the universe through these macro-divisions such as radiation, matter, and dark energy is useful to understand the dynamics of the universe in a first approximation, the Big Bang theory proposes that our universe originated from a singularity at very high temperatures in a very small volume and from there our universe was expanding and at the same time cooling. According to this last idea, the physics that governs the components in the first moments of the universe is very different from what we observe today (where gravity is the most important interaction at large scales). Using kinetic theory treatments and the interactions that we know (through nuclear and particle physics) between the particles that make up the universe, we can understand the history of the universe from its first moments to the present day.

Before presenting the most relevant milestones in the thermal history of the universe, it is important to clarify that, although we had mentioned that in cosmology, time was usually represented with the scale factor or the redshift. To study the universe's thermal history is a lot more convenient to use temperature to characterize the passage of time. The convenience comes from the fact that when treating the dynamics of the particles from their distributions and interactions, the temperature (or the energy if we make the combination $k_B T$ with k_B the Boltzmann constant) will give us a better intuition for when specific processes or reactions are or are not occurring in the universe. The relationship between the temperature of the universe and the scale can be deduced merely from the fact that the wavelengths of the signals have a dependence $\lambda \propto a$ so that the frequencies will obey an inverse law $\nu \propto 1/\lambda \propto 1/a$ while the temperature will be linearly related to the frequency for the black bodies. With this, we have a simple and intuitive relationship between temperature (or energy) and the scale factor $T \sim 1/a$ so that as we refer to higher temperatures, we will be talking about earlier moments in the history of the universe.

Let us also note that for all this treatment, it is assumed that we are studying a universe that, from its early stages, was in thermal equilibrium, which is well justified by its homogeneity and isotropy at present, but it is not a trivial assumption to explain for the initial conditions (inflation theory solves this question satisfactorily).

Below we will present a summary of the thermal history of the universe starting from the universe at a temperature of approximately 100Gev ($10^{-10}s$). Since that moment, the dynamics of the elements of the universe are well justified and understood based on what we know in particle physics, gravity, and nuclear physics. If we go further back in time, we enter an area where physics' known laws begin not to work well, and we have to start speculating.

- When the universe lowers its temperature below 100Gev, the electro-weak transition occurs. That is, the symmetry between the weak and electromagnetic fields is lost. In these early stages of the universe, it was composed mainly of protons, neutrons, neutrinos, photons, electrons, and positrons.

- Then, when the universe is at a temperature of 1 Mev, the neutrinos are decoupled from the rest of the particles. By not interacting with the rest of the particles, the neutrinos' distribution stops varying, and their temperature evolves independently of the rest of the particles. Furthermore, as long as the temperature of the universe is above 0.5 Mev, the creation/annihilation of positron-electron pairs were quite frequent.

- When the temperature drops below 0.5Mev, it is no longer efficient to create and destroy positrons and electrons, occurring a final pair annihilation. Since the universe is electrically neutral, several electrons must have survived this process to compensate for the charge on the protons.

- The next milestone occurs when the temperature is about 0.07Mev (~ 3 min), as this is when the first Deuterium Helium and Lithium atomic nuclei are formed. This moment in the history of the universe is called the Big Bang Nucleosynthesis (BBN). After nuclei were formed, the universe's only relevant process (from a thermodynamic perspective) is Thomson scattering.

- Long after all this, the next advance in thermal history occurs the moment of equality between radiation and matter, when the universe was at a temperature of 0.8eV (or was 10,000 years old). Within this analysis, this particular moment does not have much relevance. It is going to be important in the study of the perturbations to the various relevant physical quantities since from this moment on, the perturbations of matter begin to grow due to gravity, growth that will give rise to the structure (stars, galaxies, etc.) that we see today.

- Since the universe has a temperature lower than 20eV, the photons stop exchanging energy with the electrons, but they change their direction due to the interaction, so the universe continues to be opaque to the photons. At about 0.26eV, the photons interact for the last time with the rest of the particles and begin to travel freely, and the universe becomes transparent. This milestone is called the Last Scattering, and these are the photons that we receive as the radiation from the microwave background. To put this milestone in perspective, it occurs when the universe is (approximately) 400,000 years old (redshift 1100) and was at a temperature of around 3000K.

- Almost at the same time as the last scattering, recombination occurs. Where the protons join with the electrons forming Hydrogen atoms. It is important to emphasize that the Big Bang theory manages to accurately predict the abundances of Hydrogen, Helium, and Lithium in the universe that we observe, this being one of the clearest tests of the validity of the theory.

After the decoupling of the photons, the universe became dark, and from the perturbations in the various physical quantities, the first stars and galaxies began to form. We will discuss this topic in detail later when we discuss the theory of cosmological perturbations.

2.4. Conformal Time and Horizons

For most of the history of cosmology and astronomy, almost all observation of the sky was due to light signals coming from the universe[3]. The light that propagates through this space-time that we characterize as FLRW and this being in expansion modifies in a certain way the path of light. In particular, if we were to graph a diagram of a null geodesic $ds^2 = 0$

[3] Without considering the recent observations of gravitational waves.

using absolute time t as a temporal variable, we would have a curve modified by the scale factor in time. To understand FLRW spaces with the intuitions of Minkowski spaces, we define the conformal time as:

$$\tau = \int \frac{dt}{a(t)}. \tag{2.34}$$

One way to understand conformal time simply is to see it like a clock that moves slower as the universe expands [37]. Using the conformal time as a temporal variable, the FLRW metric (2.7) becomes:

$$ds^2 = a(\tau)\left[-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right], \tag{2.35}$$

in this way, the trajectories of light rays in a space-time diagram with time as they are lines with angles of $\pm 45°$ as in a Minkowski space. This is extremely useful for studying horizons and causal connections between events for the FLRW metric.

Working within the framework that the Big Bang theory is the best theory to explain the origin of the universe and the processes that have occurred in it from its early stages to the present day, the universe has a finite age. So, along with the fact that photons travel at a finite speed through a vacuum, we cannot access the entire universe from an observation point. If we define t_i as the moment of the origin of the universe, the particle horizon is the maximum comoving distance in which light can propagate from the origin of the universe to today, that is:

$$\chi_p = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)} = \int_{a_i}^{a} \frac{da}{a^2 H} = \int_{a_i}^{a} \frac{d\ln a}{aH}. \tag{2.36}$$

Later, in the context of studying the evolution of modes for perturbations in the dynamic quantities of the universe, in order to separate the types of regimes in the modes, we will define a length scale associated with the Hubble radius $\sim 1/aH$ which is also called the Hubble horizon. Although for a universe dominated by matter and with null curvature (as the observations in our universe show), the particle horizon is of the order of $\sim 1/aH$ also, they are not the same quantity. Since the Hubble horizon appears in the dynamic study of perturbations, while the particle horizon is a kinematic quantity that accounts for photons' trajectories [38].

2.5. Problems with the Big Bang Theory

With the clear concept of particle horizon, we can already present specific problems that the Big Bang theory has that require a complementary theory that explains the dynamics of the universe in its very early stages. We call them problems because there are observations today that are not fully explained by the Big Bang theory.

2.5.1. The Horizon Problem

Consider the following situation [39]: we are studying the signal from the cosmic microwave background radiation in two diametrically opposite directions. The physical emission distance of one of these signals can be understood as the particle horizon distance (evaluated in at the emission time t_e) multiplied by the scale factor (since the particle horizon is a comoving quantity), so it will be:

$$d_{CMB}(t_e) = a(t_e) \int_{t_e}^{t_0} \frac{dt}{a(t)}. \tag{2.37}$$

While the separation distance between the two signals is twice the distance mentioned above.

We can then ask ourselves if these two areas were in causal contact at the radiation emission time. To do this, we calculate the ratio between the separation distance and the size of the horizon. If this ratio is greater than 2, there could be no causal contact between the two points. We write the ratio between the separation and the size of the horizon as:

$$\frac{d_{\mathrm{sep}(t_e)}}{d_H(t_e)} = \frac{2 \int_{t_e}^{t_0} \frac{dt}{a(t)}}{\int_0^{t_0} \frac{dt}{a(t)}}. \tag{2.38}$$

To simplify the calculation, we consider a universe dominated by matter, and with zero curvature, we obtain:

$$\frac{d_{\mathrm{sep}(t_e)}}{d_H(t_e)} = 2\Big((1+z)^{1/2} - 1\Big). \tag{2.39}$$

The redshift associated with the CMB emission is approximately z = 1100, which tells us that (at the moment of emission) the separation between these points is approximately 80 times the distance from the horizon, in evident causal disconnection. Even without being so exaggerated, if we consider two signals from the CMB at an angular distance of 2°, these two signals will also be causally disconnected. The fact that our night sky is made up of thousands of patches not causally connected at the time of its formation and that it is still an almost homogeneous spectrum (beyond small inhomogeneities) is what we call the horizon problem.

2.5.2. Flatness Problem

From the definition of the critical density, we can write the Friedmann equation (2.20) separating the energy density components of the universe with the curvature:

$$|1 - \Omega| = \frac{K}{(aH)^2}. \tag{2.40}$$

As our universe is expanding, the quantity $1/aH$ grows as a function of time (the horizon increases as the universe expands), so the observed solution of $\Omega \sim 1$) today is an unstable equilibrium point. Any small deviation is amplified by expansion, no matter how small it is. If we look back in time, if today we can say that the universe is flat with a confidence of 1%, that is, that our $|1 - \Omega| < 0.01$ today. For moments of recombination, we should enforce a much stricter condition, ($|1 - \Omega| < 10^{-5}$) [40]. So as we go to an early universe, we must demand an extremely flat universe to be consistent with current observations. This is what we call the flatness problem, since the Big Bang theory does not give us a dynamic reason why the universe from its origins was so flat.

2.6. Inflation (General Forumlation)

From the two problems associated with the Big Bang, we can deduce a common element that generates them; the fact that the Hubble radius ($1/aH$) grows with time. The natural way to solve these two problems (together with the problem of the initial conditions that we will discuss later) is considering that in the very early stages of the universe, it expanded rapidly, to then give way to the evolution that we know of the universe. This period is called inflation.

If the expansion is accelerated, this tells us that the Hubble radius decreases with time, so this would generate that zones that were in causal contact (i.e., in thermodynamic equilibrium) would cease to be in a moment of inflation, and then return to be in contact when the Hubble radius began to grow again. Thus solving the horizon problem since, if we return to the previous example of diametrically opposite emissions from the CMB, these will not have been in causal contact at the time of recombination, but they were already thermalized from inflation. Another way of looking at it is that with an accelerated expansion, the integral that defines the particle horizon (2.36) is dominated by the early stages of the universe.

On the other hand, if we revisit the Friedmann equation (2.40) In inflation, the quantity $1/aH$ decreases with time, making the solution of $\Omega \sim 1$ an attractor, thus solving the universe Flatness problem.

Mathematically speaking, we have three equivalent conditions for inflation: As we had previously presented, to solve the horizon and flatness problem, we must force the Hubble radius to shrink over time:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0. \tag{2.41}$$

By expanding the derivative of the previous equation, we have that the condition becomes:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{(aH)^2} < 0 \qquad \Rightarrow \qquad \ddot{a} > 0, \tag{2.42}$$

which results in an accelerated expansion of the universe. Finally, if we ask ourselves some dynamic effect that generates this accelerated expansion, say which energy-momentum tensor causes this acceleration, we can use the equation(2.21) and the fact of imposing $\ddot{a} > 0$ we have:

$$p < -\frac{1}{3}\rho, \tag{2.43}$$

that is, negative pressure could generate the accelerated expansion necessary in inflation.

A particular case of accelerated expansion is that described by “De Sitter” space [41, 42], which consists of an expansion characterized by a constant Hubble parameter: ($H = const$), so the expansion will be exponential:

$$a(t) = e^{H(t-t_0)}, \tag{2.44}$$

while the scale factor is a function of time as we can express it as:

$$a(\tau) = -\frac{1}{H\tau}. \tag{2.45}$$

It is important to note that for this type of expansion, the moment of the singularity that gives rise to the universe (a = 0) is characterized by $\tau = -\infty$. The fact that the scale factor becomes infinite for the case of ($\tau = 0$) should not concern us since for the De sitter space $\tau = 0$ corresponds to the infinite future $t \to \infty$ for the coordinate time.

Why is it relevant to present the De Sitter space in the context of inflation? A good inflationary model can be understood as a quasi-De Sitter process, where the Hubble parameter is almost a constant. The way that we quantify this condition is by rewriting the scale factor acceleration $\ddot{a}/a$ and introducing the parameter $\varepsilon = -\dot{H}/H^2$, we get

$$\frac{\ddot{a}}{a} = H^2(1-\varepsilon). \tag{2.46}$$

For the inflation requirements to be met, we must impose that $\varepsilon < 1$. Later on, we will be able to rewrite this condition for this parameter based on the system's dynamic variables and thus have more clarity of what happens in inflation.

To close this part, we must mention that whatever the dynamic form that generates inflation, in the end, its energy density must be transferred to radiation in the form of particles of the standard model. This process that ends inflation is called reheating.

2.6.1. Single Field Inflation

Although the idea of a shrinking Hubble sphere (i.e., an accelerated expansion), for a few fractions of a second at the beginning of the universe directly solves the problems associated with the Big Bang, the dynamic origin of inflation is not so simple to imagine with the classical intuitions. Needless to emphasize what we mentioned in the equation (2.43), we would need a component of the universe that has negative pressure for inflation to occur.

Without specifying its physical nature, we can model inflation with a scalar field called an inflaton. In the simplest case, this field is minimally coupled to gravity through the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{2.47}$$

The first term is the part of the Einstein Hilbert action and $M_{Pl}^2 = 1/8\pi G$ is the reduced Planck mass[4].By varying this action with respect to the metric, we can derive the Einstein equations (2.5). While the rest is the contribution of the scalar field.

If we vary the action with respect to the inflaton, we will obtain the equation of motion

[4] When it is useful, we're going to use this definition. Besides, it is common to set this term to 1 to normalize the discussion in terms of the Planck mass

of the field:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{dV}{d\phi} = 0. \tag{2.48}$$

From this equation, particularly from the second term, we can deduce that the expansion of the universe acts as a friction force for the field's dynamics.

The energy-momentum tensor associated with the scalar field is:

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}\partial^\lambda\phi\partial_\lambda\phi + V(\phi)\Big). \tag{2.49}$$

Whereas if we assume that we are working on an FLRW metric at all times, the inflaton will keep the symmetries of the metric, so this field can only be time-dependent. With this assumption, the term $\nabla^2\phi$ in the equation (2.48) vanishes, and the energy-momentum tensor of the inflaton is that of a perfect fluid, with:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.50}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.51}$$

From the energy density, it can be clearly seen how we have the kinetic energy of the field in the first term while the second is the potential. We also note that if the contribution of the potential to the energy density exceeds that of the kinetic energy, we will have the accelerated expansion necessary for inflation to occur. If this condition is fulfilled ($V \gg \dot{\phi}$), we will have the so-called slow-roll inflation.

With equations (2.50) and (2.51), we can write the Friedmann and the acceleration equations in terms of the inflaton field and its potential:

$$H^2 = \frac{8\pi G}{3}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big]. \tag{2.52}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{8\pi G}{3}\Big(\dot{\phi}^2 - V(\phi)\Big) \tag{2.53}$$

If we equalize equation (2.46) with (2.53), we will have an expression of the parameter ε as a function of the field, which will be very useful to describe the inflation conditions:

$$\varepsilon = 4\pi G\frac{\dot{\phi}^2}{H^2} = \frac{1}{2M_{Pl}^2}\frac{\dot{\phi}^2}{H^2}. \tag{2.54}$$

For the case of slow-roll inflation, we can approximate the Friedmann equation to:

$$H^2 \simeq \frac{8\pi G}{3}V(\phi). \tag{2.55}$$

For inflation to last a sufficient amount of time to be able to solve the problems associated with the Big Bang, we must impose a second condition; the acceleration of the field should be small. If we see the equation of motion for the field (2.48), we can deduce that the field's acceleration has to be small to fulfill the request:

$$\left|\ddot{\phi}\right| \ll \left|3H\dot{\phi}\right|, \left|\frac{dV}{d\phi}\right|. \tag{2.56}$$

To qualify this condition more intuitively, we define a second slow-roll parameter:

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{2.57}$$

With this, slow-roll inflation would be characterized by ($\epsilon < 1$ and $\eta < 1$). With this second condition, we can write an approximate version for the inflaton equation of motion in the same way as we did with (2.55), obtaining:

$$3H\dot{\phi} + \frac{dV}{d\phi} \simeq 0. \tag{2.58}$$

We can also use the slow roll conditions, to define parameters that depend only of the shape of the potential associated with the inflation:

$$\varepsilon_V = \frac{1}{16\pi G}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 = \frac{M_{Pl}^2}{2}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \qquad \eta_V = \frac{1}{8\pi G}\frac{1}{V}\frac{d^2V}{d\phi^2} = M_{Pl}^2\left(\frac{1}{V}\frac{d^2V}{d\phi^2}\right) \tag{2.59}$$

Note that there is no single potential that satisfy the conditions to generate inflation. And there is a diverse classification of types of inflation differentiated by various characteristics of their potentials [43, 44].

To close the discussion of this part, we need to consider that inflation has a beginning and an end. To characterize the duration of this period, which are fractions of fractions of a second, we define the number of e-folds:

$$N \equiv \ln\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} H dt \tag{2.60}$$

Where the indices f and i are for the end and the beginning of inflation respectively. In the other hand, for the flatness problem to be solved, we must require that ($|\Omega_f - 1| < 10^{-60}$), so from the ratio between the initial and final curvature:

$$\frac{|\Omega_f - 1|}{|\Omega_f - 1|} \simeq \left(\frac{a_i}{a_f}\right)^2 = e^{-2N} \tag{2.61}$$

If we impose that the initial condition is that the difference between $|\Omega_i - 1|$ is of the order of unity, that is, an initial condition very far from a flat space, we would need N$>$ 70 for the universe to be approximately flat from its origins after inflation.

Chapter 3

Perturbations in Inflation

3.1. Linear Perturbation Theory

So far, we have talked about the dynamics of the universe as a homogeneous fluid full of various components (radiation, matter, and dark energy). From the assumption of homogeneity, it is possible to understand, with a reasonably high degree of certainty, the dynamics and history of the universe. We can understand how its components evolved, how these elements affects its evolution by identifying the different eras. And there is even a theory that allows solving the Big Bang theory's specific problems by introducing an accelerated expansion in its first fractions of existence.

As we presented previously, the assumption of the homogeneity of the universe is a helpful tool to understand it. But, it is a matter of looking at the sky at night, and we will realize that our universe is full of structure (clusters of galaxies, stars, planets, etc.), not dust of perfectly distributed matter. All this without forgetting that on large scales, the cosmological principle is perfectly maintained. So it is the natural step for modern cosmology to try to understand these inhomogeneities, at least in a first approximation, and see how they relate to the large-scale dynamics of the universe modeled by the FLRW metric.

The theory that we are going to develop in this chapter is called the theory of cosmological perturbations, whose starting point is to perturb the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.1}$$

Where $\bar{g}_{\mu\nu}$ corresponds to the background spacetime metric, which for our case will be the Friedmann-Lemaitre-Robertson-Walker metric:

$$\bar{g}_{00} = -1 \qquad \bar{g}_{0i} = \bar{g}_{i0} = 0 \qquad \bar{g}_{ij} = a^2(t)\delta_{ij}. \tag{3.2}$$

From now on, in this chapter, all the quantities with a bar on them will correspond to the background quantities whose evolution we studied in the previous section.

On the other hand, we will keep all this perturbative treatment in a linear order for $h_{\mu\nu}$, that is, we will neglect higher orders of h. With this in mind, the inverse of the perturbation will be $h^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}$, whose components we write as:

$$h^{00} = -h_{00}, \qquad h^{i0} = \frac{1}{a^2(t)}h_{i0}, \qquad h^{ij} = -\frac{1}{a^4(t)}h_{ij}. \tag{3.3}$$

Since the cosmological principle only applies to the background metric, we cannot make many assumptions about the perturbation tensor $h_{\mu\nu}$, other than the fact that it is symmetric. To work better with this tensor, we are going to write its 10 degrees of freedom as a combination of 4 scalar perturbations (A, B, C, E), two vector perturbations (F_i and G_i), and one pure tensor perturbation γ_{ij}[5]:

$$h_{00} = -2A, \tag{3.4}$$

$$h_{i0} = a(t)(\partial_i C + F_i), \tag{3.5}$$

$$h_{ij} = a^2(t)\Big(2B\delta_{ij} + \partial_i\partial_j E - \frac{1}{3}\delta_{ij}\nabla^2 E + \partial_j G_i + \gamma_{ij}\Big). \tag{3.6}$$

Vector and tensor perturbations must also fulfill the following conditions in order not to have additional hidden scalar or vector degrees of freedom:

$$\delta^{ij}\partial_i F_j = 0, \tag{3.7}$$

$$\delta^{ij}\partial_i G_j = 0, \tag{3.8}$$

$$\delta^{ij}\partial_i \gamma_{kj} = 0, \tag{3.9}$$

$$\delta^{ij}\gamma_{ij} = 0. \tag{3.10}$$

The convenience of this way of writing the perturbation is that each component transforms and evolves independently. This fact will come in handy when discussing Gauge transformations and the equations of motion for the perturbations.

The next step in this discussion is to consider perturbations in the energy-momentum tensor. We do this by considering the following separation:

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \tag{3.11}$$

where $\bar{T}_{\mu\nu}$ is the background energy-momentum tensor also expressed as a function of background quantities:

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}_\mu\bar{u}_\nu + \bar{p}\bar{g}_{\mu\nu}. \tag{3.12}$$

To identify the perturbed part of the energy-moment tensor, we start by considering that the total shape of the energy-moment tensor is also that of a perfect fluid.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{3.13}$$

Then each quantity is separated into part of background and perturbation:

$$\rho = \bar{\rho} + \delta\rho, \qquad p = \bar{p} + \delta p, \tag{3.14}$$

[5] With this decomposition it is fulfilled that the different elements transform independently under changes of coordinates, without contributing to the other elements of the decomposition. For example, the vector part transforms as a vector without adding scalar quantities.

and the 4-velocity, when doing the separation: $u^\mu = \bar{u}^\mu + \delta u^\mu$, the spatial part can be separated into a scalar part and another purely vector part:

$$\delta u_i \equiv \partial_i \delta u + \delta u_i^V, \qquad \partial_i \delta u^{i(V)} = 0. \tag{3.15}$$

We replace (3.14), (3.15) and (3.1) in (3.13) leaving everything in linear order in the perturbations and we obtain:

$$\delta T_{\mu\nu} = \delta p \bar{g}_{\mu\nu} + \bar{p} h_{\mu\nu} + (\delta\rho + \delta p)\bar{u}_\mu \bar{u}_\nu + (\bar{\rho} + \bar{p})\delta u_\mu \bar{u}_\nu + (\bar{\rho} + \bar{p})\bar{u}_\mu \delta u_\nu. \tag{3.16}$$

Splitting the components in the same way as in (3.4) - (3.6), we have:

$$\delta T_{00} = \delta\rho - \bar{\rho} h_{00}, \tag{3.17}$$
$$\delta T_{0i} = \bar{p} h_{0i} - (\bar{p} + \bar{\rho})\Big(\partial_i \delta u + \delta u_i^V\Big), \tag{3.18}$$
$$\delta T_{ij} = a^2(t)\delta_{ij}\delta p + \bar{p} h_{ij}. \tag{3.19}$$

Since we started under the assumption that the shape of the momentum energy tensor was a perfect fluid, the spatial part of the tensor has only one scalar degree of freedom. To consider all the degrees of freedom, we must introduce an anisotropic stress tensor π_{ij}, which represents dissipative effects due to the inertia of the system. For what would remain:

$$T_{ij} = a^2(t)\delta_{ij}\delta p + \bar{p} h_{ij} + a^2(t)\pi_{ij}, \tag{3.20}$$

The anisotropic stress tensor can also be split down into scalar, vector and tensor parts:

$$\pi_{ij} = \partial_i \partial_j \pi^S - \frac{1}{3}\delta_{ij}\nabla^2 \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + a^2(t)\pi_{ij}, \tag{3.21}$$

where each component must fulfill the conditions analogous to those presented in the equations (3.7-3.10).

3.2. Gauge Transformations

In the discussion of perturbation theory, we have to consider that by adding a perturbation to the metric, we break in a certain way the invariance to change between reference frames of the metric. To be more precise, if we have the metric decomposition of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, nothing prohibits us from that in a given reference frame the system behaves like a background space plus a different perturbation, or even a reference system where the perturbation doesn't exist. That is, the decomposition proposed in the equation (3.1) is not unique. The main complication that this brings is that when trying to solve the Einstein equations for a perturbed metric, we will have to take into account that there will be mixed physical quantities (that we want to study) and quantities associated only with the choice of coordinates.

If we consider infinitesimal transformations between coordinate systems $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, an arbitrary tensor $A_{\mu\nu\ldots\rho}$ will transform as:

$$A'_{\mu\nu\ldots\rho}(x') = A_{\mu\nu\ldots\rho}(x) - \partial_\mu \xi^\lambda A_{\lambda\nu\ldots\rho}(x) - \partial_\nu \xi^\lambda A_{\mu\lambda\ldots\rho}(x) - \ldots - \partial_\rho \xi^\lambda A_{\mu\nu\ldots\lambda}(x), \tag{3.22}$$

these types of transformations are called gauge transformations.

If we want to keep the study in a linear order, the gauge transformation for the perturbation in the metric and the energy-momentum tensor will be:

$$h'_{\mu\nu} = h_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu, \tag{3.23}$$
$$\delta T'_{\mu\nu} = \delta T_{\mu\nu} - \xi_\lambda \bar{\nabla}^\lambda \bar{T}_{\mu\nu} - \bar{T}^\lambda_\mu \bar{\nabla}_\nu \xi_\lambda - \bar{T}^\lambda_\nu \bar{\nabla}_\mu \xi_\lambda, \tag{3.24}$$

where $\bar{\nabla}$ corresponds to the covariant derivative defined in (2.2) but with the Christoffel symbols derived from the background metric. If we separate the previous equation by components, in the same way as in (3.4)-(3.6) and (3.17)-(3.19) we have a transformation rule for each component of the perturbed metric scalar, vector and tensor decomposition:

$$A' = A + \dot{\xi}_0, \tag{3.25}$$
$$C' = C - \frac{1}{a}\Big(\dot{\xi}^S + \xi_0 - 2H\xi^S\Big), \tag{3.26}$$
$$F'_i = F_i - \frac{1}{a}\Big(\dot{\xi}^V_i - 2H\xi^V_i\Big), \tag{3.27}$$
$$E' = E - \frac{2}{a^2}\xi^S, \tag{3.28}$$
$$B' = B + H\xi_0 - \frac{1}{3a^2}\nabla^2\xi^S, \tag{3.29}$$
$$G'_j = G_j - \frac{1}{a^2}\xi^V_j, \tag{3.30}$$
$$\gamma'_{ij} = \gamma_{ij} \tag{3.31}$$

where ξ^S and ξ^V_i are the scalar and vector decomposition of ξ_i:

$$\xi_i \equiv \partial_i \xi^S + \xi^V_i, \qquad \delta^{ij}\partial_i \xi^V_j = 0. \tag{3.32}$$

Then, the transformation of the elements of the energy-momentum tensor results:

$$\delta\rho' = \delta\rho + \xi_0 \dot{\bar{\rho}}, \tag{3.33}$$
$$\delta u' = \delta u - \xi_0, \tag{3.34}$$
$$\delta p' = \delta p + \xi_0 \dot{\bar{p}}, \tag{3.35}$$
$$\delta u'^V_i = \delta u^V_i \tag{3.36}$$
$$\pi'_{ij} = \pi_{ij}, \tag{3.37}$$

We can see right away that δu^V_i and π_{ij} are gauge invariants[6].

From a simple inspection, we can identify gauge-invariant quantities (i.e., physical quantities), for example:

[6] For a perfect and irrotational fluid.

$$\Phi_{\rm GI} = A + \frac{d}{dt}\left(aC - \frac{1}{2}a^2\dot{E}\right), \tag{3.38}$$

$$\Psi_{\rm GI} = -B + \frac{1}{6}\nabla^2 E - aHC + \frac{a^2}{2}H\dot{E}, \tag{3.39}$$

$$\delta\rho_{\rm GI} = \delta\rho + \dot{\bar{\rho}}\left(aC - \frac{a^2}{2}\dot{E}\right), \tag{3.40}$$

$$\delta u_{GI} = \delta u - aC + \frac{a^2}{2}\dot{E} \tag{3.41}$$

The gauge-invariant quantities that will be of particular interest for our study of the inflationary period are the gauge invariant comoving curvature perturbation $\mathcal{R}$ and the curvature perturbation on uniform-density hypersurfaces ζ defined as:

$$\mathcal{R} = B - \frac{1}{6}\nabla^2 E + H\delta u, \tag{3.42}$$

$$\zeta = B - \frac{1}{6}\nabla^2 E - \frac{H}{\dot{\bar{\rho}}}\delta\rho. \tag{3.43}$$

3.2.1. Gauge Election

Due to the fact that there are different ways to represent the metric perturbation, we are free to choose the transformation to a particular reference frame, that is, selecting ξ^μ. Although this is a priori arbitrary, a good selection of Gauge can simplify the equations to study certain problems in particular. Next, we will present the most common gauge choices.

- **Newtonian Gauge**
  The Newtonian (or longitudinal) Gauge consists of locating ourselves in a reference frame such that:

$$E' = 0 \qquad\qquad C' = 0 \qquad\qquad F_i' = 0. \tag{3.44}$$

  In this gauge the scalar perturbation in the temporal part of the metric coincides with the gauge-invariant potential defined in (3.38). This gauge's main use is to describe the evolution of the large-scale structure after recombination.

- **Synchronous Gauge**
  This gauge is characterized by the fact that the temporal parts of the metric (g_{00} and g_{0i}) are not perturbed. This characteristic makes the synchronous gauge very useful in discussing perturbations in the photon-baryon fluid in the primordial universe since we can more easily connect certain ideas and derivations of kinetic theory. To characterize the perturbations of the metric in this gauge, we take the choice of ξ^μ such that:

$$A' = 0, \qquad\qquad C' = 0, \qquad\qquad F_i' = 0. \tag{3.45}$$

  It is important to mention that this gauge has what is known as a residual gauge, that is, the condition expressed in (3.45) is fulfilled for an infinite amount of ξ^μ. This residual gauge is mainly fixed in discussions about fluid velocity associated with dark matter.

- **Comoving Gauge**
  This gauge is particularly useful for studying perturbations in the inflationary period. This gauge has the characteristic such that the 3-speed of the fluid we are studying disappears. For this, we must choose ξ^μ such that:

$$\delta u' = 0, \qquad E' = 0, \qquad G_i' = 0. \tag{3.46}$$

  With the conditions mentioned above, for this gauge, we have $B = \mathcal{R}$ (which will be relevant in the discussion about initial conditions for our universe).

3.3. Einstein Equations (scalar perturbations)

Once we identify the perturbations with which we are going to work and ideate a tool to detect which quantities are physical and which are not, we must solve Einstein's equations to see their evolution in time and space:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{3.47}$$

As we have already mentioned previously, dividing the perturbations into a scalar-vector and tensorial part enables us to study the equations of motion (Eqn. (3.47)) separately for per type of perturbation. Scalar modes are the most complex to understand since they present a larger number of degrees of freedom and have a greater gauge freedom also. On the other hand, vector modes decay with time, and tensor modes are gauge-invariant, so their treatment is a bit more standard [7].

3.3.1. Equations in Fourier Space

To study the different perturbations, it will be more convenient to work the equations in Fourier space. In which, for the case of scalar modes, the perturbations in the physical space will be defined by a superposition of plane waves with a comoving wavenumber $\vec{k}$. In addition, because the perturbations are invariant for spatial translations, the modes will evolve independently. In this way, given a certain perturbation $\psi(\vec{x}, t)$, we define its Fourier transform as:

$$\tilde{\psi}(\vec{k}, t) = \int_{\vec{k}} \psi(\vec{x}, t) e^{-i\vec{k}\cdot\vec{x}}, \tag{3.48}$$

where $\int_{\vec{k}} \equiv \int \frac{d^3k}{(2\pi)^3}$. Then we can rewrite the equations of motion for the modes in Fourier space, making the following correspondences:

$$\psi(\vec{x}, t) \to \psi(\vec{k}, t) \qquad \partial_i \psi(\vec{x}, t) \to i k_i \psi(\vec{k}, t) \qquad \nabla^2 \psi(\vec{x}, t) \to -k^2 \psi(\vec{k}, t) \tag{3.49}$$

This is particularly useful for handling the different equations in perturbation theory. Since in cosmology we have a fundamental scale to separate physical phenomena, the Hubble horizon scale ($\propto H^{-1}$). As we mentioned in section 2.4, the Hubble length informs us if two zones are causally connected in the universe. By working in Fourier space, we can compare

[7] In order to go directly to the equations that are relevant to the work of this thesis, in appendix A is a compilation of the expressions for the Einstein equations for the scalar modes in different gauges.

the scales of the modes characterized by their wavelength[8] $\lambda \propto k^{-1}$, with the Hubble horizon and simplify the equations for these cases.

A fairly illustrative example is the equation of motion of a scalar field in an expanding universe (eq. (2.48)), which without considering a potential for the field and in Fourier space is:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{k^2}{a^2}\phi = 0. \tag{3.50}$$

According to the relationship between k and H, we can separate the equation for each mode and solve them independently. If $k \gg H$ we can neglect the dissipative term associated with the expansion. We called this the sub-horizon mode. On the other hand, we have the case of the modes with $k \ll H$, called superhorizon, where we can neglect the third term.

Finally, it will be of vital importance to note that the modes in this decomposition in Fourier space and the Hubble horizon do not evolve in the same way. Since the physical wavelengths will evolve linearly with the scale factor ($\lambda_{\text{phys}} \sim a$), while the evolution of the Hubble parameter will strongly depend on the time of the universe in which we are, where:

$$\frac{1}{H_r(a)} \sim a^2, \qquad \frac{1}{H_m(a)} \sim a^{3/2}, \tag{3.51}$$

where the subindices r,m comes from the radiation and matter era (see our discussion in chapter 2), so the evolution of the different modes and how we approximate the equations we work with will not be trivial.

3.4. Initial Conditions for the Perturbations

As we can see from Appendix A, where we derive the equations of motion for the first-order perturbations, we have a sufficient number of equations to solve the evolution of the perturbations (of matter and metric) and be able to better understand various phenomena on a smaller scale (than the cosmological scale), such as the formation of structure, the anisotropies of the CMB among others. Then, to have a deterministic solution, we must know the initial conditions for these perturbations.

Following the discussion in the previous section, due to the homogeneity of the universe, we are going to understand the initial conditions in Fourier space. The basis of our argument to identify the nature of the initial conditions was also mentioned earlier. As the Hubble horizon (in times dominated by radiation and matter) grows faster than the wavelength of the perturbation modes. So, a reasonable assumption to begin to understand the initial conditions for the perturbations is to assume that all the modes were outside the horizon at very early times.

We can get more information from this assumption if we put ourselves in the following case: we want to study a scalar quantity χ of our universe and its perturbations in the usual way , i. e. $\chi(\vec{x}, t) = \bar{\chi}(t) + \delta\chi(\vec{x}, t)$. But, under the assumption that the relevant wavelengths

[8] It is important to note that this is the comoving wavelength, which is related to the physical one via the scale factor $\lambda_{\text{phys}} = a(t)\lambda$.

of the initial conditions for the evolution of the χ perturbations are much larger than the Hubble radius, it should not be possible to differentiate between the background quantity and the total quantity (except due to a phase that indicates that we are in a certain Hubble patch), that is:

$$\chi(\vec{x},t) = \bar{\chi}(t + \delta t(\vec{x},t)). \tag{3.52}$$

At first order we can obtain the following relation for δt:

$$\delta t(\vec{x},t) = \frac{\delta\chi(\vec{x},t)}{\dot{\bar{\chi}}(t)}. \tag{3.53}$$

This relationship should hold for all perturbed scalar quantities (density, pressure, temperature, etc.). If we additionally impose that the species that command the background solutions are in local thermodynamic equilibrium, then it must be satisfied that δt must be common for all species:

$$\frac{\delta\rho_a(\vec{x},t)}{\dot{\bar{\rho}}_a(t)} = \frac{\delta p_a(\vec{x},t)}{\dot{\bar{p}}_a(t)} = \frac{\delta T(\vec{x},t)}{\dot{\bar{T}}(t)} = \delta t(\vec{x},t) \qquad \forall a. \tag{3.54}$$

We will call the perturbations that fit this condition adiabatic. With this, we can say that all adiabatic scalar perturbations are determined by a single primordial perturbation [9]. The challenge that follows will be to connect this perturbation (which we will now call primordial) with inflation. The logic is as follows: the accelerated expansion of the universe during the inflationary epoch makes the modes of the perturbations generated in said period (whose nature we will deal with later) exit the horizon and re-enter after inflation ends, and the subsequent ages of the universe begin.

To make the connection between the inflationary period and the primordial perturbation more direct, it will be useful to study a perturbation (remember that we have a gauge freedom) where its superhorizon modes are constant. Let's prove this last statement, if we work with the curvature perturbation in hypersurfaces of uniform density ζ, defined in the equation (3.43) and use the Newtonian Gauge:

$$\zeta = -\Psi + \frac{\delta\rho}{3(\bar{\rho}+\bar{p})}, \tag{3.55}$$

in the second term we use the continuity equation (eq. (2.9)) to replace $\dot{\bar{\rho}}$. Do not forget that ζ is a gauge-invariant quantity, so although we are doing this treatment in the Newtonian gauge, it is valid for any other. If we derive this definition and use the equation (A.16) for the superhorizon case ($k^2/a^2 \ll 1$), we are left

$$\dot{\zeta} = \frac{\dot{\bar{\rho}}\,\dot{\bar{p}}}{3(\bar{\rho}+\bar{p})^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} - \frac{\delta p}{\dot{\bar{p}}}\right). \tag{3.56}$$

But, if the perturbations we are studying are adiabatic, then the parentheses of this equation is canceled by the condition expressed in equation (3.54), with this we conclude that ζ is conserved outside the horizon.

[9] In Appendix B, we use the equations for perturbations in the synchronous gauge and show this explicitly.

Additionally, for purposes of understanding the physics that originate these perturbations, we can relate ζ with $\mathcal{R}$ (comoving curvature perturbation) through their definitions (equations (3.42) and (3.43)) and the equation (A.9), where we obtain

$$\zeta = \mathcal{R} - \frac{k^2}{a^2} \frac{\Psi_{GI}}{12\pi G(\bar{\rho} + \bar{p})}, \tag{3.57}$$

where for superhorizon modes $\zeta = \mathcal{R}$. For this reason, it is common not to differentiate these quantities when we speak of a primordial curvature perturbation.

3.4.1. Statistics of $\mathcal{R}$

The fact that the initial conditions for the perturbations in the universe are adiabatic, and even the fact that they came from a single field of perturbations, are ideas prior to the inflationary theory that explains their origin [16]. How the assumptions associated with these conditions were justified, ignoring the fundamental physics that originated them, is closely connected to how we do astronomical observation to understand the universe. 99 % of our knowledge about the universe is derived fundamentally from the detection of multiple signals that come to us from millions of sources in space. Where then, statistics are made with these observations[10] in order to connect observable values with the theory that explains the phenomena. As the connection between observation and theory is through statistical quantities, a valid assumption to work with the primordial perturbation is to assume that $\mathcal{R}$ is a random field.

The information that we will obtain from $\mathcal{R}$ will be obtained from understanding its statistics, mainly from n-point correlations. But $\mathcal{R}$ cannot be any random field. Instead, it must meet specific basic rules associated with the cosmological principle and the background behavior of the universe on which this perturbation exists. For example, since $\mathcal{R}$ is a perturbation in the metric, its spatial average must be zero so as not to modify the background spacetime. Furthermore, because the homogeneity and isotropy of the universe, the ergodic theorem gives us a condition for the correlations of two points

$$\langle \mathcal{R}(\vec{x}), \mathcal{R}(\vec{y}) \rangle \sim f(|\vec{x} - \vec{y}|), \tag{3.58}$$

condition that, in Fourier space, is translated into

$$\left\langle \mathcal{R}(\vec{k}), \mathcal{R}(\vec{k}') \right\rangle \equiv (2\pi)^3 \delta^{(3)}\left(\vec{k} + \vec{q}\right) P_{\mathcal{R}}(k). \tag{3.59}$$

Where $k = \left|\vec{k}\right|$ and $P_{\mathcal{R}}(k)$ is the power spectrum of $\mathcal{R}$. Additionally, it is common to define the dimensionless power spectrum $\Delta_{\mathcal{R}}$, through

$$P_{\mathcal{R}}(k) \equiv \frac{2\pi^2}{k^3} \Delta_{\mathcal{R}}(k). \tag{3.60}$$

The simplest way to understand the statistics of primordial perturbations is to assume that it is gaussian. Where the main characteristic that a gaussian $\mathcal{R}$ fulfills is that all its statistical information lies only in the power spectrum, since the n-point correlations would

[10] The justification for why it is reasonable to do statistics with the observations of the universe is through the ergodic theorem in conjunction with the cosmological principle [16].

follow the following rule:

$$\langle \mathcal{R}_1 \mathcal{R}_2 ... \mathcal{R}_n \rangle = \begin{cases} \langle \mathcal{R}_1 \mathcal{R}_2 \rangle \, ... \, \langle \mathcal{R}_{n-1} \mathcal{R}_n \rangle + \text{perms} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases} \tag{3.61}$$

So the question that automatically continues is: What is the shape of this primordial power spectrum? The simplest answer is that it should have a power-law shape

$$\Delta_{\mathcal{R}}(k) = A_{\mathcal{R}} \left(\frac{k}{k_*} \right)^{n_s - 1}, \tag{3.62}$$

where $A_{\mathcal{R}}$ and n_s are the amplitude and spectral index of the power spectrum, while k_* is a pivot scale to fix the power spectrum observationally. This form for $\Delta_{\mathcal{R}}$ was proposed in the first instance to justify the theory of galaxy formation, where a scale-invariant power spectrum ($n_s \approx 1$) was introduced [45, 46]. Various observations have made it possible to constrain the parameters $A_{\mathcal{R}}$, n_s and k_* of the power spectrum, for example, the observations of the Planck satellite of the anisotropies of the CMB give us the following values (with $k_* = 0.002 Mpc^{-1}$):

$$\ln \left(10^{10} A_{\mathcal{R}} \right) = 3.044 \pm 0.014, \qquad n_s = 0.9649 \pm 0.0042. \tag{3.63}$$

The primordial power spectrum will be a fundamental quantity to understand, in a first approximation, the behavior of physics that gives rise to the primordial perturbations and how this is connected with the observations. We say first approximation since by studying higher-order correlations (non-gaussianities), we can continue to obtain greater detail in the fundamental physics behind inflation and the first moments of the universe. Unfortunately, the last point remains in the speculative part since the observations today are not precise enough to measure non-gaussianities [47].

