# QUASI-RANDOM WORDS AND LIMITS OF WORD SEQUENCES 

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#### Abstract

Words are sequences of letters over a finite alphabet. We study two intimately related topics for this object: quasi-randomness and limit theory. With respect to the first topic we investigate the notion of uniform distribution of letters over intervals, and in the spirit of the famous Chung-Graham-Wilson theorem for graphs we provide a list of word properties which are equivalent to uniformity. In particular, we show that uniformity is equivalent to counting 3 -letter subsequences.

Inspired by graph limit theory we then investigate limits of convergent word sequences, those in which all subsequence densities converge. We show that convergent word sequences have a natural limit, namely Lebesgue measurable functions of the form $f:[0,1] \rightarrow[0,1]$. Via this theory we show that every hereditary word property is testable, address the problem of finite forcibility for word limits and establish as a byproduct a new model of random word sequences.

Along the lines of the proof of the existence of word limits, we can also establish the existence of limits for higher dimensional structures. In particular, we obtain an alternative proof of the result by Hoppen, Kohayakawa, Moreira, Ráth and Sampaio [J. Combin. Theory Ser. B 103(1):93-113, 2013] establishing the existence of permutons.


## 1. Introduction

Roughly speaking, quasi-random structures are deterministic objects which share many characteristic properties of their random counterparts. Formalizing this concept has turned out to be tremendously fruitful in several areas, among others, number theory, graph theory, extremal combinatorics, the design of algorithms and complexity theory. This often follows from the fact that if an object is quasi-random, then it immediately enjoys many other properties satisfied by its random counterpart.

Seminal work on quasi-randomness concerned graphs [13, 35, 38]. Subsequently, other combinatorial objects were considered, which include subsets of $\mathbb{Z}_{n}[12,21]$, hypergraphs [1, 11, 22, 39], finite groups [23], and permutations [15]. Curiously, in the rich history of quasi-randomness, words, i.e., sequences of letters from a finite alphabet, one of the most basic combinatorial object with many applications, do not seem to have been explicitly investigated. We overcome this apparent neglect, put forth a notion of quasi-random words and show it is equivalent to several other properties.

In contrast to the classical topic of quasi-randomness, the research of limits for discrete structures was launched rather recently by Chayes, Lovász, Sós, Szegedy and Vesztergombi [10, 30], and has become a very active topic of research since. Central to the area is the notion of convergent graph sequences $\left(G_{n}\right)_{n \rightarrow \infty}$, i.e., sequences of graphs which, roughly speaking, become more and more "similar" as $\left|V\left(G_{n}\right)\right|$ grows. For convergent graph sequences, Lovász and Szegedy [30] show the existence of natural limit objects, called graphons, endow the space of these structures with a metric and establish the equivalence of their notion of convergence and convergence on such a metric. Among many other consequences, it follows that quasi-random graph sequences, with edge density $p+o(1)$, converge to the constant $p$ graphon.

[^0]In this paper, we continue the lines of previously mentioned investigations and study quasirandomness for words and limits of convergent word sequences. Not only in the literature of quasi-randomness but also in the one concerning limits of discrete structures, explicit investigation of this fundamental object has not been considered so far.

## 2. Main contributions

A word $\boldsymbol{w}$ of length $n$ is an ordered sequence $\boldsymbol{w}=\left(w_{1}, w_{2}, \ldots, w_{n}\right)$ of letters $w_{i} \in \Sigma$ from a fixed size alphabet $\Sigma$. For the sake of presentation, unless explicitly said otherwise, we restrict our discussion to the two letter alphabet $\Sigma=\{0,1\}$, but most of our results and their proofs have straightforward generalizations to finite size alphabets.
2.1. Quasi-random words. Concerning quasi-randomness for words, our central notion is that of uniform distribution of letters over intervals. Specifically, a word $\boldsymbol{w}=\left(w_{1} \ldots w_{n}\right) \in\{0,1\}^{n}$ is called $(d, \varepsilon)$-uniform if for every interval $I \subseteq[n]$ we have ${ }^{1}$

$$
\begin{equation*}
\sum_{i \in I} w_{i}=\left|\left\{i \in I: w_{i}=1\right\}\right|=d|I| \pm \varepsilon n . \tag{1}
\end{equation*}
$$

We say that $\boldsymbol{w}$ is $\varepsilon$-uniform if $\boldsymbol{w}$ is $(d, \varepsilon)$-uniform for some $d$. Thus, uniformity states that up to an error term of $\varepsilon n$ the number of 1-entries of $\boldsymbol{w}$ in each interval $I$ is roughly $d|I|$, a property which binomial random words with parameter $d$ satisfy with high probability. This notion of uniformity has been studied by Axenovich, Person and Puzynina in [5], where a regularity lemma for words was established and applied to the problem of finding twins in words. In a different context, it has been studied by Cooper [15] who gave a list of equivalent properties. A word $\left(w_{1}, \ldots, w_{n}\right) \in\{0,1\}^{n}$ can also be seen as the set $W=\left\{i: w_{i}=1\right\} \subseteq \mathbb{Z}_{n}$ and from this point of view uniformity should be compared to the classical notion of quasi-randomness of subsets of $\mathbb{Z}_{n}$, studied by Chung and Graham in [12] and extended to the notion of $U_{k}$-uniformity by Gowers in [21]. With respect to this line of research we note that our notion of uniformity is strictly weaker than all of the ones studied in $[12,21]$. Indeed, the weakest of them concerns $U_{2}$-uniformity and may be rephrased as follows: $W \subseteq \mathbb{Z}_{n}$ has $U_{2}$-norm at most $\varepsilon>0$ if for all $A \subseteq \mathbb{Z}_{n}$ and all but $\varepsilon n$ elements $x \in \mathbb{Z}$ we have $|W \cap(A+x)|=|W| \frac{|A|}{n} \pm \varepsilon n$ where $A+x=\{a+x: a \in A\}$. Thus, e.g., the word $0101 \ldots 01$ is uniform in our sense but its corresponding set does not have small $U_{2}$-norm.

Analogous to the graph case there is a counting property related to uniformity. Given a word $\boldsymbol{w}=\left(w_{1} \ldots w_{n}\right)$ and a set of indices $I=\left\{i_{1}, \ldots, i_{\ell}\right\} \subseteq[n]$, where $i_{1}<i_{2}<\cdots<i_{\ell}$, let $\operatorname{sub}(I, \boldsymbol{w})$ be the length $\ell$ subsequence $\boldsymbol{u}=\left(u_{1} \ldots u_{\ell}\right)$ of $\boldsymbol{w}$ such that $u_{j}=w_{i_{j}}$. We show that uniformity implies adequate subsequence count, i.e., for any fixed $\boldsymbol{u}$ the number of subsequences equal to $\boldsymbol{u}$ in a large uniform word $\boldsymbol{w}$, denoted by $\binom{\boldsymbol{w}}{\boldsymbol{u}}$, is roughly as expected from a random word with same density of 1 -entries as $\boldsymbol{w}$. It is then natural to ask whether the converse also holds and one of our main results concerning quasi-random words states that uniformity is indeed already enforced by counting of subsequences of length three. If we let $\|\boldsymbol{w}\|_{1}=\sum_{i \in[n]} w_{i}$ denote the number of 1-entries in $\boldsymbol{w}$, then our result reads as follows.

Theorem 1. For every $\varepsilon>0, d \in[0,1]$, and $\ell \in \mathbb{N}$, there is an $n_{0}$ such that for all $n>n_{0}$ the following holds.

- If $\boldsymbol{w} \in\{0,1\}^{n}$ is (d, $\varepsilon$ )-uniform, then for each $\boldsymbol{u} \in\{0,1\}^{\ell}$

$$
\binom{\boldsymbol{w}}{\boldsymbol{u}}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{\ell-\|\boldsymbol{u}\|_{1}}\binom{n}{\ell} \pm 5 \varepsilon n^{\ell} .
$$

- Conversely, if $\boldsymbol{w} \in\{0,1\}^{n}$ is such that for all $\boldsymbol{u} \in\{0,1\}^{3}$ we have

$$
\binom{\boldsymbol{w}}{\boldsymbol{u}}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{3-\|\boldsymbol{u}\|_{1}}\binom{n}{3} \pm \varepsilon n^{3},
$$

[^1]then $\boldsymbol{w}$ is $\left(d, 42 \varepsilon^{1 / 3}\right)$-uniform.
Note that in the second part of the theorem the density of 1-entries is implicitly given. This is because $\binom{\boldsymbol{w}}{(111)}=\binom{\|\boldsymbol{w}\|_{1}}{3}$, and therefore the condition $\binom{\boldsymbol{w}}{(111)} \approx d^{3}\binom{n}{3}$ implies that $\|\boldsymbol{w}\|_{1} \approx d n$. We also note that length three subsequences in the theorem cannot be replaced by length two subsequences and in this sense the result is best possible. Indeed, the word ( $0 \ldots 01 \ldots 10 \ldots 0$ ) consisting of $(1-d) \frac{n}{2}$ zeroes followed by $d n$ ones followed by $(1-d) \frac{n}{2}$ zeroes contains the "right" number of every length two subsequences without being uniform.

We also study a property called Equidistribution and show that it is equivalent to uniformity. Together with Theorem 1 (and its direct consequences) and a result from Cooper [15, Theorem 2.2] this yields a list of equivalent properties stated in Theorem 2. To state the result let $\boldsymbol{w}[j]$ denote the $j$-th letter of the word $\boldsymbol{w}$. Furthermore, by the Cayley digraph $\Gamma(\boldsymbol{w})$ of a word $\boldsymbol{w}=\left(w_{1}, \ldots, w_{n}\right)$ we mean the digraph on the vertex set $\mathbb{Z}_{2 n}{ }^{2}$ in which $v$ is connected to $(v+i)(\bmod 2 n)$ for any $i$ with $w_{i}=1$. Given a word $\boldsymbol{u} \in\{0,1\}^{\ell}$, a sequence of vertices $\left(v_{1}, \ldots, v_{\ell+1}\right)$ is an induced $\boldsymbol{u}$-walk in $\Gamma(\boldsymbol{w})$ if the numbers $i_{1}, \ldots, i_{\ell} \in[n]$ defined by $v_{k+1}=v_{k}+i_{k}(\bmod n)$ satisfy $i_{1}<\cdots<i_{\ell}$ and for each $k \in[\ell]$ the pair $\left(v_{k}, v_{k+1}\right)$ is an edge in $\Gamma(\boldsymbol{w})$ if and only if $u_{k}=1$. Note that the number of induced $\boldsymbol{u}$-walks in $\Gamma(\boldsymbol{w})$ is precisely $2 n\binom{\boldsymbol{w}}{\boldsymbol{u}}$.
Theorem 2. For a sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words $\boldsymbol{w}_{n} \in\{0,1\}^{n}$ such that $\left\|\boldsymbol{w}_{n}\right\|_{1}=d n+o(n)$ for some $d \in[0,1]$, the following are equivalent:

- (Uniformity) $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is (d,o(1))-uniform.
- (Counting) For all $\ell \in \mathbb{N}$ and all $\boldsymbol{u} \in\{0,1\}^{\ell}$ we have

$$
\binom{\boldsymbol{w}_{n}}{\boldsymbol{u}}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{\ell-\|\boldsymbol{u}\|_{1}}\binom{n}{\ell}+o\left(n^{\ell}\right) .
$$

- (Minimizer) For all $\boldsymbol{u} \in\{0,1\}^{3}$ we have

$$
\binom{\boldsymbol{w}_{n}}{\boldsymbol{u}}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{3-\|\boldsymbol{u}\|_{1}}\binom{n}{3}+o\left(n^{3}\right) .
$$

- (Exponential sums) For any fixed $k \in \mathbb{N}, k \neq 0$, we have

$$
\frac{1}{n} \sum_{j \in[n]} \boldsymbol{w}_{n}[j] \cdot \exp \left(\frac{2 \pi i}{n} k j\right)=o(1)
$$

- (Equidistribution) For every Lipschitz function $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$

$$
\frac{1}{n} \sum_{j \in[n]} \boldsymbol{w}_{n}[j] \cdot f\left(\frac{j}{n}\right)=d \int_{\mathbb{R} / \mathbb{Z}} f+o(1)
$$

- (Cayley graph) For all $\boldsymbol{u} \in\{0,1\}^{3}$ the number of induced $\boldsymbol{u}$-walks in $\Gamma\left(\boldsymbol{w}_{n}\right)$ is

$$
d^{\|\boldsymbol{u}\|_{1}}(1-d)^{3-\|\boldsymbol{u}\|_{1}} 2 n\binom{n}{3}+o\left(n^{4}\right) .
$$

We will say that a word sequence is quasi-random if it satisfies one of (hence all) the properties of Theorem 2.
2.2. Convergent word sequences and word limits. Over the last two decades it has been recognized that quasi-randomness and limits of discrete structures are intimately related subjects. Being interesting in their own right, limit theories have also unveiled many connections between various branches of mathematics and theoretical computer science. Thus, as a natural continuation of the investigation on quasi-randomness, we study convergent word sequences and their limits, a topic which, to the best of our knowledge, has only been briefly mentioned by Szegedy [36].

The notion of convergence we consider is specified in terms of convergence of subsequence densities. Given $\boldsymbol{w} \in\{0,1\}^{n}$ and $\boldsymbol{u} \in\{0,1\}^{\ell}$, let $t(\boldsymbol{u}, \boldsymbol{w})$ be the density of occurrences of $\boldsymbol{u}$ in $\boldsymbol{w}$, i.e.,

$$
t(\boldsymbol{u}, \boldsymbol{w})=\binom{\boldsymbol{w}}{\boldsymbol{u}}\binom{n}{\ell}^{-1}
$$

[^2]Alternatively, if we define $\operatorname{sub}(\ell, \boldsymbol{w}):=\operatorname{sub}(I, \boldsymbol{w})$ for $I$ uniformly chosen among all subsets of $[n]$ of size $\ell$, then $t(\boldsymbol{u}, \boldsymbol{w})=\mathbb{P}(\operatorname{sub}(\ell, \boldsymbol{w}))=\boldsymbol{u})$.

A sequence of words $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is called convergent if for every finite word $\boldsymbol{u}$ the sequence $\left(t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)\right)_{n \rightarrow \infty}$ converges. In what follows, we will only consider sequences of words such that the length of the words tend to infinity. This, however, is not much of a restriction since convergent word sequences with bounded lengths must be constant eventually and limits considerations for these sequences are simple. ${ }^{3}$

We show that convergent word sequences have natural limit objects, which turn out to be Lebesgue measurable functions of the form $f:[0,1] \rightarrow[0,1]$. Formally, write $f^{1}=f$ and $f^{0}=1-f$ for a function $f:[0,1] \rightarrow[0,1]$ and for a word $\boldsymbol{u} \in\{0,1\}^{\ell}$ define

$$
\begin{equation*}
t(\boldsymbol{u}, f)=\ell!\int_{0 \leq x_{1}<\cdots<x_{\ell} \leq 1} \prod_{i \in[\ell]} f^{u_{i}}\left(x_{i}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell} \tag{2}
\end{equation*}
$$

We say that $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $f$ and that $f$ is the limit of $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$, if for every word $\boldsymbol{u}$ we have

$$
\lim _{n \rightarrow \infty} t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)=t(\boldsymbol{u}, f) .
$$

In particular, $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is convergent in this case. Furthermore, let $\mathcal{W}$ be the set of all Lebesgue measurable functions of the form $f:[0,1] \rightarrow[0,1]$ in which, moreover, functions are identified when they are equal almost everywhere. We show that each convergent word sequence converges to a unique $f \in \mathcal{W}$ and that, conversely, for each $f \in \mathcal{W}$ there is a word sequence which converges to $f$.

Theorem 3 (Limits of convergent word sequences).

- For each convergent word sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ there is an $f \in \mathcal{W}$ such that $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $f$. Moreover, if $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $g$ then $f$ and $g$ are equal almost everywhere.
- Conversely, for every $f \in \mathcal{W}$ there is a word sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ which converges to $f$.

Theorem 3 can be phrased in topological terms as follows. Given a word $\boldsymbol{u}$, one can think of $t(\boldsymbol{u}, \cdot)$ as a function from $\mathcal{W}$ to $[0,1]$. Then, endow $\mathcal{W}$ with the initial topology with respect to the family of maps $t(\boldsymbol{u}, \cdot)$, with $\boldsymbol{u} \in\{0,1\}^{\ell}$ and $\ell \in \mathbb{N}$, that is, the smallest topology that makes all these maps continuous. We show that this topology is actually metrizable and, moreover, compact (thereby proving Theorem 3).

The overall approach we follow is in line with what has been done for graphons [30] and permutons [24]. Nevertheless, there are important technical differences, specially concerning the (in our case, more direct) proofs of the equivalence between distinct notions of convergence which avoid compactness arguments. Instead, we rely on Bernstein polynomials and their properties as used in the (constructive) proof the Stone-Weierstrass approximation theorem.

In contrast with other technically more involved limit theories, say the ones concerning graph sequences [30] and permutation sequences [24], the simplicity of the underlying combinatorial objects we consider (words) yields concise arguments, elegant proofs, simple limit objects, and requires the introduction of far fewer concepts. Yet despite the technically comparatively simpler theory, many interesting aspects common to other structures and some specific to words appear in our investigation. As an illustration, we work out the implications for testing of the class of so-called hereditary word properties and address the question concerning finite forcibility for words, i.e., which word limits are completely determined by a finite number of prescribed subsequence densities.

[^3]2.3. Testing hereditary word properties. The concept of self-testing/correcting programs was introduced by Blum et al. [8, 9] and greatly expanded by the concept of graph property testing proposed by Goldreich, Goldwasser and Ron [20] (for an in depth coverage of the property testing paradigm, the reader is referred to the book by Goldreich [19]). An insightful connection between testable graph properties and regularity was established by Alon and Shapira [3] and further refined in $[2,4]$. It was then observed that similar and related results can be obtained via limit theories (for the case of testing graph properties, the reader is referred to [31], and for the case of (weakly) testing permutation properties, to [25]). Thus, it is not surprising that analogue results can be established for word properties. On the other hand, it is noteworthy that such consequences can be obtained very concisely and elegantly.

We next state our main result concerning testing word properties. Formally, for $\boldsymbol{u}, \boldsymbol{w} \in\{0,1\}^{n}$ let $d_{1}(\boldsymbol{w}, \boldsymbol{u})=\frac{1}{n} \sum_{i \in[n]}\left|w_{i}-u_{i}\right|$. A word property is simply a collection of words. A word property $\mathcal{P}$ is said to be testable if there is another word property $\mathcal{P}^{\prime}$ (called test property for $\mathcal{P}$ ) satisfying the following conditions:
(Completeness) For every $\boldsymbol{w} \in \mathcal{P}$ of length $n$ and every $\ell \in[n], \mathbb{P}\left(\operatorname{sub}(\ell, \boldsymbol{w}) \in \mathcal{P}^{\prime}\right) \geq \frac{2}{3}$.
(Soundness) For every $\varepsilon>0$ there is an $\ell(\varepsilon) \geq 1$ such that if $\boldsymbol{w} \in\{0,1\}^{n}$ with $d_{1}(\boldsymbol{w}, \mathcal{P})=$ $\min _{\boldsymbol{u} \in \mathcal{P} \cap\{0,1\}^{n}} d_{1}(\boldsymbol{w}, \boldsymbol{u}) \geq \varepsilon$, then $\mathbb{P}\left(\operatorname{sub}(\ell, \boldsymbol{w}) \in \mathcal{P}^{\prime}\right) \leq \frac{1}{3}$ for all $\ell(\varepsilon) \leq \ell \leq n$.
If completeness holds with probability 1 instead of $2 / 3$ one says that the property is testable with perfect completeness. Variants of the notion of testability can be considered. However, the one stated is sort of the most restrictive. On the other hand, the notion can be strengthened by replacing the $2 / 3$ in the completeness part by $1-\varepsilon$ and $1 / 3$ in the soundness part by $\varepsilon$. The notion can be weakened letting the test property $\mathcal{P}^{\prime}$ depend on $\varepsilon$.

A word property $\mathcal{P}$ is called hereditary if for each $\boldsymbol{w} \in \mathcal{P}$, every subsequence $\boldsymbol{u}$ of $\boldsymbol{w}$ also belongs to $\mathcal{P}$.

Theorem 4. Every hereditary word property is testable with perfect completeness.
Since the notion of testability given above is very restrictive (it consists in sampling uniformly a constant number of characters from the word being tested) it straightforwardly yields efficient (polynomial time) testing procedures.