3.5. Scalar Perturbations in Single Field Inflation

Now we are going to focus on understanding the perturbations in inflation, which we are going to consider as the source of the perturbations in the matter. These perturbations in the inflaton field, we write them as any perturbation of a scalar field:

$$\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t). \tag{3.64}$$

Where if we want to relate the perturbations in the inflaton with the energy-momentum tensor, this must be through the equation (2.50). Connecting to a perfect fluid, we have that the density, pressure, and 4-velocity will be:

$$\rho_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi), \tag{3.65}$$

$$p_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{3.66}$$

$$u^\mu_\phi = -[-g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi]^{-1/2} g^{\mu\tau} \partial_\tau \phi. \tag{3.67}$$

Now, we can split these quantities into a background part and the perturbation. Where the behavior of the background quantities is those explained in section 2.6.1.. Whereas if we expand (3.65, 3.66, 3.67) up to first order, we will obtain the matter perturbations:

$$\delta\rho = \dot{\bar{\phi}}\delta\dot{\phi} + \frac{dV}{d\phi}\delta\phi + \frac{h_{00}}{2}\dot{\bar{\phi}}^2, \tag{3.68}$$

$$\delta p = \dot{\bar{\phi}}\delta\dot{\phi} - \frac{dV}{d\phi}\delta\phi + \frac{h_{00}}{2}\dot{\bar{\phi}}^2, \tag{3.69}$$

$$\delta u = -\frac{\delta\phi}{\dot{\bar{\phi}}}. \tag{3.70}$$

We are interested in studying the comoving curvature perturbation $\mathcal{R}$ in inflation. For this, we are going to use the comoving gauge defined by the conditions in (3.46). The advantage of this gauge in this discussion comes from the fact that if δu vanishes, this implies by the equation (3.70), that $\delta\phi$ is also zero. In other words, in the comoving gauge, the perturbations of the inflaton are "carried over" to perturbations in the metric.

Then, at this gauge, the perturbation in the energy density and pressure become equal:

$$\delta\rho = \delta p = \frac{\delta\mathcal{N}\dot{H}}{4\pi G}, \tag{3.71}$$

to arrive to the last equality, we used the identity $\dot{\bar{\phi}}^2 = -\dot{H}/4\pi G$ that is obtained by combining the equations (2.52) and (2.53). If we replace what is obtained in the equation (3.71) in the equations (A.28), (A.29) and (A.32) of the appendix A , we will obtain the following system of equations:

$$0 = H\delta\dot{\mathcal{N}} + 2\Big(3H^2 + \dot{H}\Big)\delta\mathcal{N} - \frac{k^2}{a^2}\mathcal{R} - \ddot{\mathcal{R}} - 6H\dot{\mathcal{R}} - \frac{H}{a}k^2C, \tag{3.72}$$

$$0 = -H\delta\mathcal{N} + \dot{\mathcal{R}}, \tag{3.73}$$

$$0 = -\frac{d}{dt}\Big(\delta\mathcal{N}\dot{H}\Big) - 6H\dot{H}\delta\mathcal{N} + \frac{k^2}{a}\dot{H}C + 3\dot{H}\dot{\mathcal{R}}, \tag{3.74}$$

remember that in this gauge $A_C \equiv \delta\mathcal{N}$ and $B_C = \mathcal{R}$. From the third equation we can isolate C in terms of $\mathcal{R}$ and $\delta\mathcal{N}$. And form the second equation, we know that $H\delta\mathcal{N} = \dot{\mathcal{R}}$. Replacing all of this in the first equation, we get to a differential equation for $\mathcal{R}$:

$$\ddot{\mathcal{R}} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right)\dot{\mathcal{R}} + \frac{k^2}{a^2}\mathcal{R} = 0. \tag{3.75}$$

This equation can be written in a compact form if we define the following quantity:

$$z^2 \equiv \left(\frac{a\dot{\bar{\phi}}}{H}\right)^2 = 2a^2\varepsilon. \tag{3.76}$$

Replacing this definition of z in the equation (3.75) and using the conformal time, we are left with:

$$\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' + k^2\mathcal{R} = 0, \tag{3.77}$$

this equation is the Mukhanov-Sasaki equation [48], which describes the comoving curvature perturbation.

3.6. Quantum fluctuations in Inflation

The physical explanation that we will give to the origin of these perturbations in the inflationary period is that they correspond to quantum fluctuations of the inflaton. Due to their nature, these perturbations would only have relevance at scales similar to Planck's scale. Still, since they occur in the inflationary period, these scales undergo an expansion until they correspond to cosmological scales. With this argument, we could explain the origin of the primordial perturbations as a consequence of the existence of the Heisenberg uncertainty principle. But it still does not solve the fundamental nature of inflation. However, it is a great advance since it establishes a point of contact between quantum theory and general relativity.

The first step in interpreting curvature perturbations as a quantum field (and thus using the tools of quantum field theory) is to have an action that describes those perturbations. To get this action, the usual treatment is to expand, up to second order, the Einstein Hilbert action with a scalar field (equation (2.47)), apply the background equations, and extract the quadratic terms[11].

In our case, we can take the easy path and figure out an action from the Mukhanov-Sasaki equation (3.77), for this we define the following variable:

$$v \equiv z\mathcal{R}. \tag{3.78}$$

Using this variable, the equation of motion becomes:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0. \tag{3.79}$$

In this last expression, we made explicit the fact that the field v_k depends on the proper time and that the wavenumber is a parameter associated with the expansion in Fourier space. The essential thing is to note that this equation can be obtained by varying the following action with respect to v[12]:

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x \left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right] \tag{3.80}$$

Then, we promote the field v and at its conjugate moment v' to a quantum operator, where, in real space, it will be

[11] This derivation is usually done using the ADM formalism of general relativity and the comoving gauge (see Appendix C for more details). We will also use this formalism in chapter 6 for the case of multifield inflation.

[12] From now on, for simplicity, we will consider $M_{Pl} = 1$

$$v(\vec{x},\tau) \rightarrow \hat{v}(\vec{x},\tau) = \int \frac{dk^3}{(2\pi)^3}\Big[v_k(\tau)\hat{a}_k e^{i\vec{k}\cdot\vec{x}} + v_k^*(\tau)\hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\Big]. \tag{3.81}$$

While in Fourier space it translates to

$$v(k,\tau) \rightarrow \hat{v}_k(\tau) = v_k(\tau)\hat{a}_k + v_{-k}^*(\tau)\hat{a}_{-k}^\dagger \tag{3.82}$$

In the above expressions, $v_k(\tau)$ are the mode functions that obey the classical equation of motion (3.79), while $\hat{a}_{-k}^\dagger$ and $\hat{a}_k$ are the creation and annihilation operators. The mode functions must comply with the following normalization

$$\langle v_k,\, v_k\rangle \equiv \frac{i}{\hbar}\Big(v_k^* v_k' - v_k^{*'} v_k\Big) = 1, \tag{3.83}$$

also the creation and annihilation operators satisfy with the canonical commutation relations:

$$\Big[\hat{a}_k, \hat{a}_{k'}^\dagger\Big] = (2\pi)^3\delta^{(3)}\Big(\vec{k} - \vec{k'}\Big). \tag{3.84}$$

3.6.1. Vaccum Selection

A fundamental part of this treatment, in which we promote fields to quantum operators, is to define a vacuum state, in which

$$\hat{a}_k\,|0\rangle = 0, \tag{3.85}$$

this choice will imply an additional restriction for the mode functions [49]. The most common choice to satisfy this condition is to assume that long in the past ($\tau \rightarrow -\infty$), all relevant modes[13] were well inside the horizon. This is equivalent to choosing an initial condition that all modes are sub-horizon ($k \gg aH$). Using this approximation in the equation (3.79), we have

$$v_k'' + k^2 v_k = 0. \tag{3.86}$$

The solution of this equation is the usual one for a harmonic oscillator, normalizing and taking the solution that minimizes the energy, we are left as an initial condition:

$$\lim_{\tau\rightarrow-\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{3.87}$$

We can also interpret this condition in the following way: modes much smaller than the characteristic scale of the horizon of the universe do not "observe" a curved space, so we can locally approximate spacetime as a flat Minkowski space, where the modes obey the equation (3.86) [38].

[13] In this theory, there will be scales in which this does not apply, those that even at the beginning of inflation were outside the horizon, but these scales are not relevant since they are very large scales that mix with the background of the universe.

3.7. Quantum Fluctuations in De Sitter Space

As inflation is a process that we called "quasi de Sitter", it is important to solve the equation of motion for the modes (equation (3.79)) in the de Sitter limit and understand its behavior[14]. In this limit $\varepsilon \to 0$ but

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}. \tag{3.88}$$

Replacing in the equation (3.79), we have:

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right)v_k = 0. \tag{3.89}$$

Where, the general solution is

$$v_k = \alpha\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{3.90}$$

If we impose the condition that we present in the equation (3.83) and we impose the Minkowski limit for the subhorizon modes (equation (3.87)), we can set the parameters $\alpha = 1$ and $\beta = 0$. Leaving us the so-called Bunch Davies mode functions [49]:

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{3.91}$$

3.7.1. Power Spectrum

With the solution for the modes obtained in (3.91), we can calculate the two-point correlation function for $\mathcal{R}$, since

$$\langle \mathcal{R}(k)\mathcal{R}(k')\rangle = \frac{1}{z^2}\left\langle \hat{v}_k\, \hat{v}_{k'}\right\rangle. \tag{3.92}$$

If we replace in this expression, the decomposition of $\hat{v}_k$ that we write in the equation (3.82) and also use the commutation relation for the creation/annihilation operators (eq. (3.84)), we get:

$$\langle \mathcal{R}(k)\mathcal{R}(k')\rangle = \frac{|v_k|^2}{z^2}(2\pi)^3\delta\left(\vec{k} + \vec{k}'\right), \tag{3.93}$$

from where we can deduce the spectrum of powers of the first term of the multiplication. Calculating $|v_k|^2$ and replacing z defined in the equation (3.76), the spectrum results

$$P_{\mathcal{R}}(k) = \frac{H^2}{2k^3}\left(\frac{H}{\dot{\phi}}\right)^2\left(1 + k^2\tau^2\right). \tag{3.94}$$

Where, in the superhorizon limit ($|k\tau| \ll 1$) we will have a dimensionless power spectrum (equation (3.60)) that is scale invariant. Let's not forget that since $\mathcal{R}$ is conserved outside the horizon, we can evaluate H and $\dot{\phi}$ at the moment of crossing the horizon t_{hc} for each

[14] In the appendix D, we solve the modes for the slow roll inflation case

mode, so that $a(t_{hc})H(t_{hc}) = k$. In this way, in the approximation of a de Sitter space, the dimensionless power spectrum will have the following scale-invariant amplitude:

$$\Delta_{\mathcal{R}}(k) = \frac{H^2}{(2\pi)^2}\frac{H^2}{\dot{\phi}^2}\bigg|_{k=aH}, \tag{3.95}$$

if we replace in the previous expression the slow-roll parameter ε, we get

$$\Delta_{\mathcal{R}}(k) = \frac{H^2}{8\pi^2}\frac{1}{\varepsilon}\bigg|_{k=aH}. \tag{3.96}$$

To close this part, in the appendix D we compute the primordial power spectrum assuming a slow-roll inflation to compute the mode functions, obtaining:

$$P_{\mathcal{R}}(k) = \frac{H^2}{2k^3}\frac{\pi^{-1}}{2\varepsilon}2^{2\nu-1}\Gamma^2(\nu)\left(\frac{k}{aH}\right)^{-2\nu+3}, \qquad \nu = \frac{3}{2} + 3\varepsilon - \eta \tag{3.97}$$

3.8. Conections with Observations

We have already raised the fact that in a first-order approximation and assuming adiabatic initial conditions, we can connect the origin of all scalar perturbations, both of matter and metric, to a single perturbation. But, unfortunately, we do not observe directly these perturbations. Instead, we see the result of their evolution in the universe in time. What's more, these perturbations interact with each other once they are all within the Hubble horizon, making the analysis even more complex.

For example, although dark matter does not interact with baryon matter directly, it generates potential wells into which the plasma photons-baryons will fall. The desirable thing is to be able to understand an observable quantity by separating the part that comes from the primordial perturbations (evaluated at the moment τ_{hc} of the departure from the inflation horizon) and another that comes only from the evolution of the perturbations from when they re-enters the horizon until it is observed. In a schematic way, we will find that an observable quantity $\mathcal{O}$ can be written as

$$\mathcal{O}_k(\tau) = \mathcal{T}_{\mathcal{O}}(k, \tau, \tau_{hc})\mathcal{R}_k(\tau_{hc}), \tag{3.98}$$

where $\mathcal{T}_{\mathcal{O}}$ is a transfer function.

Next we will explain how this reasoning is applied to the calculation of the CMB angular spectrum and to the matter power spectrum.

3.8.1. CMB anisotropies

As we discussed in section 2.3, in the early stages of the universe, photons were coupled with baryonic matter. Then, as the temperature drops, they travel freely, becoming the photons of the CMB. But, continuing the discussion in this chapter, curvature perturbations also generate density perturbations in this primordial plasma and thus also left their mark on the anisotropies of the CMB.

The common way of making a map of the anisotropy in the temperature of the CMB, is written from a decomposition into spherical harmonics:

$$a_{lm}^{\text{obs}} = \int d\Omega \left(\frac{\Delta T}{T_{\text{CMB}}}(\hat{n}) \right)_{\text{obs}} Y_{lm}(\hat{n}), \tag{3.99}$$

where $\hat{n}$ is the unit vector in the observation direction. Moreover, if we consider that we are doing statistics with an isotropic background spectrum, we can combine the moments to have the angular spectrum (invariant before rotations):

$$C_\ell^{obs} \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \left| a_{lm}^{obs} \right|^2. \tag{3.100}$$

This same angular spectrum can be obtained theoretically[15], Where we can make a definition equivalent to eq. (3.99):

$$a_{lm}^{th} = 4\pi i^\ell \int_{\vec{k}} Y_{lm}(\hat{k}) \mathcal{T}_l(k) \mathcal{R}(k). \tag{3.101}$$

We will obtain the transfer function, in this case, by solving the Boltzmann equations for the photons temperature perturbations $\Theta \equiv \Delta T/T$ and leaving everything in gauge-invariant terms. The transfer function is generally separated into two contributions:

$$\mathcal{T}_\ell(k) = \mathcal{T}_\ell^{\text{LS}} + \mathcal{T}_\ell^{\text{ISW}}, \tag{3.102}$$

where "LS", which comes from Last Scattering, is the contribution of the physics prior to the last scattering in the transfer function. In contrast, the second contribution is called "Integrated Sach Wolfe effect" (ISW), which considers the effects of photons passing through gravitational potential wells after decoupling from matter[16], each part can be written as:

$$\mathcal{T}_\ell^{LS}(k)\mathcal{R}(\vec{k}) = j_\ell(kr_L)\left[\Theta_{0,GI}^\tau(\vec{k}, t_L) + \Phi_{GI}(\vec{k}, t_L) + \frac{1}{4}\Theta_2^\tau(\vec{k}, t_L)\right] \tag{3.103}$$

$$+ 3j_\ell'(kr_L)\Theta_{1,GI}^\tau(\vec{k}, t_L) + \frac{3}{4}j_\ell''(kr_L)\Theta_2^\tau(\vec{k}, t_L), \tag{3.104}$$

$$\mathcal{T}_\ell^{\text{ISW}}(k)\mathcal{R}(\vec{k}) = \int_{t_L}^{t_0} dt' j_\ell(kr(t'))\left(\dot{\Phi}_{GI}(\vec{k}, t') + \dot{\Psi}_{GI}(\vec{k}, t')\right). \tag{3.105}$$

In the previous expressions, j_l are the spherical Bessel functions, t_L and r_L are the physical time and the comoving distance at the time of the last scattering. Additionally, $\Theta_{\ell,GI}^\tau$ are the multipole expansions of the temperature perturbation expressed in its gauge-invariant form[17]. The superscript τ tells us that we are considering the effect of photon scattering for

[15] We are going to state only the relevant results, see reference [34] for the complete derivation.

[16] This effect is generally considered less relevant than the contribution of the effects that are before the last scattering since potential wells don't change much over time [35]

[17] It can be shown that the monopole $\ell = 0$, the dipole $\ell = 1$ and the quadrupole $\ell = 2$ terms of the expansion are proportional to the perturbation in the density $\delta\rho$, the velocity perturbation δu, and the anisotropic stress tensor π^S respectively. This is why we can write them in a Gauge invariant form following expressions equivalent to the equations (3.40, 3.41), while the quadrupole is automatically Gauge invariant.

the last time before free travel:

$$\Theta^{\tau}_{\ell,GI}(\vec{k}, t_L) \equiv \int_{t_{ini}}^{\infty} dt \rho_{\text{scatt}}(t) \Theta_{\ell,GI}(\vec{k}, t),$$

with $\rho_{\text{scatt}}(t)$ is the probability density for the scattering between a photon and an electron by Thomson scattering. Finally Φ_{GI} and Ψ_{GI} are the Gauge invariant potentials defined in (3.38) and (3.39).

With all this in mind, the theoretical angular spectrum will be

$$C_\ell^{th} = 4\pi \int \frac{dk}{k} |\mathcal{T}_\ell(k)|^2 \Delta_{\mathcal{R}}(k), \tag{3.106}$$

with $\Delta_{\mathcal{R}}(k)$ is the dimensionless primordial power spectrum, defined in the equation (3.60).

3.8.2. Matter Power Spectrum

In an oversimplified way, the large-scale structure is formed from the accumulation of matter (mainly dark matter) in gravitational potentials wells. Therefore, by theoretically studying the evolution of perturbations in the dark matter density, we could connect observations of galaxies and their structure with primordial physics. By convention, the dark matter power spectrum is related to the primordial power spectrum as

$$P_\delta(k, \tau) = \frac{4}{25} \left(\frac{k}{aH} \right)^4 \mathcal{T}_\delta^2(k, \tau) P_{\mathcal{R}}. \tag{3.107}$$

Contrary to the angular spectrum of the CMB, one can intuit the behavior of the transfer function for dark matter $\mathcal{T}_\delta(k)$ qualitatively from the general physics of matter perturbations as it enters the horizon. A characteristic of the radiation dominated era is the existence of radiation pressure, a pressure that prevents the growth of perturbations, so that in this period, the density contrast for dark matter $\delta_m \equiv \delta\rho_m/\bar{\rho}_m$ only grows logarithmically $\delta \sim \ln a$. Then, during the period dominated by matter, this radiation pressure can be neglected, so the growth of matter perturbations and gravitational collapse operate more efficiently. Hence, the density contrast grows with the factor of scale $\delta_m \sim a$ [50]. Putting these ideas together, the transfer function for matter can be approximated to

$$\mathcal{T}_\delta(k) \approx \begin{cases} 1 & k < k_{eq} \\ \left(\frac{k_{eq}}{k} \right)^2 & k > k_{eq}. \end{cases} \tag{3.108}$$

This expression is too simple to be able to do a deeper analysis or a more predictive study on the large-scale structure. An analytical expression that approximates this transfer function in a much better way was derived by Bardeen et al. [51]:

$$\mathcal{T}_\delta(q) = \frac{\ln(1 + 2.34q)}{2.34q} \left(1 + 3.89q + (1.61q)^2 + (5.46q)^3 + (6.71q)^2 \right)^{-1/4}, \tag{3.109}$$

where

$$q = \frac{k}{\Gamma h} Mpc^{-1}, \qquad \Gamma \equiv \Omega h \exp\left(-\Omega_b - \sqrt{2h}\Omega_b/\Omega \right). \tag{3.110}$$

More exact transfer function can be derived numerically with CAMB or CMBFAST for example [52].

Chapter 4

Beyond the Basic Principles

Although the evidence that connects the primordial perturbations physics with the universe that we can observe today seems quite satisfactory, as we've repeatedly said before, they don't explain the fundamental physics behind inflation. Let's not forget that single-field inflation is a model that was initially thought of as the simplest that generated an accelerated expansion for the universe. Furthermore, as we have seen throughout chapter 3, the first-order study of the scalar perturbations of the inflaton (interpreted as quantum fluctuations of the vacuum) generate conditions so that $\mathcal{R}$, in the superhorizon case, corresponds to the assumption (and observations) that the initial conditions for the perturbations are adiabatic and obey a scale-invariant gaussian statistic. This chapter will focus on these last points and discuss how to connect possible departures to these principles with much more detailed primordial physics.

4.1. Isocurvature Perturbations

Until now, we have focused on the study of the adiabatic modes of perturbations. That is, they obey the equation (3.54) and are well justified in the homogeneity of the universe. Moreover, we can understand adiabaticity from the fact that the relative number between components of the universe varies uniformly between all species of the universe. With this idea in mind, we are going to define the isocurvature (or entropic) modes as the variation of the relative quantity between the component i of the universe, with respect to radiation $\rho_r = \rho_\nu + \rho_\gamma$ [53]:

$$S_i = \frac{\delta(n_i/n_r)}{n_i/n_r}. \tag{4.1}$$

So, for each component of the universe model (ΛCDM) we will have a different isocurvature mode, where the previous relation translates to:

$$S_{DM} = \delta_{DM} - \frac{3}{4}\delta_r, \tag{4.2}$$

$$S_B = \delta_B - \frac{3}{4}\delta_r, \tag{4.3}$$

$$S_\nu = \frac{3}{4}\delta_\nu - \frac{3}{4}\delta_r. \tag{4.4}$$

In general, the evolution of perturbations is studied considering an isocurvature mode different than zero, to later see its effects on the different observables of the universe [54].

Then, it is necessary to ask the question if these isocurvature modes affect our understanding of the evolution of adiabatic modes, mainly if they have an influence on the connection that we can make between inflationary physics and primordial perturbations from $\mathcal{R}$. It can be shown that a general isocurvature mode $\mathcal{S}$ (we omit the index i):

$$\mathcal{S} \equiv H\left(\frac{\delta p}{\dot{\bar{p}}} - \frac{\delta \rho}{\dot{\bar{\rho}}}\right) \tag{4.5}$$

affects the temporal variation of the curvature perturbation (in the superhorizon limit) $\mathcal{R}$ (equation (3.56))[18][55] as:

$$\dot{\mathcal{R}} \approx -3H\frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\mathcal{S}. \tag{4.6}$$

As we did with the curvature perturbation $\mathcal{R}$, to connect the isocurvature modes with the observables, in the first instance, we need the two-point correlation functions for $\mathcal{S}$ and the correlation between adiabatic and isocurvature mode:

$$\langle \mathcal{S}(k)\mathcal{S}(k')\rangle = (2\pi)^2\delta^{(3)}\left(\vec{k}+\vec{k}'\right)P_{\mathcal{S}}(k), \tag{4.7}$$

$$\langle \mathcal{S}(k)\mathcal{R}(k')\rangle = (2\pi)^2\delta^{(3)}\left(\vec{k}+\vec{k}'\right)P_{\mathcal{S}\mathcal{R}}(k). \tag{4.8}$$

The quantity that is relevant to us to notice the presence of the isocurvature modes (or the adiabaticity of our initial conditions) is the ratio between the power spectra $P_{\mathcal{S}}/P_{\mathcal{R}}$ through the following α parameter:

$$\frac{\alpha}{1-\alpha} \equiv \frac{P_{\mathcal{S}}}{P_{\mathcal{R}}}. \tag{4.9}$$

Without going into much detail, the observational constraints are divided based on how correlated the modes are, this is quantified with a parameter defined as

$$\beta \equiv \frac{P_{\mathcal{S}\mathcal{R}}}{\sqrt{P_{\mathcal{R}}P_{\mathcal{S}}}}. \tag{4.10}$$

Current constraints (with 95 % of confidence level) for dark matter isocurvature modes (4.2) are $\alpha_0 < 0.067$ in case that the modes are not correlated ($\beta = 0$) and $\alpha_{-1} < 0.0037$ in the fully anti-correlated case ($\beta = -1$) [56][19]

What will be relevant for this work is how isocurvature modes are connected with an inflationary theory. In particular, if the eventual observation of these modes would give us more information about the fundamental physics behind the inflationary period.

It can be shown by studying first-order cosmological perturbations that we will always have two independent adiabatic solutions (i.e. obey the relations of equations (3.54)) for the scalar modes [62]. In the case of Single Field Inflation, we have only two independent solutions for scalar perturbations, a solution that decays with time and another that remains

[18] Remember that the gauge-invariant quantities ζ and $\mathcal{R}$ are equal in the superhorizon limit (equation 3.57)
[19] Check [57–61] for more details on the constraints for isocurvature modes both for CDM and for the rest of the components of the universe.

constant outside the horizon. This is the reason why single-field scalar perturbations are automatically adiabatic.

Another way to corroborate this last point is by replacing in the definition of $\mathcal{S}$ (eq. (4.5)) the density and pressure perturbations for inflation perturbations (3.68, 3.69) together with the background quantities (2.50, 2.51), where we have:

$$\mathcal{S} = \frac{2H\frac{dV}{d\phi}}{\dot{\bar{\phi}}^2\left(\ddot{\bar{\phi}}^2 - \left(\frac{dV}{d\phi}\right)^2\right)}\left[\dot{\bar{\phi}}\left(\dot{\delta\phi} - A\dot{\bar{\phi}}\right) - \ddot{\bar{\phi}}\delta\phi\right]. \tag{4.11}$$

The parentheses can be related to the gauge-invariant quantity Ψ_{GI} through the equation (A.9)[20], resulting:

$$\mathcal{S} = \frac{H\frac{dV}{d\phi}}{\dot{\bar{\phi}}^2\left(\ddot{\bar{\phi}}^2 - \left(\frac{dV}{d\phi}\right)^2\right)}\frac{k^2}{a^2}\Psi_{GI}. \tag{4.12}$$

From this last equation, it is evident that for scalar perturbations in single-field inflation, the isocurvature modes are approximately zero at large scales.

To conclude this subsection, let us note that if we add more degrees of freedom to the physical model behind inflation, we will automatically find ourselves in a scenario with isocurvature modes. In fact, the formalism that we will explore in this thesis, which considers inflation generated by multiple fields, contemplates $N-1$ isocurvature modes additional to adiabatic modes (with N the number of fields) [63].

4.2. Non Gaussianities

Returning to the discussion made in section 4.1, we had proposed that a gaussian statistic for the primordial perturbations was a simple assumption and corresponded in a good way (at a first-order approximation) to the observations.

Going into more detail with this idea, as we discussed in sections 3.6 and 3.7, by studying the perturbations in inflation and associating them with a quantum field, one can give (according to an inflationary model) an origin and value to the power spectrum for the primordial curvature perturbation (for example, equation (3.95) for the De Sitter case). Furthermore, the property of gaussianity (eq. (3.61)) is automatically fulfilled by the decomposition of the modes for the curvature perturbation (eq. (3.82)).

In the same way, at the end of section 3.4.1, we mentioned that studying higher-order correlations would give us more information on the physics behind inflation. A good analogy, taken from [19], is to take the case of particle physics, where the two-point correlation functions correspond to free particles in Minkowski spacetime. More interesting elements of the theory, such as new particles and interactions, come out of higher-order correlations. It is not surprising then that the first-order study of perturbations gave us an action (eq. (3.80))

[20] To make the correspondence properly you have to replace $\delta\rho$ and δp of the equations (3.68, 3.69) and remember that we have imposed $M_{pl} = 1/\sqrt{8\pi G} = 1$

very similar to the free particle one. So to break down degeneracies between theories, we must explore these non-gaussianities and their observational implications, in the same way, that particle colliders explore the standard model at higher and higher energies.

4.2.1. Modeling Non-Gaussianity

The most common way to begin to deal with these departures to gaussian statistics is through the next order correlation, that is, the three-point correlation function, defined as

$$\left\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\right\rangle \equiv \mathcal{B}_{\mathcal{R}}(\vec{k}_1,\vec{k}_2,\vec{k}_3). \tag{4.13}$$

Under the same assumptions for which we define the power spectrum for the two-point correlation (homogeneity and isotropy), we can further restrict the shape of the function $\mathcal{B}_{\mathcal{R}}$. In this case, we define the bispectrum through:

$$\mathcal{B}_{\mathcal{R}}(\vec{k}_1,\vec{k}_2,\vec{k}_3) = (2\pi)^3\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)B_{\mathcal{R}}(k_1,k_2,k_3). \tag{4.14}$$

One of the first ways to characterize non-gaussianities is through a correction for the gaussian perturbation, which in real space we define as [64–66]:

$$\mathcal{R}(\vec{x}) = \mathcal{R}_g(\vec{x}) + \frac{3}{5}f_{NL}^{\text{local}}\left(\mathcal{R}_g(\vec{x})^2 - \left\langle \mathcal{R}(\vec{x})^2\right\rangle\right). \tag{4.15}$$

This way of writing the curvature perturbation[21] is nonlinear and local in real space, for this reason it is called local non-gaussianity.

With this definition of the curvature perturbation, the bispectrum (4.14) results us as a function of f_{NL}^{local} and the power spectrum that comes from the two-point correlation function:

$$B_{\mathcal{R}}(k_1,k_2,k_3) = \frac{6}{5}f_{NL}^{\text{local}}[P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_2) + P_{\mathcal{R}}(k_2)P_{\mathcal{R}}(k_3) + P_{\mathcal{R}}(k_3)P_{\mathcal{R}}(k_1)]. \tag{4.16}$$

Without going into much detail about the complexity of the methods, to date there are observational restrictions for the bispectrum of local non-gaussianity. Planck, for example, in its 2018 results set, with a 68% confidence level, $f_{nl}^{\text{local}} = -0.9 \pm 5.1$ [47].

The delta function that appears in the definition (4.14) tells us that in order to measure a nonzero bispectrum, the three momenta must form a triangle. This is relevant since different inflationary models show higher non-gaussianities for certain configurations. The three most relevant configurations are: squeezed triangle ($k_1 \approx k_2 \gg k_3$), equilateral triangle ($k_1 \approx k_2 \approx k_3$) y orthogonal triangle (defined in such a way that it is orthogonal to local and equilateral configurations [67])[22], where the current observational restrictions for these last two configurations (with a 68% confidence level) are $f_{nl}^{\text{equil}} = -26 \pm 47$ y $f_{nl}^{\text{ortho}} = -38 \pm 24$ respectively [47].

[21] The 3/5 factor comes from the fact that, conventionally, local non-gaussianity is defined from the Newtonian potential Φ, which is related to $\mathcal{R}$ with this factor during the epoch dominated by matter.

[22] For more details on these configurations and the inflationary models with which they are associated, review[68–76].

4.2.2. in-in formalism

A very popular method for studying the quantum origin of non-gaussianities is the in-in formalism, initially proposed by [77, 78] and revisited in the last two decades [79]. Without going into many technicalities, for us, the in-in formalism is the counterpart applied to cosmology of the methods used in particle physics to calculate the matrix of transitions between states (S-matrix). In quantum field theory applied to particle physics, the transitions between particles between two states are studied, one $|in\rangle$ evaluated in a long time in the past, with one $\langle out|$ evaluated in the far future:

$$\langle out|S|in\rangle = \langle out(+\infty)|in(-\infty)\rangle \, . \tag{4.17}$$

This is useful in that case, since it can be assumed that in these asymptotic states the particles are free and in a minkowski space.

If we want to bring these ideas into our objective of computing correlations of n-points (non-gaussianities) for quantum perturbations in inflation, we must bear in mind that the boundary conditions are not the same. When studying perturbations in inflation, we have a moment in the universe's evolution in which these perturbations cross the horizon, losing their quality of being quantum perturbations and begin to evolve classically. In other words, we are interested in calculating the n-point correlation function in a given time. In the in-in formalism, we compute n-point correlations through expectation values between two states $|in\rangle$ that correspond to the Bunch-Davies vacuum. For a product of operators $J(\tau)$[23], evaluated at a time τ, its expectation value in this formalism would be:

$$\langle J(\tau)\rangle \equiv \frac{\langle in|J(\tau)|in\rangle}{\langle in|in\rangle}. \tag{4.18}$$

To work the Hamiltonian of the interactions and the temporal evolution of the operators, we use the interaction picture for quantum mechanics. In this picture (see [50] and [19] for more details), the Hamiltonian is separated into a free part and an interaction part

$$H = H_0 + H_{int}. \tag{4.19}$$

Then we can write the expectation value (4.18) using the interaction Hamiltonian as:

$$\langle J(\tau)\rangle = \langle 0|\left(\bar{T}e^{i\int_{-\infty^-}^{\tau} H_{int}(\tau')d\tau'}\right)J(\tau)\left(Te^{-i\int_{-\infty^+}^{\tau} H_{int}(\tau'')d\tau''}\right)|0\rangle \, . \tag{4.20}$$

Where T and $\bar{T}$ are the time-ordering and anti-time-ordering operators and we use the notation $-\infty^{\pm} \equiv -\infty(1 \mp i\epsilon)$. This is the main equation of the in-in formalism, to use it with a given theory, we must identify its interaction Hamiltonian and express $J(\tau)$ perturbatively.

4.2.3. Single-field non-gaussianities

One of the most remarkable works on this topic is the calculation of the three-point correlations for perturbations in single-field inflation by Maldacena [80]. Next, we will state the

[23] For example $J(\tau) = \mathcal{R}_{k_1}(\tau)\mathcal{R}_{k_2}(\tau)\mathcal{R}_{k_3}(\tau)$ to compute the bispectrum

most relevant points of this discussion.

Using the ADM formalism (see appendix C) and the comoving gauge, the action is expanded to the second and third order for the curvature perturbation. With:

$$S_2 = \int dtd^3x\Big[a^3\varepsilon\dot{\zeta}^2 - a\varepsilon(\partial_i\zeta)^2\Big], \tag{4.21}$$

$$S_3 = \int dtd^3x\left[a^3\varepsilon^2\zeta\dot{\zeta}^2 + a\varepsilon^2\zeta(\partial_i\zeta)^2 - 2a\varepsilon\dot{\zeta}(\partial_i\zeta)(\partial_i\chi) + f(\zeta)\frac{\delta L}{\delta\zeta} + \mathcal{O}(\varepsilon^3)\right] \tag{4.22}$$

where

$$\chi = a^2\varepsilon\partial^{-2}\dot{\zeta}, \tag{4.23}$$

$$\frac{\delta L}{\delta\zeta} = 2a\left(\frac{d}{dt}\partial^2\chi + H\partial^2\chi - \varepsilon\partial^2\zeta\right), \tag{4.24}$$

$$f(\zeta) = \frac{\eta}{4}\zeta^2 + \text{terms with derivatives on } \zeta. \tag{4.25}$$

Where ∂^{-2} is the inverse Laplacian and $\delta L/\delta\zeta$ is the variation of the quadratic Lagrangian with respect to ζ. We study the quadratic action in section 3.6, from which we obtain the power spectrum presented in the equation (3.96).

If we want to calculate the three-point correlation function using the in-in formalism, we must identify the interaction Hamiltonian in this case as $H_I = -L_3$. It can be shown that we can get rid of the term proportional to $f(\zeta)$ by redefining the field $\zeta \to \zeta_n + f(\zeta_n)$. This redefinition adds an extra term in the three-point correlation that we must take into account:

$$\langle\zeta(x_1)\zeta(x_2)\zeta(x_3)\rangle = \langle\zeta_n(x_1)\zeta_n(x_2)\zeta_n(x_3)\rangle + \frac{\eta}{2}(\langle\zeta(x_1)\zeta(k_2)\rangle\langle\zeta(k_1)\zeta(k_3)\rangle + \text{cyclic}) + ... \tag{4.26}$$

With all this, we can use the equation (4.20) to calculate the three-point correlation function. However, to do this, we must do a very long calculation, and it does not add much to our discussion in this section. On the other hand, we can estimate the order of magnitude for the bispectrum. For this, first of all, let's note that we can expand the exponential functions in the equation (4.20) up to first order and study the leading term:

$$\left\langle\zeta_n^3\right\rangle = -i\,\langle 0|\int_{t_0}^{t}[\zeta_n(k_1)\zeta_n(k_2)\zeta_n(k_3), H_I]|0\rangle\,. \tag{4.27}$$

If we study only the first term of the interaction hamiltonian, that is

$$\int dtH_I(t) \supset -\int dx^3d\tau a^2\varepsilon^2\zeta\zeta'^2. \tag{4.28}$$

Using the fact that $a \propto H^{-1}$ and $\zeta \propto \zeta' \propto \Delta_\zeta \sim H/\sqrt{\varepsilon}$, we can do the following rough estimation:

$$\left\langle\zeta^3\right\rangle = -i\int d\tau\left\langle\left[\zeta^3, H_I\right]\right\rangle \propto \frac{H^3}{\varepsilon} \propto \mathcal{O}(\varepsilon)\Delta_\zeta^4 \sim f_{NL}\Delta_\zeta^4. \tag{4.29}$$

This same estimation is valid for the first three terms in the cubic action (4.22), while the term proportional to $f(\zeta)$ contributes an amount proportional to η in non-gaussianity. Putting all this together, we can conclude that for single-field slow-roll inflation, non-gaussianity is suppressed by the same slow-roll parameters:

$$f_{NL} \sim \mathcal{O}(\varepsilon, \eta) \ll 1. \tag{4.30}$$

This amount of f_{NL} is too small to be detected, since even non-linearities in the evolution of the CMB would generate $f_{NL} \sim \mathcal{O}(1)$ for example [81].

4.2.4. Consistency Relation

To close this discussion on non-gaussianities in single-field inflation, we must state the consistency relation. Creminelli and Zaldarriaga [82] proved that if we consider a single-field inflation scenario and no other additional assumptions, the following relation must be fulfilled in the squeezed limit[24]:

$$\lim_{k_1 \to 0} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^3 \delta\Big(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\Big)(1 - n_s) P_\zeta(k_1) P_\zeta(k_3). \tag{4.31}$$

Why this relation is so relevant is for two things:

i) As we mentioned earlier, it does not require further assumptions beyond an inflation characterized by a single degree of freedom. Independent of the shape of its kinetic term, the shape of its potential or the initial empty state.

ii) It tells us that in the squeezed limit the three-point correlation function is suppressed by a factor $(1 - n_s)$ and is zero for a completely scale-invariant power spectrum.

These conditions are so powerful that any detection of non-gaussianities in the squeezed could rule out single-field inflation.

4.3. Power Spectrum non-scale-invariant

In section 3.4, we mentioned the fact that the simplest way to model the primordial power spectrum is with a power law (eq. (3.62)) and that this assumption corresponds quite well to the observations [13]. In particular, these observations give us a power spectrum with a spectral index n_s close to 1, that is, almost scale-invariant. In this section, we will extend the discussion on this topic.

First, what does a scale-invariant power spectrum physically mean? The simple answer to this question refers to the fact that in the Fourier space decomposition of the different modes for the perturbations, all modes have the same amplitude[25]. In the inflationary case, this means that the accelerating expansion of the universe affects all these modes equally. For the

[24] See [83] for a good discussion on the consistency relation and a much more explanatory proof.

[25] In more formal mathematical language, scale invariance is the symmetry under rescaling the system, in the same way as fractals, for example.

case of slow-roll inflation, the spectral index, derived from the following general relationship:

$$n_s - 1 \equiv \frac{d \ln \Delta_{\mathcal{R}}}{d \ln k}, \tag{4.32}$$

can be written as a function of the slow-roll parameters, where the first order is [26]:

$$n_s - 1 = -6\varepsilon + 2\eta. \tag{4.33}$$

With this relationship, we immediately prove that slow-roll inflation produces a nearly scale-invariant power spectrum for the primordial perturbations.

As we also mentioned in section 3.4, assuming a scale-invariant power spectrum precedes knowledge of inflation and was justified mainly in galaxy observations. The latter is a key point since we do not have access to all cosmological scales in these observations. Furthermore, in the observations of the angular spectrum of the CMB, we can make the following relation between the multipole ℓ and the scale as $\ell \sim (10^4 Mpc)k$, where it is important to remember that the maximum resolution of these measurements is approximately up to multipole $\ell \sim 3000$. In other words, the amplitude and the spectral index observed by Planck (eq. (3.63)) [13] are valid only in the range of scales that we consider "large":

$$10^{-4} Mpc^{-1} \leq k \leq 1 Mpc^{-1}, \tag{4.34}$$

while for smaller scales the constraints are much weaker.

Our paradigm on which we settle the connection between inflationary physics and the initial conditions of cosmological perturbations in the fact that as the different scales go out of the horizon due to the accelerated expansion of the Inflating universe, they freeze and later reenter the horizon. For this reason, we can characterize each scale of the primordial perturbations with a characteristic time in which they exit the horizon in inflation. Under this logic, the scales that we observe in the CMB and the large-scale structure can be placed at a time quite far from the end of inflation.

So, if we wanted to explore small scales in the primordial spectrum, we would be working on scales that left the horizon near the end of inflation, a highly non-linear process, where the slow roll assumption for the inflaton does not necessarily apply. The bet we make is that by understanding these final moments of inflation, we would be closer to being able to understand the fundamental physics behind this period. We close this chapter with the question: Is there an observable that allows us to explore small scales?

[26] See appendix D for the derivation

Chapter 5

Primordial Black Holes

As we have mentioned in previous chapters, historically, our main method to observe and study the Universe was through the signals we received from it. Where these signals were limited only to electromagnetic waves in a wide range of frequencies. In other words, we were able to study the objects of the Universe through the light they emitted or how they interacted with it (gravitational lensing, for example).

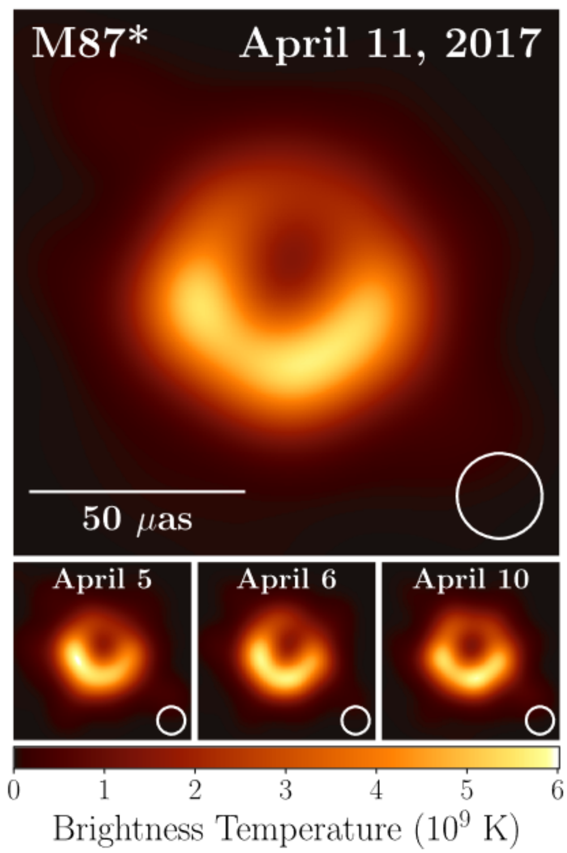


Figure 5.1: First image of a black hole, observed by the event horizon telescope [84].

Under this precept, so-called "dark" objects have always been elusive to direct observations. For example, although the existence of black holes was more than recognized for many years in general knowledge (due to being a fundamental pillar in the theory of general relativity), they had never been observed. Its existence was only deduced through other observations (accretion discs, X-ray emissions, among others). This was until a couple of years ago with the observations of the Event Horizon Telescope, where for the first time, an image of the event horizon of a black hole could be obtained [84] (Fig. 5.1). While on the other hand, dark matter falls into a much more speculative and open terrain. Since, as we have mentioned previously, there is evidence of the existence of this type of matter, but there has not been any direct observation of it.

The astronomical observation paradigm took a significant leap in the last ten years with the launch of gravitational wave detector interferometers such as LIGO and VIRGO (and it will continue to grow with the launching of new observatories of this type). The study of gravitational waves allows us to explore the Universe through a signal whose nature is, in principle, very different from light. As far as we know, the main source of these signals are collisions between massive objects, such as black holes. So the detection of gravitational waves is unequivocal proof of the existence of black holes.

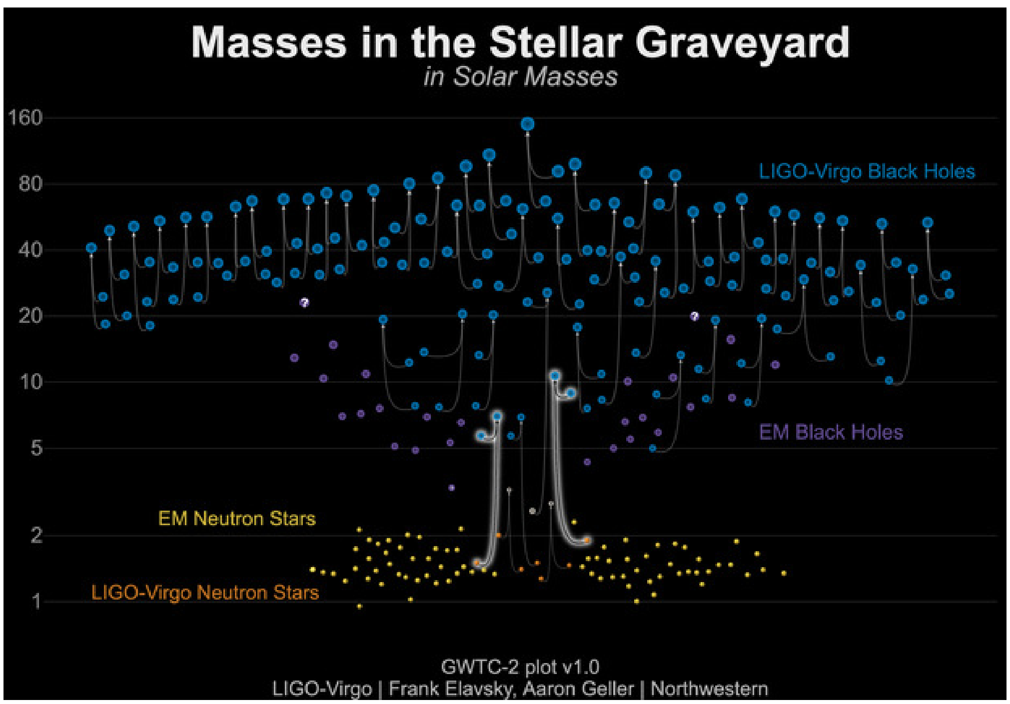


Figure 5.2: Diagram of the masses of the objects associated with the detections made by LIGO-VIRGO, **Source:** LIGO webpage.

Since gravitational-wave observations have been started, the observations have not been exactly how we theoretically expected them. Mainly in two aspects: the number of events detected and the masses of the black holes that originate them (see Fig. 5.2). These two points converge on the assumption that the vast majority of events are caused by collisions of astrophysical black holes, that is, black holes formed by the gravitational collapse of stars. The impressive thing is that the number of gravitational wave events observed is much more than expected. Furthermore, black hole collisions where at least one of them has a mass located in the Mass Gap have been observed (e.g., [85]). The mass gap is a range of masses where a black hole could not be formed by the gravitational collapse of stars. Without going into much detail, it can be estimated that the range is between ~ 2 and $\sim 5M_\odot$ and between ~ 50 and $\sim 150M_\odot$ [86].

These unexpected observations have revived the idea of the existence of another type of black hole, the Primordial Black Holes (PBH) [87], initially proposed in the 60s. These would differ from their astrophysics counterpart in their origin since these would come from the gravitational collapse of large curvature perturbations in the early Universe. For this reason, their denomination of primordial. Beyond a different origin, these black holes cannot be distinguished from astrophysical black holes. Theoretically, primordial black holes have no restrictions on their mass, and there could be a greater abundance than astrophysical black holes, managing to explain anomalous observations of gravitational waves as collisions between PBHs.

In this chapter, we will make a brief historical introduction about primordial black holes and then point out their relevance as a possible observable that helps us understand primordial physics on scales other than those of the CMB. Although the objective of this thesis will not be to study the primordial black holes in-depth, they will be a fundamental motivation for the inflation proposal that we will work on in the following chapters.

5.1. Historical Overview

The idea of the existence of black holes in the early Universe is not new. Recall our discussion on the initial conditions for cosmological perturbations, where we had mentioned that even prior to inflationary theory, there was the assumption of adiabatic, gaussian, and scale-invariant initial conditions. Then a straightforward logic tells us that since there are no good reasons to cut small scales for perturbations[27]. It is possible to have areas where these curvature overdensities are large enough to collapse into a black hole. These ideas were first formulated in the 60s by Zel'Dovich and Novikov [24], and then Hawking and Carr in the 70s delved into the idea [25].

Immediately after the beginning of this idea, it was shown that, although these black holes are generated in a FRLW space-time background in early times of the Universe history, they do not grow or accrete matter [88]. In fact, a rough estimate for the mass of a primordial black hole is the mass contained in the horizon at a given moment, that is, at a certain moment in the history of the Universe, a PBH formed in a said instant will have a mass of the order of:

$$M_{PBH} \sim M_H \sim \frac{c^3 t}{G} \sim 10^{15}\left(\frac{t_i}{10^{-23}s}\right)g. \tag{5.1}$$

Almost at the same time these proposals were being developed, the theory of black hole radiation (i. e., Hawking radiation) arose [89], so one of the first certainties about PBHs was its minimum mass. A primordial black hole to be observed today must have formed with a mass of at least $10^{15}g$. Every lower-mass black hole has been evaporated by Hawking radiation during the history of the universe [90].

[27] Recall that we had argued that we could cut extremely large scales since they mix into the background.

Continuing with this brief chronology, due to their dark nature, in the 1970s, the proposal for primordial black holes as a dark matter candidate immediately emerged. This with studies on how stellar formation would occur in a Universe where PBH replace dark matter [91, 92]. It is important to emphasize that under the standard principles on initial conditions (adiabaticity, gaussianity, and scale invariance), the gravitational collapse to form primordial black holes is a rare phenomenon. So, a priori, the number of primordial black holes would be negligible. Taking this last point into account, in these first proposals on PBH as DM, a fundamental fact arises: to have a PBH production necessary to be comparable with the whole (or a fraction) of dark matter, we must have a perturbation in the matter density of order $\delta \sim \mathcal{O}(1)$. The latter can be contrasted with the measurements of the CMB temperature anisotropies, which are of the order of $\mathcal{O}(10^{-5})$.

In the 80s and 90s, the study of primordial black holes was focused on studying the different theoretical mechanisms that would generate an increase in the amplitude of the curvature perturbation. Mainly different inflationary models such as [93–98, 30, 31].

Also in the 90s, progress was made in the attempt to connect primordial black holes with some observable. At that time, PBHs was classified as a possible Massive Astrophysical Compact Halo Object (MACHO) [99, 100]. During this decade, the MACHO project was carried out, which sought to observe the effect of gravitational microlensing in the milky way. Unfortunately, the results obtained were not conclusive to identify the dark matter around our galaxy as MACHOs [101]. So these observations serve as a restriction for the amount of PBH with a certain mass that could exist (we will go into more detail on observational constraints later).

To close this historical panorama on PBHs, prior to the great commotion that occurred after the detections of gravitational waves in recent years, at the end of the 90s and the first decade of this century, the progress of two essential elements in the study continued of these black holes: the refinement of the numerical value that the critical density must have for gravitational collapse to occur [102–107], and the effect of non-gaussianities on PBH production [108–112]. Although we will detail these two topics later, they converge on understanding the non-linear nature behind the formation of primordial black holes, mainly how sensitive it is to changes in the tails of the primordial perturbations distributions.

5.2. PBH Formation

Although it does not seem so far-fetched to think that large inhomogeneities in curvature collapse in black holes, it is necessary to specify what perturbation is considered large enough to produce a PBH.

A scale that helps us to identify if there is a gravitational collapse is the Jeans length, defined as:

$$L_J = \sqrt{\frac{\pi c_s^2}{G\rho}}, \tag{5.2}$$

with c_s^2 is the sound speed in the medium. Another way to interpret Jeans length is the distance in which a sound wave travels in the collapsing zone. With this in mind, the first intuition to estimate if an overdensity $\delta\rho$ collapses and forms a black hole is if its scale is greater than the scale defined by the Jeans length. Following the work of Carr [90], with the ideas presented above, we can derive a critical density contrast $\delta \equiv \delta\rho/\bar{\rho}$ where a PBH is generated, such as $\delta_c \approx \omega$, where ω is the equation of state for the fluid in which we are considering collapse[28].

Carr also proposed an upper bound for these overdensities, where $\delta < 1$ so that the region that collapsed into a black hole would continue to be connected to our Universe, if the overdensity is greater than unity, it will evolve as an independent universe.

However, Carr's proposed treatment failed to oversimplify the gravitational collapse that generated primordial black holes (e.g., assuming a spherical collapse). This fact is very significant because we're working with a highly non-linear process and dealing with the tails of the distributions. Consequently, a change by a factor of 2 in δ_c generate a large difference in the prediction of the amount of PBH (we will discuss this in the other section).

In the last 20 years and the development of techniques in numerical simulations, this threshold for the overdensity has been studied in much more detail [102–106]. Also in this work [107], a new analytic quantity was derived for δ_c, where:

$$\delta_c = \frac{3(1+\omega)}{5+3\omega}\sin^2\left(\frac{\pi\sqrt{2}}{1+3\omega}\right). \tag{5.3}$$

For the era dominated by radiation (i. e. $\omega = 1/3$) the threshold for the density takes the value of $\delta_c \approx 0.41$[29].

5.3. PBH Mass and Abundance

We had previously proposed the fact that the mass of a primordial black hole will be of the order of the mass contained within the horizon approximately, whose time dependence we write in the equation (5.1). Using this equation, we can connect the horizon mass to the characteristic scale where the perturbation re-enters the horizon [114]:

$$M_H \simeq 17\left(\frac{g}{10.75}\right)^{-1/6}\left(\frac{k}{10^6 Mpc^{-1}}\right)^{-2} M_{\odot}, \tag{5.4}$$

where g is the number of relativistic degrees of freedom.

However, mainly for numerical calculations, the discussion about the mass of PBHs is extended. It can be shown that the mass of black holes will not be precisely the mass of the horizon but will also depend on the amplitude of the perturbation. This extended function

[28] This argument falls apart if we consider the collapse in the matter-dominated epoch where $\omega \approx 0$, i. e., every overdensity, regardless of its scale, would collapse into a black hole. On works of gravitational collapse and generation of PBH in the epoch dominated by matter, check [113].

[29] For more details on the derivation of the equation(5.3), we highly recommend look at [107], in particular the discussion on the different results of numerical simulations.

for the mass of PBHs has the following form [102]:

$$M = \kappa M_H (\delta - \delta_c)^{\gamma}. \tag{5.5}$$

Where the constants κ and γ in the above expression can be obtained numerically. For the epoch dominated by radiation, we have that $\kappa = 3.3$ and $\gamma = 0.36$.

Once we can connect the mass of black holes with a specific scale, we are interested in having a numerical value with which we can characterize the abundance of primordial black holes with a certain mass. This quantity, which we will identify as β, is usually described by the mass fraction of the Universe that collapses into primordial black holes at the time of its formation. Using the Press-Schechter formalism, which was originally used to study gravitational collapse in galaxy formation [115], the function β is calculated as:

$$\beta = 2 \int_{\delta_c}^{\infty} P(\delta) d\delta, \tag{5.6}$$

where $P(\delta)$ is the probability density function of the density contrast and δ_c is the critical density we mentioned in the previous section.

If we consider that the perturbations follow a gaussian statistic, this means that the probability density $P(\delta)$ can be written as:

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right), \tag{5.7}$$

where the variance for the density perturbations $\sigma(k)$, is defined from the power spectrum:

$$\sigma^2(k) = \int_0^{\infty} W^2(kR) P_\delta(k) \frac{dk}{k}. \tag{5.8}$$

In the above expression, $W(kR)$ is a smoothing window function, usually chosen one with a gaussian shape[30], and R is the radius of the horizon at a given time. Let us also remember that the relation between the density contrast and the primordial curvature perturbation is through:

$$\delta(k,t) = \frac{2(1+\omega)}{5+3\omega} \left(\frac{k}{aH}\right)^2 \mathcal{R}(k,t). \tag{5.9}$$

Through the relation between the horizon mass and the scale, mentioned in the equation (5.4) and the expression for the variance (eq. (5.8)), we can deduce that the dependence of the function β with the mass will be through the power spectrum.

Additionally, if we integrate $\beta(M)$ in the entire range of masses allowed for PBHs to exist, we obtain the total abundance of PBHs in the universe:

[30] This choice can be considered arbitrary, the dependence of the mass functions in the selection of the window function has been studied, see [116].

$$\Omega_{PBH} = \int\limits_{M_{min}}^{M_{max}} d\ \ln M \left(\frac{M_{eq}}{M_{PBH}}\right)^{1/2} \beta(M). \tag{5.10}$$

Where M_{eq} is the horizon mass at the time of the matter-radiation equality. The term inside the parentheses gives us the fact that to obtain the total fraction of the Universe that is PBH, we must consider that since its formation, $\beta(M)$ evolves as matter during the epoch dominated by radiation. So we get the total abundance of PBHs by integrating β at the time of equality between radiation and matter.

Generally, the essential quantity for discussing observational constraints on the abundance of PBH in our universe is the mass function $f(M)$. This corresponds to the fraction of dark matter that is formed by PBH of a certain mass M:

$$f(M) = \frac{1}{\Omega_{CDM}} \frac{d\Omega_{PBH}}{d\ \ln M}. \tag{5.11}$$

Later, we will discuss the various observational constraints on this mass function's value.

For extended mass distributions (eq. (5.5)), the function $\beta(M)$ is modified to take into account the collapse of a volume with mass other than that of the horizon:

$$\beta = 2 \int\limits_{\delta_c}^{\infty} \frac{M}{M_H} P(\delta) d\delta = 2 \int\limits_{\delta_c}^{\infty} \kappa(\delta - \delta_c)^{\gamma} P(\delta) d\delta. \tag{5.12}$$

If we use this equation in conjunction with the relation between δ and mass, obtained by inverting the equation (5.5), we have an expression for a mass function that depends on the mass of the PBH [117]:

$$f(M) = \frac{1}{\Omega_{CDM}} \int\limits_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi\sigma^2(M_H)}} \exp\left(-\frac{\left(\mu^{1/\gamma} + \delta_c\right)}{2\sigma^2(M_H)}\right) \frac{M}{\gamma M_H} \mu^{1/\gamma} \sqrt{\frac{M_{eq}}{M_H}} d\ \ln M_H, \tag{5.13}$$

where $\mu \equiv \frac{M}{\kappa M_H}$. With this expression above, the mass function for PBHs can be calculated from the Power Spectrum alone. The latter will be very relevant since, in section 5.5, we will discuss the various observational constraints that exist for this function, so we can connect these restrictions with the parameters that characterize the inflationary models that originate the primordial perturbations.

5.4. Primordial Power Spectrum Constraints For PBH

In the previous section, we mentioned that the fraction of the Universe that collapses into primordial black holes β strongly depends on the variance of the primordial perturbations. Now we will clarify this idea, if we consider a gaussian probability density for the perturbations (i.e., they follow the given form in the equation (5.7)), we can analytically integrate the function β for the case in which the mass of the black holes is the same as the mass of the horizon $M_{PBH} = M_H$ (monochromatic case):

$$\beta(M_H) \approx \text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma(M_H)}\right). \tag{5.14}$$

With this we can do the following mental exercise to understand the impact of the variance on the value of the fraction β (extracted from [118]). If we assume that the power spectrum that is valid for large scales governs all cosmological scales, the amplitude of this power spectrum is $A_{\mathcal{R}} = 2.1 \times 10^{-9}$ [13]. Making a rough estimation where we consider $\delta_c = 0.5$ and $\sigma^2 = A_{\mathcal{R}}$, the function β would be approximately $\beta \approx \text{erfc}(7000) \approx \frac{1}{7000}\exp(-7000)$, which is an absolute negligible amount. Following the logic presented in [118], for all dark matter to be primordial black holes with mass $M_{PBH} \sim M_{\odot}$, the fraction must be approximately $\beta \sim 10^{-8}$, this necessarily implies that the variance must be of the order of $\sigma \sim 0.1$. For the variance of the primordial perturbations to reach this value, the power spectrum must be of the order $A_{\mathcal{R}} \sim 0.01$, almost 7 orders of magnitude above what is observed on large scales.

Beyond the scales in which we observe the CMB, where the primordial power spectrum is well restricted, there are other observational constraints for smaller scales. Still, as we mentioned earlier, these are not as strict. Recalling what was proposed at the end of section 4.3, the fact that we do not have many restrictions for the power spectrum at small scales means that we do not know precisely the physics behind the end of inflation, and the observation (direct or indirect) of primordial black holes would provide us with valuable information about this period. The figure (5.3), extracted from [118] shows the current constraints for the power spectrum, where the solid areas are constraints based on observations, while the dotted lines are potentially measured constraints in the future.

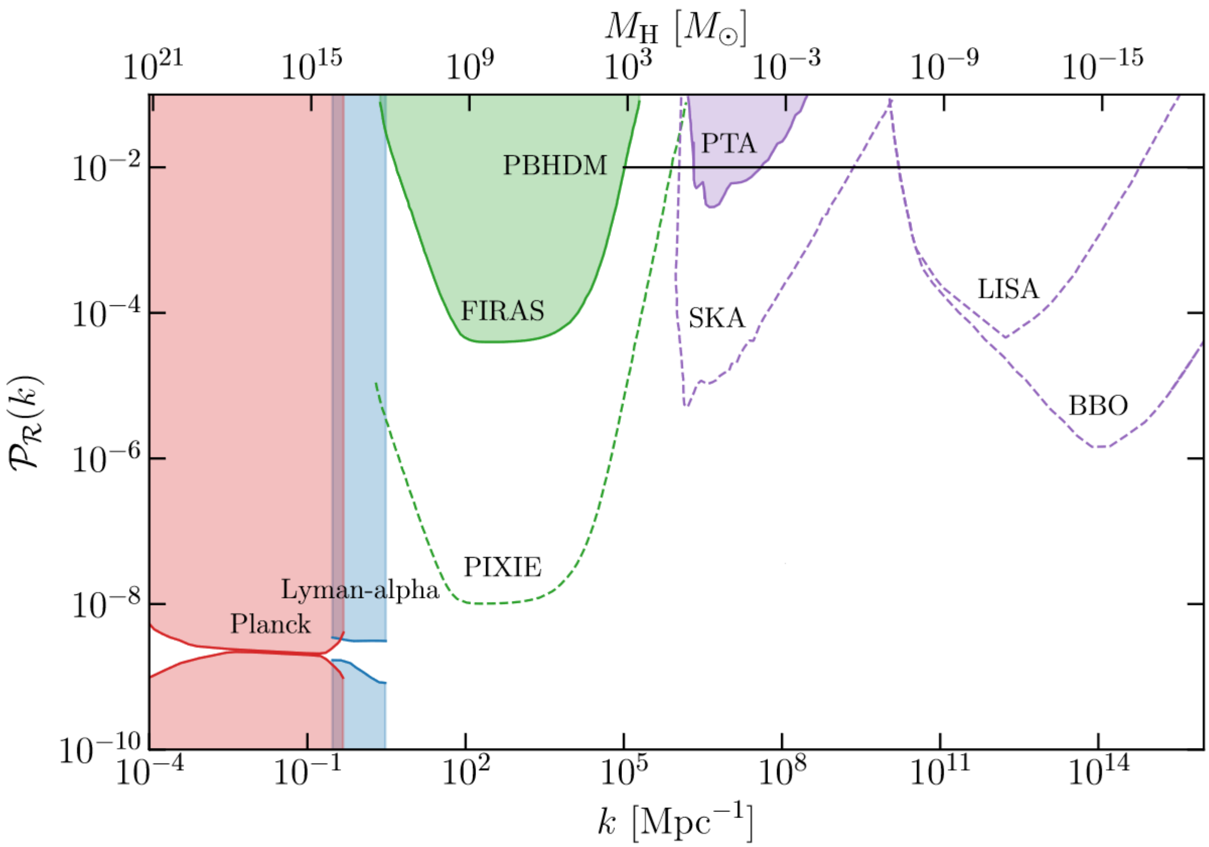


Figure 5.3: Constraints on the primordial power spectrum $P_\mathcal{R}(k)$, extracted from [118]. The red, blue, green and magenta constraints correspond to the angular spectrum observations of the CMB [13], the Lyman - α forest [119], CMB spectral distortions [120, 121] and pulsar timing array limits on gravitational waves [122] respectively. In dotted green and magenta lines are the potential constraints from future PIXIE-like spectral distortion experiment [123] and limits in gravitational waves (by SKA, LISA and BBO) [124, 123] respectively. The black line corresponds to the amplitude necessary hypothetically for a significant amount of dark matter to be primordial black holes.

5.5. Constraints on the Mass Function for PBH

Previously we emphasized the fact that black holes radiate, they lose mass over time, so since the existence of primordial black holes was proposed, it was common sense to restrict their mass to $M_{PBH} > 10^{15} g$ so they could be seen today. Additionally, PBHs cover a relatively wide range of possible masses, and these have not yet been observed with a high enough degree of certainty. This is why it is natural to use the observations we have today to restrict the possible existence of primordial black holes of specific masses. The logic is quite simple: if there were a fraction f^* of PBH with mass M^*, this would have an impact on a particular observation X, as this does not happen, we must restrict the existence of PBHs of mass M^* in such a way that they do not affect said observation.

At first, to make these restrictions for the PBH fraction, a monochromatic formation was assumed and in recent years the discussion for extended mass functions [125] has been developed. Conventionally an alternative form of the mass function is used than the one we write in the equation (5.11), where $f_{PBH}(M) \equiv \Omega_{PBH}/\Omega_{DM}$.

Currently, there are many observations, so the list of constraints for the fraction $f_{PBH}(M)$ (i. e. PBH as DM) is quite broad, and this discussion can get out of hand very easily. For up-to-date state-of-the-art on these constraints, we highly recommend checking out [27]. To have a general notion, following what is proposed by [118], we can group and order the different constraints to the mass function according to an increasing order in the mass of the PBH[31]:

- **Evaporation**: They correspond to the evaporation effects of PBHs by Hawking radiation.
- **Interactions with stars**: This restriction is based on observations of stars and possible interaction with PBHs. If a PBH interacts with a star, it would lose energy or could even destroy it, and these effects should be captured.
- **Gravitational Lensin**g: These are restrictions associated with gravitational lensing between PBH and background objects. Depending on the size of the black holes, we can separate these restrictions in the effects in microlensing, mililensing, or femtolensing.
- **Gravitational Waves**: Possible observations of gravitational waves from binary systems of solar mass PBH-DM are taken into account.
- **Dynamical Effects**: These correspond to the effects that PBHs would generate with astrophysical systems by gravitational interactions, such as dwarf galaxies or wide binaries.
- **Accretion**: The effects of gas accretion by PBH in the early Universe are taken into account. This generates radiation that could have effects on the anisotropies of the CMB. Or also the impact of interstellar gas accretion in primordial black holes, which could generate X-ray and radio emissions.
- **Large Scale Structure**: It takes into account the fact that if PBHs are a relevant fraction of dark matter, the Poisson fluctuations in number density will increase in the matter power spectrum at small scales.

The Figure (5.4), extracted from [118], shows the different restrictions for the fraction of DM in the form of PBH that we mentioned earlier. At first glance, it seems that there is not much space to satisfy the idea that all dark matter exists in the form of primordial black holes (i.e., $f_{PBH} = 1$) [32]. We maintain an optimistic mentality that allows us to say that if we can understand even a ten percent of the content of dark matter, it is a huge leap in our understanding of the components of the Universe.

[31] For more details, check out [118] and the references therein

[32] This is for the monochromatic case. In the case of PBH as DM with a wide mass range, there is a greater chance of satisfy this condition.

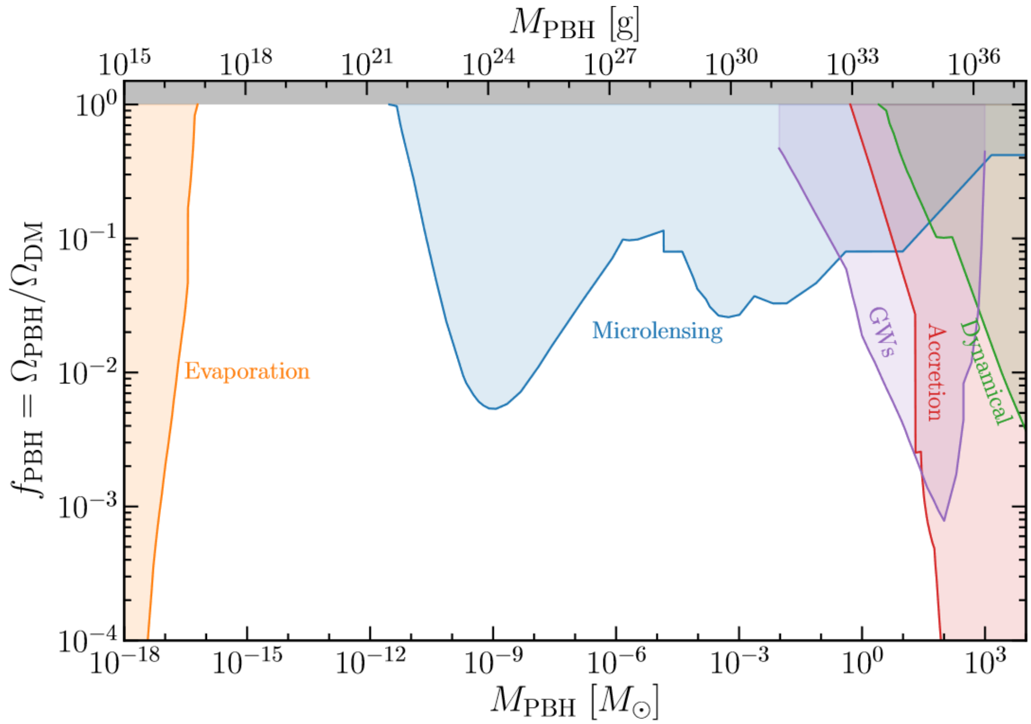


Figure 5.4: Constraints for f_{PBH}, the fraction of dark matter in the form of PBH. Due to PBH evaporation, microlensing, gravitational waves, accretion, and dynamical constraints. Each region is a mantle that unites different observational constraints of the same type. Figure from [118].

Chapter 6

Multi-field Inflation and The Enhancement of the Primordial Power Spectrum

This chapter will be based on the article published in Physical Review Letters, entitled "Seeding primordial black holes in multifield inflation" written by G. Palma, S. Sypsas and the author of this thesis [126].

As we mentioned in the previous chapter, to give an explanation to the anomalous number of gravitational wave events, or solve a part of the dark matter mystery, we need a significant amount of primordial black holes in our universe. This require a physical mechanism that can generate an amplification of the power spectrum for the primordial curvature Δ_ζ for small scales ($k \geq 10^8\ Mpc^{-1}$) of the order of 10^7 with respect to the power spectrum in CMB scales.

This amplification can be achieved from various models, such as: single-field models with special potentials [127–133]; single field models with resonant backgrounds [134, 135]; models with light spectator fields [100, 136–138]; models where the inflaton is coupled with gauge fields [139, 140]; models with resonant instabilities during the preheating inflation's decay [141–143]. In our work we are interested in studying the amplification of the primordial power spectrum $\Delta_\zeta(k)$ in the multifield inflation paradigm [29–31, 144, 145].

In UV complete systems[33], such as supergravity and string theory, these provide us models with a variety of fields that can be mapped into multidimensional target spaces with curved geometries [146, 147]. The effective theory[34] of these models, which is valid during inflation, includes potential substantial interactions between the curvature perturbation ζ and other perturbations (isocurvature [35]) [150–153], an idea that has recently gained relevance.

[33] Where ultraviolet divergences do not occur.
[34] Review [148, 149] for a good introduction on the subject.
[35] Remember our discussion in section 4.1

6.1. Multifield Inflation

In our case, we are going to focus on the particular case of multifield inflation with two fields. The general action that we consider has the form:

$$S = S_{EH} - \int d^4x\sqrt{-g}\Big[\frac{1}{2}\gamma_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + V(\phi)\Big], \tag{6.1}$$

where S_{EH} is the Einstein-Hilbert action, defined in the equation (2.47), from the space-time metric $g_{\mu\nu}$ with determinant g. Furthermore, γ_{ab} is the metric that characterizes the geometry of the space defined by the fields $\phi^a = (\phi^1, \phi^2)$.

If we want to describe the background equations for a flat spacetime described by this action, we consider the FLRW metric proposed in equation (2.7) with $K = 0$:

$$ds^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j. \tag{6.2}$$

If we vary the action (6.1) with respect to the fields, we will have the following equation of motion:

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi^a + \Gamma^a_{bc}g^{\mu\nu}\partial_\mu\phi^b\partial_\nu\phi^c - \gamma^{ab}\frac{\partial V}{\partial\phi^b} = 0, \tag{6.3}$$

where γ^{ab} is the inverse metric of γ_{ab} and

$$\Gamma^a_{bc} = \frac{1}{2}\gamma^{ad}\left(\frac{\partial\gamma_{dc}}{\partial\phi^b} + \frac{\partial\gamma_{db}}{\partial\phi^c} - \frac{\partial\gamma_{bc}}{\partial\phi^d}\right) \tag{6.4}$$

are the Christoffel symbols for the field space. Then follows the fact that the background fields only depend on time, so the scalar fields $\phi^a_0(t)$ satisfy:

$$D_t\dot{\phi}^a_0 + 3H\dot{\phi}^a_0 + \gamma^{ab}V_b = 0, \tag{6.5}$$

where $V_a \equiv \partial V/\partial\phi^a$. Additionally we define the covariant derivative D_t through the action in a vector A^a, where

$$D_tA^a = \dot{A}^a + \Gamma^a_{bc}\dot{\phi}^b_0 A^c. \tag{6.6}$$

To solve the background equation for the fields (6.5) we need to include the Friedmann equation, which in this case is:

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2_0 + V\right), \tag{6.7}$$

where $\dot{\phi}^2_0 \equiv \gamma_{ab}\dot{\phi}^a_0\dot{\phi}^b_0$.

With appropriate initial conditions, this two-field system follows a path ϕ^a_0 with tangent and normal unit vectors defined as

$$T^a \equiv \frac{\dot{\phi}^a_0}{\dot{\phi}_0}, \qquad N^a \equiv -\frac{1}{\Omega}D_tT^a, \tag{6.8}$$

where $\Omega \equiv -N_aD_tT^a$ is the angular velocity with which the path bends.

As we said in chapter 2, in order for us to have an inflationary period that solves the Big Bang problems, we must impose that H remain constant (i. e. $\varepsilon \ll 1$). For simplicity, we are going to assume that ε can be considered as a constant. However, we will allow the angular velocity Ω to depend on time. That is a situation where the inflationary trajectory in the field space experiences turns without significantly affecting the accelerated expansion of the universe[36][37].

By separating the system into a background part and a perturbation, we can use the definitions of unit vectors (equation (6.8)) to separate the perturbation into two parts:

$$\phi^a(\vec{x},t) = \phi_0^a + T^a(t)\varphi(\vec{x},t) + N^a(t)\psi(\vec{x},t), \tag{6.9}$$

where φ corresponds to the adiabatic perturbation, while ψ is the isocurvature perturbation [63, 155].

The objective that follows is to obtain the Lagrangian for the perturbations. For this, we will do a procedure analogous to the one used in appendix C to derive the equation (3.80). In the co-moving gauge ($\varphi = 0$), using the ADM formalism, we define the primordial curvature perturbation ζ through the perturbed metric as:

$$ds^2 = -\mathcal{N} dt^2 + a^2 e^{2\zeta}\delta_{ij}\Big(dx^i + \mathcal{N}^i dt\Big)\Big(dx^j + \mathcal{N}^j dt\Big), \tag{6.10}$$

where $\mathcal{N}$ and $\mathcal{N}^i$ are the lapse and the shift functions. Putting these definitions in (6.1) and solving the constraint equations[38], we arrive at a Lagrangian which can be separated in $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{iso}$, with

$$\mathcal{L}_{kin} = \frac{a^3}{2}\Big[(D_t\zeta_c)^2 - \frac{1}{a^2}(\nabla\zeta_c)^2 + \dot{\psi}^2 - \frac{1}{a^2}(\nabla\psi)^2\Big], \tag{6.11}$$

and

$$\mathcal{L}_{iso} = -a^3 U(\psi), \tag{6.12}$$

where $U(\psi)$ is a potential for ψ. In the equation (6.11), $\zeta_c \equiv \sqrt{2\varepsilon}\zeta$ is the canonically normalized version of ζ, and the covariant derivative acting on ζ_c is defined as :

$$D_t\zeta_c \equiv \dot{\zeta}_c - \lambda H \psi, \tag{6.13}$$

with $\lambda \equiv 2\Omega/H$. This quantity, $\lambda(t)$, will be nonzero as long as the trajectory experiences turns, and plays a prominent role in multifield inflation: it couples ζ and ψ to quadratic order in a way that cannot be trivially removed by field redefinition [156]. In subsequent calculations, we will omit self-interactions from ψ by making $U = 0$.

[36] In any case, the results obtained will not depend on these assumptions.
[37] See section 2 of [154] for the discussion about the slow-roll condition in the multifield scenario.
[38] See Appendix E for more details

6.2. Mild mixing between ζ and ψ

In particular, the limit $\lambda \ll 1$, the interaction between ζ and ψ can be analytically understood [157]. At this limit, in the equation (6.11), we can neglect the terms proportional to $\sim \lambda^2$ and it can be divided as $\mathcal{L}_{kin} = \mathcal{L}_{free} + \mathcal{L}_{mix}$, where $\mathcal{L}_{free}$ is

$$\mathcal{L}_{free} = \frac{a^3}{2}\Big[\dot{\zeta}_c^2 - \frac{1}{a^2}(\nabla\zeta_c)^2 + \dot{\psi}^2 - \frac{1}{a^2}(\nabla\psi)^2\Big], \tag{6.14}$$

and $\mathcal{L}_{mix}$ is the interaction part, given by

$$\mathcal{L}_{mix} \equiv -a^3\lambda H\dot{\zeta}_c\psi. \tag{6.15}$$

In this limit, ζ_c and ψ are massless scalar fields, interacting through a mixing term proportional to $\dot{\zeta}_c\psi$. For this reason we can decompose the solutions of the linear equations in Fourier space, derived from (6.14) in the same way as in (3.82), that is

$$\tilde{\zeta}_0^c(\vec{k}, t) = u(k,t)a_\zeta(\vec{k}) + u^*(k,t)a_\zeta^\dagger(-\vec{k}), \tag{6.16}$$

$$\tilde{\psi}_0(\vec{k}, t) = u(k,t)a_\psi(\vec{k}) + u^*(k,t)a_\psi^\dagger(-\vec{k}), \tag{6.17}$$

where $a_{\zeta,\psi}$ and $a^\dagger_{\zeta,\psi}$ are the annihilation and creation operators, and these must comply with the usual commutation relation (eq. (3.84)):

$$[a_a(\vec{k}), a_b^\dagger(\vec{q})] = (2\pi)^3\delta_{ab}\delta^{(3)}(\vec{k}-\vec{q}), \tag{6.18}$$

where δ_{ab} is the Kronecker delta. Also $u(k,t)$ are the mode functions associated with Bunch-Davies initial conditions (eq. (3.91)):

$$u(k,t) = \frac{iH}{\sqrt{2k^3}}[1 + ik\tau(t)]e^{-ik\tau(t)}, \tag{6.19}$$

where $\tau(t)$ follows what is expressed in (2.45). If $\lambda = 0$, the dynamics of ζ is that of a single-field with a solution given by (6.16), the dimensionless power spectrum (defined in the equations (3.59) (3.60)) in this case it will not be different from the one expressed in (3.96), where there is a small correction of the scale invariance from the slow-roll of inflation:

$$\Delta_{\zeta,0} = \frac{H^2}{8\pi^2\varepsilon}. \tag{6.20}$$

On the other hand, if λ is small but nonzero, ψ sources ζ_c via the mixing in eq. (6.15), where the solution for ζ_c will be [157]

$$\tilde{\zeta}_c(\vec{k}) = \tilde{\zeta}_0^c(\vec{k}) + 2\Delta\theta(k)\tilde{\psi}_0(\vec{k}), \tag{6.21}$$

and $\tilde{\psi}(k) = \tilde{\psi}_0(\vec{k})$, where

$$\Delta\theta(k) = \frac{1}{2}\int_{t_{hc}}^{t_{end}} \lambda H dt \tag{6.22}$$

is the total angle swept by a mode that crosses the horizon at a time $t_{hc} = \ln(k/H)/H$. With this solution, the power spectrum Δ_ζ results in [157]:

$$\Delta_\zeta = \Delta_{\zeta,0} \times \Big(1 + 4\Delta\theta^2(k)\Big), \tag{6.23}$$

which is greater than the power spectrum of eq. (6.20) by a factor $1+4\Delta\theta^2(k)$. The equation (6.21) shows us that for a small λ, all superhorizon modes are equally amplified while the turn is taking place, making the power spectrum (eq. (6.23)) have a greater amplitude for long wavelengths, independent of the form of $\lambda(t)$. This scenario is incompatible with a large enhancement of Δ_ζ in small wavelengths (with respect to the CMB scales) that we expect for PBH production, this forces us to consider the regime with $\lambda \gg 1$.

6.3. Strong mixing between ζ and ψ

For the case where the turns are short, we can study the system analytically even if $\lambda \gg 1$. This is considered the fast turn regime [150, 151, 154, 158–166], where $\Omega \gg H$. The question we must ask ourselves is whether this does not compromise the perturbativity of the system? To answer this question, let us note that the origin of λ comes only from the kinetic term in the Lagrangian (eq. (6.1)), that a quadratic order makes the covariant derivative $D_t\zeta_c$ appear. This implies that at higher orders λ appears in the Lagrangian through operators of the order $(D_t\zeta_c)^2$ gravitationally coupled to ζ_c. The condition for the splitting to remain weakly coupled is

$$\frac{\mathcal{L}_\lambda^{(3)}}{\mathcal{L}_\lambda^{(2)}} \sim \frac{\dot{\zeta}_c}{\sqrt{2\varepsilon}H} \ll 1, \tag{6.24}$$

evaluated at horizon crossing[39]. Since $(\dot{\zeta}_c/\sqrt{2\varepsilon}H)^2 \sim \Delta_\zeta$ is satisfied at the crossing of the horizon, the requirement is equivalent to demanding $\Delta_\zeta \ll 1$. So while this is true we can keep the mixing term in the complete kinetic term of the equation (6.11). Then the equations of motion for the fluctuations that result from varying eq. (6.11) are

$$\frac{d}{dt}D_t\tilde{\zeta}_c + 3HD_t\tilde{\zeta}_c + \frac{k^2}{a^2}\tilde{\zeta}_c = 0, \tag{6.25}$$

$$\ddot{\tilde{\psi}} + 3H\dot{\tilde{\psi}} + \frac{k^2}{a^2}\tilde{\psi} + \lambda H D_t\tilde{\zeta}_c = 0. \tag{6.26}$$

To continue, in this work we consider the case where λ consists of a top-hat function[40], of the form

$$\lambda(t) = \lambda_0[\theta(t-t_1) - \theta(t-t_2)], \tag{6.27}$$

with a small width $\delta t \equiv t_2 - t_1 \ll H^{-1}$. This makes the trajectory profile describe a short turn with an angular velocity $\Omega = H\lambda_0/2$ between t_1 and t_2.

Now, if $\delta t \ll H^{-1}$, during this brief period of time we can ignore the friction terms in the equations (6.25) and (6.26), and treat the fluctuations as if they were evolving in a Minkowski

[39] See Appendix F for details.

[40] The choice of this function is mainly based on the fact that it allows us to simplify the resolution of the equations of motion analytically. For numerical solutions, other functions that have the same characteristics (located in time) can be used, such as a gaussian profile for example.

space-time [167].