Hereditary properties can be characterized as collections $\mathcal{P}_{\mathcal{F}}$ of words that do not contain as subsequence any word in $\mathcal{F}$ where $\mathcal{F}$ is a family of words ( $\mathcal{F}$ might even be infinite). For instance, given $\mathcal{P}_{1}, \ldots, \mathcal{P}_{k}$ hereditary word properties, the collection $\mathcal{P}_{\text {col }}$ of words that can be $k$-colored (i.e., each of its letters assigned a color in $[k]$ ) so that for all $c \in[k]$ the induced $c$ colored sub-word is in $\mathcal{P}_{c}$ is an example of a hereditary word property.
2.4. Finite forcibility. Finite forcibility was introduced by Lovász and Sós [29] while studying a generalization of quasi-random graphs. For an in depth investigation of finitely forcible graphons we refer to the work of Lovász and Szegedy [32]. We say that $f \in \mathcal{W}$ is finitely forcible if there is a finite list of words $\boldsymbol{u}_{1}, \ldots \boldsymbol{u}_{m}$ such that any function $h:[0,1] \rightarrow[0,1]$ which satisfies $t\left(\boldsymbol{u}_{i}, h\right)=t\left(\boldsymbol{u}_{i}, f\right)$ for all $i \in[m]$ must agree with $f$ almost everywhere. A direct consequence of Theorem 1 concerning quasi-random words is that the constant functions are finitely forcible (by words of length three). We can generalize this result as follows:

Theorem 5. Piecewise polynomial functions are finitely forcible. Specifically, if there is an interval partition $\left\{I_{1}, \ldots, I_{k}\right\}$ of $[0,1]$, polynomials $P_{1}(x), \ldots, P_{k}(x)$ of degrees $d_{1}, \ldots, d_{k}$, respectively, and $f \in$ $\mathcal{W}$ is such that $f(x)=P_{i}(x)$ for all $i \in[k]$ and $x \in I_{i}$, then there is a list of words $\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{m}$, with $m \leq(k+1)^{2 k^{2}\left(1+\max _{i} \operatorname{deg} P_{i}\right)}$ such that any function $h:[0,1] \rightarrow[0,1]$ which satisfies $t\left(\boldsymbol{u}_{i}, h\right)=t\left(\boldsymbol{u}_{i}, f\right)$ for all $i \in[m]$ must agree with $f$ almost everywhere.
2.5. Extensions. We have considered quasi-randomness for words and limits of convergent word sequences. Our results are formulated for words over the alphabet $\{0,1\}$. However, our results (except for the ones concerning testing word properties) can be easily extended to any alphabet of finite size. Also, note that a word of length $n$ can be viewed as a 1 -dimensional $\{0,1\}$ array $A:[n] \rightarrow\{0,1\}$, which labels each element of $[n]$ with 0 or 1 . Thus, a natural generalization of the 1 dimensional binary word object is a $d$-dimensional $\{0,1\}$-array, $d$-array for short, $A:[n]^{d} \rightarrow\{0,1\}$. Our approach can also be generalized to handle $d$-arrays. Indeed, the natural extension to $d$-arrays of the notion of convergence of 1-arrays yields a notion of convergent $d$-array sequence $\left(A_{n}\right)_{n \rightarrow \infty}$, where $A_{n}:[n]^{d} \rightarrow\{0,1\}$ for all $n \in \mathbb{N}$, whose limit is a Lebesgue measurable functions mapping $[0,1]^{d}$ to $[0,1]$ and where each such mapping is the limit of a convergent $d$-array sequence.
2.6. Permutons from words limits. Given $n \in \mathbb{N}$, we denote by $\mathfrak{S}_{n}$ the set of permutations of order $n$ and $\mathfrak{S}=\bigcup_{n>1} \mathfrak{S}_{n}$ the set of all finite permutations. Also, for $\sigma \in \mathfrak{S}_{n}$ and $\tau \in \mathfrak{S}_{k}$ we let $\Lambda(\tau, \sigma)$ be the number of copies of $\tau$ in $\sigma$, that is, the number of $k$-tuples $1 \leq x_{1}<\cdots<x_{k} \leq n$ such that for every $i, j \in[k]$

$$
\sigma\left(x_{i}\right) \leq \sigma\left(x_{j}\right) \quad \text { iff } \quad \tau(i) \leq \tau(j) .
$$

The density of copies of $\tau$ in $\sigma$, denoted by $t(\tau, \sigma)$, is the probability that $\sigma$ restricted to a randomly chosen $k$-tuple of $[n]$ yields a copy of $\tau$. A sequence $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ of permutations, with $\sigma_{n} \in \mathfrak{S}_{n}$ for each $n \in \mathbb{N}$, is said to be convergent if $\lim _{n \rightarrow \infty} t\left(\tau, \sigma_{n}\right)$ exists for every permutation $\tau \in \mathfrak{S}$. Hoppen et al. [24] proved that every convergent sequence of permutations converges to a suitable analytic object called permuton, which are probability measures on the Borel $\sigma$-algebra on $[0,1] \times[0,1]$ with uniform marginals, the collection of which they denote by $\mathcal{Z}$, and also extend the map $t(\tau, \cdot)$ to the whole of $\mathcal{Z}$. Then, they define a metric $d_{\square}$ on $\mathcal{Z}$ so that for all $\tau \in \mathfrak{S}$ the maps $t(\tau, \cdot)$ are continuous with respect to $d_{\square}$. They also show that $\left(\mathcal{Z}, d_{\square}\right)$ is compact and, as a consequence, establish that convergence as defined above and convergence in $d_{\square}$ are equivalent. In particular, they prove that for every convergent sequence of permutations $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ there is a permuton $\mu \in \mathcal{Z}$ such that $t\left(\tau, \sigma_{n}\right) \rightarrow t(\tau, \mu)$ for all $\tau \in \mathfrak{S}$. We give new proofs (see Proposition 29 and Theorem 30) of these two results by using a more direct approach based on Theorem 3 .
2.7. Organization. We discuss quasi-randomness in Section 3, proving Theorem 2 concerning the equivalent characterizations of quasi-random words and the second part of Theorem 1, that uniformity is implied by the counting property of length three subsequences. The first part of Theorem 1, which claims that uniformity entails the counting property of all subsequences, follows from the more general Lemma 11 from Section 4.

In Section 4 we develop the limit theory of convergent word sequences. Besides proving Theorem 3 , thus establishing the existence of word limits, among others, we also prove the uniqueness of such limits and that the initial topology of $\mathcal{W}$ is metrizable and complete.

Section 5 is dedicated to the study of testable word properties, in particular to the proof of Theorem 4 concerning testability of hereditary word properties. Finite forcibility is addressed in Section 6 where we prove Theorem 5 concerning forcibility of piecewise polynomial functions. The proof also yields an alternative proof of the second part of Theorem 1 which is moreover formulated in the language of word limits, see Remark 26. Section 7 is devoted to an alternative derivation of two key results of Hoppen et al. [24] about permutons. In Section 8, we discuss generalizations of our results to words over non-binary alphabets and extensions to higher dimensional objects, specifically multi-dimensional arrays. We conclude in Section 9 with a brief discussion of potential future research directions.

## 3. QuAsI-RANDOMNESS

In this section we give the proof of the second part of Theorem 1 and Theorem 2. We start by establishing an inverse form of the Cauchy-Schwarz inequality which is used to prove the second part of Theorem 1, that controlling the density of subsequences of length three is enough to guarantee uniformity. An alternative demonstration of the second part of Theorem 1 can be extracted from the proof of Theorem 5 (see Remark 26).

Then, after recalling some basic facts and terminology about Fourier analysis and Lipschitz functions, we proceed to prove the equivalence of the quasi-random properties listed in Theorem 2.

Lemma 6. If $\boldsymbol{g}=\left(g_{1}, \ldots, g_{n}\right), \boldsymbol{h}=\left(h_{1}, \ldots, h_{n}\right) \in \mathbb{R}^{n}$ and $\varepsilon \in(0,1)$ are such that

$$
\langle\boldsymbol{g}, \boldsymbol{h}\rangle^{2} \geq\|\boldsymbol{g}\|^{2}\|\boldsymbol{h}\|^{2}-\varepsilon n^{3}\|\boldsymbol{h}\|^{2},
$$

then all but at most $\varepsilon^{1 / 3} n$ indices $i \in[n]$ satisfy $g_{i}=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} h_{i} \pm \varepsilon^{1 / 3} n$.
Proof. Let $\boldsymbol{z}$ be the projection of $\boldsymbol{g}$ onto the plane orthogonal to $\boldsymbol{h}$, i.e., $\boldsymbol{z}=\boldsymbol{g}-\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} \boldsymbol{h}$. As $\boldsymbol{z}$ and $\boldsymbol{h}$ are orthogonal, it follows that

$$
\|\boldsymbol{g}\|^{2}=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle^{2}}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle^{2}}\|\boldsymbol{h}\|^{2}+\|\boldsymbol{z}\|^{2}=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle^{2}}{\|\boldsymbol{h}\|^{2}}+\|\boldsymbol{z}\|^{2} .
$$

The assumption then yields

$$
\begin{equation*}
\varepsilon n^{3} \geq\|\boldsymbol{z}\|^{2}=\sum_{i \in[n]}\left(g_{i}-\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} h_{i}\right)^{2} . \tag{3}
\end{equation*}
$$

Thus, the conclusion of the lemma must hold, otherwise $\|\boldsymbol{z}\|^{2}>\varepsilon^{1 / 3} n\left(\varepsilon^{1 / 3} n\right)^{2}=\varepsilon n^{3}$, contradicting (3).

Proof (of the second part of Theorem 1). Given $\varepsilon>0$ let $n>n_{0}$ be sufficiently large. By a word containing $*$ we mean the family of words obtained by replacing $*$ by 0 or 1, e.g., $\boldsymbol{u}=\left(* u_{2} u_{3}\right)$ denotes the family $\left\{\left(0 u_{2} u_{3}\right),\left(1 u_{2} u_{3}\right)\right\}$. For a word $\boldsymbol{u}$ containing $*$, let $\binom{\boldsymbol{w}}{\boldsymbol{u}}=\sum_{\boldsymbol{u}^{\prime}}\binom{\boldsymbol{w}}{\boldsymbol{u}^{\prime}}$ where the sum ranges over the family mentioned above. Given a word $\boldsymbol{w}=\left(w_{1} \ldots w_{n}\right) \in\{0,1\}^{n}$ which satisfies the assumption of the theorem we have

$$
\begin{equation*}
\binom{\boldsymbol{w}}{11 *} \leq d^{2}\binom{n}{3}+2 \varepsilon n^{3} \quad \text { and } \quad\binom{\boldsymbol{w}}{* 1 *}+\binom{\boldsymbol{w}}{1 * *} \geq 2 d\binom{n}{3}-8 \varepsilon n^{3} . \tag{4}
\end{equation*}
$$

We may also assume that $d \geq \varepsilon$, otherwise the first condition yields $\|\boldsymbol{w}\|_{1} \leq 3 \varepsilon^{1 / 3} n$ due to $\binom{\left\|\boldsymbol{w}_{1}\right\|_{1}}{3}=$ $\binom{\boldsymbol{w}}{111}$ and the result follows trivially.
Note that by assumption $\binom{\|\boldsymbol{w}\|_{1}}{3}=d^{3}\binom{n}{3} \pm \varepsilon n^{3}$, implying that $\|\boldsymbol{w}\|_{1}^{3}=d^{3} n^{3} \pm 7 \varepsilon n^{3}$, whence $\|\boldsymbol{w}\|_{1}=d n \pm 3 \varepsilon^{1 / 3} n$. Next, let $\boldsymbol{g}=\left(g_{1}, \ldots, g_{n}\right)$ where $g_{\ell}=\sum_{i \in[\ell]} w_{i}$ and let $\boldsymbol{h}=(1,2, \ldots, n)$. It is easily seen that $\boldsymbol{w}$ is $42 \varepsilon^{1 / 3}$-uniform if $g_{\ell}=d \ell \pm 21 \varepsilon^{1 / 3} n$ for every $\ell \in[n]$ and, since $g_{n}=\|\boldsymbol{w}\|_{1}=$ $d n \pm 3 \varepsilon^{1 / 3} n$, that the latter follows from

$$
\begin{equation*}
g_{\ell}=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} \ell \pm 9 \varepsilon^{1 / 3} n \quad \text { for every } \ell \in[n] . \tag{5}
\end{equation*}
$$

To show (5) note first that

$$
g_{\ell}^{2}=\left|\left\{(i, j) \in[\ell]^{2}: w_{i}=w_{j}=1\right\}\right| \leq\left|\left\{(i, j) \in[\ell-1]^{2}: w_{i}=w_{j}=1, i \neq j\right\}\right|+3(\ell-1)+1 .
$$

Hence, up to an additive error of $3(\ell-1)+1$ the quantity $g_{\ell}^{2}$ is twice the number of subsequences of $\boldsymbol{w}$ of the form ( $11 *$ ) ending at $\boldsymbol{w}$ 's $\ell$-th letter. Summing over all $\ell \in[n]$ we obtain from (4)

$$
\begin{equation*}
\|\boldsymbol{g}\|^{2}=\sum_{\ell \in[n]} g_{\ell}^{2} \leq 2\binom{\boldsymbol{w}}{11 *}+\frac{3}{2} n^{2} \leq 2 d^{2}\binom{n}{3}+5 \varepsilon n^{3} . \tag{6}
\end{equation*}
$$

Consider next, for an $\ell \in[n]$, the family $S_{\ell}$ of pairs $(i, j) \in[\ell-1], i \neq j$, such that $\left(w_{i} w_{j}\right)$ is a subsequence of $\left(w_{1}, \ldots, w_{\ell-1}\right)$ and either $w_{i}=1$ or $w_{j}=1$. Then, we have $\left|S_{\ell}\right| \leq g_{\ell} \cdot \ell$, since there are at most $g_{\ell}$ choices for $i$ and each such choice of $i$ gives rise to $(i-1)+(\ell-i-1) \leq \ell$ choices for $j$. On the other hand, $\sum_{\ell \in[n]}\left|S_{\ell}\right|$ counts all subsequences of $\boldsymbol{w}$ of the form ( $* 1 *$ ) and ( $1 * *$ ). Hence, (4) together with $\boldsymbol{h}=(1,2, \ldots, n)$ yields

$$
\langle\boldsymbol{g}, \boldsymbol{h}\rangle^{2}=\left(\sum_{\ell \in[n]} g_{\ell} \cdot \ell\right)^{2} \geq\left(\sum_{\ell \in[n]}\left|S_{\ell}\right|\right)^{2}=\left(\binom{\boldsymbol{w}}{* 1 *}+\binom{\boldsymbol{w}}{1 * *}\right)^{2} \geq 4 d^{2}\binom{n}{3}^{2}-32 \varepsilon\binom{n}{3} n^{3} .
$$

As $\|\boldsymbol{h}\|^{2}=\sum_{i \in[n]} i^{2}=\frac{1}{6} n(n+1)(2 n+1)=2\binom{n}{3}+\frac{3}{2} n^{2}-\frac{n}{2}$ from (6) we obtain

$$
\begin{aligned}
\langle\boldsymbol{g}, \boldsymbol{h}\rangle^{2}-\|\boldsymbol{g}\|^{2}\|\boldsymbol{h}\|^{2} & \geq 4 d^{2}\binom{n}{3}^{2}-32 \varepsilon\binom{n}{3} n^{3}-\left(2 d^{2}\binom{n}{3}+5 \varepsilon n^{3}\right)\|\boldsymbol{h}\|^{2} \\
& \geq 2 d^{2}\binom{n}{3}\left(\|\boldsymbol{h}\|^{2}-\frac{3}{2} n^{2}\right)-16 \varepsilon n^{3}\|\boldsymbol{h}\|^{2}-\left(2 d^{2}\binom{n}{3}+5 \varepsilon n^{3}\right)\|\boldsymbol{h}\|^{2} \\
& \geq-22 \varepsilon n^{3}\|\boldsymbol{h}\|^{2} .
\end{aligned}
$$

By Lemma 6 all but at most $(22 \varepsilon)^{1 / 3} n$ indices $i \in[n]$ satisfy $g_{i}=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} i \pm(22 \varepsilon)^{1 / 3} n$. In particular, for every $\ell \in[n]$ there is such an index $i$ with $i=\ell \pm(22 \varepsilon)^{1 / 3} n$. Thus

$$
g_{\ell}=g_{i} \pm(22 \varepsilon)^{1 / 3} n=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} i \pm 2(22 \varepsilon)^{1 / 3} n=\frac{\langle\boldsymbol{g}, \boldsymbol{h}\rangle}{\langle\boldsymbol{h}, \boldsymbol{h}\rangle} \ell \pm 3(22 \varepsilon)^{1 / 3} n
$$

which shows (5) and the second part of Theorem 1 follows.
Remark 7. The previous proof shows something stronger than what is claimed. Specifically, that instead of requiring the right count of all subsequences of length three it is sufficient to have (4), i.e., the correct upper bound for the count of $(11 *)$ and the correct lower bound for the sum of the count of $(* 1 *)$ and $(1 * *)$.

We now turn our attention to Theorem 2 and recall here some facts from Fourier analysis on the circle. Letting $\mathrm{d} x$ correspond to the Lebesgue measure on the unit circle, for $k \in \mathbb{Z}$, the Fourier transform $\widehat{f}(k)$ of a function $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$ is defined by

$$
\widehat{f}(k)=\int_{\mathbb{R} / \mathbb{Z}} f(x) e^{-2 \pi i k x} \mathrm{~d} x .
$$

Given $N \in \mathbb{N}$, the Fejér approximation of order $N$ of $f$ is defined by

$$
\sigma_{N} f(x)=\sum_{|n| \leq N}\left(1-\frac{|n|}{N+1}\right) \widehat{f}(n) e^{2 \pi i n x}
$$

Finally, we define the Lipschitz-norm of $f$ as $\|f\|_{\text {Lip }}=\|f\|_{\infty}+\sup _{x \neq y} \frac{|f(x)-f(y)|}{d(x, y)}$, where $d(x, y)=$ $\min \{1-|x-y|,|x-y|\}$ is the usual distance in $\mathbb{R} / \mathbb{Z}$.
Lemma 8 (Proposition 1.2.12 from [34]). There is a constant $C>0$ such that for any Lipschitz function $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$ and for every $M \geq 2$ one has

$$
\left\|f-\sigma_{M} f\right\|_{\infty} \leq C\|f\|_{\operatorname{Lip}} \frac{\log M}{M}
$$

Lemma 9 (Theorem 1.5.3 from [34]). There is a constant $c>0$ such that for any Lipschitz function $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$ and for every $m \neq 0$ one has

$$
|\widehat{f}(m)| \leq \frac{c\|f\|_{\text {Lip }}}{|m|}
$$

We are now in the position to prove Theorem 2.
Proof (of Theorem 2). The equivalence between the Uniformity, Counting, and Minimizer properties follow from Theorem 1. The Cayley graph and Counting properties are the same property since there is a one-to- $n$ correspondence between subsequences in $\boldsymbol{w}_{n}$ equal to $\boldsymbol{u}$ and induced $\boldsymbol{u}$-walks in $\Gamma\left(\boldsymbol{w}_{n}\right)$. To see this, simply note that $\left(v_{1}, \ldots, v_{\ell+1}\right)$ is an induced $\boldsymbol{u}$-walk in $\Gamma\left(\boldsymbol{w}_{n}\right)$ if and only if $\left(v_{1}+a, \ldots, v_{\ell+1}+a\right)$ is an induced $\boldsymbol{u}$-walk in $\Gamma\left(\boldsymbol{w}_{n}\right)$, for all $a \in[n]$ (where arithmetic over vertices is modulo $n$ ). The equivalence between the properties Uniformity and Exponential sums was shown by Cooper in [15, Theorem 2.2]. ${ }^{4}$ We next show that the properties Exponential sums and Equidistribution are equivalent. Since $f(x)=\exp (2 \pi i k x)$ integrates to 0 and has Lipschitz norm at most $2|k|$, it is clear that the Equidistribution property implies the Exponential sums property. To show the converse, let $f: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$ be given. We will show that for any $\varepsilon>0$ and for large $n$, the following holds for $d=\left\|\boldsymbol{w}_{n}\right\|_{1} / n$ :

$$
\left|\frac{1}{n} \sum_{j: \boldsymbol{w}_{n}[j]=1} f(j / n)-d \int_{\mathbb{R} / \mathbb{Z}} f\right| \leq \varepsilon\|f\|_{\text {Lip }}
$$

Let $C$ and $c$ be the absolute constants from Lemma 8 and Lemma 9, respectively. Choose $M$ large enough so that $M / \log M \geq 2 C / \varepsilon$ and $n$ large enough so that for all $|m| \leq M$ we have $\left|\sum_{j: \boldsymbol{w}_{n}[j]=1} \exp \left(\frac{2 \pi i}{n} m j\right)\right|<\frac{\varepsilon}{2 c M} n|m|$. Applying this bound we obtain

$$
\begin{aligned}
\sum_{j: \boldsymbol{w}_{n}[j]=1} \sigma_{M} f(j / n) & =\sum_{j: \boldsymbol{w}_{n}[j]=1} \sum_{m \mid \leq M}\left(1-\frac{|m|}{M+1}\right) \widehat{f}(m) \exp \left(\frac{2 \pi i}{n} m j\right) \\
& =\sum_{|m| \leq M}\left(1-\frac{|m|}{M+1}\right) \widehat{f}(m) \sum_{j: \boldsymbol{w}_{n}[j]=1} \exp \left(\frac{2 \pi i}{n} m j\right) \\
& =\widehat{f}(0) \cdot d n \pm \frac{\varepsilon}{2 c M} n \sum_{0<|m| \leq M}\left|\left(1-\frac{|m|}{M+1}\right) \widehat{f}(m)\right||m| .
\end{aligned}
$$

As $\widehat{f}(0)=\int_{\mathbb{R} / \mathbb{Z}} f$, we obtain from Lemma 9 that

$$
\left|\frac{1}{n} \sum_{j: \boldsymbol{w}_{n}[j]=1} \sigma_{M} f(j / n)-d \int_{\mathbb{R} / \mathbb{Z}} f\right| \leq \frac{\varepsilon}{2 c M} \sum_{0<|m| \leq M}\left|\left(1-\frac{|m|}{M+1}\right) \widehat{f}(m)\right||m| \leq \frac{\varepsilon}{2}\|f\|_{\text {Lip }}
$$

By Lemma 8 , triangle inequality and the choice of $M$ we conclude

$$
\begin{aligned}
\left|\frac{1}{n} \sum_{j: \boldsymbol{w}_{n}[j]=1} f(j / n)-d \int_{\mathbb{R} / \mathbb{Z}} f\right| & \leq\left|\frac{1}{n} \sum_{j: \boldsymbol{w}_{n}[j]=1} \sigma_{M} f(j / n)-d \int_{\mathbb{R} / \mathbb{Z}} f\right|+C\|f\|_{\text {Lip }} \frac{\log M}{M} \\
& \leq \frac{\varepsilon}{2}\|f\|_{\text {Lip }}+\frac{\varepsilon}{2}\|f\|_{\text {Lip }}=\varepsilon\|f\|_{\text {Lip }}
\end{aligned}
$$

This finishes the proof.

[^4]
## 4. Limits of word sequences

In this section we give the proof of Theorem 3 concerning word limits. Although the overall approach is in line with what has been done for graphons [30] and permutons [24], there are important technical differences which we will stress below. Central concepts and auxiliary results involved in the proof will be introduced along the way. The section is divided into four subsections. We start by a simple reformulation of the notion of convergent word sequences in terms of convergence of a function sequence in $\mathcal{W}$. This notion is called $t$-convergence and we in Lemma 10 show that the limit of a $t$-convergent function sequence is unique, if it exists. In the second subsection, we endow $\mathcal{W}$ with the interval-distance $d_{\square}$ and show in Lemma 11 that convergence with respect to $d_{\square}$ implies $t$-convergence. Proposition 14 from the same subsection gives a direct proof of the converse. In the third subsection, we specify a third and last notion of convergence (convergence in distribution) based on sampling of $f$-random letters for a given $f \in \mathcal{W}$. We prove in Lemma 16 that this notion of convergence is equivalent to the two previously defined, and deduce the compactness of the metric space ( $\mathcal{W}, d_{\square}$ ) in Theorem 17. In the fourth and last part, we show in Lemma 18 and Corollary 19 that every element of $f \in \mathcal{W}$ is, a.s., the limit of a convergent random word sequence.
4.1. Uniqueness and $t$-convergence. Given the nature of the limit it is convenient to first reformulate the notion of convergence in analytic terms. For a given word $\boldsymbol{w}_{n}=\left(w_{1}, \ldots, w_{n}\right)$ define the function associated to $\boldsymbol{w}_{n}$ to be the $n$-step 0-1-function $f_{\boldsymbol{w}_{n}} \in \mathcal{W}$ given by $f_{\boldsymbol{w}_{n}}(x)=w_{\lceil n x\rceil}$. It is then easy to see that $t\left(\boldsymbol{u}, f_{\boldsymbol{w}_{n}}\right)$, as defined in (2), satisfies ${ }^{5}$

$$
\begin{equation*}
t\left(\boldsymbol{u}, f_{\boldsymbol{w}_{n}}\right)=t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)+O\left(n^{-1}\right) \quad \text { for every word } \boldsymbol{u} \tag{7}
\end{equation*}
$$

Thus the following, applied to $f_{n}=f_{\boldsymbol{w}_{n}}$, yields a reformulation of convergence of $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$. Given a sequence $\left(f_{n}\right)_{n \rightarrow \infty}$ in $\mathcal{W}$ and $f \in \mathcal{W}$, we say that

$$
f_{n} \xrightarrow{t} f \quad \text { if } \quad \lim _{n \rightarrow \infty} t\left(\boldsymbol{u}, f_{n}\right)=t(\boldsymbol{u}, f) \quad \text { for all finite words } \boldsymbol{u} .
$$

The next lemma implies that the limit, if it exists, is guaranteed to be unique. The idea of the proof goes back to a remark of Král' and Pikhurko concerning permutons (see [28, Remark 6]).