Before t_1 the system solutions will be exactly those expressed in the equations (6.16), (6.17) and (6.19). Between t_1 and t_2 the solutions, which we will denote as $\tilde{\Phi}^a \equiv \left(\tilde{\zeta}_c, \tilde{\psi}\right)$, turn out to be:

$$\tilde{\Phi}^a(\vec{k},t) = \left(A^a_\pm e^{+i\omega_\pm t} + B^a_\pm e^{-i\omega_\pm t}\right)a_\zeta(\vec{k}) + \left(C^a_\pm e^{+i\omega_\pm t} + D^a_\pm e^{-i\omega_\pm t}\right)a_\psi(\vec{k}) + H.c.(-\vec{k}), \quad (6.28)$$

where $A^a_\pm$, $B^a_\pm$, $C^a_\pm$ and $D^a_\pm$ are amplitudes which satisfy

$$kA^\zeta_\pm = \mp i\omega_\pm A^\psi_\pm, \qquad kB^\zeta_\pm = \pm i\omega_\pm B^\psi_\pm, \qquad kC^\zeta_\pm = \mp i\omega_\pm C^\psi_\pm, \qquad kD^\zeta_\pm = \pm i\omega_\pm D^\psi_\pm, \quad (6.29)$$

and $\omega_\pm$ are the dispersion relations, given by

$$\omega_\pm = \sqrt{k^2 \pm kk_0\lambda_0}, \quad (6.30)$$

in this equation, $k_0 \equiv He^{H(t_1+t_2)/2}$, is the wave number of the modes that cross the horizon during the turn. Finally, the solutions after t_2 are of the form

$$\tilde{\Phi}^a(\vec{k},t) = [E^a u(k,t) + F^a u^*(k,t)]a_\zeta(\vec{k}) + [G^a u(k,t) + H^a u^*(k,t)]a_\psi(\vec{k}) + H.c.(-\vec{k}), \quad (6.31)$$

where $u(k,t)$ is expressed in the equation (6.19). The amplitudes shown in the equations (6.28) and (6.31) can be determined by imposing continuity of $\tilde{\zeta}_c(\vec{k},t)$, $D_t\tilde{\zeta}_c(\vec{k},t)$, $\tilde{\psi}(\vec{k},t)$ and $\dot{\tilde{\psi}}(\vec{k},t)$ at times t_1 and t_2. This is achieved if the following conditions are fulfilled:

$$\tilde{\zeta}_c(\vec{k},t_1^-) = \tilde{\zeta}_c(\vec{k},t_1^+), \quad \dot{\tilde{\zeta}}_c(\vec{k},t_1^-) = D_t\tilde{\zeta}_c(\vec{k},t_1^+), \quad \tilde{\psi}(\vec{k},t_1^-) = \tilde{\psi}(\vec{k},t_1^+), \quad \dot{\tilde{\psi}}(\vec{k},t_1^-) = \dot{\tilde{\psi}}(\vec{k},t_1^+),$$

$$\tilde{\zeta}_c(\vec{k},t_2^-) = \tilde{\zeta}_c(\vec{k},t_2^+), \quad \dot{\tilde{\zeta}}_c(\vec{k},t_2^-) = D_t\tilde{\zeta}_c(\vec{k},t_2^+), \quad \tilde{\psi}(\vec{k},t_2^-) = \tilde{\psi}(\vec{k},t_2^+), \quad \dot{\tilde{\psi}}(\vec{k},t_2^-) = \dot{\tilde{\psi}}(\vec{k},t_2^+), \quad (6.32)$$

where $t_i^\pm \equiv t_i \pm \epsilon$ with $\epsilon \to 0$. As a result, at the end of inflation, $\tilde{\zeta}_c(\vec{k})$ becomes a linear combination of quanta created and destroyed by the operators a_ζ and a_ψ, whose contributions are modulated by high and low frequencies[41]:

$$\begin{aligned}\tilde{\zeta}_c(\vec{k}) = \frac{iH}{\sqrt{2k^3}} e^{2i\frac{k}{k_0}\sinh\left(\frac{\delta N}{2}\right)} \sum_\pm \Bigg\{ \frac{1}{2}\Bigg[\cos\left(\frac{\omega_\pm\delta N}{k_0}\right) - i\frac{k_0^2+k^2+\omega_\pm^2}{2k\omega_\pm}\sin\left(\frac{\omega_\pm\delta N}{k_0}\right) \\ - ie^{2i\frac{k}{k_0}\exp\left(-\frac{\delta N}{2}\right)}\frac{\left(ik_0+k\right)^2-\omega_\pm^2}{2k\omega_\pm}\sin\left(\frac{\omega_\pm\delta N}{k_0}\right)\Bigg]a_\zeta(\vec{k}) \pm \frac{i}{4}\Bigg[-\left(2+\frac{k_0^2}{k^2}\right)\cos\left(\frac{\omega_\pm\delta N}{k_0}\right) \\ + \frac{(k_0+ik)k^2-(k_0-ik)\omega_\pm^2}{k^2\omega_\pm}\sin\left(\frac{\omega_\pm\delta N}{k_0}\right) + e^{2i\frac{k}{k_0}\exp\left(-\frac{\delta N}{2}\right)}\left(\frac{k_0}{k^2}(k_0-2ik)\cos\left(\frac{\omega_\pm\delta N}{k_0}\right)\right) \\ + \frac{(k_0-ik)\left(\omega_\pm^2-k^2\right)}{k^2\omega_\pm}\sin\left(\frac{\omega_\pm\delta N}{k_0}\right)\Bigg]a_\psi(\vec{k})\Bigg\} + H.c.(-\vec{k}), \quad (6.33)\end{aligned}$$

[41] See Appendix G for the complete derivation.

where $\delta N \equiv H\delta t$ is the duration of the turn in e-folds. A similar solution is obtained for $\tilde{\psi}(\vec{k})$. With a WKB approximation, we can follow the same steps used to arrive at the equation (6.33), this should allow us to obtain solutions with more general functions for $\lambda(t)$.

The equation (6.33) is our main result. It shows us how, regardless of the value of λ_0, ζ and ψ are combined after the turn. Another important characteristic in the equation (6.33) is the fact that $\omega_- = \sqrt{k^2 - kk_0\lambda_0}$ becomes imaginary for $k < \lambda_0 k_0$, generating an instability that induces an exponential amplification of $\tilde{\zeta}_c(\vec{k})$. On scales where $k < \lambda_0 k_0$ the fluctuation has an amplitude $\tilde{\zeta}_c(\vec{k}) \propto e^{\sqrt{\lambda_0 k_0 k - k^2}\delta N/k_0}$ with a maximum value at $k_{max} = \lambda_0 k_0/2$. While $2\delta\theta \equiv \lambda_0 \delta N > 1$ instability generates large enhancements in the power spectrum of ζ at the end of inflation.

6.4. PBH from strong multifield mixing

Now we will use the analytical solution obtained in the equation (6.33) to study the origin of primordial black holes due to multifield inflation effects. Let us first notice that, if we are in scales where $k \gg k_0\lambda_0/2$, eq. (6.33) translates into

$$\tilde{\zeta}_{\rm short}(\vec{k}) = ik^{-3/2}\Delta_{\zeta,0}^{1/2} a_\zeta(\vec{k}) + H.c.(-\vec{k}), \tag{6.34}$$

from where the single field power spectrum of the equation (6.20) is recovered. This tells us that modes that are far in the horizon are not affected by spin. On the other hand, in scales where $k \ll k_0\lambda_0/2$ we have

$$\tilde{\zeta}_{\rm long}(\vec{k}) = ik^{-3/2}\Delta_{\zeta,0}^{1/2}\Big[a_\zeta(\vec{k}) + 2\delta\theta a_\psi(\vec{k})\Big] + H.c.(-\vec{k}), \tag{6.35}$$

i.e. $\Delta_\zeta = (1 + 4\delta\theta^2) \times \Delta_{\zeta,0}$, confirming that long wavelengths all receive the same amplification shown previously in the equation (6.23). Finally, for $\delta\theta \sim \mathcal{O}(1)$ or higher, around $k \sim k_0\lambda_0/2$ the curvature fluctuation is dominated by (without the oscillatory phases)

$$\tilde{\zeta}_{\rm bump}(\vec{k}) \sim ik^{-3/2}\Delta_{\zeta,0}^{1/2} e^{\sqrt{\lambda_0 k_0 k - k^2}\delta N/k_0} \times \frac{1}{4}\Big[(1-i)a_\zeta(\vec{k}) - (1+i)a_\psi(\vec{k})\Big] + H.c.(-\vec{k}), \tag{6.36}$$

from where we can conclude that the power spectrum will have a bump centered at $k = k_0\lambda_0/2$ of amplitude $\Delta_{\zeta,0}e^{2\delta\theta/4}$, with a width

$$\Delta N_k \simeq \ln\left[\frac{4\delta\theta^2}{\ln^2(16\delta\theta^2)}\right], \tag{6.37}$$

where $N_k = \ln(k/H)$ is the wave number in units of e-folds. In summary we have

$$\frac{\Delta_\zeta}{\Delta_{\zeta,0}} \sim \begin{cases} 1 + 4\delta\theta^2 & \text{if}\, k \ll k_0\lambda_0/2 \\ \frac{1}{4}e^{2\delta\theta} & \text{if}\, k \sim k_0\lambda_0/2 \\ 1 & \text{if}\, k \gg k_0\lambda_0/2 \end{cases}. \tag{6.38}$$

This behavior is quite characteristic: for large $\delta\theta$, the ratio of the power spectrum at long and short wavelengths determines the bump height. In particular, the increase of Δ_ζ with respect to the long-wavelength value can be predicted as $\sim e^{2\delta\theta}/(4\delta\theta)^2$. This tells us that for us to have an increase in the power spectrum of the order of 10^7, it is enough to have $\delta\theta \sim 4\pi$.

We can check Δ_ζ from the equation (6.33) with the numerical solutions of the equations (6.25) and (6.26). The results are shown in the figure (6.1). As expected, we found a good agreement between the results for the cases with $\delta N = 0.1$ and also for $\delta N = 1$. For the case with $\delta N = 0.1$ a difference is noted in the decay of the power spectrum, this is only because numerical inaccuracies due to the fact of solving differential equations with large numbers.

A characteristic of this mechanism is that this rapid growth of the power spectrum evades the limitations for enhancements in single-fields models [122, 168, 169]. If the proposal by [170, 171] is followed, there must be a dependence of the parameters λ_0 and δN in the computation of the abundance of PBH as a function of mass.

To close this discussion, let us note that the power spectrum Δ_ζ describes a characteristic band structure resulting from the oscillating parts in the equation (6.33). However, it has recently been proven that this band structure does not translate to the mass function [172]. The reason why this occurs is mainly due to the integral used in this type of computation (see for example eq. (5.5)). On the contrary, it has been shown that these oscillations are transferred to the spectrum of primordial gravitational waves generated in the period dominated by radiation [173, 174]. In addition, a non-zero potential for the isocurvature field has recently been worked out, where for a cubic interaction (i. e. $\sim \psi^3$) in the Lagrangian, it has been shown that the bispectrum presents the same type of amplification that we presented in this work[42].

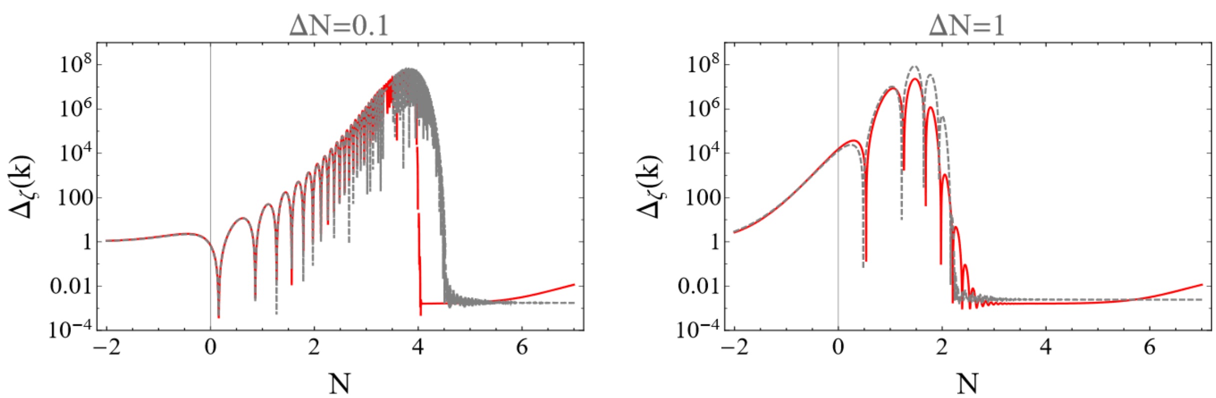


Figure 6.1: Plot of the primordial power spectrum as a function of $N(k) = \ln(k/k_0)$ for $\delta N = 0.1$ and $\lambda_0 = 251.3$ (left panel), and $\delta N = 1$ and $\lambda_0 = 25.1$ (right panel). Solid/dashed lines correspond to numerical/analytical solutions. The spectra is normalized with respect to the values at CMB scales.

[42] This work has not yet been published, thanks to Nicolás Parra for sharing his early results.

Chapter 7

Conclusions

The main objective of this thesis was to connect characteristics of multifield inflation with an observable that has gained importance in the last years: primordial black holes.

From the beginning of this work (chapters 2 and 3), we have made it clear that the most straightforward inflationary theory, single-field slow-roll inflation, helps us to solve the classic problems of the Big Bang theory (the horizon problem and the problem of the flatness for example). Whereas the perturbations in this type of inflation, physically interpreted as quantum fluctuations, solves the origin of the primordial perturbations that justify the origin of the large-scale structure and the anisotropies of the microwave background radiation. Unfortunately, we close the analysis in those chapters by emphasizing that single field inflation is a phenomenological theory that cannot be directly connected to a fundamental physics theory (like the standard model or GR).

In Chapter 4, we delve into the fact that the characteristics that have been observed on primordial perturbations (adiabaticity, gaussianity, and scale invariance) are restricted to various factors. For example, the exploration of non-gaussianities in primordial perturbations falls mainly on the difficulty of studying correlations of order higher than 2. Additionally, the current observations that allow us to reach these conclusions are valid only on scales that we identify as large. These large scales have their origin in perturbations that escaped the horizon at times significantly before the end of inflation, a stage in which their phenomenology was well justified by slow-roll single-field inflation. This is why it is difficult to obtain more information on the physics behind inflation through observations of large-scale perturbations.

Primordial black holes, which was the core of our discussion in Chapter 5, objects that were initially theorized in the 1960s and returned to the eye of the cosmology community in recent years due to observations of gravitational waves, they have the goodness of being an element of the universe whose origin would be related mainly to physics on small scales. In that chapter, we detail PBH formation characteristics, how its abundance could be predicted in such a way that it generates observable effects in the current universe (gravitational wave events, for example), and various observational constraints that we have for the existence of these black holes. A very interesting scenario in which PBH takes a leading role is the one in which they make up a fraction (or all) of the dark matter. It has been shown that for this to occur, there needs to be a jump in the power spectrum of at least seven orders of magnitude on small scales. Therefore, the observation of phenomena in which the existence of PBH is justified would give us valuable information on new physics associated with small scales. These physics could justify the physical nature behind inflation.

In our work, we focused on an inflationary model with multiple fields, whose origin is justified in various UV-complete theories. This inflationary scenario has the property where isocurvature modes can be coupled with adiabatic curvature modes, generating enhancements in their amplitude, providing an ideal scenario for the production of primordial black holes or other signals that require an increase in the amplitude of the small-scale perturbations.

In our model, the λ interaction between these modes is modeled by a top hat function in time, thus modeling a rapid turn in the inflationary path in the field space. Although it was known that these rapid turns in the inflationary trajectory generate features in the power spectrum. We obtained analytical solutions for ζ in the regime of $\lambda \gg 1$. If we recall the discussion at the end of section 6.2, large Δ_ζ enhancements require fast turns, but also require non-trivial geometry in the field space (see discussion in Appendix H). For example, if we consider the growth of Δ_ζ in a canonical multifield inflation scheme due to a turn with a duration δN with Ω constant, this tells us that $\lambda = 2\delta\theta/\delta N$ However, in these canonical models, $\delta\theta < \pi$, so there is a maximum value for the mixing $\lambda_{max} = 2\pi/\delta N$. So, if $\delta N \gg 1$, we would be in the slow turn regime where, as we showed in section 6.2, it does not produce great enhancements. On the other hand, if $\delta N \leq 1$, we fall into the fast turn regime, where it applies a behavior of the form given by the equation (6.38), where we obtained that for an increase of 10^7 in Δ_ζ, we require $\delta\theta \sim 4\pi$, moving away from canonical models.

One of the usual problems of these inflationary models is that they require a high fine-tuning between their parameters in order to generate the desired scenarios. One of the bonanzas that our results have is that the main parameter that produces the increase in the power spectrum is $\delta\theta$, the total angle swept by the trajectory. In this sense, our mechanism, which is exponentially sensitive to $\delta\theta$, gives us a significant enhancement without a greater hierarchy. However, like many other inflation-based proposals, our model does not necessarily give us the range of scales where the enhancement occurs since this depends directly on the moment in which the turn occurs.

Finally, let's remember that for simplicity, we ignore the potential U. A nonzero U could introduce a mass for ψ, modifying the dispersion relations that we express in the equation (6.30) and the amplitudes in (6.28), so the results for the enhancement in the power spectrum will be altered (eq. (6.38)). Another effect is the generation of non-gaussianities, following the guidelines of the references [175–177]. A large λ will induce distortions in the ζ statistic that could change the details in the PBH formation [178–187]. This last point directly impacts our discussion in section 5.3, where for example, we should consider a much more complex form for the probability distribution for δ than that expressed in (5.7). As we mentioned at the end of chapter 6, although it has been studied that the oscillating features are not transferred to the mass function, they are translated into the primordial gravitational wave spectrum generated in the period dominated by radiation. But the dynamics studied in our model occur mainly in the inflationary period, so exploring the spectrum of gravitational waves generated in inflation could generate more significant signals than those generated in the radiation era (see for example [188, 189]).

Bibliography

[1] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik*, vol. 354, pp. 769–822, Jan. 1916.

[2] E. Hubble, "A relation between distance and radial velocity among extra-galactic nebulae," *Proc. Nat. Acad. Sci.*, vol. 15, pp. 168–173, 1929.

[3] A. A. Penzias and R. W. Wilson, "A Measurement of excess antenna temperature at 4080-Mc/s," *Astrophys. J.*, vol. 142, pp. 419–421, 1965.

[4] G. F. Smoot, "COBE observations and results," *AIP Conf. Proc.*, vol. 476, no. 1, pp. 1–10, 1999.

[5] A. Friedmann, "Über die Krümmung des Raumes," *Zeitschrift fur Physik*, vol. 10, pp. 377–386, Jan. 1922.

[6] G. Lemaitre, "A Homogeneous Universe of Constant Mass and Growing Radius Accounting for the Radial Velocity of Extragalactic Nebulae," *Annales Soc. Sci. Bruxelles A*, vol. 47, pp. 49–59, 1927.

[7] H. P. Robertson, "Kinematics and World-Structure," , vol. 82, p. 284, Nov. 1935.

[8] A. G. Walker, "On Milne's Theory of World-Structure," *Proceedings of the London Mathematical Society*, vol. 42, pp. 90–127, Jan. 1937.

[9] G. F. Smoot, C. L. Bennett, A. Kogut, E. L. Wright, J. Aymon, N. W. Boggess, E. S. Cheng, G. de Amici, S. Gulkis, M. G. Hauser, G. Hinshaw, P. D. Jackson, M. Janssen, E. Kaita, T. Kelsall, P. Keegstra, C. Lineweaver, K. Loewenstein, P. Lubin, J. Mather, S. S. Meyer, S. H. Moseley, T. Murdock, L. Rokke, R. F. Silverberg, L. Tenorio, R. Weiss, and D. T. Wilkinson, "Structure in the COBE Differential Microwave Radiometer First-Year Maps," , vol. 396, p. L1, Sept. 1992.

[10] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure.* Cambridge University Press, 2000.

[11] R. K. Sachs and A. M. Wolfe, "Perturbations of a cosmological model and angular variations of the microwave background," *Astrophys. J.*, vol. 147, pp. 73–90, 1967.

[12] S. Dodelson, *Modern cosmology.* 2003.

[13] N. Aghanim *et al.*, "Planck 2018 results. I. Overview and the cosmological legacy of Planck," *Astron. Astrophys.*, vol. 641, p. A1, 2020.

[14] W. Rindler, "Visual horizons in world models," , vol. 116, p. 662, Jan. 1956.

[15] R. Dicke, *Gravitation and the Universe: Robert H. Dicke ... Jayne Lectures for 1969.* Pa., American philosophical Society, 1970.

[16] P. J. E. Peebles, *The large-scale structure of the universe.* 1980.

[17] A. H. Guth, “Inflationary universe: A possible solution to the horizon and flatness problems,” *Phys. Rev. D*, vol. 23, pp. 347–356, Jan 1981.

[18] D. Wands, “Local non-gaussianity from inflation,” *Classical and Quantum Gravity*, vol. 27, p. 124002, May 2010.

[19] X. Chen, “Primordial non-gaussianities from inflation models,” *Advances in Astronomy*, vol. 2010, p. 1–43, 2010.

[20] Y. Wang, “Inflation, cosmic perturbations and non-gaussianities,” *Communications in Theoretical Physics*, vol. 62, p. 109–166, Jul 2014.

[21] S. Renaux-Petel, “Primordial non-gaussianities after planck 2015: An introductory review,” *Comptes Rendus Physique*, vol. 16, p. 969–985, Dec 2015.

[22] B. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, and et al., “Observation of gravitational waves from a binary black hole merger,” *Physical Review Letters*, vol. 116, Feb 2016.

[23] B. P. Abbott, R. Abbott, T. D. Abbott, S. Abraham, F. Acernese, K. Ackley, C. Adams, R. X. Adhikari, V. B. Adya, C. Affeldt, and et al., “Binary black hole population properties inferred from the first and second observing runs of advanced ligo and advanced virgo,” *The Astrophysical Journal*, vol. 882, p. L24, Sep 2019.

[24] Y. B. Zel’dovich and I. D. Novikov, “The Hypothesis of Cores Retarded during Expansion and the Hot Cosmological Model,” , vol. 10, p. 602, Feb. 1967.

[25] S. Hawking, “Gravitationally collapsed objects of very low mass,” *Mon. Not. Roy. Astron. Soc.*, vol. 152, p. 75, 1971.

[26] P. S. Cole and C. T. Byrnes, “Extreme scenarios: the tightest possible constraints on the power spectrum due to primordial black holes,” *Journal of Cosmology and Astroparticle Physics*, vol. 2018, p. 019–019, Feb 2018.

[27] B. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, “Constraints on Primordial Black Holes,” 2 2020.

[28] S. Clesse and J. García-Bellido, “Seven Hints for Primordial Black Hole Dark Matter,” *Phys. Dark Univ.*, vol. 22, pp. 137–146, 2018.

[29] A. S. Sakharov and M. Y. Khlopov, “Cosmological signatures of family symmetry breaking in multicomponent inflation models,” *Phys. Atom. Nucl.*, vol. 56, pp. 412–417, 1993.

[30] L. Randall, M. Soljacic, and A. H. Guth, “Supernatural inflation: Inflation from supersymmetry with no (very) small parameters,” *Nucl. Phys. B*, vol. 472, pp. 377–408, 1996.

[31] J. Garcia-Bellido, A. D. Linde, and D. Wands, “Density perturbations and black hole formation in hybrid inflation,” *Phys. Rev. D*, vol. 54, pp. 6040–6058, 1996.

[32] R. M. Wald, *General Relativity.* Chicago, USA: Chicago Univ. Pr., 1984.

[33] S. M. Carroll, *Spacetime and Geometry.* Cambridge University Press, 7 2019.

[34] S. Weinberg, *Cosmology.* 2008.

[35] S. Dodelson, *Modern Cosmology.* Amsterdam: Academic Press, 2003.

[36] E. A. Milne, "A Newtonian Expanding Universe," *The Quarterly Journal of Mathematics*, vol. os-5, pp. 64–72, 01 1934.

[37] D. Baumann, "Inflation," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, 7 2009.

[38] V. Mukhanov, *Physical Foundations of Cosmology.* Oxford: Cambridge University Press, 2005.

[39] R. Brawer, "Inflationary cosmology and horizon and flatness problems : the mutual constitution of explanation and questions," Master's thesis, MIT, 1996.

[40] W. H. Kinney, "Tasi lectures on inflation," 2009.

[41] W. de Sitter, "On the relativity of inertia. Remarks concerning Einstein's latest hypothesis," *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences*, vol. 19, pp. 1217–1225, Mar. 1917.

[42] W. de Sitter, "On the curvature of space," *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences*, vol. 20, pp. 229–243, 1917.

[43] S. Tsujikawa, "Introductory review of cosmic inflation," in *2nd Tah Poe School on Cosmology: Modern Cosmology*, 4 2003.

[44] M. Yamaguchi, "Supergravity-based inflation models: a review," *Classical and Quantum Gravity*, vol. 28, p. 103001, Apr 2011.

[45] E. R. Harrison, "Fluctuations at the threshold of classical cosmology," *Phys. Rev. D*, vol. 1, pp. 2726–2730, May 1970.

[46] Y. B. Zel'Dovich, "Reprint of 1970A&A.....5...84Z. Gravitational instability: an approximate theory for large density perturbations.," , vol. 500, pp. 13–18, Mar. 1970.

[47] Y. Akrami *et al.*, "Planck 2018 results. IX. Constraints on primordial non-Gaussianity," *Astron. Astrophys.*, vol. 641, p. A9, 2020.

[48] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, "Theory of cosmological perturbations," , vol. 215, pp. 203–333, June 1992.

[49] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space.* Cambridge Monographs on Mathematical Physics, Cambridge, UK: Cambridge Univ. Press, 2 1984.

[50] D. Baumann, "Tasi lectures on inflation," 2012.

[51] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, "The Statistics of Peaks of Gaussian Random Fields," *Astrophys. J.*, vol. 304, pp. 15–61, 1986.

[52] U. Seljak and M. Zaldarriaga, "A Line of sight integration approach to cosmic microwave background anisotropies," *Astrophys. J.*, vol. 469, pp. 437–444, 1996.

[53] D. H. Lyth and A. R. Liddle, *The primordial density perturbation: Cosmology, inflation and the origin of structure.* 2009.

[54] M. Bucher, K. Moodley, and N. Turok, "General primordial cosmic perturbation," *Physical Review D*, vol. 62, Sep 2000.

[55] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, "New approach to the evolution

of cosmological perturbations on large scales," *Physical Review D*, vol. 62, Jul 2000.

[56] E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, and et al., "Seven-year wilkinson microwave anisotropy probe (wmap) observations: Cosmological interpretation," *The Astrophysical Journal Supplement Series*, vol. 192, p. 18, Jan 2011.

[57] W. Hu, E. F. Bunn, and N. Sugiyama, "COBE constraints on baryon isocurvature models," *The Astrophysical Journal*, vol. 447, jul 1995.

[58] E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Page, and et al., "Five-year wilkinson microwave anisotropy probe observations: Cosmological interpretation," *The Astrophysical Journal Supplement Series*, vol. 180, p. 330–376, Feb 2009.

[59] I. Sollom, A. Challinor, and M. P. Hobson, "Cold dark matter isocurvature perturbations: Constraints and model selection," *Physical Review D*, vol. 79, Jun 2009.

[60] D. Langlois and B. v. Tent, "Isocurvature modes in the cmb bispectrum," *Journal of Cosmology and Astroparticle Physics*, vol. 2012, p. 040–040, Jul 2012.

[61] M. Savelainen, J. Väliviita, P. Walia, S. Rusak, and H. Kurki-Suonio, "Constraints on neutrino density and velocity isocurvature modes from wmap-9 data," *Physical Review D*, vol. 88, Sep 2013.

[62] S. Weinberg, "Adiabatic modes in cosmology," *Physical Review D*, vol. 67, Jun 2003.

[63] C. Gordon, D. Wands, B. A. Bassett, and R. Maartens, "Adiabatic and entropy perturbations from inflation," *Physical Review D*, vol. 63, Dec 2000.

[64] E. Komatsu and D. N. Spergel, "Acoustic signatures in the primary microwave background bispectrum," *Physical Review D*, vol. 63, Feb 2001.

[65] A. Gangui, F. Lucchin, S. Matarrese, and S. Mollerach, "The Three point correlation function of the cosmic microwave background in inflationary models," *Astrophys. J.*, vol. 430, pp. 447–457, 1994.

[66] D. S. Salopek and J. R. Bond, "Nonlinear evolution of long-wavelength metric fluctuations in inflationary models," *Phys. Rev. D*, vol. 42, pp. 3936–3962, Dec 1990.

[67] L. Senatore, K. M. Smith, and M. Zaldarriaga, "Non-gaussianities in single field inflation and their optimal limits from the wmap 5-year data," *Journal of Cosmology and Astroparticle Physics*, vol. 2010, p. 028–028, Jan 2010.

[68] N. Bartolo, S. Matarrese, and A. Riotto, "Non-gaussianity from inflation," *Physical Review D*, vol. 65, Apr 2002.

[69] F. Bernardeau and J.-P. Uzan, "Non-gaussianity in multifield inflation," *Physical Review D*, vol. 66, Nov 2002.

[70] F. Bernardeau and J.-P. Uzan, "Inflationary models inducing non-gaussian metric fluctuations," *Physical Review D*, vol. 67, Jun 2003.

[71] M. Sasaki, "Multi-brid inflation and non-gaussianity," *Progress of Theoretical Physics*, vol. 120, p. 159–174, Jul 2008.

[72] G. Dvali, A. Gruzinov, and M. Zaldarriaga, "New mechanism for generating density

perturbations from inflation," *Physical Review D*, vol. 69, Jan 2004.

[73] D. H. Lyth, C. Ungarelli, and D. Wands, "Primordial density perturbation in the curvaton scenario," *Physical Review D*, vol. 67, Jan 2003.

[74] P. Creminelli and L. Senatore, "A smooth bouncing cosmology with scale invariant spectrum," *Journal of Cosmology and Astroparticle Physics*, vol. 2007, p. 010–010, Nov 2007.

[75] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, "Observational signatures and non-gaussianities of general single-field inflation," *Journal of Cosmology and Astroparticle Physics*, vol. 2007, p. 002–002, Jan 2007.

[76] R. Holman and A. J. Tolley, "Enhanced non-gaussianity from excited initial states," *Journal of Cosmology and Astroparticle Physics*, vol. 2008, p. 001, May 2008.

[77] J. Schwinger, "Brownian Motion of a Quantum Oscillator," *Journal of Mathematical Physics*, vol. 2, pp. 407–432, May 1961.

[78] P. M. Bakshi and K. T. Mahanthappa, "Expectation value formalism in quantum field theory. 1.," *J. Math. Phys.*, vol. 4, pp. 1–11, 1963.

[79] S. Weinberg, "Quantum contributions to cosmological correlations," *Physical Review D*, vol. 72, Aug 2005.

[80] J. Maldacena, "Non-gaussian features of primordial fluctuations in single field inflationary models," *Journal of High Energy Physics*, vol. 2003, p. 013–013, May 2003.

[81] N. Bartolo, S. Matarrese, and A. Riotto, "Non-gaussianity and the cosmic microwave background anisotropies," *Advances in Astronomy*, vol. 2010, p. 1–68, 2010.

[82] P. Creminelli and M. Zaldarriaga, "A single-field consistency relation for the three-point function," *Journal of Cosmology and Astroparticle Physics*, vol. 2004, p. 006–006, Oct 2004.

[83] C. Cheung, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, "On the consistency relation of the three-point function in single-field inflation," *Journal of Cosmology and Astroparticle Physics*, vol. 2008, p. 021, Feb 2008.

[84] K. Akiyama *et al.*, "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," *Astrophys. J. Lett.*, vol. 875, p. L1, 2019.

[85] R. Abbott *et al.*, "GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object," *Astrophys. J. Lett.*, vol. 896, no. 2, p. L44, 2020.

[86] B. P. Abbott *et al.*, "Binary Black Hole Population Properties Inferred from the First and Second Observing Runs of Advanced LIGO and Advanced Virgo," *Astrophys. J. Lett.*, vol. 882, no. 2, p. L24, 2019.

[87] S. Bird, I. Cholis, J. B. Muñoz, Y. Ali-Haïmoud, M. Kamionkowski, E. D. Kovetz, A. Raccanelli, and A. G. Riess, "Did LIGO detect dark matter?," *Phys. Rev. Lett.*, vol. 116, no. 20, p. 201301, 2016.

[88] B. J. Carr and S. W. Hawking, "Black holes in the early Universe," , vol. 168, pp. 399–416, Aug. 1974.

[89] S. W. Hawking, “Black hole explosions,” *Nature*, vol. 248, pp. 30–31, 1974.

[90] B. J. Carr, “The primordial black hole mass spectrum.,” , vol. 201, pp. 1–19, Oct. 1975.

[91] P. Meszaros, “Primeval black holes and galaxy formation.,” , vol. 38, pp. 5–13, Jan. 1975.

[92] G. F. Chapline, “Cosmological effects of primordial black holes,” *Nature*, vol. 253, no. 5489, pp. 251–252, 1975.

[93] M. Khlopov, B. A. Malomed, and I. B. Zeldovich, “Gravitational instability of scalar fields and formation of primordial black holes,” *Mon. Not. Roy. Astron. Soc.*, vol. 215, pp. 575–589, 1985.

[94] J. Silk and M. S. Turner, “Double Inflation,” *Phys. Rev. D*, vol. 35, p. 419, 1987.

[95] A. Polnarev and R. Zembowicz, “Formation of Primordial Black Holes by Cosmic Strings,” *Phys. Rev. D*, vol. 43, pp. 1106–1109, 1991.

[96] J. D. Barrow, E. J. Copeland, E. W. Kolb, and A. R. Liddle, “Baryogenesis in extended inflation. 2. Baryogenesis via primordial black holes,” *Phys. Rev. D*, vol. 43, pp. 984–994, 1991.

[97] P. Ivanov, P. Naselsky, and I. Novikov, “Inflation and primordial black holes as dark matter,” *Phys. Rev. D*, vol. 50, pp. 7173–7178, 1994.

[98] B. J. Carr, J. H. Gilbert, and J. E. Lidsey, “Black hole relics and inflation: Limits on blue perturbation spectra,” *Phys. Rev. D*, vol. 50, pp. 4853–4867, 1994.

[99] S. Zylberajch, “MACHOS, WIMPS or DUST: What is dark matter?,” in *11th Moriond Workshop: Tests of Fundamental Laws in Physics*, 1991.

[100] J. Yokoyama, “Formation of MACHO primordial black holes in inflationary cosmology,” *Astron. Astrophys.*, vol. 318, p. 673, 1997.

[101] C. Alcock, R. A. Allsman, D. R. Alves, T. S. Axelrod, A. C. Becker, D. P. Bennett, K. H. Cook, N. Dalal, A. J. Drake, K. C. Freeman, and et al., “The macho project: Microlensing results from 5.7 years of large magellanic cloud observations,” *The Astrophysical Journal*, vol. 542, p. 281–307, Oct 2000.

[102] J. C. Niemeyer and K. Jedamzik, “Near-critical gravitational collapse and the initial mass function of primordial black holes,” *Phys. Rev. Lett.*, vol. 80, pp. 5481–5484, 1998.

[103] J. C. Niemeyer and K. Jedamzik, “Dynamics of primordial black hole formation,” *Phys. Rev. D*, vol. 59, p. 124013, 1999.

[104] M. Shibata and M. Sasaki, “Black hole formation in the Friedmann universe: Formulation and computation in numerical relativity,” *Phys. Rev. D*, vol. 60, p. 084002, 1999.

[105] I. Musco, J. C. Miller, and L. Rezzolla, “Computations of primordial black hole formation,” *Class. Quant. Grav.*, vol. 22, pp. 1405–1424, 2005.

[106] I. Musco, J. C. Miller, and A. G. Polnarev, “Primordial black hole formation in the radiative era: Investigation of the critical nature of the collapse,” *Class. Quant. Grav.*, vol. 26, p. 235001, 2009.

[107] T. Harada, C.-M. Yoo, and K. Kohri, “Threshold of primordial black hole formation,”

Phys. Rev. D, vol. 88, no. 8, p. 084051, 2013. [Erratum: Phys.Rev.D 89, 029903 (2014)].

[108] P. Pina Avelino, "Primordial black hole constraints on non-gaussian inflation models," *Phys. Rev. D*, vol. 72, p. 124004, 2005.

[109] D. Seery and J. C. Hidalgo, "Non-Gaussian corrections to the probability distribution of the curvature perturbation from inflation," *JCAP*, vol. 07, p. 008, 2006.

[110] J. C. Hidalgo, "The effect of non-Gaussian curvature perturbations on the formation of primordial black holes," 8 2007.

[111] E. Bugaev and P. Klimai, "Formation of primordial black holes from non-Gaussian perturbations produced in a waterfall transition," *Phys. Rev. D*, vol. 85, p. 103504, 2012.

[112] C. T. Byrnes, E. J. Copeland, and A. M. Green, "Primordial black holes as a tool for constraining non-Gaussianity," *Phys. Rev. D*, vol. 86, p. 043512, 2012.

[113] T. Kokubu, K. Kyutoku, K. Kohri, and T. Harada, "Effect of Inhomogeneity on Primordial Black Hole Formation in the Matter Dominated Era," *Phys. Rev. D*, vol. 98, no. 12, p. 123024, 2018.

[114] T. Nakama, J. Silk, and M. Kamionkowski, "Stochastic gravitational waves associated with the formation of primordial black holes," *Phys. Rev. D*, vol. 95, no. 4, p. 043511, 2017.

[115] W. H. Press and P. Schechter, "Formation of Galaxies and Clusters of Galaxies by Self-Similar Gravitational Condensation," , vol. 187, pp. 425–438, Feb. 1974.

[116] K. Tokeshi, K. Inomata, and J. Yokoyama, "Window function dependence of the novel mass function of primordial black holes," *JCAP*, vol. 12, p. 038, 2020.

[117] C. T. Byrnes, M. Hindmarsh, S. Young, and M. R. S. Hawkins, "Primordial black holes with an accurate QCD equation of state," *JCAP*, vol. 08, p. 041, 2018.

[118] A. M. Green and B. J. Kavanagh, "Primordial Black Holes as a dark matter candidate," *J. Phys. G*, vol. 48, no. 4, p. 043001, 2021.

[119] S. Bird, H. V. Peiris, M. Viel, and L. Verde, "Minimally parametric power spectrum reconstruction from the lyman forest," *Monthly Notices of the Royal Astronomical Society*, vol. 413, p. 1717–1728, Mar 2011.

[120] D. J. Fixsen, E. S. Cheng, J. M. Gales, J. C. Mather, R. A. Shafer, and E. L. Wright, "The Cosmic Microwave Background spectrum from the full COBE FIRAS data set," *Astrophys. J.*, vol. 473, p. 576, 1996.

[121] J. C. Mather, E. S. Cheng, D. A. Cottingham, J. Eplee, R. E., D. J. Fixsen, T. Hewagama, R. B. Isaacman, K. A. Jensen, S. S. Meyer, P. D. Noerdlinger, S. M. Read, L. P. Rosen, R. A. Shafer, E. L. Wright, C. L. Bennett, N. W. Boggess, M. G. Hauser, T. Kelsall, J. Moseley, S. H., R. F. Silverberg, G. F. Smoot, R. Weiss, and D. T. Wilkinson, "Measurement of the Cosmic Microwave Background Spectrum by the COBE FIRAS Instrument," , vol. 420, p. 439, Jan. 1994.

[122] C. T. Byrnes, P. S. Cole, and S. P. Patil, "Steepest growth of the power spectrum and primordial black holes," *JCAP*, vol. 06, p. 028, 2019.

[123] J. Chluba *et al.*, "New Horizons in Cosmology with Spectral Distortions of the Cosmic Microwave Background," 9 2019.

[124] K. Inomata and T. Nakama, "Gravitational waves induced by scalar perturbations as probes of the small-scale primordial spectrum," *Phys. Rev. D*, vol. 99, no. 4, p. 043511, 2019.

[125] B. Carr, M. Raidal, T. Tenkanen, V. Vaskonen, and H. Veermäe, "Primordial black hole constraints for extended mass functions," *Physical Review D*, vol. 96, Jul 2017.

[126] G. A. Palma, S. Sypsas, and C. Zenteno, "Seeding primordial black holes in multifield inflation," *Phys. Rev. Lett.*, vol. 125, no. 12, p. 121301, 2020.

[127] S. M. Leach, I. J. Grivell, and A. R. Liddle, "Black hole constraints on the running mass inflation model," *Phys. Rev. D*, vol. 62, p. 043516, 2000.

[128] L. Alabidi and K. Kohri, "Generating Primordial Black Holes Via Hilltop-Type Inflation Models," *Phys. Rev. D*, vol. 80, p. 063511, 2009.

[129] K. Kohri, D. H. Lyth, and A. Melchiorri, "Black hole formation and slow-roll inflation," *JCAP*, vol. 04, p. 038, 2008.

[130] J. Garcia-Bellido and E. Ruiz Morales, "Primordial black holes from single field models of inflation," *Phys. Dark Univ.*, vol. 18, pp. 47–54, 2017.

[131] C. Germani and T. Prokopec, "On primordial black holes from an inflection point," *Phys. Dark Univ.*, vol. 18, pp. 6–10, 2017.

[132] H. Motohashi and W. Hu, "Primordial Black Holes and Slow-Roll Violation," *Phys. Rev. D*, vol. 96, no. 6, p. 063503, 2017.

[133] M. P. Hertzberg and M. Yamada, "Primordial Black Holes from Polynomial Potentials in Single Field Inflation," *Phys. Rev. D*, vol. 97, no. 8, p. 083509, 2018.

[134] Y.-F. Cai, X. Tong, D.-G. Wang, and S.-F. Yan, "Primordial Black Holes from Sound Speed Resonance during Inflation," *Phys. Rev. Lett.*, vol. 121, no. 8, p. 081306, 2018.

[135] R.-G. Cai, Z.-K. Guo, J. Liu, L. Liu, and X.-Y. Yang, "Primordial black holes and gravitational waves from parametric amplification of curvature perturbations," *JCAP*, vol. 06, p. 013, 2020.

[136] M. Kawasaki, N. Kitajima, and T. T. Yanagida, "Primordial black hole formation from an axionlike curvaton model," *Phys. Rev. D*, vol. 87, no. 6, p. 063519, 2013.

[137] K. Kohri, C.-M. Lin, and T. Matsuda, "Primordial black holes from the inflating curvaton," *Phys. Rev. D*, vol. 87, no. 10, p. 103527, 2013.

[138] S. Pi, Y.-l. Zhang, Q.-G. Huang, and M. Sasaki, "Scalaron from R^2-gravity as a heavy field," *JCAP*, vol. 05, p. 042, 2018.

[139] A. Linde, S. Mooij, and E. Pajer, "Gauge field production in supergravity inflation: Local non-Gaussianity and primordial black holes," *Phys. Rev. D*, vol. 87, no. 10, p. 103506, 2013.

[140] E. Bugaev and P. Klimai, "Axion inflation with gauge field production and primordial black holes," *Phys. Rev. D*, vol. 90, no. 10, p. 103501, 2014.

[141] A. M. Green and K. A. Malik, "Primordial black hole production due to preheating,"

Phys. Rev. D, vol. 64, p. 021301, 2001.

[142] B. A. Bassett and S. Tsujikawa, "Inflationary preheating and primordial black holes," *Phys. Rev. D*, vol. 63, p. 123503, 2001.

[143] J. Martin, T. Papanikolaou, and V. Vennin, "Primordial black holes from the preheating instability in single-field inflation," *JCAP*, vol. 01, p. 024, 2020.

[144] M. Kawasaki, N. Sugiyama, and T. Yanagida, "Primordial black hole formation in a double inflation model in supergravity," *Phys. Rev. D*, vol. 57, pp. 6050–6056, 1998.

[145] S. Clesse and J. García-Bellido, "Massive Primordial Black Holes from Hybrid Inflation as Dark Matter and the seeds of Galaxies," *Phys. Rev. D*, vol. 92, no. 2, p. 023524, 2015.

[146] J. Wess and B. Zumino, "Supergauge transformations in four dimensions," *Nuclear Physics B*, vol. 70, no. 1, pp. 39–50, 1974.

[147] P. Nath and R. L. Arnowitt, "Generalized Supergauge Symmetry as a New Framework for Unified Gauge Theories," *Phys. Lett. B*, vol. 56, pp. 177–180, 1975.

[148] A. Pich, "Effective field theory: Course," in *Les Houches Summer School in Theoretical Physics, Session 68: Probing the Standard Model of Particle Interactions*, 6 1998.

[149] R. Penco, "An Introduction to Effective Field Theories," 6 2020.

[150] S. Cremonini, Z. Lalak, and K. Turzynski, "Strongly Coupled Perturbations in Two-Field Inflationary Models," *JCAP*, vol. 03, p. 016, 2011.

[151] A. Achucarro, J.-O. Gong, S. Hardeman, G. A. Palma, and S. P. Patil, "Features of heavy physics in the CMB power spectrum," *JCAP*, vol. 01, p. 030, 2011.

[152] X. Chen and Y. Wang, "Quasi-Single Field Inflation and Non-Gaussianities," *JCAP*, vol. 04, p. 027, 2010.

[153] S. Pi and M. Sasaki, "Curvature Perturbation Spectrum in Two-field Inflation with a Turning Trajectory," *JCAP*, vol. 10, p. 051, 2012.

[154] S. Cespedes, V. Atal, and G. A. Palma, "On the importance of heavy fields during inflation," *JCAP*, vol. 05, p. 008, 2012.

[155] S. Groot Nibbelink and B. J. W. van Tent, "Scalar perturbations during multiple field slow-roll inflation," *Class. Quant. Grav.*, vol. 19, pp. 613–640, 2002.

[156] E. Castillo, B. Koch, and G. Palma, "On the integration of fields and quanta in time dependent backgrounds," *JHEP*, vol. 05, p. 111, 2014.

[157] A. Achúcarro, V. Atal, C. Germani, and G. A. Palma, "Cumulative effects in inflation with ultra-light entropy modes," *JCAP*, vol. 02, p. 013, 2017.

[158] A. Achucarro, V. Atal, S. Cespedes, J.-O. Gong, G. A. Palma, and S. P. Patil, "Heavy fields, reduced speeds of sound and decoupling during inflation," *Phys. Rev. D*, vol. 86, p. 121301, 2012.

[159] V. Assassi, D. Baumann, D. Green, and L. McAllister, "Planck-Suppressed Operators," *JCAP*, vol. 01, p. 033, 2014.

[160] A. R. Brown, "Hyperbolic Inflation," *Phys. Rev. Lett.*, vol. 121, no. 25, p. 251601, 2018.

[161] A. V. Iyer, S. Pi, Y. Wang, Z. Wang, and S. Zhou, "Strongly Coupled Quasi-Single Field Inflation," *JCAP*, vol. 01, p. 041, 2018.

[162] H. An, M. McAneny, A. K. Ridgway, and M. B. Wise, "Quasi Single Field Inflation in the non-perturbative regime," *JHEP*, vol. 06, p. 105, 2018.

[163] A. Achúcarro, E. J. Copeland, O. Iarygina, G. A. Palma, D.-G. Wang, and Y. Welling, "Shift-symmetric orbital inflation: Single field or multifield?," *Phys. Rev. D*, vol. 102, no. 2, p. 021302, 2020.

[164] J. Fumagalli, S. Garcia-Saenz, L. Pinol, S. Renaux-Petel, and J. Ronayne, "Hyper-Non-Gaussianities in Inflation with Strongly Nongeodesic Motion," *Phys. Rev. Lett.*, vol. 123, no. 20, p. 201302, 2019.

[165] T. Bjorkmo, "Rapid-Turn Inflationary Attractors," *Phys. Rev. Lett.*, vol. 122, no. 25, p. 251301, 2019.

[166] P. Christodoulidis, D. Roest, and E. I. Sfakianakis, "Scaling attractors in multi-field inflation," *JCAP*, vol. 12, p. 059, 2019.

[167] A. Achucarro, J.-O. Gong, S. Hardeman, G. A. Palma, and S. P. Patil, "Mass hierarchies and non-decoupling in multi-scalar field dynamics," *Phys. Rev. D*, vol. 84, p. 043502, 2011.

[168] P. Carrilho, K. A. Malik, and D. J. Mulryne, "Dissecting the growth of the power spectrum for primordial black holes," *Phys. Rev. D*, vol. 100, no. 10, p. 103529, 2019.

[169] O. Özsoy and G. Tasinato, "On the slope of the curvature power spectrum in non-attractor inflation," *JCAP*, vol. 04, p. 048, 2020.

[170] C. Germani and I. Musco, "Abundance of Primordial Black Holes Depends on the Shape of the Inflationary Power Spectrum," *Phys. Rev. Lett.*, vol. 122, no. 14, p. 141302, 2019.

[171] C. Germani and R. K. Sheth, "Nonlinear statistics of primordial black holes from Gaussian curvature perturbations," *Phys. Rev. D*, vol. 101, no. 6, p. 063520, 2020.

[172] J. Fumagalli, S. Renaux-Petel, J. W. Ronayne, and L. T. Witkowski, "Turning in the landscape: a new mechanism for generating Primordial Black Holes," 4 2020.

[173] J. Fumagalli, S. Renaux-Petel, and L. T. Witkowski, "Oscillations in the stochastic gravitational wave background from sharp features and particle production during inflation," 12 2020.

[174] J. Fumagalli, S. Renaux-Petel, and L. T. Witkowski, "Resonant features in the stochastic gravitational wave background," 5 2021.

[175] X. Chen, G. A. Palma, W. Riquelme, B. Scheihing Hitschfeld, and S. Sypsas, "Landscape tomography through primordial non-Gaussianity," *Phys. Rev. D*, vol. 98, no. 8, p. 083528, 2018.

[176] X. Chen, G. A. Palma, B. Scheihing Hitschfeld, and S. Sypsas, "Reconstructing the Inflationary Landscape with Cosmological Data," *Phys. Rev. Lett.*, vol. 121, no. 16, p. 161302, 2018.

[177] G. A. Palma, B. Scheihing Hitschfeld, and S. Sypsas, "Non-Gaussian CMB and LSS statistics beyond polyspectra," *JCAP*, vol. 02, p. 027, 2020.

[178] S. Young and C. T. Byrnes, "Primordial black holes in non-Gaussian regimes," *JCAP*, vol. 08, p. 052, 2013.

[179] G. Franciolini, A. Kehagias, S. Matarrese, and A. Riotto, "Primordial Black Holes from Inflation and non-Gaussianity," *JCAP*, vol. 03, p. 016, 2018.

[180] R.-g. Cai, S. Pi, and M. Sasaki, "Gravitational Waves Induced by non-Gaussian Scalar Perturbations," *Phys. Rev. Lett.*, vol. 122, no. 20, p. 201101, 2019.

[181] S. Passaglia, W. Hu, and H. Motohashi, "Primordial black holes and local non-Gaussianity in canonical inflation," *Phys. Rev. D*, vol. 99, no. 4, p. 043536, 2019.

[182] V. Atal and C. Germani, "The role of non-gaussianities in Primordial Black Hole formation," *Phys. Dark Univ.*, vol. 24, p. 100275, 2019.

[183] G. Panagopoulos and E. Silverstein, "Primordial Black Holes from non-Gaussian tails," 6 2019.

[184] G. Panagopoulos and E. Silverstein, "Multipoint correlators in multifield cosmology," 3 2020.

[185] C.-M. Yoo, J.-O. Gong, and S. Yokoyama, "Abundance of primordial black holes with local non-Gaussianity in peak theory," *JCAP*, vol. 09, p. 033, 2019.

[186] V. Atal, J. Cid, A. Escrivà, and J. Garriga, "PBH in single field inflation: the effect of shape dispersion and non-Gaussianities," *JCAP*, vol. 05, p. 022, 2020.

[187] J. M. Ezquiaga, J. García-Bellido, and V. Vennin, "The exponential tail of inflationary fluctuations: consequences for primordial black holes," *JCAP*, vol. 03, p. 029, 2020.

[188] Z. Zhou, J. Jiang, Y.-F. Cai, M. Sasaki, and S. Pi, "Primordial black holes and gravitational waves from resonant amplification during inflation," *Phys. Rev. D*, vol. 102, no. 10, p. 103527, 2020.

[189] R.-G. Cai, C. Chen, and C. Fu, "Primordial black holes and stochastic gravitational wave background from inflation with a noncanonical spectator field," 8 2021.

[190] T. Padmanabhan, *Gravitation: Foundations and frontiers*. Cambridge University Press, 12 2014.

[191] S. Garcia-Saenz, L. Pinol, and S. Renaux-Petel, "Revisiting non-Gaussianity in multifield inflation with curved field space," *JHEP*, vol. 01, p. 073, 2020.

[192] F. Larsen, J. P. van der Schaar, and R. G. Leigh, "De Sitter holography and the cosmic microwave background," *JHEP*, vol. 04, p. 047, 2002.

[193] A. Bzowski, P. McFadden, and K. Skenderis, "Holography for inflation using conformal perturbation theory," *JHEP*, vol. 04, p. 047, 2013.

[194] J. Garriga, K. Skenderis, and Y. Urakawa, "Multi-field inflation from holography," *JCAP*, vol. 01, p. 028, 2015.

[195] A. Achúcarro, S. Céspedes, A.-C. Davis, and G. A. Palma, "Constraints on Holographic Multifield Inflation and Models Based on the Hamilton-Jacobi Formalism," *Phys. Rev. Lett.*, vol. 122, no. 19, p. 191301, 2019.

Appendix A

Einstein Equations for Scalar Perturbations

A.1. Einstein Equations for General Scalar Perturbations

In this appendix, we follow the treatment worked in Weinberg's text [34]. We will study scalar perturbations, in which we write the metric as:

$$ds^2 = -(1+2A)dt^2 + a(t)\partial_i C dx^i dt + a^2(t)\Big[(1-2B)\delta_{ij} + \frac{1}{3}\nabla^2 E\delta_{ij} + \partial_i\partial_j E\Big]. \tag{A.1}$$

Together with the scalar perturbations of the energy-momentum tensor ($\delta\rho$, δp, δu and π^S), we will have the following equations of motion derived from Einstein's equation:

$$\begin{aligned}-4\pi G\Big(\delta\rho - \delta p + \frac{2}{3}k^2\pi^S\Big) = {}& H\dot{A} + 2(3H^2+\dot{H})A - \frac{1}{2}\frac{k^2}{a^2}\Big(2B + \frac{1}{3}k^2E\Big)\\ &- \frac{1}{2}\frac{d^2}{dt^2}\Big(2B+\frac{1}{3}k^2E\Big) - 3H\frac{d}{dt}\Big(2B+\frac{1}{3}k^2E\Big) + \frac{1}{2}Hk^2\dot{E} - \frac{H}{a}k^2C,\end{aligned} \tag{A.2}$$

$$-4\pi G\pi^S = \frac{1}{2a^2}A + \frac{1}{4a^2}\Big(2B+\frac{1}{3}k^2E\Big) - \frac{1}{4}\ddot{E} - \frac{3}{4}H\dot{E} + \frac{1}{2a}\dot{C} + \frac{H}{a}C, \tag{A.3}$$

$$-4\pi G(\bar{\rho}+\bar{p})\delta u = HA - \frac{1}{2}\frac{d}{dt}\Big(2B+\frac{1}{3}k^2E\Big), \tag{A.4}$$

$$\begin{aligned}-4\pi G(\delta\rho + 3\delta p) = {}& \frac{1}{a^2}k^2A - 3H\dot{A} + \frac{1}{a}k^2\dot{C} + \frac{H}{a}k^2C + \frac{3}{2}\frac{d^2}{dt^2}\Big(2B+\frac{1}{3}k^2E\Big)\\ &+ 3H\frac{d}{dt}\Big(2B+\frac{1}{3}k^2E\Big) - 6\Big(\dot{H}+H^2\Big)A - \frac{1}{2}k^2\ddot{E} - Hk^2\dot{E}.\end{aligned} \tag{A.5}$$

Additionally, we must consider the conservation of the energy moment tensor $\nabla^\mu T_{\mu\nu} = 0$ for the case without a source (to consider collisions we must add a source term C_μ). The time component gives us the conservation of energy:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = (\bar{\rho} + \bar{p})\left(\frac{k^2}{a^2}(\delta u - aC) - 3\dot{B}\right). \tag{A.6}$$

While the spatial part gives us the momentum conservation:

$$\delta p - \frac{2}{3}k^2\pi^S + \frac{d}{dt}((\bar{\rho} + \bar{p})\delta u) = -3H(\bar{\rho} + \bar{p})\delta u - (\bar{\rho} + \bar{p})A. \tag{A.7}$$

If we combine the Einstein equations for the perturbations (equations (A.2) - (A.5)), we can derive the following relation:

$$\frac{k^2}{a^2}\left(-B - \frac{1}{6} + \frac{1}{2}a^2H\dot{E} - aHC\right) = -4\pi G(\delta\rho - 3H(\bar{\rho} + \bar{p})\delta u). \tag{A.8}$$

Where the expression in the left parenthesis is a gauge-invariant quantity defined in (3.39). So we arrive at an equation analogous to the Poisson equation for that gauge invariant field:

$$\frac{k^2}{a^2}\Psi_{GI} = -4\pi G(\delta\rho - 3H(\bar{\rho} + \bar{p})\delta u). \tag{A.9}$$

Also, if we rewrite the equation (A.3), using the gauge-invariant quantities, defined in (3.38) and (3.39), we obtain:

$$\frac{\Phi_{GI} - \Psi_{GI}}{2a^2} = -4\pi G\pi^S, \tag{A.10}$$

from this relationship it follows the fact that without the presence of an anisotropic stress tensor, the gravitational dynamics is dominated by a single field (this becomes much more explicit in the Newtonian Gauge).

A.2. Einstein Equations for the Different Gauges

- **Newtonian Gauge**
  In the Newtonian Gauge, we have to make the following changes:

$$E_N = 0, \qquad C_N = 0, \qquad A_N \equiv \Phi, \qquad B_N \equiv -\Psi \tag{A.11}$$

  Einstein's equations result:

$$\begin{aligned}
-4\pi G\left(\delta\rho - \delta p + \frac{2}{3}k^2\pi^S\right) &= H\dot{\Phi} + 2\left(3H^2 + \dot{H}\right)\Phi + \ddot{\Psi} + 6H\dot{\Psi} + \frac{k^2}{a^2}\Psi, & \text{(A.12)}\\
-4\pi G\pi^S &= \frac{1}{2a^2}(\Phi - \Psi), & \text{(A.13)}\\
-4\pi G(\bar{\rho} + \bar{p})\delta u &= H\Phi + \dot{\Psi}, & \text{(A.14)}\\
-4\pi G(\delta\rho + 3\delta p) &= \frac{k^2}{a^2}\Phi - 3H\dot{\Phi} - 6\left(\dot{H} + H^2\right)\Phi - 3\ddot{\Psi} - 6H\dot{\Psi}. & \text{(A.15)}
\end{aligned}$$

While the energy-momentum conservation equations are:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = (\bar{\rho} + \bar{p})\left(\frac{k^2}{a^2}\delta u + 3\dot{\Psi}\right), \tag{A.16}$$

$$\delta p - \frac{2}{3}k^2\pi^S + \frac{d}{dt}((\bar{\rho} + \bar{p})\delta u) = -3H(\bar{\rho} + \bar{p})\delta u - (\bar{\rho} + \bar{p})\Phi. \tag{A.17}$$

- **Synchronous Gauge**
  In this gauge, we must do the following changes:

$$A_S = 0, \qquad\qquad C_S = 0, \tag{A.18}$$

The Einstein's equations become

$$\begin{aligned} -4\pi G\left(\delta\rho - \delta p + \frac{2}{3}k^2\pi^S\right) =& -\frac{1}{2}\frac{k^2}{a^2}\left(2B + \frac{1}{3}k^2 E\right) - \frac{1}{2}\frac{d^2}{dt^2}\left(2B + \frac{1}{3}k^2 E\right) \\ & - 3H\frac{d}{dt}\left(2B + \frac{1}{3}k^2 E\right) + \frac{1}{2}Hk^2\dot{E}, \end{aligned} \tag{A.19}$$

$$-4\pi G\pi^S = \frac{1}{4a^2}\left(2B + \frac{1}{3}k^2 E\right) - \frac{1}{4}\ddot{E} - \frac{3}{4}H\dot{E}, \tag{A.20}$$

$$-4\pi G(\bar{\rho} + \bar{p})\delta u = -\frac{1}{2}\frac{d}{dt}\left(2B + \frac{1}{3}k^2 E\right), \tag{A.21}$$

$$-4\pi G(\delta\rho + 3\delta p) = \frac{1}{a^2}\frac{d}{dt}\left(a^2\psi\right). \tag{A.22}$$

Where ψ in the last equation is defined by $\psi = 3\dot{B}$. We can use this variable in the energy conservation equation:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) - (\bar{\rho} + \bar{p})\left(\frac{k^2}{a^2}\delta u - \psi\right) = 0. \tag{A.23}$$

On the other hand, one of the peculiarities of this gauge is that the momentum conservation equation does not depend on the perturbations in the metric:

$$\delta p - \frac{2}{3}k^2\pi^S + \frac{d}{dt}((\bar{\rho} + \bar{p})\delta u) + 3H(\bar{\rho} + \bar{p})\delta u = 0. \tag{A.24}$$

An additional treatment that we can do with the Einstein equations in this Gauge, is to obtain an equation for the co-moving curvature perturbation $\mathcal{R}$. If we add 3 times the equation A.19 with $(-2k^2)$ the equation A.20 and the equation A.22 we will arrive at the following expression :

$$-\frac{k^2}{a^2}\left(2B + \frac{1}{3}k^2 E\right) = -8\pi G\delta\rho + 2H\psi. \tag{A.25}$$

Where, if on the left side of the equation we make appear $\mathcal{R}$, defined in the equation (3.42), we will obtain:

$$-\frac{k^2}{a^2}\mathcal{R} = H\psi - 4\pi G\delta\rho + \frac{k^2}{a^2}H\delta u. \tag{A.26}$$

- **Co-moving Gauge**

  In the co-moving Gauge, we must impose:

$$\delta u_C = 0, \qquad E_C = 0, \qquad B_C = \mathcal{R}, \qquad A_C \equiv \delta\mathcal{N}. \tag{A.27}$$

With this, Einstein's equations translate into:

$$-4\pi G\Big(\delta\rho - \delta p + \frac{2}{3}k^2\pi^S\Big) = H\delta\dot{\mathcal{N}} + 2\Big(3H^2 + \dot{H}\Big)\delta\mathcal{N} - \frac{k^2}{a^2}\mathcal{R} - \ddot{\mathcal{R}} - 6H\dot{\mathcal{R}} - \frac{H}{a}k^2 C \tag{A.28}$$

$$-4\pi G\pi^S = \frac{1}{2a^2}\delta\mathcal{N} + \frac{1}{2a^2}\mathcal{R} + \frac{1}{2a}\dot{C} + \frac{H}{a}C, \tag{A.29}$$

$$0 = H\delta\mathcal{N} - \dot{\mathcal{R}}, \tag{A.30}$$

$$\begin{aligned}-4\pi G(\delta\rho + 3\delta p) = &\frac{k^2}{a^2}\delta\mathcal{N} - 3H\delta\dot{\mathcal{N}} + \frac{1}{a}k^2\dot{C} + \frac{H}{a}k^2 C + 3\ddot{\mathcal{R}}\\ &+ 6H\dot{\mathcal{R}} - 6\Big(\dot{H} + H^2\Big)\delta\mathcal{N}.\end{aligned} \tag{A.31}$$

While the equations for the conservation of energy and momentum result:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = -(\bar{\rho} + \bar{p})\left(\frac{k^2}{a^2}C + 3\dot{\mathcal{R}}\right), \tag{A.32}$$

$$\delta p - \frac{2}{3}k^2\pi^S + (\bar{\rho} + \bar{p})\delta\mathcal{N} = 0. \tag{A.33}$$

Appendix B

Equations of motion for the perturbations in the synchronous gauge

In this appendix we will derive the equations for the perturbations in the synchronous gauge and show that under the assumption of adiabaticity for the initial conditions of the perturbations, all perturbations come from a single primordial perturbation.

We will take as a starting point the conservation equations of the energy-moment tensor for each component $\nabla^\nu T_{\mu\nu} = C_\mu$, where C_μ is a source that describes the collisions between different components. The equations will be:

$$\delta\dot{\rho}_a + 3H(\delta\rho_a + \delta p_a) - (\bar{\rho}_a + \bar{p}_a)\left(\frac{k^2}{a^2}\delta u_a - \psi\right) = \delta C_{0,a}, \tag{B.1}$$

$$\delta p_a - \frac{2}{3}k^2\pi^S a + \frac{d}{dt}((\bar{\rho}_a + \bar{p}_a)\delta u_a) + 3H(\bar{\rho}_a + \bar{p}_a)\delta u_a = \delta C_a^S, \tag{B.2}$$

where a is the subscript that will differentiate each component of the universe and δC_a^S is the scalar part of δC_i.
Next we will write this equations for the different components of the universe.

- **Dark Matter**
  We will model dark matter as non-relativistic particles, with this we can impose $\bar{p}_D = 0$, $\delta p_D = 0$ and $\pi_D^S = 0$. Furthermore, dark matter only interacts with the rest of the components of the universe gravitationally, so we can neglect collisional terms. With this, the equations become:

$$\delta\dot{\rho}_D + 3H\delta\rho_D - \bar{\rho}_D\frac{k^2}{a^2}\delta u_D = -\bar{\rho}_D\psi, \tag{B.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\rho}_D\delta u_D) + 3H\bar{\rho}_D\delta u_D = 0. \tag{B.4}$$

From the continuity equation for the background quantities, it is satisfied that $\dot{\bar{\rho}}_D =$

$-3H\bar{\rho}_D$. If we plug this relation into the second equation, this implies

$$\delta\dot{u}_D = 0. \tag{B.5}$$

We can use the residual gauge that we have in the synchronous gauge, thus imposing $\delta u_D = 0$. Putting all this together, the first equation results:

$$\delta\dot{\rho}_D + 3H\delta\rho_D + \bar{\rho}_D\psi = 0. \tag{B.6}$$

- **Neutrinos**
  Like dark matter, neutrinos do not interact with the rest of the components of the universe. Additionally we must impose $\delta p = 1/3\delta\rho_\nu$ and a priori we cannot neglect the anisotropic stress tensor. The equations result:

$$\delta\dot{\rho}_\nu + 4H\delta\rho_\nu - \frac{4}{3}\bar{\rho}_\nu\frac{k^2}{a^2}\delta u_\nu = -\frac{4}{3}\bar{\rho}_\nu\psi, \tag{B.7}$$

$$\delta\rho_\nu - 2k^2\pi_\nu^S + 4\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\rho}_\nu\delta u_\nu) + 12H\bar{\rho}_\nu\delta u_\nu = 0. \tag{B.8}$$

  In the above equation we can neglect the anisotropic stress tensor only if we are considering large scales.

- **Photons**
  For photons, we consider that the only relevant interaction (other than gravitational) is with electrons via Thomson scattering. In these collisions we will consider that the photons change their momentum but not their energy, i.e $\delta C_{\gamma,0} = 0$, so

$$\delta C_\gamma^S = \frac{4}{3}\dot{\tau}\bar{\rho}_\gamma(\delta u_\gamma - \delta u_e). \tag{B.9}$$

  The previous expression is derived using QFT tools, in addition, $\dot{\tau}$ is called the optical depth and characterizes the amount of interactions that occur between photons and electrons in this type of scattering. Including this in the equations, we are left with:

$$\delta\dot{\rho}_\gamma + 4H\delta\rho_\gamma - \frac{4}{3}\bar{\rho}_\gamma\frac{k^2}{a^2}\delta u_\gamma = -\frac{4}{3}\bar{\rho}_\gamma\psi, \tag{B.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\rho}_\gamma\delta u_\gamma) + H\bar{\rho}_\gamma\delta u_\gamma - \frac{1}{2}k^2\pi_\gamma^S + \frac{1}{4}\delta\rho_\gamma = \dot{\tau}\bar{\rho}_\gamma(\delta u_\gamma - \delta u_e). \tag{B.11}$$

  In this case we can neglect the anisotropic stress tensor, because its effect is less relevant than Thomson's scattering. If we want to include its effect in the system, we must consider an equation that takes into account the temporal variation of the anisotropic tensor, obtained from the quadrupole moment of the multipolar expansion of the Boltzmann equation for photons.

- **Baryons**
  To study baryons, we will consider electrons and protons as a single fluid. Like photons, we will consider that baryons interact only with photons using Thomson scattering, that

is,

$$\delta C_{B,0} = 0, \qquad \delta C_B^S = -\delta C_\gamma^S = \frac{4}{3}\dot{\tau}\bar{\rho}_\gamma(\delta u_e - \delta u_\gamma). \tag{B.12}$$

Additionally, we consider baryonic matter as non-relativistic, so we neglect its anisotropic stress tensor in the same way as dark matter. With all this, the equations result:

$$\delta\dot{\rho}_B + 3H\delta\rho_B - \bar{\rho}_B\frac{k^2}{a^2}\delta u_B = -\bar{\rho}_B, \tag{B.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\rho}_B\delta u_B) + 3H\bar{\rho}_B\delta u_B = \frac{4}{3}\dot{\tau}\bar{\rho}_\gamma(\delta u_B - \delta u_\gamma). \tag{B.14}$$

- **Photon-Baryon Plasma**
  Due to their interaction through Thomson scattering, it is convenient to study photons with baryons together. If we group the equations in a convenient way, we have:

$$\delta\dot{\rho}_\gamma + 4H\delta\rho_\gamma - \frac{4}{3}\bar{\rho}_\gamma\frac{k^2}{a^2}\delta u_{\gamma B} = -\frac{4}{3}\bar{\rho}_\gamma\psi, \tag{B.15}$$

$$\delta\dot{\rho}_B + 3H\delta\rho_B - \bar{\rho}_B\frac{k^2}{a^2}\delta u_{\gamma B} = -\bar{\rho}_B, \tag{B.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B\right)\delta u_{\gamma B}\right] + 3H\left(\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B\right)\delta u_{\gamma B} + \frac{1}{3}\delta\bar{\rho}_\gamma = 0. \tag{B.17}$$

  In the previous equations we introduce $\delta u_{\gamma B}$, defined as:

$$\delta u_{\gamma B} \equiv \frac{\frac{4}{3}\bar{\rho}_\gamma\delta u_\gamma + \bar{\rho}_B\delta u_B}{\frac{4}{3}\bar{\rho}_\gamma + \bar{\rho}_B}. \tag{B.18}$$

  We also made the assumption where, if $\dot{\tau} \gg 1$ then the difference between the velocity perturbations between the baryons and photons must be small (so as not to break the hierarchy of perturbations), so it must comply $\delta u_\gamma \simeq \delta u_B \simeq \delta u_{\gamma B}$.

In addition to these equations obtained from the conservation of the energy-moment tensor of each component, we must consider an equation for the potential ψ, obtained from Einstein's equations:

$$\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\psi\right) = -4\pi G(2\delta\rho_\gamma + 2\delta\rho_\nu + \delta\rho_B + \delta\rho_D). \tag{B.19}$$

To work the equations more easily, we will work with the density contrast, defined as $\delta_a \equiv \delta\rho_a/\bar{\rho}_a$. With this variable, the time derivatives fulfill the following property:

$$\bar{\rho}_a\dot{\delta}_a = \delta\dot{\rho}_a + 3H(\omega + 1)\delta\rho_a. \tag{B.20}$$

Then, the equations for δ_D, δ_ν, δ_γ, δ_B, δu_γ and δu_ν becomes:

$$\dot{\delta}_D = -\psi, \tag{B.21}$$

$$\dot{\delta}_\nu - \frac{4}{3}\frac{k^2}{a^2}\delta u_\nu = -\frac{4}{3}\psi, \tag{B.22}$$

$$\dot{\delta}_\gamma - \frac{4}{3}\frac{k^2}{a^2}\delta u_{\gamma B} = -\frac{4}{3}\psi, \tag{B.23}$$

$$\dot{\delta}_B - \frac{k^2}{a^2}\delta u_{\gamma B} = -\psi, \tag{B.24}$$

$$\frac{1}{4}\delta_\gamma + a\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+R)\frac{\delta u_{\gamma B}}{a}\right] = 0, \tag{B.25}$$

$$\frac{1}{4}\delta_\nu + a\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta u_\nu}{a}\right) = 0, \tag{B.26}$$

where $R = 3\bar{\rho}_B/4\bar{\rho}_\gamma$. Besides, the equation for ψ becomes:

$$\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\psi\right) = -4\pi G(2\bar{\rho}_\gamma\delta_\gamma + 2\bar{\rho}_\nu\delta_\nu + \bar{\rho}_B\delta_B + \bar{\rho}_D\delta_D). \tag{B.27}$$

Our goal will be to solve these equations at early times. If we impose the adiabaticity condition for the perturbations in the different components (eq. (3.54)), this translates to the following relation

$$\frac{\delta\rho_\nu}{\dot{\bar{\rho}}_\nu} = \frac{\delta\rho_\gamma}{\dot{\bar{\rho}}_\gamma} = \frac{\delta\rho_D}{\dot{\bar{\rho}}_D} = \frac{\delta\rho_B}{\dot{\bar{\rho}}_B} = \delta t. \tag{B.28}$$

If in this relation we use the continuity equation for the background quantities and the definition of the density contrast, it results

$$\frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma = \delta_D = \delta_B = -3H\delta t. \tag{B.29}$$

On the other hand, if we work at early times, so we can assume the superhorizon limit for all modes (i.e. $k/a \to 0$). With this, by combining the equations ((B.21) - (B.24)), we get

$$\frac{3}{4}\dot{\delta}_\nu = \frac{3}{4}\dot{\delta}_\gamma = \dot{\delta}_D = \dot{\delta}_B = -3H\delta t. \tag{B.30}$$

With these relations, we can rewrite the equation (B.27), where we obtain:

$$\frac{3}{4}\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\dot{\delta}_\gamma\right) = 4\pi G\delta_\gamma\bar{\rho}_R\left(1 + \frac{3}{8}\frac{\bar{\rho}_M}{\bar{\rho}_R}\right), \tag{B.31}$$

where $\bar{\rho}_R = \bar{\rho}_\gamma + \bar{\rho}_\nu$ and $\bar{\rho}_M = \bar{\rho}_D + \bar{\rho}_B$. Since we are working in the era dominated by radiation, then $a \propto t^{1/2}$ and $H = 1/2t$. Additionally, it is true that $\bar{\rho}_R \gg \bar{\rho}_M$, so we can neglect the second term in the parentheses and assume $\bar{\rho}_R \simeq \bar{\rho}_{tot}$. Using these approximations and the Friedmann equation ($3H^2 = 8\pi G\bar{\rho}_{tot}$), we obtain the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\dot{\delta}_\gamma\right) - \frac{1}{t}\delta_\gamma = 0. \tag{B.32}$$

The solutions of this equation will be $\delta_\gamma \propto t$ and $\delta_\gamma \propto 1/t$. We will keep only the increasing mode. This solution, if combined with (B.30), implies that ψ does not depend on time. With this, the solution for the increasing mode of the density contrasts of the different components will be:

$$\frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma = \delta_D = \delta_B = -t\psi_0(\vec{k}). \tag{B.33}$$

If we plug this result into the equations (B.25) and (B.26), we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t^{1/2}}\delta u_{\gamma B}\right) = \frac{1}{3}\psi_0(\vec{k})t^{1/2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t^{1/2}}\delta u_{\nu}\right) = \frac{1}{3}\psi_0(\vec{k})t^{1/2}. \tag{B.34}$$

The solution to these equations will imply that

$$\delta u_\gamma = \delta u_B = \delta u_\nu = \frac{2}{9}\psi_0(\vec{k})t^2. \tag{B.35}$$

To continue this derivation, let's recall the equation (A.26) that involved the curvature perturbation $\mathcal{R}$ in the synchronous gauge:

$$\frac{k^2}{a^2}\mathcal{R} = -H\psi_0 + 4\pi G\delta\rho + \frac{k^2}{a^2}H\delta u. \tag{B.36}$$

If we consider an initial time t_{ini} (in which primordial perturbations begin to evolve) where the universe is dominated by radiation, we can replace $\delta\rho = \bar{\rho}_\gamma\delta_\gamma + \bar{\rho}_\nu\delta_\nu + \bar{\rho}_D\delta_D + \bar{\rho}_B\delta_B \simeq \bar{\rho}_R\delta_\gamma \simeq \bar{\rho}_{tot}\delta_\gamma$. This gives us

$$\psi_o(\vec{k}) = -t\frac{k^2}{a^2}\mathcal{R}(\vec{k}, t_{ini}). \tag{B.37}$$

With this last result, we can conclude that initially every perturbation can be written as a function of the gauge-invariant quantity $\mathcal{R}$, as

$$\frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma = \delta_D = \delta_B = \frac{1}{4H^2}\frac{k^2}{a^2}\mathcal{R}(\vec{k}, t_{ini}), \tag{B.38}$$

$$\delta u_{\gamma B} = \delta u_\nu = -\frac{1}{36H^3}\frac{k^2}{a^2}\mathcal{R}(\vec{k}, t_{ini}), \tag{B.39}$$

$$\psi_0(\vec{k}) = -\frac{1}{2H}\frac{k^2}{a^2}\mathcal{R}(\vec{k}, t_{ini}). \tag{B.40}$$

Appendix C

Derivation of the Mukhanov-Sasaki equation with the ADM formalism

In this appendix we will derive the action presented in the equation (3.80) using the ADM formalism. For the initial formulation of the formalism we will mainly follow the derivation of the Padmanabhan book [190], we recommend reviewing it for more details.

The ADM (Arnowitt-Deser-Misner) formalism, or Hamiltonian formalism of general relativity, consists (in a few words) of working the equations of general relativity (the Einstein-Hilbert action in our case) separating the spatial part and the temporary part. Although it is useful to work with the metric as a whole. To study evolutions of certain elements, it is more convenient to do this separation (mainly for work in numerical relativity).

The starting point is a scalar $t(x^\alpha)$ that defines spacelike hypersurfaces, each of these hypersurfaces is parametrized by the time $\Sigma(t)$. Each hypersurface has a spatial coordinate system y^α. The next step is to connect the coordinate systems between hypersurfaces, for this we use a four-dimensional system $x^i = (t, y^\alpha)$, with a tangent unit vector $t^a = (\partial x^a/\partial t)$ and normal vector $n_a = -N\partial_a t$. Where in the last expression we define the scalar function N called lapse. If we define the projection on the hypersurfaces as $e^a_\alpha = \partial x^a/\partial y^\alpha$, we can decompose the tangent vector t^a as $t^a = Nn^a + N^\alpha e^a_\alpha$, with N^α named shift vector. With these definitions, an infinitesimal jump between hypersurfaces will be

$$dx^a = t^a dt + e^a_\alpha dy^\alpha. \tag{C.1}$$

With this, we can express the metric as:

$$ds^2 = -N^2 dt^2 + h_{\alpha\beta}(dx^\alpha + N^\alpha dt)\Big(dx^\beta + N^\beta dt\Big), \tag{C.2}$$

where $h_{\alpha\beta}$ is a three-dimensional metric that can be understood as the projection of the metric on the spatial hypersurfaces. More explicitly, the elements of the metric in four dimensions will be

$$g_{00} = h_{ij}N^i N^j - N^2, \qquad g_{0j} = h_{ij}N^i, \qquad g_{ij} = h_{ij}. \tag{C.3}$$

while the elements of the inverse metric are:

$$g^{00} = -N^{-2}, \qquad g^{0j} = N^{-2}N^j, \qquad g^{ij} = h^{ij} - N^{-2}N^iN^j. \tag{C.4}$$

Note that for the spatial metric $h_{\alpha\beta}$ it satisfy

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{C.5}$$

while we can relate the determinants, where

$$\sqrt{-g} = N\sqrt{h}. \tag{C.6}$$

With the terms we have so far, for the spatial metric and the normal vector, they fulfill the following useful properties:

$$h^a_b n^n = 0, \qquad n^s\nabla_m n_s = 0, \qquad n_{[m}\nabla_n n_{r]} = 0 \tag{C.7}$$

Continuing, we must include a definition of a covariant derivative for hypersurfaces, this will be via projecting the covariant derivative in the 3-D space, that is, its action on a vector X_n will be:

$$D_m X_n = h^a_m h^b_n \nabla_a X_b. \tag{C.8}$$

This covariant derivative satisfies $D_a h_{mn} = 0$, which makes it a consistent operation on hypersurfaces.

What follows is to have a quantity that quantifies how three-dimensional hypersurfaces behave in 4-D spacetime, we define the extrinsic curvature of the hypersurface as

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta. \tag{C.9}$$

It is desirable to be able to write this quantity in terms of the metric and derivatives of three-dimensional space. For this it can be shown that the extrinsic curvature $K_{\alpha\beta}$ is related to the Lie derivative of the 3-D metric in the normal direction, that is

$$\pounds_{\vec{n}} h_{mn} = -\frac{1}{2}K_{mn}. \tag{C.10}$$

Using the fact that $t^a = Nn^a + N^\alpha e^a_\alpha$ and applying a couple of Lie derivative identities, result the following expression for the extrinsic curvature:

$$K_{ab} = -\frac{1}{2N}\Big[D_a N_b + D_b N_a - \dot{h}_{ab}\Big]. \tag{C.11}$$

Recall that our goal is to write the Einstein-Hilbert action using this formalism, for this we will need to write the Ricci scalar in terms of the three-dimensional metric. The next step to get to this is to have an equation for the Riemann tensor in terms of K_{ab} and the curvature of 3-D space. If we start with the definition of the Riemann tensor through the commutator of the covariant derivatives of a vector

$$-\,{}^{(3)}R^a_{bmn}X_a = D_m D_n X_b - D_n D_m X_b. \tag{C.12}$$

If we work this expression, we will get to

$$^{(3)}R_{abcd} = h_a^m h_b^n h_c^s h_d^t R_{mnst} + \epsilon(K_{ac}K_{bd} - K_{ad}K_{bc}), \tag{C.13}$$

where $\epsilon = n_i n^i = -1$. The above expression is called the Gauss-Codazzi equation.

Additionally, to write the Ricci scalar, we will need to write the contraction of the Ricci tensor two normal vectors (i.e. $R_{bd}n^b n^d$). To calculate this expression, we use as a starting point:

$$R_{mnba}n^a = \nabla_m \nabla_n n_b - \nabla_n \nabla_m n_b, \tag{C.14}$$

and contract two indices, that is

$$R_{bd}n^b n^d = g^{ac}R_{abcd}n^b n^d = n^b \nabla_a \nabla_b n^a - n^b \nabla_b \nabla_a n^a. \tag{C.15}$$

If we use the following identities:

$$K \equiv K_a^a = -\nabla_a n^a, \qquad K_{ij}K^{ij} = \Big(\nabla_i n^j\Big)\Big(\nabla_j n^i\Big), \tag{C.16}$$

results

$$R_{bd}n^b n^d = \nabla_i \Big(K n^i + a^i\Big) - K_{ab}K^{ab} + K^2, \tag{C.17}$$

where $a^i = n^a \nabla_a n^i$.

In order to obtain an expression for the Ricci scalar, we will also need to write the contraction of the Einstein tensor with two normal vectors. For this, we will use the following identity

$$h^{mn}h^{ab}R_{manb} = 2n^m n^n G_{mn}. \tag{C.18}$$

Then, if in the equation (C.13) we contract the indices, we will obtain

$$^{(3)}R = h^{sm}h^{tn}R_{mnst} - K^2 + K_{cd}K^{cd}, \tag{C.19}$$

replacing the identity, we are left with

$$2n^m n^n G_{mn} = {}^{(3)}R + k^2 - K_{mn}K^{mn}. \tag{C.20}$$

Making a convenient identity appear from the Ricci scalar and using the definition of the Einstein tensor, we have

$$R = -2\frac{1}{2}Rg_{ab}n^a n^b = 2(G_{ab} - R_{ab})n^a n^b. \tag{C.21}$$

Where we already know the expressions for $2n^m n^n G_{mn}$ and $R_{ab}n^a n^b$, so we get

$$R = {}^{(3)}R + K_{ab}K^{ab} - K^2 - 2\nabla_i (K n^i - a^i). \tag{C.22}$$

When writing the action for this formalism we can omit the last term since it is a total derivative, so the Einstein Hilbert action is:

$$S_{ADM,EH} = \int dt dx^3 N\sqrt{h}\Big({}^{(3)}R + K_{ab}K^{ab} - K^2\Big) \tag{C.23}$$

Now we must incorporate the part of the action associated with the inflaton. If we replace the metric of this formalism, which we express by components in (C.3), in the part associated with the kinetic term of the inflaton, we obtain the following action:

$$S_{ADM} = \frac{1}{2}\int dtd^3x\sqrt{h}\Big[N^{(3)}R + \frac{1}{N}\Big(E_{ij}E^{ij} - E^2\Big) + \frac{1}{N}\Big(\dot{\phi} - N^i\partial_i\phi\Big)^2 - Nh^{ij}\partial_i\phi\partial_j\phi - 2NV\Big], \tag{C.24}$$

where we use the relation for the determinant of g expressed in the equation (C.6) and we define the tensor E_{ij} which is the dimensionless extrinsic curvature $E_{ij} = NK_{ij}$.

One of the great benefits of this formalism is that N and N^i can be interpreted as Lagrange multipliers, so that by varying the action with respect to these variables we will obtain the constraint equations. If we vary, in the first place, the action with respect to N and we minimize, this is

$$\frac{\delta S_{ADM}}{\delta N} = 0 \Leftrightarrow {}^{(3)}R - \frac{1}{N^2}\Big(E_{ij}E^{ij} - E^2\Big) - \frac{1}{N^2}\Big(\dot{\phi} - N^i\partial_i\phi\Big)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V = 0. \tag{C.25}$$

With the variation with respect to the shift vector N^i we must be careful with the fact that the extrinsic curvature also depends on this variable, in the procedure we must integrate by parts and we get

$$\frac{\delta S_{ADM}}{\delta N^i} = 0 \Leftrightarrow -\frac{1}{N}\partial_i\phi\Big(\dot{\phi} - N^j\partial_j\phi\Big) + D_j\left(\frac{1}{N}\Big(E_i^j - h_i^jE\Big)\right) = 0, \tag{C.26}$$

remember that in the previous expression $E^i_j = h^{ik}E_{kj}$ and $E = E^i_i$.

Our next step is to place ourselves in the comoving gauge, where $\delta\phi = 0$ so all the spatial derivatives of the inflaton vanish (since $\phi = \bar{\phi}(t)$). With this the constraint equations will be

$${}^{(3)}R - \frac{1}{N^2}\Big(E_{ij}E^{ij} - E^2\Big) - \frac{1}{N^2}\dot{\bar{\phi}}^2 - 2V = 0, \tag{C.27}$$

$$D_j\left(\frac{1}{N}\Big(E_i^j - h_i^jE\Big)\right) = 0. \tag{C.28}$$

On the other hand, in this gauge, the spatial metric h_{ij} takes the following form[43]

$$h_{ij} = a^2(1 - 2\mathcal{R})\delta_{ij}, \qquad h^{ij} = \frac{1}{a^2(1 - 2\mathcal{R})}\delta^{ij}. \tag{C.29}$$

Additionally, this gauge fulfills that the Ricci scalar is

$${}^{(3)}R = \frac{4}{a^2}\vec{\nabla}^2\mathcal{R}, \tag{C.30}$$

where $\vec{\nabla}^2 \equiv \partial^i\partial_i$.

[43] This is the spatial metric on the comoving gauge that we defined in Chapter 3 and worked in Appendix A. For the derivations shown in Chapter 6, that we detailed in Appendix E, we use an equivalent definition with $h_{ij} = a^2e^{2\zeta}\delta_{ij}$

What continues is to solve the equations (C.27, C.28) for N and N^i in a first order perturbatively in $\mathcal{R}$, since in this way we obtain an action quadratic in $\mathcal{R}$ when replacing the solutions in (C.24). For this, first of all, we separate the shift vector as:

$$N_i = \partial_i\psi + \tilde{N}_i, \qquad \partial^i\tilde{N}_i = 0. \tag{C.31}$$

While we define the perturbation in the lapse as $N = 1+\alpha$ and expand α, ψ and $\tilde{N}_i$ as

$$\psi = \psi_1 + \psi_2 + ..., \qquad \alpha = \alpha_1 + \alpha_2 + ..., \qquad \tilde{N}_i = \tilde{N}_i^{(1)} + \tilde{N}_i^{(2)} + \ldots . \tag{C.32}$$

Where in the previous expression it is true for example that $\mathcal{O}(\psi_i) = \mathcal{O}(\mathcal{R}^i)$.