Lemma 10. Let $f, g \in \mathcal{W}$. If $t(\boldsymbol{u}, f)=t(\boldsymbol{u}, g)$ for all words $\boldsymbol{u}$, then $f=g$ almost everywhere.
Proof. Given $k \in \mathbb{N}$, note that

$$
\begin{aligned}
\int_{0}^{1} f(x) x^{k} \mathrm{~d} x & =\int_{0}^{1} f(x)\left(\int_{0}^{x} \mathrm{~d} y\right)^{k} \mathrm{~d} x=\int_{y_{1}, \ldots, y_{k} \leq x} f(x) \mathrm{d} y_{1} \ldots \mathrm{~d} y_{k} \mathrm{~d} x \\
& =k!\int_{y_{1}<\cdots<y_{k}<x} f(x) \mathrm{d} y_{1} \ldots \mathrm{~d} y_{k} \mathrm{~d} x=\frac{1}{k+1} \sum_{u \in\{0,1\}^{k}} t\left(u_{1} \ldots u_{k} 1, f\right) \\
& =\frac{1}{k+1} \sum_{u \in\{0,1\}^{k}} t\left(u_{1} \ldots u_{k} 1, g\right)=\int_{0}^{1} g(x) x^{k} \mathrm{~d} x .
\end{aligned}
$$

Thus, for each polynomial $P(x) \in \mathbb{R}[x]$ we get $\int_{0}^{1} f(x) P(x) \mathrm{d} x=\int_{0}^{1} g(x) P(x) \mathrm{d} x$, and by the Stone-Weierstrass theorem $\int_{0}^{1} f(x) h(x) \mathrm{d} x=\int_{0}^{1} g(x) h(x) \mathrm{d} x$ holds for every continuous function $h:[0,1] \rightarrow \mathbb{R}$. This implies that $f=g$ almost everywhere.

[^5]4.2. Interval-metric and the metric space $\left(\mathcal{W}, d_{\square}\right)$. In view of the equivalence of uniformity and subsequence counts shown in Theorem 1, it is natural to consider the following notions of norm, distance and convergence, which are all analogues of the notions of cut-norm, cut-distance and convergence in graph limit theory. Given $h:[0,1] \rightarrow[-1,1]$ define the interval-norm
$$
\|h\|_{\square}=\sup _{I \subseteq[0,1]}\left|\int_{I} h(x) \mathrm{d} x\right|,
$$
where the supremum is taken over all intervals $I \subseteq[0,1]$. The interval-metric $d_{\square}$ is then defined by
$$
d_{\square}(f, g)=\|f-g\|_{\square} \quad \text { for every } f, g:[0,1] \rightarrow[0,1],
$$
and we write
$$
f_{n} \xrightarrow{\square} f \quad \text { if } \quad \lim _{n \rightarrow \infty} d_{\square}\left(f_{n}, f\right)=0 .
$$

The following result states that the interval-norm controls subsequence counts, in particular, $f_{n} \xrightarrow{\square} f$ implies $f_{n} \xrightarrow{t} f$. As a byproduct of the lemma, we obtain the first part of Theorem 1 concerning counting subsequences in uniform words.

Lemma 11. For $f, g \in \mathcal{W}$ and $\boldsymbol{u} \in\{0,1\}^{\ell}$ we have

$$
|t(\boldsymbol{u}, f)-t(\boldsymbol{u}, g)| \leq \ell^{2} \cdot d_{\square}(f, g) .
$$

In particular, if $\boldsymbol{w} \in\{0,1\}^{n}$ is $(d, \varepsilon)$-uniform and $n=n(\varepsilon, \ell)$ is sufficiently large, then for some $d \in[0,1]$ we have for each $\boldsymbol{u} \in\{0,1\}^{\ell}$

$$
\binom{\boldsymbol{w}}{\boldsymbol{u}}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{\ell-\|\boldsymbol{u}\|_{1}}\binom{n}{\ell} \pm 5 \varepsilon n^{\ell} .
$$

Proof. We first show that the second part follows from the first. Given a $(d, \varepsilon)$-uniform word $\boldsymbol{w} \in\{0,1\}^{n}$, let $f:[0,1] \rightarrow[0,1]$ be the function associated to $\boldsymbol{w}$. Define $g:[0,1] \rightarrow[0,1]$ to be constant equal to $d$ and recall that $g^{1}=g$ and $g^{0}=1-g$. Then, for each $\boldsymbol{u} \in\{0,1\}^{\ell}$

$$
t(\boldsymbol{u}, g)=\ell!\int_{0 \leq x_{1}<\cdots<x_{\ell} \leq 1} \prod_{i \in[\ell]} g^{u_{i}}\left(x_{i}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{\ell-\|\boldsymbol{u}\|_{1}} .
$$

Since $d_{\square}(f, g) \leq 2 \varepsilon$ due to uniformity of $\boldsymbol{w}$, for large $n$, the second part of the lemma follows from the first part and (7) as

$$
\binom{\boldsymbol{w}}{\boldsymbol{u}}=t(\boldsymbol{u}, f)\binom{n}{\ell} \pm \varepsilon n^{\ell}=t(\boldsymbol{u}, g)\binom{n}{\ell} \pm 5 \varepsilon n^{\ell}=d^{\|\boldsymbol{u}\|_{1}}(1-d)^{\ell-\|\boldsymbol{u}\|_{1}}\binom{n}{\ell} \pm 5 \varepsilon n^{\ell} .
$$

Now we turn to the proof of the first part. Let

$$
X_{j}\left(x_{1}, \ldots, x_{\ell}\right)=\left(f^{u_{j}}\left(x_{j}\right)-g^{u_{j}}\left(x_{j}\right)\right) \prod_{i=1}^{j-1} f^{u_{i}}\left(x_{i}\right) \prod_{i=j+1}^{\ell} g^{u_{i}}\left(x_{i}\right)
$$

Making use of a telescoping sum we write

$$
\begin{aligned}
|t(\boldsymbol{u}, f)-t(\boldsymbol{u}, g)| & =\ell!\left|\int_{x_{1}<\cdots<x_{\ell}}\left(\prod_{j \in[\ell]} f^{u_{j}}\left(x_{j}\right)-\prod_{j \in[\ell]} g^{u_{j}}\left(x_{j}\right)\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell}\right| \\
& =\ell!\left|\int_{x_{1}<\cdots<x_{\ell}} \sum_{j \in[\ell]} X_{j}\left(x_{1}, \ldots, x_{\ell}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell}\right| \\
& \leq \ell!\sum_{j \in[\ell]}\left|\int_{x_{1}<\cdots<x_{\ell}} X_{j}\left(x_{1}, \ldots, x_{\ell}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell}\right| .
\end{aligned}
$$

Since $\left|\int_{x_{j-1}}^{x_{j+1}}\left(f^{u_{j}}\left(x_{j}\right)-g^{u_{j}}\left(x_{j}\right)\right) \mathrm{d} x_{j}\right| \leq d_{\square}(f, g)$ and $0 \leq f, g \leq 1$, for $j \in[\ell]$ we have

$$
\left|\int_{x_{j-1}}^{x_{j+1}} X_{j}\left(x_{1}, \ldots, x_{\ell}\right) \mathrm{d} x_{j}\right| \leq d_{\square}(f, g) \prod_{i=1}^{j-1} f^{u_{i}}\left(x_{i}\right) \prod_{i=j+1}^{\ell} g^{u_{i}}\left(x_{i}\right) \leq d_{\square}(f, g) .
$$

Hence,

$$
\begin{aligned}
& \left|\int_{x_{1}<\cdots<x_{\ell}} X_{j}\left(x_{1}, \ldots, x_{\ell}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell}\right| \\
& \quad \leq d_{\square}(f, g) \int_{\substack{x_{1}<\cdots<x_{j-1} \\
\leq x_{j+1}<\cdots<x_{\ell}}} \mathrm{d} x_{1} \ldots \mathrm{~d} x_{j-1} \mathrm{~d} x_{j+1} \ldots \mathrm{~d} x_{\ell} \\
& \quad \leq \frac{1}{(\ell-1)!} d_{\square}(f, g)
\end{aligned}
$$

and the first part of the lemma follows.
Remark 12. We note that the same argument extends without change to larger size alphabets in the following sense. Given an alphabet $\Sigma=\left\{a_{1}, \ldots, a_{k}\right\}$, let $\boldsymbol{f}=\left(f^{a_{1}}, \ldots, f^{a_{k}}\right)$ and $\boldsymbol{g}=\left(g^{a_{1}}, \ldots, g^{a_{k}}\right)$ be two tuples of functions $f^{a_{i}}, g^{a_{i}}:[0,1] \rightarrow[0,1]$, for $i \in[k]$, such that

$$
f^{a_{1}}(x)+\cdots+f^{a_{k}}(x)=1 \text { and } g^{a_{1}}(x)+\cdots+g^{a_{k}}(x)=1 \text { almost everywhere. }
$$

For a word $\boldsymbol{u} \in \Sigma^{\ell}$, define the density of $\boldsymbol{u}$ in $\boldsymbol{f}$ in similar manner as in (2), namely

$$
t(\boldsymbol{u}, \boldsymbol{f})=\ell!\int_{0 \leq x_{1}<\cdots<x_{\ell} \leq 1} \prod_{i \in[k]} f^{u_{i}}\left(x_{i}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell} .
$$

Then, the proof from above yields

$$
|t(\boldsymbol{u}, \boldsymbol{f})-t(\boldsymbol{u}, \boldsymbol{g})| \leq \ell^{2} \cdot \max _{i \in[k]} d_{\square}\left(f^{a_{i}}, g^{a_{i}}\right) .
$$

Note that Lemma 11 implies that if $f_{n} \xrightarrow{\square} f$, then $f_{n} \xrightarrow{t} f$. Our goal now is to show that the converse also holds. Let $\left(f_{n}\right)_{n \rightarrow \infty}$ be a sequence such that $f_{n} \xrightarrow{t} f$. Following the proof of Lemma 10, we will use that for any polynomial $P(x) \in \mathbb{R}[x]$ we can write $\int_{0}^{1}\left(f_{n}(x)-f(x)\right) P(x)$ as a linear combination of subsequence densities. By approximating $\mathbf{1}_{[a, b]}(x)$ by a polynomial $P_{a, b}(x) \in \mathbb{R}[x]$, with error term uniform in $0 \leq a<b \leq 1$, we may show that $\int_{0}^{1}\left(f_{n}(x)-f(x)\right) \mathbf{1}_{[a, b]}(x)$ can be approximated by $\int_{0}^{1}\left(f_{n}(x)-f(x)\right) P_{a, b}(x)$, thence by a linear combination of subsequence densities, implying our claim. In order to prove this approximation result, we introduce next the class of Bernstein polynomials.

Even though here we only need to approximate functions on $[0,1]$, we will consider the general case of functions on $[0,1]^{k}$ since it will later be useful in our study of higher dimensional combinatorial structures. For $k, t \in \mathbb{N} \backslash\{0\}$, let $\boldsymbol{i}=\left(i_{1}, \ldots i_{k}\right) \in[t]^{k}$. Given a function $J:[0,1]^{k} \rightarrow \mathbb{R}$, define its Bernstein polynomial evaluated at $\boldsymbol{x}=\left(x_{1}, \ldots, x_{k}\right) \in[0,1]^{k}$ by

$$
B_{t, J}(\boldsymbol{x})=\sum_{0 \leq i_{1}, \ldots, i_{k} \leq t} J\left(\frac{i}{t}\right) \prod_{j \in[k]}\binom{t}{i_{j}} x_{j}^{i_{j}}\left(1-x_{j}\right)^{t-i_{j}} .
$$

We can now formally state the approximation of indicator functions we use.
Lemma 13. For $\boldsymbol{a}=\left(a_{1}, \ldots, a_{k}\right) \in[0,1]^{k}$ let $J=\mathbf{1}_{\left[0, a_{1}\right] \times \cdots \times\left[0, a_{k}\right]}$. If $r \in \mathbb{N}$ and $\boldsymbol{x} \in[0,1]^{k}$ satisfy $\left|x_{i}-a_{i}\right|>r^{-1 / 4}$ for all $i \in[k]$, then $\left|B_{r, J}(\boldsymbol{x})-J(\boldsymbol{x})\right| \leq k r^{-1 / 2}$.

Proof. Let $B=B_{r, J}$ and let $\boldsymbol{X}=\left(X_{1}, \ldots, X_{r}\right)$ be such that $X_{1}, \ldots, X_{r}$ are independent random variables where $X_{j}$ follows a binomial distribution with parameters $r$ and $x_{j}$, so $\mathbb{P}\left(X_{j}=i\right)=$ $\binom{r}{i} x_{j}^{i}\left(1-x_{j}\right)^{r-i}$. Note that $B_{r, J}(\boldsymbol{x})=\mathbb{E}\left(J\left(\frac{1}{r} \boldsymbol{X}\right)\right)$. Let $L=\left\{\boldsymbol{i}:\left\|\boldsymbol{x}-\frac{\boldsymbol{i}}{r}\right\|_{\infty}>r^{-1 / 4}\right\} \subseteq(\{0\} \cup[r])^{k}$. As $\left|x_{j}-a_{j}\right|>r^{-1 / 4}$ for all $j \in[k]$, for each $\boldsymbol{i} \notin L$ we have that $J\left(\frac{\boldsymbol{i}}{r}\right)=J(\boldsymbol{x})$ and thus

$$
\mathbb{E}\left(\left|J\left(\frac{1}{r} \boldsymbol{X}\right)-J(\boldsymbol{x})\right| \mathbf{1}_{\bar{L}}(\boldsymbol{X})\right)=0
$$

Due to $\left|J\left(\frac{\boldsymbol{i}}{r}\right)-J(\boldsymbol{x})\right| \leq 1$ we have

$$
\begin{equation*}
\mathbb{E}\left(\left|J\left(\frac{1}{r} \boldsymbol{X}\right)-J(\boldsymbol{x})\right|\right)=\mathbb{E}\left(\left|J\left(\frac{1}{r} \boldsymbol{X}\right)-J(\boldsymbol{x})\right| \mathbf{1}_{L}(\boldsymbol{X})\right) \leq \mathbb{P}(\boldsymbol{X} \in L) \leq \sum_{\ell \in[k]} \mathbb{P}\left(\left|\frac{1}{r} X_{\ell}-x_{\ell}\right|>r^{-1 / 4}\right) \tag{8}
\end{equation*}
$$

Since $\mathbb{E}\left(X_{\ell}\right)=r x_{\ell}$, by Chebyshev's inequality,

$$
\mathbb{P}\left(\left|\frac{1}{r} X_{\ell}-x_{\ell}\right|>r^{-1 / 4}\right)=\mathbb{P}\left(\left|X_{\ell}-\mathbb{E}\left(X_{\ell}\right)\right|>r^{3 / 4}\right) \leq \frac{1}{r^{3 / 2}} r x_{\ell}\left(1-x_{\ell}\right) \leq r^{-1 / 2}
$$

Since the bound holds for every $\ell \in[k]$, the RHS of (8) is at most $k r^{-1 / 2}$, as required.
Given two functions $f, g \in \mathcal{W}$, we have the inequality

$$
\begin{equation*}
\sup _{b \in[0,1]}\left|\int_{0}^{b} f(x) \mathrm{d} x-\int_{0}^{b} g(x) \mathrm{d} x\right| \leq d_{\square}(f, g) \leq 2 \sup _{b \in[0,1]}\left|\int_{0}^{b} f(x) \mathrm{d} x-\int_{0}^{b} g(x) \mathrm{d} x\right| \tag{9}
\end{equation*}
$$

The first inequality in (9) is direct from the definition of $d_{\square}$, and the second inequality follows from the identity $\int_{0}^{b}(f(x)-g(x))=\int_{a}^{b}(f(x)-g(x))+\int_{0}^{a}(f(x)-g(x))$.

The following proposition states that $t$-convergence implies convergence with respect to $d_{\square}$, and thus, together with Lemma 11, establishes that both notions of convergence are equivalent.

Proposition 14. If $\left(f_{n}\right)_{n \rightarrow \infty}$ is a sequence in $\mathcal{W}$ which ist-convergent, then it is a Cauchy sequence with respect to $d_{\square}$. Moreover, if $f_{n} \xrightarrow{t}$ f for some $f \in \mathcal{W}$, then $f_{n} \xrightarrow{\square} f$.

Proof. Given $\varepsilon>0$, let $r=\left\lceil(20 / \varepsilon)^{4}\right\rceil$. For $\delta=\varepsilon / 2^{3 r+2}$, let $n_{0}$ be sufficiently large so that for all $n, m \geq n_{0}$ we have

$$
\begin{equation*}
\left|t\left(\boldsymbol{u}, f_{n}\right)-t\left(\boldsymbol{u}, f_{m}\right)\right| \leq \delta \quad \text { for all } \boldsymbol{u} \in \bigcup_{s \in[r]}\{0,1\}^{s} \tag{10}
\end{equation*}
$$

Recall from the proof of Lemma 10, that for each $k \in \mathbb{N}$ we have

$$
\int_{0}^{1} f(x) x^{k} \mathrm{~d} x=\frac{1}{k+1} \sum_{\boldsymbol{u} \in\{0,1\}^{k}} t\left(u_{1} \ldots u_{k} 1, f\right)
$$

Thus, for $k \leq r$ and $h=f_{n}-f_{m}$, we have

$$
\left|\int_{0}^{1} h(x) x^{k} \mathrm{~d} x\right|=\frac{1}{k+1}\left|\sum_{\boldsymbol{u} \in\{0,1\}^{k}}\left(t\left(u_{1} \ldots u_{k} 1, f_{n}\right)-t\left(u_{1}, \ldots, u_{k} 1, f_{m}\right)\right)\right| \leq 2^{k} \delta
$$

For $a \in[0,1]$, let $J_{a}=\mathbf{1}_{[0, a]}$ and $j_{a}$ be the largest integer such that $\frac{j_{a}}{r} \leq a$. Then,

$$
\left|\int_{0}^{1} h(x) B_{r, J_{a}}(x) \mathrm{d} x\right| \leq \sum_{i=0}^{j_{a}}\binom{r}{i}\left|\int_{0}^{1} h(x) x^{i}(1-x)^{r-i} \mathrm{~d} x\right| \leq 2^{3 r} \delta
$$

Thus, since $|h| \leq 1$ and $\left|\mathbf{1}_{[0, a]}(x)-B_{r, J_{a}}\right| \leq 2$, by Lemma 13, we have

$$
\begin{aligned}
\left|\int_{0}^{1} h(x) \mathbf{1}_{[0, a]}(x) \mathrm{d} x\right| & \leq\left|\int_{0}^{1} h(x) B_{r, J_{a}}(x) \mathrm{d} x\right|+\left|\int_{0}^{1} h(x)\left(\mathbf{1}_{[0, a]}(x)-B_{r, J_{a}}(x)\right) \mathrm{d} x\right| \\
& \leq 2^{3 r} \delta+\left(4 r^{-1 / 4}+r^{-1 / 2}\right)
\end{aligned}
$$

The desired conclusion follows from (9) and by our choice of $t$ and $\delta$ observing that

$$
d_{\square}\left(f_{n}, f_{m}\right) \leq 2 \sup _{a \in[0,1]}\left|\int_{0}^{1} h(x) \mathbf{1}_{[0, a]}(x) \mathrm{d} x\right| \leq 2^{3 r+1} \delta+10 r^{-1 / 4} \leq \varepsilon .
$$

The second part follows by replacing $f_{m}$ by $f$ in (10), taking $h=f_{n}-f$, and repeating the above argument.

The compactness of the metric space ( $\mathcal{W}, d_{\square}$ ) can be easily established via the Banach-Alaoglu theorem in $L^{\infty}([0,1])$. Instead, we follow a different strategy laid out in the following section. This strategy has the advantage that it emphasizes the probabilistic point of view of convergence. It is based on a new model of random words that naturally arises from the theory and that may be of independent interest.

We note that one can also establish the compactness of ( $\mathcal{W}, d_{\square}$ ) by using the regularity lemma for words [5]. This approach has the advantage of being more constructive and for the sake of completeness we include it in the Appendix A.
4.3. Random letters from limits and compactness of $\left(\mathcal{W}, d_{\square}\right)$. Consider the Euclidean metric on $[0,1]$ and the discrete metric on $\{0,1\}$. Let $\Omega=[0,1] \times\{0,1\}$ be equipped with the $L_{\infty^{-}}$ distance, which thus assigns to a pair of points in $\Omega$ the standard distance of their first coordinates if the second coordinates agree and one otherwise. Let $\mathcal{B}$ denote the Borel $\sigma$-algebra of $\Omega$, let $f:[0,1] \rightarrow[0,1]$ be a Borel measurable function and recall that $f^{1}=f$ and $f^{0}=1-f$. Also, denote by $\mathrm{U}([0,1])$ and $\mathrm{B}(p)$ the uniform distribution over $[0,1]$ and the Bernoulli distribution with expected value $p \in[0,1]$, respectively. We say that

$$
(X, Y) \in \Omega \text { is an } f \text {-random letter } \quad \text { if } \quad X \sim \mathrm{U}([0,1]) \text { and } Y \sim \mathrm{~B}(f(X))
$$

Observe that an $f$-random letter $(X, Y)$ is a pair of mixed ${ }^{6} \operatorname{random}^{\text {variables where } Y \text { is distributed }}$ according to the conditional pmf

$$
f_{Y \mid X}(\varepsilon \mid x)=\mathbb{P}(Y=\varepsilon \mid X=x)=f^{\varepsilon}(x) \quad \varepsilon \in\{0,1\} \text { and } x \in[0,1] .
$$

Then, $(X, Y)$ has the mixed joint probability distribution

$$
\begin{equation*}
F(x, \varepsilon)=\mathbb{P}(X \leq x, Y=\varepsilon)=\int_{0}^{x} f^{\varepsilon}(t) \mathrm{d} t \tag{11}
\end{equation*}
$$

and thus the mixed joint pmf $f_{X, Y}(x, \varepsilon)=f^{\varepsilon}(x)$. The marginal probability distribution of $Y$ is

$$
\mathbb{P}(Y=\varepsilon)=F(1, \varepsilon)=\int_{0}^{1} f^{\varepsilon}(t) \mathrm{d} t, \quad \varepsilon \in\{0,1\}
$$

hence $Y \sim \mathrm{~B}(p)$ with $p=\int_{0}^{1} f(t) \mathrm{d} t$. Furthermore, conditioned on $Y$ the variable $X$ is distributed according to the conditional pmf $f_{X \mid Y}$ which satisfies

$$
\begin{equation*}
f_{X \mid Y}(x \mid \varepsilon) \cdot \mathbb{P}(Y=\varepsilon)=f_{X, Y}(x, \varepsilon)=f^{\varepsilon}(x) . \tag{12}
\end{equation*}
$$

One may therefore equivalently sample ( $X, Y$ ) by first choosing $Y \sim \mathrm{~B}(p)$ with $p=\int_{0}^{1} f(t) \mathrm{d} t$, and then choose $X$ (conditional on $Y$ ) according to the conditional pmf $f_{X \mid Y}$ satisfying (12). By means

[^6]of this sampling procedure a sequence $\left(f_{n}\right)_{n \rightarrow \infty}$ gives rise to a sequence $\left(\left(X_{n}, Y_{n}\right)\right)_{n \rightarrow \infty}$, where each $\left(X_{n}, Y_{n}\right)$ is the $f_{n}$-random letter, and the corresponding sequence of probability distributions $\left(\mathbb{P}_{n}\right)_{n \rightarrow \infty}$ is as defined in (11). As usual for general metric spaces (see, e.g., [7, Chapter 5]), we say that $\left(\left(X_{n}, Y_{n}\right)\right)_{n \rightarrow \infty}$ converges to $(X, Y)$ in distribution if $\left(\mathbb{P}_{n}\right)_{n \rightarrow \infty}$ weakly converges to $\mathbb{P}$, i.e., if for all bounded continuous functions $h: \Omega \rightarrow \mathbb{R}$ we have
\[

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\Omega} h \mathbb{d}_{n}=\int_{\Omega} h \mathrm{~d} \mathbb{P} \tag{13}
\end{equation*}
$$

\]

From this definition we immediately have the following.
Fact 15. If $\left(\left(X_{n}, Y_{n}\right)\right)_{n \rightarrow \infty}$ converges to $(X, Y)$ in distribution, then $\left(X_{n}\right)_{n \rightarrow \infty}$ (resp., $\left.\left(Y_{n}\right)_{n \rightarrow \infty}\right)$ converges to $X$ (resp. $Y$ ) in distribution.