At first order, we can approximate $D_iN_j \approx \partial_iN_j$ in the extrinsic curvature E_{ij}, then

$$\begin{aligned} E_{ij} &\approx \frac{1}{2}\left(\dot{h}_{ij} - \partial_iN_j - \partial_jN_i\right) \\ &= \frac{1}{2}\left(2a\dot{a}(1-2\mathcal{R})\delta_{ij} - 2\dot{\mathcal{R}}\delta_{ij} - \partial_iN_j - \partial_jN_i\right) \\ &= a^2H\left(1 - 2\mathcal{R} - \frac{\dot{\mathcal{R}}}{H}\right)\delta_{ij} - \frac{1}{2}(\partial_iN_j + \partial_jN_i). \end{aligned} \tag{C.33}$$

With this, we can calculate $E_{ij}E^{ij}$, also approaching the first order h^{ij}

$$h^{ij} \approx \frac{1}{a^2}(1+2\mathcal{R})\delta^{ij}, \tag{C.34}$$

so

$$E_{ij}E^{ij} = E_{ij}h^{il}h^{jm}E_{lm} = \frac{1}{a^4}(1+2\mathcal{R})^2E_{ij}E_{lm}\delta^{il}\delta^{jm} \approx \frac{1}{a^4}(1+4\mathcal{R})E_{ij}E_{lm}\delta^{il}\delta^{jm}, \tag{C.35}$$

replacing the approximation for E_{ij}

$$\begin{aligned} E_{ij}E^{ij} &\approx \frac{1}{a^4}(1+4\mathcal{R})E_{ij}E_{lm}\delta^{il}\delta^{jm} \\ &= \frac{\delta^{il}\delta^{jm}}{a^4}(1+4\mathcal{R})\left[a^2H\left(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta_{ij} - \frac{1}{2}(\partial_iN_j+\partial_jN_i)\right]\times \\ &\qquad \left[a^2H\left(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta_{lm} - \frac{1}{2}(\partial_lN_m+\partial_mN_l)\right] \\ &\approx \frac{(1+4\mathcal{R})}{a^4}\left[3a^4H^2\left(1-4\mathcal{R}-\frac{2\dot{R}}{H}\right) - 2a^2H\partial_iN_j\delta^{ij}\right] \\ &\approx 3H^2\left(1-\frac{2\dot{R}}{H}\right) - \frac{2H}{a^2}\partial_iN_j\delta^{ij}. \end{aligned} \tag{C.36}$$

Additionally

$$
\begin{aligned}
E^i_j &= h^{ik}E_{kj} \\
&\approx \frac{1}{a^2}(1+2\mathcal{R})\delta^{ik}\left[a^2H\left(1-2\mathcal{R}-\frac{\dot{R}}{H}\right)\delta_{kj}-\frac{1}{2}(\partial_k N_j+\partial_j N_k)\right] \\
&\approx H\left(1-\frac{\dot{\mathcal{R}}}{H}\right)\delta^i_j-\frac{\delta^{ik}}{2a^2}(\partial_k N_j+\partial_j N_k)
\end{aligned} \tag{C.37}
$$

with this we can also deduce

$$
\begin{aligned}
E &= E^i_i \\
&\approx 3H\left(1-\frac{\dot{\mathcal{R}}}{H}\right)-\frac{1}{a^2}\partial_i N_j\delta^{ij},
\end{aligned} \tag{C.38}
$$

which implies

$$
\begin{aligned}
E^2 &\approx \left[3H\left(1-\frac{\dot{\mathcal{R}}}{H}\right)-\frac{1}{a^2}\partial_i N_j\delta^{ij}\right]^2 \\
&\approx 9H^2\left(1-2\frac{\dot{\mathcal{R}}}{H}\right)-\frac{6H}{a^2}\partial_i N_j\delta^{ij},
\end{aligned} \tag{C.39}
$$

putting together the results for $E_{ij}E^{ij}$ and E we obtain that

$$
\begin{aligned}
E_{ij}E^{ij}-E^2 &\approx 3H^2\left(1-\frac{2\dot{R}}{H}\right)-\frac{2H}{a^2}\partial_i N_j\delta^{ij}-9H^2\left(1-2\frac{\dot{\mathcal{R}}}{H}\right)+\frac{6H}{a^2}\partial_i N_j\delta^{ij} \\
&= -6H^2\left(1-\frac{2\dot{R}}{H}\right)+\frac{4H}{a^2}\partial_i N_j\delta^{ij}
\end{aligned} \tag{C.40}
$$

Replacing the equations (C.40) and (C.30) in the first constraint equation (eq. (C.27)), we obtain

$$
\frac{4}{a^2}\vec{\nabla}^2\mathcal{R}-\frac{1}{(1+\alpha_1)^2}\left(-6H^2\left(1-\frac{2\dot{R}}{H}\right)+\frac{4H}{a^2}\partial_i N_j\delta^{ij}\right)-\frac{\dot{\phi}^2}{(1+\alpha_1)^2}-2V=0, \tag{C.41}
$$

where we replace $N=1+\alpha_1$. At first order, the above equation results:

$$
\frac{4}{a^2}\vec{\nabla}^2\mathcal{R}-12H\dot{\mathcal{R}}-\frac{4H}{a^2}\partial_i N_j\delta^{ij}-12H^2\alpha_1+2\dot{\phi}^2\alpha_1-\dot{\phi}^2-2V+6H^2=0, \tag{C.42}
$$

if we use the Friedmann equation the last three terms vanish and we can rewrite the equation in a simpler way

$$
\frac{4}{a^2}\vec{\nabla}^2\mathcal{R}-12H\dot{\mathcal{R}}-\frac{4H}{a^2}\partial_i N_j\delta^{ij}-12H^2\alpha_1+2\dot{\phi}^2\alpha_1=0 \tag{C.43}
$$

While the second constraint equation, making the substitutions according to the expansion, remains

$$\partial_j\left[\frac{1}{1+\alpha_1}\left(-2H\left(1-\frac{\dot{\mathcal{R}}}{H}\right)\delta_i^j-\frac{\delta^{jk}}{2a^2}(\partial_k N_i+\partial_i N_k)+\frac{\delta_i^j}{a^2}\partial_l N_m\delta^{lm}\right)\right]=0, \tag{C.44}$$

where in a first order we get

$$\partial_j\left[-2H\left(1-\alpha_1-\frac{\dot{\mathcal{R}}}{H}\right)\delta_i^j-\frac{\delta^{jk}}{2a^2}(\partial_k N_i+\partial_i N_k)+\frac{\delta_i^j}{a^2}\partial_l N_m\delta^{lm}\right]=0. \tag{C.45}$$

If we make the substitution for the shift vector that we write in the equation (C.31) to first order, in the constraint equations (C.43, C.45), it results

$$\frac{4}{a^2}\vec{\nabla}^2\mathcal{R}-12H\dot{\mathcal{R}}-4H\vec{\nabla}^2\psi_1-4\alpha_1 V=0, \tag{C.46}$$

$$2H\partial_i\left(\alpha_1+\frac{\dot{\mathcal{R}}}{H}\right)-\frac{1}{2}\vec{\nabla}^2\tilde{N}_i^{(1)}=0, \tag{C.47}$$

where in both equations we incorporate the fact that first order $\vec{\nabla}^2=\partial^i\partial_i\approx a^{-2}\delta^{ij}\partial_i\partial_j$. We can solve the second constraint equation directly since α_1 and $\tilde{N}_i^{(1)}$ are independent parameters, we obtain

$$\alpha_1=-\frac{\dot{\mathcal{R}}}{H}, \qquad \tilde{N}_i^{(1)}=0, \tag{C.48}$$

the solution for $\tilde{N}_i^{(1)}$ can be deduced since we are free to choose appropriate boundary conditions. Replacing the solution for α_1 in the first constraint equation, we solve for ψ_1

$$\vec{\nabla}^2\psi_1=\frac{\vec{\nabla}^2\mathcal{R}}{a^2H}+\dot{\mathcal{R}}\left(\frac{V}{H^2}-3\right), \tag{C.49}$$

the term in the parentheses corresponds to the slow roll parameter $\varepsilon=\dot{\phi}^2/2H^2=3-V/H^2$. If we define the inverse Laplacian $\vec{\nabla}^{-2}$ in such a way that it satisfies $\vec{\nabla}^{-2}\vec{\nabla}^2=1$, we solve for ψ_1

$$\psi_1=\frac{\mathcal{R}}{a^2H}-\varepsilon\vec{\nabla}^{-2}\dot{\mathcal{R}}. \tag{C.50}$$

What follows is to separate the Lagrangian by order of magnitude up to second order, that is

$$\begin{aligned}\mathcal{L}&=N^{(3)}R+\frac{1}{N}\left(E_{ij}E^{ij}-E^2\right)+\frac{\dot{\phi}^2}{N}-2NV\\&=\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2+...,\end{aligned} \tag{C.51}$$

to do this, the terms we need to expand are

$$\frac{1}{N}=\frac{1}{1+\alpha}\approx 1-\alpha+\alpha^2=1-\alpha_1-\alpha_2+(\alpha_1+\alpha_2)^2\approx 1-\alpha_1-\alpha_2-\alpha_1^2, \tag{C.52}$$

and

$$\Big(E_{ij}E^{ij}-E^2\Big)\approx\Big(E_{ij}E^{ij}-E^2\Big)^{(0)}+\Big(E_{ij}E^{ij}-E^2\Big)^{(1)}+\Big(E_{ij}E^{ij}-E^2\Big)^{(2)}. \tag{C.53}$$

So the second order Lagrangian remains

$$\begin{aligned}\mathcal{L}=(1+\alpha_1+\alpha_2)^{(3)}R+\Big(1-\alpha_1-\alpha_2-\alpha_1^2\Big)\Bigg[\Big(E_{ij}E^{ij}-E^2\Big)^{(0)}+\Big(E_{ij}E^{ij}-E^2\Big)^{(1)}+\\ \Big(E_{ij}E^{ij}-E^2\Big)^{(2)}\Bigg]+\Big(1-\alpha_1-\alpha_2-\alpha_1^2\Big)\dot{\phi}^2-2(1+\alpha_1+\alpha_2)V,\end{aligned} \tag{C.54}$$

that separated by order of magnitude in $\mathcal{R}$ (we make $\alpha_2=0$) is

$$\mathcal{L}_0=\dot{\phi}^2-2V+\Big(E_{ij}E^{ij}-E^2\Big)^{(0)}, \tag{C.55}$$

$$\mathcal{L}_1={}^{(3)}R-\alpha_1\dot{\phi}^2-2\alpha_1 V+\Big(E_{ij}E^{ij}-E^2\Big)^{(1)}-\alpha_1\Big(E_{ij}E^{ij}-E^2\Big)^{(0)}, \tag{C.56}$$

$$\mathcal{L}_2=\alpha_1{}^{(3)}R+\alpha_1^2\dot{\phi}^2+\Big(E_{ij}E^{ij}-E^2\Big)^{(2)}-\alpha_1\Big(E_{ij}E^{ij}-E^2\Big)^{(1)}+\alpha_1^2\Big(E_{ij}E^{ij}-E^2\Big)^{(0)}. \tag{C.57}$$

It is very important not to forget that we must also expand the square root of the determinant of the spatial metric, so additional contributions will appear to the Lagrangian up to second order. The expansion is

$$\sqrt{h}=\sqrt{a^6(1-2\mathcal{R})^3}\approx a^3\Big(1-3\mathcal{R}+\frac{3}{2}\mathcal{R}^2\Big). \tag{C.58}$$

Now we must expand the expression $(E_{ij}E^{ij}-E^2)$ to second order (remember that to solve the constraint equations we expand said expression to first order), we start from the definition of E_{ij}:

$$\begin{aligned}E_{ij}&=\frac{1}{2}\Big(\dot{h}_{ij}-D_iN_j-D_jN_j\Big)\\ &=a^2H\Bigg(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\Bigg)\delta_{ij}-\frac{1}{2}(D_iN_j+D_jN_i),\end{aligned} \tag{C.59}$$

follows that at order two, the inverse spatial metric h^{ij} is

$$h^{ij}\approx\frac{1}{a^2}\Big(1+2\mathcal{R}+4\mathcal{R}^2\Big)\delta^{ij}, \tag{C.60}$$

so

$$\begin{aligned}E_{ij}E^{ij}\approx\frac{\delta^{il}\delta^{jm}}{a^4}\Big(1+2\mathcal{R}+4\mathcal{R}^2\Big)^2\Bigg[a^2H\Bigg(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\Bigg)\delta_{ij}-\frac{1}{2}(D_iN_j+D_jN_i)\Bigg]\times\\ \Bigg[a^2H\Bigg(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\Bigg)\delta_{lm}-\frac{1}{2}(D_lN_m+D_mN_l)\Bigg],\end{aligned}$$

we define $F_{ij}\equiv D_iN_j+D_jN_j$ to simplify the calculations (noting that $\delta^{ij}F_{ij}=2\delta^{ij}D_iN_j$), we have

$$E_{ij}E^{ij} \approx \frac{\left(1+2\mathcal{R}+4\mathcal{R}^2\right)^2}{a^4}\Bigg[3a^4H^2\left(1+4\mathcal{R}^2+\frac{\dot{\mathcal{R}}^2}{H^2}-4\mathcal{R}-\frac{2\dot{\mathcal{R}}}{H}+4\mathcal{R}\frac{\dot{\mathcal{R}}}{H}\right)-$$
$$2a^2H\left(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta^{ij}D_iN_j+\frac{1}{4}\delta^{il}\delta^{jm}F_{ij}F_{lm}\Bigg]$$
$$\approx 3H^2\left(1-\frac{2\dot{\mathcal{R}}}{H}+\frac{\dot{\mathcal{R}}^2}{H^2}-\frac{4\mathcal{R}\dot{\mathcal{R}}}{H}\right)-\frac{2H}{a^2}\left(1+2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta^{ij}D_iN_j+\frac{1}{4a^4}\delta^{il}\delta^{jm}F_{ij}F_{lm}, \tag{C.61}$$

then

$$E^i_j = h^{ik}E_{kj}$$
$$\approx \frac{1}{a^2}\left(1+2\mathcal{R}+4\mathcal{R}^2\right)\delta^{ik}\left[a^2H\left(1-2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta_{kj}-\frac{1}{2}(D_kN_j+D_jN_k)\right]$$
$$\approx H\left(1-\frac{\dot{\mathcal{R}}}{H}-\frac{2\mathcal{R}\dot{\mathcal{R}}}{H}\right)\delta^i_j-\frac{(1+2\mathcal{R})}{2a^2}(D_kN_j+D_jN_k)\delta^{ik}, \tag{C.62}$$

this implies

$$E = E^i_i$$
$$\approx 3H\left(1-\frac{\dot{\mathcal{R}}}{H}-\frac{2\mathcal{R}\dot{\mathcal{R}}}{H}\right)-\frac{(1+2\mathcal{R})}{a^2}\delta^{ik}D_kN_i \tag{C.63}$$

$$\Rightarrow E^2 \approx \left[3H\left(1-\frac{\dot{\mathcal{R}}}{H}-\frac{2\mathcal{R}\dot{\mathcal{R}}}{H}\right)-\frac{(1+2\mathcal{R})}{a^2}\delta^{ik}D_kN_i\right]^2$$
$$\approx 9H^2\left(1-\frac{2\dot{\mathcal{R}}}{H}+\frac{\dot{\mathcal{R}}^2}{H^2}-\frac{4\mathcal{R}\dot{\mathcal{R}}}{H}\right)-\frac{6H}{a^2}\left(1+2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta^{ij}D_iN_j+\frac{1}{a^4}\left(\delta^{ij}D_iN_j\right)^2. \tag{C.64}$$

Putting all of the above together, we have a version of $E_{ij}E^{ij}-E^2$ up to second order

$$E_{ij}E^{ij}-E^2 \approx -6H^2\left(1-\frac{2\dot{\mathcal{R}}}{H}+\frac{\dot{\mathcal{R}}^2}{H^2}-\frac{4\mathcal{R}\dot{\mathcal{R}}}{H}\right)+\frac{4H}{a^2}\left(1+2\mathcal{R}-\frac{\dot{\mathcal{R}}}{H}\right)\delta^{ij}D_iN_j+$$
$$\frac{1}{4a^4}\delta^{il}\delta^{jm}F_{ij}F_{lm}-\frac{1}{a^4}\left(\delta^{ij}D_iN_j\right)^2 \tag{C.65}$$

To continue, we must write the spatial covariant derivative D_iN_j explicitly, from its definition as

$$D_iN_j = \partial_iN_j+\Gamma^k_{ij}N_k, \tag{C.66}$$

where $\Gamma^k_{ij}=\frac{1}{2}h^{kl}(\partial_ih_{lj}+\partial_jh_{li}-\partial_lh_{ij})$. If we approximate up to order two, we have

$$D_iN_j \approx \partial_iN_j+(\partial_i\mathcal{R})N_j+(\partial_j\mathcal{R})N_i-\delta_{ij}\delta^{kl}N_k(\partial_l\mathcal{R}). \tag{C.67}$$

With this expression, we can write the terms that have a covariant derivative of the equation (C.65)

$$\delta^{ij} D_i N_j = \delta^{ij}\partial_i N_j - \delta^{ij}(\partial_i \mathcal{R}) N_j, \tag{C.68}$$

$$\delta^{il}\delta^{jm} F_{ij} F_{lm} \approx \delta^{il}\delta^{jm}(\partial_i N_j \partial_l N_m + \partial_i N_j \partial_m N_l + \partial_j N_i \partial_l N_m + \partial_j N_i \partial_m N_l), \tag{C.69}$$

$$\left(\delta^{ij} D_i N_j\right)^2 \approx \left(\delta^{ij}\partial_i N_j\right)^2. \tag{C.70}$$

if we replace $N_j = \partial_j \psi_1$, we get

$$\delta^{ij} D_i N_j = \delta^{ij}\partial_i\partial_j\psi_1 - \delta^{ij}\partial_i\psi_1\partial_j\mathcal{R} = \delta^{ij}\partial_i\partial_j\psi_1(1+\mathcal{R}) = a^2\vec{\nabla}^2\psi_1(1+\mathcal{R}), \tag{C.71}$$

where in the second equality we integrate by parts while in the third we use that $\delta^{ij}\partial_i\partial_j = a^2\vec{\nabla}^2$. By replacing these last results in the equation (C.65), we can omit the last two terms since they cancel, we are left with

$$E_{ij}E^{ij} - E^2 \approx -6H^2\left(1 - \frac{2\dot{\mathcal{R}}}{H} + \frac{\dot{\mathcal{R}}^2}{H^2} - \frac{4\mathcal{R}\dot{\mathcal{R}}}{H}\right) + 4H\left(1 + 2\mathcal{R} - \frac{\dot{\mathcal{R}}}{H}\right)(1+\mathcal{R})\vec{\nabla}^2\psi_1. \tag{C.72}$$

Replacing the solution for ψ_1 obtained from the constraint equation (C.50), we obtain

$$\begin{aligned} E_{ij}E^{ij} - E^2 \approx\ & -6H^2 + 12H\dot{\mathcal{R}} - 6\dot{\mathcal{R}}^2 + 24H\mathcal{R}\dot{\mathcal{R}} + \\ & 4H\left[\frac{\vec{\nabla}^2\mathcal{R}}{a^2H} - \varepsilon\dot{\mathcal{R}} - \frac{\dot{\mathcal{R}}\vec{\nabla}^2\mathcal{R}}{a^2H^2} - \frac{\varepsilon\dot{\mathcal{R}}^2}{H} + \frac{3\mathcal{R}\vec{\nabla}^2\mathcal{R}}{a^2H} - 3\varepsilon\mathcal{R}\dot{\mathcal{R}}\right]. \end{aligned} \tag{C.73}$$

In order not to get confused, we are going to separate this expression by order of magnitude first, that is,

$$\left(E_{ij}E^{ij} - E^2\right)^{(0)} = -6H^2, \tag{C.74}$$

$$\left(E_{ij}E^{ij} - E^2\right)^{(1)} = \frac{4}{a^2}\vec{\nabla}^2\mathcal{R} + 12H\dot{\mathcal{R}} - 4H\varepsilon\dot{\mathcal{R}}, \tag{C.75}$$

$$\left(E_{ij}E^{ij} - E^2\right)^{(2)} = -6\dot{\mathcal{R}}^2 + 24H\mathcal{R}\dot{\mathcal{R}} - \frac{4}{a^2H}\dot{\mathcal{R}}\vec{\nabla}^2\mathcal{R} + 4\varepsilon\dot{\mathcal{R}}^2 + \frac{12}{a^2}\mathcal{R}\vec{\nabla}^2\mathcal{R} - 12H\varepsilon\mathcal{R}\dot{\mathcal{R}}. \tag{C.76}$$

Replacing these expressions in the equations (C.55), (C.56) and (C.57), together with the solution for α_1 and the Friedmann equation for ϕ, we obtain

$$\mathcal{L}_0 = -6H^2 + \dot{\phi}^2 - 2V = -4V, \tag{C.77}$$

$$\begin{aligned} \mathcal{L}_1 &= \frac{8}{a^2}\vec{\nabla}^2\mathcal{R} + 12H\dot{\mathcal{R}} - 4H\varepsilon\dot{\mathcal{R}} + \frac{\mathcal{R}}{H}\left(-6H^2 + \dot{\phi}^2 + 2V\right) \\ &= \frac{8}{a^2}\vec{\nabla}^2\mathcal{R} + 12H\dot{\mathcal{R}} - 4H\varepsilon\dot{\mathcal{R}}, \end{aligned} \tag{C.78}$$

$$\mathcal{L}_2 = -\frac{4}{a^2H}\dot{\mathcal{R}}\vec{\nabla}^2\mathcal{R} + 24H\mathcal{R}\dot{\mathcal{R}} + \frac{12}{a^2}\mathcal{R}\vec{\nabla}^2\mathcal{R} - 12H\varepsilon\mathcal{R}\dot{\mathcal{R}} + \dot{\mathcal{R}}^2\frac{\dot{\phi}^2}{H^2}. \tag{C.79}$$

The first term in $\mathcal{L}_1$ is an boundary term so we can neglect it. Now, we must consider the contributions in the quadratic Lagrangian from the combination $\sqrt{h}(\mathcal{L}_0 + \mathcal{L}_1)$. The contribution of $\mathcal{L}_0$ is

$$\left(\frac{3}{2}\mathcal{R}^2\right)\mathcal{L}_0 = -6\mathcal{R}^2 V, \tag{C.80}$$

while the quadratic contribution of $\mathcal{L}_1$ is

$$(-3\mathcal{R})\mathcal{L}_1 = -36H\mathcal{R}\dot{\mathcal{R}} + 12\varepsilon H\mathcal{R}\dot{\mathcal{R}}. \tag{C.81}$$

Putting these contributions together, the quadratic Lagrangian results

$$\begin{aligned}\mathcal{L}^{(2)}_{ADM} &= a^3\left[\left(\frac{3}{2}\mathcal{R}^2\right)\mathcal{L}_0 + (-3\mathcal{R})\mathcal{L}_1 + \mathcal{L}_2\right] \\ &= a^3\left[-6\mathcal{R}^2 V - 12H\mathcal{R}\dot{\mathcal{R}} - \frac{4}{a^2 H}\dot{\mathcal{R}}\vec{\nabla}^2\mathcal{R} + \frac{12}{a^2}\mathcal{R}\vec{\nabla}^2\mathcal{R} + \dot{\mathcal{R}}^2\frac{\dot{\phi}^2}{H^2}\right].\end{aligned} \tag{C.82}$$

To conclude, we have to arrange this action in a convenient way. First we will focus on the second term noting that we can rewrite it as

$$(*) \equiv -12a^3 H\mathcal{R}\dot{\mathcal{R}} = 6\partial_t\left(a^3 H\right)\mathcal{R}^2 + \frac{d}{dt}(...), \tag{C.83}$$

in this equation we get rid of the total derivative, then we do the time derivative and we are left

$$(*) = 6a^3\left(3H^2 + \dot{H}\right), \tag{C.84}$$

then, we replace with the background equation $\dot{H} = -\dot{\phi}^2/2$, this is

$$(*) = 6a^3\left(3H^2 + \frac{1}{2}\dot{\phi}^2\right)\mathcal{R}^2 = 6a^3\mathcal{R}^2 V. \tag{C.85}$$

In the last equality we use the Friedmann equation $3H^2 = \dot{\phi}^2/2 + V$. We notice then that the second term of the action in the equation (C.82) cancels the first one perfectly. It follows to do an equivalent procedure for the third term, where

$$(**) \equiv -\frac{4a}{H}\dot{\mathcal{R}}\vec{\nabla}^2\mathcal{R} = -\frac{4}{aH}\dot{\mathcal{R}}\delta^{ij}\partial_i\partial_j\mathcal{R} = \frac{4}{aH}\delta^{ij}\partial_i\dot{\mathcal{R}}\partial_j\mathcal{R}, \tag{C.86}$$

where in the first equality we use the definition of $\vec{\nabla}^2\mathcal{R} = a^{-2}\delta^{ij}\partial_i\partial_j$, while in the second equality we integrate by parts in space. As in the procedure for $(*)$, we integrate by parts in time

$$(**) = -2\partial_t\left(\frac{1}{aH}\right)\delta^{ij}\partial_i\mathcal{R}\partial_j\mathcal{R} = \frac{2}{(aH)^2}\left(\dot{a}H + a\dot{H}\right)\delta^{ij}\partial_i\mathcal{R}\partial_j\mathcal{R} = \frac{2}{a}\left(1 + \frac{\dot{H}}{H^2}\right)\delta^{ij}\partial_i\mathcal{R}\partial_j\mathcal{R}. \tag{C.87}$$

In the above expression, we can rearrange the first term by integrating by parts back into space. Additionally we replace $\dot{H}$ from the background equations

$$(**) = -2a\mathcal{R}\vec{\nabla}^2\mathcal{R} + \frac{1}{a}\frac{\dot{\phi}^2}{H^2}\delta^{ij}\partial_i\mathcal{R}\partial_j\mathcal{R}. \tag{C.88}$$

The second order Lagrangian remains (we use $h^{ij} = a^{-2}\delta^{ij}$ at first order)

$$\mathcal{L}^{(2)}_{ADM} = a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - \frac{1}{a^2}\partial^i\mathcal{R}\partial_i\mathcal{R}\right] + 10a\mathcal{R}\vec{\nabla}^2\mathcal{R}. \tag{C.89}$$

The last term disappears as it is a boundary term. Finally, if we use the Mukhanov variables, that is

$$v = z\mathcal{R}, \qquad z = a\frac{\dot{\phi}}{H}, \tag{C.90}$$

and using the conformal time, we arrive at the Mukhanov-Sasaki action that we write in the equation (3.80)

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x\left((v')^2 + \frac{z''}{z}v^2 + (\partial_i v)^2\right) \tag{C.91}$$

Appendix D

Power Spectrum in Slow-Roll Single-Field inflation

In this appendix we will derive the solutions for the mode fucntions in the case of slow roll inflation and thus obtain the power spectrum expressed in the equation (3.97).

We will start by writing the slow roll parameters ε and η, where

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}, \qquad \eta \equiv 2\varepsilon - \frac{\dot{\varepsilon}}{2H\varepsilon}. \tag{D.1}$$

From the definition of the parameter ε, we can solve an approximation for the scale factor as a function of the proper time, since

$$\frac{d}{d\tau}\left(\frac{1}{aH}\right) = -(1-\varepsilon). \tag{D.2}$$

If ε is considered a constant for this case, we integrate directly and obtain the following approximate expression

$$a(\tau) \approx -\frac{1}{\tau H(1-\varepsilon)}. \tag{D.3}$$

Now we must solve the Mukhanov-Sasaki equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, \tag{D.4}$$

for the case of slow-roll inflation. We should write z''/z conveniently taking advantage of the fact that ε and η are small parameters. With a little direct algebra (and getting rid of the elements of order higher than two) we will arrive to

$$\frac{z''}{z} \approx \frac{1}{\tau^2}\left(\frac{9}{4} + 22\varepsilon^2 + 9\varepsilon - 11\varepsilon\eta - 3\eta + \eta^2 - \frac{1}{4}\right) \approx \frac{1}{\tau^2}\left(\left(\frac{3}{2} + 3\varepsilon - \eta\right)^2 - \frac{1}{4}\right), \tag{D.5}$$

where in the second approximation we assemble the perfect square by removing second order elements. Writing z''/z in this way is convenient since if we define

$$\nu = \frac{3}{2} + 3\varepsilon - \eta, \tag{D.6}$$

the solution for the equation (D.4) can be written as a linear combination of Hankel functions

$$v_k(\tau) = \sqrt{x}\Big[c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x)\Big], \tag{D.7}$$

where $x \equiv k|\tau|$.

It follows to impose the boundary condition for a very early time ($x = k|\tau| \to \infty$), where the mode functions are set equal to the ones of a minkowski space (eq. (3.87)). In this limit for x the Hankel functions can be approximated as

$$H_\nu^{(1,2)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big)\right]. \tag{D.8}$$

Taking care that $|\tau| = -\tau$ in inflation, when imposing this limit c_2 should disappear. The solution for v_k will be

$$v_k = a_1\sqrt{\frac{\pi x}{4k}} H_\nu^{(1)}(x), \tag{D.9}$$

whit $a_1 = \exp[i(2\nu + 1)\pi/4]$ a scale-invariant phase.

We are interested in obtaining the power spectrum of $\mathcal{R}$ at the time of the end of inflation. For this, we can approximate $H_\nu^{(1)}(x)$ for the limit where $x \to 0$ as

$$H_\nu^{(1)}(x) \approx \frac{i}{\pi}\Gamma(\nu)\left(\frac{k\tau}{2}\right)^{-\nu}, \tag{D.10}$$

with $\Gamma(\nu)$ the Gamma function. In this limit, the mode function v_k (moving the terms a bit) we are left with

$$v_k(\tau) = i\frac{a_1\pi^{-1/2}}{\sqrt{2k}} 2^{\nu-1/2}\Gamma(\nu)(k\tau)^{-\nu+1/2}. \tag{D.11}$$

We obtain the power spectrum using the equation (3.93), if we additionally use the approximation of the proper time $\tau = 1/aH$, we get

$$P_{\mathcal{R}}^{(SR)}(k) = \frac{H^2}{2k^3}\frac{\pi^{-1}}{2\varepsilon} 2^{2\nu-1}\Gamma^2(\nu)\left(\frac{k}{aH}\right)^{-2\nu+3}. \tag{D.12}$$

If we use the fact that $\Gamma(3/2) = \sqrt{\pi}/2$, the power spectrum takes a rather convenient form

$$P_{\mathcal{R}}^{(SR)} = P_{\mathcal{R}} 2^{2\nu-3}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{k}{aH}\right)^{-2\nu+3}, \tag{D.13}$$

where $P_{\mathcal{R}}$ is the power spectrum derived in the de Sitter space (eq. (3.96)).

First of all, let us note that it is straightforward to observe that this expression for the power spectrum (eq. (D.13)) returns the scale-invariant solution for Sitter's case if we impose the limit $\varepsilon = \eta = 0$ ($\nu = 3/2$). On the other hand, the spectral index presents corrections due to the Slow-Roll parameters

$$\frac{d\ln\Delta_{\mathcal{R}}^{(SR)}}{d\ln k} \equiv n_s - 1 = -6\varepsilon + 2\eta, \tag{D.14}$$

which is the final result that we expected to obtain in this appendix.

Appendix E

Multifield Inflation Derivations

In this appendix we will make the derivations of multifield inflation, with the main objective of obtaining the kinetic part of the second order Lagrangian expressed in the equation (6.11)[44]

We will start by explicitly calculating the equations of motion for the background fields in multifield inflation. First, let us remember that the action with which we will start has the following form

$$S = S_{EH} - \frac{1}{2}\int d^4x\sqrt{-g}\Big[\gamma_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + 2V(\phi)\Big], \tag{E.1}$$

where γ_{ab} is the metric in the field space. If we vary this action with respect to the fields we obtain

$$\begin{aligned}\frac{\delta S}{\delta\phi^c} &= \frac{1}{2}\sqrt{-g}\bigg(\frac{\delta\gamma_{ab}}{\delta\phi^c}\bigg)\partial_\mu\phi^a\partial^\mu\phi^b + \sqrt{-g}\gamma_{ab}\frac{\delta(\partial_\mu\phi^a)}{\delta\phi^c}\partial^\mu\phi^b + \sqrt{-g}\frac{dV(\phi)}{d\phi^c}\\ &= \frac{1}{2}\sqrt{-g}\tilde{\partial}_c\gamma_{ab}\partial_\mu\phi^a\partial^\mu\phi^b - \partial_\mu\Big(\gamma_{cb}\sqrt{-g}\partial^\mu\phi^b\Big) + \sqrt{-g}V_c,\end{aligned} \tag{E.2}$$

where in the second equality we integrate by parts the second term of the sum and define $\tilde{\partial}_c \equiv \partial/\partial\phi^c$ and $V_c \equiv dV/d\phi^c$ [45]. If in the same second term we use that, for the background FLRW metric, $\partial_\mu\sqrt{-g} = \sqrt{-g}\,3H\delta^0_\mu$ is satisfied, and additionally we note that the background fields depend only on time, it turns out

$$\frac{\delta S}{\delta\phi^c} = \sqrt{-g}\Big[\frac{1}{2}\tilde{\partial}_c\gamma_{ab}\dot{\phi}^a_0\dot{\phi}^b_0 + \gamma_{cb}\,3H\dot{\phi}^b_0 - \tilde{\partial}_a\gamma_{cb}\dot{\phi}^a_0\dot{\phi}^b_0 - \gamma_{cb}\ddot{\phi}^b_0 + V_c\Big]. \tag{E.3}$$

Due to the symmetry of the indices a and b, we can group the first and third terms with the definition of the Christoffel symbols for the field space metric $\tilde{\Gamma}^a_{bc} = \frac{1}{2}\gamma^{af}\Big(\tilde{\partial}_b\gamma_{fc} + \tilde{\partial}_c\gamma_{fb} - \tilde{\partial}_f\gamma_{bc}\Big)$. By imposing that the variation of the action vanishes, we obtain the equation of motion for the background fields:

$$D_t\dot{\phi}^a_0 - 3H\dot{\phi}^a_0 - V^a = 0, \tag{E.4}$$

where we use the definition $D_t\dot{\phi}^a_0 \equiv \ddot{\phi}^a_0 + \tilde{\Gamma}^a_{bc}\dot{\phi}^b_0\dot{\phi}^c_0$ and $V^a = \gamma^{ab}V_b$.

[44] For more details on the derivation and the terms associated with the potential, check the references [154, 191] and the references therein.

[45] In this appendix we will use $\tilde{()}$ to identify operations in the field space

It follows to note from the definition of the energy-momentum tensor associated with this action[46], that for the backround quantities is fulfilled

$$\rho = \frac{1}{2}\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b + V(\phi) \equiv \frac{1}{2}\dot{\phi}_0^2 + V(\phi). \tag{E.5}$$

Then, with the definition of $\dot{\phi}_0^2$, the Friedmann equation for the multifield inflation case takes the usual form of $3H^2 = \dot{\phi}_0^2/2 + V$.

With the definition of $\dot{\phi}_0^2$, we can define the unitary tangent and normal vectors[47] to the inflationary path followed in the field space

$$T^a \equiv \frac{\dot{\phi}_0^a}{\dot{\phi}_0}, \qquad n^a \equiv -\frac{1}{\Omega}D_t T^a \tag{E.6}$$

where $\Omega \equiv n_a D_t T^a$. With this last definition, it is straightforward to deduce that the relation $D_t n^a = \Omega T^a$ holds for the covariant derivative of the normal vector.

What follows is to rewrite the action using the ADM formalism. Using the definitions we introduce in Appendix C, the components of the 4-D metric are

$$g^{00} = -\frac{1}{N^2} \qquad g^{0j} = \frac{N^j}{N^2} \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{E.7}$$

With this, the kinetic term of the action for the case of multifield inflation is

$$\gamma_{ab} g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b = -\frac{1}{N^2}\gamma_{ab}\dot{\phi}^a\dot{\phi}^b + \frac{2N^i}{N^2}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b + \left(h^{ij} - \frac{N^iN^j}{N^2}\right)\gamma_{ab}\partial_i\phi^a\partial_j\phi^b. \tag{E.8}$$

To make it easier to write the constraints equations, we will separate the action into two parts. The first part with the terms associated with the curvature of space-time that is not modified with respect to our derivation in Appendix C

$$S^g_{ADM} = \frac{1}{2}\int dt d^3x\sqrt{h}\Big[N^{(3)}R + \frac{1}{N}\Big(E_{ij}E^{ij} - E^2\Big)\Big], \tag{E.9}$$

while the other part of the action is the one that contains the multifield terms

$$S^\phi_{ADM} = -\frac{1}{2}\int dt d^3x\sqrt{h}\left[-\frac{1}{N}\gamma_{ab}\dot{\phi}^a\dot{\phi}^b + \frac{2N^i}{N}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b + \left(Nh^{ij} - \frac{N^iN^j}{N}\right)\gamma_{ab}\partial_i\phi^a\partial_j\phi^b + 2NV\right]. \tag{E.10}$$

First let's see the variation of the action with respect to the lapse function N, where

$$\frac{\delta S^g_{ADM}}{\delta N} = \frac{1}{2}\int dt d^3x\sqrt{h}\Big[{}^{(3)}R - \frac{1}{N^2}\Big(E_{ij}E^{ij} - E^2\Big)\Big], \tag{E.11}$$

[46] i. e. $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}}$

[47] In this appendix, we will use the lowercase letter n^a to describe the normal vector so as not to confuse it with the shift vector of the ADM formalism.

while

$$\frac{\delta S^{\phi}_{ADM}}{\delta N} = -\frac{1}{2}\int dtd^3x\sqrt{h}\Big[\frac{1}{N^2}\gamma_{ab}\dot{\phi}^a\dot{\phi}^b - \frac{2N^i}{N^2}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b + \Big(h^{ij} + \frac{N^iN^j}{N^2}\Big)\gamma_{ab}\partial_i\phi^a\partial_j\phi^b + 2V\Big]. \tag{E.12}$$

Putting these two results together, the first constraint equation (i. e. $\delta S/\delta N = 0$) remains

$$^{(3)}R - \frac{1}{N^2}\Big(E_{ij}E^{ij} - E^2\Big) - \frac{1}{N^2}\gamma_{ab}\dot{\phi}^a\dot{\phi}^b + \frac{2N^i}{N^2}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b - \Big(h^{ij} + \frac{N^iN^j}{N^2}\Big)\gamma_{ab}\partial_i\phi^a\partial_j\phi^b - 2V = 0. \tag{E.13}$$

On the other hand, the variation with respect to the shift vector of the action contains

$$\frac{\delta S^{g}_{ADM}}{\delta N^i} = \frac{1}{2}\int dtd^3x\sqrt{h}\Big[D_j\Big(\frac{1}{N}\Big(E^j_i - h^h_iE\Big)\Big]\Big), \tag{E.14}$$

and

$$\frac{\delta S^{\phi}_{ADM}}{\delta N^i} = -\frac{1}{2}\int dtd^3x\sqrt{h}\Big[\frac{2}{N}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b - \frac{N^j}{N}\gamma_{ab}\partial_i\phi^a\partial_j\phi^b\Big]. \tag{E.15}$$

With this, the second constraint equation is

$$D_j\Big[\frac{1}{N}\Big(E^j_i - h^h_iE\Big)\Big] - \frac{2}{N}\gamma_{ab}\partial_i\phi^a\dot{\phi}^b + \frac{N^j}{N}\gamma_{ab}\partial_i\phi^a\partial_j\phi^b = 0 \tag{E.16}$$

What follows is to define the spatial metric and the gauge that we will use. According to the notation presented in Chapter 6, we will use the comoving gauge, where the spatial metric (and its second-order approximation) is

$$h_{ij} = a^2e^{2\zeta}\delta_{ij} \approx a^2\Big(1 + 2\zeta + 2\zeta^2\Big)\delta_{ij}, \quad h^{ij} = a^{-2}e^{-2\zeta}\delta^{ij} \approx a^{-2}\Big(1 - 2\zeta + 2\zeta^2\Big)\delta^{ij}, \tag{E.17}$$

with its determinant

$$\sqrt{h} = a^3e^{3\zeta} \approx a^3\Big(1 + 3\zeta + \frac{9}{2}\zeta^2\Big). \tag{E.18}$$

We immediately write the second order approximation since we know that our objective will be to write the second order action in the perturbations. A priori, we will separate the perturbations from the fields as

$$\phi^a(\vec{x}, t) = \phi^a_0(t) + \delta\phi^a. \tag{E.19}$$

With this division for the fields, we will have to consider the second-order expansion for the quantities that depend on the fields (as opposed to the derivation in Appendix C). That is, we expand the field space metric and the potential as

$$\gamma_{ab} \approx \gamma_{ab} + \tilde{\partial}_c\gamma_{ab}\delta\phi^c + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\delta\phi^c\delta\phi^d, \tag{E.20}$$

$$V(\phi) \approx V + V_c\delta\phi^c + \frac{1}{2}V_{c,d}\delta\phi^c\delta\phi^d, \tag{E.21}$$

where $V_{c,d} \equiv \partial V_c/\partial\phi^d$. Later we will explain the fact that the perturbation of the fields is only in the direction normal to the trajectory (we use this notation to simplify the calculations), that is, the following identity is fulfilled:

$$\gamma_{ab}\dot{\phi}_0^a\delta\phi^b = 0. \tag{E.22}$$

Only with this orthogonality rule, the terms of the action that contain the perturbations of the fields can be written as

$$\begin{aligned}
\gamma_{ab}\dot{\phi}^a\dot{\phi}^b &= \left(\gamma_{ab} + \tilde{\partial}_c\gamma_{ab}\delta\phi^c + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\delta\phi^c\delta\phi^d\right)\left(\dot{\phi}_0^a + \delta\dot{\phi}^a\right)\left(\dot{\phi}_0^b + \delta\dot{\phi}^b\right) \\
&= \dot{\phi}_0^2 + 2\gamma_{ab}\dot{\phi}_0^a\delta\dot{\phi}^b + \tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c + \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b + 2\tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^a\delta\dot{\phi}^b\delta\phi^c + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d \\
&= \dot{\phi}_0^2 - 2\left(\tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^c\dot{\phi}_0^a\delta\phi^b + \tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^a\delta\dot{\phi}^c\delta\phi^b + \gamma_{ab}\ddot{\phi}_0^a\delta\phi^b\right) + \tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c + \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b \\
&\qquad + 2\tilde{\partial}_c\gamma_{ab}\dot{\phi}_0^a\delta\dot{\phi}^b\delta\phi^c + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d \\
&= \dot{\phi}_0^2 - 2\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\gamma_{cd}\Gamma^d_{ab} - 2\gamma_{ab}\ddot{\phi}_0^a\delta\phi^b + \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d \\
&= \dot{\phi}_0^2 - 2\gamma_{ab}(D_t\dot{\phi}_0^a)\delta\phi^b + \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d \\
&= \dot{\phi}_0^2 + 2V_a\delta\phi^a + \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d \\
&\equiv \dot{\phi}_0^2 + A + B^2,
\end{aligned} \tag{E.23}$$

where

$$A \equiv 2V_a\delta\phi^a, \tag{E.24}$$

$$B^2 \equiv \gamma_{ab}\delta\dot{\phi}^a\delta\dot{\phi}^b + \frac{1}{2}\tilde{\partial}_c\tilde{\partial}_d\gamma_{ab}\dot{\phi}_0^a\dot{\phi}_0^b\delta\phi^c\delta\phi^d. \tag{E.25}$$

To obtain this result, in the third equality, we replace an identity obtained by taken the time derivative of the orthogonality rule, while in the sixth equality, we use the equation of motion for the background fields and the orthogonality condition.