We now write

$$
f_{n} \xrightarrow{\mathrm{~d}} f \quad \text { if } \quad\left(\left(X_{n}, Y_{n}\right)\right)_{n \rightarrow \infty} \text { converges to }(X, Y) \text { in distribution. }
$$

The next lemma shows the equivalences of convergence in $d_{\square}$ and convergence in distribution.
Lemma 16. Let $f_{1}, f_{2}, \ldots$ and $f$ be functions in $\mathcal{W}$. Then, $f_{n} \xrightarrow{\square} f$ if and only if $f_{n} \xrightarrow{\mathrm{~d}} f$.
Proof. Let $\left(X_{n}, Y_{n}\right)$ be an $f_{n}$-random letter (resp. $(X, Y)$ be an $f$-random letter) with the associated probability measure $\mathbb{P}_{n}$ and distribution $F_{n}$ (resp. $\mathbb{P}$ and $F$ ). Recall that $\Omega=[0,1] \times\{0,1\}$, and for $(x, \varepsilon) \in \Omega$ let $F_{n}(x, \varepsilon)=\int_{0}^{x} f_{n}^{\varepsilon}(t) \mathrm{d} t$ and $F(x, \varepsilon)=\int_{0}^{x} f^{\varepsilon}(t) \mathrm{d} t$. Since,

$$
\left\|F_{n}-F\right\|_{\infty}=\sup _{(x, \varepsilon) \in \Omega}\left|F_{n}(x, \varepsilon)-F(x, \varepsilon)\right|
$$

it follows that

$$
\left\|F_{n}-F\right\|_{\infty}=\sup _{x \in[0,1]}\left|F_{n}(x, 0)-F(x, 0)\right|=\sup _{x \in[0,1]}\left|F_{n}(x, 1)-F(x, 1)\right|=\sup _{x \in[0,1]}\left|\int_{0}^{x}\left(f_{n}-f\right)(t) \mathrm{d} t\right| .
$$

Now observe that

$$
\begin{equation*}
\left\|F_{n}-F\right\|_{\infty} \leq d_{\square}\left(f_{n}, f\right) \leq 2\left\|F_{n}-F\right\|_{\infty}, \tag{14}
\end{equation*}
$$

where the first inequality is obvious and the second one follows because for all $\varepsilon \in\{0,1\}$ and $0 \leq a<b \leq 1$ it holds that $\int_{[a, b]}\left(f_{n}-f\right)(t) \mathrm{d} t=\left(F_{n}-F\right)(b, \varepsilon)-\left(F_{n}-F\right)(a, \varepsilon)$. Thus, $f_{n} \xrightarrow{\square} f$ if and only if $\lim _{n \rightarrow \infty}\left\|F_{n}-F\right\|_{\infty}=0$ which we claim holds if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} F_{n}(x, \varepsilon)=F(x, \varepsilon) \quad \text { for all } \varepsilon \in\{0,1\} \text { and } x \in[0,1] . \tag{15}
\end{equation*}
$$

Indeed, it is clear that $\lim _{n \rightarrow \infty}\left\|F_{n}-F\right\|_{\infty}=0$ implies (15). For the converse note that for each $\varepsilon \in\{0,1\}$ we have $\left|f^{\varepsilon}\right| \leq 1$, thus for every $x, y \in[0,1]$

$$
\begin{equation*}
|F(x, \varepsilon)-F(y, \varepsilon)|=\left|\int_{0}^{x} f^{\varepsilon}(t) \mathrm{d} t-\int_{0}^{y} f^{\varepsilon}(t) \mathrm{d} t\right| \leq|x-y| . \tag{16}
\end{equation*}
$$

Given an integer $k>0$, by (15), there is an $n_{k}$ such that $\max _{i \in[k]}\left|F_{n}\left(\frac{i}{k}, \varepsilon\right)-F\left(\frac{i}{k}, \varepsilon\right)\right|<\frac{1}{k}$ for each $n>n_{k}$. For an $x \in[0,1]$ let $i_{x} \in[k]$ be such that $\left|x-\frac{i_{x}}{k}\right| \leq \frac{1}{k}$. Then, by triangle inequality and (16), for any $x \in[0,1]$

$$
\left|F_{n}(x, \varepsilon)-F(x, \varepsilon)\right| \leq\left|F_{n}\left(\frac{i_{x}}{k}, \varepsilon\right)-F\left(\frac{i_{x}}{k}, \varepsilon\right)\right|+2\left|x-\frac{i_{x}}{k}\right| \leq \frac{3}{k}
$$

which thus establishes that (15) implies $\lim _{n \rightarrow \infty}\left\|F_{n}-F\right\|_{\infty}=0$.
To prove the lemma we now show that (15) holds if and only if $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots$ converges to $(X, Y)$ in distribution, i.e., $\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots$ weakly converges to $\mathbb{P}$ as defined in (13). For an $h: \Omega \rightarrow \mathbb{R}$ and an $\varepsilon \in\{0,1\}$ define the projection $h_{\varepsilon}:[0,1] \rightarrow \mathbb{R}$ via $h_{\varepsilon}(x)=h(x, \varepsilon)$. Thus, $F_{\varepsilon}(x)=F(x, \varepsilon)$,
$F_{n, \varepsilon}(x)=F_{n}(x, \varepsilon)$ and we also define $\mathbb{P}_{\varepsilon}$ via $\mathbb{P}_{\varepsilon}(A)=\mathbb{P}(A \times\{\varepsilon\})$ for any $A \in \mathcal{B}([0,1])$ and in the same manner define $\mathbb{P}_{n, \varepsilon}$.

For a metric space ( $M, d$ ), we denote by $C(M)$ the set of continuous functions $h: M \rightarrow \mathbb{R}$. As $\Omega$ is equipped with $L_{\infty}$-distance $d_{\Omega}$ we have $d_{\Omega}((x, \alpha),(y, \beta))=\delta<1$ if an only if $\alpha=\beta$ and $|x-y|=\delta$. Hence, $h \in C(\Omega)$ if and only if $h_{0}, h_{1} \in C([0,1])$. Moreover, by verifying the following for step functions $h$ and then extending to all $h \in C(\Omega)$ by a standard limiting argument we have

$$
\int_{\Omega} h \mathrm{~d} \mathbb{P}_{n}=\sum_{\varepsilon} \int_{[0,1]} h_{\varepsilon} \mathrm{d} \mathbb{P}_{n, \varepsilon} \quad \text { and } \quad \int_{\Omega} h \mathrm{~d} \mathbb{P}=\sum_{\varepsilon} \int_{[0,1]} h_{\varepsilon} \mathrm{d} \mathbb{P}_{\varepsilon}
$$

In particular,

$$
\lim _{n \rightarrow \infty} \int_{\Omega} h \mathrm{~d}_{n}=\int_{\Omega} h \mathrm{~d} \mathbb{P} \quad \text { for all } h \in C(\Omega)
$$

holds if and only if

$$
\lim _{n \rightarrow \infty} \int_{\Omega} h \mathrm{~d}_{n, \varepsilon}=\int_{\Omega} h \mathbb{d}_{\varepsilon} \quad \text { for all } \varepsilon \in\{0,1\}, \text { and all } h \in C([0,1])
$$

In other words, $\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots$ converges weakly to $\mathbb{P}$ if and only if $\mathbb{P}_{1, \varepsilon}, \mathbb{P}_{2, \varepsilon}, \ldots$ converges weakly to $\mathbb{P}_{\varepsilon}$ for all $\varepsilon \in\{0,1\}$. As the underlying space is $[0,1]$ it is well known that weak convergence of $\mathbb{P}_{1, \varepsilon}, \mathbb{P}_{2, \varepsilon}, \ldots$ to $\mathbb{P}_{\varepsilon}$ is equivalent to the fact that $\lim _{n \rightarrow \infty} F_{n, \varepsilon}(x)=F_{\varepsilon}(x)$ holds for all $x$ where $F_{\varepsilon}(x)$ is continuous. As seen from (16), $F_{\varepsilon}$ is continuous on the entirety of $[0,1]$. This thus shows that weak convergence of $\mathbb{P}_{1}, \mathbb{P}_{2}, \ldots$ to $\mathbb{P}$ is equivalent to (15) and the lemma follows.

The compactness of ( $\mathcal{W}, d_{\square}$ ) now follows from Lemma 16 and classical results from measure theory, namely Prokhorov's theorem concerning the existence of weak convergent subsequences for a given sequence of measures over compact measurable spaces and Radon-Nikodym theorem concerning the existence of derivatives of measures which are absolutely continuous with respect to the Lebesgue measure.

Theorem 17. The metric space $\left(\mathcal{W}, d_{\square}\right)$ is compact.
Proof. Given a sequence $\left(f_{n}\right)_{n \rightarrow \infty}$ of functions $f_{n} \in \mathcal{W}$. Consider the sequence of $f_{n}$-random letters $\left(\left(X_{n}, Y_{n}\right)\right)_{n \rightarrow \infty}$ with the corresponding sequence of probabilities $\left(\mathbb{P}_{n}\right)_{n \rightarrow \infty}$ on $(\Omega, \mathcal{B})$ defined by (11). As $\Omega$ is compact we conclude from Prokhorov's theorem (see Chapter 1, Section 5 of [7]) that there is a pair of random variables $(X, Y)$ with joint probability measure $\mathbb{P}$ such that $\left(\mathbb{P}_{n}\right)_{n \rightarrow \infty}$ contains a subsequence $\left(\mathbb{P}_{n_{i}}\right)_{i \rightarrow \infty}$ which weakly converges to $\mathbb{P}$. By Fact 15 we know that $X \sim \mathrm{U}[0,1]$ while $Y$ is Bernoulli. Denoting by $\lambda$ the Lebesgue measure, the restriction of $\mathbb{P}$ to $Y=1$ yields a measure $\mu$ which satisfies $\mu(A)=\mathbb{P}(X \in A, Y=1) \leq \mathbb{P}(X \in A)=\lambda(A)$ for every measurable set $A$. In particular, $\mu$ is absolutely continuous with respect to the Lebesgue measure $\lambda$ (i.e., $\mu(A)=0$ whenever $\lambda(A)=0$ ) and the Radon-Nikodym theorem guarantees the existence of a function $f$ such that

$$
\mu([0, x])=\int_{0}^{x} f(t) \mathrm{d} t=\mathbb{P}(X \leq x, Y=1)
$$

and thus

$$
\mathbb{P}(X \leq x, Y=0)=x-\mu([0, x])=\int_{0}^{x}(1-f(t)) \mathrm{d} t .
$$

In other words, $f_{X, Y}(x, \varepsilon)=f^{\varepsilon}(x)$ is the pmf of $(X, Y)$ and we thus have $f_{n_{i}} \xrightarrow{\text { d }} f$. Lemma 16 guarantees that $f_{n_{i}} \xrightarrow{\square} f$ as well. Lastly, it is easily seen that $f(x) \in[0,1]$ almost everywhere and we may therefore assume that $f \in \mathcal{W}$.

The last theorem thus establishes the existence of the limit object claimed in the first part of Theorem 3.
4.4. Random words from limits. To establish the second part of Theorem 3 we consider, for any $f \in \mathcal{W}$, a suitable sequence of random words arising from $f$ and show that it converges to $f$ almost surely. For $f \in \mathcal{W}$ and $\boldsymbol{x}=\left(x_{1}, \ldots, x_{\ell}\right) \in[0,1]^{\ell}$ such that $x_{1}<x_{2}<\ldots<x_{\ell}$ let $\boldsymbol{w}=\operatorname{sub}(\boldsymbol{x}, f)$ be the word obtained by choosing $w_{i}=1$ with probability $f\left(x_{i}\right)$ and $w_{i}=0$ with probability $1-f\left(x_{i}\right)$ (making independent decisions for different $i$ 's). Consider now $n$ independent $f$-random letters $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$. After reordering the first coordinate, i.e., taking a permutation $\sigma:[n] \rightarrow[n]$ so that $X_{\sigma(1)}<\cdots<X_{\sigma(n)}$, the $f$-random word $\operatorname{sub}(n, f)$ is given by

$$
\operatorname{sub}(n, f)=\left(Y_{\sigma(1)}, \ldots, Y_{\sigma(n)}\right) .
$$

Lemma 18. Let $f \in \mathcal{W}$ and let $f_{n}$ be the function associated to the $f$-random word $\operatorname{sub}(n, f)$. For all $n \in \mathbb{N}$ and $a \geq \frac{1}{n}$ we have

$$
\mathbb{P}\left(d_{\square}\left(f_{n}, f\right) \geq 8 a\right) \leq 4 n e^{-2 a^{2} n}
$$

Proof. For $x \in[0,1]$ let

$$
W_{n}(x)=\int_{0}^{x} f_{n}(t) \mathrm{d} t \quad \text { and } \quad W(x)=\int_{0}^{x} f(t) \mathrm{d} t .
$$

Recall that by (9) we have $d_{\square}\left(f_{n}, f\right) \leq 2\left\|W_{n}-W\right\|_{\infty}$. Therefore, we only need to bound $\mathbb{P}\left(\| W_{n}-\right.$ $\left.W \|_{\infty} \geq 5 a\right)$.

Given $i \in[n]$ and $x \in\left[\frac{i-1}{n}, \frac{i}{n}\right)$, since $\left|f_{n}\right|,|f| \leq 1$, we have that $\left|W_{n}(x)-W(x)\right| \leq \left\lvert\, W_{n}\left(\frac{i}{n}\right)-\right.$ $\left.W\left(\frac{i}{n}\right) \right\rvert\,+\frac{2}{n}$, and thus

$$
\left\|W_{n}-W\right\|_{\infty} \leq \frac{2}{n}+\max _{i \in[n]}\left|W_{n}\left(\frac{i}{n}\right)-W\left(\frac{i}{n}\right)\right| .
$$

For $i \in[n]$, we next bound the probability that $\left|W_{n}\left(\frac{i}{n}\right)-W\left(\frac{i}{n}\right)\right|$ is at least $2 a$. Consider the sequence $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ of $f$-random letters that define $\operatorname{sub}(n, f)$, and suppose that $X_{\sigma(1)}<\cdots<$ $X_{\sigma(n)}$ for some permutation $\sigma:[n] \rightarrow[n]$. Since $f_{n}$ is the function associated to $\operatorname{sub}(n, f)$ we have

$$
W_{n}\left(\frac{i}{n}\right)=\frac{1}{n} \sum_{j=1}^{i} Y_{\sigma(j)}
$$

and thus, letting $Z_{i}=\frac{1}{n} \sum_{j=1}^{n} \mathbf{1}\left\{X_{j} \leq \frac{i}{n}\right\}$ and $S_{i}=\frac{1}{n} \sum_{j=1}^{n} Y_{j} \mathbf{1}\left\{X_{j} \leq \frac{i}{n}\right\}=\frac{1}{n} \sum_{j=1}^{n Z_{i}} Y_{\sigma(j)}$, we get

$$
\begin{equation*}
\left|W_{n}\left(\frac{i}{n}\right)-S_{i}\right| \leq\left|\frac{i}{n}-Z_{i}\right| . \tag{17}
\end{equation*}
$$

On the other hand, for every $j \in[n]$ we have that

$$
\mathbb{E}\left(Y_{j} \mathbf{1}\left\{X_{j} \leq \frac{i}{n}\right\}\right)=\int_{0}^{\frac{i}{n}} f(t) \mathrm{d} t=W\left(\frac{i}{n}\right)
$$

so $\mathbb{E}\left(S_{i}\right)=W\left(\frac{i}{n}\right)$. Using Chernoff's bound (see Theorem 2.8 and Remark 2.5 from [26]) we get

$$
\mathbb{P}\left(\left|Z_{i}-\frac{i}{n}\right| \geq a\right) \leq 2 e^{-2 a^{2} n} \quad \text { and } \quad \mathbb{P}\left(\left|S_{i}-W\left(\frac{i}{n}\right)\right| \geq a\right) \leq 2 e^{-2 a^{2} n}
$$

which together with (17) and the fact that $a \geq \frac{1}{n}$, implies that

$$
\left.\left.\mathbb{P}\left(\left|W_{n}\left(\frac{i}{n}\right)-W\left(\frac{i}{n}\right)\right| \geq 2 a\right) \leq \mathbb{P}\left(\left\lvert\, S_{i}-W\left(\frac{i}{n}\right)\right.\right) \right\rvert\, \geq a\right)+\mathbb{P}\left(\left|Z_{i}-\frac{i}{n}\right| \geq a\right) \leq 4 e^{-2 a^{2} n} .
$$

Putting everything together we conclude that

$$
\mathbb{P}\left(d_{\square}\left(f_{n}, f\right) \geq 8 a\right) \leq \mathbb{P}\left(\left\|W_{n}-W\right\|_{\infty} \geq 4 a\right) \leq \sum_{i=1}^{n} \mathbb{P}\left(\left|W_{n}\left(\frac{i}{n}\right)-W\left(\frac{i}{n}\right)\right| \geq 2 a\right) \leq 4 n e^{-2 a^{2} n}
$$

As an immediate consequence we obtain the following.
Corollary 19. For all $f \in \mathcal{W}$, the sequence of $f$-random words $(\operatorname{sub}(n, f))_{n \rightarrow \infty}$ converges to $f$ a.s. Proof. For $n \in \mathbb{N}$ let $f_{n}=\operatorname{sub}(n, f)$. Taking $a=n^{-\frac{1}{4}}$ in Lemma 18 and using the Borel-Cantelli lemma, it follows that $f_{n} \xrightarrow{\square} f$ almost surely. Then, by Lemma 11 we conclude that $f_{n} \xrightarrow{t} f$ almost surely, and therefore, by $(7),(\operatorname{sub}(n, f))_{n \rightarrow \infty}$ converges to $f$ almost surely.

Equipped with the results from above we now establish the second main result of this section.
Proof (of Theorem 3). The uniqueness of the limit, if it exists, follows from Lemma 10. The second part of the theorem concerning the existence of word sequences converging to any given $f \in \mathcal{W}$ follows from Corollary 19.

It is thus left to establish the existence of a limit. Consider a convergent sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words and let $\left(f_{n}\right)_{n \rightarrow \infty}$ be the sequence of associated functions $f_{n}=f_{\boldsymbol{w}_{n}} \in \mathcal{W}$. Because of (7) the sequence $\left(f_{n}\right)_{n \rightarrow \infty}$ is $t$-convergent and thus, by Proposition $14,\left(f_{n}\right)_{n \rightarrow \infty}$ is a Cauchy sequence with respect to $d_{\square}$. The compactness of $\left(\mathcal{W}, d_{\square}\right)$, as guaranteed by Theorem 17 , implies that there exists $f \in \mathcal{W}$ such that $d_{\square}\left(f_{n}, f\right) \rightarrow 0$. Finally, because of Lemma 11 we have that $f_{n} \xrightarrow{t} f$ and therefore $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $f$.

Concluding this section, and in preparation for the next one, we show that a tail bound on $d_{\square}\left(f_{\boldsymbol{u}}, f_{\boldsymbol{w}}\right)$ similar to the one of Lemma 18 holds if instead of sampling an $f_{\boldsymbol{w}}$-random word for some word $\boldsymbol{w}$, we sample a subsequence $\boldsymbol{u}=\operatorname{sub}(\ell, \boldsymbol{w})$.

Lemma 20. For all $\varepsilon>0$ there is an $\ell_{0}$ such that for any $\boldsymbol{w} \in\{0,1\}^{n}$ and $n \geq \ell \geq \ell_{0}$ the random word $\boldsymbol{u}=\operatorname{sub}(\ell, \boldsymbol{w})$ satisfies

$$
\mathbb{P}\left(d_{\square}\left(f_{\boldsymbol{u}}, f_{\boldsymbol{w}}\right) \geq \varepsilon\right) \leq 4 \ell e^{-\varepsilon^{2} \ell / 300}
$$

Proof. For given $\varepsilon>0$ we choose $\ell_{0}=\ell_{0}(\varepsilon)$ to be sufficiently large and let $n \geq \ell>\ell_{0}$. Let $J \in\binom{[n]}{\ell}$ chosen uniformly at random and define $\boldsymbol{u}=\operatorname{sub}(J, \boldsymbol{w})=u_{1} \ldots u_{\ell}$. Consider the intervals $I_{i}=\left\{1, \ldots,\left\lfloor i \cdot \frac{n}{\ell}\right\rfloor\right\}, i \in[\ell]$. By definition of cut-distance and that of $f_{\boldsymbol{u}}$ and $f_{\boldsymbol{w}}$ we have

$$
\begin{aligned}
d_{\square}\left(f_{\boldsymbol{u}}, f_{\boldsymbol{w}}\right) & \leq 2 \sup _{x \in[0,1]}\left|\int_{0}^{x}\left(f_{\boldsymbol{u}}-f_{\boldsymbol{w}}\right)(t) \mathrm{d} t\right| \leq 2 \max _{i \in[\ell]}\left|\int_{0}^{\frac{\left|i_{i}\right|}{n}}\left(f_{\boldsymbol{u}}-f_{\boldsymbol{w}}\right)(t) \mathrm{d} t\right|+\frac{2}{\ell} \\
& \leq 2 \max _{i \in[\ell]}\left|\frac{1}{\ell} \sum_{j=1}^{i} u_{j}-\frac{1}{n} \sum_{j=1}^{\left|I_{i}\right|} w_{j}\right|+\frac{4}{\ell} .
\end{aligned}
$$

Thus the lemma follows once we have shown that

$$
\begin{equation*}
\mathbb{P}\left(\left|\sum_{j=1}^{i} u_{j}-\frac{\ell}{n} \sum_{j=1}^{\left|I_{i}\right|} w_{j}\right|>\frac{\varepsilon \ell}{3}\right) \leq 4 \ell e^{-\varepsilon^{2} \ell / 300} \quad \text { for all } i \in[\ell] . \tag{18}
\end{equation*}
$$

To show (18) consider for an $i \in[\ell]$

$$
Z_{i}=\left|J \cap I_{i}\right| \quad \text { and } \quad U_{i}=\sum_{j \in I_{i}} w_{j} \mathbf{1}\{j \in J\}=\sum_{j=1}^{Z_{i}} u_{j} .
$$

Both are hypergeometric random variables with expectations

$$
\mathbb{E}\left(Z_{i}\right)=\frac{\ell}{n}\left|I_{i}\right|=i \pm 1 \quad \text { and } \quad \mathbb{E}\left(U_{i}\right)=\frac{\ell}{n} \sum_{j=1}^{\left|I_{i}\right|} w_{j} .
$$

Moreover, $X \in\left\{Z_{i}, U_{i}\right\}$ satisfies the concentration bound (see (2.5), (2.6) and Theorem 2.10 from [26])

$$
\mathbb{P}(|X-\mathbb{E}(X)| \geq t) \leq 2 \exp \left(-\frac{t^{2}}{2(\mathbb{E}(X)+t / 3)}\right) \quad t \geq 0
$$

Thus, for $t=\varepsilon \ell / 10$ we conclude that with probability at least $1-4 \exp \left(-\varepsilon^{2} \ell / 300\right)$ we have

$$
\left|Z_{i}-i\right| \leq \frac{\varepsilon \ell}{10}+1 \quad \text { and } \quad\left|U_{i}-\frac{\ell}{n} \sum_{j=1}^{\left|I_{i}\right|} w_{j}\right| \leq \frac{\varepsilon \ell}{10} \quad \text { for all } i \in[\ell] .
$$

Finally, for a choice $J \in\binom{[n]}{\ell}$ which satisfies both of these properties we have for large $\ell_{0}$

$$
\sum_{j=1}^{i} u_{j}=\sum_{j=1}^{Z_{i}} u_{j} \pm \frac{\varepsilon \ell}{9}=\frac{\ell}{n} \sum_{j=1}^{\left|I_{i}\right|} w_{j} \pm \frac{\varepsilon \ell}{3}
$$

This proves (18) and the lemma follows.