In addition to these definitions for A and B, note that $A = \mathcal{O}(\delta\phi)$ and $B^2 = \mathcal{O}(\delta\phi^2)$. We continue with the terms of the fields that have spatial derivatives:

$$\gamma_{ab}\partial_i\phi^a\dot{\phi}^b = \gamma_{ab}\partial_i\delta\phi^a\left(\dot{\phi}_0^b + \delta\dot{\phi}^b\right) = \gamma_{ab}\partial_i\delta\phi^a\delta\dot{\phi}^b + \tilde{\partial}_c\gamma_{ab}\delta\phi^c\partial_i\delta\phi^a\dot{\phi}_0^b, \tag{E.26}$$

$$\gamma_{ab}\partial_i\phi^a\partial_j\phi^b = \gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b. \tag{E.27}$$

What follows will be to calculate the terms of the constraints equations up to second order and then neglect the quadratic terms to solve the equations. In this way, we avoid repeating calculations when we compute the second order action.

Let's start with E_{ij}, where at second order we get

$$E_{ij} = \frac{1}{2}\dot{h}_{ij} - \frac{1}{2}(D_i N_j + D_j N_i)$$
$$\approx a^2 H\left(1 + 2\zeta + \frac{\dot{\zeta}}{H} + 2\zeta^2 + \frac{2\zeta\dot{\zeta}}{H}\right)\delta_{ij} - \frac{F_{ij}}{2}, \tag{E.28}$$

where $F_{ij} \equiv D_i N_j + D_j N_i$. Then

$$E_{ij}E^{ij} = h^{il}h^{jm}E_{ij}E_l m$$
$$\approx 3H^2\left(1 + \frac{2\dot{\zeta}}{H} + \frac{\zeta^2}{H^2}\right) - \frac{2H}{a^2}\left(1 - 2\zeta + \frac{\dot{\zeta}}{H}\right)\delta^{ij}D_i N_j + \frac{1}{4a^4}\delta^{il}\delta^{jm}F_{ij}F_{lm}. \tag{E.29}$$

On the other hand, to obtain E^2 we must calculate

$$E^i_j = h^{ik}E_{kj}$$
$$\approx H\left(1 + \frac{\dot{\zeta}}{H}\right)\delta^i_j - \frac{1}{2a^2}(1 - 2\zeta)\delta^{ik}F_{kj}, \tag{E.30}$$

this implies

$$E = E^i_i \approx 3H\left(1 + \frac{\dot{\zeta}}{H}\right) - \frac{1}{a^2}(1 - 2\zeta)\delta^{lm}D_l N_m, \tag{E.31}$$

and

$$E^2 \approx 9H^2\left(1 + \frac{2\dot{\zeta}}{H} + \frac{\dot{\zeta}^2}{H^2}\right) - \frac{6H}{a^2}\left(1 - 2\zeta + \frac{\dot{\zeta}}{H}\right)\delta^{lm}D_l N_m + \frac{1}{a^4}\left(\delta^{lm}D_l N_m\right)^2. \tag{E.32}$$

Putting these results together, we have

$$E_{ij}E^{ij} - E^2 \approx -6H^2\left(1 + \frac{2\dot{\zeta}}{H} + \frac{\dot{\zeta}^2}{H}\right) + \frac{4H}{a^2}\left(1 - 2\zeta + \frac{\dot{\zeta}}{H}\right)\delta^{lm}D_l N_m +$$
$$\frac{1}{4a^4}\delta^{il}\delta^{jm}F_{ij}F_{lm} - \frac{1}{a^4}\left(\delta^{lm}\partial_l N_m\right)^2. \tag{E.33}$$

As in Appendix C, we will solve the second constraint equation first. If we separate the lapse function as $N = 1 + \alpha = 1 + \alpha_1 + \alpha_2 + ...$, we must use the following approximations

$$\frac{1}{N} = \frac{1}{N + \alpha} \approx 1 - \alpha + \alpha^2 \approx 1 - \alpha_1 + \alpha_1^2, \tag{E.34}$$

$$\frac{1}{N^2} = \frac{1}{(1 + \alpha)^2} \approx 1 - 2_1 + 3\alpha_1^2. \tag{E.35}$$

If we look at the equalities that we present in the equations (E.26) and (E.27), we note that these terms are quadratic in the perturbations, so they do not contribute to the second constraint equation. Up to first order, we must solve

$$\partial_j\left\{(1-\alpha_1)\left[H\left(1+\frac{\dot\zeta}{H}\right)\delta_i^j-\frac{1}{2a^2}(1-2\zeta)\delta^{jk}F_{ki}-\delta_i^j\left(3H\left(1+\frac{\dot\zeta}{H}\right)-\delta^{ij}\partial_iN_j\right)\right]\right\}=0 \tag{E.36}$$

$$\begin{aligned}0&=\partial_j\left\{(1-\alpha_1)\left[H\left(1+\frac{\dot\zeta}{H}\right)\delta_i^j-\frac{1}{2a^2}(1-2\zeta)\delta^{jk}F_{ki}-\delta_i^j\left(3H\left(1+\frac{\dot\zeta}{H}\right)-\delta^{ij}\partial_iN_j\right)\right]\right\}\\&=\partial_j\left\{-2H(1-\alpha_1)\left(1+\frac{\dot\zeta}{H}\right)\delta_i^j-\frac{1}{a^2}\left(\frac{1}{2}\delta^{jk}(\partial_kN_i+\partial_iN_k)-\delta_i^j\delta^{lm}\partial_lN_m\right)\right\}.\end{aligned} \tag{E.37}$$

If we use the fact that $N_i=\partial_i\chi$ (we will not take into account vector perturbations), the right parenthesis is canceled, and we obtain the following equation for α_1

$$-2H\partial_i\left(\frac{\dot\zeta}{H}-\alpha_1\right)=0. \tag{E.38}$$

So, imposing appropriate initial conditions, the solution for α_1 is

$$\alpha_1=\frac{\dot\zeta}{H}. \tag{E.39}$$

To write the first constraint equation (eq. (E.13)), we will write up to first order each term separately, this is

$$-\frac{1}{N^2}\gamma_{ab}\dot\phi^a\dot\phi^b\approx-(1-2\alpha_1)\left(\dot\phi_0^2+A\right)\approx-\dot\phi_0^2+2\frac{\dot\zeta}{H}\dot\phi_0^2-A, \tag{E.40}$$

$${}^{(3)}R=\frac{2}{a^2}e^{2\zeta}\left(2\delta^{ij}\partial_i\partial_j\zeta-\delta^{ij}\partial_i\zeta\partial_j\zeta\right)\approx\frac{4}{a^2}\delta^{ij}\partial_i\partial_j\zeta \tag{E.41}$$

$$-\frac{1}{N^2}\left(E_{ij}E^{ij}-E^2\right)\approx-(1-2\frac{\dot\zeta}{H})\left(-6H^2\left(1+2\frac{\dot\zeta}{H}\right)+\frac{4H}{a^2}\delta^{ij}\partial_i\partial_j\chi\right)\approx6H^2-\frac{4H}{a^2}\delta^{ij}\partial_i\partial_j\chi, \tag{E.42}$$

$$2V\approx2V+2V_c\delta\phi^c \tag{E.43}$$

As we observe directly from the equations (E.26) and (E.27), the other terms associated with the fields are of higher order, so we do not consider them in the solution of this constraint equation. Replacing this approximations in the first constraint equation we have:

$$\frac{4}{a^2}\delta^{ij}\partial_i\partial_j\zeta-\frac{4H}{a^2}\delta_{ij}\partial_i\partial_j\chi+2\frac{\dot\zeta}{H}\dot\phi_0^2+6H^2-\dot\phi_0^2-2V-2V_c\delta\phi^c-A=0. \tag{E.44}$$

Using Friedmann's equation we cancel terms. Then, we solve for an expression for $\delta^{ij}\partial_i\partial_j\chi$, where

$$\delta^{ij}\partial_i\partial_j\chi=\delta^{ij}\partial_i\partial_j\frac{\zeta}{H}+a^2\varepsilon\dot\zeta-\frac{a^2}{H}V_c\delta\phi^c. \tag{E.45}$$

With the solutions for α_1 and χ, continue to write the action to second order. We will start with the part of the curvature S^g_{ADM}, where at second order its constituents are

$$N^{(3)}R \approx \frac{2}{a^2}\bigg(2\delta^{ij}\partial_i\partial_j\zeta + 2\delta^{ij}\frac{\dot{\zeta}}{H}\partial_i\partial_j\zeta + 4\delta^{ij}\zeta\partial_i\partial_j\zeta - \delta^{ij}\partial_i\zeta\partial_j\zeta\bigg), \tag{E.46}$$

$$\begin{aligned}\frac{1}{N}\Big(E_{ij}E^{ij} - E^2\Big) \approx -6H^2 - 6H\dot{\zeta} + \frac{4}{a^2}\delta^{ij}\partial_i\partial_j\zeta + 4H\varepsilon\dot{\zeta} \\ - 4V_c\delta\phi^c - \frac{4}{a^2}\delta^{ij}\zeta\partial_i\partial_j\zeta - 4H\varepsilon\dot{\zeta}\zeta + 4\zeta V_c\delta\phi^c.\end{aligned} \tag{E.47}$$

Putting these results together with the second order expression for $\sqrt{h}$, we obtain a Lagrangian expression associated with this part of the action

$$\begin{aligned}2\mathcal{L}^g = a^3\bigg(&\frac{8}{a^2}\delta^{ij}\partial_i\partial_j\zeta + \frac{8}{a^2H}\delta^{ij}\dot{\zeta}\partial_i\partial_j\zeta - \frac{2}{a^2}\delta^{ij}\partial_i\zeta\partial_j\zeta + \frac{28}{a^2}\delta^{ij}\zeta\partial_i\partial_j\zeta - 6H^2 - 6H\dot{\zeta} + 4H\varepsilon\dot{\zeta} \\ &- 4V_c\delta\phi^c - 18H^2\zeta - 18H\zeta\dot{\zeta} + 8H\varepsilon\zeta\dot{\zeta} - 8\zeta V_c\delta\phi^c - 27H^2\zeta^2\bigg).\end{aligned} \tag{E.48}$$

For the next part of the action, we note that the only terms contributing [48] are

$$\frac{1}{N}\gamma_{ab}\dot{\phi}^a\dot{\phi}^b \approx \dot{\phi}_0^2 + A + B^2 - \dot{\phi}_0^2\frac{\dot{\zeta}}{H} - A\frac{\dot{\zeta}}{H} + \dot{\phi}_0^2\frac{\dot{\zeta}^2}{H^2}, \tag{E.49}$$

$$-Nh^{ij}\gamma_{ab}\partial_i\phi^a\partial_j\phi^b \approx -\frac{\delta_{ij}}{a^2}\gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b, \tag{E.50}$$

$$\begin{aligned}-2NV &= -2\bigg(1 + \frac{\dot{\zeta}}{H}\bigg)\bigg(V + V_c\delta\phi^c + \frac{1}{2}V_{c,d}\delta\phi^c\delta\phi^d\bigg) \\ &\approx -2\bigg(V + V_c\delta\phi^c + \frac{\dot{\zeta}}{H}V + \frac{\dot{\zeta}}{H}V_c\delta\phi^c + \frac{1}{2}V_{c,d}\delta\phi^c\delta\phi^d\bigg).\end{aligned} \tag{E.51}$$

In the same way as $\mathcal{L}^g$, we replace the expression for $\sqrt{h}$ and we have the Lagrangian up to second order of this part:

$$\begin{aligned}2\mathcal{L}^\phi = a^3\bigg(&\dot{\phi}_0^2 + B^2 - \dot{\phi}_0^2\frac{\dot{\zeta}}{H} + \dot{\phi}_0^2\frac{\dot{\zeta}^2}{H^2} - \frac{\delta_{ij}}{a^2}\gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b - 2V - V_{c,d}\delta\phi^c\delta\phi^d - 2\frac{\dot{\zeta}}{H}V \\ &+ 3\zeta\dot{\phi}_0^2 - 3\dot{\phi}_0^2\zeta\frac{\dot{\zeta}}{H} - 6\zeta V - 6\zeta\frac{\dot{\zeta}}{H}V - 4\frac{\dot{\zeta}}{H}V_c\delta\phi^c + \frac{9}{2}\zeta^2\dot{\phi}_0^2 - 9\zeta^2V\bigg).\end{aligned} \tag{E.52}$$

[48] The others are of higher order.

Putting both parts of the Lagrangian together, we have

$$2\mathcal{L} = a^3\Bigg(B^2 - \frac{\delta_{ij}}{a^2}\gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b + \frac{8}{a^2}\delta^{ij}\partial_i\partial_j\zeta - \frac{2}{a^2}\delta^{ij}\partial_i\zeta\partial_j\zeta - 4V - 4V_c\delta\phi^c - V_{c,d}\delta\phi^c\delta\phi^d$$
$$+ a\zeta + \frac{28}{a^2}\delta^{ij}\zeta\partial_i\partial_j\zeta - 3\zeta\Big(6H^2 - \dot{\phi}_0^2 + 2V\Big) - 8\zeta V_c\delta\phi^c - \frac{9}{2}\zeta^2\Big(6H^2 - \dot{\phi}_0^2 + 2V\Big)$$
$$+ \frac{8}{a^2H}\delta^{ij}\dot{\zeta}\partial_i\partial_j\zeta - \frac{\dot{\zeta}}{H}\Big(6H^2 + \dot{\phi}_0^2 + 2V\Big) - 4\frac{\dot{\zeta}}{H}V_c\delta\phi^c - 3\zeta\frac{\dot{\zeta}}{H}\Big(6H^2 + \dot{\phi}_0^2 + 2V\Big)$$
$$+ \dot{\zeta}^2\frac{\dot{\phi}_0^2}{H^2} + 4H\varepsilon\dot{\zeta} + 8H\varepsilon\dot{\zeta}\zeta\Bigg). \quad \text{(E.53)}$$

Separating by order of magnitude, we have

$$\mathcal{L}^{(0)} = -2a^3V \quad \text{(E.54)}$$

$$\mathcal{L}^{(1)} = \frac{a^3}{2}\left(\frac{8}{a^2}\delta^{ij}\partial_i\partial_j\zeta - 12V\zeta - 12H\dot{\zeta} + 4H\varepsilon\dot{\zeta} - 2V_c\delta\phi^c\right) \quad \text{(E.55)}$$

$$\mathcal{L}^{(2)} = \frac{a^3}{2}\left(\dot{\zeta}^2\frac{\dot{\phi}_0^2}{H^2} + \frac{28}{a^2}\delta^{ij}\zeta\partial_i\partial_j\zeta - 18V\zeta^2 - \frac{2}{a^2}\delta^{ij}\partial_i\zeta\partial_j\zeta + \frac{8}{a^2H}\delta^{ij}\dot{\zeta}\partial_i\partial_j\zeta - 36H\dot{\zeta}\zeta + 8H\varepsilon\dot{\zeta}\zeta\right)$$
$$+ \frac{a^3}{2}\left(-\frac{\delta_{ij}}{a^2}\gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b - 8\zeta V_c\delta\phi^c - 4\frac{\dot{\zeta}}{H}V_c\delta\phi^c + B^2 - V_{c,d}\delta\phi^c\delta\phi^d\right). \quad \text{(E.56)}$$

In this quadratic Lagrangian, we note that the terms within the first parentheses are those that depend on the curvature perturbation. We can do the same integrations by parts that we did in Appendix C to rewrite this part of the quadratic Lagrangian as

$$\mathcal{L}_\zeta^{(2)} = \frac{a^3}{2}\frac{\dot{\phi}_0^2}{H^2}\left(\dot{\zeta}^2 - \frac{1}{a^2}(\nabla\zeta)^2\right) = \frac{a^3}{2}\left(\dot{\zeta}_c^2 - \frac{1}{a^2}(\nabla\zeta_c)^2\right), \quad \text{(E.57)}$$

where in the last equality we use the definition of $\zeta_c \equiv \sqrt{2\varepsilon}\zeta$. If we integrate by parts, we can rewrite the part of the Lagrangian that depends on the fields perturbations as

$$\mathcal{L}_{\delta\phi}^{(2)} = \frac{a^3}{2}\left(-\frac{\delta_{ij}}{a^2}\gamma_{ab}\partial_i\delta\phi^a\partial_j\delta\phi^b - 4V_c\delta\phi^c\frac{\dot{\zeta}}{H} + B^2 - V_{c,d}\delta\phi^c\delta\phi^d\right). \quad \text{(E.58)}$$

If we write the isocurvature perturbation as $\delta\phi^a \equiv n^a\psi$, and we use the fact that in the first order it is satisfied that $\dot{n}^a = \Omega T^a$ and $n^cV_c = \dot{\phi}_0\Omega$, we have [49]

$$\mathcal{L}_{\delta\phi}^{(2)} = \frac{a^3}{2}\left(-\frac{1}{a^2}(\nabla\psi)^2 - 4\dot{\zeta}_c\Omega\psi + 4\Omega^2\psi^2 + \dot{\psi}^2 - V_{c,d}n^cn^d\psi^2 + \frac{1}{2}\tilde{R}\dot{\phi}_0^2\psi^2\right). \quad \text{(E.59)}$$

Where $\tilde{R}$ is the Ricci scalar in the field space.

To conclude this appendix, we will put together the parts $\mathcal{L}_\zeta^{(2)}$ and $\mathcal{L}_{\delta\phi}^{(2)}$ of the Lagrangian

[49] See [154] for more details

and also use the definition of the covariant derivative $D_t\zeta_c = \dot{\zeta}_c - 2\Omega\psi \equiv \dot{\zeta}_c - \lambda H\psi$. Obtaining, on the one hand, the kinetic part corresponding to the equation presented in (6.11):

$$\mathcal{L}_{kin} = \frac{a^3}{2}\left[(D_t\zeta_c)^2 - \frac{1}{a^2}(\nabla\zeta_c)^2 + \dot{\psi}^2 - \frac{1}{a^2}(\nabla\psi)^2\right], \tag{E.60}$$

while we group the terms proportional to ψ^2 in the potential $U(\psi)$.

Appendix F

Analysis of the Cubic Lagrangian

In this appendix we will show that the condition for a strong mixing between the fields (i. e. $\Lambda \gg 1$) does not break the perturbativity of the system.

For this, we must analyze the cubic Lagrangian for this case of inflation with two fields. This derivation has already been made in the work presented in [191]. If we want to analyze the relation between the cubic and quadratic Lagrangian, we must look only at the predominant terms of the cubic part and compare them with $D_t\zeta_c$.

If we make the correspondence between our notation and the one used in [191], we must consider the relation: $\lambda\mathcal{F} = \dot{\zeta}_c - D_t\zeta_c$. With this, we can write the leading order of the cubic Lagrangian as

$$\frac{1}{a^3}\mathcal{L}^{(3)} \supset -\frac{\varepsilon V}{3}\frac{\dot{\zeta}^3}{H^3} + \frac{\varepsilon V}{H}\frac{\dot{\zeta}^2}{H^2}D_T\zeta - \frac{2\varepsilon V}{3H^2}\frac{\dot{\zeta}}{H}(D_t\zeta)^2 - \frac{\varepsilon H}{6}(D_t\zeta)^3. \tag{F.1}$$

So, the relation between the cubic and quadratic Lagrangian as

$$\begin{aligned}
\frac{\mathcal{L}^{(3)}}{\mathcal{L}^{(2)}} &= -\frac{\dot{\zeta}}{H}\frac{\dot{\zeta}^2}{(D_t\zeta)^2} + 3\frac{\dot{\zeta}^2}{H^2}\frac{H}{D_t\zeta} - 2\frac{\dot{\zeta}}{H} - \frac{H^2}{6}\frac{D_t\zeta}{H} \\
&= -\frac{\dot{\zeta}}{H}\left(2 + \frac{\dot{\zeta}^2}{(D_t\zeta)^2} - 3\frac{\dot{\zeta}}{D_t\zeta} + \frac{H^2}{6}\frac{D_t\zeta}{\dot{\zeta}}\right).
\end{aligned} \tag{F.2}$$

From where we conclude what is expressed in the equation (6.24)

$$\frac{\mathcal{L}^{(3)}}{\mathcal{L}^{(2)}} \sim \frac{\dot{\zeta}_c}{\sqrt{2\varepsilon}H} \ll 1, \tag{F.3}$$

also valid for $\lambda \gg 1$.

Appendix G

Top-Hat Analytics

In this appendix, we will derive equation (6.33) in more detail. First, we are going to rewrite the equations of motion for the fields using the number of e-folds as the variable through $N = Ht$. Equations (6.26) - (6.26) can be written as[50]

$$e^{-3N}\left(e^{3N}D\zeta\right)' + e^{2(N_k-N)}\zeta = 0, \tag{G.1}$$

$$e^{-3N}\left(e^{3N}\psi'\right)' + e^{2(N_k-N)}\psi + \lambda D\zeta = 0. \tag{G.2}$$

In the above equations we use $d/dN = ()'$, additionally, we define $e^{2N_k} \equiv k^2/H^2$ and $D\zeta \equiv \zeta' - \lambda\psi$.

Then, we impose the shape of the λ coupling as a top-hat function

$$\lambda(N) = \frac{\lambda_0}{2}[\theta(N - N_1) - \theta(N - N_2)]. \tag{G.3}$$

Where, we will also define

$$\delta N = N_2 - N_1, \qquad N_0 = \frac{1}{2}(N_2 + N_1) \tag{G.4}$$

Thanks to the form of the $\lambda(N)$ function, we separate the temporary solutions for ζ and ψ into three regions as expressed in the figure G.1.

In region I, the equations of motion become

$$e^{-3N}\left(e^{3N}\zeta'\right)' + e^{2(N_k-N)}\zeta = 0, \tag{G.5}$$

$$e^{-3N}\left(e^{3N}\psi'\right)' + e^{2(N_k-N)}\psi = 0. \tag{G.6}$$

With solutions that take the usual form

$$\zeta_k(N) = Z(N)a_\zeta(\vec{k}) + h.c.(-\vec{k}), \tag{G.7}$$

$$\psi_k(N) = P(N)a_\psi(\vec{k}) + h.c.(-\vec{k}). \tag{G.8}$$

[50] For simplicity, we are going to omit the subscript in ζ_c from the canonically normalized version of ζ

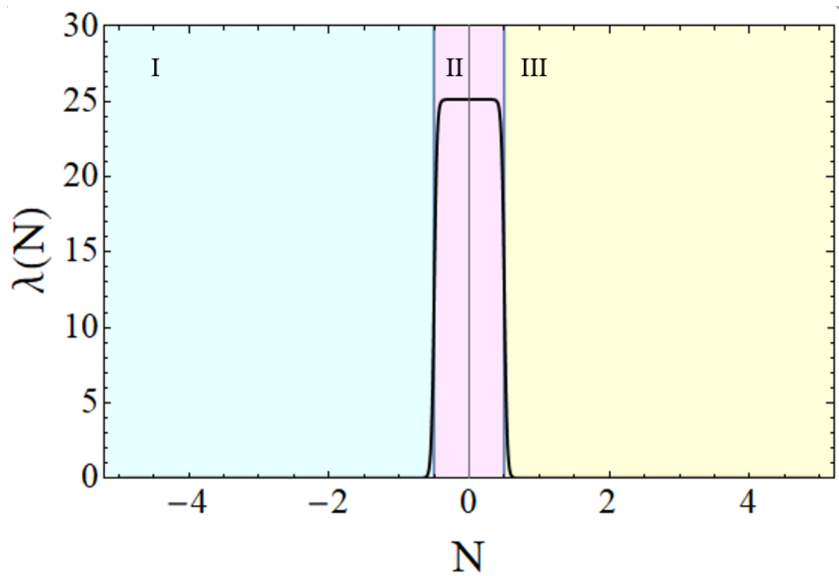


Figure G.1: Representation of the function $\lambda(N)$ in our model, and the different temporal regions in which we solve the equations.

Where $Z(N)$ and $P(N)$ correspond to the Bunch-Davies mode functions

$$Z(N) = P(N) = \frac{iH}{\sqrt{2k^3}}\Big(1 - ie^{N_k - N}\Big)e^{ie^{N_k - N}} \tag{G.9}$$

In region II, we write the equations as

$$(D\zeta)' + \kappa^2\zeta = 0, \tag{G.10}$$

$$\psi'' + \kappa^2\psi + \lambda D\zeta = 0, \tag{G.11}$$

With $\kappa = e^{N_k - N_0}$. We write the solutions in this region according to the ansatz

$$\begin{aligned}\zeta_k(N) = &\Big(A_+e^{+i\omega_- N} + A_-e^{-i\omega_- N} + B_+e^{+i\omega_+ N} + B_-e^{-i\omega_+ N}\Big)a_\zeta(\vec{k})+ \\ &\Big(C_+e^{+i\omega_- N} + C_-e^{-i\omega_- N} + D_+e^{+i\omega_+ N} + D_-e^{-i\omega_+ N}\Big)a_\psi(\vec{k}) + h.c.(-\vec{k}),\end{aligned} \tag{G.12}$$

$$\begin{aligned}\psi_k(N) = &\Big(E_+e^{+i\omega_- N} + E_-e^{-i\omega_- N} + F_+e^{+i\omega_+ N} + F_-e^{-i\omega_+ N}\Big)a_\zeta(\vec{k})+ \\ &\Big(G_+e^{+i\omega_- N} + G_-e^{-i\omega_- N} + H_+e^{+i\omega_+ N} + H_-e^{-i\omega_+ N}\Big)a_\psi(\vec{k}) + h.c.(-\vec{k}).\end{aligned} \tag{G.13}$$

To find $\omega_\pm$, we plug this ansatz into the equation of motion for this region and obtain

$$\Big(\omega_\pm^2 + \lambda^2 - \kappa^2\Big)\Big(\omega_\pm^2 - \kappa^2\Big) = \lambda^2\omega_\pm^2. \tag{G.14}$$

We choose the positive solution for $\omega_\pm$ (to match in the case $\lambda = 0$), this is

$$\omega_\pm = \sqrt{\kappa^2 \pm \kappa\lambda} = \sqrt{k^2 \pm kk_0\lambda}. \tag{G.15}$$

Where in the last equality we use the definition of $k_0 = He^{H(t_1+t_2)/2}$.

In region III, the form of the equations of motion will be the same as that expressed in the equations (G.5) and (G.6), but the solutions are expressed as follows:

$$\begin{aligned}\zeta_k(N) = &\Big(I_+\Big(1 - ie^{N_k-N}\Big)e^{+e^{N_k-N}} + I_-\Big(1 + ie^{N_k-N}\Big)e^{-e^{N_k-N}}\Big)a_\zeta(\vec{k}) + \\ &\Big(J_+\Big(1 - ie^{N_k-N}\Big)e^{+e^{N_k-N}} + J_-\Big(1 + ie^{N_k-N}\Big)e^{-e^{N_k-N}}\Big)a_\psi(\vec{k}) + h.c.(-\vec{k}), \quad (G.16)\end{aligned}$$

$$\begin{aligned}\psi_k(N) = &\Big(K_+\Big(1 - ie^{N_k-N}\Big)e^{+e^{N_k-N}} + K_-\Big(1 + ie^{N_k-N}\Big)e^{-e^{N_k-N}}\Big)a_\zeta(\vec{k}) + \\ &\Big(L_+\Big(1 - ie^{N_k-N}\Big)e^{+e^{N_k-N}} + L_-\Big(1 + ie^{N_k-N}\Big)e^{-e^{N_k-N}}\Big)a_\psi(\vec{k}) + h.c.(-\vec{k}). \quad (G.17)\end{aligned}$$

Before dealing with the initial conditions, let us see what conditions emerge from the commutation relations (or Wronskian conditions) of the solutions in region II. First of all, recall that in this region, the derivatives of the solutions are

$$\begin{aligned}\zeta_k' &= \Big[i\omega_-\left(A_+e^{+i\omega_-N} - A_-e^{-i\omega_-N}\right) + i\omega_+\left(B_+e^{+i\omega_+N} - B_-e^{-i\omega_+N}\right)\Big]a_\zeta(\vec{k}) \\ &+\Big[i\omega_-\left(C_+e^{+i\omega_-N} - C_-e^{-i\omega_-N}\right) + i\omega_+\left(D_+e^{+i\omega_+N} - D_-e^{-i\omega_+N}\right)\Big]a_\psi(\vec{k}) + h.c.(-\vec{k}),\end{aligned} \quad (G.18)$$

$$\begin{aligned}\psi_k' &= \Big[i\omega_-\left(E_+e^{+i\omega_-N} - E_-e^{-i\omega_-N}\right) + i\omega_+\left(F_+e^{+i\omega_+N} - F_-e^{-i\omega_+N}\right)\Big]a_\zeta(\vec{k}) \\ &+\Big[i\omega_-\left(G_+e^{+i\omega_-N} - G_-e^{-i\omega_-N}\right) + i\omega_+\left(H_+e^{+i\omega_+N} - H_-e^{-i\omega_+N}\right)\Big]a_\psi(\vec{k}) + h.c.(-\vec{k}).\end{aligned} \quad (G.19)$$

We continue noticing that the following commutation relations are automatically satisfied:

$$[\zeta_k, \zeta_q] = 0, \quad (G.20)$$
$$[\psi_k, \psi_q] = 0, \quad (G.21)$$

Therefore, we must check the following commutation relations

$$[\zeta_k, \psi_q] = 0, \quad (G.22)$$
$$\Big[\zeta_k, \psi_q'\Big] = 0, \quad (G.23)$$
$$[\zeta_k' - \lambda\psi_k, \psi_q] = 0, \quad (G.24)$$
$$(G.25)$$

$$\left[\zeta'_k - \lambda\psi_k, \psi'_q\right] = 0, \tag{G.26}$$
$$\left[\zeta'_k - \lambda\psi_k, \zeta'_q - \lambda\psi_q\right] = 0, \tag{G.27}$$
$$\left[\psi'_k, \psi'_q\right] = 0, \tag{G.28}$$
$$He^{3N}\left[\zeta_k, \zeta'_q - \lambda\psi_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.29}$$
$$He^{3N}\left[\psi_k, \psi'_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.30}$$

These are equivalent to

$$[\zeta_k, \psi_q] = 0, \tag{G.31}$$
$$\left[\zeta_k, \psi'_q\right] = 0, \tag{G.32}$$
$$[\zeta'_k, \psi_q] = 0, \tag{G.33}$$
$$\left[\zeta'_k, \zeta'_q\right] = 0, \tag{G.34}$$
$$\left[\psi'_k, \psi'_q\right] = 0, \tag{G.35}$$
$$He^{3N}\left[\zeta_k, \zeta'_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.36}$$
$$He^{3N}\left[\psi_k, \psi'_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.37}$$
$$He^{3N}\left[\zeta'_k, \psi'_q\right] = \lambda_0 i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.38}$$

Of these, the following two are automatic

$$\left[\zeta'_k, \zeta'_q\right] = 0, \tag{G.39}$$
$$\left[\psi'_k, \psi'_q\right] = 0, \tag{G.40}$$

So we are left with

$$[\zeta_k, \psi_q] = 0, \tag{G.41}$$
$$\left[\zeta_k, \psi'_q\right] = 0, \tag{G.42}$$
$$[\zeta'_k, \psi_q] = 0, \tag{G.43}$$
$$He^{3N}\left[\zeta_k, \zeta'_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.44}$$
$$He^{3N}\left[\psi_k, \psi'_q\right] = i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.45}$$
$$He^{3N}\left[\zeta'_k, \psi'_q\right] = \lambda_0 i(2\pi)^3\delta(\vec{k}+\vec{q}), \tag{G.46}$$

Let's check them one by one. The first one is

$$\begin{aligned}
0 \;=\; & \left[A_+e^{+i\omega_-N} + A_-e^{-i\omega_-N} + B_+e^{+i\omega_+N} + B_-e^{-i\omega_+N}\right]\times \\
& \qquad \left[E_+^*e^{-i\omega_-N} + E_-^*e^{+i\omega_-N} + F_+^*e^{-i\omega_+N} + F_-^*e^{+i\omega_+N}\right] \\
& -\left[A_+^*e^{-i\omega_-N} + A_-^*e^{+i\omega_-N} + B_+^*e^{-i\omega_+N} + B_-^*e^{+i\omega_+N}\right]\times \\
& \qquad \left[E_+e^{+i\omega_-N} + E_-e^{-i\omega_-N} + F_+e^{+i\omega_+N} + F_-e^{-i\omega_+N}\right] \\
& +\left[C_+e^{+i\omega_-N} + C_-e^{-i\omega_-N} + D_+e^{+i\omega_+N} + D_-e^{-i\omega_+N}\right]\times \\
& \qquad \left[G_+^*e^{-i\omega_-N} + G_-^*e^{+i\omega_-N} + H_+^*e^{-i\omega_+N} + H_-^*e^{+i\omega_+N}\right] \\
& -\left[C_+^*e^{-i\omega_-N} + C_-^*e^{+i\omega_-N} + D_+^*e^{-i\omega_+N} + D_-^*e^{+i\omega_+N}\right]\times \\
& \qquad \left[G_+e^{+i\omega_-N} + G_-e^{-i\omega_-N} + H_+e^{+i\omega_+N} + H_-e^{-i\omega_+N}\right]. \qquad \text{(G.47)}
\end{aligned}$$

They imply the following conditions:

$$A_+E_-^* + C_+G_-^* - E_+A_-^* - G_+C_-^* = 0, \qquad \text{(G.48)}$$

$$B_+F_-^* + D_+H_-^* - F_+B_-^* - H_+D_-^* = 0, \qquad \text{(G.49)}$$

$$\begin{aligned}
&(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*)- \\
&(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) = 0, \qquad \text{(G.50)}
\end{aligned}$$

$$\begin{aligned}
&(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*)+ \\
&(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) = 0, \qquad \text{(G.51)}
\end{aligned}$$

$$\begin{aligned}
&(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c.})+ \\
&(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c.}) = 0. \qquad \text{(G.52)}
\end{aligned}$$

Next, we go for the second commutation. This is:

$$
\begin{aligned}
0 \; = \; & \Big[A_+e^{+i\omega_-N} + A_-e^{-i\omega_-N} + B_+e^{+i\omega_+N} + B_-e^{-i\omega_+N}\Big] \times \\
& \qquad \Big[- i\omega_- \left(E_+^*e^{-i\omega_-N} - E_-^*e^{+i\omega_-N}\right) - i\omega_+ \left(F_+^*e^{-i\omega_+N} - F_-^*e^{+i\omega_+N}\right)\Big] \\
& - \Big[A_+^*e^{-i\omega_-N} + A_-^*e^{+i\omega_-N} + B_+^*e^{-i\omega_+N} + B_-^*e^{+i\omega_+N}\Big] \times \\
& \qquad \Big[i\omega_- \left(E_+e^{+i\omega_-N} - E_-e^{-i\omega_-N}\right) + i\omega_+ \left(F_+e^{+i\omega_+N} - F_-e^{-i\omega_+N}\right)\Big] \\
& + \Big[C_+e^{+i\omega_-N} + C_-e^{-i\omega_-N} + D_+e^{+i\omega_+N} + D_-e^{-i\omega_+N}\Big] \times \\
& \qquad \Big[- i\omega_- \left(G_+^*e^{-i\omega_-N} - G_-^*e^{+i\omega_-N}\right) - i\omega_+ \left(H_+^*e^{-i\omega_+N} - H_-^*e^{+i\omega_+N}\right)\Big] \\
& - \Big[C_+^*e^{-i\omega_-N} + C_-^*e^{+i\omega_-N} + D_+^*e^{-i\omega_+N} + D_-^*e^{+i\omega_+N}\Big] \times \\
& \qquad \Big[i\omega_- \left(G_+e^{+i\omega_-N} - G_-e^{-i\omega_-N}\right) + i\omega_+ \left(H_+e^{+i\omega_+N} - H_-e^{-i\omega_+N}\right)\Big].
\end{aligned}
\tag{G.53}
$$

They imply the following conditions (we only list new relations):

$$
\begin{aligned}
\omega_+(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) + & \\
\omega_-(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, \qquad (G.54) \\
\omega_+(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) + & \\
\omega_-(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0, \qquad (G.55) \\
\omega_-(A_-E_-^* + A_-^*E_- - A_+E_+^* - A_+^*E_+ + & \\
G_-C_-^* + C_-G_-^* - G_+C_+^* - C_+G_+^*) + & \\
\omega_+(B_-F_-^* + F_-B_-^* - B_+F_+^* - F_+B_+^* + & \\
H_-D_-^* + D_-H_-^* - H_+D_+^* - D_+H_+^*) &= 0. \qquad (G.56)
\end{aligned}
$$

Following the same steps, the third commutation relation leads to (we only list new relations)

$$
\begin{aligned}
\omega_-(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) + & \\
\omega_+(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, \qquad (G.57) \\
\omega_-(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) + & \\
\omega_+(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0. \qquad (G.58)
\end{aligned}
$$

The fourth commutation relation implies:

$$\begin{aligned} B_+A_+^* - A_-B_-^* + D_+C_+^* - C_-D_-^* &= 0, && \text{(G.59)} \\ B_-A_+^* - A_-B_+^* + D_-C_+^* - C_-D_+^* &= 0, && \text{(G.60)} \\ 2\omega_-(|A_-|^2 - |A_+|^2 + |C_-|^2 - |C_+|^2) + & \\ 2\omega_+(|B_-|^2 - |B_+|^2 + |D_-|^2 - |D_+|^2) &= \frac{1}{H}e^{-3N_0}. && \text{(G.61)} \end{aligned}$$

The fifth commutation relation gives:

$$\begin{aligned} F_+E_+^* - E_-F_-^* + H_+G_+^* - G_-H_-^* &= 0, && \text{(G.62)} \\ F_-E_+^* - E_-F_+^* + H_-G_+^* - G_-H_+^* &= 0, && \text{(G.63)} \\ 2\omega_-(|E_-|^2 - |E_+|^2 + |G_-|^2 - |G_+|^2) + & \\ 2\omega_+(|F_-|^2 - |F_+|^2 + |H_-|^2 - |H_+|^2) &= \frac{1}{H}e^{-3N_0}. && \text{(G.64)} \end{aligned}$$

Finally, the sixth relation only gives the following new relation:

$$\begin{aligned} \omega_-^2(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c}) + & \\ \omega_+^2(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) &= \frac{i\lambda_0}{H}e^{-3N_0}. && \text{(G.65)} \end{aligned}$$

For convenience, let's list all of the wronskian conditions

$$\begin{aligned} A_+E_-^* + C_+G_-^* - E_+A_-^* - G_+C_-^* &= 0, && \text{(G.66)} \\ B_+F_-^* + D_+H_-^* - F_+B_-^* - H_+D_-^* &= 0, && \text{(G.67)} \\ (A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) - & \\ (E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, && \text{(G.68)} \\ (A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) + & \\ (B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0, && \text{(G.69)} \\ (A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c.}) + & \\ (B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c.}) &= 0, && \text{(G.70)} \\ \omega_+(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) + & \\ \omega_-(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, && \text{(G.71)} \\ \omega_+(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) + & \\ \omega_-(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0, && \text{(G.72)} \\ \omega_-(A_-E_-^* + A_-^*E_- - A_+E_+^* - A_+^*E_+ + & \\ G_-C_-^* + C_-G_-^* - G_+C_+^* - C_+G_+^*) & \\ +\omega_+(B_-F_-^* + F_-B_-^* - B_+F_+^* - F_+B_+^* + & \\ H_-D_-^* + D_-H_-^* - H_+D_+^* - D_+H_+^*) &= 0, && \text{(G.73)} \\ \omega_-(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) + & \\ \omega_+(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, && \text{(G.74)} \end{aligned}$$

$$\begin{aligned}
\omega_-(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*)+ & & \\
\omega_+(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0, & \text{(G.75)}\\
B_+A_+^* - A_-B_-^* + D_+C_+^* - C_-D_-^* &= 0, & \text{(G.76)}\\
B_-A_+^* - A_-B_+^* + D_-C_+^* - C_-D_+^* &= 0, & \text{(G.77)}\\
2\omega_-(|A_-|^2 - |A_+|^2 + |C_-|^2 - |C_+|^2)+ & & \\
2\omega_+(|B_-|^2 - |B_+|^2 + |D_-|^2 - |D_+|^2) &= \frac{1}{H}e^{-3N_0}, & \text{(G.78)}\\
F_+E_+^* - E_-F_-^* + H_+G_+^* - G_-H_-^* &= 0, & \text{(G.79)}\\
F_-E_+^* - E_-F_+^* + H_-G_+^* - G_-H_+^* &= 0, & \text{(G.80)}\\
2\omega_-(|E_-|^2 - |E_+|^2 + |G_-|^2 - |G_+|^2)+ & & \\
2\omega_+(|F_-|^2 - |F_+|^2 + |H_-|^2 - |H_+|^2) &= \frac{1}{H}e^{-3N_0}, & \text{(G.81)}\\
\omega_-^2(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c})+ & & \\
\omega_+^2(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) &= \frac{i\lambda_0}{H}e^{-3N_0}. & \text{(G.82)}
\end{aligned}$$

Now, equations (G.71) and (G.74) imply

$$\begin{aligned}
(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) &= 0, & \text{(G.83)}\\
(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, & \text{(G.84)}
\end{aligned}$$

whereas equations (G.72) and (G.75) imply

$$\begin{aligned}
(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) &= 0, & \text{(G.85)}\\
(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0. & \text{(G.86)}
\end{aligned}$$

These equations imply that (G.68) and (G.69) are automatically satisfied. In addition, (G.70) and (G.82) imply:

$$\begin{aligned}
(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_-^2 - \omega_+^2)}e^{-3N_0}, & \text{(G.87)}\\
(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_+^2 - \omega_-^2)}e^{-3N_0}. & \text{(G.88)}
\end{aligned}$$

We are then left with

$$\begin{aligned}
A_+E_-^* + C_+G_-^* - E_+A_-^* - G_+C_-^* &= 0, & \text{(G.89)}\\
B_+F_-^* + D_+H_-^* - F_+B_-^* - H_+D_-^* &= 0, & \text{(G.90)}\\
(A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^*) &= 0, & \text{(G.91)}\\
(E_+B_+^* - B_-E_-^* + G_+D_+^* - D_-G_-^*) &= 0, & \text{(G.92)}\\
(A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^*) &= 0, & \text{(G.93)}\\
(B_-E_+^* - E_-B_+^* + D_-G_+^* - G_-D_+^*) &= 0, & \text{(G.94)}
\end{aligned}$$

$$\begin{aligned}
&\omega_-(A_-E_-^* + A_-^*E_- - A_+E_+^* - A_+^*E_+ + \\
&\qquad G_-C_-^* + C_-G_-^* - G_+C_+^* - C_+G_+^*) \\
&+\omega_+(B_-F_-^* + F_-B_-^* - B_+F_+^* - F_+B_+^* + \\
&\qquad H_-D_-^* + D_-H_-^* - H_+D_+^* - D_+H_+^*) = 0, \qquad \text{(G.95)}
\end{aligned}$$

$$B_+A_+^* - A_-B_-^* + D_+C_+^* - C_-D_-^* = 0, \qquad \text{(G.96)}$$

$$B_-A_+^* - A_-B_+^* + D_-C_+^* - C_-D_+^* = 0, \qquad \text{(G.97)}$$

$$\begin{aligned}
&2\omega_-(|A_-|^2 - |A_+|^2 + |C_-|^2 - |C_+|^2) + \\
&\qquad 2\omega_+(|B_-|^2 - |B_+|^2 + |D_-|^2 - |D_+|^2) = \frac{1}{H}e^{-3N_0}, \qquad \text{(G.98)}
\end{aligned}$$

$$F_+E_+^* - E_-F_-^* + H_+G_+^* - G_-H_-^* = 0, \qquad \text{(G.99)}$$

$$F_-E_+^* - E_-F_+^* + H_-G_+^* - G_-H_+^* = 0, \qquad \text{(G.100)}$$

$$\begin{aligned}
&2\omega_-(|E_-|^2 - |E_+|^2 + |G_-|^2 - |G_+|^2) + \\
&\qquad 2\omega_+(|F_-|^2 - |F_+|^2 + |H_-|^2 - |H_+|^2) = \frac{1}{H}e^{-3N_0}, \qquad \text{(G.101)}
\end{aligned}$$

$$(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c}) = \frac{i\lambda_0}{H(\omega_-^2 - \omega_+^2)}e^{-3N_0}, \qquad \text{(G.102)}$$

$$(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) = \frac{i\lambda_0}{H(\omega_+^2 - \omega_-^2)}e^{-3N_0}. \qquad \text{(G.103)}$$

Additionally, we have the eigenvector relations (coming from the equations of motion):

$$A_+ = -i\lambda_0\frac{\omega_-}{\omega_-^2 - \kappa^2}E_+, \qquad \text{(G.104)}$$

$$A_- = +i\lambda_0\frac{\omega_-}{\omega_-^2 - \kappa^2}E_-, \qquad \text{(G.105)}$$

$$B_+ = -i\lambda_0\frac{\omega_+}{\omega_+^2 - \kappa^2}F_+, \qquad \text{(G.106)}$$

$$B_- = +i\lambda_0\frac{\omega_+}{\omega_+^2 - \kappa^2}F_-, \qquad \text{(G.107)}$$

$$C_+ = -i\lambda_0\frac{\omega_-}{\omega_-^2 - \kappa^2}G_+, \qquad \text{(G.108)}$$

$$C_- = +i\lambda_0\frac{\omega_-}{\omega_-^2 - \kappa^2}G_-, \qquad \text{(G.109)}$$

$$D_+ = -i\lambda_0\frac{\omega_+}{\omega_+^2 - \kappa^2}H_+, \qquad \text{(G.110)}$$

$$D_- = +i\lambda_0\frac{\omega_+}{\omega_+^2 - \kappa^2}H_-. \qquad \text{(G.111)}$$

These relations imply that (G.89) is trivial. They also imply that the pairs (G.91) and (G.92), (G.93) and (G.94), (G.96) and (G.99), and (G.97) and (G.100) are equivalent. They also imply that (G.95) is trivial. And that (G.98) and (G.101) follow from (G.102) and (G.103). We are then left with

$$
\begin{aligned}
A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^* &= 0, & \text{(G.112)}\\
A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^* &= 0, & \text{(G.113)}\\
B_+A_+^* - A_-B_-^* + D_+C_+^* - C_-D_-^* &= 0, & \text{(G.114)}\\
B_-A_+^* - A_-B_+^* + D_-C_+^* - C_-D_+^* &= 0, & \text{(G.115)}\\
(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_-^2 - \omega_+^2)} e^{-3N_0}, & \text{(G.116)}\\
(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_+^2 - \omega_-^2)} e^{-3N_0}. & \text{(G.117)}
\end{aligned}
$$

Moreover (G.112) is equivalent to (G.114), and (G.113) is equivalent to (G.115). We therefore have reduced the wronskian conditions to

$$
\begin{aligned}
A_+F_+^* - F_-A_-^* + C_+H_+^* - H_-C_-^* &= 0, & \text{(G.118)}\\
A_-F_+^* - F_-A_+^* + C_-H_+^* - H_-C_+^* &= 0, & \text{(G.119)}\\
(A_-E_-^* + A_+E_+^* + C_-G_-^* + C_+G_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_-^2 - \omega_+^2)} e^{-3N_0}, & \text{(G.120)}\\
(B_-F_-^* + B_+F_+^* + D_-H_-^* + D_+H_+^* - \text{c.c}) &= \frac{i\lambda_0}{H(\omega_+^2 - \omega_-^2)} e^{-3N_0}. & \text{(G.121)}
\end{aligned}
$$

We now use the eigenvector relations to express the wronskian conditions only in terms of E, F, G, and H:

$$
\begin{aligned}
-E_+F_+^* + F_-E_-^* - G_+H_+^* + H_-G_-^* &= 0, & \text{(G.122)}\\
E_-F_+^* - F_-E_+^* + G_-H_+^* - H_-G_+^* &= 0, & \text{(G.123)}\\
2\omega_-(|E_-|^2 - |E_+|^2 + |G_-|^2 - |G_+|^2) &= \frac{1}{2H} e^{-3N_0}, & \text{(G.124)}\\
2\omega_+(|F_-|^2 - |F_+|^2 + |H_-|^2 - |H_+|^2) &= \frac{1}{2H} e^{-3N_0}. & \text{(G.125)}
\end{aligned}
$$

These are the basic wronskian conditions.