## 5. Testing hereditary word properties

We now turn our focus to algorithmic considerations, specifically, to the study of testable word properties via word limits. The presentation below is heavily influenced by the derivation of analogous results for graphons by Lovász and Szegedy [31] (for related results concerning testability of permutation properties and limit objects see [25, 27]).

Let us briefly discuss the approach and the organization of this section. While the distance of choice for word limit is the interval-metric $d_{\square}$, property testing inherently deals with the normalized Hamming metric (between words $\boldsymbol{w} \in\{0,1\}^{n}$ and a word property $\mathcal{P}$ ) which we recall to be

$$
\begin{equation*}
d_{1}(\boldsymbol{w}, \mathcal{P})=\min _{\boldsymbol{u} \in \mathcal{P} \cap\{0,1\}^{n}} d_{1}(\boldsymbol{w}, \boldsymbol{u}) \quad \text { where } \quad d_{1}(\boldsymbol{w}, \boldsymbol{u})=\frac{1}{n} \sum_{i \in[n]}\left|w_{i}-u_{i}\right| . \tag{19}
\end{equation*}
$$

Being able to relate these two metrics by means of the limit theory is the essence of our approach and formally this is done via the notion of closure $\overline{\mathcal{P}}$ of a word property $\mathcal{P}$, defined as

$$
\overline{\mathcal{P}}=\left\{f \in \mathcal{W}: \text { There is a sequence }\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty} \text { in } \mathcal{P} \text { which converges to } f\right\} .
$$

Note that $\overline{\mathcal{P}}$ may not contain $f_{\boldsymbol{w}}$ for a $\boldsymbol{w} \in \mathcal{P}$. However, compactness of ( $\mathcal{W}, d_{\square}$ ) immediately implies that $f_{\boldsymbol{w}}$ is close to $\overline{\mathcal{P}}$ in the interval-metric if $\boldsymbol{w} \in \mathcal{P}$ is large enough (see Lemma 21). Alternative characterizations of $\overline{\mathcal{P}}$ for a hereditary word property $\mathcal{P}$ will be given in Proposition 22 which moreover shows that non-trivial hereditary $\mathcal{P}$ admits only $0-1$ valued $f \in \overline{\mathcal{P}}$ (up to a null measure set).

The $d_{1}$-metric in (19) can be analogously defined for functions $f$ and $\overline{\mathcal{P}}$ as follows (as usual let $d_{1}(f, \overline{\mathcal{P}})=\infty$ if $\left.\overline{\mathcal{P}}=\emptyset\right):$

$$
d_{1}(f, \overline{\mathcal{P}})=\inf _{g \in \overline{\mathcal{P}}} d_{1}(f, g) \quad \text { where } \quad d_{1}(f, g)=\|f-g\|_{1}=\int|f(x)-g(x)| \mathrm{d} x .
$$

With respect to the closure $\overline{\mathcal{P}}$ of a hereditary word property $\mathcal{P}$ Lemma 24 will allow us to "switch" from convergence in $d_{\square}$ to convergence in $d_{1}$, while Lemma 23 shows $d_{1}(\boldsymbol{w}, \mathcal{P}) \leq d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)$, thus allowing to pass from $\overline{\mathcal{P}}$ back to $\mathcal{P}$. With these results at hand we then give the proof of Theorem 4 at the end of this section.

We start with the following.

Lemma 21. For all $\delta>0$ there is an $n_{0}$ such that any $\boldsymbol{w} \in \mathcal{P}$ of length $n>n_{0}$ satisfies

$$
d_{\square}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)=\inf _{g \in \overline{\mathcal{P}}} d_{\square}\left(f_{\boldsymbol{w}}, g\right)<\delta .
$$

Proof. Let $\delta>0$ be given and for a contradiction suppose there is a sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ in $\mathcal{P}$ such that every $\boldsymbol{w}_{n}$ satisfies $d_{\square}\left(f_{\boldsymbol{w}_{n}}, \overline{\mathcal{P}}\right) \geq \delta$. By compactness of $\left(\mathcal{W}, d_{\square}\right)$ there is an $f \in \mathcal{W}$ such that (by passing to a subsequence $\left.{ }^{7}\right) f_{\boldsymbol{w}_{n}} \xrightarrow{马} f$. In particular, $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is a sequence in $\mathcal{P}$ which converges to $f$ and thus $f \in \overline{\mathcal{P}}$. However, this implies that $d_{\square}\left(f_{\boldsymbol{w}_{n}}, \overline{\mathcal{P}}\right) \leq d_{\square}\left(f_{\boldsymbol{w}_{n}}, f\right)<\delta$ for large enough $n$, a contradiction.

Recall that a property $\mathcal{P}$ is hereditary if $\operatorname{sub}(I, \boldsymbol{w}) \in \mathcal{P}$ for every $\boldsymbol{w} \in \mathcal{P}$ of length $n$ and every $I \subseteq[n]$.
Proposition 22. If $\mathcal{P}$ is a hereditary word property, then
$\overline{\mathcal{P}}=\{f \in \mathcal{W}: \mathbb{P}(\operatorname{sub}(\ell, f) \in \mathcal{P})=1$ for all $\ell \geq 1\}=\{f \in \mathcal{W}: t(\boldsymbol{u}, f)=0$ for all $\boldsymbol{u} \notin \mathcal{P}\}$.
Moreover, if $\mathcal{P}$ does not contain all words, then every $f \in \overline{\mathcal{P}}$ is 0-1 valued except maybe on a set of null measure.
Proof. The second equality holds since for each integer $\ell \geq 1$ we have

$$
\begin{equation*}
0=\mathbb{P}(\operatorname{sub}(\ell, f) \notin \mathcal{P})=\sum_{\boldsymbol{u} \in\{0,1\}^{\ell} \backslash \mathcal{P}} \mathbb{P}(\operatorname{sub}(\ell, f)=\boldsymbol{u})=\sum_{\boldsymbol{u} \in\{0,1\}^{\ell} \backslash \mathcal{P}} t(\boldsymbol{u}, f) . \tag{20}
\end{equation*}
$$

To show the first equality recall from Corollary 19 that $(\operatorname{sub}(\ell, f))_{\ell \rightarrow \infty}$ converges to $f$ a.s. Hence, if moreover $\mathbb{P}(\operatorname{sub}(\ell, f) \in \mathcal{P})=1$ holds for every $\ell$, then there is a sequence of words from $\mathcal{P}$ which converges to $f$, showing that $f \in \overline{\mathcal{P}}$. For the converse, let $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ be a sequence of words in $\mathcal{P}$ that converges to $f \in \overline{\mathcal{P}}$, i.e., $\lim _{n \rightarrow \infty} t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)=t(\boldsymbol{u}, f)$ for every word $\boldsymbol{u}$. In particular, if $\boldsymbol{u} \notin \mathcal{P}$ then $t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)=0$ by heredity of $\mathcal{P}$ and thus $t(\boldsymbol{u}, f)=0$. By (20) we then obtain $\mathbb{P}(\operatorname{sub}(\ell, f) \notin \mathcal{P})=0$.

Finally, suppose that $f \in \overline{\mathcal{P}}$ and that there is a $\boldsymbol{u} \in\{0,1\}^{\ell} \backslash \mathcal{P}$ for some $\ell$. Let $\boldsymbol{X}=\left(X_{1}, \ldots, X_{\ell}\right)$ be uniformly chosen in $[0,1]^{\ell}$, then the characterization of $\overline{\mathcal{P}}$ and (2) yields

$$
0=\mathbb{P}(\operatorname{sub}(\ell, f) \notin \mathcal{P}) \geq t(\boldsymbol{u}, f)=\ell!\int_{\substack{x_{1}, \ldots, x_{\ell} \in \in^{-1}(\mathrm{l} 0,1 \mathrm{D}) \\ 0 \leq x_{1}<\ldots<x_{\ell} \leq 1}} \prod_{i \in[\ell]} f^{u_{i}}\left(x_{i}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell} .
$$

Thus, $f^{-1}(] 0,1[)$ has null Lebesgue measure.
Lemma 23. If $\mathcal{P}$ is a hereditary word property and $\boldsymbol{w}$ is a word, then $d_{1}(\boldsymbol{w}, \mathcal{P}) \leq d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)$.
Proof. The claim is obvious if $\overline{\mathcal{P}}$ is empty or $\boldsymbol{w} \in \mathcal{P}$. Let $\delta>0$ and let $\boldsymbol{w} \notin \mathcal{P}$ be a word of length $n$. By definition there is a $g \in \overline{\mathcal{P}}$ such that $d_{1}\left(f_{\boldsymbol{w}}, g\right) \leq d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)+\delta$ and by Proposition $22 g$ is $0-1$ valued and $\mathbb{P}(\operatorname{sub}(n, g) \in \mathcal{P})=1$ for all $n \geq 1$. In particular, $\mathbb{P}\left(\boldsymbol{w}^{\prime} \in \mathcal{P}\right)=1$ when $\boldsymbol{w}^{\prime}=\operatorname{sub}(\boldsymbol{X}, g)$, with $\boldsymbol{X}=\left(X_{1}, \ldots, X_{n}\right)$ and $X_{i}$ is uniformly chosen in the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. Since the probability (conditioned on $X_{i}$ ) that index $i$ contributes to $d_{1}\left(\boldsymbol{w}, \boldsymbol{w}^{\prime}\right)$ is $g\left(X_{i}\right)$ if $w_{i}=0$ and $1-g\left(X_{i}\right)$ if $w_{i}=1$ we have

$$
\mathbb{E}\left(d_{1}\left(\boldsymbol{w}, \boldsymbol{w}^{\prime}\right)\right)=\left\|f_{\boldsymbol{w}}-g\right\|_{1}=d_{1}\left(f_{\boldsymbol{w}}, g\right) \leq d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)+\delta .
$$

In particular, there exists $\widetilde{\boldsymbol{w}} \in \mathcal{P}$ for which $d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)+\delta \geq d_{1}(\boldsymbol{w}, \widetilde{\boldsymbol{w}}) \geq d_{1}(\boldsymbol{w}, \mathcal{P})$ holds. Since $\delta$ is arbitrary, the desired conclusion follows.
Lemma 24. If $\mathcal{P}$ is a hereditary word property and $\left(f_{n}\right)_{n \rightarrow \infty}$ is a sequence of functions in $\mathcal{W}$ such that $d_{\square}\left(f_{n}, \overline{\mathcal{P}}\right) \rightarrow 0$, then $d_{1}\left(f_{n}, \overline{\mathcal{P}}\right) \rightarrow 0$.

[^7]Proof. We may assume that $\mathcal{P}$ is not the set of all words, otherwise $\overline{\mathcal{P}}=\mathcal{W}$ and the lemma follows. Let $\left(f_{n}\right)_{n \rightarrow \infty}$ with $d_{\square}\left(f_{n}, \overline{\mathcal{P}}\right) \rightarrow 0$ be given. By definition there is a sequence $\left(\varepsilon_{n}\right)_{n \rightarrow \infty}$ that converges to 0 , and a sequence $\left(g_{n}\right)_{n \rightarrow \infty}$ in $\overline{\mathcal{P}}$ such that

$$
d_{\square}\left(f_{n}, g_{n}\right) \leq d_{\square}\left(f_{n}, \overline{\mathcal{P}}\right)+\varepsilon_{n} \quad \text { for all } n
$$

Since $\left(\mathcal{W}, d_{\square}\right)$ is compact we may assume (by passing to a subsequence) that $g_{n} \rightrightarrows f$ for some $f \in \mathcal{W}$. By definition of closure there is an increasing function $m=m(n)$ such that for every $g_{n} \in \overline{\mathcal{P}}$ there is a word $\boldsymbol{w}_{m} \in \mathcal{P}$ with $d_{\square}\left(f_{\boldsymbol{w}_{m}}, g_{n}\right) \leq \varepsilon_{n}$. Since $d_{\square}\left(f_{\boldsymbol{w}_{m}}, f\right) \leq d_{\square}\left(f_{\boldsymbol{w}_{m}}, g_{n}\right)+d_{\square}\left(g_{n}, f\right) \leq$ $2 \varepsilon_{n} \rightarrow 0$ when $n \rightarrow \infty$, it follows that $f_{\boldsymbol{w}_{m}} \xrightarrow{G} f$ or in other words, that $\left(\boldsymbol{w}_{m}\right)_{m \rightarrow \infty}$ converges to $f$ and thus $f \in \overline{\mathcal{P}}$. Moreover, by Proposition 22, we get that $f$ is $0-1$ valued. Consider the Lebesgue measurable sets $\Omega_{b}=f^{-1}(b)$ for $b \in\{0,1\}$. Then

$$
d_{1}\left(f_{n}, f\right)=\left\|f_{n}-f\right\|_{1}=\int_{\Omega_{0}} f_{n}+\int_{\Omega_{1}}\left(1-f_{n}\right) .
$$

In case $\Omega_{0}, \Omega_{1}$ are intervals we conclude from $\lim _{n \rightarrow \infty} d_{\square}\left(f_{n}, f\right)=0$ that

$$
\lim _{n \rightarrow \infty} \int_{\Omega_{0}} f_{n}=\int_{\Omega_{0}} f=0 \quad \text { and } \quad \lim _{n \rightarrow \infty} \int_{\Omega_{1}}\left(1-f_{n}\right)=\int_{\Omega_{1}}(1-f)=0 .
$$

By standard limiting arguments this extends to finite unions of intervals and finally to all Lebesgue measurable sets, so we conclude that $d_{1}\left(f_{n}, f\right) \rightarrow 0$ when $n \rightarrow \infty$ and the lemma follows since $f \in \overline{\mathcal{P}}$.

Finally, we are ready to derive the main result of this section, that any hereditary word property is testable.

Proof of Theorem 4. Let $\mathcal{P}$ be a hereditary word property and $\overline{\mathcal{P}}$ its closure. Let $\mathcal{P}^{\prime}\left(\frac{1}{i}\right)$ be the collection of words $\boldsymbol{v}$ that either belong to $\mathcal{P}$ or that satisfy $d_{\square}\left(f_{\boldsymbol{v}}, \overline{\mathcal{P}}\right) \leq \frac{1}{i}$. Let $\mathcal{P}^{\prime}=\bigcap_{i \in \mathbb{N}} \mathcal{P}^{\prime}\left(\frac{1}{i}\right)$ which we claim to be a test property for $\mathcal{P}$. Perfect completeness is clearly satisfied since $\mathcal{P}$ is hereditary, so when $\boldsymbol{w} \in \mathcal{P}$ then $\operatorname{sub}(\ell, \boldsymbol{w}) \in \mathcal{P} \subseteq \mathcal{P}^{\prime}$ with probability 1 .

To prove soundness let $\varepsilon>0$ be given. We need to show the existence of an $\ell(\varepsilon)$ such that any $\boldsymbol{w} \in\{0,1\}^{n}$ with $d_{1}(\boldsymbol{w}, \mathcal{P}) \geq \varepsilon$ satisfies $\mathbb{P}\left(\operatorname{sub}(\ell, \boldsymbol{w}) \in \mathcal{P}^{\prime}\right) \leq \frac{1}{3}$ for all $\ell(\varepsilon) \leq \ell \leq n$. We apply Lemma 24 and conclude that there is a $\delta=\delta(\varepsilon)>0$ such that $d_{1}(f, \overline{\mathcal{P}})<\varepsilon$ whenever $d_{\square}(f, \overline{\mathcal{P}})<\delta$. Further, we apply Lemma 20 and Lemma 21 with $\delta / 4$ to obtain $n_{0}$ and $\ell_{0}$. Finally, choose $\ell(\varepsilon) \geq \max \left\{n_{0}, \ell_{0}, 2^{15} \varepsilon^{-3}\right\}$.

With this choice of constants note that a word $\boldsymbol{v} \in \mathcal{P}^{\prime}$ of length $\ell \geq \ell(\varepsilon)$ satisfies $d_{\square}\left(f_{\boldsymbol{v}}, \overline{\mathcal{P}}\right) \leq \delta / 2$. Indeed, by definition of $\mathcal{P}^{\prime}$ this is clear if $\boldsymbol{v} \in \mathcal{P}^{\prime} \backslash \mathcal{P}$ and for $\boldsymbol{v} \in \mathcal{P}$ this follows from Lemma 21 and $\ell \geq \ell(\varepsilon)$. Now let $\boldsymbol{w} \in\{0,1\}^{n}$ with $d_{1}(\boldsymbol{w}, \mathcal{P}) \geq \varepsilon$ be given, let $\boldsymbol{u}=\operatorname{sub}(\ell, \boldsymbol{w})$ for some $\ell(\varepsilon) \leq \ell \leq n$ and for a contradiction assume that soundness does not hold. Then we conclude from the above that

$$
\mathbb{P}\left(d_{\square}\left(f_{\boldsymbol{u}}, \overline{\mathcal{P}}\right) \leq \delta / 2\right) \geq \mathbb{P}\left(\boldsymbol{u} \in \mathcal{P}^{\prime}\right)>1 / 3
$$

Further, by Lemma 20 and the choice of $\ell(\varepsilon)$ we have $\mathbb{P}\left(d_{\square}\left(f_{\boldsymbol{u}}, f_{\boldsymbol{w}}\right)<\delta / 4\right)>2 / 3$ and thus, there is a word $\boldsymbol{v}$ such that $d_{\square}\left(f_{\boldsymbol{v}}, f_{\boldsymbol{w}}\right)<\delta / 4$ and $d_{\square}\left(f_{\boldsymbol{v}}, \overline{\mathcal{P}}\right) \leq \delta / 2$ hold simultanously. Triangle inequality then gives $d_{\square}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)<\delta$, which by Lemma 24 and the choice of $\delta$, implies $d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)<\varepsilon$. Finally, Proposition 23 yields $d_{1}(\boldsymbol{w}, \mathcal{P}) \leq d_{1}\left(f_{\boldsymbol{w}}, \overline{\mathcal{P}}\right)<\varepsilon$ which is the desired contradiction.

## 6. Finite forcibility

In this section we investigate word limits that are prescribed by a finite number of subsequence densities. In particular, we prove Theorem 5 showing that piecewise polynomial functions are forcible. The proof relies on the following lemma which shows, among other, that moments of
cumulative distributions can be characterized by a finite number of subsequence densities of the distribution's mass density function.
Lemma 25. If $f:[0,1] \rightarrow[0,1]$ is a Lebesgue measurable function and $F(x)=\int_{0}^{x} f(t) \mathrm{d} t$, then for each $i, j \in \mathbb{N}$ we have

$$
\int x^{i} F(x)^{j} d x=\frac{i!j!}{(i+j+1)!} \sum_{\substack{\boldsymbol{u} \in\{0,1\}^{i+j+1} \\ u_{1}+\ldots+u_{i+j} \geq j}}\left({\left.\underset{j}{1+\ldots+u_{i+j}}\right) t(\boldsymbol{u}, f) . . ~ . ~ . ~}_{j} .\right.
$$

Proof. Observe that

$$
\begin{aligned}
\int x^{i} F(x)^{j} \mathrm{~d} x & =\int\left(\int_{0}^{x} d y\right)^{i}\left(\int_{0}^{x} f(z) d z\right)^{j} \mathrm{~d} x \\
& =\int\left(\int_{0 \leq y_{1}, \ldots, y_{i} \leq x} \mathrm{~d} y_{1} \ldots \mathrm{~d} y_{i}\right)\left(\int_{0 \leq z_{1}, \ldots, z_{j} \leq x} \prod_{k=1}^{j} f\left(z_{k}\right) \mathrm{d} z_{1} \ldots \mathrm{~d} z_{j}\right) \mathrm{d} x \\
& =i!j!\int\left(\int_{0 \leq y_{1}<\ldots<y_{i} \leq x} \mathrm{~d} y_{1} \ldots \mathrm{~d} y_{i}\right)\left(\int_{0 \leq z_{1}<\ldots<z_{j} \leq x} \prod_{k=1}^{j} f\left(z_{k}\right) \mathrm{d} z_{1} \ldots \mathrm{~d} z_{j}\right) \mathrm{d} x \\
& =i!j!\sum_{S \subseteq[i+j]:|S|=j} \int_{0 \leq x_{1}<\ldots<x_{i+j} \leq x} \prod_{s \in S} f\left(x_{s}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{i+j} \mathrm{~d} x .
\end{aligned}
$$

Since

$$
1=\prod_{s \in[i+j] \backslash S}\left(f\left(x_{s}\right)+\left(1-f\left(x_{s}\right)\right)\right)=\sum_{U \subseteq[i+j]: S \subseteq U}\left(\prod_{s \in U \backslash S} f\left(x_{s}\right)\right)\left(\prod_{s \notin U}\left(1-f\left(x_{s}\right)\right)\right),
$$

we get

$$
\begin{aligned}
\int x^{i} F(x)^{j} \mathrm{~d} x & =i!j!\sum_{U \subseteq[i+j]:|U| \geq j}\binom{|U|}{j} \int_{0 \leq x_{1}<\ldots<x_{i+j} \leq x} \prod_{s \in U} f\left(x_{s}\right) \prod_{s \notin U}\left(1-f\left(x_{s}\right)\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{i+j} \mathrm{~d} x \\
& =\frac{i!j!}{(i+j+1)!} \sum_{\substack{u \in\{0,1\}^{i+j+1} \\
u_{1}+\ldots+u_{i+j} \geq j}}\binom{u_{1}+\ldots+u_{i+j}}{j} t(\boldsymbol{u}, f) .
\end{aligned}
$$

We next prove this section's main result concerning the finite forcibility of piecewise polynomial functions.

Proof of Theorem 5. Let $f$ be a piecewise polynomial function given with the corresponding polynomials $P_{1}(x), \ldots, P_{k}(x)$ and the corresponding intervals $\left(I_{1}, \ldots, I_{k}\right)$ ordered by their natural appearance in $[0,1]$. Thus, for any $i \in[k]$ and $x \in I_{i}$ we have $f(x)=P_{i}(x)$.

For each $i \in[k]$ and $x \in I_{i}$ we define

$$
Q_{i}(x)=\int_{I_{i} \cap[0, x]} P_{i}(t) \mathrm{d} t+\sum_{j=1}^{i-1} \int_{I_{j}} P_{j}(t) \mathrm{d} t .
$$

Then $Q_{i}$ is a polynomial on the interval $I_{i}$ and we extend it to the whole interval $[0,1]$. Further, $F(x)=\int_{0}^{x} f(t) \mathrm{d} t$ satisfies $F(x)=Q_{i}(x)$ for all $i \in[k]$ and $x \in I_{i}$. Next, let $d=\sum_{i \in[k]} \operatorname{deg}\left(Q_{i}\right)$ and define the polynomial

$$
P(x, y)=\left(y-Q_{1}(x)\right)^{2}\left(y-Q_{2}(x)\right)^{2} \ldots\left(y-Q_{k}(x)\right)^{2}=\sum_{1 \leq i+j \leq 2 d} c_{i j} x^{j} y^{i}
$$

for some coefficients $c_{i j}$. Since $P(x, F(x))=0$ for all $x \in[0,1]$ we have

$$
\begin{equation*}
\int_{0}^{1} P(x, F(x)) \mathrm{d} x=0 . \tag{21}
\end{equation*}
$$

This remains true when we remove duplicated $Q_{i}$ 's in the definition of $P(x, y)$, hence we may assume that the $Q_{i}$ 's are pairwise distinct.

By Lemma 25 we conclude that there is a list of words of length at most $2 d+1$, say, $\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{s}$ with $s \leq 2^{2 d+1}$, such that (21) already follows from the prescription of the values $t\left(\boldsymbol{u}_{i}, f\right), i \in[s]$. In particular, if $h \in \mathcal{W}$ satisfies $t\left(\boldsymbol{u}_{i}, h\right)=t\left(\boldsymbol{u}_{i}, f\right)$ for all $i \in[s]$, then $H(x)=\int_{0}^{x} h(t) \mathrm{d} t$ is continuous and satisfies $0=\int_{0}^{1} P(x, H(x)) \mathrm{d} x$. Since $P \geq 0$ this implies that $P(x, H(x))=0$ everywhere, and by the definition of $P(x, y)$ we conclude that for each $x \in[0,1]$ there is an $\ell=\ell_{H}(x) \in[k]$ such that $H(x)=Q_{\ell}(x)$.

Let $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{t}, b_{t}\right)$ be the intersection points of $Q_{1}, \ldots, Q_{k}$ ordered by their first coordinate (with ties broken arbitrarily) and let $a_{0}=0$ and $a_{t+1}=1$. Note that $t \leq\binom{ k}{2} \max _{i \in[k]} \operatorname{deg}\left(Q_{i}\right)$ as two distinct polynomials $Q_{i}$ and $Q_{j}$ have at $\operatorname{most} \max \left\{\operatorname{deg}\left(Q_{i}\right), \operatorname{deg}\left(Q_{j}\right)\right\}$ intersection points. Further, for an interval $\left(a_{i-1}, a_{i}\right), i \in[t+1]$, the function $\ell_{H}(x)$ must be constant on this interval. This is because $H$ is continuous and therefore if $x^{\prime}>x$ and $\ell_{H}(x) \neq \ell_{H}\left(x^{\prime}\right)$, then there must exist an intersection point in the interval $\left[x, x^{\prime}\right]$. We infer that $H$ is uniquely determined by the $(t+1)$ values $\ell_{H}(\cdot) \in[k]$ on the intervals $\left(a_{i-1}, a_{i}\right), i \in[t+1]$. Hence, there are at most $k^{t+1}$ such functions $H$, implying at most that many functions $h:[0,1] \rightarrow[0,1]$ such that $t\left(\boldsymbol{u}_{i}, h\right)=t\left(\boldsymbol{u}_{i}, f\right)$ for all $i \in[s]$.

To finish the proof note that by uniqueness of word limits, see Theorem 3, we can find for each $h$, which differs from $f$ by a non-zero measure set, a word $\boldsymbol{u}_{h}$ such that $t\left(\boldsymbol{u}_{h}, f\right) \neq t\left(\boldsymbol{u}_{h}, h\right)$. Thus, $f$ is uniquely determined by the densities of at most $s+k^{t+1} \leq(k+1)^{2 k^{2}\left(1+\max _{i} \operatorname{deg}\left(P_{i}\right)\right)}$ words.

Remark 26. The same proof for $k=1$ and $P_{1}(x)=a$ being constant yields an alternative proof of the second part of Theorem 1. In this case

$$
P(x, F(x))=(F(x)-a x)^{2}=F(x)^{2}-2 a x F(x)+a^{2} x^{2}
$$

and by Lemma 25, the fact $\int_{0}^{1} P(x, F(x)) \mathrm{d} x=0$ is determined by densities of words of length three.

## 7. Permutons from words Limits

In this section we use our results concerning word limits to give an alternative proof of two key results by Hoppen et al [24] concerning permutons, limits of permutation sequences, see Proposition 29 and Theorem 30. Overall, our approach gives a simpler proof for the existence of permutons, Theorem 30, due to the simpler objects (words and measurable transformations of the unit interval) on which our analysis is carried out. Moreover, in Proposition 29 we give a direct proof (avoiding compactness arguments) of the equivalence between $t$-convergence and convergence in the respective cut-distance, which we believe is both technically original and of independent interest.

For $n \in \mathbb{N}$ we write $\mathfrak{S}_{n}$ for the set of permutations of order $n$ and $\mathfrak{S}$ for the set of all finite permutations. Also, for $\sigma \in \mathfrak{S}_{n}$ and $\tau \in \mathfrak{S}_{k}$ we write $\Lambda(\tau, \sigma)$ for the number of copies of $\tau$ in $\sigma$, that is, the number of $k$-tuples $1 \leq x_{1}<\cdots<x_{k} \leq n$ such that for every $i, j \in[k]$

$$
\sigma\left(x_{i}\right) \leq \sigma\left(x_{j}\right) \quad \text { iff } \quad \tau(i) \leq \tau(j) .
$$

The density of copies of $\tau$ in $\sigma$, denoted by $t(\tau, \sigma)$, was defined as the probability that $\sigma$ restricted to a randomly chosen $k$-tuple of $[n]$ yields a copy of $\tau$, that is

$$
t(\tau, \sigma)= \begin{cases}\binom{n}{k}^{-1} \Lambda(\tau, \sigma) & \text { if } n \geq k \\ 0 & \text { otherwise }\end{cases}
$$

Following [24, Definition 1.2], a sequence $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ of permutations, with $\sigma_{n} \in \mathfrak{S}_{n}$ for each $n \in \mathbb{N}$, is said to be convergent if $\lim _{n \rightarrow \infty} t\left(\tau, \sigma_{n}\right)$ exists for every permutation $\tau \in \mathfrak{S}$. A permuton is a probability measure $\mu$ on the Borel $\sigma$-algebra on $[0,1] \times[0,1]$ that has uniform marginals, that is, for every measurable set $A \subseteq[0,1]$ one has

$$
\mu(A \times[0,1])=\mu([0,1] \times A)=\lambda(A) .
$$

The collection of permutons is denoted by $\mathcal{Z}$. It turns out that every permutation may be identified with a permuton which preserves the sub-permutation densities. Indeed, given a permutation $\sigma \in \mathfrak{S}_{n}$ we define the permuton $\mu_{\sigma}$ associated to $\sigma$ in the following way. First, for $i, j \in[n]$ define

$$
B_{i, j}=B_{i} \times B_{j} \quad \text { where } \quad B_{i}= \begin{cases}{\left[\frac{i-1}{n}, \frac{i}{n}\right)} & \text { if } i \neq n, \\ {\left[\frac{n-1}{n}, 1\right]} & \text { otherwise } .\end{cases}
$$

and note that $B_{i, j}$ has Lebesgue measure $\lambda^{(2)}\left(B_{i, j}\right)=1 / n^{2}$ for every $i, j \in[n]$. For every measurable set $E \subseteq[0,1]^{2}$ we let

$$
\mu_{\sigma}(E)=\sum_{i=1}^{n} n \lambda^{(2)}\left(B_{i, \sigma(i)} \cap E\right)=\int_{E} n \mathbf{1}\{\sigma(\lceil n x\rceil)=\lceil n y\rceil\} \mathrm{d} x \mathrm{~d} y
$$

It is easy to see that $\mu_{\sigma} \in \mathcal{Z}$.
We next argue that the densities of sub-permutations is preserved by $\mu_{\sigma}$. First, let us explain what we mean by sub-permutation densities for a permuton. Given $\mu \in \mathcal{Z}$ and $k \in \mathbb{N}$, we sample $k$ points $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{k}, Y_{k}\right)$, where each $\left(X_{i}, Y_{i}\right)$ is sampled independently accordingly to $\mu$. Then, if $\sigma, \pi \in \mathfrak{S}_{k}$ are two permutations such that

$$
X_{\pi(1)} \leq \cdots \leq X_{\pi(k)} \quad \text { and } \quad Y_{\sigma(1)} \leq \cdots \leq Y_{\sigma(k)}
$$

we define the random sub-permutation $\operatorname{sub}(k, \mu) \in \mathfrak{S}_{k}$ by $\operatorname{sub}(k, \mu)=\sigma^{-1} \pi$.
Henceforth, let $\mu^{(k)}=\mu \otimes \cdots \otimes \mu$ be the $k$-fold product measure on $([0,1] \times[0,1])^{k}$. Given a permutation $\tau \in \mathfrak{S}_{k}$, the density of $\tau$ in $\mu$, denoted by $t(\tau, \mu)$, is defined as the probability that $\operatorname{sub}(k, \mu)$ equals $\tau$, that is

$$
t(\tau, \mu)=k!\int \mathbf{1}\left\{x_{1}<\cdots<x_{k}, y_{\tau^{-1}(1)}<\cdots<y_{\tau^{-1}(k)}\right\} \mathrm{d} \mu^{(k)} .
$$

It is easily shown (see [24, Lemma 3.5] for a proof) that given any permutations $\sigma \in \mathfrak{S}_{n}$ and $\tau \in \mathfrak{S}_{k}$ we have

$$
\begin{equation*}
\left|t(\tau, \sigma)-t\left(\tau, \mu_{\sigma}\right)\right| \leq\binom{ k}{2} \frac{1}{n} \tag{22}
\end{equation*}
$$

In particular, (22) implies that a sequence of permutations $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ converges if and only if $\left(t\left(\tau, \mu_{\sigma_{n}}\right)\right)_{n \rightarrow \infty}$ is convergent for every permutation $\tau \in \mathfrak{S}$, and thus we may talk about permutations and permutons as the "same" object. We say that a sequence of permutons $\left(\mu_{n}\right)_{n \rightarrow \infty}$ is $t$-convergent if $\left.\left(t\left(\tau, \mu_{n}\right)\right)_{n \rightarrow \infty}\right)$ converges for every $\tau \in \mathfrak{S}$.

As in the case of words one can define a metric $d_{\square}$ on $\mathcal{Z}$ so that for all $\tau \in \mathfrak{S}$ the maps $t(\tau, \cdot)$ are Lipschitz continuous with respect to $d_{\square}$. Indeed, given two permutons $\mu, \nu \in \mathcal{Z}$ define

$$
d_{\square}(\mu, \nu)=\sup _{I, J \subseteq[0,1]}|\mu(I \times J)-\nu(I \times J)|,
$$

where the supremum is taken over all intervals in $[0,1]$. Next, we establish that $t(\tau, \cdot)$ is Lipschitz continuous with respect to $d_{\square}$ via the following result, which is the permuton analogue of Lemma 11.
Lemma 27. Given a permutation $\tau \in \mathfrak{S}_{k}$, for all permutons $\mu, \nu \in \mathcal{Z}$ we have

$$
|t(\tau, \mu)-t(\tau, \nu)| \leq k_{24}^{2} d_{\square}(\mu, \nu) .
$$

Proof. Given $\vec{x}, \vec{y} \in[0,1]^{k}$, we denote by $(\vec{x}, \vec{y})$ the vector of pairs $\left(x_{j}, y_{j}\right)$ for $j \in[k]$. Define

$$
\begin{equation*}
E^{\tau}=\left\{(\vec{x}, \vec{y}) \in([0,1] \times[0,1])^{k}: x_{1}<\cdots<x_{k}, y_{\tau^{-1}(1)}<\cdots<y_{\tau^{-1}(k)}\right\} . \tag{23}
\end{equation*}
$$

Then, we have $t(\tau, \mu)=k!\mu^{(k)}\left(E^{\tau}\right)$ and $t(\tau, \nu)=k!\nu^{(k)}\left(E^{\tau}\right)$. For $j \in[k]$, let

$$
Q_{j}=\mu^{(j)} \otimes \nu^{(k-j)}-\mu^{(j-1)} \otimes \nu^{(k-j+1)}
$$

and note that

$$
\frac{1}{k!}|t(\tau, \mu)-t(\tau, \nu)|=\left|\mu^{(k)}\left(E^{\tau}\right)-\nu^{(k)}\left(E^{\tau}\right)\right|=\left|\sum_{j=1}^{k} Q_{j}\left(E^{\tau}\right)\right| \leq \sum_{j=1}^{k}\left|Q_{j}\left(E^{\tau}\right)\right| .
$$

Let $(\vec{x}, \vec{y}) \in([0,1] \times[0,1])^{k}$. We define

$$
E_{j}^{\tau}(\vec{x}, \vec{y})= \begin{cases}{\left[0, x_{2}\right] \times\left[0, y_{\tau^{-1}(2)}\right]} & \text { for } j=1, \\ {\left[x_{j-1}, x_{j+1}\right] \times\left[y_{\tau^{-1}(j-1)}, y_{\tau^{-1}(j+1)}\right]} & \text { for } 2 \leq j \leq k-1, \\ {\left[x_{k-1}, 1\right] \times\left[y_{\tau^{-1}(k-1)}, 1\right]} & \text { for } j=k\end{cases}
$$

if $x_{1}<\cdots<x_{j-1}<x_{j+1}<\cdots<x_{k}$ and $y_{\tau^{-1}(1)}<\cdots<y_{\tau^{-1}(j-1)}<y_{\tau^{-1}(j+1)}<\cdots<y_{\tau^{-1}(k)}$, and $E_{j}^{\tau}(\vec{x}, \vec{y})=\emptyset$ otherwise. Thus $\left|\mu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)-\nu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)\right| \leq d_{\square}(\mu, \nu)$ for all $(\vec{x}, \vec{y})$. Letting $\vec{x}_{-j} \in[0,1]^{k-1}$ be the vector obtained by removing $x_{j}$ from $\vec{x} \in[0,1]^{k}$, for $2 \leq j \leq k-1$ we have that

$$
\begin{aligned}
\left|Q_{j}\left(E^{\tau}\right)\right| & =\left|\int\left(\mu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)-\nu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)\right) \mathrm{d} \mu^{(j-1)} \otimes \nu^{(k-j)}\left(\vec{x}_{-j}, \vec{y}_{-j}\right)\right| \\
& \leq \int\left|\mu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)-\nu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)\right| \mathrm{d} \mu^{(j-1)} \otimes \nu^{(k-j)}\left(\vec{x}_{-j}, \vec{y}_{-j}\right) \\
& \leq \int_{x_{1}<\cdots<x_{j-1}<x_{j+1}<\cdots<x_{k}}\left|\mu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)-\nu\left(E_{j}^{\tau}(\vec{x}, \vec{y})\right)\right| \mathrm{d} \mu^{(j-1)} \otimes \nu^{(k-j)}\left(\vec{x}_{-j}, \vec{y}_{-j}\right) \\
& \leq \frac{1}{(k-1)!} d_{\square}(\mu, \nu),
\end{aligned}
$$

and for $j=1$ and $j=k$ the same bound holds. Finally, summing for each $j \in[k]$ we obtain the bound.

In Hoppen et al. [24], the compactness of $\left(\mathcal{Z}, d_{\square}\right)$ is established and, as a consequence, also the equivalence between $t$-convergence and convergence in $d_{\square}$. In particular, they prove that for every convergent sequence of permutations $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ there is a permuton $\mu \in \mathcal{Z}$ such that $t\left(\tau, \sigma_{n}\right) \rightarrow$ $t(\tau, \mu)$ for all $\tau \in \mathfrak{S}$. The goal of this section is to give a new proof of these two results by using a more direct approach based on Theorem 3 and the permuton analogue of Proposition 14 based on Bernstein polynomials.

We start with a permuton analogue of Lemma 10.
Lemma 28. Let $\mu \in \mathcal{Z}$ be a permuton and let $i, j \in \mathbb{N}$. There exist a set $S_{i, j}$ of permutations of order $i+j+1$ and positive numbers $\left(C_{\tau}^{i, j}\right)_{\tau \in S_{i, j}}$ such that

$$
\int_{[0,1]^{2}} x^{i} y^{j} \mathrm{~d} \mu(x, y)=\sum_{\tau \in S_{i, j}} C_{\tau}^{i, j} t(\tau, \mu) .
$$

We discovered that a similar result was proved by Glebov, Grzesik, Klimošová and Král' [18, Theorem 3]. As the proofs are rather different we decide to include our proof here.

Proof. We proceed as in the proof of Lemma 10. First, since $\mu$ has uniform marginals we have that

$$
x^{i}=\left(\int_{[0, x] \times[0,1]} \mathrm{d} \mu\left(x^{\prime}, y^{\prime}\right)\right)^{i}=\int_{[0,1]^{2 i}} 1\left\{x_{1}, \ldots, x_{i} \leq x\right\} \mathrm{d} \mu\left(x_{1}, y_{1}\right) \ldots \mathrm{d} \mu\left(x_{i}, y_{i}\right)
$$

and similarly

$$
y^{j}=\int_{[0,1]^{2 j}} \mathbf{1}\left\{y_{i+1}, \ldots, y_{i+j} \leq y\right\} \mathrm{d} \mu\left(x_{i+1}, y_{i+1}\right) \ldots \mathrm{d} \mu\left(x_{i+j}, y_{i+j}\right) .
$$

Whence, setting

$$
G_{U}(\vec{x}, x)=\mathbf{1}\left\{x_{1}, \ldots, x_{i} \leq x\right\} \prod_{u \in U} \mathbf{1}\left\{x_{i+u} \leq x\right\} \prod_{u \notin U} \mathbf{1}\left\{x<x_{i+u}\right\}
$$

and

$$
H_{S}(\vec{y}, y)=1\left\{y_{i+1}, \ldots, y_{i+j} \leq y\right\} \prod_{s \in S} 1\left\{y_{s} \leq y\right\} \prod_{s \notin S} 1\left\{y<y_{s}\right\}
$$

by the Fubini-Tonelli theorem, we have

$$
\begin{aligned}
x^{i} y^{j} & =\int_{[0,1]^{2(i+j)}} \mathbf{1}\left\{x_{1}, \ldots, x_{i} \leq x\right\} \mathbf{1}\left\{y_{i+1}, \ldots, y_{i+j} \leq y\right\} \mathrm{d} \mu^{(i+j)}(\vec{x}, \vec{y}) \\
& =\sum_{U \subseteq[j]} \sum_{S \subseteq[i]} \int_{[0,1]^{2(i+j)}} G_{U}(\vec{x}, x) H_{S}(\vec{y}, y) \mathrm{d} \mu^{(i+j)}(\vec{x}, \vec{y}) .
\end{aligned}
$$

Finally, by reordering the position of the coordinates below and above $x$, respectively, we have

$$
\int_{[0,1]^{2}} x^{i} y^{j} \mathrm{~d} \mu(x, y)=\sum_{k \in[j]} \sum_{\ell \in[i]}\binom{j}{k}\binom{i}{\ell} \frac{(i+k)!(j-k)!}{(i+j+1)!} \sum_{\sigma \in \mathfrak{S}_{i+j+1}: \sigma(i+k+1) \geq j+1} t(\sigma, \mu),
$$

where given distinct values $x_{1}, \ldots, x_{i+j} \in[0,1]$ and a $k$ element set $U \subseteq[j]$ the factor $(i+k)$ ! represents all the possible orderings of the $(i+k)$ element set $\left\{x_{m}: m \in[i] \vee m-i \in U\right\}$, the factor $(j-k)$ ! represents all possible orderings of the $(j-k)$ element set $\left\{x_{m}: m \in[j] \backslash U\right\}$, and the $(i+j+1)$ ! term in the denominator comes from the definition of $t(\sigma, \mu)$ for $\sigma \in \mathfrak{S}_{i+j+1}$.

As pointed out in [28], the previous result can be used to prove the uniqueness of the limit of a sequence of permutations as we did for limits of words by using Lemma 10. Indeed, suppose that $\mu, \nu \in \mathcal{Z}$ are two permutons such that $t(\sigma, \mu)=t(\sigma, \nu)$ for every finite permutation $\sigma \in \mathfrak{S}$. By Lemma 28 and the Stone-Weierstrass theorem we deduce that for every continuous function $h:[0,1]^{2} \rightarrow \mathbb{R}$ we have

$$
\int_{[0,1]^{2}} h(x, y) \mathrm{d} \mu(x, y)=\int_{[0,1]^{2}} h(x, y) \mathrm{d} \nu(x, y),
$$

which implies that $\mu=\nu$. On the other hand, Lemma 28 can also be used to establish the permuton analogue of Proposition 14, that $t$-convergence implies the convergence with respect to $d_{\square}$.

Proposition 29. If $\left(\mu_{n}\right)_{n \rightarrow \infty}$ is a sequence in $\mathcal{Z}$ which is $t$-convergent, then it is a Cauchy sequence with respect to $d_{\square}$. Moreover, if $\mu_{n} \xrightarrow{t} \mu$ for some $\mu \in \mathcal{Z}$, then $\mu_{n} \xrightarrow{马} \mu$.