Now, these conditions should be satisfied automatically, after imposing the initial conditions together with the eigenvector relations. We shall verify this later on. Lets write down the solutions with the eigenvector relations:

$$
\begin{aligned}
\zeta_k &= \Bigg[-i\lambda_0 \frac{\omega_-}{\omega_-^2-\kappa^2}(E_+ e^{+i\omega_- N} - E_- e^{-i\omega_- N}) - \\
&\qquad\qquad i\lambda_0 \frac{\omega_+}{\omega_+^2-\kappa^2}(F_+ e^{+i\omega_+ N} - F_- e^{-i\omega_+ N})\Bigg] a_\zeta(\vec{k}) \\
&+ \Bigg[-i\lambda_0 \frac{\omega_-}{\omega_-^2-\kappa^2}(G_+ e^{+i\omega_- N} - G_- e^{-i\omega_- N}) - \\
&\qquad\qquad i\lambda_0 \frac{\omega_+}{\omega_+^2-\kappa^2}(H_+ e^{+i\omega_+ N} - H_- e^{-i\omega_+ N})\Bigg] a_\psi(\vec{k}) \\
&+ \Bigg[+i\lambda_0 \frac{\omega_-}{\omega_-^2-\kappa^2}(E_+^* e^{-i\omega_- N} - E_-^* e^{+i\omega_- N}) + \\
&\qquad\qquad i\lambda_0 \frac{\omega_+}{\omega_+^2-\kappa^2}(F_+^* e^{-i\omega_+ N} - F_-^* e^{+i\omega_+ N})\Bigg] a_\zeta^\dagger(-\vec{k}) \\
&+ \Bigg[+i\lambda_0 \frac{\omega_-}{\omega_-^2-\kappa^2}(G_+^* e^{-i\omega_- N} - G_-^* e^{+i\omega_- N}) + \\
&\qquad\qquad i\lambda_0 \frac{\omega_+}{\omega_+^2-\kappa^2}(H_+^* e^{-i\omega_+ N} - H_-^* e^{+i\omega_+ N})\Bigg] a_\psi^\dagger(-\vec{k}), \qquad \text{(G.126)}
\end{aligned}
$$

$$
\begin{aligned}
\psi_k &= \Bigg[E_+ e^{+i\omega_- N} + E_- e^{-i\omega_- N} + F_+ e^{+i\omega_+ N} + F_- e^{-i\omega_+ N}\Bigg] a_\zeta(\vec{k}) \\
&+ \Bigg[G_+ e^{+i\omega_- N} + G_- e^{-i\omega_- N} + H_+ e^{+i\omega_+ N} + H_- e^{-i\omega_+ N}\Bigg] a_\psi(\vec{k}) \\
&+ \Bigg[E_+^* e^{-i\omega_- N} + E_-^* e^{+i\omega_- N} + F_+^* e^{-i\omega_+ N} + F_-^* e^{+i\omega_+ N}\Bigg] a_\zeta^\dagger(-\vec{k}) \\
&+ \Bigg[G_+^* e^{-i\omega_- N} + G_-^* e^{+i\omega_- N} + H_+^* e^{-i\omega_+ N} + H_-^* e^{+i\omega_+ N}\Bigg] a_\psi^\dagger(-\vec{k}). \qquad \text{(G.127)}
\end{aligned}
$$

Then

$$\begin{aligned}
\zeta_k' &= \lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(E_+e^{+i\omega_-N}+E_-e^{-i\omega_-N}\right)+ \\
&\qquad\qquad \frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(F_+e^{+i\omega_+N}+F_-e^{-i\omega_+N}\right)\Bigg]a_\zeta(\vec{k}) \\
&\quad +\lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(G_+e^{+i\omega_-N}+G_-e^{-i\omega_-N}\right)+ \\
&\qquad\qquad \frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(H_+e^{+i\omega_+N}+H_-e^{-i\omega_+N}\right)\Bigg]a_\psi(\vec{k}) \\
&\quad +\lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(E_+^*e^{-i\omega_-N}+E_-^*e^{+i\omega_-N}\right)+ \\
&\qquad\qquad \frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(F_+^*e^{-i\omega_+N}+F_-^*e^{+i\omega_+N}\right)\Bigg]a_\zeta^\dagger(-\vec{k}) \\
&\quad +\lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(G_+^*e^{-i\omega_-N}+G_-^*e^{+i\omega_-N}\right)+ \\
&\qquad\qquad \frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(H_+^*e^{-i\omega_+N}+H_-^*e^{+i\omega_+N}\right)\Bigg]a_\psi^\dagger(-\vec{k}), \qquad \text{(G.128)}
\end{aligned}$$

$$\begin{aligned}
\psi_k' &= \Bigg[i\omega_-\left(E_+e^{+i\omega_-N}-E_-e^{-i\omega_-N}\right)+ \\
&\qquad\qquad i\omega_+\left(F_+e^{+i\omega_+N}-F_-e^{-i\omega_+N}\right)\Bigg]a_\zeta(\vec{k}) \\
&\quad +\Bigg[i\omega_-\left(G_+e^{+i\omega_-N}-G_-e^{-i\omega_-N}\right)+ \\
&\qquad\qquad i\omega_+\left(H_+e^{+i\omega_+N}-H_-e^{-i\omega_+N}\right)\Bigg]a_\psi(\vec{k}) \\
&\quad +\Bigg[-i\omega_-\left(E_+^*e^{-i\omega_-N}-E_-^*e^{+i\omega_-N}\right)- \\
&\qquad\qquad i\omega_+\left(F_+^*e^{-i\omega_+N}-F_-^*e^{+i\omega_+N}\right)\Bigg]a_\zeta^\dagger(-\vec{k}) \\
&\quad +\Bigg[-i\omega_-\left(G_+^*e^{-i\omega_-N}-G_-^*e^{+i\omega_-N}\right)- \\
&\qquad\qquad i\omega_+\left(H_+^*e^{-i\omega_+N}-H_-^*e^{+i\omega_+N}\right)\Bigg]a_\psi^\dagger(-\vec{k}). \qquad \text{(G.129)}
\end{aligned}$$

We now have the solutions in terms of E, F, G and H. Let's impose on them the boundary conditions in N_1. Recall that these are:

$$\zeta(N_1-\epsilon) = \zeta(N_1+\epsilon), \tag{G.130}$$
$$\psi(N_1-\epsilon) = \psi(N_1+\epsilon), \tag{G.131}$$
$$\zeta'(N_1-\epsilon) = (\zeta'-\lambda\psi)(N_1+\epsilon), \tag{G.132}$$
$$\psi'(N_1-\epsilon) = \psi'(N_1+\epsilon). \tag{G.133}$$

As we did before, let us say that the solution from region I, evaluated at N_1 have the form[51]

$$\zeta_k = Za_\zeta + \text{h.c}(-\vec{k}), \tag{G.134}$$
$$\zeta'_k = Z'a_\zeta + \text{h.c}(-\vec{k}), \tag{G.135}$$
$$\psi_k = Pa_\psi + \text{h.c}(-\vec{k}), \tag{G.136}$$
$$\psi'_k = P'a_\psi + \text{h.c}(-\vec{k}). \tag{G.137}$$

Then, we obtain

$$\begin{aligned}
-i\lambda_0\frac{\omega_-}{\omega_-^2-\kappa^2}(E_+e^{+i\omega_-N_1}-E_-e^{-i\omega_-N_1})- \\
i\lambda_0\frac{\omega_+}{\omega_+^2-\kappa^2}(F_+e^{+i\omega_+N_1}-F_-e^{-i\omega_+N_1}) &= Z, \qquad (G.138)\\
-i\lambda_0\frac{\omega_-}{\omega_-^2-\kappa^2}(G_+e^{+i\omega_-N_1}-G_-e^{-i\omega_-N_1})- \\
i\lambda_0\frac{\omega_+}{\omega_+^2-\kappa^2}(H_+e^{+i\omega_+N_1}-H_-e^{-i\omega_+N_1}) &= 0, \qquad (G.139)\\
\left[E_+e^{+i\omega_-N_1}+E_-e^{-i\omega_-N_1}+F_+e^{+i\omega_+N_1}+F_-e^{-i\omega_+N_1}\right] &= 0, \qquad (G.140)\\
\left[G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}+H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}\right] &= P, \qquad (G.141)\\
\lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(E_+e^{+i\omega_-N_1}+E_-e^{-i\omega_-N_1}\right)+ \\
\frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(F_+e^{+i\omega_+N_1}+F_-e^{-i\omega_+N_1}\right)\Bigg] \\
-\lambda_0\left[E_+e^{+i\omega_-N_1}+E_-e^{-i\omega_-N_1}+F_+e^{+i\omega_+N_1}+F_-e^{-i\omega_+N_1}\right] &= Z', \qquad (G.142)\\
\lambda_0\Bigg[\frac{\omega_-^2}{\omega_-^2-\kappa^2}\left(G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}\right)+ \\
\frac{\omega_+^2}{\omega_+^2-\kappa^2}\left(H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}\right)\Bigg] \\
-\lambda_0\left[G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}+H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}\right] &= 0, \qquad (G.143)
\end{aligned}$$

[51] From now on, the quantities Z, Z', P and P' are evaluated at N_1

$$\left[i\omega_- \left(E_+ e^{+i\omega_- N_1} - E_- e^{-i\omega_- N_1}\right) + i\omega_+ \left(F_+ e^{+i\omega_+ N_1} - F_- e^{-i\omega_+ N_1}\right)\right] = 0, \quad \text{(G.144)}$$

$$\left[i\omega_- \left(G_+ e^{+i\omega_- N_1} - G_- e^{-i\omega_- N_1}\right) + i\omega_+ \left(H_+ e^{+i\omega_+ N_1} - H_- e^{-i\omega_+ N_1}\right)\right] = P'. \quad \text{(G.145)}$$

These are 8 equations for 8 unknown variables. We can group them in 2 set of equations. The first set is to solve E and F:

$$-i\lambda_0 \frac{\omega_-}{\omega_-^2 - \kappa^2}(E_+ e^{+i\omega_- N_1} - E_- e^{-i\omega_- N_1}) - i\lambda_0 \frac{\omega_+}{\omega_+^2 - \kappa^2}(F_+ e^{+i\omega_+ N_1} - F_- e^{-i\omega_+ N_1}) = Z, \quad \text{(G.146)}$$

$$\left[E_+ e^{+i\omega_- N_1} + E_- e^{-i\omega_- N_1} + F_+ e^{+i\omega_+ N_1} + F_- e^{-i\omega_+ N_1}\right] = 0, \quad \text{(G.147)}$$

$$\lambda_0 \left[\frac{\kappa^2}{\omega_-^2 - \kappa^2}\left(E_+ e^{+i\omega_- N_1} + E_- e^{-i\omega_- N_1}\right) + \frac{\kappa^2}{\omega_+^2 - \kappa^2}\left(F_+ e^{+i\omega_+ N_1} + F_- e^{-i\omega_+ N_1}\right)\right] = Z', \quad \text{(G.148)}$$

$$\left[i\omega_- \left(E_+ e^{+i\omega_- N_1} - E_- e^{-i\omega_- N_1}\right) + i\omega_+ \left(F_+ e^{+i\omega_+ N_1} - F_- e^{-i\omega_+ N_1}\right)\right] = 0. \quad \text{(G.149)}$$

They lead to

$$E_+ e^{+i\omega_- N_1} - E_- e^{-i\omega_- N_1} = \frac{iZ}{\lambda_0 \omega_-} \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{(\omega_+^2 - \omega_-^2)}, \quad \text{(G.150)}$$

$$E_+ e^{+i\omega_- N_1} + E_- e^{-i\omega_- N_1} = \frac{Z'}{\lambda_0 \kappa^2} \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{(\omega_+^2 - \omega_-^2)}, \quad \text{(G.151)}$$

$$F_+ e^{+i\omega_+ N_1} - F_- e^{-i\omega_+ N_1} = -\frac{iZ}{\lambda_0 \omega_+} \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{(\omega_+^2 - \omega_-^2)}, \quad \text{(G.152)}$$

$$F_+ e^{+i\omega_+ N_1} + F_- e^{-i\omega_+ N_1} = -\frac{Z'}{\lambda_0 \kappa^2} \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{(\omega_+^2 - \omega_-^2)}. \quad \text{(G.153)}$$

Combining them we find

$$E_+ = \left(\frac{Z'}{\kappa^2} + \frac{iZ}{\omega_-}\right) \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{2\lambda_0(\omega_+^2 - \omega_-^2)} e^{-i\omega_- N_1}, \quad \text{(G.154)}$$

$$E_- = \left(\frac{Z'}{\kappa^2} - \frac{iZ}{\omega_-}\right) \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{2\lambda_0(\omega_+^2 - \omega_-^2)} e^{+i\omega_- N_1}, \quad \text{(G.155)}$$

$$F_+ = -\left(\frac{Z'}{\kappa^2} + \frac{iZ}{\omega_+}\right) \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{2\lambda_0(\omega_+^2 - \omega_-^2)} e^{-i\omega_+ N_1}, \quad \text{(G.156)}$$

$$F_- = -\left(\frac{Z'}{\kappa^2} - \frac{iZ}{\omega_+}\right) \frac{(\omega_-^2 - \kappa^2)(\omega_+^2 - \kappa^2)}{2\lambda_0(\omega_+^2 - \omega_-^2)} e^{+i\omega_+ N_1}. \quad \text{(G.157)}$$

We now go for the next set:

$$\begin{aligned}
\frac{\omega_-}{\omega_-^2-\kappa^2}(G_+e^{+i\omega_-N_1}-G_-e^{-i\omega_-N_1})+ & & \\
\frac{\omega_+}{\omega_+^2-\kappa^2}(H_+e^{+i\omega_+N_1}-H_-e^{-i\omega_+N_1}) &= 0, & \text{(G.158)}\\
\left[G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}+H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}\right] &= P, & \text{(G.159)}\\
\frac{\kappa^2}{\omega_-^2-\kappa^2}\left(G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}\right)+ & & \\
\frac{\kappa^2}{\omega_+^2-\kappa^2}\left(H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}\right) &= 0, & \text{(G.160)}\\
\left[i\omega_-\left(G_+e^{+i\omega_-N_1}-G_-e^{-i\omega_-N_1}\right)+i\omega_+\left(H_+e^{+i\omega_+N_1}-H_-e^{-i\omega_+N_1}\right)\right] &= P'. & \text{(G.161)}
\end{aligned}$$

This implies

$$G_+e^{+i\omega_-N_1}+G_-e^{-i\omega_-N_1}=-\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}P, \quad \text{(G.162)}$$

$$G_+e^{+i\omega_-N_1}-G_-e^{-i\omega_-N_1}=\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}\frac{iP'}{\omega_-}, \quad \text{(G.163)}$$

$$H_+e^{+i\omega_+N_1}+H_-e^{-i\omega_+N_1}=\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}P, \quad \text{(G.164)}$$

$$H_+e^{+i\omega_+N_1}-H_-e^{-i\omega_+N_1}=-\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}\frac{iP'}{\omega_+}. \quad \text{(G.165)}$$

So

$$\begin{aligned}
G_+ &= -\frac{1}{2}\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P-\frac{iP'}{\omega_-}\right)e^{-i\omega_-N_1}, & \text{(G.166)}\\
G_- &= -\frac{1}{2}\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P+\frac{iP'}{\omega_-}\right)e^{+i\omega_-N_1}, & \text{(G.167)}\\
H_+ &= \frac{1}{2}\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P-\frac{iP'}{\omega_+}\right)e^{-i\omega_+N_1}, & \text{(G.168)}\\
H_- &= \frac{1}{2}\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P+\frac{iP'}{\omega_+}\right)e^{+i\omega_+N_1}. & \text{(G.169)}
\end{aligned}$$

Thus, the complete list of solutions are

$$E_{+} = \left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_-}\right)\frac{(\omega_-^2-\kappa^2)(\omega_+^2-\kappa^2)}{2\lambda_0(\omega_+^2-\omega_-^2)}e^{-i\omega_- N_1}, \tag{G.170}$$

$$E_{-} = \left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_-}\right)\frac{(\omega_-^2-\kappa^2)(\omega_+^2-\kappa^2)}{2\lambda_0(\omega_+^2-\omega_-^2)}e^{+i\omega_- N_1}, \tag{G.171}$$

$$F_{+} = -\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_+}\right)\frac{(\omega_-^2-\kappa^2)(\omega_+^2-\kappa^2)}{2\lambda_0(\omega_+^2-\omega_-^2)}e^{-i\omega_+ N_1}, \tag{G.172}$$

$$F_{-} = -\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_+}\right)\frac{(\omega_-^2-\kappa^2)(\omega_+^2-\kappa^2)}{2\lambda_0(\omega_+^2-\omega_-^2)}e^{+i\omega_+ N_1}, \tag{G.173}$$

$$G_{+} = -\frac{1}{2}\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P-\frac{iP'}{\omega_-}\right)e^{-i\omega_- N_1}, \tag{G.174}$$

$$G_{-} = -\frac{1}{2}\frac{\omega_-^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P+\frac{iP'}{\omega_-}\right)e^{+i\omega_- N_1}, \tag{G.175}$$

$$H_{+} = \frac{1}{2}\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P-\frac{iP'}{\omega_+}\right)e^{-i\omega_+ N_1}, \tag{G.176}$$

$$H_{-} = \frac{1}{2}\frac{\omega_+^2-\kappa^2}{\omega_+^2-\omega_-^2}\left(P+\frac{iP'}{\omega_+}\right)e^{+i\omega_+ N_1}. \tag{G.177}$$

We can simplify them

$$E_{+} = -\frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_-}\right)e^{-i\omega_- N_1}, \tag{G.178}$$

$$E_{-} = -\frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_-}\right)e^{+i\omega_- N_1}, \tag{G.179}$$

$$F_{+} = \frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_+}\right)e^{-i\omega_+ N_1}, \tag{G.180}$$

$$F_{-} = \frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_+}\right)e^{+i\omega_+ N_1}, \tag{G.181}$$

$$G_{+} = \frac{1}{4}\left(P-\frac{iP'}{\omega_-}\right)e^{-i\omega_- N_1}, \tag{G.182}$$

$$G_{-} = \frac{1}{4}\left(P+\frac{iP'}{\omega_-}\right)e^{+i\omega_- N_1}, \tag{G.183}$$

$$H_{+} = \frac{1}{4}\left(P-\frac{iP'}{\omega_+}\right)e^{-i\omega_+ N_1}, \tag{G.184}$$

$$H_{-} = \frac{1}{4}\left(P+\frac{iP'}{\omega_+}\right)e^{+i\omega_+ N_1}. \tag{G.185}$$

We should now check whether these solutions satisfy the wronskian conditions. The first, (G.122), reads

$$\kappa^2\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_-}\right)\left(\frac{Z'^*}{\kappa^2}-\frac{iZ^*}{\omega_+}\right)-\kappa^2\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_+}\right)\left(\frac{Z^{*\prime}}{\kappa^2}+\frac{iZ^*}{\omega_-}\right) \tag{G.186}$$

$$-\left(P-\frac{iP'}{\omega_-}\right)\left(P^*+\frac{iP^{*\prime}}{\omega_+}\right)+\left(P+\frac{iP'}{\omega_+}\right)\left(P^*-\frac{iP^{*\prime}}{\omega_-}\right)=0, \tag{G.187}$$

which is equivalent to

$$ZZ'^* - Z'Z^* - (PP'^* - P'P^*) = 0, \tag{G.188}$$

which is true. The same holds for (G.123). The last two conditions read

$$\frac{i}{4}(P'P^* - PP'^* + Z'Z^* - ZZ'^*) = \frac{1}{2H}e^{-3N_0}. \tag{G.189}$$

Indeed, we have

$$\begin{aligned} Z'Z^* - ZZ'^* &= \frac{H^2}{2k^3}\left(1-ie^{N_k-N_1}\right)e^{2(N_k-N_1)} - \frac{H^2}{2k^3}\left(1+ie^{N_k-N_1}\right)e^{2(N_k-N_1)} \\ &= -i\frac{H^2}{k^3}e^{3(N_k-N_1)} = -\frac{ie^{-3N_1}}{H}. \end{aligned} \tag{G.190}$$

Then (G.124) and (G.125) are satisfied since we have $N_0 \sim N_1$.

We will continue toward obtaining a complete solution of the system at the end of inflation. After N_2, the exact solutions are of the form

$$\begin{aligned} \zeta_k &= \left[I_+\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}} + I_-\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}}\right]a_\zeta(\vec{k}) \\ &\quad+\left[J_+\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}} + J_-\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}}\right]a_\psi(\vec{k}) \\ &\quad+\left[I_+^*\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}} + I_-^*\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}}\right]a_\zeta^\dagger(-\vec{k}) \\ &\quad+\left[J_+^*\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}} + J_-^*\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}}\right]a_\psi^\dagger(-\vec{k}) \\ \psi_k &= \left[K_+\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}} + K_-\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}}\right]a_\zeta(\vec{k}) \\ &\quad+\left[L_+\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}} + L_-\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}}\right]a_\psi(\vec{k}) \\ &\quad+\left[K_+^*\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}} + K_-^*\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}}\right]a_\zeta^\dagger(-\vec{k}) \\ &\quad+\left[L_+^*\left(1+ie^{N_k-N}\right)e^{-i\frac{k}{H}e^{-N}} + L_-^*\left(1-ie^{N_k-N}\right)e^{+i\frac{k}{H}e^{-N}}\right]a_\psi^\dagger(-\vec{k}) \end{aligned} \tag{G.191}$$

with derivatives

$$\begin{aligned}
\zeta_k' &= -e^{2(N_k-N)}\left(I_+e^{+i\frac{k}{H}e^{-N}}+I_-e^{-i\frac{k}{H}e^{-N}}\right)a_\zeta(\vec{k})\\
&\quad -e^{2(N_k-N)}\left(J_+e^{+i\frac{k}{H}e^{-N}}+J_-e^{-i\frac{k}{H}e^{-N}}\right)a_\psi(\vec{k})\\
&\quad -e^{2(N_k-N)}\left(I_+^*e^{-i\frac{k}{H}e^{-N}}+I_-^*e^{+i\frac{k}{H}e^{-N}}\right)a_\zeta^\dagger(-\vec{k})\\
&\quad -e^{2(N_k-N)}\left(J_+^*e^{-i\frac{k}{H}e^{-N}}+J_-^*e^{+i\frac{k}{H}e^{-N}}\right)a_\psi^\dagger(-\vec{k})\\
\psi_k' &= -e^{2(N_k-N)}\left(K_+e^{+i\frac{k}{H}e^{-N}}+K_-e^{-i\frac{k}{H}e^{-N}}\right)a_\zeta(\vec{k})\\
&\quad -e^{2(N_k-N)}\left(L_+e^{+i\frac{k}{H}e^{-N}}+L_-e^{-i\frac{k}{H}e^{-N}}\right)a_\psi(\vec{k})\\
&\quad -e^{2(N_k-N)}\left(K_+^*e^{-i\frac{k}{H}e^{-N}}+K_-^*e^{+i\frac{k}{H}e^{-N}}\right)a_\zeta^\dagger(-\vec{k})\\
&\quad -e^{2(N_k-N)}\left(L_+^*e^{-i\frac{k}{H}e^{-N}}+L_-^*e^{+i\frac{k}{H}e^{-N}}\right)a_\psi^\dagger(-\vec{k})
\end{aligned}\tag{G.192}$$

The initial conditions for the region III are

$$\zeta(N_2+\epsilon)=\zeta(N_2-\epsilon),\tag{G.193}$$
$$\psi(N_2+\epsilon)=\psi(N_2-\epsilon),\tag{G.194}$$
$$\zeta'(N_2+\epsilon)=(\zeta'-\lambda\psi)(N_2-\epsilon),\tag{G.195}$$
$$\psi'(N_2+\epsilon)=\psi'(N_2-\epsilon).\tag{G.196}$$

It follows that

$$\left[I_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+I_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}}\right] = M_1,\tag{G.197}$$
$$\left[J_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+J_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}}\right] = M_2,\tag{G.198}$$
$$\left[K_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+K_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}}\right] = M_3,\tag{G.199}$$
$$\left[L_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+L_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}}\right] = M_4,\tag{G.200}$$
$$-e^{2(N_k-N_2)}\left(I_+e^{+i\frac{k}{H}e^{-N_2}}+I_-e^{-i\frac{k}{H}e^{-N_2}}\right) = M_5,\tag{G.201}$$
$$-e^{2(N_k-N_2)}\left(J_+e^{+i\frac{k}{H}e^{-N_2}}+J_-e^{-i\frac{k}{H}e^{-N_2}}\right) = M_6,\tag{G.202}$$

$$-e^{2(N_k-N_2)}\left(K_+e^{+i\frac{k}{H}e^{-N_2}}+K_-e^{-i\frac{k}{H}e^{-N_2}}\right) = M_7, \quad (G.203)$$

$$-e^{2(N_k-N_2)}\left(L_+e^{+i\frac{k}{H}e^{-N_2}}+L_-e^{-i\frac{k}{H}e^{-N_2}}\right) = M_8, \quad (G.204)$$

where we have defined:

$$M_1 = -i\lambda_0\left[\frac{\omega_-}{\omega_-^2-\kappa^2}(E_+e^{+i\omega_-N_2}-E_-e^{-i\omega_-N_2})+\frac{\omega_+}{\omega_+^2-\kappa^2}(F_+e^{+i\omega_+N_2}-F_-e^{-i\omega_+N_2})\right], \quad (G.205)$$

$$M_2 = -i\lambda_0\left[\frac{\omega_-}{\omega_-^2-\kappa^2}(G_+e^{+i\omega_-N_2}-G_-e^{-i\omega_-N_2})+\frac{\omega_+}{\omega_+^2-\kappa^2}(H_+e^{+i\omega_+N_2}-H_-e^{-i\omega_+N_2})\right], \quad (G.206)$$

$$M_3 = \left[E_+e^{+i\omega_-N_2}+E_-e^{-i\omega_-N_2}+F_+e^{+i\omega_+N_2}+F_-e^{-i\omega_+N_2}\right], \quad (G.207)$$

$$M_4 = \left[G_+e^{+i\omega_-N_2}+G_-e^{-i\omega_-N_2}+H_+e^{+i\omega_+N_2}+H_-e^{-i\omega_+N_2}\right], \quad (G.208)$$

$$M_5 = \lambda_0\left[\frac{\kappa^2}{\omega_-^2-\kappa^2}\left(E_+e^{+i\omega_-N_2}+E_-e^{-i\omega_-N_2}\right)+\frac{\kappa^2}{\omega_+^2-\kappa^2}\left(F_+e^{+i\omega_+N_2}+F_-e^{-i\omega_+N_2}\right)\right], \quad (G.209)$$

$$M_6 = \lambda_0\left[\frac{\kappa^2}{\omega_-^2-\kappa^2}\left(G_+e^{+i\omega_-N}+G_-e^{-i\omega_-N}\right)+\frac{\kappa^2}{\omega_+^2-\kappa^2}\left(H_+e^{+i\omega_+N}+H_-e^{-i\omega_+N}\right)\right], \quad (G.210)$$

$$M_7 = \left[i\omega_-\left(E_+e^{+i\omega_-N_2}-E_-e^{-i\omega_-N_2}\right)+i\omega_+\left(F_+e^{+i\omega_+N_2}-F_-e^{-i\omega_+N_2}\right)\right], \quad (G.211)$$

$$M_8 = \left[i\omega_-\left(G_+e^{+i\omega_-N_2}-G_-e^{-i\omega_-N_2}\right)+i\omega_+\left(H_+e^{+i\omega_+N_2}-H_-e^{-i\omega_+N_2}\right)\right]. \quad (G.212)$$

Now we are only left to find expressions for the quantities I, J, K and L in terms of the M's.

$$I_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+I_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}} = M_1, \quad (G.213)$$

$$J_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+J_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}} = M_2, \quad (G.214)$$

$$K_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+K_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}} = M_3, \quad (G.215)$$

$$L_+\left(1-ie^{N_k-N_2}\right)e^{+i\frac{k}{H}e^{-N_2}}+L_-\left(1+ie^{N_k-N_2}\right)e^{-i\frac{k}{H}e^{-N_2}} = M_4, \quad (G.216)$$

$$I_+ e^{+i\frac{k}{H}e^{-N_2}} + I_- e^{-i\frac{k}{H}e^{-N_2}} = -M_5 e^{-2(N_k-N_2)}, \tag{G.217}$$
$$J_+ e^{+i\frac{k}{H}e^{-N_2}} + J_- e^{-i\frac{k}{H}e^{-N_2}} = -M_6 e^{-2(N_k-N_2)}, \tag{G.218}$$
$$K_+ e^{+i\frac{k}{H}e^{-N_2}} + K_- e^{-i\frac{k}{H}e^{-N_2}} = -M_7 e^{-2(N_k-N_2)}, \tag{G.219}$$
$$L_+ e^{+i\frac{k}{H}e^{-N_2}} + L_- e^{-i\frac{k}{H}e^{-N_2}} = -M_8 e^{-2(N_k-N_2)}. \tag{G.220}$$

They all have the same form, so it is enough to deal with one. In the end we obtain

$$I_+ = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_5 e^{-2(N_k-N_2)} + M_1\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.221}$$
$$I_- = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_5 e^{-2(N_k-N_2)} + M_1\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}, \tag{G.222}$$
$$J_+ = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_6 e^{-2(N_k-N_2)} + M_2\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.223}$$
$$J_- = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_6 e^{-2(N_k-N_2)} + M_2\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}, \tag{G.224}$$
$$K_+ = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_7 e^{-2(N_k-N_2)} + M_3\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.225}$$
$$K_- = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_7 e^{-2(N_k-N_2)} + M_3\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}, \tag{G.226}$$
$$L_+ = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_8 e^{-2(N_k-N_2)} + M_4\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.227}$$
$$L_- = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_8 e^{-2(N_k-N_2)} + M_4\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}. \tag{G.228}$$
$$\tag{G.229}$$

This completes the computation.

So, equation (6.33) and the analytic version of the power spectrum that we showed in Figure (6.1) are computed using

$$\zeta_k = \left[I_+ + I_-\right]a_\zeta(\vec{k}) + \left[J_+ + J_-\right]a_\psi(\vec{k}) + h.c.(-\vec{k}), \tag{G.230}$$

and

$$P_\zeta(k) = |I_+ + I_-|^2 + |J_+ + J_-|^2. \tag{G.231}$$

Where

$$I_{+} = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_5e^{-2(N_k-N_2)}+M_1\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.232}$$

$$I_{-} = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_5e^{-2(N_k-N_2)}+M_1\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}, \tag{G.233}$$

$$J_{+} = \frac{i}{2}\left[\left(1+ie^{N_k-N_2}\right)M_6e^{-2(N_k-N_2)}+M_2\right]e^{-(N_k-N_2)}e^{-i\frac{k}{H}e^{-N_2}}, \tag{G.234}$$

$$J_{-} = -\frac{i}{2}\left[\left(1-ie^{N_k-N_2}\right)M_6e^{-2(N_k-N_2)}+M_2\right]e^{-(N_k-N_2)}e^{+i\frac{k}{H}e^{-N_2}}, \tag{G.235}$$

and where

$$M_1 = -i\lambda_0\left[\frac{\omega_-}{\omega_-^2-\kappa^2}(E_+e^{+i\omega_-N_2}-E_-e^{-i\omega_-N_2})+ \frac{\omega_+}{\omega_+^2-\kappa^2}(F_+e^{+i\omega_+N_2}-F_-e^{-i\omega_+N_2})\right], \tag{G.236}$$

$$M_2 = -i\lambda_0\left[\frac{\omega_-}{\omega_-^2-\kappa^2}(G_+e^{+i\omega_-N_2}-G_-e^{-i\omega_-N_2})+ \frac{\omega_+}{\omega_+^2-\kappa^2}(H_+e^{+i\omega_+N_2}-H_-e^{-i\omega_+N_2})\right], \tag{G.237}$$

$$M_5 = \lambda_0\left[\frac{\kappa^2}{\omega_-^2-\kappa^2}\left(E_+e^{+i\omega_-N_2}+E_-e^{-i\omega_-N_2}\right)+ \frac{\kappa^2}{\omega_+^2-\kappa^2}\left(F_+e^{+i\omega_+N_2}+F_-e^{-i\omega_+N_2}\right)\right], \tag{G.238}$$

$$M_6 = \lambda_0\left[\frac{\kappa^2}{\omega_-^2-\kappa^2}\left(G_+e^{+i\omega_-N}+G_-e^{-i\omega_-N}\right)+ \frac{\kappa^2}{\omega_+^2-\kappa^2}\left(H_+e^{+i\omega_+N}+H_-e^{-i\omega_+N}\right)\right], \tag{G.239}$$

with

$$E_+ = -\frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_-}\right)e^{-i\omega_-N_1}, \tag{G.240}$$

$$E_- = -\frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_-}\right)e^{+i\omega_-N_1}, \tag{G.241}$$

$$F_+ = \frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}+\frac{iZ}{\omega_+}\right)e^{-i\omega_+N_1}, \tag{G.242}$$

$$F_- = \frac{\kappa}{4}\left(\frac{Z'}{\kappa^2}-\frac{iZ}{\omega_+}\right)e^{+i\omega_+N_1}, \tag{G.243}$$

$$G_+ = \frac{1}{4}\left(P-\frac{iP'}{\omega_-}\right)e^{-i\omega_-N_1}, \tag{G.244}$$

$$G_- = \frac{1}{4}\left(P + \frac{iP'}{\omega_-}\right) e^{+i\omega_- N_1}, \tag{G.245}$$

$$H_+ = \frac{1}{4}\left(P - \frac{iP'}{\omega_+}\right) e^{-i\omega_+ N_1}, \tag{G.246}$$

$$H_- = \frac{1}{4}\left(P + \frac{iP'}{\omega_+}\right) e^{+i\omega_+ N_1}, \tag{G.247}$$

and finally with

$$Z = \frac{iH}{\sqrt{2k^3}}\left(1 - ie^{N_k - N_1}\right) e^{+ie^{N_k - N_1}}, \tag{G.248}$$

$$P = \frac{iH}{\sqrt{2k^3}}\left(1 - ie^{N_k - N_1}\right) e^{+ie^{N_k - N_1}}, \tag{G.249}$$

$$Z' = -\frac{iH}{\sqrt{2k^3}} e^{2(N_k - N_1)} e^{+ie^{N_k - N_1}}, \tag{G.250}$$

$$P' = -\frac{iH}{\sqrt{2k^3}} e^{2(N_k - N_1)} e^{+ie^{N_k - N_1}}. \tag{G.251}$$

Appendix H

A top-down example

In this appendix we will show a concrete example of a multifield action, where the inflationary trajectory can experience sudden turns, but keeping ε small and constant and $U(\psi) = 0$. This can be achieved in the context of holographic inflation [192–194], where the potential V in equation (6.1) is determined by a "fake" superpotential W like

$$V = 3W^2 - 2\gamma^{ab} W_a W_b, \tag{H.1}$$

with $W_a = \partial W / \partial \phi^a$. In this case, the background quantities obey the Hamilton-Jacobi equations:

$$\dot{\phi}^a = -2\gamma^{ab} W_b, \qquad H = W. \tag{H.2}$$

It turns out that U is related to W [195] via:

$$\partial_\psi^2 U|_{\psi=0} = 6HW_{NN} - 4W_{NN}^2 + 2\dot{W}_{NN}, \tag{H.3}$$

where $W_{NN} \equiv N^a N^b \nabla_a W_b$. Then we can define appropriate expressions for γ_{ab} and W such that ε and λ have the desired time dependence. For instance, let us use the fields $(\phi^1, \phi^2) = (\phi, \chi)$ and consider the following dependency on the metric:

$$\gamma_{ab} = \begin{pmatrix} e^{2f(\phi,\chi)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{H.4}$$

If we take a superpotential W that only depends on ϕ, then the equation (H.1) implies that $\dot{\phi} = -2e^{2f}\dot{W}_\phi$ and therefore $\dot{\chi} = 0$ regardless of the location in the field space. However, assuming $\dot{\phi} > 0$, the tangent and normal vectors are

$$T^a = \left(e^{-f}, 0\right), \qquad N^a = (0, 1), \tag{H.5}$$

and the turning rate becomes

$$\Omega = -N_a D_t T^a = \dot{\phi} e^f f_\chi, \tag{H.6}$$

implying that $\lambda H = 2\dot{\phi} e^f f_\chi$. Furthermore, it follows that $W_{NN} = 0$ and, thanks to the equation (H.3), $\partial_\psi^2 U|_{\psi=0} = 0$. Since this result is independent of χ, and ψ is a preturbation in the direction of χ, then $U = 0$ exactly.

Next, let us note that the function f can be expanded as

$$f(\phi, \chi) = \sum_n \frac{1}{n!} (\chi - \chi_0)^n f_n(\phi). \tag{H.7}$$

A redefinition of ϕ allows us to set $f_0(\phi) = 0$. Doing this, along the specific path $\chi = \chi_0$, we find that $\dot{\phi} = -2W_\phi$ and $\lambda = \sqrt{8\varepsilon} f_1[\phi(t)]$. As a consequence, $\varepsilon = 2W_\phi^2/W^2$ and the expansion rate depends only on $W(\phi)$, remaining unchanged by the turning rate λ. Finally, one can always define $W(\phi)$ and $f_1(\phi)$ to get the desired expressions for ε and λ.