Proof. Let $\varepsilon>0$ be fixed and let $r=\left\lceil(80 / \varepsilon)^{4}\right\rceil$. Let $S_{i, j} \subseteq \mathfrak{S}_{i+j+1}$ and $C_{\tau}^{i, j}$ be as in the statement of Lemma 28, define $C=\max \left\{C_{\tau}^{i, j}: \tau \in S_{i, j}, i, j \leq r\right\}$, and let

$$
\delta=\frac{\varepsilon}{C(2 r+1)!2^{4 r+3}}
$$

Let $n_{0}$ be sufficiently large so that for all $n, m \geq n_{0}$ we have

$$
\begin{equation*}
\left|t\left(\tau, \mu_{n}\right)-t\left(\tau, \mu_{m}\right)\right| \leq \delta \quad \text { for all } \tau \in \bigcup_{i \in[r]} \mathfrak{S}_{i} . \tag{24}
\end{equation*}
$$

Hence, for $i, j \leq r$ and $\nu=\mu_{n}-\mu_{m}$, by Lemma 28 and since $\left|\mathfrak{S}_{i+j+1}\right| \leq(2 r+1)$ !, we have

$$
\left|\int_{[0,1]^{2}} x^{i} y^{j} \mathrm{~d} \nu(x, y)\right|=\left|\sum_{\tau \in S_{i, j}} C_{\tau}^{i, j}\left(t\left(\tau, \mu_{n}\right)-t\left(\tau, \mu_{m}\right)\right)\right| \leq C(2 r+1)!\delta
$$

For $a, b \in[0,1]$, let $J_{a, b}=\mathbf{1}_{[0, a] \times[0, b]}$ and let $j_{a}, j_{b}$ be the largest integers such that $\frac{j_{a}}{r} \leq a$ and $\frac{j_{b}}{r} \leq b$. Recall that the Bernstein polynomial of $J_{a, b}$ is denoted by $B_{r, J_{a, b}}$ and observe that

$$
\begin{aligned}
\left|\int B_{r, J_{a, b}}(x, y) \mathrm{d} \nu(x, y)\right| & \leq \sum_{i=0}^{j_{a}} \sum_{j=0}^{j_{b}}\binom{r}{i}\binom{r}{j}\left|\int x^{i}(1-x)^{r-i} y^{j}(1-y)^{r-j} \mathrm{~d} \nu(x, y)\right| \\
& \leq \sum_{0 \leq i, j \leq r} \sum_{k=0}^{r-i} \sum_{\ell=0}^{r-j}\binom{r}{i}\binom{r}{j}\binom{r-i}{k}\binom{r-j}{\ell}\left|\int x^{i+k} y^{j+\ell} \mathrm{d} \nu(x, y)\right| \\
& \leq C 2^{4 r}(2 r+1)!\delta .
\end{aligned}
$$

Now, by Lemma 13 we have

$$
\begin{aligned}
|\nu([0, a] \times[0, b])| & =\left|\int \mathbf{1}_{[0, a] \times[0, b]}(x, y) \mathrm{d} \nu(x, y)\right| \\
& \leq\left|\int B_{r, J_{a, b}}(x, y) \mathrm{d} \nu(x, y)\right|+\left|\int\left(\mathbf{1}_{[0, a] \times[0, b]}(x, y)-B_{r, J_{a, b}}(x, y)\right) \mathrm{d} \nu(x, y)\right| \\
& \leq C 2^{4 r}(2 r+1)!\delta+\left(8 r^{-1 / 4}+2 r^{-1 / 2}\right),
\end{aligned}
$$

where the last inequality follows since $\mu_{n}$ and $\mu_{m}$ have uniform marginals. Putting everything together, by our choice of $r, \delta$ and $\nu$, we have

$$
d_{\square}\left(\mu_{n}, \mu_{m}\right) \leq 4 \sup _{a, b \in[0,1]}|\nu([0, a] \times[0, b])| \leq C 2^{4 r+2}(2 r+1)!\delta+40 r^{-1 / 4} \leq \varepsilon .
$$

For the second part just replace $\mu_{m}$ by $\mu$ in (24) and choose $\nu=\mu_{n}-\mu$. Then, repeat the above argument.

We can now give the alternative proof of the result of Hoppen et al [24] concerning the existence of a limit (permuton) for a convergent permutation sequence. Note that this limit is unique as discussed right after the proof of Lemma 28.

Theorem 30 (Hoppen et al. [24, Theorem 1.6]). For every convergent sequence of permutations $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ there exists a permuton $\mu \in \mathcal{Z}$ such that $\sigma_{n} \xrightarrow{t} \mu$.

Proof. Let $\left(\sigma_{n}\right)_{n \rightarrow \infty}$ be given and let $\left(\mu_{n}\right)_{n \rightarrow \infty}$ be the sequence of corresponding permutons. Given $x \in[0,1]$ and $n \in \mathbb{N}$, we define

$$
f_{n, x}(y)=\int_{0}^{x} n \mathbf{1}\left\{\sigma_{n}(\lceil n t\rceil)=\lceil n y\rceil\right\} \mathrm{d} t \quad \text { for all } y \in[0,1] .
$$

It is easy to see that
(i) $f_{n, x}(\cdot) \leq f_{n, x^{\prime}}(\cdot)$ a.e. for all $x \leq x^{\prime}$,
(ii) $f_{n, 0}(\cdot)=0$ a.e. for all $n \in \mathbb{N}$, and
(iii) $f_{n, 1}(\cdot)=1$ a.e. for all $n \in \mathbb{N}$.

We claim that $\left(f_{n, x}\right)_{n \rightarrow \infty}$ converges in $d_{\square}$ for all $x \in[0,1]$. Indeed, by Proposition $29,\left(\mu_{n}\right)_{n \rightarrow \infty}$ is a Cauchy sequence with respect to $d_{\square}$, and for every interval $I \subseteq[0,1]$

$$
\left|\int_{I}\left(f_{n, x}-f_{m, x}\right)(t) \mathrm{d} t\right|=\left|\mu_{n}([0, x] \times I)-\mu_{m}([0, x] \times I)\right| \leq d_{\square}\left(\mu_{n}, \mu_{m}\right) .
$$

Thus $\left(f_{n, x}\right)_{n \rightarrow \infty}$ is a Cauchy sequence in $\left(\mathcal{W}, d_{\square}\right)$ and therefore, by Theorem 17, it has a limit $f_{x} \in \mathcal{W}$. Furthermore, by the dominated convergence theorem, for all $x \in[0,1]$ we have

$$
\begin{equation*}
\int_{0}^{1} f_{x}(t) \mathrm{d} t=\lim _{n \rightarrow \infty} \int_{0}^{1} f_{n, x}(t) \mathrm{d} t=\lim _{n \rightarrow \infty} \frac{\lceil n x\rceil}{n}=x \tag{25}
\end{equation*}
$$

and, because of (i), for all $a, x, x^{\prime} \in[0,1]$,

$$
\begin{equation*}
\left|\int_{0}^{a} f_{x}(t) \mathrm{d} t-\int_{0}^{a} f_{x^{\prime}}(t) \mathrm{d} t\right| \leq\left|\int_{0}^{1}\left(f_{x}-f_{x^{\prime}}\right)(t) \mathrm{d} t\right|=\left|x-x^{\prime}\right| \tag{26}
\end{equation*}
$$

Given $0 \leq a<b \leq 1$ and $0 \leq c<d \leq 1$, we set

$$
\begin{equation*}
\tilde{\mu}([a, b) \times[c, d))=\int_{c}^{d} f_{[a, b)}(t) \mathrm{d} t, \tag{27}
\end{equation*}
$$

where $f_{[a, b)}(t)=\left(f_{b}-f_{a}\right)(t)$ for all $t \in[0,1]$. We also set $\tilde{\mu}([a, b) \times[c, 1])=\tilde{\mu}([a, b) \times[c, 1))$ and $\tilde{\mu}([a, 1] \times[c, d))=\tilde{\mu}([a, 1) \times[c, d))$ for all $0 \leq a<b \leq 1$ and $0 \leq c<d \leq 1$.

We claim that $\tilde{\mu}$ extends to a unique measure $\mu \in \mathcal{Z}$. Let $\mathcal{F}$ be the semiring on $[0,1] \times[0,1]$ consisting of boxes of the form $[a, b) \times[c, d),[a, 1] \times[c, d),[a, b) \times[c, 1]$ and $[a, 1] \times[c, 1]$, for $0 \leq a<b \leq 1$ and $0 \leq c<d \leq 1$. Due to (i) and (ii) we see that $\tilde{\mu} \geq 0$ and that $\tilde{\mu}$ is monotone on $\mathcal{F}$. Moreover, it is clear that $\tilde{\mu}$ is finitely additive since $f_{x}(t)$ is integrable for all $x \in[0,1]$ and in the following we show that $\tilde{\mu}$ is $\sigma$-additive on $\mathcal{F}$. Let $\left(I_{n} \times J_{n}\right)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint boxes such that $\bigcup_{n \in \mathbb{N}} I_{n} \times J_{n}=I \times J \in \mathcal{F}$. Without loss of generality, we assume that $I_{n}=\left[a_{n}, b_{n}\right)$ and $J_{n}=\left[c_{n}, d_{n}\right)$ for all $n \in \mathbb{N}$. Since $\tilde{\mu}$ is monotone we have $\sum_{i=0}^{n} \tilde{\mu}\left(I_{i} \times J_{i}\right) \leq \tilde{\mu}(I \times J)$ for all $n \in \mathbb{N}$ and thus $\sum_{n \in \mathbb{N}} \tilde{\mu}\left(I_{n} \times J_{n}\right) \leq \tilde{\mu}(I \times J)$. In order to prove the upper bound, let $\delta>0$ be arbitrary and define $I_{n}^{\prime}=\left[a_{n}-2^{-n} \delta, b_{n}+2^{-n} \delta\right) \cap[0,1]$ and $J_{n}^{\prime}=\left[c_{n}-2^{-n} \delta, d_{n}+2^{-n} \delta\right) \cap[0,1]$ for each $n \in \mathbb{N}$. Note that the closure of $I \times J$ is contained in $\bigcup_{n \in \mathbb{N}}^{\prime} I_{n}^{\prime} \times J_{n}^{\prime}$ and thus, as $[0,1] \times[0,1]$ is compact, there exists a finite covering $I \times J \subseteq\left(I_{n_{1}}^{\prime} \times J_{n_{1}}^{\prime}\right) \cup \cdots \cup\left(I_{n_{\ell}}^{\prime} \times J_{n_{\ell}}^{\prime}\right)$. Observe that

$$
\tilde{\mu}\left(I_{n_{i}}^{\prime} \times J_{n_{i}}^{\prime}\right)-\tilde{\mu}\left(I_{n_{i}} \times J_{n_{i}}\right)=\int_{I_{n_{i}}^{\prime} \backslash I_{n_{i}}} f_{J_{n_{i}}^{\prime}}(t) \mathrm{d} t+\int_{I_{n_{i}}}\left(f_{J_{n_{i}}^{\prime}}-f_{J_{n_{i}}}\right)(t) \mathrm{d} t \leq 2 \cdot 2^{-n_{i}} \delta+2 \cdot 2^{-n_{i}} \delta,
$$

where the inequality is due to $\left\|f_{J_{n_{i}}}\right\|_{\infty} \leq 1$ and (26). Then we have

$$
\tilde{\mu}(I \times J) \leq \sum_{i=0}^{\ell} \tilde{\mu}\left(I_{n_{i}}^{\prime} \times J_{n_{i}}^{\prime}\right) \leq \sum_{n \in \mathbb{N}}\left(\tilde{\mu}\left(I_{n} \times J_{n}\right)+4 \cdot 2^{-n} \delta\right) \leq \sum_{n \in \mathbb{N}} \tilde{\mu}\left(I_{n} \times J_{n}\right)+4 \delta,
$$

which implies $\tilde{\mu}(I \times J)=\sum_{n \in \mathbb{N}} \tilde{\mu}\left(I_{n} \times J_{n}\right)$ as $\delta>0$ was arbitrary. Therefore $\tilde{\mu}$ is a pre-measure on $\mathcal{F}$ and thus there exists a measure $\mu$ on the Borel sets extending $\tilde{\mu}$ (see Theorem 11.3 from [6]). Moreover, since $\tilde{\mu}$ is finite, it follows that $\mu$ is unique (see Theorem 10.3 from [6]). Finally, due to (iii) we have $f_{1}(\cdot)=1$ a.e. which, together with (25), imply that $\mu$ has uniform marginals and therefore $\mu \in \mathcal{Z}$.

To conclude that $\sigma_{n} \xrightarrow{t} \mu$, we note that by Lemma 27 it is enough to show that $d_{\square}\left(\sigma_{n}, \mu\right) \rightarrow 0$. If not, then there is an $\varepsilon>0$ and sequences $\left(x_{n}\right)_{n \rightarrow \infty}$ and $\left(a_{n}\right)_{n \rightarrow \infty}$ such that, without loss of generality, for all $n$ sufficiently large we have

$$
\int_{0}^{a_{n}} f_{n, x_{n}}(t) \mathrm{d} t \geq \mu\left(\left[0, x_{n}\right) \times\left[0, a_{n}\right]\right)+\varepsilon=\int_{0}^{a_{n}} f_{x_{n}}(t) \mathrm{d} t+\varepsilon
$$

Moreover, by compactness of $[0,1]$ we can find $a \in[0,1]$ such that (passing to a subsequence) $\left(a_{n}\right)_{n \rightarrow \infty}$ converges to $a$ and for all $n$ sufficiently large we have

$$
\left|\int_{0}^{a} f_{n, x_{n}}(t) \mathrm{d} t-\int_{0}^{a_{n}} f_{n, x_{n}}(t) \mathrm{d} t\right|,\left|\int_{0}^{a_{n}} f_{x_{n}}(t) \mathrm{d} t-\int_{0}^{a} f_{x_{n}}(t) \mathrm{d} t\right| \leq\left|a-a_{n}\right| \leq \frac{\varepsilon}{8} .
$$

Thus,

$$
\int_{0}^{a} f_{n, x_{n}}(t) \mathrm{d} t-\int_{0}^{a} f_{x_{n}}(t) \mathrm{d} t \geq \frac{3 \varepsilon}{4}
$$

Again by compactness, there exists an $x \in[0,1]$ such that (passing to a subsequence) $\left(x_{n}\right)_{n \rightarrow \infty}$ converges to $x$ and, by (26) applied with $x^{\prime}=x_{n}$, for all $n$ sufficiently large we can assume that $\int_{0}^{a}\left(f_{x}-f_{x_{n}}\right)(t) \mathrm{d} t \leq \frac{\varepsilon}{8}$. Finally, observing that $\int_{0}^{a}\left(f_{n, x}-f_{n, x_{n}}\right)(t) \mathrm{d} t \leq\left|x-x_{n}\right|$, taking $n$ sufficiently large so that $\left|x-x_{n}\right| \leq \frac{\varepsilon}{8}$ we conclude that

$$
\int_{0}^{a} f_{n, x}(t) \mathrm{d} t \geq \int_{0}^{a} f_{x}(t) \mathrm{d} t+\frac{\varepsilon}{2}
$$

contradicting the fact that $\left(f_{n, x}\right)_{n \rightarrow \infty}$ converges to $f_{x}$.

## 8. Extensions

In this section we consider two generalizations of our limit theory for binary words. First, to non-binary words, and then to higher dimensional array structures.
8.1. Non-binary words. Let $\Sigma$ be a finite alphabet. For a word $\boldsymbol{w} \in \Sigma^{n}$ and an interval $I \subseteq[n]$ let $N_{a}(\boldsymbol{w}, I)$ denote the number of occurrences of $a \in \Sigma$ in $\operatorname{sub}(I, \boldsymbol{w})$ and let $N_{a}(\boldsymbol{w})=N_{a}(\boldsymbol{w},[n])$. Moreover, as for the binary alphabet case, denote by $\binom{\boldsymbol{w}}{\boldsymbol{u}}$ the number of subsequences of $\boldsymbol{w}$ which coincide with $\boldsymbol{u}$ and, assuming the length of $\boldsymbol{u}$ is $\ell$, let $t(\boldsymbol{u}, \boldsymbol{w})$ be the probability that a randomly chosen $\ell$-subsequence of $\boldsymbol{w}$ yields a copy of $\boldsymbol{u}$. A sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words $\boldsymbol{w}_{n} \in \Sigma^{n}$ is called $o(1)$-uniform if for each $a \in \Sigma$ there is a density $d_{a}$ such that $N_{a}\left(\boldsymbol{w}_{n}, I\right)=d_{a}|I|+o(1) n$ holds for each interval $I \subseteq[n]$. We obtain the following analogue (generalization) of Theorem 1 for finite size alphabets.
Theorem 31. Given a sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words $\boldsymbol{w}_{n} \in \Sigma^{n}$ over the finite size alphabet $\Sigma$. If $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is o(1)-uniform, then for each $a \in \Sigma$ there is a density $d_{a} \in[0,1]$ such that for every $\ell \in \mathbb{N}$ and every word $\boldsymbol{u} \in \Sigma^{\ell}$ we have $\binom{\boldsymbol{w}_{n}}{\boldsymbol{u}}=\prod_{a \in \Sigma} d_{a}^{N_{a}(\boldsymbol{u})}\binom{n}{\ell}+o\left(n^{\ell}\right)$. Conversely, if for some collection of densities $\left\{d_{a} \in[0,1]: a \in \Sigma\right\}$ we have $\binom{\boldsymbol{w}_{n}}{\boldsymbol{u}}=\prod_{a \in \Sigma} d_{a}^{N_{a}(\boldsymbol{u})}\binom{n}{3}+o\left(n^{3}\right)$ for all words $\boldsymbol{u} \in \Sigma^{3}$, then $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is o(1)-uniform.

Proof. The first part of the theorem follows from Remark 12 (choosing, for each $a \in \Sigma$, the $g^{a}$ 's therein as the constant function $g^{a}=d_{a}$ ) by an argument similar to the one used in the first part of the proof of Lemma 11. For the second part, let us consider a letter $a \in \Sigma$ and a word $\boldsymbol{w}$ over $\Sigma$. We define the binary word $\boldsymbol{w}^{a}$ as the word obtained by replacing each letter $a$ in $\boldsymbol{w}$ by 1 and the remaining letters by 0 . Moreover, for $\boldsymbol{u} \in\{0,1\}^{\ell}$ we let $\Sigma_{a}(\boldsymbol{u})$ be the set of words $\boldsymbol{v} \in \Sigma^{\ell}$ such that $\boldsymbol{v}^{a}=\boldsymbol{u}$. Then, it is easy to see that

$$
\begin{equation*}
t\left(\boldsymbol{u}, \boldsymbol{w}^{a}\right)=\sum_{\boldsymbol{v} \in \Sigma_{a}(\boldsymbol{u})} t(\boldsymbol{v}, \boldsymbol{w}) . \tag{28}
\end{equation*}
$$

For each $a \in \Sigma$ we can thus define the sequence $\left(\boldsymbol{w}_{n}^{a}\right)_{n \rightarrow \infty}$ of words over the alphabet $\{0,1\}$ which, because of (28) and since $\sum_{b \neq a} d_{b}=1-d_{a}$, satisfies the counting property for subsequences of length 3 . From Theorem 1 and our working hypothesis we conclude that $\left(\boldsymbol{w}_{n}^{a}\right)_{n \rightarrow \infty}$ is $o(1)$-uniform over the alphabet $\{0,1\}$ and thus we deduce that $N_{a}\left(\boldsymbol{w}_{n}, I\right)=N_{1}\left(\boldsymbol{w}_{n}^{a}, I\right)=d_{a}|I|+o(1) n$ for all intervals $I \subseteq[n]$. By repeating the above argument for each letter in $\Sigma$ we conclude that $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is $o(1)$-uniform.

Similarly, one can obtain an analog of Theorem 3 concerning limits of convergent word sequences for larger alphabets. A sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words over the alphabet $\Sigma=\left\{a_{1}, \ldots, a_{k}\right\}$ is convergent if for all $\ell \in \mathbb{N}$ and $\boldsymbol{u} \in \Sigma^{\ell}$ the subsequence density $\left(\binom{\boldsymbol{w}_{n}}{u} /\binom{n}{\ell}\right)_{n \rightarrow \infty}$ converges. Moreover, given a $k$-tuple of functions $\boldsymbol{f}=\left(f^{a_{1}}, \ldots, f^{a_{k}}\right) \in \mathcal{W}^{k}$ such that $f^{a_{1}}(x)+\cdots+f^{a_{k}}(x)=1$ for almost all $x \in[0,1]$, we say that $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $\boldsymbol{f}=\left(f^{a_{1}}, \ldots, f^{a_{k}}\right)$ if for all $\ell \in \mathbb{N}$ and $\boldsymbol{u} \in \Sigma^{\ell}$ the subsequence density $\left(\binom{\boldsymbol{w}_{n}}{\boldsymbol{u}} /\binom{n}{\ell}\right)_{n \rightarrow \infty}$ converges to

$$
t(\boldsymbol{u}, \boldsymbol{f})=\ell!\int_{0 \leq x_{1}<\cdots<x_{\ell} \leq 1} \prod_{i \in \ell \ell]} f^{u_{i}}\left(x_{i}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{\ell} .
$$

For the case of non-binary alphabets, we obtain the following limit theorem.
Theorem 32 (Limits of convergent $k$-letter word sequences). Let $\Sigma=\left\{a_{1}, \ldots, a_{k}\right\}$.

- Each convergent sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words, $\boldsymbol{w}_{n} \in \Sigma^{n}$, converges to some vector $\boldsymbol{f}=$ $\left(f^{a_{1}}, \ldots, f^{a_{k}}\right) \in \mathcal{W}^{k}$ and $f^{a_{1}}(x)+\cdots+f^{a_{k}}(x)=1$ for almost all $x \in[0,1]$. Moreover, if $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $\boldsymbol{g}=\left(g^{a_{1}}, \ldots, g^{a_{k}}\right)$, then $f^{a_{i}}=g^{a_{i}}$ almost everywhere, for all $i \in[k]$.
- Conversely, for every vector $\boldsymbol{f}=\left(f^{a_{1}}, \ldots, f^{a_{k}}\right) \in \mathcal{W}^{k}$ which satisfies $f^{a_{1}}(x)+\cdots+f^{a_{k}}(x)=$ 1 for almost all $x \in[0,1]$ there is a sequence $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ of words $\boldsymbol{w}_{n} \in \Sigma^{n}$ which converges to $\boldsymbol{f}$.

Proof. The first part follows by reducing to the size two alphabet case. Indeed, fix $a_{i} \in \Sigma$. For each $n \in \mathbb{N}$ we define the word $\boldsymbol{w}_{n}^{a_{i}}$ as in the proof of Theorem 31 and thus we obtain a sequence $\left(\boldsymbol{w}_{n}^{a_{i}}\right)_{n \rightarrow \infty}$ of words over the binary alphabet, which we claim is convergent. Indeed, since $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ is convergent then each term in the RHS in (28) is convergent and thus $\left(t\left(\boldsymbol{u}, \boldsymbol{w}_{n}^{a_{i}}\right)\right)_{n \rightarrow \infty}$ is convergent. Therefore, Theorem 3 implies that $\left(\boldsymbol{w}_{n}^{a_{i}}\right)_{n \rightarrow \infty}$ converges to a (unique) $f^{a_{i}} \in \mathcal{W}$. In particular, by (7), the sequence $\left(f_{n}^{a_{i}}\right)_{n \rightarrow \infty}$ of functions associated to $\left(\boldsymbol{w}_{n}^{a_{i}}\right)_{n \rightarrow \infty}$ satisfies $f_{n}^{a_{i}} \xrightarrow{t} f^{a_{i}}$ and Proposition 14 implies that $f_{n}^{a_{i}} \xrightarrow{\square} f^{a_{i}}$ as well. The $k$-letters analog of Lemma 11, see Remark 12, and the analog of (7) for $k$-letters ${ }^{8}$ then yields that $\left(\boldsymbol{w}_{n}\right)_{n \rightarrow \infty}$ converges to $\boldsymbol{f}=\left(f^{a_{1}}, \ldots, f^{a_{k}}\right)$ and it is not hard to see that $f^{a_{1}}(x)+\cdots+f^{a_{k}}(x)=1$ for almost all $x \in[0,1]$.

To prove the second part, we exhibit a sequence of words which converges to a given $f=$ $\left(f^{a_{1}}, \ldots, f^{a_{k}}\right)$. Consider the $\boldsymbol{f}$-random letter $(X, Y) \in[0,1] \times \Sigma$ obtained by choosing $X$ uniformly in $[0,1]$ and, conditioned on $X=x$, choosing $Y$ to be $a_{i} \in \Sigma$ with probability $f^{a_{i}}(x)$. Next, for each positive integer $n$ choose $\boldsymbol{f}$-random letters $\left(X_{1}, Y_{1}\right), \ldots\left(X_{n}, Y_{n}\right)$ and a permutation $\sigma:[n] \rightarrow[n]$ such that $X_{\sigma(1)} \leq \cdots \leq X_{\sigma(n)}$. Then, define the $\boldsymbol{f}$-random word $\boldsymbol{w}_{n}=Y_{\sigma(1)} \ldots Y_{\sigma(n)}$. By fixing a letter $a_{i} \in \Sigma$ and replacing the $\boldsymbol{w}_{n}$ 's by $\boldsymbol{w}_{n}^{a_{i}}$ 's as above we obtain a sequence of $f^{a_{i}}$-random words over size two alphabets whose associated functions converge in the interval-norm to $f^{a_{i}}$ a.s. due to Corollary 19. Then, the argument shown in Lemma 11, see Remark 12, and the $k$-letters analog of (7) imply that the $\boldsymbol{f}$-random word sequence converges to $\boldsymbol{f}$.
8.2. Multidimensional arrays. For $n, d \geq 1$, a $d$-dimensional $\{0,1\}$-array, $d$-array for short, of size $n$ is a function $A:[n]^{d} \rightarrow\{0,1\}$ which labels each element of $[n]^{d}$ with a 0 or 1 . Note that for $d=1$ a 1 -array of size $n$ is just an $n$-letter word, and for $d=2$ a 2 -dimensional array is just a $n$-by- $n$ zero-one matrix. In general, given $d \geq 1$ and $\vec{m}=\left(m_{1}, \ldots, m_{d}\right) \in \mathbb{N}^{d}$ a $d$-array of index $\vec{m}$ is a labeling $B:\left[m_{1}\right] \times \cdots \times\left[m_{d}\right] \rightarrow\{0,1\}$. As in the other cases considered so far, we need to say what will be the notion of sub-array. First, consider the $d=2$ case, that is, the case of matrices. We say that a matrix $A$ contains a copy of a matrix $B$ if by deleting rows and columns from $A$ one ends with the matrix $B$. In other words, we say that $B \in\{0,1\}^{k \times m}$ is a sub-array of $A \in\{0,1\}^{n \times n}$ if there are indices $1 \leq i_{1}<\cdots<i_{k} \leq n$ and $1 \leq j_{1}<\cdots<j_{m} \leq n$ such that $A_{i_{r}, j_{s}}=B_{r, s}$ for all

[^8]$r \in[k]$ and $s \in[m]$. For higher dimensional arrays the idea is similar. We say that a $d$-array $A$ of size $n$ contains a copy of a $d$-array $B$ of index $\vec{m} \in[n]^{d}$ if there exists a set of indices
$$
L=\left\{\left(i_{j_{1}}^{1}, \ldots, i_{j_{d}}^{d}\right) \in[n]^{d}: j_{1} \in\left[m_{1}\right], \ldots, j_{d} \in\left[m_{d}\right]\right\},
$$
with $i_{1}^{k}<\cdots<i_{m_{k}}^{k}$ for each $k \in[d]$, such that $\left.A\right|_{L}=B$. We denote by $\binom{A}{B}$ the number of copies of $B$ in $A$ and write $t(B, A)$ for the density of $B$ in $A$, i.e.,
$$
t(B, A)=\frac{\binom{A}{B}}{\binom{n}{m_{1}} \ldots\binom{n}{m_{d}}} .
$$

As we did for words, we can define a notion of convergence for $d$-arrays in terms of sub-array densities. We say that a sequence $\left(A_{n}\right)_{n \rightarrow \infty}$ of $d$-arrays, with $A_{n} \in\{0,1\}^{[n]^{d}}$ for each $n \in \mathbb{N}$, is $t$-convergent if for every $d$-array $B$ the sequence $\left.\left(t\left(B, A_{n}\right)\right)\right)_{n \rightarrow \infty}$ converges. Along the same lines of the proof of Theorem 3 , one can show that $t$-convergence is "equivalent" to a higher order intervaldistance and thus one can prove that every $t$-convergent sequence of $d$-arrays $\left(A_{n}\right)_{n \rightarrow \infty}$ converges to a Lebesgue measurable function $f:[0,1]^{d} \rightarrow[0,1]$. Moreover, for every Lebesgue measurable function $f:[0,1]^{d} \rightarrow[0,1]$ there exists a sequence of $d$-arrays, which arise from a random sampling from $f$, that converges to $f$ a.s.

## 9. Concluding Remarks

We conclude with a discussion on some potential future research directions. A variety of applications use data structures and algorithms on strings/words. In many settings, it is reasonable to assume that strings are generated by a random source of known characteristics. Several basic (generic) probabilistic models have been proposed and are often encountered in the analysis of problems on words, among others; memoryless Markov, mixing and ergodic sources (for a detailed discussion see [37]). Our investigations suggest that a new probabilistic model for generating strings under which to analyze the behavior of algorithms on words is the random words from limits model of Section 4.4 (i.e., for $f \in \mathcal{W}$, the sequence of distributions on words $\left.(\operatorname{sub}(n, f))_{n \in \mathbb{N}}\right)$. For instance, one may consider variants of classical long-standing open problems on words such as the Longest Common Subsequence (LCS) problem, for which it was shown [14] in the mid 70's that two random words uniformly chosen in $\{0,1\}^{n}$ have a LCS of size proportional to $n$ plus low order terms. The exact value of the proportionality constant remains unknown, although good upper and lower bounds have been established [33]. Generalizing this model, one may consider two random strings $\operatorname{sub}\left(n, f_{1}\right)$ and $\operatorname{sub}\left(n, f_{2}\right)$ and ask for conditions on $f_{1}, f_{2} \in \mathcal{W}$ so that the expected length of the longest common subsequence is of size $o(n)$.

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## Appendix A.

In this section we give an alternative proof of Theorem 17 based on the regularity lemma for words which was introduced by Axenovich, Puzynina and Person in [5] to study decomposition of words into identical subsequences. For completeness, we give an (analytic) proof of the regularity lemma.

A measurable partition $\mathcal{P}$ of $[0,1]$ is a partition in which each atom is a measurable set of positive measure. Moreover, we say that $\mathcal{P}$ is an interval partition if every atom in $\mathcal{P}$ is a non-degenerate interval. In what follows, we will only consider measurable partitions $\mathcal{P}$ with a finite number of atoms which we denote by $|\mathcal{P}|$. Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ we say that $\mathcal{Q}$ refines $\mathcal{P}$, which we denote by $\mathcal{Q} \preceq \mathcal{P}$, if for every $P \in \mathcal{P}$ there are atoms $Q_{1}, \ldots, Q_{k} \in \mathcal{Q}$ such that $P=Q_{1} \cup \cdots \cup Q_{k}$. The common refinement of $\mathcal{P}$ and $\mathcal{Q}$ is the partition

$$
\mathcal{P} \wedge \mathcal{Q}=\{A \cap B: A \in \mathcal{P}, B \in \mathcal{Q} \text { such that } A \cap B \neq \emptyset\} .
$$

Moreover, given a measurable set $A$ we define the refinement of $\mathcal{P}$ by $A$ as the common refinement of $\mathcal{P}$ and the partition $\left\{A, A^{c}\right\}$.

Let $f:[0,1] \rightarrow \mathbb{R}$ be an integrable function and let $\mathcal{P}$ be a partition. As usual, let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. The conditional expectation of $f$ with respect to $\mathcal{P}$ is the function $\mathbb{E}(f \mid \mathcal{P})$ defined as

$$
\mathbb{E}(f \mid \mathcal{P})(x)=\sum_{P \in \mathcal{P}} \frac{\mathbf{1}_{P(x)}}{\lambda(P)} \int_{P} f(t) \mathrm{d} t
$$

for all $x \in[0,1]$. The energy of $\mathcal{P}$ with respect to $f$ is defined by

$$
\mathcal{E}_{f}(\mathcal{P})=\int_{0}^{1}(\mathbb{E}(f \mid \mathcal{P})(x))^{2} \mathrm{~d} x
$$

Note that $\mathcal{E}_{f}(\mathcal{P}) \leq\|f\|_{\infty}^{2}$. The following is a well known (and easily derived) result about conditional expectations.

Lemma 33. Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions such that $\mathcal{Q} \preceq \mathcal{P}$. Given any integrable function $f:[0,1] \rightarrow \mathbb{R}$, we have

$$
\int_{0}^{1} \mathbb{E}(f \mid \mathcal{P})(t) \mathbb{E}(f \mid \mathcal{Q})(t) \mathrm{d} t=\int_{0}^{1}(\mathbb{E}(f \mid \mathcal{P})(t))^{2} \mathrm{~d} t
$$

Our next result shows that every $[0,1]$-valued integrable function over the interval $[0,1]$ can be approximated by a step function, which is supported on a partition of "bounded complexity" (a somewhat related result by Feige et al., the so called Local Repetition Lemma, was obtained in [16, Lemma 2.4]).

Theorem 34. (Weak regularity lemma) Let $\varepsilon>0$ and let $\mathcal{P}$ be an interval partition of $[0,1]$. For every integrable function $f:[0,1] \rightarrow[0,1]$ there exists an interval partition $\mathcal{P}_{\varepsilon} \preceq \mathcal{P}$ such that $\left\|f-\mathbb{E}\left(f \mid \mathcal{P}_{\varepsilon}\right)\right\|_{\square} \leq \varepsilon$ and $\left|\mathcal{P}_{\varepsilon}\right| \leq|\mathcal{P}|+2 \varepsilon^{-2}$.
Proof. Set $\mathcal{P}_{1}=\mathcal{P}$ and suppose that $\left\|f-\mathbb{E}\left(f \mid \mathcal{P}_{1}\right)\right\|_{\square}>\varepsilon$, as otherwise the result is trivial. For $k \geq 1$, assume we have defined a sequence of interval partitions $\mathcal{P}_{k} \preceq \cdots \preceq \mathcal{P}_{1}$ such that $\left\|f-\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)\right\|_{\square}>$
$\varepsilon$. This implies that there is an interval $I_{k+1} \notin \mathcal{P}_{k}$ such that

$$
\begin{equation*}
\left|\int_{I_{k+1}}\left(f-\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)\right)(t) \mathrm{d} t\right|>\varepsilon \tag{29}
\end{equation*}
$$

Define $\mathcal{P}_{k+1}$ as the refinement of $\mathcal{P}$ by $I_{k+1}$. Since either $I_{k+1}$ can split two distinct intervals of $\mathcal{P}_{k}$ into two subintervals each, or split a single interval of $\mathcal{P}_{k}$ into three subintervals, we have $\left|\mathcal{P}_{k+1}\right| \leq\left|\mathcal{P}_{k}\right|+2$. From (29) and by the Cauchy-Schwarz inequality, we deduce that

$$
\begin{aligned}
\varepsilon^{2} & <\left(\int_{I_{k+1}}\left(\mathbb{E}\left(f \mid \mathcal{P}_{k+1}\right)(t)-\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)(t)\right) \mathrm{d} t\right)^{2} \\
& \leq \int_{0}^{1}\left(\mathbb{E}\left(f \mid \mathcal{P}_{k+1}\right)(t)-\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)(t)\right)^{2} \mathrm{~d} t \\
& =\int_{0}^{1}\left(\mathbb{E}\left(f \mid \mathcal{P}_{k+1}\right)(t)\right)^{2} \mathrm{~d} t-\int_{0}^{1}\left(\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)(t)\right)^{2} \mathrm{~d} t
\end{aligned}
$$

where the last equality follows from Lemma 33. Thus we have

$$
1 \geq\|f\|_{\infty}^{2} \geq \mathcal{E}_{f}\left(\mathcal{P}_{k+1}\right) \geq \mathcal{E}_{f}\left(\mathcal{P}_{k}\right)+\varepsilon^{2}
$$

and so, after at most $\varepsilon^{-2}$ iterations, one finds some $\ell \leq \varepsilon^{-2}+1$ which satisfies $\left\|f-\mathbb{E}\left(f \mid \mathcal{P}_{\ell}\right)\right\|_{\square} \leq \varepsilon$. Since $\left|\mathcal{P}_{k}\right| \leq\left|\mathcal{P}_{k+1}\right|+2$ for every $k \in[\ell]$, we get the claimed upper bound for $\left|\mathcal{P}_{\ell}\right|$.
Lemma 35 (Theorem 35.5 from [6]). Let $f:[0,1] \rightarrow \mathbb{R}$ be an integrable function, and let $\left(\mathcal{P}_{i}\right)_{i \in \mathbb{N}}$ be a sequence of partitions such that $\mathcal{P}_{i+1} \preceq \mathcal{P}_{i}$ for all $i \in \mathbb{N}$. Then the sequence $\left(\mathbb{E}\left(f \mid \mathcal{P}_{i}\right)\right)_{i \in \mathbb{N}}$ converges a.e. to $\mathbb{E}\left(f \mid \mathcal{P}_{\infty}\right)$, where $\mathcal{P}_{\infty}$ is the smallest $\sigma$-algebra containing each atom in $\left(\mathcal{P}_{i}\right)_{i \in \mathbb{N}}$.

Before providing an alternative proof of Theorem 17, we state some basic results from functional analysis. Given a normed vector space $(X,\|\cdot\|)$, the dual space $X^{*}$ of $X$ is the vector space of all linear and continuos functions from $X$ to $\mathbb{R}$. It turns out that $X^{*}$ is a normed vector space endowed with the operator norm $\|\varphi\|^{*}=\sup \{|\varphi(x)|:\|x\|=1\}$. On the other hand, the weak* topology on $X^{*}$ is defined as the smallest topology that makes the functionals $\varphi \mapsto \varphi(x)$ continuous for all $x \in X$. In particular, a sequence $\left(\varphi_{n}\right)_{n \in \mathbb{N}} \subseteq X^{*}$ converges to $\varphi \in X^{*}$ in the weak* topology if and only if $\varphi_{n}(x) \rightarrow \varphi(x)$ for all $x \in X$. One of the main reasons to use the weak* topology instead of the operator norm topology is the Banach-Alaoglu theorem (see Theorem 5.18 from [17]), which states that the unit ball $B=\left\{\varphi \in X^{*}:\|\varphi\|^{*} \leq 1\right\}$ is compact in the weak* topology.

A classical result in functional analysis states that $L^{1}([0,1])^{*}$ is isomorphic to $L^{\infty}([0,1])$ (see Theorem 6.15 from [17]). Thus, a sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ in $L^{\infty}([0,1])$ converges in the weak* topology if for every $g \in L^{1}([0,1])$ we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{0}^{1} f_{n}(x) g(x) \mathrm{d} x=\int_{0}^{1} f(x) g(x) \mathrm{d} x \tag{30}
\end{equation*}
$$

It is easily shown that $\mathcal{W}$ is a weak* closed subset of the unit ball in $L^{\infty}([0,1])$, i.e., if a sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ in $\mathcal{W}$ converges to $f \in L^{\infty}([0,1])$ in the weak ${ }^{*}$ topology then $f \in \mathcal{W}$. Indeed, letting $U_{\varepsilon}=\{x \in[0,1]: f(x) \geq 1+\varepsilon\}$ and $g=\mathbf{1}_{U_{\varepsilon}}$ in (30) we see that $(1+\varepsilon) \lambda\left(U_{\varepsilon}\right) \leq \lambda\left(U_{\varepsilon}\right)$ which implies that $\lambda\left(U_{\varepsilon}\right)=0$ for any $\varepsilon>0$ and so $f(x) \leq 1$ almost everywhere. On the other hand, letting $V_{\varepsilon}=\{x \in[0,1]: f(x) \leq-\varepsilon\}$ and $g=\mathbf{1}_{V_{\varepsilon}}$ we have $0 \leq-\varepsilon \lambda\left(V_{\varepsilon}\right)$ which implies $\lambda\left(V_{\varepsilon}\right)=0$ for any given $\varepsilon>0$, and thus $0 \leq f(x) \leq 1$ almost everywhere.

With these facts at hand we now give an alternative proof of the compactness of ( $\mathcal{W}, d_{\square}$ ).
Proof of Theorem 17. Let $\left(f_{n}\right)_{n \in \mathbb{N}}$ be any sequence in $\mathcal{W}$. Since $\mathcal{W}$ is a weak* closed subset of the unit ball in $L^{\infty}([0,1])$, by the Banach-Alaoglu theorem we may assume that $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges in
the weak* topology to some $f \in \mathcal{W}$. We claim that there are a collection of subsequences $\left(f_{n, k}\right)_{n \in \mathbb{N}}$, for $k \in \mathbb{N}$, satisfying the following properties.
(i) $f_{n, 0}=f_{n}$ for all $n \in \mathbb{N}$ and $\mathcal{P}_{0}=\{[0,1]\}$.
(ii) For every $k \geq 1$, the sequence $\left(f_{n, k}\right)_{n \in \mathbb{N}}$ is a subsequence of $\left(f_{n, k-1}\right)_{n \in \mathbb{N}}$.
(iii) For $k \geq 1$, there is an interval partition $\mathcal{P}_{k} \preceq \mathcal{P}_{k-1}$ such that $\left|\mathcal{P}_{k}\right| \leq 3 k^{3}$ and $\| f_{n, k}-$ $\mathbb{E}\left(f_{n, k} \mid \mathcal{P}_{k}\right) \|_{\square} \leq \frac{1}{k}$ for every $n \in \mathbb{N}$.
(iv) For all $k \geq 1$, the sequence $\left(\mathbb{E}\left(f_{n, k} \mid \mathcal{P}_{k}\right)\right)_{n \in \mathbb{N}}$ converges a.e. to $f_{k}^{*}=\mathbb{E}\left(f \mid \mathcal{P}_{k}\right)$.

Clearly, properties (ii), (iii), and (iv) hold vacuously for $k=0$. Assume we have constructed the sequence up to step $k$. We apply Theorem 34 , with $\varepsilon=\frac{1}{k+1}$ and initial partition $\mathcal{P}_{k}$, to each function in the sequence $\left(f_{n, k}\right)_{n \in \mathbb{N}}$ so that for every $n \in \mathbb{N}$ we get an interval partition $\mathcal{P}_{n, k} \preceq \mathcal{P}_{k}$, with $\left|\mathcal{P}_{n, k}\right| \leq\left|\mathcal{P}_{k}\right|+2(k+1)^{2} \leq 3(k+1)^{3}$ and such that $\left\|f_{n, k}-\mathbb{E}\left(f_{n, k} \mid \mathcal{P}_{n, k}\right)\right\|_{\square} \leq \frac{1}{k+1}$. For $n \in \mathbb{N}$, let $J_{n, k}=\left\{a_{n, 1}=0<\cdots<a_{n, \ell_{n}}=1\right\}$ be the set of points that define the intervals of $\mathcal{P}_{n, k}$. Note that $\ell_{n} \leq 1+3(k+1)^{3}$. By the pigeonhole principle there is an integer $\ell \leq 1+3(k+1)^{3}$ and a subsequence $\left(f_{n, k+1}\right)_{n \in \mathbb{N}}$ of $\left(f_{n, k}\right)_{n \in \mathbb{N}}$ such that $\ell_{n}=\ell$ for all $n \in \mathbb{N}$. Moreover, since $[0,1]$ is compact we may even assume that $a_{n, i} \rightarrow a_{i}$ for each $i \in[\ell]$, where $a_{1}=0 \leq \cdots \leq a_{\ell}=1$. Let $\mathcal{P}_{k+1} \preceq \mathcal{P}_{k}$ be the partition defined by $J_{k}=\left\{a_{1} \leq \cdots \leq a_{\ell}\right\}=\left\{a_{1}^{\prime}<\cdots<a_{\ell^{\prime}}^{\prime}\right\}$ for some $\ell^{\prime} \leq \ell$. Note that (ii) and (iii) hold because of the definition of $\left(f_{n, k+1}\right)_{n \in \mathbb{N}}$. Furthermore, because $\mathcal{P}_{k+1}$ is finite and since $\left(f_{n, k+1}\right)_{n \in \mathbb{N}}$ converges in the weak ${ }^{*}$ topology to $f$ we conclude that (iv) also holds.

Finally, by Lemma 35 we deduce that the sequence $\left(f_{k}^{*}\right)_{k \in \mathbb{N}}$ converges a.e. to $f_{\infty}=\mathbb{E}\left(f \mid \mathcal{P}_{\infty}\right)$. We claim that $\lim _{k \rightarrow \infty} d_{\square}\left(f_{k, k}, f_{\infty}\right)=0$. Indeed, given $\eta>0$, using the dominated convergence theorem, (iii) and (iv), we have for large $k \geq m \geq 3 \eta^{-1}$

$$
d_{\square}\left(f_{\infty}, f_{k, k}\right) \leq d_{\square}\left(f_{\infty}, f_{m}^{*}\right)+d_{\square}\left(f_{m}^{*}, \mathbb{E}\left(f_{k, k} \mid \mathcal{P}_{m}\right)\right)+d_{\square}\left(\mathbb{E}\left(f_{k, k} \mid \mathcal{P}_{m}\right), f_{k, k}\right) \leq \frac{\eta}{3}+\frac{1}{m}+\frac{\eta}{3} \leq \eta,
$$

where the second inequality follow from the fact that $\left(f_{k, k}\right)_{k \geq m}$ is a subsequence of $\left(f_{n, m}\right)_{n \in \mathbb{N}}$.
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[^1]:    ${ }^{1}$ We write $a \pm x$ to denote a number contained in the interval $[a-x, a+x]$.

[^2]:    ${ }^{2}$ Choosing the vertex set to be $\mathbb{Z}_{2 n}$ instead of $\mathbb{Z}_{n}$ avoids the graph having loops.

[^3]:    ${ }^{3}$ Word sequences with bounded lengths contain a subsequence of infinite length which is constant and due to convergence all members of the original sequence must agree with this constant eventually.

[^4]:    ${ }^{4}$ In [15], Cooper works in the context of subsets of $\mathbb{Z}_{n}$, calling a subset $S \subseteq \mathbb{Z}_{n} \varepsilon$-balanced if $D(S)=$ $\sup _{I \subseteq \mathbb{Z}_{n}}| | S \cap I\left|-\frac{|S|}{n}\right| I| | \leq \varepsilon n$, where the supremum is taken over all the intervals $I \subseteq \mathbb{Z}_{n}$. Defining the $n$-letter word $W_{S}=w_{1} \ldots w_{n}$, where $w_{i}=\mathbf{1}_{S}\{i\}$ for each $i \in[n]$, is easily seen that $S$ is $\varepsilon$-balanced if and only if $W_{S}$ is (|S|/n, $\varepsilon$ )-uniform.

[^5]:    ${ }^{5}$ To see (7), split $[0,1]$ into $n$ intervals of equal lengths. Let $A$ denote the event that $\ell$ independent uniform random points of $[0,1]$ land in different intervals and let $B$ be the event that, after reordering these points, say $x_{1}<\cdots<x_{\ell}$, we have $\left(f_{\boldsymbol{w}_{n}}\left(x_{1}\right), \ldots, f_{\boldsymbol{w}_{n}}\left(x_{\ell}\right)\right)=\boldsymbol{u}$. Then, $t\left(\boldsymbol{u}, f_{\boldsymbol{w}_{n}}\right)=\mathbb{P}(B \mid A) \mathbb{P}(A)+\mathbb{P}(B \mid \bar{A}) \mathbb{P}(\bar{A})$ and we further have $\mathbb{P}(B \mid A)=t\left(\boldsymbol{u}, \boldsymbol{w}_{n}\right)$ and $\mathbb{P}(A)=\prod_{i=1}^{\ell-1}(1-i / n)=1-O\left(n^{-1}\right)$.

[^6]:    ${ }^{6}$ Mixed in the sense that $X$ is continuous while $Y$ is discrete.

[^7]:    ${ }^{7}$ The term "passing to a subsequence" means considering a subsequence instead of the original sequence. However, to avoid making the notation more cumbersome, the subsequence keeps the same name as the original sequence.

[^8]:    ${ }^{8}$ Note that the proof of (7) given in the footnote 5 extends without change to the $k$-letters case.

